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A Galerkin method to strongly nonlinear KdV equations and Schrödinger equations
Internal capillary-gravity waves of a two-layer fluid with free surface over an obstruction—Forced extended KdV equation

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In this paper we study steady capillary-gravity waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation, called a forced extended KdV equation (FEKdV) governing interfacial wave forms, is obtained by a refined asymptotic method. Various solutions and numerical results of this equation are presented. © 1996 American Institute of Physics.

I. INTRODUCTION

We consider a two-layer medium of immiscible, inviscid, and incompressible fluids having different but constant densities. The medium is bounded above by a free surface and below by a horizontal rigid boundary with an interface in between (Fig. 1). The surface tension effect is taken into consideration at both the free surface and the interface. We assume that a two-dimensional object is moving along the lower boundary at a constant speed, and in reference to a coordinate system moving with the object, the fluid flow is steady. Two critical speeds are obtained. When the object is moving at a speed near either one of them, a FKdV for steady flow can be derived and has been extensively investigated in Refs. 1 and 2. We note that numerical studies of steady flow of a two-layer fluid over a bump or a step bounded by a free or rigid upper boundary were carried out by Forbes, Belward and Forbes, Sha and Vanden-Broeck, and Moni and King, among others, and an asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen on the basis of the FKdV theory, and with surface tension by Choi, Sun, and Shen, where a forced modified KdV equation (FMKdV) was obtained. The FKdV theory fails when the coefficient of the nonlinear term or that of the third derivative in the FKdV vanishes. In the case considered here, when the wave speed is near the smaller critical speed for internal waves, the amplitude of which is larger at the interface than at the free surface, the coefficient of the nonlinear term in the FKdV may vanish. Furthermore, at a wave speed near either one of the critical speeds, the coefficient of the third-order derivative may also vanish. To overcome the difficulty of a vanishing nonlinear term in the FKdV, we shall develop a refined asymptotic method to derive a new equation, called the forced extended KdV equation (FEKdV), in the following form:

\[(F_1 \eta_2^2 + F_2 \eta_2 + F_3) \eta_{2x} + F_4 \eta_{2xxx} = -F_5 b_x,\]

where \(F_1 \sim F_5\) are constants depending on several parameters and \(z = -H + b(x)\) is the equation of the obstruction. The objective of this paper is to investigate solutions of the FEKdV, which represent possible interfacial wave forms. We remark that when the coefficient of the third-order derivative in the FEKdV vanishes, a forced perturbed KdV equation with a fourth-order derivative could be derived, and is deferred to a subsequent study.

In Sec. II, we formulate the problem and develop the asymptotic scheme to derive the FEKdV. Section III consists of two sections. The supercritical case of \(F_1 F_4 > 0\) and the subcritical case of \(F_1 F_4 < 0\) are studied in Secs. III A and III B, respectively. In general, we can find three types of solutions. The first-type solution consists of symmetric solitary-wave-like solutions. The second-type solution is one that is a part of a free solitary wave behind the bump and a periodic wave solution ahead of the bump. The free solitary wave is a solitary wave solution of the extended KdV equation without forcing. By a third-type solution we mean a solution that is constant behind the bump and periodic ahead of the bump. In many cases both second- and third-type solutions do satisfy the conservation of mass, even if they do not tend to zero far upstream. In both Secs. III A and III B, analytical and numerical results, which indicate the appearance of various types of solutions, are presented. It is found that four branches of first-type solutions can appear in the supercritical case, and there are no first- and second-type solutions in the subcritical case. The third-type solutions appear in both supercritical and subcritical cases. In both cases, symmetric solutions without a periodic part are embedded in the third-type solutions at discrete values of a parameter, and a hydraulic jump wave solution appears as a limiting case of third-type solutions in the subcritical case.
II. Formulation and Successive Approximate Equations

We consider steady internal capillary-gravity waves between two immiscible, inviscid, and incompressible fluids of constant but different densities, bounded above by a free surface and below by a horizontal rigid boundary, with a small obstruction of compact support. The domains of the upper fluid with a constant density $\rho^{*+}$ and the lower fluid with a constant density $\rho^{*-}$ are denoted by $\Omega^{*+}$ and $\Omega^{*-}$, respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed $C$ in reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed $C$ far upstream. The governing equations and boundary conditions are the following: In $\Omega^{*\pm}$,

$$u^{*\pm}_{x\pm} + w^{*\pm}_{z\pm} = 0,$$

$$u^{*\pm}_{x\pm} u^{*\pm}_{x\pm} + w^{*\pm}_{z\pm} u^{*\pm}_{z\pm} = -\rho^{*\pm} \frac{\partial \rho^{*\pm}}{\partial x^{\pm}},$$

