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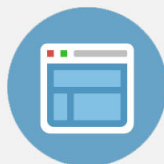
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# The effect of thermal expansion on nonlinear first and second sound in the whole temperature region of He II

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We consider one-dimensional, weakly nonlinear first and second sound waves in He II. The first correction to the shock speed is computed for each sound mode. The expressions obtained are exact with respect to the coefficient of thermal expansion  $\beta$ . It is shown that the commonly made assumption of negligible  $\beta$  can lead to significant error in the shock speeds for first sound away from the lambda-line. This contrasts with the calculation of the linear sound speeds and the shock speed for second sound where the  $\beta=0$  approximation yields accurate results. Near the lambda-line, the exact expressions for both modes are seen to contain fundamentally different singularities than those found in the commonly employed  $\beta=0$  theory. © 1996 American Institute of Physics. [S1070-6631(96)02009-0]

## I. INTRODUCTION

At ordinary pressures the transition from ordinary liquid helium (He I) to superfluid helium (He II) occurs at approximately 2.17 K. The remarkable properties of He II include the existence of quantized vorticity, the ability to flow through microscopic passages with virtually no resistance and the existence of multiple sound modes. The primary focus of the present investigation is the influence of the coefficient of thermal expansion,

$$\beta := - \frac{1}{\rho} \left. \frac{\partial \rho}{\partial T} \right|_{p, w^2}, \quad (1)$$

on the first and second sound modes. Here  $\rho$  is the mass density of He II,  $T$  is the absolute temperature,  $p$  is the pressure and  $w$  is the counterflow or slip velocity. First sound is the direct analog of ordinary sound in any liquid or gas in that it carries nearly isentropic perturbations in the density and particle velocity. Second sound has no analog in ordinary fluids and carries perturbations in the temperature, entropy and counterflow velocity; in second sound both the pressure and bulk velocity of the fluid are nearly constant.

A commonly made simplification in the study of sound waves in superfluid helium is the assumption that the thermal expansion coefficient (1) is negligibly small. As pointed out by Putterman,<sup>1</sup> such an assumption would seem reasonable because the size of  $\beta$  is on the order of  $-(10^{-3} - 10^{-2}) \text{ K}^{-1}$  at pressures and temperatures of practical interest. In fact, if one simply neglects  $\beta$  everywhere it appears, the resultant linearized sound speeds agree well with the exact, i.e.,  $\beta \neq 0$ , versions; further discussion of this point can be found in Ref. 1 and in Section III of the present study. However, a recent study by Cramer and Kluwick<sup>11</sup> reveals that the  $\beta=0$  approximation leads to significant error in the calculation of many of the properties of fourth sound. The latter

sound mode occurs in clamped He II, i.e., that where the velocity of the normal component is rendered stationary in flows through densely packed powders or narrow capillaries. In particular, Cramer and Kluwick demonstrated that the expressions for the Doppler shifts and the nonlinear wave speed are inaccurate when the  $\beta=0$  condition is used indiscriminately. As one might expect, the accuracy of the  $\beta=0$  assumption was seen to depend on the size of the terms pre-multiplying  $\beta$ . In both the present case and that of fourth sound  $\beta$  tends to appear in the nondimensional combinations

$$\beta T \quad \text{and} \quad G := \frac{\beta a^2}{c_p}, \quad (2)$$

where  $G$  is the well-known Grüneisen parameter,

$$a := \left( \left. \frac{\partial p}{\partial \rho} \right|_{s, w^2} \right)^{1/2} \quad \text{and} \quad c_p := T \left. \frac{\partial s}{\partial T} \right|_{p, w^2}$$

are the  $\beta=0$  versions of the linearized speed of first sound and the specific heat at constant pressure. The quantity  $s$  is the fluid entropy. The first quantity in (2) was found to be relatively small at most temperatures and pressures. However, because the size of  $a$  and  $c_p$  are of the order of  $(200-300) \text{ m} \cdot \text{s}^{-1}$  and  $10^3 \text{ m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$ , the Grüneisen parameter  $G$  can be of order one at many pressures and temperatures. For example, if we use the data of Maynard,<sup>9</sup> we find that  $\beta T \approx -0.03$  and  $G \approx -1.05$  at a temperature of 1.6 K and a pressure of 20 bar. Thus, at this temperature and pressure, the assumption of negligible  $\beta$  is by no means obvious and the purpose of the present investigation is to re-examine its validity for first and second sound with particular attention being paid to nonlinear effects.

In the present study we focus on small disturbances in the form of simple right-running waves. Nonlinear effects will be taken into account by applying the well-known multiple scales technique of Taniuti and Wei.<sup>3</sup> The governing equations will be taken to be the Landau two-fluid equations, the exact form of which are given in Section II. The general

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multiple-scales approach is described in Section III A and the linear theory is examined in Sections III B–III C. In Section III C we show that even the linear theory is inaccurate if the  $\beta=0$  approximation is applied universally to problems involving first sound. In particular, first sound cannot be regarded as isothermal if the Grüneisen parameter (2) is of order one.

In Section III D, we go on to compute the quadratic steepening coefficients for first and second sound. There it is shown that nonlinear second sound is accurately modeled by the commonly made  $\beta=0$  approximation, at least away from the  $\lambda$ -line. On the other hand, when first sound is considered, noticeable numerical differences between the exact values of the steepening parameter and that obtained in the full  $\beta \rightarrow 0$  limit were found. As in the case of fourth sound these differences are small at low pressures but become more pronounced as the pressure increases or the  $\lambda$ -line is approached.

We complete our analysis of the influence of thermal expansion on the nonlinear sound modes by examining the limiting behaviours at the  $\lambda$ -line and as the temperature approaches absolute zero; this is carried out in Sections IV and V, respectively. The results of our analysis are that the asymptotic behaviour predicted by the  $\beta=0$  theory is incorrect for both first and second sound as either the  $\lambda$ -line or absolute zero is approached.

In Section VI, we point out an error in the fourth sound  $\lambda$ -line analysis of Cramer and Kluwick.<sup>11</sup> In this section we provide the correction along with the limiting results for the fourth sound steepening parameter as the absolute temperature approaches zero.

## II. FORMULATION

We take the governing equations of the superfluid He II to be inviscid, unsteady and one dimensional. For the present purposes it is convenient to use the density  $\rho = \rho_n + \rho_s$ , the bulk velocity  $v = (\rho_n/\rho)v_n + (\rho_s/\rho)v_s$ , the entropy  $s$  and the slip or counterflow velocity between the normal and superfluid component  $w = v_n - v_s$  as dependent variables. If we use the thermodynamic relation for the chemical potential  $\mu$ :

$$d\mu = \frac{dp}{\rho} - sdT + \frac{1}{2}(\alpha - 1)dw^2, \quad (3)$$

where  $p = p(\rho, s, w^2)$  is the pressure,  $T = T(\rho, s, w^2)$  is the absolute temperature and  $\alpha := \rho_s/\rho$  is the superfluid mass fraction. If we further take into account that

$$\rho_n = \rho(1 - \alpha), \quad \rho_s = \rho\alpha,$$

$$v_n = v + \alpha w, \quad v_s = v + (\alpha - 1)w,$$

the exact two-fluid equations according to Putterman<sup>1</sup> in dimensional form are

$$\rho_t + v\rho_x + \rho v_x = 0, \quad (4)$$

$$\begin{aligned} v_t + \frac{\rho_x}{\rho} \{ p_\rho + w^2[\rho\alpha_\rho(1 - 2\alpha) + \alpha(1 - \alpha)] \} + vv_x \\ + s_x \left[ \frac{p_s}{\rho} + w^2\alpha_s(1 - 2\alpha) \right] \\ + 2ww_x \left[ \frac{p_w}{\rho} + w^2\alpha_w(1 - 2\alpha) + \alpha(1 - \alpha) \right] = 0, \end{aligned} \quad (5)$$

$$\begin{aligned} s_t + \frac{\rho_x}{\rho} (\rho sw\alpha_\rho + s\alpha w) + s_x(v + \alpha w + sw\alpha_s) \\ + w_x(2sw^2\alpha_w + s\alpha) = 0, \end{aligned} \quad (6)$$

$$\begin{aligned} w_t [(\alpha - 1) + 2w^2\alpha_w] + \rho_x \left\{ -sT_\rho + w^2 \left[ \frac{\alpha}{\rho}(\alpha - 1) \right. \right. \\ \left. \left. - \alpha_\rho(2 - 3\alpha + s\alpha_s) - \frac{s\alpha\alpha_s}{\rho} \right] \right\} + v_x w [(\alpha - 1) - \rho\alpha_\rho] \\ + s_x [-sT_s + w^2\alpha_s(2\alpha - 2 - s\alpha_s)] \\ + w_x \{ v(\alpha - 1 + 2w^2\alpha_w) + w[3\alpha(\alpha - 1) \\ - 2w^2\alpha_w(2 - 3\alpha + s\alpha_s) - s(\alpha\alpha_s + 2T_w)] \} = 0, \end{aligned} \quad (7)$$

where the subscripts  $t, x, \rho, s$  and  $w$  denote differentiation with respect to time, propagation distance, density, entropy and the square of the counterflow velocity, i.e.,  $w^2$ . Equations (4)–(6) are the mass, momentum and entropy balance. Equation (7) will be referred to as the slip equation.

