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Internal capillary-gravity waves of a two-layer fluid with free surface over an obstruction—Forced extended KdV equation

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In this paper we study steady capillary-gravity waves in a two-layer fluid bounded above by a free surface and below by a horizontal rigid boundary with a small obstruction. Two critical speeds for the waves are obtained. Near the smaller critical speed, the derivation of the usual forced KdV equation (FKdV) fails when the coefficient of the nonlinear term in the FKdV vanishes. To overcome this difficulty, a new equation, called a forced extended KdV equation (FEKdV) governing interfacial wave forms, is obtained by a refined asymptotic method. Various solutions and numerical results of this equation are presented. © *1996 American Institute of Physics.* [S1070-6631(96)01202-3]

I. INTRODUCTION

We consider a two-layer medium of immiscible, inviscid, and incompressible fluids having different but constant densities. The medium is bounded above by a free surface and below by a horizontal rigid boundary with an interface in between (Fig. 1). The surface tension effect is taken into consideration at both the free surface and the interface. We assume that a two-dimensional object is moving along the lower boundary at a constant speed, and in reference to a coordinate system moving with the object, the fluid flow is steady. Two critical speeds are obtained. When the object is moving at a speed near either one of them, a FKdV for steady flow can be derived and has been extensively investigated in Refs. 1 and 2. We note that numerical studies of steady flow of a two-layer fluid over a bump or a step bounded by a free or rigid upper boundary were carried out by Forbes,[3] Belward and Forbes,[4] Sha and Vanden-Broeck,[5] and Moni and King,[6] among others, and an asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen[7] on the basis of the FKdV theory, and with surface tension by Choi, Sun, and Shen,[8] where a forced modified KdV equation (FMKdV) was obtained. The FKdV theory fails when the coefficient of the nonlinear term or that of the third derivative in the FKdV vanishes. In the case considered here, when the wave speed is near the smaller critical speed for internal waves, the amplitude of which is larger at the interface than at the free surface, the coefficient of the nonlinear term in the FKdV may vanish. Furthermore, at a wave speed near either one of the critical speeds, the coefficient of the third-order derivative may also vanish. To overcome the difficulty of a vanishing nonlinear term in the FKdV, we shall develop a refined asymptotic method to derive a new equation, called the forced extended KdV equation (FEKdV), in the following form:

$$(F_1\eta_2^2+F_2\eta_2+F_3)\eta_{2x}+F_4\eta_{2xxx}=-F_5b_x,$$

where F_1-F_5 are constants depending on several parameters and $z=-H^-+b(x)$ is the equation of the obstruction. The objective of this paper is to investigate solutions of the FEKdV, which represent possible interfacial wave forms. We remark that when the coefficient of the third-order derivative in the FEKdV vanishes, a forced perturbed KdV equation with a fourth-order derivative could be derived, and is deferred to a subsequent study.

In Sec. II, we formulate the problem and develop the asymptotic scheme to derive the FEKdV. Section III consists of two sections. The supercritical case of $F_1F_4>0$ and the subcritical case of $F_1F_4<0$ are studied in Secs. III A and III B, respectively. In general, we can find three types of solutions. The first-type solution consists of symmetric solitary-wave-like solutions. The second-type solution is one that is a part of a free solitary wave behind the bump and a periodic wave solution ahead of the bump. The free solitary wave is a solitary wave solution of the extended KdV equation without forcing. By a third-type solution we mean a solution that is constant behind the bump and periodic ahead of the bump. In many cases both second- and third-type solutions do satisfy the conservation of mass, even if they do not tend to zero far upstream. In both Secs. III A and III B, analytical and numerical results, which indicate the appearance of various types of solutions, are presented. It is found that four branches of first-type solutions can appear in the supercritical case, and there are no first- and second-type solutions in the subcritical case. The third-type solutions appear in both supercritical and subcritical cases. In both cases, symmetric solutions without a periodic part are embedded in the third-type solutions at discrete values of a parameter, and a hydraulic jump wave solution appears as a limiting case of third-type solutions in the subcritical case.

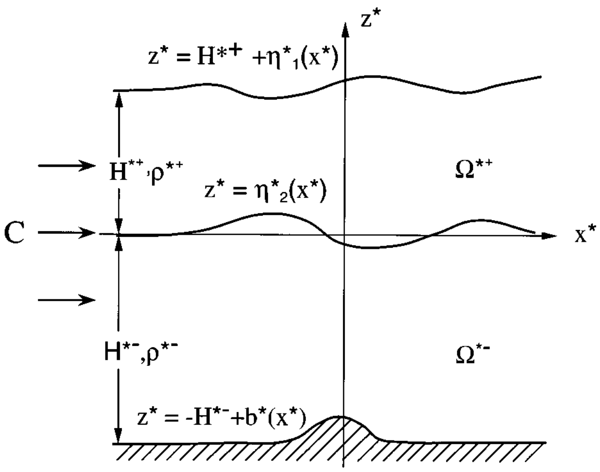

FIG. 1. Fluid domain.

II. FORMULATION AND SUCCESSIVE APPROXIMATE EQUATIONS

We consider steady internal capillary-gravity waves between two immiscible, inviscid, and incompressible fluids of constant but different densities, bounded above by a free surface and below by a horizontal rigid boundary, with a small obstruction of compact support. The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-}, respectively (Fig. 1). Assume that the small obstruction is moving with a constant speed C. In reference to a coordinate system moving with the obstruction, the flow is steady and moving with the speed C far upstream. The governing equations and boundary conditions are the following: In $\Omega^{*\pm}$,

$$u^{*\pm}_{x^*}+w^{*\pm}_{z^*}=0,$$

$$u^{*\pm}u^{*\pm}_{x^*}+w^{*\pm}u^{*\pm}_{z^*}=-p^{*\pm}_{x^*}/\rho^{*\pm},$$

$$u^{*\pm}w^{*\pm}_{x^*}+w^{*\pm}w^{*\pm}_{z^*}=-p^{*\pm}_{z^*}/\rho^{*\pm}-g;$$

at the free surface, $z^* = H^{*+} + \eta_1^*$,

$$u^{*+}\eta^*_{1x^*}-w^{*+}=0,$$

$$p^{*+}=-T_1^*\eta^*_{1x^*x^*}/(1+\eta^{*2}_{1x^*})^{3/2};$$

at the interface, $z^* = \eta_2^*$,

$$p^{*+}-p^{*-}=T_2^*\eta^*_{2x^*x^*}/(1+\eta^{*2}_{2x^*})^{3/2},$$

$$u^{*\pm}\eta^*_{2x^*}-w^{*\pm}=0;$$

at the rigid bottom, $z^*=-H^{*-}+b^*(x^*)$,

$$w^{*-}-b^*_{x^*}u^{*-}=0,$$

where $u^{*\pm}$ and $w^{*\pm}$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant, and T_1^* and T_2^* are surface tension constants at the free surface and the interface, respectively. We define the following nondimensional variables:

