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Stability of a layer of viscous magnetic fluid flow down an inclined plane

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This paper concerns the linear stability of a layer of viscous magnetic fluid flow down an inclined plane under the influence of gravity and a tangential magnetic field. The stability of a magnetic fluid in a three-dimensional space is first reduced to the stability of the flow in a two-dimensional space by using Squire's transformation. The stability of long waves and short waves is analyzed asymptotically. The stability of waves with intermediate length is obtained numerically. It is found that the magnetic field has a stabilizing effect on both the surface and shear modes and can be used to postpone the instability of such flows.

I. INTRODUCTION

There has been a great deal of interest in magnetic fluids in recent years. A magnetic fluid, also known in the literature as a ferromagnetic fluid or simply a ferrofluid, is a stable colloidal system consisting of rather small solid surfactant-coated ferromagnetic particles dispersed in a liquid carrier. Generally, there is almost no electric current flowing in magnetic fluids and the body force is due to a polarization force which in turn requires material magnetization in the presence of magnetic field gradients. The fluids display considerable magnetic response so that the motion of the fluids can be dominated by an applied magnetic field. This has attracted increasing attention with the promise of applications in many areas, for example, see Rosensweig[1] and Bashtovoy *et al.*,[2] and has led to many investigations. A very important area is the stability of magnetic fluid flows. Zelazo and Melcher[3] studied the linear stability of an ideal magnetic fluid on a rigid horizontal plane under a tangential magnetic field theoretically, as well as experimentally. They found that the magnetic field has a stabilizing influence on the stability of the fluid surface. In contrast, when the magnetic field is normal to the fluid surface, Cowley and Rosensweig[4] found existence of an instability leading to a pattern of spikes on the fluid surface. Bashtovoy[5] also discussed the convective and surface instability of such waves under a magnetic field normal to the fluid surface. Malik and Singh[6] considered a magnetic fluid of infinite depth in the presence of a tangential magnetic field and obtained the modulational instability on the free surface as an extension of the work by Benjamin and Feir[7] and Hasimoto and Ono[8] in inviscid water waves. They also found that the magnetic field has a stabilizing influence on the nonlinear stability. Recently Shen *et al.*[9] studied weakly nonlinear waves on a viscous magnetic fluid flow down an inclined plane under gravity and a tangential magnetic field and obtained the critical Reynolds number for the stability of long waves.

This paper concerns flows in magnetic fluid layers and films, which may be used in many practical situations. The application of a magnetic fluid as a heat carrier is efficient in the devices which already have strong magnetic fields and may considerably increase the permissible transformer loads. It may also increase the mass transfer rate in the course of liquid–liquid extraction when the carrier fluid is chosen so as to absorb particular chemical materials and the fluid is flowing down the surface of permanent magnets. A magnetic fluid may also be an effective means of controlling the flow of ordinary fluids and reducing hydraulic resistance since magnetic fluids can be easily controlled by external magnetic fields and coating streamlined bodies with a layer of low-viscous magnetic fluid retained by a magnetic field significantly reduces shear stresses at flow boundaries. However, the effective functioning of a magnetic film in such roles is greatly limited by the stability of the flow and it is essential to know when the flows become unstable and whether a strong magnetic field can be applied to postpone the instability of such films.

As we have noted above, the linear stability for inviscid magnetic fluid flows under magnetic fields has been studied. However, in order to study flows down an inclined plane we include viscous effects so that the flow can support a nonzero equilibrium state. The configuration of the flow is as follows (Fig. 1). Assume that a magnetic fluid with constant viscosity moves down under gravity between two parallel, inclined, perfectly conducting planes, and the fluid is incompressible. There exists a vacuum region between the free surface of the fluid and the upper boundary. The equilibrium state for the magnetic field is a constant vector parallel to the planes permeating through the fluid and the vacuum region. If a small disturbance is prescribed in the fluid or on the free surface, we would like to know its linear stability and how to stabilize the waves using magnetic fields.

Many works on stability analysis are restricted to the inviscid magnetic fluids. However, the viscosity of a magnetic fluid is usually greater than the viscosity of the carrier liquid due to the presence of suspended particles and the tendency of the magnetic particles in the fluid to remain rigidly aligned with the direction of the applied magnetic field. Therefore, it is important to include the viscosity in the analysis of such flows. Also the choice of vacuum above the magnetic fluid is prompted by the fact that very different fluids ride on the film in different applications. As a starting point for further modifications, we have chosen to concentrate on the basic influence of the magnetic field on the film stability.

A viscous fluid flow down an inclined plane without a magnetic field has been studied rather extensively in the past. The linear stability theory for long waves was developed by

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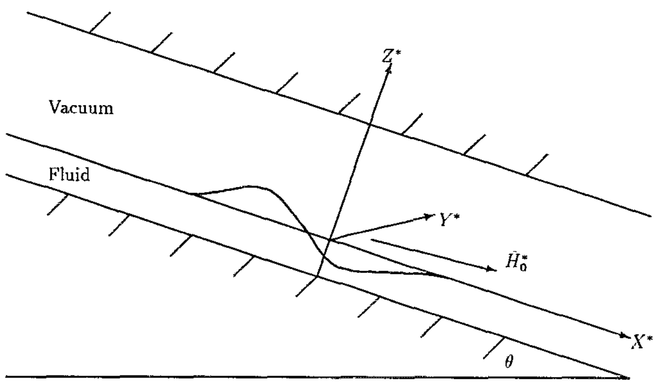

FIG. 1. Magnetic fluid flow down an inclined plane.

Benjamin[10] and Yih.[11] Lin[12] first studied stability of waves with intermediate length by using numerical computation. De Bruin[13] pointed out an error in the equations which Lin had used, and calculated the neutral curves for the linear stability and instability again by using corrected governing equations. It is found that when the inclination angle becomes small the instability occurs first for the shear mode rather than the surface mode. Floryan *et al.*[14] also studied the stability of such fluid flows with surface tension and obtained the neutral curves for different inclination angles and surface tension using numerical integration. The corresponding stability and instability of two viscous shearing fluids with surface tension at the interface were considered by Yih,[15] Hooper and Boyd,[16] and Renardy.[17] It is of interest to extend these works and to ask what will happen if magnetic fluids are considered and magnetic fields are introduced, and how the magnetic fields affect the stability of the fluid flows. To obtain the answers to such questions, we first introduce a transformation to reduce the linear stability problem in a three-dimensional space to the problem in a two-dimensional space. Then asymptotic methods are used to derive the stability criteria for short waves and long waves in terms of Reynolds numbers. The long-wave criterion is consistent with the one obtained by Shen *et al.*[9] using a different method. In order to complete the picture for waves of intermediate length, it is necessary to use numerical methods. We use a spectral scheme[18] to solve the corresponding eigenvalue problem for the linearized governing equations and obtain the stability of such waves.

In Sec. II, the governing equations are formulated and the nondimensional variables are defined by using appropriate units to obtain the starting equations. In Sec. III, the stability problem in a three-dimensional space is reduced to the problem in a two-dimensional space by introducing the so-called Squire's transformation. In Sec. IV, the stability of short waves and long waves is determined using asymptotic methods. In Sec. V, a numerical method is used to find the linear stability for the waves with intermediate length. In Sec. VI, the results are summarized and some possible applications of our results are indicated. In the Appendix, the stability for the two degenerate cases of Squire's transformation is proved.

II. FORMULATION

We consider a layer of viscous incompressible magnetic fluid down an inclined plane under the influence of an applied magnetic field and gravity. The inclination angle is denoted by θ, $0<\theta<\pi/2$. The vacuum is chosen above the fluid and the vacuum and fluid are bounded by two rigid parallel perfectly conducting planes. The coordinate system is chosen such that x^* axis is pointing down the planes, y^* axis is transverse to the planes, and z^* axis is the normal direction of the planes pointing upward. Let the free surface be $z^*=\eta^*(t^*,x^*,y^*)$ and the upper and lower boundaries be $z^* = h_0^* > 0$ and $z^*=-h^*<0$, respectively. The equations and boundary conditions governing the motion of the fluid are the following. In the fluid region $-h^*<z^*<\eta^*(t^*,x^*,y^*)$,

$$\nabla\cdot\bar{q}^*=0, \tag{1}$$

$$\rho\left(\frac{\partial\bar{q}^*}{\partial t^*}+\bar{q}^*\cdot\nabla\bar{q}^*\right)=\rho\bar{g}+\nabla\cdot\Sigma^*, \tag{2}$$

$$\nabla\cdot\bar{B}^*=0, \quad \nabla\times\bar{H}^*=0, \tag{3}$$

where ρ is the density of the fluid. At the free surface $\zeta^*=-z^*+\eta^*(t^*,x^*,y^*)=0$,

