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Neutral stability calculations for boundary-layer flows

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An analysis is presented of the parallel neutral stability of three-dimensional incompressible, isothermal boundary-layer flows. A Taylor-series expansion of the dispersion relation is used to derive the general eigenvalues. These equations are functions of the complex group velocity. These relations are verified by numerical results obtained for two- and three-dimensional disturbances in two- and three-dimensional flows.

I. INTRODUCTION

Recent interest in laminar flow control of wings has demanded extensive analysis of the stability of boundary-layer flows. The first step in this type of analysis is to distinguish unstable from stable disturbances for a given flow. The line or surface separating the group of stable disturbances from the group of unstable disturbances is called the neutral stability curve or surface. Extensive work conducted by Kurtz,[1] Kaplan,[2] Osborne,[3] and Wazzan *et al.*[4] have resulted in the determination of the neutral stability curve for two-dimensional incompressible boundary-layer flows given by the Blasius and Falkner–Skan solutions. Mack[5] determined neutral stability curves for two-dimensional compressible flows. All these results agree with the experimental results of Schubauer and Skramstad,[6] and Ross *et al.*[7] to a certain degree. Except for those in Ref. 5, the calculations were always based on the tedious method of scanning the growth rate-wavenumber space. Although, this does not pose a major problem in the analysis of two-dimensional flows, the task of analyzing three-dimensional flows becomes prohibitively difficult.

In this paper, we present a formal method of obtaining neutrally stable disturbances. Equations are derived that relate the neutral disturbances to the non-neutral disturbances. These equations depend on the components of the complex group velocity calculated by using quadratures.[8]

II. PROBLEM FORMULATION

We consider the stability of a three-dimensional, incompressible boundary-layer flow. We introduce dimensionless quantities using the chordwise velocity U^*, the density ρ^*, and the kinematic viscosity ν^* at the edge of the boundary layer, and the reference length $L^* = (\nu^* x^*/U^*)^{1/2}$, where x^* is the distance measured from the initiation of the boundary layer. Solving dimensionless steady boundary-layer equations yields the velocity components

$$U(y) \text{ and } W(y) \tag{1}$$

in the x and z directions, respectively. Let P be the pressure. We superpose the small unsteady disturbances $u\exp(i\theta)$, $v\exp(i\theta)$, $w\exp(i\theta)$, and $p\exp(i\theta)$ on this basic flow. Here θ is such that

$$\frac{\partial\theta}{\partial t} = -\omega,\quad \frac{\partial\theta}{\partial x} = \alpha,\quad \frac{\partial\theta}{\partial z} = \beta\,.$$

The dimensionless linearized equations satisfied by these disturbances are

$$L_1(u,v,w,p) = i\alpha u + i\beta w + Dv = 0\,, \tag{2}$$

$$L_2(u,v,w,p) = i(\alpha U) + \beta W - \omega)u + vDU + i\alpha p - R^{-1}(D^2 - \alpha^2 - \beta^2)u = 0\,, \tag{3}$$

$$L_3(u,v,w,p) = i(\alpha U + \beta W - \omega)v + Dp - R^{-1}(D^2 - \alpha^2 - \beta^2)v = 0\,, \tag{4}$$

$$L_4(u,v,w,p) = i(\alpha U + \beta W - \omega)w + vDW + i\beta p - R^{-1}(D^2 - \alpha^2 - \beta^2)w = 0\,. \tag{5}$$

The boundary conditions satisfied by u, v, and w are

$$u,\ v,\ \text{and } w \to \infty\,, \tag{6}$$

and

$$u = v = w = 0,\ \text{ at } y = 0\,. \tag{7}$$

Equations (2)–(5) along with the boundary conditions (6) and (7) constitute an eigenvalue problem. For temporal stability, R, α, and β are real numbers and ω is complex; whereas for spatial stability, R and ω are real numbers and α and β are complex. In either case the dispersion relation can be written in the form