$$\epsilon=H/L\ll 1,\quad \eta_1=\frac{\epsilon^{-1}\eta_1^*}{H^{*-}},\quad \eta_2=\frac{\epsilon^{-1}\eta_2^*}{H^{*-}},$$

$$p^{\pm}=\frac{p^{*\pm}}{gH^{*-}\rho^{*-}},$$

$$(x,z)=(\epsilon x^*,z^*)/H^{*-},$$

$$(u^{\pm},w^{\pm})=(gH^{*-})^{-1/2}(u^{*\pm},\epsilon^{-1}w^{*\pm}),$$

$$\rho^+=\frac{\rho^{*+}}{\rho^{*-}}<1,\quad \rho^-=\frac{\rho^{*-}}{\rho^{*-}}=1,\quad U=\frac{C}{(gH^{*-})^{1/2}},$$

$$T_i=T_i^*/\rho^{*-}g(H^{*-})^2,\quad i=1,2,$$

$$h=H^{*+}/H^{*-},\quad b(x)=b^*(x)(H^{*-}\epsilon^3)^{-1},$$

where L is the horizontal scale, H is the vertical scale, $b(x)=b^*(x)(H^{*-}\epsilon^3)^{-1}$, H^{*+} and H^{*-} are the equilibrium depths of the upper and lower fluids at $x^*=-\infty$, respectively, and $z^*=-H^{*-}+b^*(x)$ is the equation of the obstruction. In terms of the nondimensional quantities, the above equations become, in $\Omega^{\pm}$,

$$u_x^{\pm}+w_z^{\pm}=0, \tag{1}$$

$$u^{\pm}u_x^{\pm}+w^{\pm}u_z^{\pm}=-p_x^{\pm}/\rho^{\pm}, \tag{2}$$

$$\epsilon^2u^{\pm}w_x^{\pm}+\epsilon^2w^{\pm}w_z^{\pm}=-p_z^{\pm}/\rho^{\pm}-1; \tag{3}$$

at $z=h+\epsilon\eta_1$,

$$p^+=-\epsilon^3T_1\eta_{1xx}/(1+\epsilon^4\eta_{1x}^2)^{3/2}, \tag{4}$$

$$\epsilon u^+\eta_{1x}-w^+=0; \tag{5}$$

at $z=\epsilon\eta_2$,

$$\epsilon u^-\eta_{2x}-w^-=0, \tag{6}$$

$$\epsilon u^+\eta_{2x}-w^+=0, \tag{7}$$

$$p^+-p^-=\epsilon^3T_2\eta_{2xx}/(1+\epsilon^4\eta_{1x}^2)^{3/2}; \tag{8}$$

at $z=-1+\epsilon^3b(x)$,

$$w^-=\epsilon^3ub_x, \tag{9}$$

where $b(x)$ has a compact support.

In the following, we use a unified asymptotic method to derive an approximate equation for the interface $\eta_2(x)$. We assume that $u^{\pm}$, $w^{\pm}$, and $p^{\pm}$ are functions of x,z near the equilibrium state $u^{\pm}=u_0$, $w^{\pm}=0$, $p^+=-\rho^+z+\rho^+h$, and $p^-=-\rho^-z+\rho^+h$, where u_0 is a constant, and possess asymptotic expansions:

$$(u^{\pm},w^{\pm},p^{\pm})=(u_0,0,-\rho^{\pm}z+\rho^+h)+\epsilon(u_1^{\pm},w_1^{\pm},p_1^{\pm})$$
$$+e^2(u_2^{\pm},w_2^{\pm},p_2^{\pm})+\epsilon^3(u_3^{\pm},w_3^{\pm},p_3^{\pm})$$
$$+O(\epsilon^4). \tag{10}$$

By inserting (10) into (1)–(4) and (7)–(9) and arranging the resulting equations according to the powers of ϵ, it follows that $(u_0,0,-\rho^{\pm}z+\rho^+h)$ are the solutions of the zeroth-order system of equations, and the equations of the order ϵ are as follows:

$$u_{1x}^{\pm}+w_{1z}^{\pm}=0, \tag{11}$$

$$u_0 u^{\pm}_{1x} = -p^{\pm}_{1x}/\rho^{\pm}, \tag{12}$$

$$p^{\pm}_{1z} = 0; \tag{13}$$

at $z=h$,

$$p^{+}_1 + \eta_1 p^{+}_{0z} = 0; \tag{14}$$

at $z=0$,

$$p^{+}_1 - p^{-}_1 + \eta_2(p^{+}_{0z} - p^{-}_{0z}) = 0, \tag{15}$$

$$u_0 \eta_{2x} - w^{+}_1 = 0; \tag{16}$$

at $z=-1$,

$$w^{-}_1 = 0. \tag{17}$$

Hereafter, for the sake of convenience we shall use ρ to denote ρ^+ and set ρ^- equal to 1. From (13), $p^{\pm}_1$ are functions of x only. It follows that $p^{+}_1 = \rho\eta_1$ by (14) and $p^{-}_1 = \rho\eta_1 + \eta_2(1-\rho)$ by (15). We can find $w^{\pm}_1$ by using (11), (12), (16), and (17) so that

$$\begin{aligned} w^{+}_1 &= z(\eta_{1x}/u_0) + u_0\eta_{2x}, \\ w^{-}_1 &= (z+1)[\rho\eta_{1x} + (1-\rho)\eta_{2x}]/u_0. \end{aligned} \tag{18}$$

Here $u^{\pm}_1$ are also derived from (11),

$$\begin{aligned} u^{+}_1 &= (-\eta_1 + H_1)/u_0, \\ u^{-}_1 &= [-\rho\eta_1 - (1-\rho)\eta_2 + \rho H_1 + (1-\rho)H_2]/u_0, \end{aligned} \tag{19}$$

where we assume $\eta_1(x=-\infty) = H_1$, $\eta_2(x=-\infty) = H_2$, and $u^{\pm}_1(x=-\infty) = 0$.

