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Interaction of surface gravity waves on a nonuniformly periodic seabed

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The interaction of surface gravity waves on a nonuniformly periodic seabed is considered. The method of multiple scales is used to analyze the interaction leading to two coupled-mode equations. The power reflection coefficient is used as an indicator to evaluate the filter action of the bottom corrugation. The results show that the modes are strongly coupled when the Bragg resonant condition is satisfied. The characteristics of the filtration are found to be enhanced by imposing special types of amplitude and phase modulated periodic nonuniformities, including amplitude taper and chirped periodic variations.

I. INTRODUCTION

The resonant interaction of surface gravity waves with a periodic seabed is a subject of fundamental importance. This stems from the fact that a considerable reflection of incident waves takes place when the resonant (Bragg) condition is satisfied. The reflection is due to the multiple interferences of the waves from the periodic structure of the bed, which leads to a strong stop-band interaction. This phenomenon might constitute an efficient mechanism for coastal protection. An important application of the Bragg effect is the construction of tuned periodic patches of sandbars in order to reflect incident waves with appropriate wave number.

Early works include the papers by McGoldrick[1] and Rhines and Bretherton[2] on long waves in shallow water over a periodic topography. Their formulation led to the Mathieu equation, and consequently, to the conclusion that a resonant reflection is possible in the subharmonic instability region. These studies were followed by the work of Davies[3] who examined the interaction between harmonic waves and a patch of bottom ripples of constant mean depth. He used a regular perturbation expansion in terms of the ratio of the bottom amplitude to the mean water depth to show that few ripples might be needed to produce a substantial backscattered wave at the Bragg condition. These theoretical predictions were supported by the experimental measurements of Heathershaw.[4] Davies and Heathershaw[5] reconsidered the problem in order to determine the amount of incident wave energy that is reflected by the ripple patch. They compared the theoretical predictions with an extensive set of laboratory observations made in a wave tank. Their theory is limited to weak reflection and fails at resonance because the reflection coefficient becomes unbounded and the perturbation expansion breaks down.

The resonant interaction phenomenon was considered by Mei[6] who explained why a strong reflection could be induced by periodic sandbars when the Bragg resonance condition is met. In his analysis, he used the method of multiple scales, allowing the incident waves to vary slowly in time and space, and found that the scattering process depends critically on whether the modulation frequency lies above or below a threshold frequency.

Kirby[7] obtained a general wave equation for linear surface waves over periodic ripples superimposed on a bed with a slowly varying mean depth. Hara and Mei[8] extended the linearized theory on Bragg scattering of surface waves by periodic sandbars to include second-order effects of the free surface and bars. Measured responses were compared with corresponding theoretical results. Mei, Hara, and Naciri[9] examined the case of an oblique incidence on a strip of periodic bars and a seabed with a mean slope. Guazzelli, Rey, and Belzons[10] carried out experiments to describe the higher-order Bragg resonant interactions between linear gravity waves and doubly sinusoidal beds. Kirby[11] has recently considered the reflection of waves on sinusoidal bars which are deposited on a mild-sloped seabed.

In the present work, the general problem of surface gravity waves on a seabed with nonuniform periodicities, including amplitude taper and chirped corrugations, is considered. The method of multiple scales is used to analyze the interaction of the propagating modes when the Bragg condition is satisfied. The coupled-mode equations together with suitable boundary conditions constitute a two-point boundary-value problem, which is solved numerically. The results are utilized for calculating the response in terms of the power reflection coefficient. The nonuniform corrugations are found to enhance certain desirable features.

II. PROBLEM FORMULATION

We consider the propagation of surface gravity waves in an inviscid, incompressible, and irrotational fluid over a nonuniformly corrugated bottom with mean wave number k_b^*. We introduce dimensionless variables by using k_b^{*-1} as a reference length. The motion can be described by a dimensionless potential function $\phi(x,z)$, which is governed by the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0. \quad (1)$$

It is assumed that the seabed is periodically corrugated in the domain extending from $x=0$ to $x=L$ and can be described by

$$h(x)=d\{1+\varepsilon f(\varepsilon x)\sin[x+g(\varepsilon x)]\}, \quad (2)$$

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where $\varepsilon = a^*/d^*$ is a small dimensionless parameter that is equal to the ratio of the average amplitude a^* of the corrugated bottom to the mean depth d^* of undisturbed fluid, $f(\varepsilon x)$ is a periodic taper function, and $g(\varepsilon x)$ is a linear chirp function (see Asfar and Hawwa[12]). Hence, the boundary condition at the bottom is given by

$$\frac{\partial \phi}{\partial z} = -h'(x)\frac{\partial \phi}{\partial x}, \quad \text{at } z=-h(x). \tag{3}$$