$$u^{*\pm}_{x\pm} w^{*\pm}_{x\pm} + w^{*\pm}_{z\pm} w^{*\pm}_{z\pm} = -\rho^{*\pm} \frac{\partial \rho^{*\pm}}{\partial z^{\pm}} - g;$$

at the free surface, $z^* = H^{*+} + \eta_1^*$,

$$u^{*+}_{x+} + \eta_{1x+}^* - w^{*+}_{z+} = 0,$$

$$p^{*+} = -T_1^* \eta_{1x+}^*/(1 + \eta_{1x+}^*)^{3/2};$$

at the interface, $z^* = \eta_2^*$,

$$p^{*+} - p^{*-} = T_2^* \eta_{2x+}^*/(1 + \eta_{2x+}^*)^{3/2},$$

$$u^{*+} \eta_{2x+}^* - w^{*+}_{z+} = 0;$$

at the rigid bottom, $z^* = -H^{*-} + b^*(x^*)$,

$$w^{*-} - b_{x+} u^{*-} = 0,$$

where $u^{*\pm}$ and $w^{*\pm}$ are horizontal and vertical velocities, $\rho^{*\pm}$ are pressures, $g$ is the gravitational acceleration constant, and $T_1^*$ and $T_2^*$ are surface tension constants at the free surface and the interface, respectively. We define the following nondimensional variables:

$$\epsilon = H/L \ll 1, \quad \eta_1 = \frac{\epsilon^{-1} \eta_1^*}{H^{*+}}, \quad \eta_2 = \frac{\epsilon^{-1} \eta_2^*}{H^{*+}},$$

$$p^{*\pm} = \frac{p^{*\pm}}{g H^{*\pm} \rho^{*\pm}},$$

$$(x,z) = (e x^*, z^*)/H^{*+},$$

$$(u^{*\pm}, w^{*\pm}) = (g H^{*+})^{-1/2} (u^{*\pm}, e^{-1} w^{*\pm}),$$

$$\rho^+ = \frac{\rho^{*+}}{\rho^{*+}} < 1, \quad \rho^- = \frac{\rho^{*-} - \rho_{x+}^*}{\rho^{*-}} = 1, \quad U = \frac{C}{(g H^{*+})^{1/2}},$$

$$T_i = T_i^*/\rho^{*+} - g (H^{*+})^2, \quad i = 1, 2,$$

$$h = H^{*+}/H^{*+}, \quad b(x) = b^*(x) (H^{*+} - b^*)^{-1},$$

where $L$ is the horizontal scale, $H$ is the vertical scale, $b(x) = b^*(x) (H^{*+} - b^*)^{-1}$, $H^{*+}$ and $H^{*-}$ are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$, respectively, and $z^* = -H^{*-} + b^*(x)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above equations become, in $\Omega^{\pm}$,

$$u^{\pm}_{x\pm} + w^{\pm}_{z\pm} = 0, \quad u^{\pm}_{x\pm} u^{\pm}_{x\pm} + w^{\pm}_{z\pm} u^{\pm}_{z\pm} = -\rho^{\pm} \frac{\partial \rho^{\pm}}{\partial x^{\pm}},$$

$$e^2 u^{\pm}_{x\pm} w^{\pm}_{x\pm} + e^2 w^{\pm}_{z\pm} w^{\pm}_{z\pm} = -\rho^{\pm} \frac{\partial \rho^{\pm}}{\partial z^{\pm}} - 1;$$

at $z = h + \epsilon \eta_1$,

$$p^+ = -e^3 T_1 \eta_{1x+}/(1 + e^4 \eta_{1x+}^2)^{3/2},$$

$$\epsilon u^{\pm} \eta_{1x+}^* - w^{\pm} = 0;$$

at $z = \epsilon \eta_2$,

$$\epsilon u^{\pm} \eta_{2x+}^* - w^{\pm} = 0, \quad \epsilon u^{\pm} \eta_{2x+}^* - w^{\pm} = 0, \quad p^+ - p^- = e^3 T_2 \eta_{2x+}/(1 + e^4 \eta_{1x+}^2)^{3/2},$$

at $z = -1 + e^3 b(x)$,

$$w^* - e^3 \epsilon u^* b_{x+} = 0;$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive an approximate equation for the interface $\eta(x)$. We assume that $u^{\pm}, w^{\pm},$ and $p^{\pm}$ are functions of $x, z$ near the equilibrium state $u^\pm = u_0$, $w^\pm = 0$, $p^\pm = p^0 z + p^0 h$, and $p^- = -\rho^- z^2 + p^0 h$, where $u_0$ is a constant, and possess asymptotic expansions:

$$(u^\pm, w^\pm, p^\pm) = (u_0, 0, -\rho^- z + p^0 h) + \epsilon (u^\pm_1, w^\pm_1, p^\pm_1)$$