Similarly, we can find $p^{\pm}_2, w^{\pm}_2, u^{\pm}_2, p^{\pm}_3, w^{\pm}_3, u^{\pm}_3$ in terms of η_1 and η_2 without using the kinematic conditions (5) and (6). From (5) and (6), and the asymptotic expansion of u^- and w^-, we have at $z=h$,

$$\begin{aligned} &u_0\eta_{1x} - w^{+}_1 + \epsilon(u^{+}_1\eta_{1x} - \eta_1 w^{+}_{1z} - w^{+}_2) + \epsilon^2[u^{+}_2\eta_{1x} \\ &\quad + \eta_1\eta_{1x}u^{+}_{1z} - (\tfrac{1}{2})w^{+}_{1zz}\eta_1^2 - \eta_1 w^{+}_{2z} - w^{+}_3] + O(\epsilon^3) = 0; \end{aligned} \tag{20}$$

and at $z=0$,

$$\begin{aligned} &u_0\eta_{2x} - w^{-}_1 + \epsilon(u^{-}_1\eta_{2x} - \eta_2 w^{-}_{1z} - w^{-}_2) + \epsilon^2[u^{-}_2\eta_{2x} \\ &\quad + \eta_2\eta_{2x}u^{-}_{1z} - (\tfrac{1}{2})w^{-}_{1zz}\eta_2^2 - \eta_2 w^{-}_{2z} - w^{-}_3] + O(\epsilon^3) = 0. \end{aligned} \tag{21}$$

Making use of these equations, we can find an approximate equation for the interface η_2. We substitute $u^{\pm}_1, w^{\pm}_1, u^{\pm}_2, w^{\pm}_2, w^{\pm}_3$ into (20), (21), eliminate η_1 from (21) by finding a relation between η_1 and η_2 up to $O(\epsilon^2)$ from (20), and obtain

$$\begin{aligned} &[u_0 - \rho c_1/u_0 - (1-\rho)/u_0]\eta_{2x} + \epsilon(E_1\eta_2\eta_{2x} + E_2\eta_{2x}) \\ &\quad + \epsilon^2(F_1\eta_2^2\eta_{2x} + F_2\eta_2\eta_{2x} + F_3\eta_{2x} + F_4\eta_{2xxx} + F_5 b_x) \\ &\quad + O(\epsilon^3) = 0, \end{aligned} \tag{22}$$

where, if we let $c_1 = [2u_0^2 - (1-\rho)]/(\rho + u_0^2 - h)$, $D_1 = u_0/(\rho + u_0^2 - h)$, $\lambda = u_2^{\pm}(-\infty)$, $H = \rho H_1 + (1-\rho)H_2$, and $R = \rho c_1 + 1 - \rho$, then

$$\begin{aligned} E_1 &= -(R^2 + 2Ru_0^2)u_0^{-3} - \rho D_1[(hc_1^2 - R^2)u_0^{-4} \\ &\quad + (2c_1^2 - 2R - 2c_1)u_0^{-2}], \\ E_2 &= [-\rho(H_1 - c_1H_2) + H]u_0^{-1} - R[-\rho c_1 H_2 \\ &\quad - (1-\rho)H_2]u_0^{-3} - \rho\, D_1\{(RH_2 + c_1H - 2 + c_1H_1 \\ &\quad - 2c_1^2H_2)u_0^{-2} + [Rc_1\rho H_2 + R(1-\rho)H_2 \\ &\quad - hc_1^2H_2]u_0^{-4}\}, \\ F_1 &= -\rho\, D_1u_0^{-1}(3c_1^3 - 3c_1^2 + R^2/2)u_0^{-3} + (3hc_1^3/2 \\ &\quad - 3R^3/2)u_0^{-5} + 3D_1(\rho u_0^{-1} + \rho R u_0^{-3})[(3R/2 + c_1 \\ &\quad - c_1^2)u_0^{-1} + (R^2/2 - hc_1^2/2)u_0^{-3}] - 3R^2u_0^{-3}/2 \\ &\quad - 3R^3u_0^{-5}/2, \\ F_2 &= (-\rho\, D_1u_0^{-1})[u_0^{-3}(2HR - 2H_1c_1^2 + 2H_1c_1) \\ &\quad + u_0^{-5}(3HR^2 - 3hH_1c_1^2)] - 2D_1(\rho u_0^{-1} + \rho R u_0^{-3}) \\ &\quad \times [u_0^{-1}(H_2R - 2H_2c_1^2 + H_2c_1 + c_1H_1) + u_0^{-3}(R^2H_2 \\ &\quad - hH_2c_1^2)] + 2HRu_0^{-3} + 3HR^2u_0^{-5}, \\ F_3 &= (-\rho\, D_1u_0^{-1})[(\lambda(2 + Ru_0^{-2} - c_1 - hc_1u_0^{-2}) \\ &\quad + u_0^{-5}(3hc_1H_1^2/2 - 3RH^2/2) + u_0^{-3}(c_1H_1^2/2 \\ &\quad - hc_1H_1^2/2 - H^2/2)] - D_1(\rho u_0^{-1} + \rho R u_0^{-3}) \\ &\quad \times [u_0^{-1}(H_2^2c_1^2/2 - H_1H_2c_1) + u_0^{-3}(hc_1^2H_2^2/2 \\ &\quad - H_2^2R/2)] + \lambda(1 + Ru_0^{-2}) - H^2u_0^{-3}/2 \\ &\quad - 3RH^2u_0^{-5}/2, \\ F_4 &= (-\rho\, D_1u_0^{-1})\{-c_1(\rho h^2/2 - T_1 + \rho/3)u_0^{-1} - [u_0^2\rho h \\ &\quad - T_2 + (1-\rho)/3]u_0^{-1} + [c_1(\rho h^3/3 - hT_1)/u_0\rho] \\ &\quad + u_0h^2/2\} - c_1(\rho h^2/2 - T_1 + \rho/3)u_0^{-1} \\ &\quad - [u_0^2\rho h - T_2 + (1-\rho)/3]u_0^{-1}, \\ F_5 &= \rho\, D_1 - u_0. \end{aligned}$$

III. EXTENDED KdV EQUATION WITH FORCING

From the zeroth-order term of (22), we obtain

$$u_0 - (\rho c_1/u_0) - (1-\rho)/u_0 = 0,$$

and by the expression for c_1 in (22), it follows that

$$u_0^4 - (1+h)u_0^2 + h(1-\rho) = 0 \tag{23}$$

and

$$u_0^2 = \{1 + h \pm [(1-h)^2 + 4\rho h]^{1/2}\}/2.$$

We denote the two values of u_0^2 by u_{01}^2 and u_{02}^2, respectively, corresponding to the plus and minus signs. Without loss of generality we assume u_{01} and u_{02} are both positive and call them critical speeds, near each of which a nonlinear theory for the motion of the interface has to be developed.