The linearized boundary condition at the free surface is

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0, \quad \text{at } z=0 \tag{4}$$

Using a Taylor-series expansion to transfer the boundary condition in Eq. (3) from $z=-h(x)$ to $z=-d$, we obtain

$$\frac{\partial \phi}{\partial z} - \{\varepsilon d f(\varepsilon x)\sin[x+g(\varepsilon x)]\}\frac{\partial^2 \phi}{\partial z^2} = -\{\varepsilon d f(\varepsilon x)\cos[x+g(\varepsilon x)]\}\frac{\partial \phi}{\partial x} + \cdots, \quad \text{at } z=-d. \tag{5}$$

III. METHOD OF SOLUTION

Employing the method of multiple scales (Nayfeh[13,14]), we seek a first-order perturbation expansion for ϕ in powers of ε in the form

$$\phi(x,z) = \phi_0(X_0, X_1, \ldots, z) + \varepsilon \phi_1(X_0, X_1, \ldots, z) + \cdots, \tag{6}$$

where $X_0 = x$ is a short length scale, which is the order of the wavelength of the corrugation, and $X_1 = \varepsilon x$ is a long length scale, which characterizes the spatial amplitude and phase modulations due to the seabed periodicity. The derivatives with respect to x are expanded in terms of ε as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X_0} + \varepsilon \frac{\partial}{\partial X_1} + \cdots, \tag{7}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial X_0^2} + \varepsilon \frac{\partial^2}{\partial X_0\, \partial X_1} + \cdots. \tag{8}$$

Substituting Eqs. (6)–(8) into Eqs. (1), (4), and (5), and equating the coefficients of ε^0 and ε^1 on both sides, we obtain $O(1)$:

$$\frac{\partial^2 \phi_0}{\partial X_0^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0, \tag{9}$$

$$\frac{\partial^2 \phi_0}{\partial t^2} + g\frac{\partial \phi_0}{\partial z} = 0, \quad \text{at } z=0, \tag{10}$$

$$\frac{\partial \phi_0}{\partial z} = 0, \quad \text{at } z=-d; \tag{11}$$

$O(\varepsilon)$:

$$\frac{\partial^2 \phi_1}{\partial X_0^2} + \frac{\partial^2 \phi_1}{\partial z^2} = -2\frac{\partial^2 \phi_0}{\partial X_0\, \partial X_1}, \tag{12}$$

$$\frac{\partial^2 \phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z} = 0, \quad \text{at } z=0, \tag{13}$$

$$\frac{\partial \phi_1}{\partial z} = -\{d f(X_1)\sin[X_0+g(X_1)]\}\frac{\partial^2 \phi_0}{\partial z^2} + \{d f(X_1)\cos[X_0+g(X_1)]\}\frac{\partial \phi_0}{\partial X_0}, \quad \text{at } z=-d. \tag{14}$$

Had we attempted a straightforward asymptotic solution, corresponding to $\partial\phi/\partial X_1 = 0$, in Eqs. (12)–(14), we would have found that it breaks down when the wave number of the corrugation is equal or nearly equal to twice the wave number k of the propagating mode; that is,

$$2k \approx 1. \tag{15}$$

This condition is known in the literature as a Bragg condition (Brillouin[15]), at which steady-state harmonic waves are unstable (resonance occurs). Physically, a Bragg condition implies that two contradirectional modes interact strongly with each other by exchanging energy, resulting in a high level of attenuation.

A. The zeroth-order solution

Equation (12) admits a solution in the form of a linear combination of incident and reflected modes. These are given by

$$\phi_0 = [A^+(X_1)e^{ikX_0} + A^-(X_1)e^{-ikX_0}] \times [\cosh kz + (\omega^2/gk)\sinh kz]e^{-i\omega t}, \tag{16}$$

where the superscripts + and − indicate an incident and a reflected mode, respectively, and the $A^{\mp}(X_1)$ are unknown functions at this level of approximation. They are determined by imposing the appropriate solvability conditions at the next level of approximation.