The knowledge of the constitutive relations,

$$\alpha = \alpha(\rho, s, w^2), \quad p = p(\rho, s, w^2), \quad T = T(\rho, s, w^2),$$

is necessary to close the system (4)–(7). For the present calculations, the following differential relations (Cramer and Sen<sup>2</sup>) will be useful:

$$\begin{aligned} dp = \frac{\beta T \rho a^2}{c_p} ds + a^2 dp \\ + \frac{\beta T \rho a^2}{2c_p} \left( \frac{\partial \alpha}{\partial T} \Big|_{p, w^2} + \frac{c_p \rho}{\beta T} \frac{\partial \alpha}{\partial p} \Big|_{T, w^2} \right) dw^2, \end{aligned} \quad (8)$$

$$\begin{aligned} dT = \frac{T}{c_v} ds + \frac{\beta T a^2}{\rho c_p} dp \\ + \frac{T}{2c_v} \left( \frac{\partial \alpha}{\partial T} \Big|_{p, w^2} + \frac{\beta a^2 \rho}{\gamma} \frac{\partial \alpha}{\partial p} \Big|_{T, w^2} \right) dw^2, \end{aligned} \quad (9)$$

where

$$\beta := - \frac{1}{\rho} \frac{\partial \rho}{\partial T} \Big|_{p, w^2}, \quad a := \left( \frac{\partial p}{\partial \rho} \Big|_{s, w^2} \right)^{1/2}, \quad (10)$$

$$c_v := T \frac{\partial s}{\partial T} \Big|_{p, w^2}, \quad c_p := T \frac{\partial s}{\partial T} \Big|_{p, w^2},$$

and

$$\gamma := \frac{c_p}{c_v} = 1 + \frac{\beta^2 T a^2}{c_p} \quad (11)$$

is the ratio of the specific heats. The quantities  $\beta$ ,  $a$ ,  $c_v$  and  $c_p$  are referred to as the coefficient of thermal expansion, the first sound speed, the specific heat at constant volume and the specific heat at constant pressure, respectively.

The nonlinear hyperbolic system of conservation equations can be written in the compact matrix-vector notation (the Einstein summation convention is used)

$$\frac{\partial u_i}{\partial t} + A_{ij}(u_k) \frac{\partial u_j}{\partial x} = 0 \quad (i, j, k = 1, 2, 3, 4), \quad (12)$$

where the dependent variables are

$$u_1 = \rho, \quad u_2 = v, \quad u_3 = s, \quad u_4 = w$$

and the components of the speed matrix  $A_{ij}$  are given by

$$A_{11} = v, \quad A_{12} = \rho, \quad A_{13} = 0, \quad A_{14} = 0,$$

$$A_{21} = \frac{p_\rho}{\rho} + \frac{w^2}{\rho} [\rho \alpha_\rho (1 - 2\alpha) + \alpha(1 - \alpha)],$$

$$A_{22} = v, \quad A_{23} = \frac{p_s}{\rho} + w^2(1 - 2\alpha)\alpha_s,$$

$$A_{24} = 2w \left[ \alpha(1 - \alpha) + \frac{p_w}{\rho} + (1 - 2\alpha)\alpha_w w^2 \right],$$

$$A_{31} = \frac{sw}{\rho} (\alpha + \rho \alpha_\rho), \quad A_{32} = 0, \quad (13)$$

$$A_{33} = v + w(s\alpha)_s, \quad A_{34} = s(\alpha + 2\alpha_w w^2),$$

$$A_{41} = \frac{1}{Z} \left[ \frac{sT_\rho}{1 - \alpha} + \frac{w^2}{\rho} \left( \alpha + \frac{s\alpha\alpha_s}{1 - \alpha} + \rho\alpha_\rho \frac{2 + s\alpha_s - 3\alpha}{1 - \alpha} \right) \right],$$

$$A_{42} = \frac{w}{Z} \left( 1 + \frac{\rho\alpha_\rho}{1 - \alpha} \right),$$

$$A_{43} = \frac{1}{Z} \left[ \frac{sT_s}{1 - \alpha} + w^2\alpha_s \left( \frac{s\alpha_s}{1 - \alpha} - 2 \right) \right],$$

$$A_{44} = v + \frac{w}{Z} \left( 3\alpha + s \frac{2T_w + \alpha\alpha_s}{1 - \alpha} + 2w^2\alpha_w \frac{s\alpha_s - 3\alpha + 2}{1 - \alpha} \right),$$

with the abbreviation

$$Z := 1 - \frac{2w^2\alpha_w}{1 - \alpha}.$$

### III. NONLINEAR EVOLUTION EQUATIONS

#### A. Outline of derivation

We study the long-time evolution of small but finite amplitude waves governed by the quasilinear system (12), where phenomena such as wave steepening can be observed. To this end the solution  $u_i$  is assumed to have an asymptotic expansion of the form

$$u_i = u_i^{(0)} + \delta u_i^{(1)} + \delta^2 u_i^{(2)} + O(\delta^3), \quad (14)$$

where  $\delta \ll 1$  is a small perturbation parameter and the uniform base state  $u_i^{(0)}$  is given by

$$u_1^{(0)} = \rho_o, \quad u_2^{(0)} = 0, \quad u_3^{(0)} = s_o, \quad u_4^{(0)} = 0.$$

To derive the evolution equation we follow the reductive perturbation method introduced by Taniuti and Wei,<sup>3</sup> assuming that cumulative nonlinear effects are of importance at times of  $O(1/\delta)$ . Transformation from  $(x, t)$  to a wave coordinate  $X = x - \lambda t$ , where  $\lambda$  is the linear sound speed of the propagation mode under consideration, and a slow time scale  $\tau = \delta t$  then finally shows that the evolution of small disturbances is governed by an inviscid Burgers equation:

$$\frac{\partial U}{\partial \tau} + \Gamma U \frac{\partial U}{\partial X} = 0, \quad (15)$$

where the shape function  $U$  is related to the original variables by

$$u_i^{(1)} = r_i U(X, \tau) \quad (16)$$

and  $\Gamma$  is the quadratic nonlinearity coefficient given by

$$\Gamma = \frac{1}{l_m r_m} l_i \frac{\partial A_{ij}}{\partial u_k} (u_n^{(0)}) r_j r_k, \quad (17)$$

which is required to be of  $O(1)$ . The derivatives of  $A_{ij}$  are to be evaluated at the undisturbed state  $u_i^{(0)}$  and the quantities  $r_i$  and  $l_i$  are the right and left eigenvectors of the speed matrix  $A_{ij}^{(0)} \equiv A_{ij}(u_k^{(0)})$  with the linear sound speeds  $\lambda$  as eigenvalues, i.e.

$$A_{im}^{(0)} r_m = \lambda r_i, \quad l_m A_{mi}^{(0)} = \lambda l_i. \quad (18)$$

Using the method of characteristics, the formal solution to equation (15) can be written in the form

$$U = \text{const on } \frac{dx}{dt} \equiv c^* = \lambda + \delta \Gamma U + O(\delta^2), \quad (19)$$

where  $c^*$  is the convected sound speed.

In the following, the computations will be carried out simultaneously for both first and second sound. Solving the eigenvalue problem (18) one obtains the right and left eigenvectors,

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = r_1 \begin{pmatrix} 1 \\ \frac{\lambda}{\rho_o} \\ \frac{c_{po}}{\beta_o T_o a_o^2 \rho_o} (\lambda^2 - a_o^2) \\ \frac{\lambda c_{po}}{\beta_o T_o a_o^2 \rho_o s_o \alpha_o} (\lambda^2 - a_o^2) \end{pmatrix} \quad (20)$$

and

$$\vec{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}^T = l_1 \begin{pmatrix} 1 \\ \frac{\rho_o}{\lambda} \\ \frac{\beta_o T_o a_o^2 \rho_o}{c_{po} (\lambda^2 - u_T^2)} \\ \frac{\beta_o T_o a_o^2 \rho_o s_o \alpha_o}{\lambda c_{po} (\lambda^2 - u_T^2)} \end{pmatrix}^T, \quad (21)$$

where  $r_1$  and  $l_1$  are arbitrary constants.