$$\omega = \Omega(\alpha,\beta,R)\,. \tag{8}$$

For a fixed value of R, this relation reduces to

$$\omega = \Omega(\alpha,\beta)\,. \tag{9}$$

Let ω_0, α_0, and β_0 be one set of eigenvalues so that $\omega_0 = \Omega(\alpha_0,\beta_0)$. Then using a Taylor-series expansion, we have

$$\omega = \omega_0 + \omega_\alpha(\alpha - \alpha_0) + \omega_\beta(\beta - \beta_0) + \cdots\,, \tag{10}$$

where

$$\omega_\alpha = \omega_{\alpha r} + i\omega_{\alpha i} = \left.\frac{\partial\omega}{\partial\alpha}\right|_{(\alpha_0,\beta_0)} \tag{11}$$

and

$$\omega_\beta = \omega_{\beta r} + i\omega_{\beta i} = \left.\frac{\partial\omega}{\partial\beta}\right|_{(\alpha_0,\beta_0)} \tag{12}$$

are the components of the complex group velocity in the x and z directions, respectively.

To determine ω_α, we differentiate Eqs. (2)–(7) with respect to α and obtain

$$L_1\left(\frac{\partial u}{\partial \alpha},\frac{\partial v}{\partial \alpha},\frac{\partial w}{\partial \alpha},\frac{\partial p}{\partial \alpha}\right)=-iu. \tag{13}$$

$$L_2\left(\frac{\partial u}{\partial \alpha},\frac{\partial v}{\partial \alpha},\frac{\partial w}{\partial \alpha},\frac{\partial p}{\partial \alpha}\right)=-i\left(U-i\frac{2\alpha}{R}\right)u-ip+i\frac{\partial \omega}{\partial \alpha}u, \tag{14}$$

$$L_3\left(\frac{\partial u}{\partial \alpha},\frac{\partial v}{\partial \alpha},\frac{\partial w}{\partial \alpha},\frac{\partial p}{\partial \alpha}\right)=-i\left(U-i\frac{2\alpha}{R}\right)v+i\frac{\partial \omega}{\partial \alpha}v. \tag{15}$$

$$L_4\left(\frac{\partial u}{\partial \alpha},\frac{\partial v}{\partial \alpha},\frac{\partial w}{\partial \alpha},\frac{\partial p}{\partial \alpha}\right)=-i\left(U-i\frac{2\alpha}{R}\right)w+i\frac{\partial \omega}{\partial \alpha}w, \tag{16}$$

$$\frac{\partial u}{\partial \alpha},\frac{\partial v}{\partial \alpha},\frac{\partial w}{\partial \alpha}\rightarrow 0, \quad \text{as } y\rightarrow\infty, \tag{17}$$

and

$$\frac{\partial u}{\partial \alpha}=\frac{\partial v}{\partial \alpha}=\frac{\partial w}{\partial \alpha}=0, \quad \text{at } y=0. \tag{18}$$

We note that the homogeneous parts of Eqs. (13)–(18) have a nontrivial solution. Hence, the inhomogeneous equations (13)–(18) have a solution if, and only if, a solvability condition is satisfied. This solvability condition can be written as

$$\frac{\partial \omega}{\partial \alpha}=\frac{g_2}{g_1}, \tag{19}$$

where g_1 and g_2 are given in quadratures in terms of the mean-flow quantities; the eigenmodes u, v, w, and p; the eigenmodes of the adjoint-problem u^*, v^*, w^*, and p^*; and the streamwise wavenumber α (see the Appendix).

If Eqs. (2)–(7) are differentiated with respect to β, we obtain an inhomogeneous problem whose solvability condition yields

$$\frac{\partial \omega}{\partial \beta}=\frac{g_3}{g_1}. \tag{20}$$

Again, g_3 is given in quadratures (see the Appendix).

III. NEUTRAL CURVES

Equation (10) forms a three-parameter surface. For the neutral stability calculations, we are interested in locating a curve on this surface on which all the three parameters ω, α, and β are real. Since Eq. (10) is a complex equation, it provides two real relations relating three quantities. Hence we need to hold one of the three parameters fixed, while interacting on the remaining parameters. Four such possibilities exist; they are treated next.