Next, we consider the coefficients of $\eta_2\eta_{2x}$ and η_{2x} in the first-order terms of Eq. (22). Note that E_2 vanishes if $H_1=H_2=0$ or

$$H_1=H_2[(2u_0^3/\rho c_1 D_1)-2(u_0^2/c_1)+2c_1+h(c_1/u_0^2)-1]$$
$$=H_2\{-1+2u_0^2(u_0^2-h)(u_0^2+1-\rho)^{-1}+\rho^{-1}u_0^{-2}[(4-2\rho+3h)u_0^2-h(1-\rho)]\}.$$

If E_1 in (22) is not zero, a FKdV can be derived and has already been studied in Ref. 1. However, E_1 may vanish. First, let us simplify the expression of E_1,

$$E_1=-[(\rho c_1+1-\rho)^2/u_0^3]-2[(\rho c_1+1-\rho)/u_0]-\rho\ D_1\{-2[(\rho c_1+1-\rho)/u_0]-[(\rho c_1-\rho+1)^2/u_0^3]$$
$$+2(c_1^2/u_0)+h(c_1^2/u_0^3)-2(c_1/u_0)\}/u_0$$
$$=3(u_0\rho)^{-1}(u_0^2+\rho-h)[\rho(u_0^2h-u_0^4-u_0^2+1)-u_0^4+2u_0^2-1]=3u_0(1-u_0^2)[\rho h(u_0^2+\rho-h)]^{-1}[u_0^4+(1-2h)u_0^2+h^2-1],$$

where (23) has been used. When u_0 satisfies the equation (23), it is seen that u_0^2 is neither 1 nor $h-\rho$. Hence, $E_1=0$ implies $u_0^4+(1-2h)u_0^2+h^2-1=0$. Let $u_0=u_{01}$ or u_{02}. Then

$$u_{01}^4+(1-2h)u_{01}^2+h^2-1=1+h\rho+(2-h)[(1-h)^2+4\rho h]^{1/2}, \tag{24}$$

$$u_{02}^4+(1-2h)u_{02}^2+h^2-1=1+h\rho-(2-h)[(1-h)^2+4\rho h]^{1/2}. \tag{25}$$

Equation (24) tells us that E_1 does not vanish if we take u_{01} as a critical speed. Suppose both sides of (24) vanish. Then real u_{01}^2 implies $h<\frac{5}{4}$ and the right-hand side of (24) is greater than zero. This is a contradiction. Thus, the only possible case for $E_1=0$ is that u_0 is equal to the critical speed u_{02}, and it is easy to show that $E_1=O(\epsilon)=\beta\epsilon$ if $u_0^2=u_{02}^2+O(\epsilon)$, and

$$1+h\rho=(2-h)[(1-h)^2+4\rho h]^{1/2}+O(\epsilon). \tag{26}$$

With the conditions (21) and (25), we obtain a time-independent FEKdV,

$$F_1\eta_2^2\eta_{2x}+\lambda_1\eta_2\eta_{2x}+\lambda_2\eta_{2x}+F_4\eta_{2xxx}+F_5 b_x=0, \tag{27}$$

where

$$F_1=3u_0(4\rho+3h-u_0^2)(u_0^2+\rho-h)^{-1},$$
$$\lambda_1=H_2[-4u_0+5(1-\rho)(u_0^2-h)(u_0^2+\rho-h)^{-1}]+H_1\{5\rho+[(3h-5\rho^3)u_0^4+6h(1-\rho)u_0^2+3h(1-\rho)^2]$$
$$\times(u_0^2+\rho-h)^{-1}\rho^{-1}u_0^{-4}\}+\beta,$$
$$\lambda_2=\lambda[2(1+h)u_0^2-4h(1-\rho)]u_0^{-2}(u_0^2+\rho-h)^{-1}-2[\rho H_1+(1-\rho)H_2]^2(u_0^2-h)u_0^{-3}(u_0^2+\rho-h)^{-1}$$
$$-hH_1^2[(3+\rho)u_0^2-3+2\rho+\rho^2][2u_0^5(u_0^2+\rho-h)]^{-1}-2H_2^2(u_0^2+\rho-h)^{-1}\{-(1/2\rho u_0)+(2u_0^2+h)$$
$$\times[(u_0^2-1+\rho)^2/2u_0^3\rho]\}+2H_1H_2(u_0^2-1+\rho)u_0^{-1}(u_0^2+1-\rho)^{-1},$$
$$F_4=u_0^{-1}(u_0^2+\rho-h)^{-1}\{h(1+h)-u_0^2(h^2+1+3\rho h)+3[u_0^2(u_0^2+\rho-1)T_1/\rho]+3(u_0^2-h)T_2\}/3,$$
$$F_5=u_0(h-u_0^2)(u_0^2+\rho-h)^{-1}.$$

The coefficients F_1–F_5 here are the simplified forms of F_1–F_5 in the previous section by using (23). We note that, for some special choice of β, λ_1 becomes zero and (27) becomes a FMKdV equation. Similar results as given in Ref. 8 can be obtained, and will not be discussed here. The sign of F_4F_1 determines the existence of solutions of (27). In the following sections, the two cases $F_4F_1>0$ and $F_4F_1<0$ will be considered separately. We remark in passing that if the surface tension constants T_1 and T_2 satisfy $F_4=0$ for given ρ and h the coefficient of the third-order derivative vanishes and a forced perturbed KdV equation could be derived to replace the FEKdV equation.