## B. Linear wave speeds

Taking the choices

$$r_1 \equiv \rho_o \quad (\text{first sound}),$$

$$r_1 \equiv -\frac{\beta_o T_o \rho_o s_o \alpha_o}{c_{po}} \quad (\text{second sound}),$$

the common results for the nondimensional nonlinearity parameters are recovered in the limit  $\beta_o \equiv 0$ . In this connection it is interesting to note that it is not necessary to specify  $l_1$  explicitly, at least for the purposes of computing  $\Gamma$ , equation (17). Furthermore, the eigenvalues  $\lambda^\pm$  corresponding to first and second soundwaves are found to be

$$(\lambda^\pm)^2 = \frac{a_o^2 + u_T^2}{2} \pm \frac{a_o^2 - u_T^2}{2} \left[ 1 + 4 \left( \frac{u_T}{a_o} \right)^2 \frac{\beta_o T_o G_o}{\gamma_o} \times \frac{1}{(1 - (u_T/a_o)^2)^2} \right]^{1/2}, \quad (22)$$

in complete agreement with relationship (7-14) given by Putterman,<sup>1</sup> where

$$u_T := \left( \frac{s_o^2 T_o \alpha_o}{c_{vo}(1 - \alpha_o)} \right)^{1/2} \quad (23)$$

is the  $\beta_o = 0$  version of the linear second sound speed and  $G_o := \beta_o a_o^2 / c_{po}$  is the Grüneisen parameter. The subscript  $o$  denotes the values of the field quantities in the unperturbed state. Expansion of (22) for small  $\beta_o T_o$  leads to

$$\lambda^+ = \pm a_o \left[ 1 + \frac{1}{2} \frac{(u_T/a_o)^2}{1 - (u_T/a_o)^2} \frac{\beta_o T_o G_o}{\gamma_o} + O \left( \left( \frac{u_T}{a_o} \right)^4 \beta_o^2 T_o^2 \right) \right], \quad (24)$$

$$\lambda^- = \pm u_T \left[ 1 - \frac{1}{2} \frac{1}{1 - (u_T/a_o)^2} \frac{\beta_o T_o G_o}{\gamma_o} + O(\beta_o^2 T_o^2) \right]. \quad (25)$$

The  $\pm$  signs on the right-hand sides now denote right and left running waves. Since  $\beta_o T_o$  tends to be small for most temperatures and pressures, the approximations  $\lambda^+ \approx a_o$ ,  $\lambda^- \approx u_T$  are commonly (and reasonably) made. One should note, however, that the approximation for  $\lambda^+$  is expected to be more accurate than that for  $\lambda^-$  because  $u_T/a_o = O(10^{-1})$  for most pressures and temperatures.

Before closing this section we briefly discuss how the results for the eigenvalues  $\lambda$  have to be modified if the mean flow velocity  $v$  or the counterflow velocity are nonzero in the unperturbed state. If  $v_o \neq 0$  and  $w_o = 0$  the Doppler-shifted wave speeds, e.g. the eigenvalues of the modified speed matrix  $A_{ijds}^{(0)}$ , are given by

$$(\lambda_{ds}^\pm - v_o)^2 = \frac{a_o^2 + u_T^2}{2} \pm \frac{a_o^2 - u_T^2}{2} \left[ 1 + 4 \left( \frac{u_T}{a_o} \right)^2 \frac{\beta_o T_o G_o}{\gamma_o} \times \frac{1}{(1 - (u_T/a_o)^2)^2} \right]^{1/2}. \quad (26)$$

The fact that  $\lambda$  seen in equation (22) has simply to be replaced by  $(\lambda - v_o)$  is, of course, a direct consequence of the frame invariance of the governing equations. If on the other hand we allow for a small counterflow velocity  $w_o/a_o \ll 1$  but no bulk velocity  $v_o = 0$  the results for first and second sound can be written in the form

$$\lambda_{ds}^+ \approx a_o \left[ 1 + \frac{w_o}{a_o} \frac{\beta_o T_o s_o}{c_{po}} \left( \frac{\rho_o a_o^2 \alpha_p}{(1 - \alpha_o)} + 2\alpha_o \right) + \dots \right], \quad (27)$$

$$\lambda_{ds}^- \approx u_T \left[ 1 + \frac{w_o}{u_T} \left( 2\alpha_o + \frac{T_o s_o \alpha_T}{c_{vo}(1 - \alpha_o)} \right) + \dots \right], \quad (28)$$

where only terms linear in  $w_o$  have been retained. These relationships are seen to be completely consistent with the  $\beta_o = 0$  approximation, e.g. Putterman.<sup>1</sup>

## C. Perturbation relations

Next let us investigate how the relationships between the disturbances of the various field quantities caused by first and second sound waves are affected by small but finite values of  $\beta_o T_o$ .

According to equations (14), (16) and (20) the perturbations of the density, velocity, entropy and counterflow velocity are related to  $U$  by

$$\frac{\rho - \rho_o}{\rho_o} = \frac{v}{\lambda} = \frac{s - s_o}{s_o} \frac{\beta_o T_o a_o^2 s_o}{c_{po}(\lambda^2 - a_o^2)} = \frac{w}{\lambda} \frac{\beta_o T_o a_o^2 s_o \alpha_o}{c_{po}(\lambda^2 - a_o^2)} = \delta U(X, \tau) = o(1). \quad (29)$$

If the general results (29) are specialized for first sound,  $\lambda = \lambda^+$ , we see that

$$\frac{v}{\lambda^+} = \frac{\rho - \rho_o}{\rho_o}, \quad (30)$$

$$\frac{s - s_o}{s_o} = \frac{w}{\lambda^+ \alpha_o} = \frac{(\rho - \rho_o)}{\rho_o} \frac{c_{po} G_o}{s_o \gamma_o} \frac{(u_T/a_o)^2}{1 - (u_T/a_o)^2} \times \left\{ 1 + O \left[ \left( \frac{u_T}{a_o} \right)^2 \beta_o T_o \right] \right\}. \quad (31)$$

Since  $c_{po} G_o / s_o$  is typically of order one it follows that  $(s - s_o)/s_o$  and  $w/\lambda$  are small compared to  $(\rho - \rho_o)/\rho_o$  primarily because of  $u_T \ll a_o$ . This indicates that the usually adopted assumption that the counterflow  $w$  is negligible in first sound should be thought of as a small  $u_T/a_o$  approximation rather than a small  $\beta_o$  approximation.

Similarly, evaluation of the results for second sound waves yields

$$\frac{v}{\lambda^-} = \frac{\rho - \rho_o}{\rho_o} \approx -\frac{\beta_o T_o s_o}{c_{po}} \frac{s - s_o}{s_o}, \quad (32)$$

$$\frac{w}{\lambda^-} = \frac{s - s_o}{\alpha_o s_o}. \quad (33)$$

Inspection of these expressions shows that both  $v/\lambda^-$  and  $(\rho-\rho_o)/\rho_o$  are small in comparison to  $(s-s_o)/s_o$  provided that  $\beta_o T_o$  is small.

These results are in general agreement with the conventional picture that (i) first sound waves are associated primarily with perturbations in  $v$  and  $\rho$  with  $w \approx 0$  and  $s \approx s_o$  and (ii) that second sound waves predominantly carry perturbations in  $w$  and  $s$  while  $v \approx 0$  and  $\rho \approx \rho_o$ .

In order to determine the temperature and pressure perturbations caused by first and second sound waves it is useful to write (8) and (9) in the small disturbance form, which reads as

$$\frac{p-p_o}{\rho_o a_o^2} = \frac{\beta_o T_o s_o}{c_{po}} \frac{s-s_o}{s_o} + \frac{\rho-\rho_o}{\rho_o}, \quad (34)$$

$$\frac{T-T_o}{T_o} = \frac{s_o}{c_{vo}} \frac{s-s_o}{s_o} + G_o \frac{\rho-\rho_o}{\rho_o}, \quad (35)$$

to lowest order. By substitution of the general relations (34), (35) in (29) one finds

$$\frac{p-p_o}{\rho_o a_o^2} = \left(\frac{\lambda}{a_o}\right)^2 \frac{\rho-\rho_o}{\rho_o} = -\frac{(\lambda/a_o)^2}{1-(\lambda/a_o)^2} \frac{\beta_o T_o s_o}{c_{po}} \frac{s-s_o}{s_o}, \quad (36)$$

$$\begin{aligned} \frac{T-T_o}{T_o} &= \left\{ \frac{\gamma_o}{\beta_o T_o} \left[ \left(\frac{\lambda}{a_o}\right)^2 - 1 \right] + G_o \right\} \left( \frac{\rho-\rho_o}{\rho_o} \right) \\ &= \frac{s_o}{c_{po}} \left( \gamma_o - \frac{\beta_o T_o G_o}{1-(\lambda/a_o)^2} \right) \frac{s-s_o}{s_o}. \end{aligned} \quad (37)$$

Here the expressions involving the density and entropy perturbations are most convenient for the investigation of first and second sound waves, respectively. Specialization of these general relations for first sound yields the expressions

$$\frac{p-p_o}{\rho_o a_o^2} \approx \frac{\rho-\rho_o}{\rho_o}, \quad (38)$$

$$\frac{T-T_o}{T_o} \approx G_o \left\{ \frac{1}{1-(u_T/a_o)^2} + O\left[\left(\frac{u_T}{a_o}\right)^4 \beta_o T_o\right] \right\} \frac{\rho-\rho_o}{\rho_o}, \quad (39)$$

which in the limit  $\beta_o T_o \rightarrow 0$  reduce to

$$\frac{T-T_o}{T_o} \rightarrow \frac{G_o}{1-(u_T/a_o)^2} \frac{\rho-\rho_o}{\rho_o} \approx \frac{G_o}{1-(u_T/a_o)^2} \frac{p-p_o}{\rho_o a_o^2}. \quad (40)$$

The main fact to be noticed here is that the temperature perturbations are non-negligible whenever  $G_o = O(1)$ . This is in sharp contrast with the  $\beta_o = 0$  approximation which is based on the implicit assumption  $G_o = O(\beta_o T_o)$  and thus treats first sound waves as an *isothermal* disturbance. Temperature perturbations then represent a higher order effect, Putterman and Garrett.<sup>4</sup> As an numerical example note that  $G_o = -1.05$  at 1.6 K and 20 bar. Clearly, therefore,  $G_o$  is not necessarily small and consequently  $(T-T_o)/T_o$  is not negligible in the moderate and high pressure regime.