A. The case ω fixed

Since ω is held fixed, Eq. (10) can be rewritten as

$$\omega_\alpha(\alpha-\alpha_0)+\omega_\beta(\beta-\beta_0)\simeq 0. \tag{21}$$

Since we are searching for neutral stability points, we take α and β to be real. We express α_0 and β_0 as

$$\alpha_0=\alpha_{0r}+i\alpha_{0i}, \quad \beta_0=\beta_{0r}+i\beta_{0i},$$

separate real and imaginary parts of Eq. (21), and obtain

$$\omega_{\alpha r}(\alpha-\alpha_{0r})+\omega_{\alpha i}\alpha_{0i}+\omega_{\beta r}(\beta-\beta_{0r})+\omega_{\beta i}\beta_{0i}\simeq 0, \tag{22}$$

$$\omega_{\alpha i}(\alpha-\alpha_{0r})-\omega_{\alpha r}\alpha_{0i}+\omega_{\beta i}(\beta-\beta_{0r})-\omega_{\beta r}\omega_{0i}\simeq 0. \tag{23}$$

Solving Eqs. (22) and (23) for α and β, we have

$$\alpha\simeq\alpha_{0r}-\alpha_{0i}\frac{\omega_{\alpha r}\omega_{\beta r}+\omega_{\alpha i}\omega_{\beta i}}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}}-\beta_{0i}\frac{\omega_{\beta r}^2+\omega_{\beta i}^2}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}} \tag{24}$$

and

$$\beta\simeq\beta_{0r}+\alpha_{0i}\frac{\omega_{\alpha r}^2+\omega_{\alpha i}^2}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}}+\beta_{0i}\frac{\omega_{\alpha r}\omega_{\beta r}+\omega_{\alpha i}\omega_{\beta i}}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}}. \tag{25}$$

Equations (24) and (25) can be simplified further by taking either α_0 or β_0 to be real to begin with. For example if β_0 was real, then we obtain

$$\alpha\simeq\alpha_{0r}-\alpha_{0i}\frac{\omega_{\alpha r}\omega_{\beta r}+\omega_{\alpha i}\omega_{\beta i}}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}} \tag{26}$$

and

$$\beta\simeq\beta_{0r}+\alpha_{0i}\frac{\omega_{\alpha r}^2+\omega_{\alpha i}^2}{\omega_{\alpha r}\omega_{\beta i}-\omega_{\alpha i}\omega_{\beta r}}. \tag{27}$$

To verify the validity of Eqs. (26) and (27), we calculated the neutral stability curves of the Blasius profile at $R=(U^*x^*/\nu^*)^{1/2}=800$. For the real values of ω_0 and β_0 given in columns 1 and 2 of Table I, we calculated the complex α_0 given in column 3. Then, using Eqs. (26) and (27) repeatedly, we calculated the neutral stability results given by the real α and β given in columns 4 and 5, respectively. Column 6 lists the number of iterations needed for arriving at neutral stability.

TABLE I. Neutral stability calculations for the Blasius flow at $R=800$. The calculations were made in the spatial stability mode.

ω_0	β_0	$\alpha_0=\alpha_{0r}+i\alpha_{0i}$	$\alpha=\alpha_r,\ \alpha_i=O(10^{-8})$	β	N
2.4E − 2	1.8E − 1	(5.821 03E − 2, 7.968 48E − 4)	6.211 34E − 2	1.455 64E − 1	4
2.4E − 2	1.6E − 1	(6.036 02E − 2, 3.041 40E − 4)	6.211 34E − 2	1.455 64E − 1	4
2.4E − 2	1.4E − 1	(6.283 76E − 2, −1.021 17E − 4)	6.211 34E − 2	1.455 64E − 1	4
2.4E − 2	1.2E − 1	(6.567 25E − 2, −3.811 73E − 4)	6.211 34E − 2	1.455 64E − 1	5
2.4E − 2	0.8E − 1	(7.227 15E − 2, −4.103 49E − 4)	7.745 50E − 2	4.867 58E − 2	5
2.4E − 2	0.6E − 1	(7.568 10E − 2, −1.708 97E − 4)	7.745 50E − 2	4.867 58E − 2	4
2.4E − 2	0.4E − 1	(7.866 62E − 2, 1.299 29E − 4)	7.745 51E − 2	4.867 58E − 2	3
2.4E − 2	0.2E − 1	(8.073 14E − 2, 3.690 49E − 4)	7.745 50E − 2	4.867 58E − 2	5

TABLE II. Neutral stability calculations for the Blasius flow at $R=800$. The calculations were made in the temporal stability mode.