Substitution of Eq. (16) into the boundary conditions (10) and (11) leads to the following dispersion relation:

$$k \tanh kd = \omega^2/g. \tag{17}$$

B. The solvability condition

Since the homogeneous first-order problem has a nontrivial solution, the inhomogeneous first-order problem has a solution only if the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem. To reach this solvability condition, we seek a particular solution for ϕ_1 in the form

$$\phi_1 = [B_j^+(z)e^{ikX_0} + B_j^-(z)e^{-ikX_0}]e^{-i\omega t}. \tag{18}$$

Substituting Eqs. (16) and (18) into the governing equation (12) and equating the coefficients of $\exp(\mp ikX_0)$ on both sides, we obtain

$$\left(\frac{d^2}{dz^2} - k^2\right)B^{\mp} = \pm 2ik\frac{dA^{\mp}}{dX_1}\left(\cosh kz + \frac{\omega^2}{kg}\sinh kz\right). \tag{19}$$

Next, we introduce the detuning parameter σ such that

$$2k=1+\varepsilon\sigma+g(X_1). \tag{20}$$

Substituting Eqs. (16) and (18) into the boundary conditions (13) and (14), expressing the sines and cosines in polar form, and equating the coefficients of $\exp(\mp ikX_0)$ on both sides, we obtain

$$\frac{dB^{\mp}}{dz}(0)=\frac{\omega^2}{g}B^{\mp}(0), \tag{21}$$

$$\frac{dB^{\mp}}{dz}(-d)=\mp\frac{1}{2}ikdf(X_1)(1-k)\times(\cosh kd)^{-1}A^{\pm}e^{\pm i\sigma X_1}e^{\pm ig(X_1)}. \tag{22}$$

The system of Eqs. (19), (21), and (22) forms a self-adjoint problem. Hence, the solvability condition takes the form

$$-\Psi(-d)G(X_1)=\int_{-d}^{0}\Psi(z)F(X_1,z)dz, \tag{23}$$

where Ψ is the adjoint solution and $F(X_1,z)$ and $G(X_1)$ are the right-hand sides of Eqs. (19) and (22), respectively.

The satisfaction of this solvability (consistency) condition leads to the following coupled-mode equations:

$$\frac{dA^+}{dX_1}=\left[f(X_1)kd\left(\frac{1-k}{I\cosh^2 kd}\right)\right]A^-e^{-i\sigma X_1}e^{-ig(X_1)}, \tag{24}$$

$$\frac{dA^-}{dX_1}=\left[f(X_1)kd\left(\frac{1-k}{I\cosh^2 kd}\right)\right]A^+e^{i\sigma X_1}e^{ig(X_1)}, \tag{25}$$

where

$$I=4\omega c_g/g, \tag{26}$$

where $c_g=\partial\omega/\partial k$ is the group velocity. Equations (24) and (25) reduce to the coupled-mode equations of Mei[6] and Kirby[7,11] for the special case of uniform periodicity.

IV. NUMERICAL ILLUSTRATIONS

In this section, we present the reflection characteristics of a corrugated seabed with mean wavelength $\lambda^*=100$ cm (i.e., $k_b^*=\pi/50\ \text{cm}^{-1}$), $a^*=5$ cm, $L^*=2500$, 5000, and 7500 cm. Hence, $\varepsilon=0.16$.

The modal interaction over the periodic bottom is governed by Eqs. (24) and (25) which, without any loss of generality, are provided with the following boundary conditions on both ends of the periodic section:

$$A^+=1 \quad \text{at } X_1=0 \quad \text{and} \quad A^-=0 \quad \text{at } X_1=4\lambda,8\lambda,\text{ or } 12\lambda. \tag{27}$$

We note that the first condition represents the excitation amplitude of the incident mode, and the second condition expresses the fact that the reflected mode vanishes at the end of the corrugated section.

The two-point boundary-value problem defined by Eqs. (24), (25), and (27) is solved numerically (see Asfar