$$\frac{\partial\zeta^*}{\partial t}+\bar{q}^*\cdot\nabla\zeta^*=0, \tag{4}$$

$$[\Sigma^*\cdot\bar{N}^*]=T^*S^*\bar{N}^*, \tag{5}$$

$$[\bar{H}^*\cdot\bar{T}^*]=0, \quad [\bar{B}^*\cdot\bar{N}^*]=0. \tag{6}$$

In the vacuum region $\eta^*(t^*,x^*,y^*) < z^* < h_0^*$,

$$\nabla\cdot\bar{B}^*=0, \quad \nabla\times\bar{H}^*=0. \tag{7}$$

At the lower boundary $z^*=-h^*$,

$$\bar{q}^*=0, \quad c^*=0. \tag{8}$$

At the upper boundary $z^* = h_0^*$,

$$C^*=0. \tag{9}$$

Here, $\nabla=(\partial/\partial x^*,\partial/\partial y^*,\partial/\partial z^*)$, $\bar{q}^*=(u^*,v^*,w^*) = (u_1^*,u_2^*,u_3^*)$ is the velocity vector, $\bar{g}=(g\sin\theta,0,-g\cos\theta)$ is the gravity with the constant gravitational acceleration g, Σ^* is the stress tensor defined by

$$\Sigma_{ij}^*=-\left(p^*+\frac{\mu_0^*|\bar{H}^*|^2}{2}\right)\delta_{ij}+\mu^*H_i^*H_j^*$$
$$+\mu_F^*\left(\frac{\partial u_i^*}{\partial x_j^*}+\frac{\partial u_j^*}{\partial x_i^*}\right),$$

for the magnetic fluid, p^* is the effective pressure which is the sum of thermodynamic pressure and magnetic pressure, $\bar{B}^*$ is the magnetic induction, $\bar{H}^*$ is the magnetic field, where $\bar{B}^* = \mu_0^*\bar{H}_v^* = \mu_0^*(A^*,B^*,C^*)$ in the vacuum and $\bar{B}^* = \mu^*\bar{H}_F^* = \mu^*(a^*,b^*,c^*) = \mu^*(H_1^*,H_2^*,H_3^*)$ in the fluid with μ_0^* and μ^* the positive constant permeabilities of the vacuum and fluid, respectively. Also μ_F^* is the constant viscosity, $\bar{N}^*$ and $\bar{T}^*$ are normal and tangential vectors to the free surface respectively, $[\cdot]$ denotes the jump of a quantity across the free surface, T^* is the constant surface tension coefficient, $|\bar{H}^*|$ is the amplitude of $\bar{H}^*$, and

$$S^* = (\eta^*_{x^*x^*}(1+\eta^{*2}_{y^*}) + \eta^*_{y^*y^*}(1+\eta^{*2}_{x^*}) - 2\eta^*_{x^*}\eta^*_{y^*}\eta^*_{x^*y^*})(1+\eta^{*2}_{x^*}+\eta^{*2}_{y^*})^{-3/2}.$$

It is noted that (5) and (6) imply the continuity of stress, the tangential component of the magnetic field and the normal component of the magnetic induction on the free surface, respectively.

We nondimensionalize the variables as follows. Let h^* be the length scale and H be the scale for the magnetic field. Define

$$(x,y,z) = (x^*,y^*,z^*)/h^*, \quad \hat{t} = t^*(g/h^*)^{1/2},$$

$$\bar{q} = (\hat{u},\hat{v},\hat{w}) = (u^*,v^*,w^*)/(gh^{*1/2}),$$

$$\eta = \eta^*/h^*, \quad h_0 = h_0^*/h^*,$$

$$\bar{H}_F = (\hat{a},\hat{b},\hat{c}) = (a^*,b^*,c^*)/H,$$

$$\bar{H}_v = (\hat{A},\hat{B},\hat{C}) = (A^*,B^*,C^*)/H,$$

$$\mu = \frac{\mu^* H^2}{\rho g h^*}, \quad \mu_0 = \frac{\mu_0^* H^2}{\rho g h^*}, \quad R = \frac{\rho (g h^*)^{1/2} h^*}{\mu_F},$$

$$T = T^*/[\rho g (h^*)^2], \quad \hat{p} = \frac{p^*}{\rho g h^*} + \frac{(\mu_0-\mu)}{2}|\bar{H}_F|^2,$$

where $\hat{p}$ is the nondimensional thermodynamic pressure, $(\mu-\mu_0)|\bar{H}_F|^2/2$ is the pressure from the magnetic field in the fluid, and R is the Reynolds number. In order to facilitate the transformation of the angular term in the normal stress condition and to apply Squire's theorem, we use the following variables:

$$(u,v,w) = (\hat{u},\hat{v},\hat{w})/(R \sin\theta), \quad R_n = R^2 \sin\theta,$$

$$t = \hat{t} R \sin\theta, \quad p = \hat{p}/(R^2 \sin^2\theta), \quad \tau = T/(R\sin\theta),$$

$$(a,b,c) = (\hat{a},\hat{b},\hat{c})/(R\sin\theta).$$

In terms of these nondimensional variables, (1)–(9) become: in the fluid region $-1<z<\eta(t,x,y)$,

$$u_x + v_y + w_z = 0, \tag{10}$$

$$u_t + uu_x + vu_y + wu_z = -p_x + R_n^{-1}(u_{xx}+u_{yy}+u_{zz}+1), \tag{11}$$

$$v_t + uv_x + vv_y + wv_z = -p_y + R_n^{-1}(v_{xx}+v_{yy}+v_{zz}), \tag{12}$$

$$w_t + uw_x + vw_y + ww_z = -p_z - R_n^{-1}\cot\theta + R_n^{-1}(w_{xx}+w_{yy}+w_{zz}), \tag{13}$$

$$a_x + b_y + c_z = 0, \quad a_y = b_x, \quad a_z = c_x, \quad b_z = c_y; \tag{14}$$

in the vacuum region $\eta(t,x,y)<z<h_0$,

$$A_x + B_y + C_z = 0, \quad A_y = B_x, \quad A_z = C_x, \quad B_z = C_y; \tag{15}$$

at the free surface $z=\eta(t,x,y)$,

$$\eta_t + u\eta_x + v\eta_y - w = 0, \tag{16}$$

$$\mu(a\eta_x + b\eta_y - c) = \mu_0(A\eta_x + B\eta_y - C), \tag{17}$$

$$a + c\eta_x = A + C\eta_x, \quad b + c\eta_y = B + C\eta_y, \tag{18}$$

$$R_n P\eta_x + 2u_x\eta_x + (u_y+v_x)\eta_y - (u_z+w_x) = \tau S\eta_x, \tag{19}$$

$$R_n P\eta_y + (v_x+u_y)\eta_x + 2v_y\eta_y - (v_z+w_y) = \tau S\eta_y, \tag{20}$$

$$-R_n P + (w_x+u_z)\eta_x + (w_y+v_z)\eta_y - 2w_z = -\tau S; \tag{21}$$

at $z=-1$,

$$u = v = w = 0, \quad c = 0; \tag{22}$$

at $z=h_0$,

$$C = 0, \tag{23}$$

where

$$P = -p - (\mu-\mu_0)[(a^2+b^2+c^2)/2] - (1/2)[(\mu-\mu_0)^2/\mu_0](a\eta_x + b\eta_y - c)^2(\eta_x^2+\eta_y^2+1)^{-1},$$

$$S = [\eta_{xx}(1+\eta_y^2) + \eta_{yy}(1+\eta_x^2) - 2\eta_x\eta_y\eta_{xy}] \times (1+\eta_x^2+\eta_y^2)^{-3/2}.$$

The derivation of the equations is straightforward and can be obtained from the book by Rosensweig.[1]

III. LINEARIZED EQUATIONS AND SQUIRE'S TRANSFORMATION

From (10)–(23), we can easily find an equilibrium state

$$u_0 = (1/2)(1-z^2), \quad v_0 = w_0 = 0, \quad \bar{H}_0 = (a_0,b_0,0),$$

$$p_0 = -\frac{z\cot\theta}{R_n} - (\mu-\mu_0)\frac{(a_0^2+b_0^2)}{2}. \tag{24}$$

Now we linearize (10)–(23) in terms of the equilibrium state (24) to have the following linearized equations: in $-1<z<0$,

$$u_x + v_y + w_z = 0, \tag{25}$$

$$u_t + u_0u_x + u_{0z}w = -p_x + R_n^{-1}(u_{xx}+u_{yy}+u_{zz}), \tag{26}$$

$$v_t + u_0v_x = -p_y + R_n^{-1}(v_{xx}+v_{yy}+v_{zz}), \tag{27}$$

$$w_t + u_0w_x = -p_z + R_n^{-1}(w_{xx}+w_{yy}+w_{zz}), \tag{28}$$

$$a_x + b_y + c_z = 0, \quad a_y = b_x, \quad a_z = c_x, \quad b_z = c_y; \tag{29}$$

in $0<z<h_0$,

$$A_x + B_y + C_z = 0, \quad A_y = B_x, \quad A_z = C_x, \quad B_z = C_y; \tag{30}$$

at $z=0$,

$$\eta_t + (1/2)\eta_x - w = 0, \tag{31}$$

$$\mu(a_0\eta_x + b_0\eta_y - c) = \mu_0(a_0\eta_x + b_0\eta_y - C), \tag{32}$$

$$a = A, \quad b = B, \tag{33}$$

$$u_z - \eta + w_x = 0, \quad v_z + w_y = 0, \tag{34}$$

$$R_n\left(p - \frac{\eta\cot\theta}{R_n} + (\mu-\mu_0)(a_0a + b_0b)\right) - 2w_z = -\tau(\eta_{xx}+\eta_{yy}); \tag{35}$$