A. Supercritical case ($F_4F_1>0$)

We assume $U=u_0+\lambda\epsilon^2+O(\epsilon^3)$ and consider (27) for $F_4F_1>0$. Here (27) can be rewritten as

$$\eta_{2xxx}=-A_1\eta_2^2\eta_{2x}+A_2\eta_2\eta_{2x}+A_3\eta_x+A_4 b_x, \tag{28}$$

where $A_1=F_1/F_4>0$, $A_2=-\lambda_1/F_4$, $A_3=-\lambda_2/F_4$, $A_4=-F_5/F_4$. When $b_x\equiv 0$, (28) has solitary wave solutions whose value is H_2 at $x=\pm\infty$ for $A=A_3+H_2A_2-A_1H_2^2>0$:

$$\eta_2(x)=H_2+A\{B\ \cosh^2[A^{1/2}(x-x_0)/2]$$
$$+C\ \sinh^2[A^{1/2}(x-x_0)/2]\}^{-1}, \tag{29}$$

or

$$\eta_2(x)=H_2-A\{C\cosh^2[A^{1/2}(x-x_0)/2]+B\sinh^2[A^{1/2}(x-x_0)/2]\}^{-1},$$

where

$$B=\{[(A_2-2A_1H_2)^2+6AA_1]^{1/2}-(A_2-2A_1H_2)\}/6,$$

$$C=\{[(A_2-2A_1H_2)^2+6AA_1]^{1/2}+(A_2-2A_1H_2)\}/6,$$

and x_0 is a phase shift to be determined by the initial condition. For $A\leqslant 0$, there is no solitary wave solution. The solutions in (29) are obtained as in the classical case by taking the limit of elliptic functions in the periodic solutions of (28) for $b_x=0$ when the wavelength tends to infinity.

Next, we consider (28) when $b_x\neq 0$. Since we assume supp $b(x)=[x^-,x^+]$, the intervals $(-\infty,x^-)$, $[x^-,x^+]$, and (x^+,∞) are to be discussed separately.

(1) $(-\infty,x^-)$. We have $b_x=0$, and may choose either (29) as a solution of (28) or the trivial solution $\eta_2\equiv H_2$, on $(-\infty,x^-)$.

(2) $[x^-,x^+]$. In this interval $b_x\neq 0$, and by integrating (28) from $-\infty$ to $x\in[x^-,x^+]$, we obtain

$$\eta_{2xx}=-(A_1/3)\eta_2^3+(A_2/2)\eta_2^2+A_3\eta+d+A_4b(x), \quad (30)$$

where

$$d=A_1(H_2^3/3)-(A_2H_2^2/2)-A_3H_2.$$

It can be shown[9] that (30) has a solution with a continuous third-order derivative in $[x^-,x^+]$ satisfying $\eta_2(x^-)=\alpha$ and $\eta_{2x}(x^-)=\beta$ for any fixed constants α and β.

(3) $[x^+,\infty)$. Here $b(x)=0$. By integrating (27) from x^+ to $x>x^+$, multiplying η_{2x} to the resulting equation, and integrating it from x^+ to $x>x^+$ again, it follows that

$$(\eta_{2x})^2=-(A_1/6)\eta_2^4+(A_2/3)\eta_2^3+A_3\eta_2^2+2d\eta_2+e=f(\eta_2), \quad (31)$$

where

$$e=A_1(\alpha^4/6)-A_2(\alpha^3/3)-A_3\alpha^2-2d\alpha+\beta^2,$$

$$\eta_2(x^+)=\alpha, \quad \eta_{2x}(x^+)=\beta,$$

$$\eta_{2xx}(x^+)=-A_1(\alpha^3/3)+A_2(\alpha^2/2)+A_3\alpha+d.$$

Let c_1, c_2, c_3, and c_4 be four zeros of $f(\eta_2)$ and we investigate the solutions of (31) for different cases of c_1, c_2, c_3, and c_4.

Case 1. None of the $c_1,\ldots,c_4$ are real. There is no solution since $(\eta_{2x})^2<0$.

Case 2. Two of the $c_1,\ldots,c_4$ are real. Let $c_1\leqslant c_2$ be real and c_3,c_4 are not real. Then $f(\eta_2)=(-A_1/6)(\eta_2-c_1)(\eta_2-c_2)[(\eta_2+\xi_1)^2+\xi_2^2]$ for some ξ_1,ξ_2 in $(-\infty,\infty)$, and

$$\eta_2=\{m_1-m_2\,\mathrm{cn}[m_3(x-x_0);m]\}/\{m_4-m_5\,\mathrm{cn}[m_3(x-x_0);m]\}, \quad (32)$$

where

$$m_1=c_2[(c_1-\xi_1)^2+\xi_2^2]^{1/2}+c_1[(c_2-\xi_1)^2+\xi_2^2]^{1/2},$$

$$m_2=c_2[(c_1-\xi_1)^2+\xi_2^2]^{1/2}-c_1[(c_2-\xi_1)^2+\xi_2^2]^{1/2},$$

$$m_3=\{(A_1/6)[(c_2-\xi_1)^2+\xi_2^2][(c_1-\xi_1)^2+\xi_2^2]\}^{-1/2},$$

$$m_4=[(c_1-\xi_1)^2+\xi_2^2]^{1/2}+[(c_2-\xi_1)^2+\xi_2^2]^{1/2},$$

$$m_5=[(c_1-\xi_1)^2+\xi_2^2]^{1/2}-[(c_2-\xi_1)^2+\xi_2^2]^{1/2},$$

$$m=[(c_2-c_1)^2-m_5^2]/\{4[(c_2-\xi_1)^2+\xi_2^2]^{1/2}[(c_1-\xi_1)^2+\xi_2^2]^{1/2}\},$$

if $c_1<c_2$ and $\eta_2=c_1$ if $c_1=c_2$. Here $\mathrm{sn}(u;k)$ and $\mathrm{cn}(u;k)$ are two Jacobian Elliptic functions defined in Ref. 10.

Case 3. All $c_1,\ldots,c_4$ are real. We have several subcases.

(i) $c_i\neq c_j$ for all $i\neq j$, $i,j=1,\ldots,4$. Without loss of generality, we can assume $c_1<c_2<c_3<c_4$. Then

$$\eta_2=\{k_1+k_2\,\mathrm{sn}^2[k_3(x-x_0);k]\}/\{k_4+\mathrm{sn}^2[k_3(x-x_0);k]\}, \quad (33)$$

for $c_3<\eta_2<c_4$.

$$\eta_2=\{k_5+k_6\,\mathrm{sn}^2[k_3(x-x_0);k]\}/\{k_7+\mathrm{sn}^2[k_3(x-x_0);k]\}, \quad (34)$$

for $c_1<\eta_2<c_2$, where

$$k_1=c_4(c_3-c_1)(c_4-c_3)^{-1}, \quad k_2=c_1,$$

$$k_3=[A_1c_4(c_3-c_1)/6]^{1/2}, \quad k_4=(c_3-c_1)(c_4-c_3)^{-1},$$

$$k_5=c_1(c_4-c_2)(c_2-c_1)^{-1}, \quad k_6=c_4,$$

$$k_7=(c_4-c_2)(c_2-c_1)^{-1},$$

$$k=[(c_4-c_3)(c_2-c_1)(c_4-c_2)^{-1}(c_3-c_1)^{-1}]^{1/2}.$$

(ii) $c_1<c_2=c_3<c_4$. It can be seen that this is a limiting case of (i). When the limit is taken in (33) and (34) as $c_3\to c_2$, the solutions are given by (29), where $\mathrm{sn}(u,1)=\tanh u$ and $1-\tanh^2 u=\mathrm{sech}^2 u$.