$$+ \epsilon^2 (u^\pm_2, w^\pm_2, p^\pm_2) + \epsilon^3 (u^\pm_3, w^\pm_3, p^\pm_3)$$

$$+ O(\epsilon^4).$$

By inserting (10) into (1)–(4) and (7)–(9) and arranging the resulting equations according to the powers of $\epsilon$, it follows that $(u_0, 0, -\rho^- z + p^0 h)$ are the solutions of the zeroth-order system of equations, and the equations of the order $\epsilon$ are as follows:

$$u^\pm_{1x+} + w^\pm_{z+} = 0,$$
\begin{equation}
\frac{u_0 u_1^+}{x} = -p_1^+/\rho^+,
\end{equation}
(12)
\begin{equation}
p_1^+=0;
\end{equation}
(13)
\begin{equation}
at z=h,
p_1^+ + \eta_1 p_0^+ = 0;
\end{equation}
(14)
\begin{equation}
at z=0,
p_1^+ - p_1^- + \eta_2 (p_0^+-p_0^-) = 0,
\end{equation}
(15)
\begin{equation}
u_0 \eta_{2x} - w_1^+ = 0;
\end{equation}
(16)
\begin{equation}
at z = -1,
w_1^- = 0.
\end{equation}
(17)

Hereafter, for the sake of convenience we shall use \( \rho^+ \) to denote \( \rho' \) and set \( \rho' = 1 \). From (13), \( p_1^+ \) are functions of \( x \) only. It follows that \( p_1^+ = \rho \eta_1 \) by (14) and \( p_1^+ = \rho \eta_1 + \eta_2 (1-\rho) \) by (15). We can find \( w_1^+ \) by using (11), (12), (16), and (17) so that
\begin{equation}
w_1^+ = (z + 1) [\rho \eta_1 + (1-\rho) \eta_2] / u_0.
\end{equation}
(18)

Here \( u_1^+ \) are also derived from (11),
\begin{equation}
u_1^+ = (1 + \eta_1 + H_1) / u_0,
\end{equation}
(19)
\begin{equation}
u_1^- = -(1-\rho) \eta_2 / u_0,
\end{equation}
where we assume \( \eta_1 (x=\infty) = H_1 \) and \( \eta_2 (x=\infty) = H_2 \), and
\begin{equation}
u_1^+ (x=\infty) = 0.
\end{equation}

Similarly, we can find \( p_2^+, w_2^+, u_2^+, p_3^+, w_3^+, u_3^+ \) in terms of \( \eta_1 \) and \( \eta_2 \) without using the kinematic conditions (5) and (6). From (5) and (6), and the asymptotic expansion of \( u^- \) and \( w^- \), we have at \( z=h \),
\begin{equation}
u_0 \eta_{1x} - w_1^+ + \varepsilon [u_1^+ \eta_1 - \eta_1 w_1^+ - w_1^-] + \varepsilon^2 [u_2^+ \eta_1]
+ \eta_1 \eta_2 u_{1z} - (1/2) w_{1zz} \eta_1^2 - \eta_2 w_{2z} + w_1^- + O(\varepsilon^3) = 0;
\end{equation}
(20)
and at \( z=0 \),
\begin{equation}
u_0 \eta_{2x} - w_1 + \varepsilon [u_1 \eta_2 - \eta_2 w_2 - w_2^-] + \varepsilon^2 [u_2 \eta_2]
+ \eta_2 \eta_2 u_{1z} - (1/2) w_{1zz} \eta_2^2 - \eta_2 w_{2z} + w_2^- + O(\varepsilon^3) = 0.
\end{equation}
(21)