The results for second sound can be expressed in the form

$$\frac{p-p_o}{\rho_o a_o^2} \approx -\frac{(u_T/a_o)^2}{1-(u_T/a_o)^2} \frac{\beta_o T_o s_o}{c_{po}} \frac{s-s_o}{s_o}, \quad (41)$$

$$\frac{T-T_o}{T_o} \approx \frac{s_o}{c_{vo}} \left( 1 - \frac{G_o}{\gamma_o} \frac{\beta_o T_o}{1-(u_T/a_o)^2} \right) \frac{s-s_o}{s_o} \approx \frac{s_o}{c_{vo}} \frac{s-s_o}{s_o}, \quad (42)$$

which show that the pressure perturbations are negligible in the limit  $\beta_o T_o \rightarrow 0$  in complete agreement with the  $\beta_o = 0$  predictions, i.e.,

$$\frac{T-T_o}{T_o} \approx \frac{s-s_o}{c_{vo}} \approx \frac{s_o \alpha_o w}{c_{vo} u_T}, \quad (43)$$

and  $p, \rho \approx$  constant with  $v \approx 0$ .

## D. Evaluation of the nonlinearity coefficients

We now complete our analysis by determining the effect of the thermal expansion coefficient on the quadratic nonlinearity parameters  $\Gamma^+$ ,  $\Gamma^-$  for first and second sound waves. Evaluation of equation (17) taking into account equations (20), (21), (13), (24) and (25) leads to

$$\begin{aligned} \frac{\Gamma^+}{\lambda^+} &= \frac{1}{2(1+A)} \left[ 2 + \frac{\rho_o p_{pp}}{(\lambda^+)^2} + \frac{2p_{ps} u_T^2 \beta_o}{(\lambda^+)^2 \gamma_o (1-(u_T/a_o)^2)} + \frac{u_T^4 \beta_o^2}{\rho_o \gamma_o^2 (1-(u_T/a_o)^2)^2} \left( \frac{p_{ss}}{(\lambda^+)^2} + \frac{2\rho_o(1-\alpha_o)}{s_o^2 \alpha_o} + \frac{2p_w}{s_o^2 \alpha_o^2} \right) \right. \\ &+ A \left( 4 + \frac{\rho_o \alpha_o (2-\alpha_o)}{\alpha_o (1-\alpha_o)} \right) + A \frac{u_T^2 \beta_o (5\alpha_o + 2s_o \alpha_s)}{s_o \alpha_o \gamma_o (1-(u_T/a_o)^2)} + \frac{\beta_o T_o a_o^2 \rho_o^2 s_o^2 \alpha_o}{(\lambda^+)^2 c_{po} ((\lambda^+)^2 - u_T^2)} \frac{T_{pp}(1-\alpha_o) + T_\rho \alpha_\rho}{(1-\alpha_o)^2} \\ &+ A \frac{s_o \alpha_o \rho_o}{(\lambda^+)^2} \frac{T_\rho (1-\alpha_o) + s_o T_{ps} (2-\alpha_o) + s_o T_\rho \alpha_s + s_o T_s \alpha_\rho}{(1-\alpha_o)^2} \\ &\left. + A \frac{\beta_o u_T^2 s_o \alpha_o}{(\lambda^+)^2 \gamma_o (1-(u_T/a_o)^2)} \frac{(T_s + s_o T_{ss})(1-\alpha_o) + s_o T_s \alpha_s}{(1-\alpha_o)^2} + A \frac{\beta_o u_T^2}{\gamma_o (1-(u_T/a_o)^2)} \frac{2T_w + \alpha_o \alpha_s}{\alpha_o (1-\alpha_o)} \right] \end{aligned} \quad (44)$$

and

$$\begin{aligned} \frac{\Gamma^-}{\lambda^-} = & \frac{1}{2(1+A)} \left[ -2A \frac{\beta_o T_o s_o \alpha_o}{c_{po}} - A \frac{p_{pp} \beta_o T_o \rho_o s_o \alpha_o}{(\lambda^-)^2 c_{po}} + \frac{2p_{ps} u_T^2 \beta_o^2 T_o s_o \alpha_o}{(\lambda^-)^2 \gamma_o c_{po} (1 - (u_T/a_o)^2)} - B \frac{u_T^2 \beta_o s_o \alpha_o}{\rho_o \gamma_o (1 - (u_T/a_o)^2)} \left( \frac{p_{ss}}{(\lambda^-)^2} \right. \right. \\ & + \frac{2\rho_o(1-\alpha_o)}{s_o^2 \alpha_o} + \frac{2p_w}{s_o^2 \alpha_o^2} \left. \right) - \frac{\beta_o T_o s_o}{c_{po}} \left( 4\alpha_o + \frac{\rho_o \alpha_o (2-\alpha_o)}{(1-\alpha_o)} \right) + B(5\alpha_o + 2s_o \alpha_s) + \frac{1}{B} \frac{\beta_o^2 T_o^2 \rho_o^2 s_o^3 \alpha_o^2}{(\lambda^-)^2 c_{po}^2} \frac{T_{pp}(1-\alpha_o) + T_\rho \alpha_\rho}{(1-\alpha_o)^2} \\ & - \frac{\beta_o T_o s_o^2 \alpha_o^2}{(\lambda^-)^2 c_{po}} \frac{T_p(1-\alpha_o) + s_o T_{ps}(2-\alpha_o) + s_o T_\rho \alpha_s + s_o T_s \alpha_p}{(1-\alpha_o)^2} + B \frac{s_o^2 \alpha_o^2}{(\lambda^-)^2} \frac{(T_s + s_o T_{ss})(1-\alpha_o) + s_o T_s \alpha_s}{(1-\alpha_o)^2} \\ & \left. + B s_o \frac{2T_w + \alpha_o \alpha_s}{\alpha_o(1-\alpha_o)} \right]. \end{aligned} \quad (45)$$

Herein  $A$ ,  $B$  denote the abbreviations

$$\begin{aligned} A := & \left[ 1 + \frac{\gamma_o c_{po} (1 - (u_T/a_o)^2)^2}{u_T^2 \beta_o^2 T_o} \right]^{-1} = \frac{u_T^2 \beta_o^2 T_o}{\gamma_o c_{po} (1 - (u_T/a_o)^2)^2} + \dots, \\ B := & \left[ 1 - \left( \frac{u_T}{a_o} \right)^2 + \frac{u_T^2 \beta_o^2 T_o}{\gamma_o c_{po} (1 - (u_T/a_o)^2)^2} \right], \end{aligned} \quad (46)$$

and the derivatives of  $a$ ,  $c_v$  and  $\alpha$  are evaluated at the unperturbed state.

For practical applications it is convenient to express these results also in a different form which takes into account that the constitutive equations are frequently expressed in terms of  $p$ ,  $T$ ,  $w$  rather than  $\rho$ ,  $s$ ,  $w$ . Applying the relationships,

$$\left. \frac{\partial}{\partial \rho} \right|_{s,w^2} = a^2 \left. \frac{\partial}{\partial p} \right|_{T,w^2} + \frac{\beta T a^2}{\rho c_p} \left. \frac{\partial}{\partial T} \right|_{p,w^2}, \quad (47)$$

$$\left. \frac{\partial}{\partial s} \right|_{\rho,w^2} = \frac{\beta T \rho a^2}{c_p} \left. \frac{\partial}{\partial p} \right|_{T,w^2} + \frac{T}{c_v} \left. \frac{\partial}{\partial T} \right|_{p,w^2}, \quad (48)$$