(iii) $c_1<c_2<c_3=c_4$. By taking the limit as $c_3\to c_4$, and using the fact that $\mathrm{sn}(u,0)=\sin u$, the solution becomes

$$\eta_2=[k_5+k_6\sin^2 k_3(x-x_0)]/[k_7+\sin^2 k_3(x-x_0)],$$

where k_3, k_5, k_6, and k_7 are defined as in (i).

(iv) $c_1=c_2<c_3<c_4$. By using the same derivation as in (iii), we obtain

$$\eta_2=[k_1+k_2\sin^2 k_3(x-x_0)]/[k_4+\sin^2 k_3(x-x_0)],$$

where k_1, k_2, k_3, and k_4 are defined as in (i).

(v) Either $c_1=c_2=c_3<c_4$ or $c_1<c_2=c_3=c_4$. It is the limiting case of (iii) or (iv), and by the same reasoning as in (ii), a solution like the first or second part of (29) exists.

(vi) $c_1=c_2=c_3=c_4$. $\eta_2(x)\equiv c_1$ is the only possibility. Since we have investigated the behavior of solutions ahead of and behind the bump, and shown that the solution of (28) always exists over the bump, a global solution of (28) can be constructed. In the following, we use numerical computation to find various types of solutions of (28), and the equation for the bump is given by $b(x)=(1-x^2)^{1/2}$ for $-1\leqslant x\leqslant 1$. We divide these solutions into symmetric solitary wave-like solutions, which are first-type solutions, and unsymmetric solutions, which consist of second- and third-type solutions.

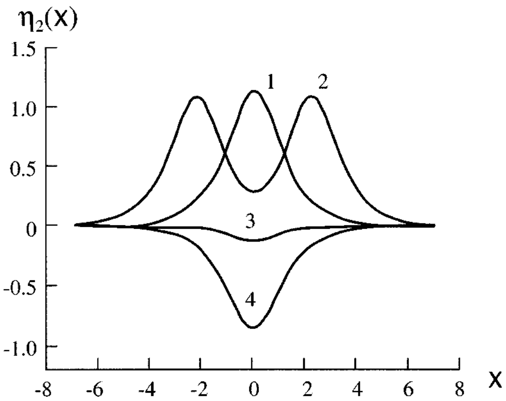

FIG. 2. Four different types of symmetric solutions. Supercritical case, $\lambda_2=-2$.

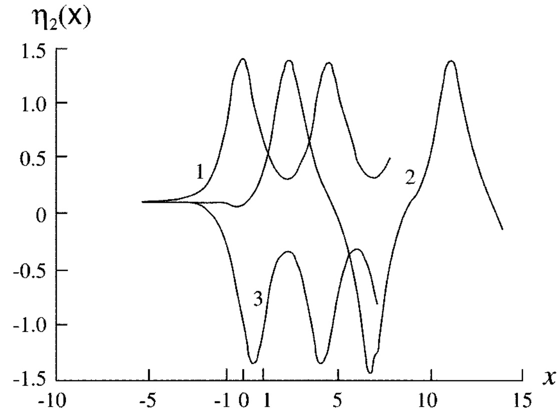

FIG. 4. Typical second-type solutions. Supercritical case, $\lambda_2=-4$, $\eta_2(-\infty)=0.1$. 1. $\eta_2(-1)=0.8$, $\eta_{2x}(-1)>0$. 2. $\eta_2(-1)=0.08$, $\eta_{2x}(-1)>0$. 3. $\eta_2(-1)=-0.25$, $\eta_{2x}(-1)<0$.

(I) Symmetric solitary-wave like solutions. Let $H_2=0$ and the free solitary wave solution as given by (29) is

$$\eta_2(x)=A_3\{B\ \cosh^2[A_3^{1/2}(x-x_0)/2] + C\ \sinh^2[A_3^{1/2}(x-x_0)/2]\}^{-1}$$

or (35)

$$\eta_2(x)=-A_3\{C\ \cosh^2[A_3^{1/2}(x-x_0)/2] + B\ \sinh^2[A_3^{1/2}(x-x_0)/2]\}^{-1},$$

where

$$B=[(A_2^2/36)+(A_3A_1/6)]^{1/2}-(A_2/6),$$

$$C=[(A_2^2/36)+(A_3A_1/6)]^{1/2}+(A_2/6).$$

By using the shooting method, we can find a symmetric solitary wave-like solution of (28), whose values ahead and behind the bump are given by (35). The numerical results are presented in Figs. 2 and 3. Four typical solitary wave-like solutions corresponding to $\lambda_1=-1$ and $\lambda_2=4$ are shown in Fig. 2. In Fig. 3, we show the relationship between $\eta_2(0)$ and λ_2 with $\lambda_1=-1$. We note that for certain pairs of (λ_1,λ_2), no solitary wave-like solution can appear.

(II) Unsymmetric solutions. Assume $\eta_2(-\infty)=H_2\neq 0$ and η_2 is periodic ahead of the bump. For $x\leqslant -1$, either η_2 is given by one of the solutions in (29) or $\eta_2\equiv H_2$.