Making use of these equations, we can find an approximite equation for the interface \( \eta_2 \). We substitute \( u_1^+, w_1^+, u_2^+, w_2^+, w_3^+ \) into (20), (21), eliminate \( \eta_1 \) from (21) by finding a relation between \( \eta_1 \) and \( \eta_2 \) up to \( O(\varepsilon^2) \) from (20), and obtain
\begin{equation}
[u_0 - \rho c_1 / u_0 - (1-\rho) / u_0] \eta_{2x} + \varepsilon [E_1 \eta_2 \eta_2 + E_2 \eta_2 x]
+ \varepsilon^2 [F_1 \eta_2^2 x + F_2 \eta_2 \eta_2 x + F_3 \eta_2 \eta_2 x + F_4 \eta_2 x + F_5 b x]
+ O(\varepsilon^3) = 0,
\end{equation}
(22)
where, if we let \( c_1 = [u_0 - (1-\rho) (1+u_0^2 - h)] / (\rho + u_0^2 h) \),
\begin{equation}
D_1 = u_0 / (\rho + u_0^2 h), \quad \lambda = u_0^- (\infty), \quad H = \rho H^2 + (1-\rho) H_2,
\end{equation}
and \( R = \rho c_1 + 1 - \rho \), then
\begin{equation}
\begin{aligned}
E_1 &= -(R^2 + 2 R u_0^2) u_0^{-3} - \rho D_1 [(hc^2 - R^2) u_0^4 + (2c^2 - 2R - 2c_1) u_0^2],
\end{aligned}
\end{equation}
\begin{equation}
E_2 = -[\rho(H_1 - c_1 H_2) + \rho H_2] u_0^{-1} - R[\rho H_2 - (1-\rho) H_2]
- 2c^2 H_2 u_0^2 + [R c_1 \rho H_2 + R (1-\rho) H_2]
- 2c^2 H_2 u_0^2 - 4],
\end{equation}
\begin{equation}
F_1 = -\rho D_1 u_0^{-5} (3c_1^2 - 3c_1^2 + 2R^2 u_0^3) + (3h c_1^2 / 2)
- 3 R^3 u_0^{-5} + 3 D_1 [\rho u_0^{-1} + \rho R u_0^{-3}][(3R^2 + c_1 - c_1^2) u_0^{-1} + (R^2 h - 2h^2 u_0^2) u_0^3] - 3 R^3 u_0^{-5} / 2,
\end{equation}
\begin{equation}
F_2 = -(\rho D_1 u_0^{-1}) [u_0^{-3} (2HR - 2H^2 c_1^2 + 2H c_1)]
+ u_0^{-5} [3H R^2 - 3H H c_1^2] - 2D_1 [\rho u_0^{-1} + \rho R u_0^{-3}]
\times [u_0^{-1} (H^2 R - 2H c_1^2 + H c_1 + c_1 H)] + u_0^{-3} (R^2 H c_1^2)
+ 2H R u_0^{-3} + 3H^2 u_0^{-5},
\end{equation}
\begin{equation}
F_3 = -(\rho D_1 u_0^{-1}) [(\lambda + H c_1 u_0^{-2})]
+ u_0^{-5} [3H c_1 H^2 - 3HR^2 u_0^{-3}] + u_0^{-3} (c_1 H^2 / 2)
- 3HR^2 u_0^{-5} + \lambda (1 + R u_0^{-2}) - 2H^2 u_0^{-3} / 2
- 3HR^2 u_0^{-5} / 2,
\end{equation}
\begin{equation}
F_4 = -(\rho D_1 u_0^{-1}) [-c_1 (\rho h^2 / 2 - T_1 + \rho / 3) u_0^{-1} - (u_0^2 h h)
- T_2 + (1-\rho) / 3 u_0^{-1} + [c_1 (\rho h^2 / 3 - h T_1) / u_0 \rho] + u_0 h / 2 - c_1 (\rho h^2 / 2 - T_1 + \rho / 3) u_0^{-1}
- [u_0^4 \rho h - T_2 + (1-\rho) / 3 u_0^{-1}]
- [u_0^2 \rho h / 2 - T_1 + (1-\rho) / 3 u_0^{-1},
\end{equation}
\begin{equation}
F_5 = \rho D_1 u_0^{-1}.
\end{equation}

III. EXTENDED KdV EQUATION WITH FORCING

From the zeroth-order term of (22), we obtain
\begin{equation}
u_0 - (\rho c_1 / u_0 - (1-\rho) / u_0) = 0,
\end{equation}
and by the expression for \( c_1 \) in (22), it follows that
\begin{equation}
u_0^4 u_0^{-1} + h (1-\rho) u_0 = 0
\end{equation}
(23)
and
\begin{equation}
u_0^4 = (1 + h \pm (1-\rho)^2 + 4 \rho x / 2)^{1/2}.
\end{equation}

We denote the two values of \( u_0^4 \) by \( u_0^{(1)} \) and \( u_0^{(2)} \), respectively, corresponding to the plus and minus signs. Without loss of generality we assume \( u_0^{(1)} \) and \( u_0^{(2)} \) are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next, we consider the coefficients of \( \eta_2 \) and \( \eta_2 \) in the
the first-order terms of Eq. (22). Note that $E_2$ vanishes if $H_1=H_2=0$ or

$$H_1 = H_2 [(2u_0^2/c_1 + 2h_1 + c_1 + h(c_1/u_0^2) - 1]$$

$$= H_2 [-1 + 2u_0^2(h_1 + 1 - \rho)^{-1} + \rho^{-1}u_0^{-2} [(4 - 2\rho + 3h)u_0^2 - h(1 - \rho)]].$$

If $E_1$ in (22) is not zero, a FKdV can be derived and has already been studied in Ref. 1. However, $E_1$ may vanish. First, let us simplify the expression of $E_1$.