at $z=-1$,

$$u = v = w = 0, \quad c = 0; \tag{36}$$

at $z=h_0$,

$$C=0. \tag{37}$$

In order to use normal modes to study the linear stability of (25)–(37), we write

$$u,v,w,p,a,b,c,A,B,C \sim f(z)\exp(\sigma t+i\alpha x+i\beta y),$$

$$\eta(t,x,y)\sim \eta \exp(\sigma t+i\alpha x+i\beta y). \tag{38}$$

From (38), Eqs. (29), (30), (32), and (33) yield

$$a=d(i\alpha/\tilde{\alpha})\cosh(\tilde{\alpha}h_0)\cosh[\tilde{\alpha}(z+1)],$$

$$A=d(i\alpha/\tilde{\alpha})\cosh\tilde{\alpha}\cosh[\tilde{\alpha}(h_0-z)],$$

$$b=d(i\beta/\tilde{\alpha})\cosh(\tilde{\alpha}h_0)\cosh[\tilde{\alpha}(z+1)],$$

$$B=d(i\beta/\tilde{\alpha})\cosh\tilde{\alpha}\cosh[\tilde{\alpha}(h_0-z)],$$

$$c=d\cosh(\tilde{\alpha}h_0)\sinh[\tilde{\alpha}(z+1)],$$

$$C=-d\cosh\tilde{\alpha}\sinh[\tilde{\alpha}(h_0-z)],$$

where d is a constant satisfying

$$d=\eta(\mu-\mu_0)(a_0\alpha+\beta b_0)i[\mu\cosh(\tilde{\alpha}h_0)\sinh\tilde{\alpha} + \mu_0\cosh\tilde{\alpha}\sinh(\tilde{\alpha}h_0)]^{-1}$$

and $\tilde{\alpha}=(\alpha^2+\beta^2)^{1/2}$. Therefore, (24)–(37) become: in $-1<z<0$,

$$i\alpha u+i\beta v+w_z=0, \tag{39}$$

$$\sigma u+i\alpha u_0 u+u_{0z}w=-i\alpha p+R_n^{-1}(u_{zz}-\tilde{\alpha}^2 u), \tag{40}$$

$$\sigma v+i\alpha u_0 v=-i\beta p+R_n^{-1}(v_{zz}-\tilde{\alpha}^2 v), \tag{41}$$

$$\sigma w+i\alpha u_0 w=-p_z+R_n^{-1}(w_{zz}-\tilde{\alpha}^2 w); \tag{42}$$

at $z=0$,

$$\sigma\eta+(i\alpha/2)\eta-w=0, \tag{43}$$

$$u_z-\eta+i\alpha w=0, \quad v_z+i\beta w=0, \tag{44}$$

$$R_n\Big(p-\frac{\eta\cot\theta}{R_n}-\eta(\mu-\mu_0)^2(\alpha a_0+\beta b_0)^2\cosh(\tilde{\alpha}h_0) \times\cosh\tilde{\alpha}\{\tilde{\alpha}[\mu\cosh(\tilde{\alpha}h_0)\sinh\tilde{\alpha} + \mu_0\cosh\tilde{\alpha}\sinh(\tilde{\alpha}h_0)]\}^{-1}\Big)-2w_z=\tau\tilde{\alpha}^2\eta; \tag{45}$$

at $z=-1$,

$$u=v=w=0. \tag{46}$$

Equations (39)–(46) are the equations we need to study the linear stability of (10)–(23). For every fixed real α and β, if (39)–(46) have nontrivial solutions only for Re $\sigma<0$, then we say that (10)–(23) are linearly stable. If (39)–(46) have nontrivial solutions for Re $\sigma>0$, then we say that (10)–(23) are linearly unstable. The eigenvalues will be determined numerically.

The three-dimensional equations (39)–(46) can be simplified to equations in a two-dimensional space by using the so-called Squire's transformation. The original Squire's transformation was for a single-layer fluid without a magnetic field, and the corresponding transformation has as yet not been introduced for the problem we are considering. Thus we write the transformation explicitly. Let $\alpha u+\beta v\not\equiv 0$ and $\alpha\neq 0$. Without loss of generality, let $\alpha>0$. Then Squire's transformation is the following:

$$\tilde{\alpha}^2=\alpha^2+\beta^2, \quad \tilde{R}=(\alpha/\tilde{\alpha})R_n, \quad \tilde{\tau}=(\tilde{\alpha}/\alpha)\tau,$$

$$\cot\tilde{\theta}=(\tilde{\alpha}/\alpha)\cot\theta,$$

$$\tilde{\alpha}\tilde{u}=\alpha u+\beta v, \quad \tilde{w}=w, \quad \tilde{p}=(\tilde{\alpha}/\alpha)p,$$

$$\tilde{\eta}\tilde{\alpha}=\eta\alpha, \quad \tilde{\alpha}\tilde{a}_0=\alpha a_0+\beta b_0, \quad \tilde{\sigma}=(\tilde{\alpha}/\alpha)\sigma. \tag{47}$$

Then (39)–(46) are: in $-1<z<0$,

$$i\tilde{\alpha}\tilde{u}+\tilde{w}_z=0, \tag{48}$$

$$\tilde{\sigma}\tilde{u}+i\tilde{\alpha}u_0\tilde{u}+u_{0z}\tilde{w}=-i\tilde{\alpha}\tilde{p}+\tilde{R}^{-1}(\tilde{u}_{zz}-\tilde{\alpha}^2\tilde{u}), \tag{49}$$

$$\tilde{\sigma}\tilde{w}+i\tilde{\alpha}u_0\tilde{w}=-\tilde{p}_z+\tilde{R}^{-1}(\tilde{w}_{zz}-\tilde{\alpha}^2\tilde{w}); \tag{50}$$

at $z=0$,

$$\tilde{\sigma}\tilde{\eta}+(i\tilde{\alpha}/2)\tilde{\eta}-\tilde{w}=0, \tag{51}$$

$$\tilde{u}_z-\tilde{\eta}+i\tilde{\alpha}\tilde{w}=0, \tag{52}$$

$$\tilde{p}-\frac{\tilde{\eta}\cot\tilde{\theta}}{\tilde{R}}-\tilde{\eta}(\mu-\mu_0)^2\tilde{\alpha}\tilde{a}_0^2\cosh(\tilde{\alpha}h_0)\cosh\tilde{\alpha} \times[\mu\cosh(\tilde{\alpha}h_0)\sinh\tilde{\alpha}+\mu_0\cosh\tilde{\alpha}\sinh(\tilde{\alpha}h_0)]^{-1} -(2/\tilde{R})\tilde{w}_z=(\tilde{\tau}/\tilde{R})\tilde{\alpha}^2\tilde{\eta}; \tag{53}$$

at $z=-1$,

$$\tilde{u}=\tilde{w}=0. \tag{54}$$

Obviously, (48)–(54) are a special case of (39)–(46) with $\beta=0$, which corresponds to a two-dimensional flow. This shows that to study the linear stability for a three-dimensional flow, it is enough to consider the stability for a two-dimensional flow except for two degenerate cases $\alpha u+\beta v\equiv 0$ and $\alpha=0$, which should be considered separately from Squire's transformation (see Hesla *et al.*[19] for a related problem). In the Appendix, we shall show that three-dimensional flows in these two degenerate cases are always stable. Thus this completes the analysis of Squire's transformation. Therefore, the linear stability of three-dimensional flows can be obtained by only studying the linear stability of two-dimensional flows.

In the subsequent sections, we focus on the linear stability for two-dimensional flows. The governing equations for the two-dimensional problem are the following: in $-1<z<0$,

$$i\alpha u+w_z=0, \tag{55}$$

$$\sigma u+i\alpha u_0 u+u_{0z}w=-i\alpha p+R_n^{-1}(u_{zz}-\alpha^2 u), \tag{56}$$

$$\sigma w+i\alpha u_0 w=-p_z+R_n^{-1}(w_{zz}-\alpha^2 w); \tag{57}$$

at $z=0$,

$$[\sigma+(i\alpha/2)]\eta-w=0, \quad u_z-\eta+i\alpha w=0, \tag{58}$$

$$p-(2/R_n)w_z=E\eta; \tag{59}$$

at $z=-1$,

$$u = w = 0, \tag{60}$$

where the nondimensional constant E is defined as

$$E = \frac{\cot\theta}{R_n} + \frac{(\mu-\mu_0)^2 \alpha a_0^2 \cosh(\alpha h_0)\cosh\alpha}{\mu\cosh(\alpha h_0)\sinh\alpha + \mu_0\cosh\alpha\sinh(\alpha h_0)} + \frac{\tau}{R_n}\alpha^2.$$

Here let us recall that θ is the inclination angle of the plane, μ_0 and μ are the constant permeabilities of the vacuum and magnetic fluid, respectively, σ is the eigenvalue which determines the stability of the solutions, α is the wave number, $R_n = R^2 \sin\theta$ where R is the Reynolds number of the fluid, and h_0 is the distance between the free surface and the upper boundary while the depth of the fluid is normalized to one. All these parameters have been nondimensionalized. It is remarkable that (55)–(60) depend only on four out of the welter of parameters, namely, σ, α, R_n, and E.