α_0	$\omega_0=\omega_{0r}+i\omega_{0i}$	α	$\omega=\omega_r$, $\omega_i=O(10^{-8})$	N
0.25	$(8.817\,58E-2, -8.082\,21E-3)$	0.195 906	$6.930\,58E-2$	5
0.20	$(7.096\,73E-2, -4.076\,60E-4)$	0.195 906	$6.930\,58E-2$	3
0.18	$(6.280\,63E-2, 1.230\,44E-3)$	0.195 906	$6.930\,58E-2$	4
0.16	$(5.462\,06E-2, 2.066\,60E-3)$	0.195 906	$6.930\,58E-2$	5
0.12	$(3.859\,43E-2, 1.803\,91E-3)$	0.084 133	$2.500\,01E-2$	4
0.10	$(3.090\,21E-2, 9.255\,49E-4)$	0.084 133	$2.500\,01E-2$	3
0.08	$(2.349\,44E-2, -2.597\,83E-4)$	0.084 133	$2.500\,01E-2$	2
0.06	$(1.640\,79E-2, -1.485\,58E-3)$	0.084 133	$2.500\,01E-2$	3

B. The case β fixed

In this case, Eq. (10) can be rewritten as

$$(\omega-\omega_0)\simeq\omega_\alpha(\alpha-\alpha_0)\,. \tag{28}$$

If we let $\omega_0=\omega_{0r}+i\omega_{0i}$ and $\alpha_0=\alpha_{0r}+i\alpha_{0i}$, then separating Eq. (28) into its real and imaginary parts yields

$$(\omega-\omega_{0r})\simeq\omega_{\alpha r}(\alpha-\alpha_{0r})+\omega_{\alpha i}\alpha_{0i}\,, \tag{29}$$

$$-\omega_{0i}\simeq\omega_{\alpha i}(\alpha-\alpha_{0r})-\omega_{\alpha r}\alpha_{0i}\,. \tag{30}$$

Equations (29) and (30) can be solved simultaneously to give

$$\omega\simeq\omega_{0r}-\omega_{\alpha r}\omega_{0i}/\omega_{\alpha i}+(\omega_{\alpha r}^2+\omega_{\alpha i}^2)\alpha_{0i}/\omega_{\alpha i}\,, \tag{31}$$

$$\alpha=\alpha_{0r}+\omega_{\alpha r}\alpha_{0i}/\omega_{\alpha i}-\omega_{0i}/\omega_{\alpha i}\,. \tag{32}$$

If we start with a real α_0, Eqs. (31) and (32) can be simplified to

$$\omega\simeq\omega_{0r}-\omega_{\alpha r}\omega_{0i}/\omega_{\alpha i}\,, \tag{33}$$

$$\alpha\simeq\alpha_{0r}-\omega_{0i}/\omega_{\alpha i}\,. \tag{34}$$

To show the validity of Eqs. (33) and (34), we calculated the temporal stability of the Blasius flow at $R=800$. The results of these calculations are presented in Table II. For $\beta_0=0$ and the real values of α_0 listed in column 1, we calculated the temporal stability defined by the complex ω_0 listed in column 2. Then, using Eqs. (33) and (34) repeatedly, we calculated the neutral stability results defined by the real α and ω listed in columns 3 and 4, respectively. Column 5 lists the number of iterations needed to reach neutral stability.