By the conditions

$$\rho = (\rho c_1 + \rho^2) u_0^2 - 2[(\rho c_1 + \rho^2)/u_0] - \rho D_1 - 2[(\rho c_1 + \rho^2)/u_0] - [(\rho c_1 + \rho^2)/u_0^3]$$

$$+ 2[(c_1^2/u_0^2) + h(c_1/u_0^2)] - 2(c_1/u_0) = 0,$$

we obtain

$$u_0^2 + (1 - 2h)u_0^2 + h^2 - 1 = 1 + \rho (1 - 2h)(1 - \rho)^2 + 4\rho h)^{1/2},$$

$$u_0^2 = (1 - 2h)u_0^2 + h^2 - 1 = 1 + \rho (1 - 2h)(1 - \rho)^2 + 4\rho h)^{1/2}.$$ (24)

Equation (24) tells us that $E_1$ does not vanish if we take $u_0$ as a critical speed. Suppose both sides of (24) vanish. Then real $u_0$ implies $h < \rho$ and the right-hand side of (24) is greater than zero. This is a contradiction. Thus, the only possible case for $E_1 = 0$ is that $u_0$ is equal to the critical speed $u_{0\text{c}}$, and it is easy to show that $E_1 = O(e) = \beta e$ if $u_0^2 = u_{0\text{c}}^2 + O(e)$, and

$$1 + \rho (1 - 2h)(1 - \rho)^2 + 4\rho h)^{1/2} + O(e).$$

With the conditions (21) and (25), we obtain a time-independent FKdV,

$$F_1 \eta_2 \eta_2 + \lambda_1 \eta_2 \eta_2 + \lambda_2 \eta_2 + F_4 \eta_2 + F_4 b_x = 0,$$ (27)

where

$$F_1 = 3u_0[4\rho + 3h - u_0^2(u_0^2 + \rho - h)^{-1}],$$

$$\lambda_1 = H_2 [ -4u_0 + 5(1 - \rho)(u_0^2 - h(u_0^2 + \rho - h)^{-1}) + H_4[5(3 - 5\rho^2)(u_0^2 + 6h(1 - \rho)u_0^2 + 3h(1 - \rho)^2]$$

$$u_0^2 - h(u_0^2 + \rho - h)^{-1} + \beta],$$

$$\lambda_2 = H_2 [(3 + \rho)u_0^2 + 3 + 2\rho + \rho^2][2u_0^2 + (u_0^2 - h)^{-1} - 2h(u_0^2 - h + h)^{-1} - (1/2u_0^2) + (2u_0^2 + h)$$

$$u_0^2 - h(u_0^2 + \rho - h)^{-1} - (1/2u_0^2) + (2u_0^2 + h)$$

$$u_0^2 - h(u_0^2 + \rho - h)^{-1} - (1/2u_0^2) + (2u_0^2 + h)$$

$$F_4 = u_0^2 - 3u_0^2 h^2 + 4h^2 + 3(3)(u_0^2 + \rho - h)T_1 / \rho] + 3(3)(u_0^2 - h)T_2 / 3,$$

$$F_5 = u_0^2 - 3u_0^2 h^2 + 4h^2 + 3(3)(u_0^2 + \rho - h)T_1 / \rho] + 3(3)(u_0^2 - h)T_2 / 3.$$

The coefficients $F_1 - F_5$ here are the simplified forms of $F_1 - F_5$ in the previous section by using (23). We note that, for some special choice of $\beta$, $\lambda_1$ becomes zero and (27) becomes a FMKVd equation. Similar results as given in Ref. 8 can be obtained, and will not be discussed here. The sign of $F_4$ determines the existence of solutions of (27). In the following sections, the two cases $F_4 > 0$ and $F_4 < 0$ will be considered separately. We remark in passing that if the surface tension constants $T_1$ and $T_2$ satisfy $F_4 = 0$ for given $\rho$ and $h$ the coefficient of the third-order derivative vanishes and a forced perturbed KdV equation could be derived to replace the FEKdV equation.

A. Supercritical case ($F_4 > 0$)

We assume $U = u_0 + \lambda \varepsilon^2 + O(\varepsilon^3)$ and consider (27) for $F_4 > 0$. Here (27) can be rewritten as

$$\eta_2 = -A_1 \eta_2 \eta_2 + A_2 \eta_2 \eta_2 + A_3 \eta_2 + A_4 b_x,$$ (28)

where $A_1 = F_4 / F_4 > 0$, $A_3 = -\lambda_1 / F_4$, $A_3 = -\lambda_2 / F_4$, $A_4 = -F_5 / F_4$. When $b_x = 0$, (28) has solitary wave solutions whose value is $H_2$ at $x = \pm \infty$ for $A = A_3 + H_2 A_2 - A_1 H_2 > 0$: $\eta_2(x) = H_2 + A [B \cosh^2(A^{1/2}(x - x_0)/2)] + C \sinh^2(A^{1/2}(x - x_0)/2)$, (29)
or
\[
\eta_2(x) = H_2 - A \{ C \cosh^2[A^{1/2}(x-x_0)/2] + B \sinh^2[A^{1/2}(x-x_0)/2] \}^{-1},
\]
where
\[
B = \{\left( (A_2 - 2A_1 H_2)^2 + 6AA_1 \right)^{1/2} - (A_2 - 2A_1 H_2) \}/6,
\]
\[
C = \{\left( (A_2 - 2A_1 H_2)^2 + 6AA_1 \right)^{1/2} + (A_2 - 2A_1 H_2) \}/6,
\]
and \(x_0\) is a phase shift to be determined by the initial condition. For \(A \neq 0\), there is no solitary wave solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for \(b = 0\) when the wavelength tends to infinity.