The main qualitative features of the film stability can, to some extent, be inferred plausibly from (55)–(60) at this stage. It is known[14] that for a fluid flow without a magnetic field, decreasing the inclination angle θ or increasing the surface tension τ, which makes E larger, stabilizes the flow. It is easy to see that increasing the magnetic field for a magnetic fluid also increases E. The term E is thus comprised of forces that tend to stabilize an inclined film: gravity, surface tension, and a magnetic pressure. However, the rate of change of E as a function of the wave number is different for each force. We note that E depends on the wave number α, which in turn appears elsewhere in the governing equations, so that a numerical approach is needed to quantify the degree to which the forces affect the growth rate Re σ at each wave number. A simple force balance argument can therefore not be developed to determine the band of unstable wave numbers, but can be used to gain some insight for the limiting cases. For α large, which corresponds to short waves, the surface tension τ has the largest effect on E while the inclination angle θ has almost no effect. For α small, which corresponds to long waves, θ has a much stronger effect on E than τ. If each of the three effects is considered separately, then the magnetic field has the best chance of increasing E over both long and short waves. We expect that the fluid is more stable for larger E so that increasing the magnetic field may effectively stabilize the waves on the free surface. These observations foreshadow our subsequent analysis using asymptotic and numerical methods.

IV. ASYMPTOTIC ANALYSIS FOR LONG WAVES AND SHORT WAVES

In this section, we study the stability of long and short waves using asymptotic methods. First, let us study the long-wave limit for the solutions of (55)–(60). Assume that α is small and u,w,p,η have the following asymptotic expansion:

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 + \cdots, \tag{61}$$

$$\sigma = \sigma_0\alpha + \sigma_1\alpha^2 + \cdots. \tag{62}$$

Substitution of (61) and (62) into (55)–(60) yields a series of successive approximations. The zeroth-order approximation is

$$u_0 = \eta_0(z+1), \quad p_0 = E_0\eta_0, \quad w_0 = 0,$$
$$E_0 = [(\cot\theta)/R_n] + (\mu-\mu_0)^2 a_0^2/(\mu+\mu_0 h_0).$$

By using the first-order approximation, we have

$$w_1 = -(i/2)(z+1)^2\eta_0, \quad \sigma_0 = -i,$$
$$u_1 = \eta_1(z+1) + i\eta_0 R_n[(z^2-1)(E_0/2) + (1/24)(z^4 - 6z^2 + 5)].$$

Then, from the second-order approximation of (55), (60), and (58), we have

$$\sigma_1 = R_n[(2/15) - (E_0/3)].$$

Therefore, the expansion of σ for the long-wave limit is

$$\sigma = -i\alpha + R_n[(2/15) - (E_0/3)]\alpha^2 + O(\alpha^3). \tag{63}$$

To have long-wave stability, we need

$$\frac{2}{15} < \frac{E_0}{3} = \frac{\cot\theta}{3R_n} + \frac{(\mu-\mu_0)^2 a_0^2}{3(\mu+\mu_0 h_0)}. \tag{64}$$

This result is consistent with the long-wave stability criterion obtained by Yih[11] without a magnetic field and by Shen *et al.*[9] with a magnetic field except for the use of different nondimensional parameters. We note that surface tension does not stabilize long waves at this order while the magnetic field has the dominant stabilizing effect. Also, when h_0 becomes sufficiently large, the motion becomes more unstable. From (64), we can see that the pressure from a horizontal magnetic field can be viewed as an additional gravitational force that can stabilize the waves. However, we note that a normal magnetic field destabilizes a magnetic fluid.[4] Therefore, in general, a magnetic pressure cannot be considered as an additional gravitational force.

Next, we shall study the stability of short waves. Let α be large, $z = \xi/\alpha$ and $\sigma = \hat{\sigma}\alpha$. Then (55)–(60) become: in $-\infty < z < 0$,

$$iu + w_\xi = 0, \tag{65}$$

$$\hat{\sigma}u + i[1 - (\xi/\alpha)^2](u/2) - (\xi/\alpha^2)w = -ip + (\alpha/R_n)(u_{\xi\xi} - u), \tag{66}$$

$$\hat{\sigma}w + i[1 - (\xi/\alpha)^2](w/2) = -p_\xi + (\alpha/R_n)(w_{\xi\xi} - w); \tag{67}$$

at $\xi = 0$,

$$[\hat{\sigma} + (i/2)]\eta = (w/\alpha), \quad u_\xi + iw - (\eta/\alpha) = 0, \tag{68}$$

$$p - (2\alpha/R_n)w_\xi = E\eta; \tag{69}$$

at $z = -\infty$,

$$u = w = 0. \tag{70}$$

Assume that u,v,p,η have asymptotic expansions

$$f = f_0 + (f_1/\alpha) + (f_2/\alpha^2) + \cdots,$$
$$\hat{\sigma} = \sigma_0 + (\sigma_0/\alpha) + \cdots,$$

with $\eta_0=0$ and $\eta_1\neq 0$. From (65)–(67), we have

$$\left(\frac{d^2}{d\xi^2}-1\right)^2 w=R_n\left[[\hat{\sigma}+(i/2)]\alpha^{-1}\left(\frac{d^2}{d\xi^2}-1\right)w\right.$$
$$\left.-i(1/2\alpha^3)\xi^2\left(\frac{d^2}{d\xi^2}-1\right)w+(i/\alpha^3)w\right]. \quad (71)$$

From the boundary conditions (68)–(70), it can be obtained that at $\xi=0$,

$$[\hat{\sigma}+(i/2)](w_{\xi\xi}+w)+(iw/\alpha^2)=0, \quad (72)$$

$$[\hat{\sigma}+(i/2)](w_{\xi\xi\xi}+w_\xi\{-3-[\hat{\sigma}+(i/2)](R_n/\alpha)\})$$
$$-(ER_n/\alpha^2)w=0; \quad (73)$$

at $\xi=-\infty$,

$$w=w_\xi=0. \quad (74)$$

Now let w_0 be the highest order in the expansion of w in terms of $1/\alpha$. The first-order approximation of (71), (73), and (74) is

$$\left(\frac{d^2}{d\xi^2}-1\right)^2 w_0=0 \quad \text{in } -1<\xi<0,$$

$$w_{0\xi\xi}+w_0=0 \quad \text{at } \xi=0,$$

$$w_0=w_{0\xi}=0 \quad \text{at } \xi=-\infty.$$

Thus $w_0=K(e^\xi-\xi e^\xi)$, where K is a nonzero constant. From (73) we have $[\hat{\sigma}+(i/2)](-2)-(ER_n/\alpha^2)=0$ or $\sigma_0+(\sigma_1/\alpha)=-(i/2)-(ER_n/2\alpha^2)$. By the definition of E, we have

$$E=\frac{\cot\theta}{R_n}+\frac{(\mu-\mu_0)^2\alpha a_0^2\cosh(\alpha h_0)\cosh\alpha}{\mu\cosh(\alpha h_0)\sinh\alpha+\mu_0\cosh\alpha\sinh(\alpha h_0)}$$
$$+\frac{\tau}{R_n}\alpha^2. \quad (75)$$

Therefore, $\hat{\sigma}\sim\sigma_0+(\sigma_1/\alpha)=(-i/2)-(1/2)(\tau+(\mu-\mu_0)^2 \times a_0^2 R_n[(\mu+\mu_0)\alpha]^{-1}+\alpha^{-2}\cot\theta)$ or

$$\sigma=-\frac{i\alpha}{2}-\frac{1}{2}\left(\alpha\tau+\frac{(\mu-\mu_0)^2a_0^2R_n}{(\mu+\mu_0)}+\frac{\cot\theta}{\alpha}\right)+O\left(\frac{1}{\alpha^2}\right). \quad (76)$$

If $\tau\geqslant O(1)$, then $\sigma=(-1/2)\alpha\tau-(i\alpha/2)+O(1)$. If $\tau\sim O(1/\alpha)$ and $(\mu-\mu_0)^2a_0^2R_n/(\mu+\mu_0)\sim O(1)$, then

$$\sigma=-\frac{i\alpha}{2}-\frac{1}{2}\left(\frac{(\mu-\mu_0)^2a_0^2R_n}{\mu+\mu_0}+\alpha\tau\right)+O\left(\frac{1}{\alpha}\right). \quad (77)$$

For $\tau\sim O(1/\alpha^2)$ and $(\mu-\mu_0)^2a_0^2R_n/(\mu+\mu_0)\sim O(1/\alpha)$,

$$\sigma=-\frac{i\alpha}{2}-\frac{1}{2\alpha}\left(\cot\theta+\frac{(\mu-\mu_0)^2a_0^2R_n\alpha}{\mu+\mu_0}+\alpha^2\tau\right)+O\left(\frac{1}{\alpha^2}\right). \quad (78)$$

From these formulas, we see that the short waves are always stable. Increasing magnetic field and surface tension or decreasing the inclination angle θ stabilizes short waves. However, the rates of these effects are different. Surface tension has the strongest stabilizing effect while the inclination angle has the least stabilizing effect.

We note that for long waves, θ and the magnetic field are both stabilizing, but surface tension has no effect on the long waves. However, surface tension has a very strong effect on short waves, while the magnetic field has less effect on such waves. Therefore, the magnetic field can simultaneously stabilize both long and short waves. In addition, the magnetic field tends to stabilize the intermediate waves, which we shall see in the numerical computation for waves of intermediate length.