one then obtains

$$\begin{aligned} \frac{\Gamma^+}{\lambda^+} = & \frac{1}{2(1+A)} \left\{ 2 + \frac{u_T^4 \beta_o^3 T_o a_o^2}{(\lambda^+)^2 \gamma_o c_{po}^2 (1 - (u_T/a_o)^2)^2} + \frac{2u_T^2 \beta_o^2 T_o}{\gamma_o c_{po} (1 - (u_T/a_o)^2)^2} + 4A + 5A \frac{u_T^2 \beta_o}{s_o \gamma_o (1 - (u_T/a_o)^2)} \right. \\ & - \frac{2u_T^2 \beta_o^2 T_o a_o^2}{(\lambda^+)^2 \gamma_o c_{po} ((\lambda^+/a_o)^2 - (u_T/a_o)^2)} + A \frac{u_T^2 \beta_o a_o^2}{(\lambda^+)^2 \gamma_o s_o} + A \frac{u_T^2 \beta_o a_o^2 (2-\alpha_o)}{(\lambda^+)^2 c_{po} (1-\alpha_o)} + A \frac{u_T^4 \beta_o}{(\lambda^+)^2 \gamma_o (1 - (u_T/a_o)^2)} \\ & \times \left( \frac{1}{s_o} + \frac{1}{c_{vo}} \right) + \left. \frac{\partial a}{\partial p} \right|_{T,w^2} \frac{2\rho_o a_o^3}{(\lambda^+)^2} \left[ 1 + \frac{u_T^2 \beta_o^2 T_o}{\gamma_o c_{po}} \left( \frac{2}{1 - (u_T/a_o)^2} + \frac{1}{(\lambda^+/a_o)^2 - (u_T/a_o)^2} \right) \right] + \left. \frac{\partial a}{\partial T} \right|_{p,w^2} \frac{2\beta_o T_o a_o^3}{(\lambda^+)^2 c_{po}} \\ & \times \left[ 1 + \left( \frac{u_T}{a_o} \right)^2 \left( \frac{2}{1 - (u_T/a_o)^2} + \frac{1}{(\lambda^+/a_o)^2 - (u_T/a_o)^2} \right) \right] - \left. \frac{\partial c_v}{\partial p} \right|_{T,w^2} \frac{u_T^2 \rho_o a_o^2}{(\lambda^+)^2} \left[ A \frac{(2-\alpha_o)}{c_{vo} (1-\alpha_o)} + \frac{u_T^2 \beta_o^2 T_o}{c_{po}^2 (1 - (u_T/a_o)^2)^2} \right. \\ & \left. + A \frac{u_T^2 \beta_o^2 T_o}{c_{po}^2 (1 - (u_T/a_o)^2)} \right] - \left. \frac{\partial c_v}{\partial T} \right|_{p,w^2} \frac{u_T^2 \beta_o T_o a_o^2}{(\lambda^+)^2 c_{po}} \left[ A \frac{(2-\alpha_o)}{c_{vo} (1-\alpha_o)} + \frac{u_T^2 \beta_o^2 T_o}{c_{po}^2 (1 - (u_T/a_o)^2)^2} + A \frac{u_T^2}{c_{vo} a_o^2 (1 - (u_T/a_o)^2)} \right] \\ & + \left. \frac{\partial \alpha}{\partial p} \right|_{T,w^2} \rho_o \left[ \frac{u_T^4 \beta_o^2 a_o^2}{\alpha_o^2 \gamma_o^2 s_o^2 (1 - (u_T/a_o)^2)^2} + A \frac{a_o^2 (2-\alpha_o)}{\alpha_o (1-\alpha_o)} + 2A \frac{u_T^2 \beta_o^2 T_o a_o^2}{\alpha_o \gamma_o c_{po} (1 - (u_T/a_o)^2)} \right. \\ & + \frac{u_T^2 \beta_o^2 T_o a_o^4}{(\lambda^+)^2 \gamma_o c_{po} (1-\alpha_o) ((\lambda^+/a_o)^2 - (u_T/a_o)^2)} + A \frac{u_T^2 \beta_o^2 T_o a_o^4}{(\lambda^+)^2 \gamma_o c_{po} (1-\alpha_o)} + A \frac{u_T^2 a_o^2}{(\lambda^+)^2 (1-\alpha_o)} \\ & \left. + A \frac{u_T^4 \beta_o^2 T_o a_o^2}{(\lambda^+)^2 \gamma_o c_{po} (1-\alpha_o) (1 - (u_T/a_o)^2)} + A \frac{u_T^2 \beta_o^2 T_o a_o^2 (1+\alpha_o)}{\alpha_o \gamma_o c_{po} (1-\alpha_o) (1 - (u_T/a_o)^2)} \right] \\ & + \left. \frac{\partial \alpha}{\partial T} \right|_{p,w^2} \frac{\beta_o T_o}{c_{po}} \left[ \frac{u_T^4 \beta_o^2 a_o^2}{\alpha_o^2 \gamma_o^2 s_o^2 (1 - (u_T/a_o)^2)^2} + A \frac{a_o^2 (2-\alpha_o)}{\alpha_o (1-\alpha_o)} + 2A \frac{u_T^2}{\alpha_o (1 - (u_T/a_o)^2)} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{u_T^2 \beta_o^2 T_o a_o^4}{(\lambda^+)^2 \gamma_o c_{po} (1 - \alpha_o) ((\lambda^+ / a_o)^2 - (u_T / a_o)^2)} + 2A \frac{u_T^2 a_o^2}{(\lambda^+)^2 (1 - \alpha_o)} + A \frac{u_T^4}{(\lambda^+)^2 (1 - \alpha_o) (1 - (u_T / a_o)^2)} \\
& + A \frac{u_T^2 (1 + \alpha_o)}{\alpha_o (1 - \alpha_o) (1 - (u_T / a_o)^2)} \Bigg\} \tag{49}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\Gamma^-}{\lambda^-} = & \frac{1}{2(1+A)} \left\{ -2A \frac{\beta_o T_o s_o \alpha_o}{c_{po}} - B \frac{u_T^2 \beta_o^2 T_o s_o \alpha_o a_o^2}{(\lambda^-)^2 c_{po}^2 (1 - (u_T / a_o)^2)} - 2B \frac{\beta_o T_o s_o \alpha_o}{c_{po} (1 - (u_T / a_o)^2)} - \frac{4\beta_o T_o s_o \alpha_o}{c_{po}} + 5B \alpha_o \right. \\
& - \frac{2}{B} \frac{u_T^2 \beta_o^3 T_o^2 s_o \alpha_o a_o^2}{(\lambda^-)^2 \gamma_o c_{po}^2} - \frac{u_T^2 \beta_o^2 T_o \alpha_o a_o^2}{(\lambda^-)^2 \gamma_o c_{po}} - \frac{u_T^2 \beta_o^2 T_o s_o \alpha_o a_o^2 (2 - \alpha_o)}{(\lambda^-)^2 c_{po}^2 (1 - \alpha_o)} + B \alpha_o \frac{u_T^2}{(\lambda^-)^2} + B \frac{u_T^2 s_o \alpha_o}{(\lambda^-)^2 c_{vo}} \\
& + \frac{\partial a}{\partial p} \Bigg|_{T,w^2} \frac{2\beta_o T_o \rho_o s_o \alpha_o a_o^3}{(\lambda^-)^2 c_{po}^2} \left[ \frac{u_T^2 \beta_o^2 T_o}{\gamma_o c_{po}} \left( \frac{2}{1 - (u_T / a_o)^2} + \frac{1}{B} \right) - A \right] + \frac{\partial a}{\partial T} \Bigg|_{p,w^2} \frac{2\beta_o^2 T_o^2 s_o \alpha_o a_o}{(\lambda^-)^2 c_{po}^2} \left[ -A a_o^2 + \frac{2u_T^2}{1 - (u_T / a_o)^2} \right. \\
& + \frac{u_T^2}{B} \Bigg] + \frac{\partial c_v}{\partial p} \Bigg|_{T,w^2} \frac{u_T^2 \beta_o T_o \rho_o s_o \alpha_o a_o^2}{(\lambda^-)^2 c_{po} c_{vo}} \left[ \frac{B}{1 - (u_T / a_o)^2} + \frac{2 - \alpha_o}{1 - \alpha_o} - B \right] + \frac{\partial c_v}{\partial T} \Bigg|_{p,w^2} \frac{u_T^2 T_o s_o \alpha_o}{(\lambda^-)^2 c_{vo}} \left[ \frac{\beta_o^2 T_o a_o^2}{c_{po}^2} \left( \frac{B}{1 - (u_T / a_o)^2} \right. \right. \\
& + \left. \left. \frac{2 - \alpha_o}{1 - \alpha_o} \right) - \frac{B}{c_{vo}} \right] + \frac{\partial \alpha}{\partial p} \Bigg|_{T,w^2} \beta_o \rho_o a_o^2 \left[ \frac{u_T^2}{\alpha_o \gamma_o s_o} \left( B(1 + \alpha_o) - \frac{B}{1 - (u_T / a_o)^2} - (2 - \alpha_o) \right) + 2B \frac{T_o s_o}{c_{po}} \right. \\
& + \left. \frac{u_T^4 \beta_o^2 T_o a_o^2}{(\lambda^-)^2 \gamma_o^2 c_{po} s_o} \left( \frac{1}{B} - 1 \right) - \frac{u_T^4}{(\lambda^-)^2 \gamma_o s_o} (1 - B) \right] + \frac{\partial \alpha}{\partial T} \Bigg|_{p,w^2} \left[ -\frac{u_T^2 \beta_o^2 T_o a_o^2}{\alpha_o \gamma_o c_{po} s_o} \left( \frac{B}{1 - (u_T / a_o)^2} + (2 - \alpha_o) + \frac{2u_T^2 \alpha_o}{(\lambda^-)^2} \right) \right. \\
& \left. + 2B \frac{T_o s_o}{c_{vo}} + B \frac{u_T^4}{(\lambda^-)^2 s_o} + B \frac{u_T^2 (1 + \alpha_o)}{\alpha_o s_o} + \frac{1}{B} \frac{u_T^4 \beta_o^4 T_o^2 a_o^4}{(\lambda^-)^2 \gamma_o^2 c_{po}^2 s_o} \right] \Bigg\}. \tag{50}
\end{aligned}$$

The commonly used results (in the following indicated by the index  $c$ ) are obtained by setting  $\beta_o \equiv 0$  and therefore also  $\gamma_o = 1$ ,  $\lambda^+ = a_o$ ,  $\lambda^- = u_T$ ,  $A = 0$ ,  $B = 1$  and, following Torczynski,<sup>5</sup>  $u_T / a_o = 0$  and  $\partial a / \partial T|_{p,w^2} = 0$ . Relationship (49) then reduces to

$$\frac{\Gamma^+}{\lambda^+} (\beta_o = 0) = \frac{\Gamma_c^+}{a_o} = 1 + a_o \rho_o \frac{\partial a}{\partial p} \Bigg|_{T,w^2} = 1 + \frac{\rho_o}{a_o} \frac{\partial a}{\partial \rho} \Bigg|_{s,w^2}, \tag{51}$$

which is equal to the well-known fundamental derivative in ordinary gas dynamics, see, e.g. Hayes<sup>6</sup> or Thompson.<sup>7</sup> For second sound one obtains

$$\begin{aligned}
\frac{\Gamma^-}{\lambda^-} (\beta_o = 0) = & \frac{\Gamma_c^-}{u_T} = 3\alpha_o + \frac{3u_T^2}{2\alpha_o s_o} \frac{\partial \alpha}{\partial T} \Bigg|_{p,w^2} + \frac{s_o \alpha_o}{2c_{vo}} \\
& - \frac{T_o s_o \alpha_o}{2c_{vo}^2} \frac{\partial c_v}{\partial T} \Bigg|_{p,w^2} \\
& = \frac{T_o s_o \alpha_o}{c_{vo}} \frac{\partial}{\partial T} \left[ \ln \left( \frac{u_T^3 c_v}{T} \right) \right] \Bigg|_{p,w^2}, \tag{52}
\end{aligned}$$

in complete agreement with Khalatnikov's form of the steepening parameter, see, e.g. Ref. 8.