It is also possible to start with real ω_0. Then Eqs. (31) and (32) reduce to

$$\omega\simeq\omega_{0r}+(\omega_{\alpha r}^2+\omega_{\alpha i}^2)\alpha_{0i}/\omega_{\alpha i}\,, \tag{35}$$

$$\alpha\simeq\alpha_{0r}+\omega_{\alpha r}\alpha_{0i}/\omega_{\alpha i}\,. \tag{36}$$

To show the validity of Eqs. (35) and (36), calculations were made for the Blasius flow at $R=800$. The results are presented in Table III. For $\beta_0=0$ and the real value of ω_0 listed in column 1, we computed the spatial stability results defined by the complex α_0 listed in column 2. Then, using Eqs. (35) and (36), we calculated the neutral stability results defined by the real ω and α listed in columns 3 and 4. Column 5 lists the number of iterations required to reach neutral stability.

C. The case α fixed

Following steps similar to those in the preceding section, we obtain

$$\omega\simeq\omega_{0r}-\omega_{\beta r}\omega_{0i}/\omega_{\beta i}+(\omega_{\beta r}^2+\omega_{\beta i}^2)\beta_{0i}/\omega_{\beta i} \tag{37}$$

and

$$\beta\simeq\beta_{0r}+\omega_{\beta r}\beta_{0i}/\omega_{\beta i}-\omega_{0i}/\omega_{\beta i}\,. \tag{38}$$

D. The case of fixed wave angle ψ

Here we define the wave angle ψ as

$$\tan\psi=\beta/\alpha,\quad \tan\psi=\beta_{0r}/\alpha_{0r} \tag{39a}$$

or

$$(\alpha-\alpha_{0r})\tan\psi=(\beta-\beta_{0r})\,. \tag{39b}$$

If ω_0, α_0, and β_0 are complex, then we rewrite Eq. (10) in the following form:

$$\omega-\omega_{0r}\simeq\omega_{\alpha r}(\alpha-\alpha_{0r})+\omega_{\alpha i}\alpha_{0i}+\omega_{\beta r}(\beta-\beta_{0r})+\omega_{\beta i}\beta_{0i}\,, \tag{40}$$

$$-\omega_{0i}\simeq\omega_{\alpha i}(\alpha-\alpha_{0r})-\omega_{\alpha r}\alpha_{0i}+\omega_{\beta i}(\beta-\beta_{0r})-\omega_{\beta r}\beta_{0i}\,. \tag{41}$$

Equations (39)–(41) provide three real equations, which

TABLE III. Neutral stability calculations for the Blasius flow at $R=800$. The calculations were made in the spatial stability mode.

ω_0	$\alpha_0=\alpha_{0r}+i\alpha_{0i}$	ω	$\alpha=\alpha_r$, $\alpha_i=O(10^{-8})$	N
$8.817\,58E-2$	$(0.235\,582, 1.513\,24E-2)$	$6.930\,58E-2$	0.195 906	4
$7.096\,70E-2$	$(0.199\,758, 9.417\,74E-4)$	$6.930\,58E-2$	0.195 906	3
$6.280\,63E-2$	$(0.180\,444, -2.957\,75E-3)$	$6.930\,58E-2$	0.195 906	4
$5.462\,06E-2$	$(0.160\,311, -5.101\,11E-3)$	$6.930\,58E-2$	0.195 906	5
$3.859\,43E-2$	$(0.116\,150, -4.614\,31E-3)$	$2.500\,01E-2$	0.084 133	4
$2.349\,44E-2$	$(0.080\,122, 6.947\,14E-4)$	$2.500\,01E-2$	0.084 133	2
$1.640\,79E-2$	$(0.060\,709, 4.209\,47E-3)$	$2.500\,01E-2$	0.084 133	3

can be solved simultaneously to yield

$$\omega \simeq \omega_{0r} - \frac{\omega_{\alpha r} + \omega_{\beta r}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}\omega_{0i} + \left(\omega_{\alpha i} + \omega_{\alpha r}\frac{\omega_{\alpha r} + \omega_{\beta r}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}\right)\alpha_{0i}$$

$$+\left(\omega_{\beta i} + \omega_{\beta r}\frac{\omega_{\alpha r} + \omega_{\beta r}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}\right)\beta_{0i}, \quad (42)$$

$$\alpha \simeq \alpha_{0r} - \frac{\omega_{0i}}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi} + \frac{\omega_{\alpha r}\alpha_{0i}}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi} + \frac{\omega_{\beta r}\beta_{0i}}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}, \quad (43)$$

and

$$\beta \simeq \beta_{0r} - \frac{\omega_{0i}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi} + \frac{\omega_{\alpha r}\alpha_{0i}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi} + \frac{\omega_{\beta r}\beta_{0i}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}. \quad (44)$$