V. NUMERICAL RESULTS

In this section, a numerical method will be used to study the eigenvalue σ of (39)–(42) with boundary conditions (43)–(46) and $\beta=0$. We use the Chebyshev-tau method to discretize the equations in the z direction. The implementation of this scheme is described in Joseph and Renardy[18] for a two-layer flow in a channel. The vertical component of the velocity is written as a combination of Chebyshev polynomials

$$w=\sum_{i=0}^{N} w_i T_i(\hat{z}), \quad \hat{z}=2z+1, \quad -1<\hat{z}<1. \quad (79)$$

Together with the free-surface perturbation η and the constant d defined after Eq. (38), there are $N+3$ unknowns. We truncate the equation of motion at the $N-3$rd Chebyshev mode, and with two boundary conditions and four free-surface conditions, we have an $N+3$ matrix eigenvalue problem. This is solved using a NAG routine in quadruple precision.

The asymptotic formulas of Sec. IV were checked numerically. We note that for the short-wave formula to be valid, we require R/α to be small. The numerical results of Floryan *et al.*[14] on the eigenvalues for the free-surface mode and the first shear mode were checked. Their Reynolds number R_F is $R_n/2$. Their surface tension parameter is ζ, where our $\tau=\zeta(\sin\theta/2R_F)^{7/6}/(3^{1/3}\sin\theta)$. Our eigenvalue is $\sigma=0.5(\alpha c_i-i\alpha c_r)$, where their k is our α. Entries in their Tables VI and VII were reproduced.

In the three-dimensional problem, if there is a contribution from both a_0 and b_0, which implies $\beta\neq 0$, then one may look at wave numbers with the relation that

$$\alpha a_0+\beta b_0=0, \quad (80)$$

allowing also for negative wave numbers. By Squire's transformation, these wave numbers are transformed to $\tilde{\alpha}=\alpha[1+(a_0/b_0)^2]^{1/2}$, and the magnetic field is transformed to $\tilde{a}_0=0$, and the problem returns to that examined by Floryan *et al.*[14] They calculated the conditions for criticality; at small slopes, the onset is at a finite wave number for the shear mode and otherwise the onset is due to long surface waves. When both a_0 and b_0 are present, the critical Reynolds number is at most the value in the two-dimensional problem with zero magnetic field, and is achieved for the wave numbers (α,β) satisfying (80).

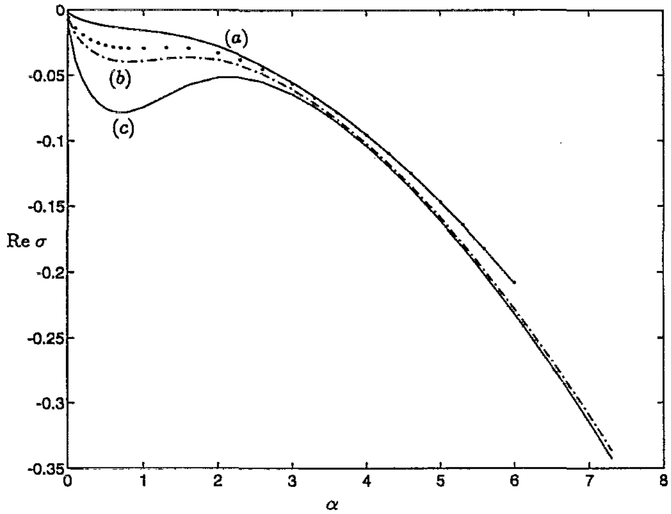

FIG. 2. (a) Growth rates versus wave numbers for the least stable modes for the carrier fluid diester; $a_0=0$, $\mu=0.1721E-5$, $\mu_0=0.2162E-8$, $R_n=306.0114$, $\theta=0.001$ rads, $\tau=1.991\,158$, $h_0=1$. The line belongs to the interfacial mode. The dots belong to an internal mode. (b) $a_0^*=4000$. (c) $a_0^*=15\,900$.

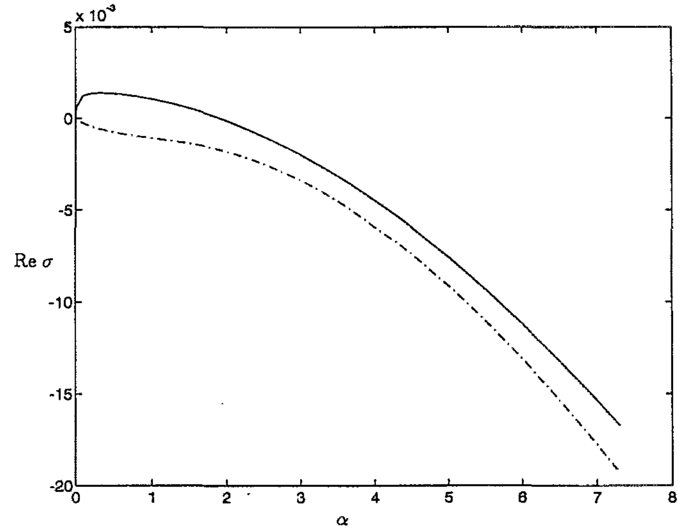

FIG. 3. Growth rates versus wave numbers for the least stable modes for the carrier fluid diester; $\mu=0.1721E-5$, $\mu_0=0.2162E-8$, $h_0=1$, $R_n=5340.725$, $\theta=0.017\,45$ rads, $\tau=0.1\,140\,888$. (a) $a_0^*=0$ (line). (b) $a_0^*=4600$ (-·-·-).

In the following, we consider typical ferrofluids described in Table 2.4 of Rosensweig.[1] The table gives the magnetic saturation (A m^{-1}), density (kg m^{-3}), viscosity (N s m^{-2}), and surface tension (mN m^{-1}), respectively, of ferrofluids with the following carrier fluids: diester (15 900, 1185, 0.075, 32), hydrocarbon (15 900, 1050, 0.003, 28), fluorocarbon (31 800, 1250, 0.006, 28), three ferrofluids in ester [(i) 7960, 2050, 2.50, 18; (ii) 15 900, 1150, 0.014, 26; (iii) 31 800, 1300, 0.030, 26], two ferrofluids in water [(i) 47 700, 1400, 0.035, 21; (ii) 15 900, 1180, 0.007, 26], and two types in polyphenylether [(i) 31 800, 1380, 0.01, 26; (ii) 7960, 2050, 7.50]. Our model concerns the situation where the magnetic field outside the fluid induces a field in the fluid linearly, leading to the relation between the magnetic induction and the magnetic field described in the paragraph after Eq. (9). In order to be consistent with this, the value of a_0^* should be much less than the value for magnetic saturation. The permeabilities are $\mu_0^*=4\pi\times10^{-7}$ Henry/m, with $\mu^*\sim10^{-3}$ to 10^{-4}. We choose the scale H to be 1, and the length scale $h^*=0.05$ m.

We begin with the diester, and choose $h_0=1$; that is, the fluid and vacuum have equal depths. For very small values of the angle $\theta<.005$ rads there is long-wave stability. We choose $\theta=0.001$ radians and examine the growth rates of the eigenvalues as a_0^* varies. The case of $\theta=0.001$ rads and $a_0=0$ is shown in Fig. 2. The Reynolds number is $R_n=306.011\,4$, the surface tension parameter is $\tau=1.991\,158$, and the dimensionless permeabilities are $\mu=0.1721E-5$, $\mu_0=0.216\,27E-8$. The Reynolds number is below criticality. Note that $a_0=a_0^*/(HR\sin\theta)$. The effect of varying a_0 from 0 to the magnetic saturation is that the growth rates are further stabilized. In Fig. 2(a) for $a_0=0$, the line represents the interfacial mode which has a negative imaginary part, and the dots represent an internal mode which has a positive imaginary part. The real parts of both modes are virtually the same for the order one wave numbers shown in the figure. The modes diverge for long waves, and for short waves. This pairing of modes with the same real parts but different imaginary parts represents waves going in opposite directions, as in the case of inviscid water waves. In contrast, very viscous flows support waves going in one direction. In Fig. 2(b), the $a_0^*=4000$ which is expected to be in the regime of applicability of the model equations. In Fig. 2(c), at a_0^* equal to the magnetic saturation value, there has been a stabilization over all the wave numbers, and the pairing of the surface mode with the least stable shear mode over order one wave numbers is still present.

When the angle is increased to $\theta>0.006$ radians, there is long-wave instability. The case of $\theta=1°$ is shown in Fig. 3. At this incline, the Reynolds number has increased to $R_n=5340.725$, the surface tension parameter is $\tau=0.114\,09$, and the permeabilities remain the same for any incline angle. With zero magnetic field, the Reynolds number is high enough to cause a long-wave instability of the surface mode but the bulk modes are stable. Figure 3 shows that the interfacial mode is unstable for wavelengths up to about 2.0. The work of Shen *et al.*[9] is relevant to the case of a nonlinear evolution of long surface waves. They derive an equation which includes the Burgers, KdV, and Kuramoto–Sivashinsky equations. When the magnetic field is absent, the works of Gumerman and Homsy[20] and Hooper and Grimshaw[21] on the K–S equation for weakly nonlinear evolution of long interfacial waves are relevant. A recent work on the K–S equation motivated by a two-layer inclined channel problem is that of Tilley *et al.*[22] In their application, they introduce a cubic nonlinearity and they examine isolated solution branches which emerge.