The exact results (49), (50) for  $\Gamma^+$  and  $\Gamma^-$  are rather lengthy and involve a number of terms the contributions of which are insignificant for most practical purposes. Similar to the investigation of fourth sound waves by Cramer and

Kluwick<sup>11</sup> it is possible, therefore, to derive a simple but accurate approximation breaking down in the immediate neighbourhood of  $T=0$  only which has to be treated separately. To this end we set

$$\frac{u_T}{a_o} \cong 0, \quad \frac{a_o}{\lambda^+} \cong 1, \quad \frac{u_T}{\lambda^-} \cong 1, \quad A \cong 0, \quad B \cong 1,$$

and neglect terms of  $O(\beta_o T_o)$  except where multiplied by derivatives of  $a$ ,  $c_v$  and  $\alpha$  with respect to  $p$  or  $T$ . We also anticipate that the simplification  $\gamma_o = 1$  is not accurate near the  $\lambda$ -line as a result of the singularity of  $c_p$ . Consequently, equations (49) and (50) can be approximated by

$$\frac{\Gamma^+}{\lambda^+} \approx \frac{\Gamma_c^+}{a_o} + \frac{\beta_o T_o a_o}{c_{po}} \frac{\partial a}{\partial T} \Bigg|_{p,w^2} \tag{53}$$

and

$$\begin{aligned}
\frac{\Gamma^-}{\lambda^-} \approx & \frac{\Gamma_c^-}{u_T} + \frac{G_o T_o \rho_o s_o \alpha_o (2 - \alpha_o)}{2c_{vo} (1 - \alpha_o)} \frac{\partial c_v}{\partial p} \Bigg|_{T,w^2} \\
& + \frac{\beta_o G_o T_o^2 s_o \alpha_o (3 - 2\alpha_o)}{2c_{po} c_{vo} (1 - \alpha_o)} \frac{\partial c_v}{\partial T} \Bigg|_{p,w^2} \\
& + \frac{u_T^2}{2s_o \alpha_o} \left[ 3 - \frac{\beta_o G_o T_o}{\gamma_o} (3 + \alpha_o) \right] \frac{\partial \alpha}{\partial T} \Bigg|_{p,w^2}. \tag{54}
\end{aligned}$$

Numerical computations of the exact, the  $\beta_o = 0$  and the simplified versions of the exact nonlinearity parameters were



performed using the tabular data given by Maynard<sup>9</sup> in the pressure range (0–25) bar and the temperature range above 1.2 K up to the vicinity of the  $\lambda$ -line. The calculations showed that the numerical differences between the exact and approximate expressions for the steepening parameter of second sound were of the same order as the uncertainties in the difference  $u_T^2 - s_o^2 T_o \alpha_o / (c_{vo}(1 - \alpha_o))$ , when the data for all quantities are taken from Maynard. As a consequence the final results depend on the choice of variables taken from Maynard's data base. In order to be consistent, therefore, the same choice was adopted for all evaluations by eliminating  $u_T$  in favour of  $s_o^2 T_o \alpha_o / (c_{vo}(1 - \alpha_o))$ .

Furthermore, the pressure and temperature derivatives were approximated by means of a second order accurate differencing scheme. The results are summarized in Fig. 1 for first sound and in Fig. 2 for second sound, respectively. In addition, Fig. 2 includes the experimental data obtained by Dessler and Fairbank.<sup>10</sup>

Inspection of Figs. 1 and 2 shows excellent agreement between the exact results (49), (50) and the simplified relationships (53), (54) for the nonlinearity parameters. Furthermore, it is seen that the influence of the thermal expansion coefficient on the nonlinearity parameter for second sound is very weak even at larger pressures. Taking into account also the results for the perturbations of the various field quantities summarized in Section III C we, therefore, conclude that the  $\beta_o = 0$  approximation is a valid approximation for second sound waves.

In contrast, the total neglect of effects associated with the thermal expansion coefficient is found to lead to significant errors in the prediction of  $\Gamma^+$ . The discrepancies from the  $\beta_o = 0$  results grow rapidly with increasing pressures and are most pronounced in the neighbourhood of the  $\lambda$ -line.

Included in Fig. 2 are also the (smoothed) experimental data reported in Dessler and Fairbank.<sup>10</sup> It is seen that the calculated and measured values of  $\Gamma^-$  are practically identical for  $T \geq 1.8$  K. Larger discrepancies which reach a maximum value of about 14% at  $T = 1.2$  K, however, occur at smaller temperatures. A comparison between theory and experiment has been carried out also in the original paper by Dessler and Fairbank. This comparison yielded slightly smaller discrepancies at moderate temperatures ( $T \approx 1.4$  K) but significantly larger ones in the higher temperature range. When comparing these observations it should be noted, however, that the accuracy of the experimentally determined values of the steepening parameter is about  $\pm 10\%$  and, furthermore, that the evaluation of the theoretical results is based on completely different procedures. While in the study of Dessler and Fairbank the exact  $\beta_o = 0$  result of  $\Gamma^-$  was approximated by a power law the data base provided by Maynard which is thought to be the most accurate available at present (the accuracy of the sound speed measurements is about  $\pm 0.2\%$ ) was used here to evaluate the Khalatnikov expression. Evaluation of the power law approximation and the full Khalatnikov expression produces differences ranging between 2% at  $T = 1.4$  K and 46% at  $T = 2$  K.

#### IV. BEHAVIOUR AT THE PHASE TRANSITION LINE ( $\lambda$ -LINE)

As mentioned already, the effect of the thermal expansion coefficient on the nonlinearity parameters is most pronounced in the vicinity of the  $\lambda$ -line where  $\beta$  diverges. It is important, therefore, to study the asymptotic behaviour of these quantities in the limit  $T \rightarrow T_\lambda$ . To this end it was necessary first to derive a self-consistent set of asymptotic expansions for the various thermodynamic quantities. Using these results (see the Appendix) one then obtains the limiting expressions for both the  $\beta_o = 0$  and the exact forms of the steepening parameters. The limiting values of the  $\beta_o = 0$  results are obtained immediately from (51) and (52)

$$\begin{aligned} \frac{\Gamma_c^+}{a_o} &\sim a_o \rho_o \left. \frac{\partial a}{\partial p} \right|_{T,w^2} \\ &\sim \frac{(C_1^\lambda)^4 \rho_\lambda^3}{2 \mathcal{A}} \left( \frac{ds_\lambda}{dp} \right)^2 \frac{dT_\lambda}{dp} \frac{1}{\varepsilon \ln^2 \varepsilon} + O\left( \frac{1}{\varepsilon \ln^3 \varepsilon} \right) \\ &\rightarrow -\infty, \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{\Gamma_c^-}{u_T} &\sim \frac{3u_T^2}{2s_o \alpha_o} \left. \frac{\partial \alpha}{\partial T} \right|_{p,w^2} \\ &\sim -\frac{ks_\lambda}{c_v^\lambda} \frac{1}{\varepsilon^{1/3}} + O\left( \frac{1}{\varepsilon^{1/3} \ln \varepsilon} \right) \rightarrow -\infty. \end{aligned} \quad (56)$$

Here the index  $\lambda$  denotes values at the  $\lambda$ -line (see the Appendix) and the parameter  $\varepsilon$  is the relative temperature distance from the phase transition, i.e.,  $\varepsilon = 1 - T/T_\lambda$ .