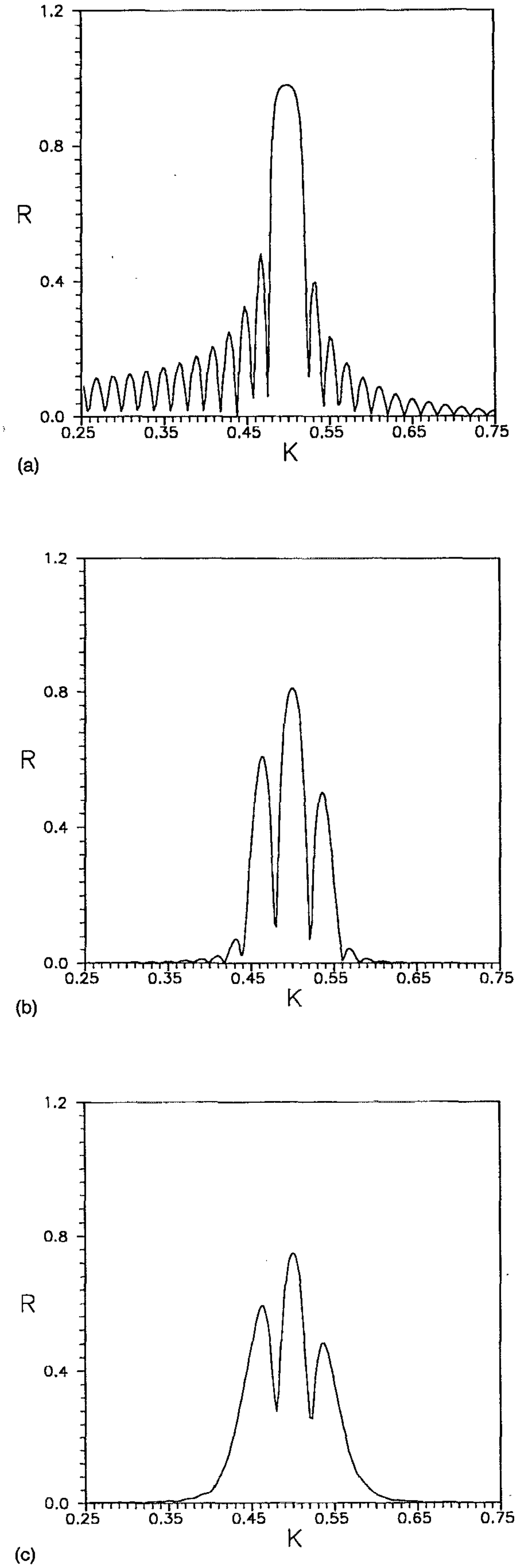

FIG. 1. Reflection coefficient R for a periodic seabed with $L=4\lambda$, (a) uniform, (b) tapered, (c) tapered and chirped.

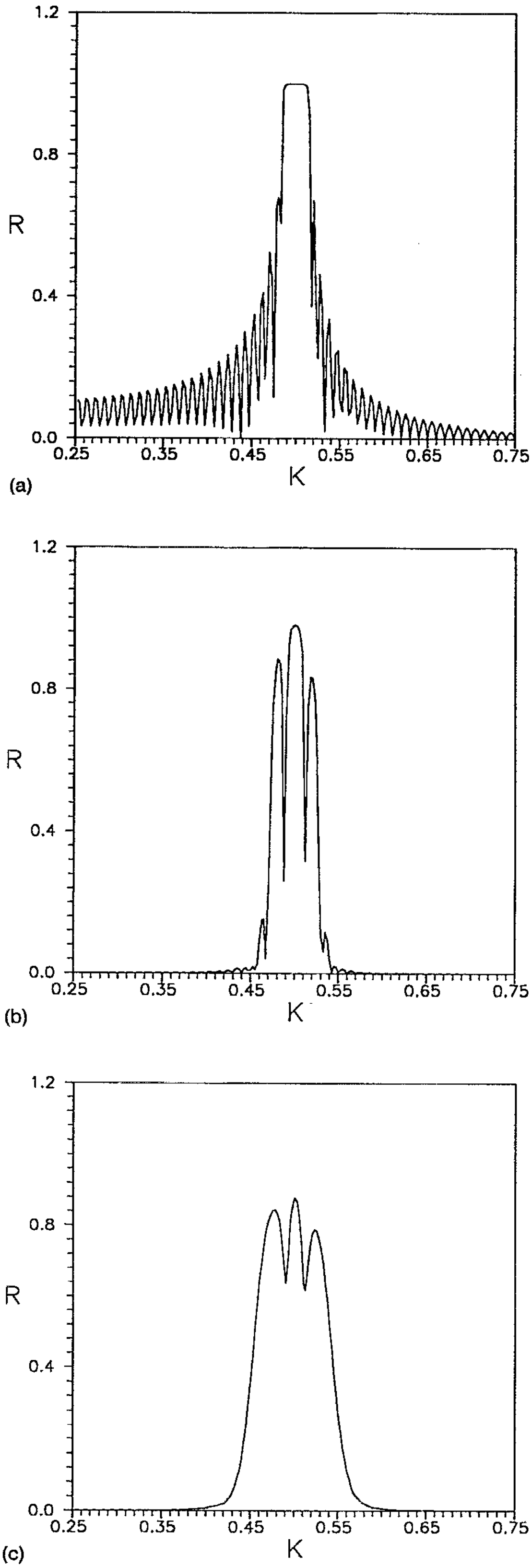

FIG. 2. Reflection coefficient R for a periodic seabed with $L=8\lambda$, (a) uniform, (b) tapered, (c) tapered and chirped.