The long-wave instability can be stabilized by introducing a magnetic field and increasing a_0^* above approximately 4500, which is well below the value for the magnetic saturation and appears to be within the range of applicability of

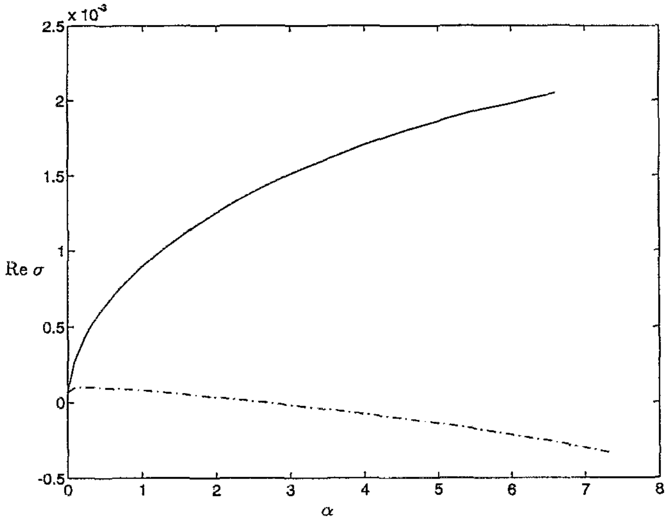

FIG. 4. Growth rates versus wave numbers for the least stable modes for the carrier fluid diester; $\theta=89°$, $R_n=305\,969.9$, $\tau=0.19\ 914E-2$. (a) $a_0=0$ (line). (b) $a_0^*=15\ 900$ (-----).

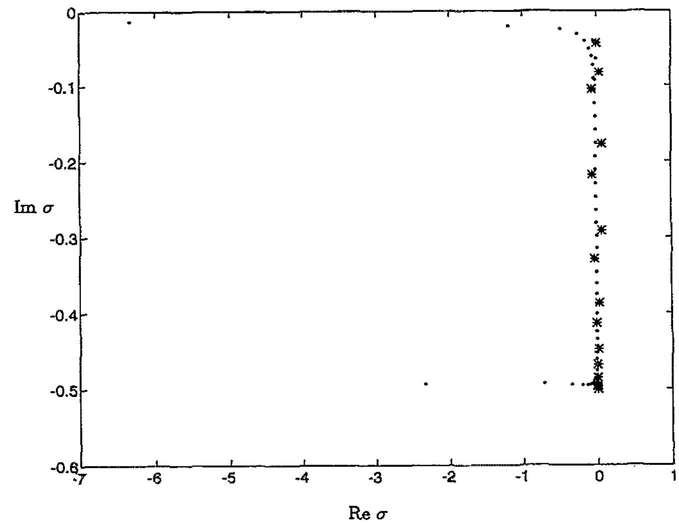

FIG. 5. The eigenvalues for $N=20$ (*) Chebyshev modes and $N=60$ (·) for the case $\theta=89°$, $\alpha=1.0$, $R_n=305\,969.9$, $\tau=0.1\ 991\ 428E-2$, the surface mode is $\sigma=0.8975E-3-0.5009i$, showing clustering toward the inviscid continuous spectrum.

the governing equations. The case of $a_0^*=4600$ is shown in Fig. 3(b), where stabilization has been achieved.

As the angle θ is increased with zero magnetic field, the range of instability grows, and the Reynolds number increases. The growth rates for the extreme case of $\theta=89°$ is shown in Fig. 4. The Reynolds number is $R_n=305\,969.9$ and the surface tension parameter is $\tau=0.199\ 14E-2$. The surface mode is unstable over the range of wave numbers shown. With the introduction of the magnetic field, the surface wave is stabilized most strongly for short waves. Figure 4(b) shows the situation for the a_0^* at the magnetic saturation value. The computed set of eigenvalues at the higher Reynolds numbers starts to cluster toward the neutrally stable continuous spectrum for inviscid flows. This continuous spectrum is calculated from Eqs. (48)–(50) by setting $R^{-1}=0$, and deriving a single equation for $\tilde{w}$, in which we set the coefficient of the highest derivative equal to zero. This yields $\sigma=-i\tilde{\alpha}(1/2)(1-z^2)=-i\tilde{\alpha}u_0$, where u_0 is defined in Eq. (24). The eigenfunctions corresponding to the continuous spectrum are not as well behaved as those of the discrete modes, and this is reflected in the poor convergence of the spectral method for those modes. For instance, the use of $N=20$ modes for $\alpha=1.0$ yields a number of slightly unstable modes where the continuous spectrum would be in inviscid flow, and $N=60$ yields stability for those modes. This is shown in Fig. 5. The neutral stability curve obtained from the long-wave formula (63) for diester in the a_0^*-θ plane is shown in Figs. 6(a) and 6(b). Numerical computations for the other modes and over other wave numbers are required to make sure that the long-wave interfacial case does govern the neutral stability for this particular fluid. For small angles, the diester-based ferrofluid is stable, and for larger angles, numerical computations show that the neutral stability curve for the case $a_0=0$ is in the regime governed by the long-wave formula. Computations at various angles for nonzero a_0 show that the stabilization of the interfacial

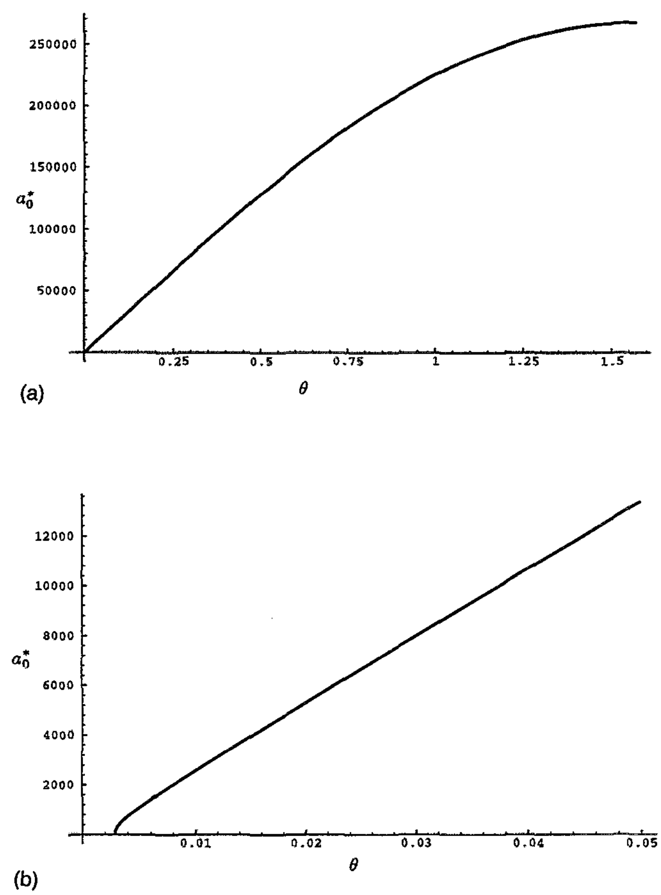

FIG. 6. (a) Neutral stability curve for the interfacial mode for long waves in the $a_0^*-\theta$ (rads) plane for the carrier fluid diester, $\mu=0.1721E-5$, $\mu_0=0.2162E-8$, $h_0=1$. $0<\theta<\pi/2$. (b) Neutral stability curve for the interfacial mode for long waves in the $a_0^*-\theta$ (rads) plane for the carrier fluid diester, $\mu=0.1721E-5$, $\mu_0=0.2162E-8$, $h_0=1$. This is a magnification of (a) for small θ.

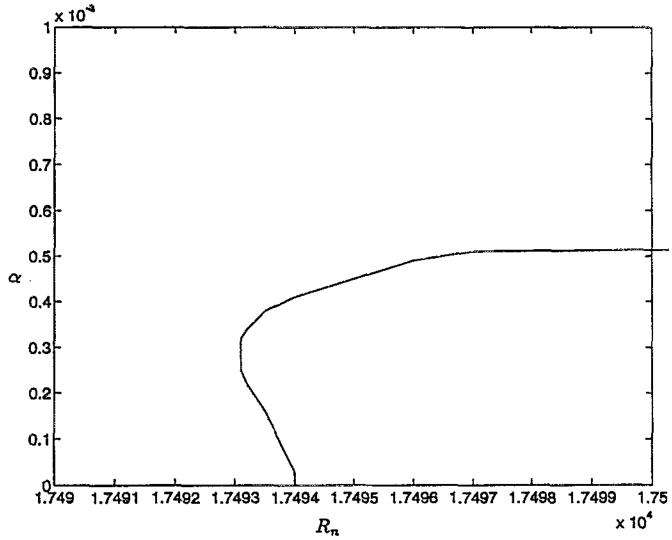

FIG. 7. Neutral stability curves for $\theta=0.5'$, $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, $\tau=0$. This is a magnification of Fig. 8(b) for small wave numbers for $R_n=17\,400$ to $17\,500$, $a_0=26.46$.