Since we have proved the existence of a solution of (28) for $[x_-,x_+]$ and derived the possible solutions of (28) in $[x_+,\infty)$, we can solve (28) numerically by using (29) or $\eta_2\equiv H_2$ behind the bump. The numerical results are presented in Figs. 4–7. Figure 4 shows the second-type solutions that have (29) as their solutions in $(-\infty,-1]$. In Fig. 5 we show a second-type solution whose mean depth of the wave ahead of the bump is $-\eta_2(-\infty)$. We also consider third-type solutions that are equal to H_2 for $x\leqslant -1$ and periodic for $x\geqslant 1$. Figure 6 shows a typical third-type solution whose mean depth ahead of the bump is $-\eta_2(-\infty)$. An interesting phenomenon appears in the process of computing third-type solutions. At discrete values of λ_2 there are symmetric solutions without a periodic part embedded in the

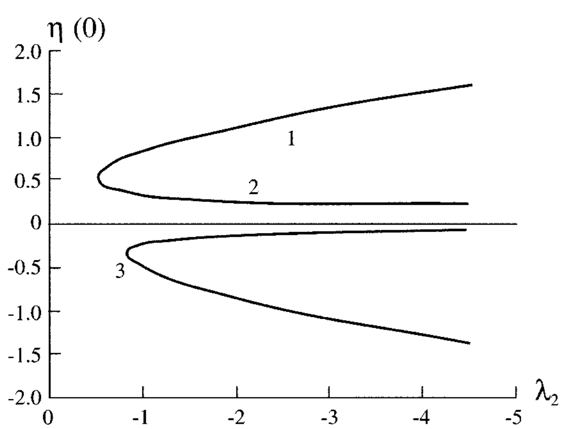

FIG. 3. Relations between λ_2 and $\eta_2(0)$ of symmetric solutions. Supercritical case. 1. Positive symmetric solutions with one crest. 2. Positive symmetric solutions with two crests. 3. Negative symmetric solutions with one trough.

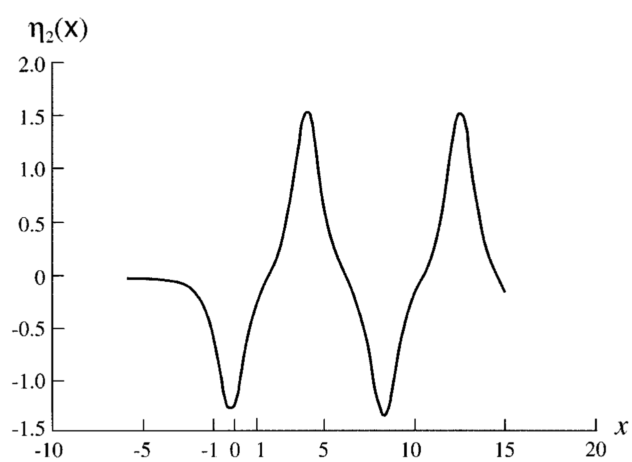

FIG. 5. Second-type solution with mean depth equal to $-\eta_2(-\infty)$ ahead of the bump. Here $\lambda_2=-0.719$, $H_2=\eta_2(-\infty)=-0.02$.

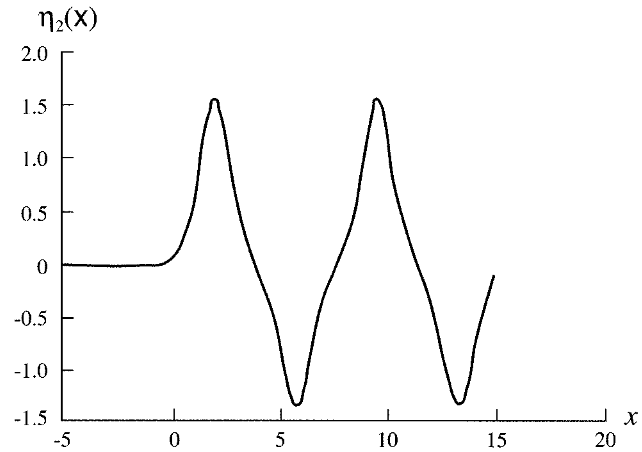

FIG. 6. Third-type solution with mean depth equal to $-\eta_2(-\infty)$ ahead of the bump. Here $\lambda_2=4$, $H_2=\eta_2(-\infty)=-0.0249$.

third-type solutions and Fig. 7 presents such a solution. In Figs. 2–7, we choose $\rho=0.2$, $h=0.6$, $\lambda_1=-1$, $T_1=2.8$, and $T_2=4$.

B. Subcritical case ($F_4F_1<0$)

The same assumptions as in Sec. III A are given in this section. The only difference here is $F_1F_4<0$. By dividing both sides of (27) by F_4, we have

$$\eta_{2xxx}=B_1\eta_2^2\eta_{2x}+A_2\eta_2\eta_{2x}+A_3\eta_{2x}+A_4b_x, \tag{36}$$

where

$$B_1=\frac{-F_1}{F_4}>0, \quad A_2=-\frac{\lambda_1}{F_4}, \quad A_3=-\frac{\lambda_2}{F_4},$$

$$A_4=-F_5/F_4.$$

If $\eta_2(x)$ is a solution of (36) and tends to H_2 at $x=-\infty$ with $\eta_{2x}(-\infty)=0$, then from $b(-\infty)=0$, η_2 satisfies

$$\eta_{2xx}=(B_1/3)\eta_2^3+(A_2/2)\eta_2^2+A_3\eta_2+d+A_4b(x), \tag{37}$$

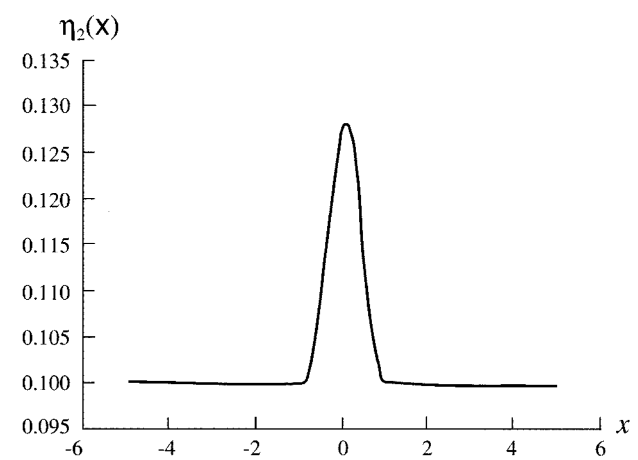

FIG. 7. Symmetric solution with one crest. Supercritical case. Here $\lambda_2=27.1$, $H_2=\eta_2(-\infty)=0.1$.

