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Temperature-jump problem with arbitrary accommodation

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A concise and accurate result for the temperature-jump coefficient based on the linearized BGK model and arbitrary accommodation is reported. The jump coefficient is expressed as a power series in (1-α), and values of the expansion coefficients are given.

We wish to give a rearranged version of some previous results[1] for the temperature-jump coefficient. As discussed earlier[1] the temperature-jump coefficient can be expressed as

$$\epsilon' = \left(\frac{2-\alpha}{\alpha}\right)\frac{5\sqrt{\pi}}{8}\epsilon_0 + \Delta(\alpha), \tag{1}$$

where α is the accommodation coefficient for the linearized BGK model,

$$\epsilon_0 = \frac{8}{5\sqrt{\pi}}\begin{vmatrix}1\\0\end{vmatrix}^t \mathbf{H}_1^{-t}\mathbf{H}_2^t\begin{vmatrix}1\\-\sqrt{\frac{2}{3}}\end{vmatrix}, \tag{2}$$

and

$$\Delta(\alpha) = \frac{(1-\alpha)}{\alpha\sqrt{\pi}}\begin{vmatrix}1\\0\end{vmatrix}^t \mathbf{H}_1^{-t}\int_0^\infty \mathbf{H}^{-1}(\eta)\begin{vmatrix}1 & 0\\0 & \sqrt{\frac{2}{3}}\end{vmatrix} \times \mathbf{A}(\eta)\,\eta\exp(-\eta^2)\,d\eta. \tag{3}$$

Here, $\mathbf{H}(\mu)$ is the 2×2 $\mathbf{H}$ matrix introduced by Kriese *et al.*,[2]

$$\mathbf{H}_\beta^t = (\pi)^{-1/2}\int_0^\infty \mathbf{H}^t(\mu)\mathbf{Q}^t(\mu)\mathbf{Q}(\mu)\,\mu^\beta\exp(-\mu^2)\,d\mu, \tag{4}$$

$$\mathbf{Q}(\mu) = \begin{vmatrix}\sqrt{\frac{2}{3}}(\mu^2-\frac{1}{2}) & 1\\ \sqrt{\frac{2}{3}} & 0\end{vmatrix}, \tag{5}$$

and the vector $\mathbf{A}(\eta)$ is the solution of the Fredholm equation

$$\mathbf{A}(\eta) = (\alpha-2)\mathbf{F}(\eta) + (1-\alpha)(\mathbf{TA})(\eta), \quad \eta > 0. \tag{6}$$

In addition, we use the superscript t to denote the transpose operation and the superscript $-t$ to denote the transpose-inverse operation. In Eq. (6)

$$\mathbf{F}(\eta) = \eta\exp(-\eta^2)\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta)\mathbf{H}_1^t\begin{vmatrix}\sqrt{\frac{3}{2}}\\-1\end{vmatrix}, \tag{7}$$

where

$$\mathbf{R}(\eta) = \frac{1}{N(\eta)}\begin{vmatrix}\sqrt{\frac{3}{2}}N_{22}(\eta) & -N_{12}(\eta)\\ -\sqrt{\frac{3}{2}}N_{12}(\eta) & N_{11}(\eta)\end{vmatrix}. \tag{8}$$

We note that Kriese *et al.*[2] have given $N_{ij}(\eta)$ and $N(\eta)$ in terms of Dawson's integral. The operator $\mathbf{T}$ in Eq. (6) is such that

$$(\mathbf{TX})(\eta) = \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}}\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta) \times\int_0^\infty \mathbf{H}^{-1}(\mu)\begin{vmatrix}\sqrt{\frac{3}{2}} & 0\\0 & 1\end{vmatrix}\mathbf{X}(\mu)\exp(-\mu^2)\,\mu\frac{d\mu}{\mu+\eta}. \tag{9}$$

We have computed the Hilbert–Schmidt[3] norm of the kernel $\mathbf{K}$ of the operator $\mathbf{T}$, where

$$\mathbf{K}(\eta,\mu) = \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}}\mathbf{R}(\eta)\mathbf{H}^{-t}(\eta) \times\mathbf{H}^{-1}(\mu)\begin{vmatrix}\sqrt{\frac{3}{2}} & 0\\0 & 1\end{vmatrix}\frac{\mu\exp(-\mu^2)}{\eta+\mu}. \tag{10}$$