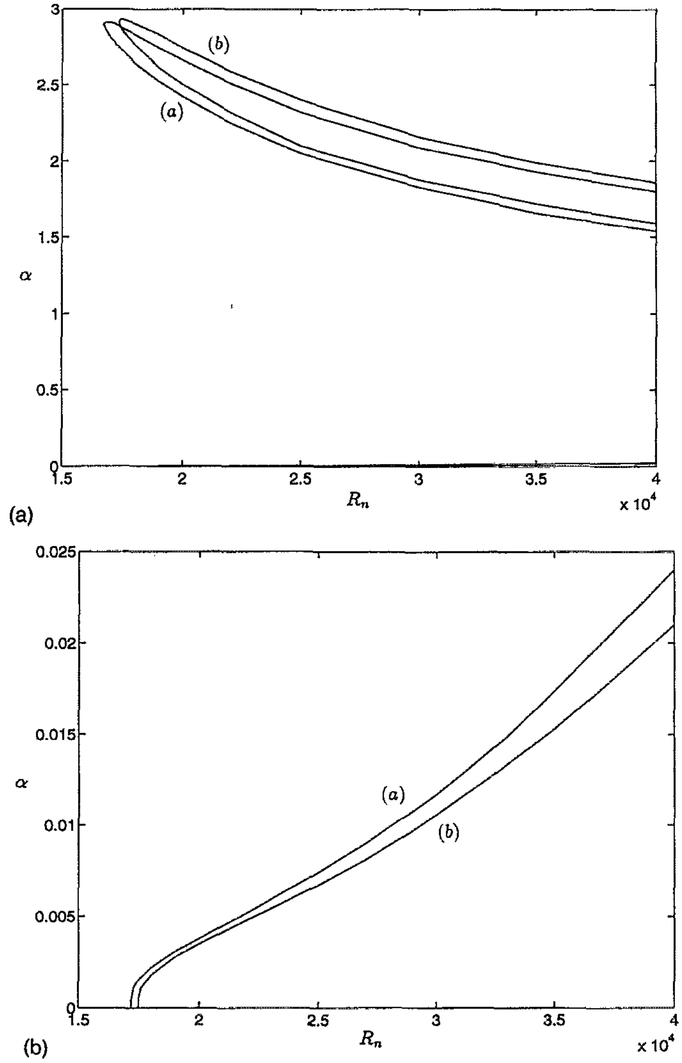

FIG. 8. (a): Neutral stability curves for $\theta=0.5'$, $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, $\tau=0$, curve (a) $a_0=0$ and curve (b) $a_0=26.46$. Wave numbers range over $0\leq\alpha\leq3$. (b): Neutral stability curves for $\theta=0.5'$, $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, $\tau=0$, curve (a) $a_0=0$ and curve (b) $a_0=26.46$. (a) Wave numbers range over $0\leq\alpha\leq3$. This is a magnification of (a) for wave numbers $0\leq\alpha\leq0.025$.

mode occurs first for finite wave numbers and then proceeds to the long waves as a_0 is increased. Hence, we expect Figs. 6(a) and 6(b) to be the neutral stability curve, apart from the effect discussed for Fig. 7.

The first of the ester-based ferrofluids is an example of a flow that has a much lower Reynolds number than the diester case. At a 1° incline, there is stability for all wavelengths with zero magnetic field. The addition of the field a_0 further stabilizes the situation. The hydrocarbon-based ferrofluid is an example of a fluid that has a higher Reynolds number than the diester case. At a 1° incline, the Reynolds number is $R_n=2\,620\,729$, $\mu=0.194\,231\,7E-5$, $\mu_0=0.244\,078\,7E-8$, $h_0=1$, $\tau=0.508\,591\,8E-2$. From Fig. 6 of Ref. 14, with their Reynolds number R_F being $R_n/2$, we see that the interfacial mode and a shear mode are unstable over a range of long and order one wave numbers if there is no magnetic field. The long-wave formula (63) shows that we require $a_0^* = 97\,234$ to stabilize the long waves, a value which is out of the range of validity of our equations. For the sake of completeness, if we examine this case, the dimensionless a_0 is 454.65. This stabilizes the long waves but leaves a range of finite wave numbers $0.5E-6<\alpha<0.3E-5$ unstable, and stabilizes the shear mode. The instability at finite wave number for the interfacial mode, with the shear modes stabilized, has not been found to occur for the case $a_0=0$. This new feature is present for situations which are close to the criticality of the interfacial mode for long waves, so that the magnetic field is just enough to stabilize them but leaves a range of rather long but finite wave numbers unstable. Closer inspection of this behavior in terms of neutral stability curves is discussed below for Fig. 7. Thus, qualitatively, there are four types of critical situations we encounter: long surface waves, a finite wave-number surface wave, a finite wave-number shear mode, or the long and finite wave numbers can be simultaneously critical.

The final flow we discuss is at a small slope $\theta=0.5'$. For angles less than this, the work of Ref. 14 for the case $a_0=0$ shows a shear mode to become unstable at a lower Reynolds number than the interfacial mode, and vice versa for angles more than this. The diester-based ferrofluid has Reynolds number $R_n=44.5$ and is stable at this angle. The hydrocarbon-based ferrofluid is at $R_n=21\,834$ and is unstable to both long and order one interfacial and shear modes. Rather than to focus on any particular fluid, we choose to display the critical wave numbers and the critical Reynolds numbers for nonzero a_0, to highlight the general shift in the shear mode and the interfacial mode as a_0 increases for $\theta=0.5'$. For this purpose, we choose $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, and zero surface tension. Figure 8(a) shows the neutral stability curves for Reynolds numbers up to 40 000 and wave numbers up to 3. On this scale, we see the stabilization of the shear mode by the magnetic field, and the neutral stability curve for the interfacial mode is close to the horizontal axis. Figure 8(b) shows the effect of

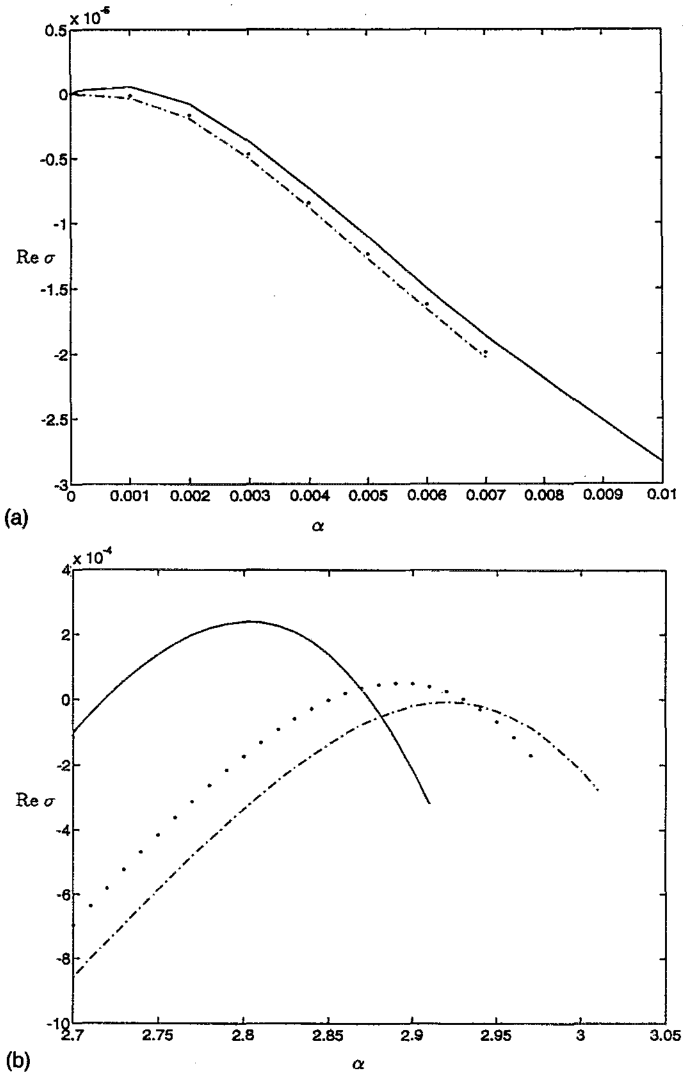

FIG. 9. (a) Growth rates for the least stable modes for $R_n=17\,500.0$, $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, $\theta=0.5'$, $\tau=0.0$, $(-)a_0=0$, $(\cdot\cdot)a_0=26.46$, $(-\cdot-)a_0=30.24$. These curves track the long wave-number instability for the surface mode. (b) Growth rates for the least stable modes for $R_n=17\,500.0$, $\mu=0.1E-4$, $\mu_0=0.1E-7$, $h_0=1$, $\theta=0.5'$, $\tau=0.0$, $(-)a_0=0$, $(\cdot\cdot)a_0=26.46$, $(-\cdot-)a_0=30.24$. These curves track the finite wave-number instability for the shear mode.

adding a_0 for the interfacial mode for wave numbers up to 0.025. This shows the same type of shift as the shear mode toward stability as a_0 is increased. The addition of the magnetic field appears to simply shift the curve. However, a qualitatively new feature appears for Reynolds numbers close to the onset of the long-wave instability. Figure 7 is a magnification of this area. The interfacial mode is unstable for finite wave numbers around $R_n=17\,494$, which is a feature that does not appear for the case of $a_0=0$. This finite wave-number instability of the interfacial mode is present for a tiny window starting at approximately $R_n=17\,493$ just below the critical Reynolds number for long waves $R_n=17\,494.02$. This type of window is found in other nonzero a_0 computations.