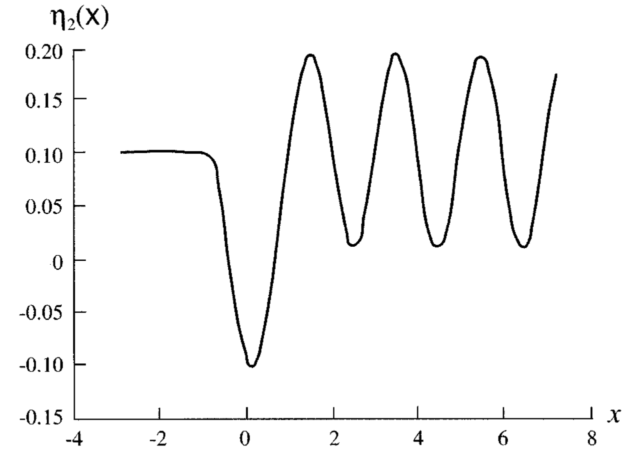

FIG. 8. Typical third-type solution. Subcritical case. Here $\lambda_2=4.0$, $H_2=\eta_2(-\infty)=0.1$.

$$\eta_2(-\infty)=H_2, \quad \eta_{2x}(-\infty)=0,$$

where

$$d=-B_1(H_2^3/3)-A_2(H_2^2/2)-A_3H_2.$$

Here we choose $\eta_2\equiv H_2$ in $(-\infty,x^-)$, since a solitary wave solution does not exist for (36) when $b(x)=0$. It can also be shown[9] that (37) possesses a solution with a continuous second-order derivative in $[x^-,x^+]$ for large $-A_3$, and by using the matching process as before, we can find the solution for all real x. We note that our numerical computation shows that no bounded solution exists for a small value of $-A_3>0$.

Now the solution in $(-\infty,x^+]$ can be connected to the solution in $[x^+,\infty)$, where $b(x)=0$ numerically by using the shooting method, as in Sec. III A. As before, we multiply (37) by η_{2x} and integrate the resulting equation from x^+ to $x>x^+$ to obtain

$$(\eta_{2x})^2=(B_1/6)\eta_2^4+(A_2/3)\eta_2^3+A_3\eta_2^2+2d\eta_2+e$$
$$=g(\eta_2),$$

where $e=-B_1(\alpha^4/6)-A_2(\alpha^3/3)-A_3\alpha^2-2d\alpha+\beta^2$, $\eta_2(x^+)=\alpha$, and $\eta_{2x}(x^+)=\beta$.

Let c_1, c_2, c_3, c_4 be zeros of $g(\eta_2)$. When none of the c_i's are real, a solution is unbounded. When two of the c_i's are real, say c_1 and c_2, and $c_1=c_2$, then $\eta_2\equiv c_1$ is the only possible solution and η_2 is unbounded otherwise. When all c_i's are real, there are several different cases. If $c_1<c_2<c_3<c_4$, the solution η_2 is periodic and $c_2\leqslant\eta_2\leqslant c_3$. If $c_1=c_2<c_3<c_4$ or $c_1<c_2<c_3=c_4$, η_2 is nonperiodic. If $c_1=c_2=c_3=c_4$, the solution is a constant. In other cases, the solutions are unbounded.

We present the numerical results of global solutions in Figs. 8–11. Figure 8 shows a typical third-type solution and Fig. 9 shows a hydraulic jump, which is the limiting solution of the third-type solution, as λ_2 being decreased and tending to some critical value. Only unbounded solutions are found when λ_2 is decreased further below the critical value. We also find multicrest symmetric solutions without a periodic

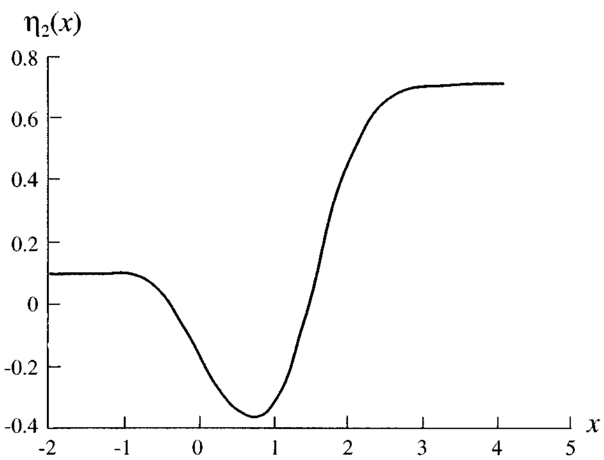

FIG. 9. Hydaulic jump wave. Subcritical case. Here $\lambda_2=2.031\,215$, $H_2=\eta_2(-\infty)=0.1$.

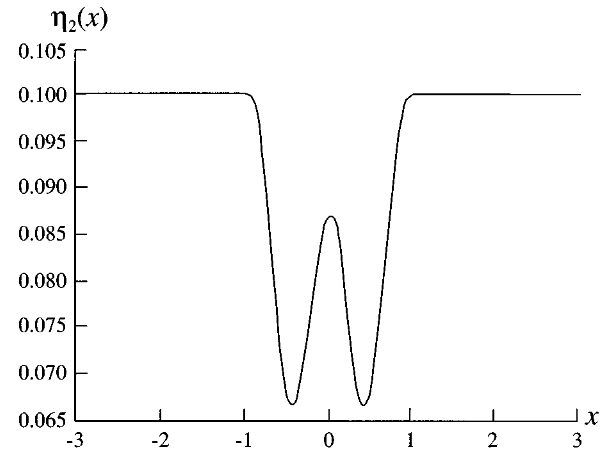

FIG. 11. Symmetric solution with two troughs. Subcritical case. Here $\lambda_2=18.8$, $H_2=\eta_2(-\infty)=0.1$.

part embedded in the third-type solutions at discrete values of λ_2, even though no first-type symmetric solutions exist in the subcritical case. Figures 10–11 present two such cases. In Figs. 8–11 we choose the same ρ and h as in Figs. 4–7, but $\lambda_1=1$ and T_1 and T_2 are equal to 10^{-2}. The obstruction is same as before.

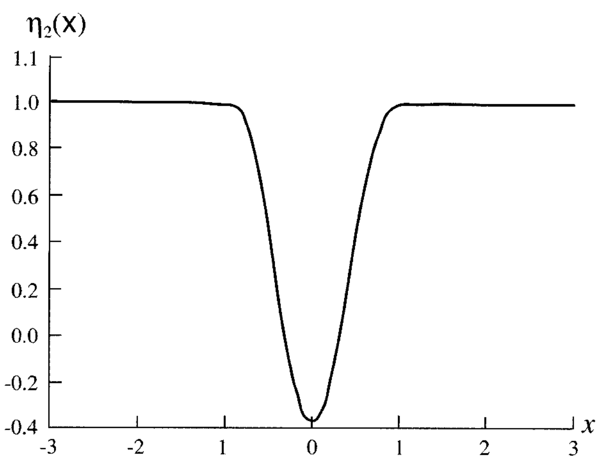

FIG. 10. Symmetric solution with one trough. Subcritical case. Here $\lambda_2=5.61$, $H_2=\eta_2(-\infty)=0.1$.

ACKNOWLEDGMENTS

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