Equations (42)–(44) are the most general form of the present theory. These equations can be simplified for a particular case of interest as desired. For example, if we start with real α_0 and β_0, then Eqs. (42)–(44) can be rewritten as

$$\omega \simeq \omega_{0r} - \frac{\omega_{\alpha r} + \omega_{\beta r}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}\omega_{0i}. \quad (45)$$

$$\alpha \simeq \alpha_{0r} - \frac{\omega_{0i}}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}, \quad (46)$$

$$\beta \simeq \beta_{0r} - \frac{\omega_{0i}\tan\psi}{\omega_{\alpha i} + \omega_{\beta i}\tan\psi}. \quad (47)$$

To show the validity of Eqs. (45)–(47), we calculated the temporal stability of the three-dimensional velocity profile of Fig. 1. This velocity was obtained for a sweptback wing having a leading edge sweepback angle of 33° and a trailing edge sweepback angle of 19°. The calculations were performed at the 14% chord location at $R = 1034.8$ for $\psi = 36.5°$. For the several real values of α_0 and β_0 listed in columns 1 and 2 of Table IV, we calculated the complex ω shown in column 3 of Table IV. Then, using Eqs. (45)–(47) repeatedly, we calculated the neutral stability results defined by the real α, β, and ω listed in columns 4, 5, and 6, respectively. Column 7 lists the number of iterations required to reach neutral stability for each case.

IV. CONCLUDING REMARKS

Equations relating neutral disturbances to non-neutral disturbances were derived. These equations are based on the components of the complex group velocity. The validity of these equations was demonstrated with several examples.

Although the equations presented here can be used under wide conditions, they break down if the group velocity is purely real. Hence, these equations cannot be used for relating neutral disturbances to a maximum amplified disturbance. To obtain a neutral disturbance from a non-neutral disturbance, repeated application of the equations is needed. The convergence of such an iterative process is guaranteed if the non-neutral disturbances are stable. Also, the iterative process converges faster for stable non-neutral disturbances than for unstable non-neutral disturbances.

It should be noted that the technique presented here is not restricted to incompressible flows. It can be used to determine the neutral stability conditions for compressible flows; this is not done in this paper.

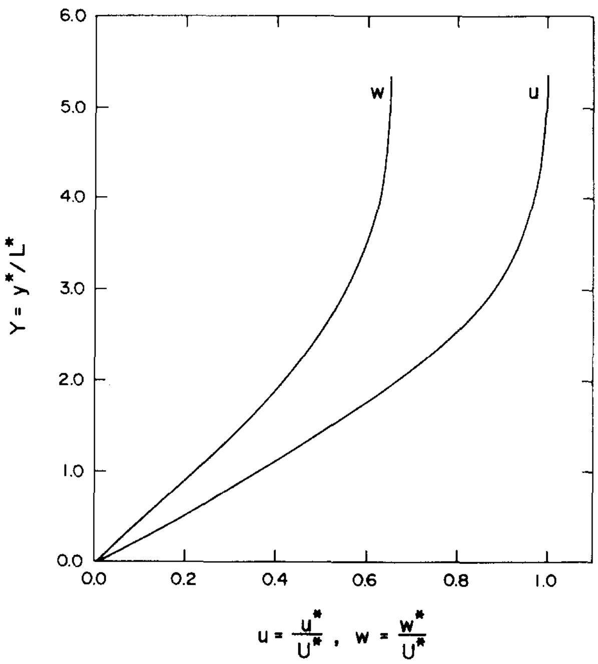

FIG. 1. Three-dimensional boundary-layer profile for sweptback wing. Leading edge sweep: 33°; Trailing edge sweep: 19°; $U_\infty = 290.43$ fps; Chord $c = 14.66'$; $x/\hat{c} = 0.14$; $R = 1034.8$.