Figures 9(a) and 9(b) show growth rates for $R_n=17\,500$ for the least stable shear mode and the interfacial mode, and the shifts as a_0 is increased. When $a_0=0$, the surface mode is unstable for long waves and the shear mode is unstable for a band of finite wave numbers. The addition of a magnetic field stabilizes the surface mode first and at $a_0=26.46$, only the shear mode is unstable. Thus traveling waves of this wave number may arise as a result of this flow instability. The weakly nonlinear analysis of this Hopf bifurcation leads to the Stuart–Landau equation. In the case of a supercritical bifurcation, the analysis of stability of the traveling wave solution, with respect to sideband perturbations, would be treated in the manner of Renardy and Renardy.[23] They have derived the criteria for stability to sideband perturbations for a problem which is qualitatively similar to this. They consider a layered Couette–Poiseuille flow with two immiscible liquids of different viscosity and density with surface tension at the interface, in which the growth rate for the surface mode approaches zero in the limit of long waves and the surface mode has a finite critical wave number. The eigenspaces at criticality thus consist of the long-wave mode, the traveling wave mode, and a pressure mode. In the present problem, the upper layer is a vacuum with zero pressure, so that we cannot add a constant to the pressure to retrieve the same flow field, and thus a pressure mode is not part of the eigenspace. As a result, the amplitude equations would have the form of their Eq. (1) with no P.

When more magnetic field is applied, e.g., $a_0=30.24$, the shear mode is also stabilized. Thus, in the two-dimensional problem, both the surface mode and the shear mode can be stabilized by using a magnetic field. This property persists for larger incline angles.

VI. CONCLUSIONS

We have presented a detailed analysis of the stability of a magnetic fluid flow down an inclined plane with surface tension under gravity and a tangential magnetic field. By introducing a transformation that is similar to Squire's transformation in hydrodynamics, the stability problem in a three-dimensional space is reduced to the problem in a two-dimensional space. However, this transformation has two degenerate cases. The stability of these degenerate cases is studied analytically and it is shown that the flows in these cases are always linearly stable. Thereafter, asymptotic methods are used to analyze the interfacial mode for short and long waves in a two-dimensional space and the stability criteria are derived for these waves in terms of Reynolds numbers. The three dominant forces at work are gravity, surface tension, and the magnetic pressure. It is found that surface tension can stabilize the short waves, but cannot stabilize the long waves. However, the magnetic field can stabilize both short and long waves. Therefore, increasing the magnetic field more effectively stabilizes the magnetic fluid flows than increasing surface tension of the fluid. Finally, a numerical method is used to obtain the stability of waves with intermediate length. It is found that the magnetic field can also stabilize these waves. Moreover, we note that at very small inclination angles, the shear modes become unstable first when the surface mode is still stable[14] if there is no magnetic field. However, such unstable shear modes can also be stabilized by introducing a magnetic field.

In summary, our results show that a horizontal magnetic field is capable of stabilizing ferrofluid films at all wavelengths, as long as the wall down which the films flow is not too steep. Moreover, the results would set a basis on which the weakly nonlinear response of magnetic fluid flows could be examined at the critical points where linear instability starts to happen.

ACKNOWLEDGMENTS

This paper is dedicated to the 75th birthday of our academic grandfather, Richard E. Meyer.

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APPENDIX: DEGENERATE CASES OF SQUIRE'S TRANSFORMATION

In this appendix, we study two degenerate cases $\alpha u+\beta v=0$ and $\alpha=0$ in Squire's transformation (47) and rigorously prove that the flows corresponding to these cases are always linearly stable.

We consider Eqs. (39)–(46). First, let us assume $\alpha u+\beta v\equiv 0$. By (39) and (46), $w\equiv 0$. Then (41) and (42) imply $p\equiv 0$ and from (45) $\eta=0$. Since u and v cannot be both identically equal to 0, we let $u\not\equiv 0$. Multiply (40) by $\bar{u}$, where $\bar{u}$ is the complex conjugate of u, and integrate it from -1 to 0 to have

$$\sigma\int_{-1}^{0}|u|^2dz+\int_{-1}^{0}i\alpha u_0|u|^2dz$$
$$=-R_n^{-1}\int_{-1}^{0}(|u_z|^2+|u|^2\tilde{\alpha}^2)dz+R_n^{-1}u_z\bar{u}|_{-1}^{0}.$$

By (44) and taking the real part, we have Re $\sigma<0$. Thus it is stable. Next, let us consider $\alpha=0$. Assume that $w\not\equiv 0$, since otherwise $\alpha u+\beta v\equiv 0$, which has been studied. From (39), (41), and (42), we have

$$(D^2-\beta^2)^2w-R_n\sigma(D^2-\beta^2)w=0\quad\text{in } -1<z<0,$$

where $D=(d/dz)$. Multiply the above equation by $\bar{w}$ and integrate it from -1 to 0. From the boundary conditions (43)–(46), we have that at $z=0$,

$$w=\sigma\eta,\quad w_{zz}+\beta^2w=0,$$
$$w_{zzz}-(3\beta^3+\sigma R_n)w_z-E\eta=0;$$

at $z=-1$, $w=w_z=0$, where

$$E=\beta^2\{\cot\theta+\tau\beta^2+R_n(\mu-\mu_0)^2\beta b_0^2\cosh(\beta h_0)$$
$$\times\cosh\beta[\mu\cosh(\beta h_0)\sinh\beta$$
$$+\mu_0\cosh\beta\sinh(\beta h_0)]^{-1}\}.$$

From these boundary conditions and the fact that $\sigma\neq 0$, we have

$$\int_{-1}^{0}|(D^2-\beta^2)w|^2dz+\sigma R_n\int_{-1}^{0}(|Dw|^2+\beta^2|w|^2)dz$$
$$+(E/\sigma)|w(0)|^2+4\beta^2\,\mathrm{Re}(w\bar{w}_z)|_{z=0}=0.$$

Since $w\not\equiv 0$, $\int_{-1}^{0}(|Dw|^2+\beta^2|w|^2)dz>0$ and $\int_{-1}^{0}|(D^2-\beta^2)w|^2dz>0$. By taking the real part, the above equation reduces to (Re $\sigma)A=-B$, where

$$A=R_n\int_{-1}^{0}(|Dw|^2+\beta^2|w|^2)dz+(E/|\sigma|^2)|w(0)|^2>0,$$

$$B=\int_{-1}^{0}|(D^2-\beta^2)w|^2dz+4\beta^2\,\mathrm{Re}(w\bar{w}_z)|_{z=0}.$$

Rewrite B as

$$B=\int_{-1}^{0}[(D^2-\beta^2)w][(D^2-\beta^2)\bar{w}]dz+4\beta^2\,\mathrm{Re}(w\bar{w}_z)|_{z=0}$$
$$=\int_{-1}^{0}(|D^2w|^2+2\beta^2|Dw|^2+\beta^4|w|^2)dz$$
$$+2\beta^2\,\mathrm{Re}(w\bar{w}_z)|_{z=0}.\tag{A1}$$

But

$$|\mathrm{Re}(w\bar{w}_z)|_{z=0}|$$
$$=\left|\mathrm{Re}\int_{-1}^{0}(w\bar{w}_z)_z\,dz\right|$$
$$\leqslant\int_{-1}^{0}|w_z|^2dz+\left|\mathrm{Re}\int_{-1}^{0}w\bar{w}_{zz}\,dz\right|$$
$$\leqslant\int_{-1}^{0}|w_z|^2dz+\int_{-1}^{0}|w\bar{w}_{zz}|dz.\tag{A2}$$

Thus by (A1) and (A2),

$$B\geqslant\int_{-1}^{0}(|D^2w|^2+2\beta^2|Dw|^2+\beta^4|w|^2)dz$$
$$-2\beta^2|\mathrm{Re}(w\bar{w}_z)|_{z=0}|$$
$$\geqslant\int_{-1}^{0}(|D^2w|-\beta^2|w|)^2dz\geqslant 0.$$

The equality $B=0$ holds only if $|D^2w|-\beta^2|w|=0$ and $\mathrm{Re}(w\bar{w}_{zz})=\pm|w\bar{w}_{zz}|$ in $(-1,0)$. Then (A1) and (A2) imply

$$B=\int_{-1}^{0}(|D^2w|^2+2\beta^2|Dw|^2+\beta^4|w|^2)dz$$
$$+2\beta^2\left(\int_{-1}^{0}|w_z|^2dz+\mathrm{Re}\int_{-1}^{0}w\bar{w}_{zz}\,dz\right)$$
$$=\int_{-1}^{0}[(|D^2w|\pm\beta^2|w|)^2+4\beta^2|w_z|^2]dz=0.$$

Thus if $B=0$ then $w\equiv 0$, which is a contradiction. Therefore, $B>0$ and Re $\sigma<0$, which implies that the flow is stable for $\alpha=0$. From this, we conclude that the degenerate cases for Squire's transformation are always stable.

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