Next, we consider (28) when \(b \neq 0\). Since we assume supp \(b(x) = [x^-, x^+]\), the intervals \((-\infty, x^-), [x^-, x^+], \) and \((x^+, \infty)\) are to be discussed separately.

1. \((-\infty, x^-)\). We have \(b(x) = 0\), and may choose either (29) as a solution of (28) or the trivial solution \(\eta_2 = H_2\), on \((-\infty, x^-)\).

2. \([x^-, x^+]\). In this interval \(b(x) \neq 0\), and by integrating (28) from \(-\infty\) to \(x \in [x^-, x^+]\), we obtain
\[
\eta_{2x} = -A_1 (A/3) \eta_2 + (A_2/2) \eta_2^2 + A_3 \eta + d + A_4 b(x),
\]
where
\[
d = A_1 (H_2^2/3) - (A_2 H_2^2/2) - A_3 H_2.
\]
It can be shown that (30) has a solution with a continuous third-order derivative in \([x^-, x^+]\) satisfying \(\eta_3(x^-) = \alpha\) and \(\eta_2(x^+) = \beta\) for any fixed constants \(\alpha\) and \(\beta\).

3. \([x^+, \infty)\). Here \(b(x) = 0\). By integrating (27) from \(x^+\) to \(x > x^+\), multiplying \(\eta_{2x}\) to the resulting equation, and integrating it from \(x^+\) to \(x > x^+\) again, it follows that
\[
(\eta_{2x})^2 = (A/6) \eta_2^4 + (A_2/3) \eta_2^2 + A_3 \eta_2 + 2d \eta_2 + e = A f(\eta_2),
\]
where
\[
e = A_1 (A^2/6) - A_2 (A^2/3) - A_3 x^2 - 2d \alpha + \beta^2,
\]
\[
\eta_2(x^-) = \alpha, \quad \eta_2(x^+) = \beta,
\]
\[
\eta_{2x}(x^+) = -A_1 (A^2/3) + A_2 (A^2/2) + A_3 \alpha + d.
\]
Let \(c_1, c_2, c_3,\) and \(c_4\) be four zeros of \(f(\eta_2)\) and we investigate the solutions of (31) for different cases of \(c_1, c_2, c_3,\) and \(c_4\).

Case 1. None of the \(c_1\) ... \(c_4\) are real. There is no solution since \((\eta_{2x})^2 < 0\).

Case 2. Two of the \(c_1\) ... \(c_4\) are real. Let \(c_1 = c_2\) be real and \(c_3, c_4\) are not real. Then \(f(\eta_2) = -A_1/6 (\eta_2 - c_1)(\eta_2 - c_2)((\eta_2 + \xi_1)^2 + \xi_2^2)\) for some \(\xi_1, \xi_2\) in \((-\infty, \infty)\), and
\[
\eta_2 = \{m_1 - m_2 \text{cn}[m_3(x-x_0)/m_3]/m_2 - m_3 \text{cn}[m_3(x-x_0)/m_3]\},
\]
where
\[
m_1 = c_2((c_1 - \xi_1)^2 + \xi_2^2)^{1/2} + c_2((c_2 - \xi_1)^2 + \xi_2^2)^{1/2},
\]
\[
m_2 = c_2((c_1 - \xi_1)^2 + \xi_2^2)^{1/2} - c_1((c_2 - \xi_1)^2 + \xi_2^2)^{1/2},
\]
\[
m_3 = \{(A/6)((c_2 - \xi_1)^2 + \xi_2^2)((c_1 - \xi_1)^2 + \xi_2^2)^{-1/2},
\]
\[
m_4 = [(c_2 - \xi_1)^2 + \xi_2^2]^{1/2} + [(c_2 - \xi_1)^2 + \xi_2^2]^{1/2},
\]
\[
m_5 = [(c_2 - \xi_1)^2 + \xi_2^2]^{1/2} - [(c_2 - \xi_1)^2 + \xi_2^2]^{1/2},
\]
\[
m = [(c_2 - c_1)^2 - m_3^2]/[4((c_2 - \xi_1)^2 + \xi_2^2)]^{1/2}(c_1 - \xi_1)^2 + \xi_2^2]^{1/2},
\]
if \(c_1 < c_2\) and \(\eta_2 = c_1\) if \(c_1 = c_2\). Here \(\text{sn}(u;k)\) and \(\text{cn}(u;k)\) are two Jacobian elliptic functions defined in Ref. 10.