The analysis starting from the exact results is more subtle. For example, investigation of (49) reveals that the correct behaviour of  $\Gamma^+$  near the  $\lambda$ -line is given by the two term expansion

$$\frac{\Gamma^+}{\lambda^+} \sim \frac{\rho_o a_o^3}{(\lambda^+)^2} \left. \frac{\partial a}{\partial p} \right|_{T,w^2} + \frac{\beta_o T_o a_o^3}{(\lambda^+)^2 c_{po}} \left. \frac{\partial a}{\partial T} \right|_{p,w^2}$$

to leading order which reduces to the  $\beta_o = 0$  result if the second term is neglected. Using the Maxwell relation (A5) the expression for  $\Gamma^+$  can be written in the equivalent form

$$\frac{\Gamma^+}{\lambda^+} \sim \frac{a_o^3 \rho_o}{(\lambda^+)^2} \left[ \left. \frac{\partial a}{\partial p} \right|_{T,w^2} - \frac{\partial a}{\partial T} \right]_{p,w^2} \frac{T_o}{c_{po}} \left. \frac{\partial s}{\partial p} \right|_{T,w^2}.$$

Insertion of the limiting relationships for  $a_p$ ,  $a_T$  and  $s_p$  given in the Appendix then leads to

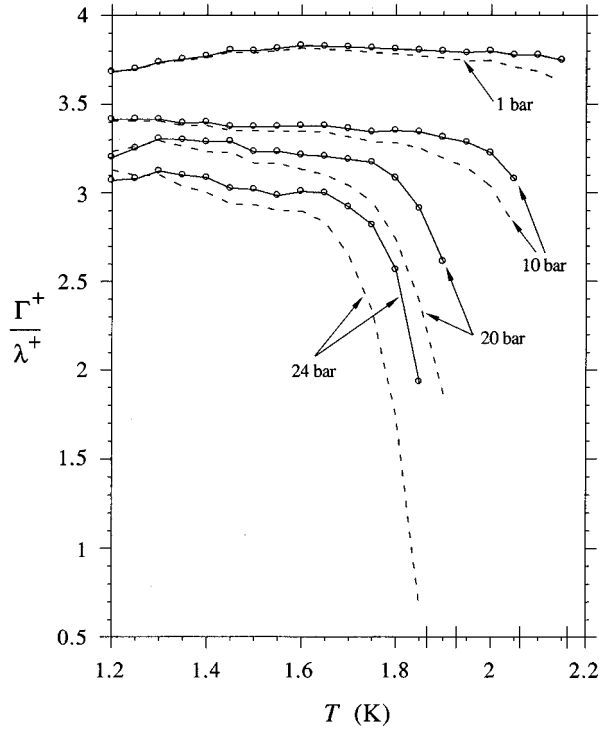


FIG. 1. Computed values of the quadratic steepening parameter for first sound. The solid lines denote the exact expression (49) and the dashed lines the results corresponding to the  $\beta_o=0$  theory (51). The open circles denote the values obtained from the simplified expression (53). Small vertical lines on the  $T$ -axis indicate the phase transition temperatures  $T_\lambda$  ( $dT_\lambda/dp < 0$ ).

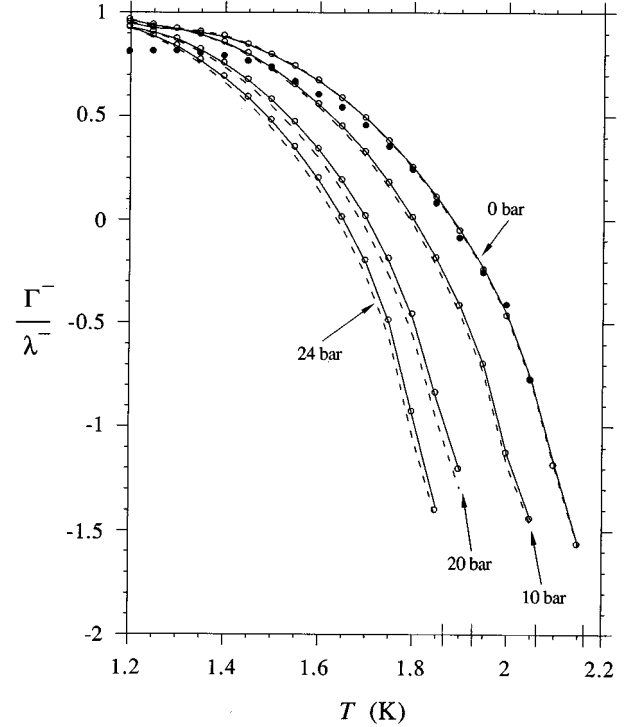


FIG. 2. Computed values of the quadratic steepening parameter for second sound. The solid lines denote the exact expression (50) and the dashed lines the results corresponding to the  $\beta_o=0$  theory (52). The open circles denote the values obtained from the simplified expression (54). Closed circles represent the experimental data taken from Dessler and Fairbank.<sup>10</sup> Small vertical lines on the  $T$ -axis indicate the phase transition temperatures  $T_\lambda$  ( $dT_\lambda/dp < 0$ ).

$$\begin{aligned}
 \frac{\Gamma^+}{\lambda^+} &\sim (C_1^\lambda \rho_\lambda + \dots) \left[ -\frac{j_1}{T_\lambda \varepsilon \ln^2 \varepsilon} \frac{dT_\lambda}{dp} \right. \\
 &\quad - \frac{2j_2}{T_\lambda \varepsilon \ln^3 \varepsilon} \frac{dT_\lambda}{dp} + \dots - \left( \frac{j_1}{T_\lambda \varepsilon \ln^2 \varepsilon} \right. \\
 &\quad \left. + \frac{2j_2}{T_\lambda \varepsilon \ln^3 \varepsilon} + \dots \right) \frac{T_\lambda (1 + \dots)}{(-\mathcal{A} \ln \varepsilon + \dots)} \left( \frac{ds_\lambda}{dp} \right. \\
 &\quad \left. + \frac{\mathcal{A} dT_\lambda}{T_\lambda dp} \ln \varepsilon + \dots \right) \Bigg] \\
 &= \frac{j_1 ds_\lambda}{\mathcal{A} dp} \frac{C_1^\lambda \rho_\lambda}{\varepsilon \ln^3 \varepsilon} + \dots \\
 &= -\frac{(C_1^\lambda)^4 \rho_\lambda^3 T_\lambda}{2\mathcal{A}^2} \left( \frac{ds_\lambda}{dp} \right)^3 \frac{1}{\varepsilon \ln^3 \varepsilon} + O\left( \frac{1}{\varepsilon \ln^4 \varepsilon} \right) \\
 &\rightarrow -\infty.
 \end{aligned} \tag{57}$$

Inspection of this expression yields the surprising result that the leading order term of the  $\beta_o=0$  approximation is exactly cancelled by the leading order term containing the thermal expansion coefficient. Consequently,  $\Gamma^+/\lambda^+$  is of order  $O(\varepsilon \ln^3 \varepsilon)^{-1}$  rather than  $O(\varepsilon \ln^2 \varepsilon)^{-1}$  as predicted by (55).

For second sound one obtains in quite a similar way

$$\begin{aligned}
 \frac{\Gamma^-}{\lambda^-} &\sim \frac{1}{2} \left[ -\frac{\beta_o^2 T_o a_o^2 u_T^2 (3 + \alpha_o)}{\gamma_o s_o \alpha_o c_{p_o}} + \frac{3u_T^2}{s_o \alpha_o} \right] \frac{\partial \alpha}{\partial T} \Bigg|_{p, w^2} \\
 &\sim \frac{3u_T^2}{2s_o \alpha_o \gamma_o} \frac{\partial \alpha}{\partial T} \Bigg|_{p, w^2} \\
 &\sim \frac{ks_\lambda}{\mathcal{A}} \frac{1}{\varepsilon^{1/3} \ln \varepsilon} + O\left( \frac{1}{\varepsilon^{1/3} \ln^2 \varepsilon} \right) \rightarrow -\infty.
 \end{aligned} \tag{58}$$

Summarizing, we conclude that taking into account the thermal expansion coefficient leaves the signs of the limiting values of  $\Gamma^+$  and  $\Gamma^-$  unchanged but reduces the strength of the singularities which form in the limit  $T \rightarrow T_\lambda$ . As a result, these singularities make themselves felt in a smaller neighbourhood of the  $\lambda$ -line than according to the  $\beta_o=0$  theory. This is seen to be in qualitative agreement with the results plotted in Fig. 1 and Fig. 2 although the available data are not sufficient to resolve these singularities.

Finally it should be noted that the limiting behaviour of  $\Gamma^+$  points to the existence of a region in the vicinity of the  $\lambda$ -line where  $\Gamma^+$  is negative, e.g. where first sound waves exhibit the phenomena associated with negative and mixed nonlinearity.

## V. BEHAVIOUR AT ABSOLUTE ZERO

As pointed out earlier it is necessary to study the behaviour of the steepening parameters for  $T \rightarrow 0$  separately. In a Bose-fluid, such as He II, the elementary excitations with small momentum are phonons, i.e. the energy of these quasi-particles is a linear function of their momentum. This linear law holds as long as the wavelength of the phonons is large compared to the intermolecular distances of the helium atoms. Near absolute zero we take He II to be an ideal Bose-gas and therefore the Bose-Einstein statistics are applied to derive a consistent set of formulas for each thermodynamic quantity (see the Appendix). It is remarkable that one *cannot* deduce  $\rho_n = 0$  and therefore  $\alpha = 1$  at  $T = 0$  from the third law of thermodynamics, but it is reasonable to make the assumption  $\alpha \equiv 1$  at absolute zero (see Ref. 1). In the following analysis we will use a tilde to denote quantities at absolute zero.