We used the definition

$$||\mathbf{T}|| = \left\{\int_0^\infty d\eta\int_0^\infty d\mu\,\mathrm{tr}[\mathbf{K}^t(\mu,\eta)\mathbf{K}(\mu,\eta)]\right\}^{1/2}, \tag{11}$$

where

$$\mathrm{tr}[\mathbf{K}^t\mathbf{K}] = K_{11}^2 + K_{12}^2 + K_{21}^2 + K_{22}^2. \tag{12}$$

We find that $||\mathbf{T}|| \simeq 0.04$; that is, for

$$|(1-\alpha)| \lesssim (0.04)^{-1},$$

or $\alpha \lesssim 25$, it should be possible to obtain a convergent Neumann–Liouville iterative solution of Eq. (6). In fact for physically interesting cases $\alpha \le 1$ and in this range, one can expect a very rapidly convergent iterative solution.

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Thus, we now express an iterative solution of Eq. (6) as

$$\mathbf{A}(\eta)=(\alpha-2)\left[\mathbf{F}(\eta)+\sum_{m=1}^{\infty}(1-\alpha)^m(\mathbf{T}^m\mathbf{F})(\eta)\right]; \tag{13}$$

then, we can write the jump coefficient as

$$\epsilon'=\frac{5\sqrt{\pi}}{8}\left(\frac{2-\alpha}{\alpha}\right)\left[\epsilon_0+\sum_{m=1}^{\infty}(1-\alpha)^m\epsilon_m\right], \tag{14}$$

where

$$\epsilon_m=-\frac{1}{\sqrt{\pi}}\,\frac{8}{5\sqrt{\pi}}\begin{vmatrix}1\\0\end{vmatrix}^t\mathbf{H}^{-t}\mathbf{W}_{m-1}\mathbf{H}_1^t\begin{vmatrix}\sqrt{\frac{3}{2}}\\-1\end{vmatrix},\quad m>0, \tag{15}$$

$$\mathbf{W}_m=\int_0^{\infty}\mathbf{H}^{-1}(\eta)\begin{vmatrix}1&0\\0&\sqrt{\frac{2}{3}}\end{vmatrix}(\mathbf{T}^m\mathbf{B})(\eta)\,\eta\exp(-\eta^2)\,d\eta, \tag{16}$$

and

$$\mathbf{B}(\eta)=\eta\exp(-\eta^2)\,\mathbf{R}(\eta)\,\mathbf{H}^{-t}(\eta). \tag{17}$$

Equation (11) is a convenient result because the ϵ_m do not depend on α. We have evaluated Eqs. (2) and (15) numerically to find the results given in Table I. A Gaussian quadrature scheme was used to evaluate all integrals, and the number of quadrature points was increased until no change in Table I was observed. The final calculation utilized 80 Gauss-Legendre points[4] in the interval (0, 1), which was mapped onto $(0,\infty)$ according to the transformation $\mu=\mu'/(1-\mu')$. The accuracy of the present results to the significant digits reported here is verified by the fact that Eqs. (15) give,

$$\sum_{i=0}^{10}\epsilon_i=1.000000;$$

that is, in the limit $\alpha\to 0$, from Eq. (11) we get

$$\epsilon'=\frac{5\sqrt{\pi}}{8}\left(\frac{2-\alpha}{\alpha}\right).$$

TABLE I. Numerical values for ϵ_m as computed from Eq. (15).

m	ϵ_m
0	1.17597
1	-1.60683×10^{-1}
2	-1.37349×10^{-2}
3	-1.38665×10^{-3}
4	-1.44586×10^{-4}
5	-1.52085×10^{-5}
6	-1.60481×10^{-6}
7	-1.69582×10^{-7}
8	-1.79329×10^{-8}
9	-1.89710×10^{-9}
10	-2.00735×10^{-10}

which is an exact result in this limit. Since in view of the expansion in $(1-\alpha)$ one would expect the maximum numerical error at $\alpha\to 0$, and since in this limit our result is exact to the number of significant digits quoted, it is clear that the present series expansion should provide exact values of ϵ' to the number of significant digits quoted here.

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