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Heat transfer in a rarefied gas enclosed between parallel plates: Role of boundary conditions

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The influence of boundary conditions of accomodation coefficients and Maxwellian diffuse specular reflection on heat transfer through a rarefied gas enclosed between two parallel plates is examined. An exact expression for heat transfer for accomodation coefficient boundary conditions and the Bhatnagar–Gross–Krook (BGK) model is constructed by using results of Cercignani and Pagani and Thomas, Chang, and Siewert. These results are compared with some variational results of Cipolla and Cercignani and some exact results of Thomas, Chang, and Siewert and Thomas for the BGK model and Maxwellian diffuse specular reflection boundary conditions. It is concluded that the two boundary conditions provide results that agree within about 3% for a range of Knudsen numbers and boundary parameters. It is found that the variational results are remarkably accurate for the BGK model and both types of boundary conditions. Further, it is noted that the heat transfer between parallel plates with different accommodation coefficients at the two surfaces can be calculated exactly by using a harmonic mean for each surface.

The problem of heat transfer in a rarefied gas enclosed between parallel plates has been considered by several authors. Some of the early work is described in Kennard,[1] and more recent work is described in the papers by Lees and Liu,[2] Cercignani and co-workers,[3–5] Loyalka,[6–8] Cipolla and Cercignani,[9] Cipolla,[10] Thomas, Chang, and Siewert,[11] and Thomas.[12] In particular, numerically accurate results for the Bhatnagar–Gross–Krook (BGK) model have been reported by Thomas, Chang,and Siewert and Thomas. For boundary conditions, the latter authors considered the Maxwellian diffuse-specular reflection. The boundary conditions of thermal accommodation coefficients were used by Bassanini, Cercignani,and Pagani,[4] who employed variational and numerical methods.

The purpose of this paper is to use the previous works to provide an appropriate set of comparisons for the heat transfer values predicted by the two different sets of boundary conditions. We begin by noting that if Q and Q_{fm} are the heat transfer rates for an arbitrary Knudsen number ($kn = l/\tilde{d} = 1/d$, where l is a mean-free-path) and free molecular limits, respectively, then for the thermal accomodation boundary condition

$$\frac{Q}{Q_{fm}} = \frac{Q_{fm1}}{Q_{fm}}\frac{Q_1}{Q_{fm1}} \times\left[1+\frac{Q_1}{Q_{fm1}}\left(\frac{1-\alpha_{t1}}{\alpha_{t1}}+\frac{1-\alpha_{t2}}{\alpha_{t2}}\right)\right]^{-1}, \tag{1}$$

which is a result of Cercignani and Pagani. Here, Q_1 and Q_{fm1} are the heat transfer rates for $\alpha_{t1}=\alpha_{t2}=1$, and α_{t1} and α_{t2} are the thermal accomodation coefficients at plates 1 and 2, respectively. Note that

$$\frac{Q_{fm1}}{Q_{fm}} = \frac{\alpha_{t1}+\alpha_{t2}-\alpha_{t1}\alpha_{t2}}{\alpha_{t1}\alpha_{t2}}. \tag{2}$$

A rational expression for Q_1/Q_{fm1} has been given by Cipolla and Cercignani

$$\frac{Q_1}{Q_{fm1}} = \frac{a+d}{a+bd+cd^2}, \tag{3}$$

where the coefficients a, b, c are given by

$$a = \left[\tfrac{9}{16}(J_2/J_1^2)-\tfrac{1}{2}\right]\left[4J_1/(3-\pi^{1/2}IJ_1)\right], \tag{4}$$

$$b = \frac{3-\pi^{1/2}IJ_1\left[\frac{9}{16}(J_2/J_1^2)+\frac{1}{2}\right]}{3-\pi^{1/2}IJ_1}, \tag{5}$$

$$c = -3/4J_1, \tag{6}$$

and for the BGK model:

$$J_1 = -\tfrac{15}{16}\sqrt{\pi}, \tag{7}$$

$$J_2 = \tfrac{13}{4}, \tag{8}$$

$$I = -\tfrac{7}{4}. \tag{9}$$

Cercignani and Cipolla have also reported heat transfer expressions for Maxwellian diffuse-specular reflection boundary conditions. Thus, if α is the coefficient of diffuse reflection at both surfaces, then

$$\frac{Q}{Q_{fm}} = \frac{a+d}{a+bd+cd^2}, \tag{10}$$

where now

$$a = (2-\alpha)\left[\tfrac{9}{16}(J_2/J_1^2)-\tfrac{1}{2}\right]\left[4J_1/(3-\sqrt{\pi}IJ_1)\right], \tag{11}$$

$$b = \frac{3-\sqrt{\pi}IJ_1\left[(2-\alpha)/2+\alpha\frac{9}{16}(J_2/J_1^2)\right]}{3-\sqrt{\pi}IJ_1}, \tag{12}$$

TABLE I. Comparison of Q/Q_{fm} for BGK model for thermal accommodation coefficient boundary condition (α_t, B.C.).