Case 3. All \(c_1, ...\) \(c_4\) are real. We have several subcases.

(i) \(c_1 \neq c_j\), for all \(i \neq j, i,j = 1, ..., 4\). Without loss of generality, we can assume \(c_1 < c_2 < c_3 < c_4\). Then
\[
\eta_2 = \{k_1 + k_2 \text{sn}[(k_3(x-x_0)/k)]/k_4 + \text{sn}^2[(k_3(x-x_0)/k)]\},
\]
for \(c_3 < \eta_2 < c_4\).

(ii) \(c_1 < c_2 < c_3 < c_4\). It can be seen that this is a limiting case of (i). When the limit is taken in (33) and (34) as \(c_3 \rightarrow c_2\), the solutions are given by (29), where \(\text{sn}(u,1) = \tan u\) and \(1-\tan^2 u = \sec^2 u\).

(iii) \(c_1 < c_2 < c_3 < c_4\). By taking the limit as \(c_1 \rightarrow c_4\), and using the fact that \(\text{sn}(u,0) = \sin u\), the solution becomes
\[
\eta_2 = \{k_5 + k_6 \text{sn}^2(k_3(x-x_0))/[k_7 + \text{sn}^2(k_3(x-x_0))]\}
\]
where \(k_5, k_6, k_7\) are defined as in (i).

(iv) \(c_1 = c_2 < c_3 < c_4\). By using the same derivation as in (iii), we obtain
\[
\eta_2 = \{k_1 + k_2 \text{sn}^2(k_3(x-x_0))/[k_4 + \text{sn}^2(k_3(x-x_0))]\}
\]
where \(k_1, k_2, k_3,\) and \(k_4\) are defined as in (i).

(v) Either \(c_1 = c_2 = c_3 < c_4\) or \(c_1 < c_2 = c_3 = c_4\). It is the limiting case of (iii) or (iv), and by the same reasoning as in (ii), a solution like the first or second part of (29) exists.

(vi) \(c_1 = c_2 = c_3 = c_4\). \(\eta_2(x) = c_1\) is the only possibility. Since we have investigated the behavior of solutions ahead of and behind the bump, and shown that the solution of (28) always exists over the bump, a global solution of (28) can be constructed. In the following, we use numerical computation to find various types of solutions of (28), and the equation for the bump is given by \(b(x) = (1-x^2)^{1/2}\) for \(-1 \leq x \leq 1\), we divide these solutions into symmetric solitary wave-like solutions, which are first-type solutions, and unsymmetric solutions, which consist of second- and third-type solutions.
FIG. 2. Four different types of symmetric solutions. Supercritical case, \( \lambda_2 = -2 \).

(I) Symmetric solitary-wave like solutions. Let \( h_2 = 0 \) and the free solitary wave solution as given by (29) is

\[
\eta_2(x) = A_2 \{B \cosh^2[A_2^{1/2}(x-x_0)/2] + C \sinh^2[A_3^{1/2}(x-x_0)/2]\}^{-1}
\]

or

\[
\eta_2(x) = -A_3 \{C \cosh^2[A_3^{1/2}(x-x_0)/2] + B \sinh^2[A_3^{1/2}(x-x_0)/2]\}^{-1},
\]

where

\[
B = [(A_2^2/36) + (A_3 A_1/6)]^{1/2} - (A_2/6),
\]

\[
C = [(A_2^2/36) + (A_3 A_1/6)]^{1/2} + (A_2/6).
\]

By using the shooting method, we can find a symmetric solitary wave-like solution of (28), whose values ahead and behind the bump are given by (35). The numerical results are presented in Figs. 2 and 3. Four typical solitary wave-like solutions corresponding to \( \lambda_1 = -1 \) and \( \lambda_2 = 4 \) are shown in Fig. 2. In Fig. 3, we show the relationship between \( \eta_2(0) \) and \( \lambda_2 \) with \( \lambda_1 = -1 \). We note that for certain pairs of \( (\lambda_1, \lambda_2) \), no solitary wave-like solution can appear.

(II) Unsymmetric solutions. Assume \( \eta_2(-\infty) = 6_2 \neq 0 \) and \( \eta_2 \) is periodic ahead of the bump. For \( x \leq -1 \), either \( \eta_2 \) is given by one of the solutions in (29) or \( \eta_2 = H_2 \).