TABLE IV. Neutral stability calculations for a three dimensional boundary-layer flow at $R = 1034.8$. The calculations were made in the temporal stability mode.

α_0	β_0	$\omega_0 = \omega_{0r} + i\omega_{0i}$	α	β	$\omega = \omega_r$; $\omega_i = O(10^{-8})$	N
0.04	0.029 598	$(1.4765E-2, -2.060\,64E-3)$	0.064 965	0.048 071	$2.6957E-2$	4
0.06	0.044 398	$(2.4459E-2, -4.441\,49E-4)$	0.064 965	0.048 071	$2.6958E-2$	3
0.08	0.059 197	$(3.4727E-2, 1.222\,77E-3)$	0.064 965	0.048 071	$2.6958E-2$	4
0.10	0.073 996	$(4.5497E-2, 2.291\,80E-3)$	0.064 965	0.048 071	$2.6958E-2$	6
0.12	0.088 795	$(5.6648E-2, 2.428\,77E-3)$	0.064 965	0.048 071	$2.6958E-2$	5
0.14	0.103 595	$(6.7990E-2, 1.366\,22E-3)$	0.152 243	0.112 654	$7.4910E-2$	5
0.16	0.118 394	$(7.9230E-2, -1.191\,95E-3)$	0.152 243	0.112 654	$7.4910E-2$	5
0.17	0.125 793	$(8.4645E-2, -3.147\,60E-3)$	0.152 243	0.112 654	$7.4910E-2$	5

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APPENDIX

$$g_1=\int_0^\infty (\zeta_1\zeta_1^* + \zeta_2\zeta_2^* + \zeta_3\zeta_3^*)dy\,, \tag{A1}$$

$$g_2=\int_0^\infty \left[\left(\zeta_1\zeta_4^* + \zeta_1^*\zeta_4 + U - \frac{2i\alpha}{R}\right) \times (\zeta_1\zeta_1^* + \zeta_2\zeta_2^* + \zeta_3\zeta_3^*)\right]dy\,, \tag{A2}$$

$$g_3=\int_0^\infty \left[(\zeta_3\zeta_4^* + \zeta_3^*\zeta_4) + \left(W - \frac{2i\beta}{R}\right) \times (\zeta_1\zeta_1^* + \zeta_2\zeta_2^* + \zeta_3\zeta_3^*)\right]dy\,, \tag{A3}$$

where $\zeta_1=u$, $\zeta_2=v$, $\zeta_3=w$, and $\zeta_4=p$, and the ζ_n^* are the solutions of the adjoint problem.

$$i\alpha\zeta_1^* + i\beta\zeta_3^* - D\zeta_2^* = 0\,, \tag{A4}$$

$$i(\alpha U + \beta W - \omega)\zeta_1^* + i\alpha\zeta_4^* - R^{-1}[D^2 - (\alpha^2+\beta^2)]\zeta_1^* = 0\,, \tag{A5}$$

$$i(\alpha U + \beta W - \omega)\zeta_2^* - D\zeta_4^* + DU\zeta_1^* + DW\zeta_3^* - R^{-1}[D^2 - (\alpha^2+\beta^2)]\zeta_2^* = 0\,, \tag{A6}$$

$$i(\zeta U + \beta W - \omega)\zeta_3^* + i\beta\zeta_4^* - R^{-1}[D^2 - (\alpha^2+\beta^2)]\zeta_3^* = 0\,, \tag{A7}$$

$$\zeta_1^* = \zeta_2^* = \zeta_3^* = 0, \text{ at } y=0\,, \tag{A8}$$

$$\zeta_1^*, \zeta_2^*, \zeta_3^* \to 0, \text{ as } y \to \infty\,. \tag{A9}$$

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