	α_t					
	1		0.826		0.759	
d	Variational	"Exact"	Variational	"Exact"	Variational	"Exact"
0.01	0.9924	0.992 484	0.9946	0.9947	0.9953	0.9954
0.1	0.9327	0.935 159	0.9517	0.9535	0.9577	0.9593
0.5	0.7653	0.768 262	0.8225	0.8250	0.8420	0.8442
1.0	0.6413	0.640 853	0.7176	0.7172	0.7451	0.7447
1.25	0.5952	0.593 761	0.6764	0.6750	0.7063	0.7050
1.5	0.5560	0.553 805	0.6402	0.6382	0.6719	0.6699
1.75	0.5219	0.519 333	0.6081	0.6056	0.6409	0.6385
2.0	0.4920	0.489 203	0.5793	0.5765	0.6130	0.6103
2.5	0.4418	0.438 66	0.5294	0.5264	0.5641	0.5612
3.0	0.4011	0.398 324	0.4877	0.4848	0.5227	0.5198
4.0	0.3390	0.336 724	0.4216	0.4191	0.4561	0.4536
5.0	0.2938	0.291 918	0.3715	0.3695	0.4048	0.4027
7.0	0.2320	0.230 810	0.3003	0.2990	0.3306	0.3291
10.0	0.1764	0.175 788	0.2334	0.2326	0.2594	0.2586

$$c = -\left(\frac{\alpha}{2-\alpha}\right)\frac{3}{4J_1}, \tag{13}$$

and J_1, J_2, and I are the same as given earlier.

Let us now consider the results obtained by Thomas, Chang and Siewert. These authors considered the boundary condition of diffuse-specular reflection ($\alpha_1 = \alpha_2$) only, but reported numerical results for $\alpha = 1$, 0.826, and 0.749. The case for $\alpha = 1$ is the case of diffuse reflection; hence, $\alpha_t = 1$, and therefore their results for this case can be used in expression (1) to provide numerically precise results for all α_{t1},α_{t2} also. The other results can provide a test of the accuracy of the results obtained by the use of expression (10) for the Maxwellian boundary condition. We have reported such comparisons in Tables I and II. It should also be noted that the integral variational results of Cercignani and Pagani and Loyalka are in excellent agreement with the Q_1/Q_{fm1} ($\alpha_{t1} = \alpha_{t2} = 1$) results of Thomas, Chang and Siewert, and hence we have not considered them here.

Let us consider next the work of Thomas, who has reported results for the Maxwellian boundary condition where $\alpha_1 \neq \alpha_2$. In Table III, we have compared these results with those obtained from the use of expression (1), with $\alpha_{t1} = \alpha_1$ and $\alpha_{t2} = \alpha_2$. Clearly, the two boundary conditions provide results that are close, provided one identifies α_t with α.

An interesting point examined by Thomas was that if one defines a harmonic mean $\bar{\alpha}$ by

$$\frac{1}{\bar{\alpha}} = \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right), \tag{14}$$

then the numerical results show that the use of such $\bar{\alpha}$ with $\alpha_1 = \bar{\alpha}$ and $\alpha_2 = \bar{\alpha}$ can provide results for Q/Q_{fm} that agree closely with those obtained by the use of α_1 and α_2. In the contrast of the use of α_{t1} and α_{t2}, it is obvious from expressions (1) and (2) that the choice of $\bar{\alpha}_t$:

$$\frac{1}{\bar{\alpha}_t} = \frac{1}{2}\left(\frac{1}{\alpha_{t1}} + \frac{1}{\alpha_{t2}}\right) \tag{15}$$

makes such a surmise exact.

In Table III we have reported some results corresponding to (α_1,α_2) and $\bar{\alpha}$ (for $d = 5.0$), as obtained by Thomas.[12] Also reported are results obtained by the use of (α_{t1},α_{t2}) or $\bar{\alpha}_t$. Clearly, in this case also the two boundary conditions provide results that are fairly close (within approximately

TABLE II. Comparison of Q/Q_{fm} for BGK model for Maxwellian diffuse-specular (α, B.C.) reflection boundary condition.