Since we have proved the existence of a solution of (28) for \( \{x_-, x_+\} \) and derived the possible solutions of (28) in \( \{x_+ \infty\} \), we can solve (28) numerically by using (29) or \( \eta_2 = H_2 \) behind the bump. The numerical results are presented in Figs. 4–7. Figure 4 shows the second-type solutions that have \( 29 \) as their solutions in \( (-\infty, -1) \). In Fig. 5 we show a second-type solution whose mean depth of the wave ahead of the bump is \( -\eta_2(-\infty) \). We also consider third-type solutions that are equal to \( H_2 \) for \( x \geq 1 \) and periodic for \( x \geq -1 \). Figure 6 shows a typical third-type solution whose mean depth ahead of the bump is \( -\eta_2(-\infty) \). An interesting phenomenon appears in the process of computing third-type solutions. At discrete values of \( \lambda_2 \) there are symmetric solutions without a periodic part embedded in the
third-type solutions and Fig. 7 presents such a solution. In Figs. 2–7, we choose $\rho=0.2$, $h=0.6$, $\lambda_1=-1$, $T_1=2.8$, and $T_2=4$.

B. Subcritical case ($F_4F_1<0$)

The same assumptions as in Sec. III A are given in this section. The only difference here is $F_1F_4<0$. By dividing both sides of (27) by $F_4$, we have

$$\eta_{2xx}=B_1 \eta_2^2 \eta_{2x} + A_2 \eta_2 \eta_{2x} + A_3 \eta_{2x} + A_4 b_x,$$

where

$$B_1=-\frac{F_1}{F_4} > 0, \quad A_2=-\frac{\lambda_1}{F_4}, \quad A_3=-\frac{\lambda_2}{F_4},$$

$$A_4=-\frac{F_5}{F_4}.$$

If $\eta_2(x)$ is a solution of (36) and tends to $H_2$ at $x=-\infty$ with $\eta_{2x}(\infty)=0$, then from $b(\infty)=0$, $\eta_2$ satisfies

$$\eta_{2xx}=(B_1/3) \eta_2^3 + (A_2/2) \eta_2^2 + A_3 \eta_2 + d + A_4 b(x),$$

(37)

where $d=-B_1(H_2^2/3) - A_2(H_2^2/2) - A_4 H_2$.

Here we choose $\eta_2=H_2$ in $(-\infty, x^-)$, since a solitary wave solution does not exist for (36) when $b(x)=0$. It can also be shown that (37) possesses a solution with a continuous second-order derivative in $[x^-, x^+]$ for large $-A_3$, and by using the matching process as before, we can find the solution for all real $x$. We note that our numerical computation shows that no bounded solution exists for a small value of $-A_3>0$.

Now the solution in $(-\infty, x^+]$ can be connected to the solution in $[x^+, \infty)$, where $b(x)=0$ numerically by using the shooting method, as in Sec. III A. As before, we multiply (37) by $\eta_{2x}$ and integrate the resulting equation from $x^-$ to $x^+$. to obtain

$$(\eta_{2x})^2=(B_1/6) \eta_2^4 + (A_2/3) \eta_2^3 + A_3 \eta_2^2 + 2d \eta_2 + e$$

$$=g(\eta_2),$$

where $e=-B_1(\alpha^2/6)-A_2(\alpha^2/3)-A_3 \alpha^2 - 2d \alpha + \beta^2$, $\eta_2(x^+)=\alpha$, and $\eta_{2x}(x^+)=\beta$.

Let $c_1$, $c_2$, $c_3$, $c_4$ be zeros of $g(\eta_2)$. When none of the $c_i$'s are real, a solution is unbounded. When two of the $c_i$'s are real, say $c_1$ and $c_2$, and $c_1=c_2$, then $\eta_2=c_1$ is the only possible solution and $\eta_2$ is unbounded otherwise. When all $c_i$'s are real, there are several different cases. If $c_1<c_2<c_3<c_4$, the solution $\eta_2$ is periodic and $c_3=\eta_2=c_3$. If $c_1=c_2<c_3<c_4$ or $c_1<c_2<c_3=c_4$, $\eta_2$ is nonperiodic. If $c_1=c_2=c_3=c_4$, the solution is a constant. In other cases, the solutions are unbounded.

We present the numerical results of global solutions in Figs. 8–11. Figure 8 shows a typical third-type solution and Fig. 9 shows a hydraulic jump, which is the limiting solution of the third-type solution, as $\lambda_2$ becomes decreased and tending to some critical value. Only unbounded solutions are found when $\lambda_2$ is decreased further below the critical value. We also find multicrest symmetric solutions without a periodic
part embedded in the third-type solutions at discrete values of $\lambda_2$, even though no first-type symmetric solutions exist in the subcritical case. Figures 10–11 present two such cases. In Figs. 8–11 we choose the same $\rho$ and $h$ as in Figs. 4–7, but $\lambda_1 = 1$ and $T_1$ and $T_2$ are equal to $10^{-2}$. The obstruction is same as before.

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