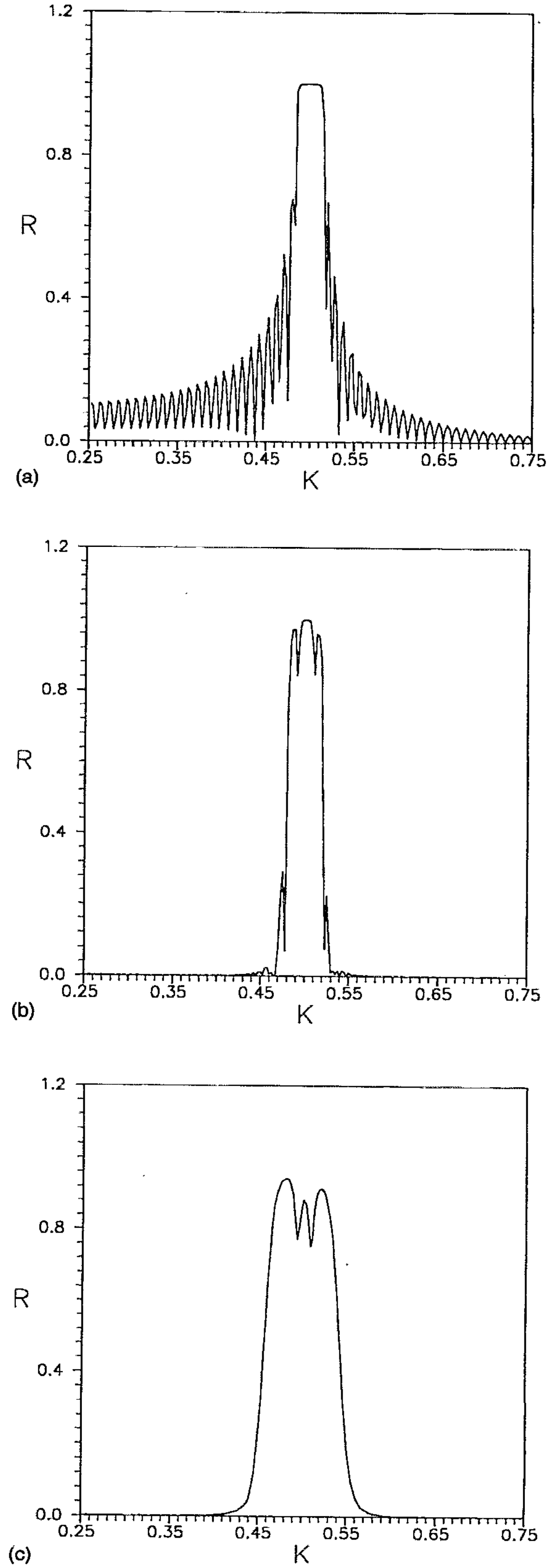

FIG. 3. Reflection coefficient R for a periodic seabed with $L=12\lambda$ (a) uniform, (b) tapered, (c) tapered and chirped.

and Hussein[16]) to obtain the missing boundary conditions, and consequently, to calculate the power reflection coefficient.

The reflection characteristics in the case of a uniformly periodic seabed [i.e., $f(X_1)=1$ and $g(X_1)=0$] are shown in Figs. 1(a), 2(a), and 3(a) for $L=4\lambda$, 8λ, and 12λ, respectively. The figures show filter responses with maximum reflection at resonance and side ripples whose level decreases away from resonance. We note that the amount of reflection increases as the number of wavelengths increases.

In order to investigate the effect of periodic nonuniformities on the response, we apply a taper of the form $f(X_1)=\sin^2(X_1/L)$ in the region $0<X_1<L/2$ and $f(X_1)=\sin^2[(L-X_1)/L]$ in the region $L/2<X_1<L$. In Figs. 1(b), 2(b), and 3(b), we show the effect of this amplitude tapering in realizing better filtration characteristics, with "clean" sides due to the disappearance of the side ripples.

When a linear chirp of the form $g(X_1)=(X_1-L/2)/L$ is imposed on the corrugation function in addition to the tapering function, wide midbands can be seen in Figs. 1(c), 2(c), and 3(c). We note that the reflection bandwidth increases but the maximum reflection coefficient decreases as L decreases. Nearly ideal filtering characteristics are realized when $L=12\lambda$.

V. CONCLUSION

The resonant interaction of surface gravity waves on a nonuniformly periodic seabed has been considered. The filtering characteristics of the bottom have been investigated. Tapered corrugations lead to a narrower midband response, while chirped corrugations lead to a wider midband response. A combination of taper and chirp has been used to realize a nearly ideal wide midband response. These theoretical results might provide the basis of new concepts for the design of tuned seabeds to protect coastal structures.

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