	α					
	1		0.826		0.759	
d	Variational	"Exact"	Variational	"Exact"	Variational	"Exact"
0.01	0.9924	0.992 484	0.9946	0.994 675	0.9953	0.995 360
0.1	0.9327	0.935 159	0.9513	0.952 722	0.9572	0.958 414
0.5	0.7653	0.768 262	0.8194	0.821 214	0.8383	0.839 639
1.0	0.6413	0.640 262	0.7133	0.712 079	0.7397	0.738 150
1.25	0.5953	0.593 761	0.6720	0.669 706	0.7006	0.698 040
1.5	0.5560	0.553 805	0.6359	0.632 830	0.6662	0.662 804
1.75	0.5219	0.519 333	0.6039	0.600 300	0.6354	0.631 460
2.0	0.4920	0.489 203	0.5752	0.571 299	0.6075	0.603 306
2.5	0.4418	0.438 866	0.5257	0.521 603	0.5591	0.554 588
3.0	0.4011	0.398 324	0.4843	0.480 375	0.5181	0.513 700
4.0	0.3390	0.336 724	0.4189	0.415 508	0.4523	0.448 465
5.0	0.2938	0.291 918	0.3693	0.366 501	0.4017	0.398 409
7.0	0.2320	0.230 810	0.2988	0.296 937	0.3284	0.326 153
10.0	0.1764	0.175 788	0.2324	0.231 353	0.2580	0.256 665

TABLE III. Normalized heat flux, Q/Q_{fm} for mean ($\bar{\alpha} = \bar{\alpha}_t$) compared to the corresponding values for ($\alpha_1 = \alpha_{t1}$, $\alpha_2 = \alpha_{t2}$), $d = 5.0$.

Coefficients α or α_t			Q/Q_{fm} for different boundary conditions		
$\alpha_1 = (\alpha_{t1})$	$\alpha_2 = (\alpha_{t2})$	$\bar{\alpha} = (\bar{\alpha}_t)$	(α_1,α_2)	$(\bar{\alpha})$	$(\alpha_{t1},\alpha_{t2}),(\bar{\alpha}_t)$
0.7	0.9	0.787 50	0.384 79	0.384 59	0.388 29
0.7	0.5	0.583 33	0.492 92	0.492 63	0.500 30
0.7	0.3	0.420 00	0.599 06	0.597 71	0.607 98
0.7	0.1	0.175 00	0.790 70	0.801 72	0.811 30
0.5	0.3	0.375 00	0.630 83	0.630 41	0.641 12
0.5	0.1	0.166 67	0.811 45	0.809 96	0.819 33
1.0	0.1	0.181 82	0.799 20	0.795 04	0.804 79

3%).

In this context let us also examine the results for Maxwellian diffuse-specular boundary condition in the temperature jump regime. Here, for the BGK model, an exact expression for the nondimensionalized jump coefficient ϵ is[13]

$$\epsilon(\alpha) = \frac{5\sqrt{\pi}}{8}\frac{2-\alpha}{\alpha}\left(\epsilon_0 + \sum_{m=1}^{\infty}(1-\alpha)^m\epsilon_m\right), \qquad (16)$$

where $\epsilon_0 = 1.17597$, $\epsilon_1 = 1.60683\times10^{-1}$, $\epsilon_2 = -1.37349\times10^{-2}$, and so forth. Now, in the jump regime

$$q(\alpha_1,\alpha_2) \hat{=} \lim_{d\gg1}\frac{Q}{Q_{fm}} = \frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{\alpha_1\alpha_2} \times\left(1-\frac{\epsilon(\alpha_1)+\epsilon(\alpha_2)}{d}\right)\frac{5\sqrt{\pi}}{4}\frac{1}{d}. \qquad (17)$$

Therefore, after some algebra, we find

$$\frac{q(\alpha_1,\alpha_2)-q(\bar{\alpha})}{q(\alpha_1,\alpha_2)} \sim \frac{5\sqrt{\pi}}{8}\frac{1}{d}\frac{(\alpha_1-\alpha_2)^2}{(\alpha_1+\alpha_2)}\epsilon_1 \leqslant \frac{5\sqrt{\pi}}{8}\frac{1}{d}\epsilon_1 \sim 10^{-2}. \qquad (18)$$

That is, in the jump regime for the Maxwellian boundary condition, approximation (14) provides results that differ from the exact results only by about 1%.

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