First we consider the  $\beta_o = 0$  versions of the nonlinearity parameters (51) and (52), which are easily shown to remain finite for  $T \rightarrow 0$ ; these are

$$\frac{\Gamma_c^+}{a_o} \sim 1 + a_o \rho_o \left. \frac{\partial a}{\partial p} \right|_{T, w^2} \rightarrow 1 + \tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}}, \quad (59)$$

$$\frac{\Gamma_c^-}{u_T} \sim 3\alpha_o + \frac{s_o \alpha_o}{2c_{vo}} + \frac{u_T^2 (1 + 2\alpha_o)}{2s_o \alpha_o} \left. \frac{\partial \alpha}{\partial T} \right|_{p, w^2} \rightarrow \frac{7}{6}. \quad (60)$$

Note, that the result (60) is independent of the pressure. Evaluation of the exact formulas (49), (50) in the limit  $T \rightarrow 0$  leads to the limiting expressions

$$\begin{aligned} \frac{\Gamma^+}{\lambda^+} &\sim \frac{\Gamma_c^+}{a_o} + \frac{A u_T^2 \beta_o (2 - \alpha_o)}{2c_{po} (1 - \alpha_o)} \\ &\rightarrow 1 + \tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}} + \frac{1}{72} \left( 1 + 3\tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}} \right)^3 \end{aligned} \quad (61)$$

and

$$\begin{aligned} \frac{\Gamma^-}{\lambda^-} &\sim \frac{\Gamma_c^-}{u_T} B - \frac{\beta_o^2 T_o s_o \alpha_o a_o^2 (2 - \alpha_o)}{2c_{po}^2 (1 - \alpha_o)} \\ &\rightarrow \frac{7}{9} - \frac{1}{18} \left( 1 + 3\tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}} \right)^2, \end{aligned} \quad (62)$$

which are fundamentally different from those of the  $\beta_o = 0$  theory, e.g. (59) and (60).

## VI. COMMENTS ON NONLINEAR FOURTH SOUND

It is interesting to apply the sets of asymptotic formulas derived in the Appendix which describe the properties of various thermodynamic quantities in the vicinity of the  $\lambda$ -line and near  $T = 0$  also to the steepening parameter of the so-called fourth sound.

Starting from the  $\beta_o = 0$  approximation given by Torczynski<sup>5</sup> one deduces that its behaviour in the neighbourhood of the  $\lambda$ -line is given by

$$\begin{aligned} \frac{\Gamma}{\lambda} (\beta_o = 0) &\equiv D \sim \frac{3\rho_o a_o^2}{2\alpha_o} \left. \frac{\partial \alpha}{\partial p} \right|_{T, w^2} \\ &\sim \rho_\lambda (C_1^\lambda)^2 \frac{d(\ln T_\lambda)}{dp} \frac{1}{\varepsilon} + O\left(\frac{1}{\varepsilon \ln \varepsilon}\right) \\ &\rightarrow -\infty. \end{aligned} \quad (63)$$

The limiting behaviour of the exact form of the nonlinearity parameter has been investigated first by Cramer and Kluwick.<sup>11</sup> Owing to the fact that the expansions of the relevant thermodynamic quantities were limited to the leading order terms listed in the paper of Maynard<sup>9</sup> a cancellation effect similar to that occurring in first and second sound waves remained unnoticed. As a consequence the calculations predicted a positive singularity in contrast to the present analysis which yields

$$\begin{aligned} \frac{\Gamma}{\lambda} &\sim \frac{3a_o^2}{2\alpha_o} \left( \left. \rho_o \frac{\partial \alpha}{\partial p} \right|_{T, w^2} + \frac{\beta_o T_o}{c_{po}} \left. \frac{\partial \alpha}{\partial T} \right|_{p, w^2} \right) \\ &\quad - \frac{3s_o T_o}{2\alpha_o c_{po}} \left( \left. \gamma_o \frac{\partial \alpha}{\partial T} \right|_{p, w^2} + \beta_o \rho_o a_o^2 \left. \frac{\partial \alpha}{\partial p} \right|_{T, w^2} \right) \\ &\sim - \frac{(C_1^\lambda)^2}{\mathcal{A}} \frac{d(s_\lambda \rho_\lambda)}{dp} \frac{1}{\varepsilon \ln \varepsilon} + O\left(\frac{1}{\varepsilon \ln^2 \varepsilon}\right) \rightarrow -\infty. \end{aligned} \quad (64)$$

As in the case of first and second sound waves the effects caused by the thermal expansion leave the sign of the steepening parameters unchanged but weaken the strength of the singularity.

Evaluation of the fourth sound nonlinearity parameter for  $\beta_o = 0$  near absolute zero leads to the expression

$$D \sim 1 + \rho_o a_o \left. \frac{\partial a}{\partial p} \right|_{T, w^2} \rightarrow 1 + \tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}}. \quad (65)$$

The exact steepening parameter of fourth sound behaves like

$$\frac{\Gamma}{\lambda} \sim \frac{a_o^2 \alpha_o}{\lambda^2} \left( 1 + \rho_o a_o \left. \frac{\partial a}{\partial p} \right|_{T, w^2} \right) \rightarrow 1 + \tilde{a} \tilde{\rho} \frac{d\tilde{a}}{d\tilde{p}}. \quad (66)$$

Equation (66) is seen to be equal to the expression for the  $\beta_o = 0$  case.

## VII. SUMMARY

The main objective of the present study was the determination of the effect of thermal expansion on nonlinear first and second sound. The quadratic steepening parameters for arbitrary values of the coefficient of thermal expansion (1) were derived through use of the multiple scales technique of Taniuti and Wei.<sup>3</sup> The exact results for first and second sound are given in equations (49) and (50), respectively. The corresponding results in terms of density and entropy (rather than pressure and temperature) derivatives are given in (44) and (45). It was found that the commonly applied  $\beta_o = 0$  approximation is an accurate estimate for second sound except in the neighbourhood of the  $\lambda$ -line. However, we found significant differences between the  $\beta_o = 0$  and exact expres-

sions in the case of first sound. These differences are small at low pressures but increase with pressure and as the  $\lambda$ -line is approached.

We have also analyzed the behaviour of the exact expressions (49) and (50) in the limits  $T \rightarrow T_\lambda$  and  $T \rightarrow 0$ . In each limit the asymptotic behaviour of both first and second sound was found to differ from the  $\beta_o = 0$  approximation. As the  $\lambda$ -line is approached, the resultant singularities for each mode was found to be weaker than those of the  $\beta_o = 0$  theory by a factor  $\ln \varepsilon$ .

Both the exact and  $\beta_o = 0$  theories reveal that  $\Gamma^+ \rightarrow -\infty$  as the  $\lambda$ -line is approached. Thus, backward steepening first sound fronts which form first sound expansion shocks are expected to occur in the vicinity of the  $\lambda$ -line, even at low pressures. At undisturbed states where  $\Gamma^+$  changes sign, the approximation scheme of Taniuti and Wei<sup>3</sup> breaks down and the scheme of Cramer and Sen<sup>18</sup> provides the appropriate extension of the Burgers equation (15). Physically, the evolution will be characterized by both backward and forward steepening (mixed nonlinearity). First sound double shock configurations analogous to those seen in the second sound experiments of Turner<sup>19</sup> and Torczynski *et al.*<sup>20</sup> are then expected to be observed.

## APPENDIX

### 1. Derivation of a set of self-consistent asymptotic formulas for the thermodynamic quantities at the $\lambda$ -line

A second order phase transition, such as that between He II and He I, involves discontinuous changes of the second order derivatives of the Gibbs free energy  $g$ , e.g. the specific heat at constant pressure  $c_p = -T(\partial^2 g / \partial T^2)|_p$  and the coefficient of thermal expansion  $\beta = \rho(\partial^2 g / \partial p \partial T)$ , whereas  $g$  and its first order derivatives (the entropy  $s$  and the density  $\rho$ ) vary smoothly when crossing the  $\lambda$ -curve. To derive consistent expansions for all thermodynamic quantities in the vicinity of the  $\lambda$ -line used in our analysis, we take the singularity law for the specific heat at constant pressure  $c_p$  and the power law for the superfluid fraction  $\alpha = \rho_s / \rho$  given by Ahlers and co-workers<sup>12-14</sup> as fundamental equations and, in addition, definitions and thermodynamic cross relations of the quantities we are interested in. These laws are in agreement with the predictions of the renormalization group theory, an exact theory of critical phenomena. In the following treatment the index  $\lambda$  denotes values at the  $\lambda$ -line and  $\varepsilon$  is the relative temperature distance to the  $\lambda$ -line in the He II region defined by

$$\varepsilon(p, T) := 1 - \frac{T}{T_\lambda(p)} \rightarrow 0, \quad (\text{A1})$$

and it should be noted that the corresponding expansions are valid for the range  $\varepsilon < O(10^{-2})$ . Following Refs. 12, 13 and 14 the asymptotic behaviour of  $c_p$  and  $\alpha$  in the limit  $\varepsilon \rightarrow 0$  is assumed to be given by

$$c_p(p, \varepsilon) \sim -\mathcal{A}(p) \ln \varepsilon + \mathcal{B}(p) + O(\varepsilon \ln \varepsilon) \rightarrow \infty, \quad (\text{A2})$$

$$\alpha(p, \varepsilon) \sim k(p) \varepsilon^{2/3} (1 + b(p) \varepsilon^{1/2}) + O(\varepsilon^{5/3}) \rightarrow 0, \quad (\text{A3})$$

with the pressure dependent coefficients  $\mathcal{A}(p) > 0$ ,  $\mathcal{B}(p)$ ,  $k(p) > 0$  and  $b(p)$  given in Refs. 12, 13 and 14. The cross relations between all other thermodynamic quantities are the definition of the specific heat,

$$c_p := T \left. \frac{\partial s}{\partial T} \right|_{p, w^2}, \quad (\text{A4})$$

the Maxwell relation,

$$\left. \frac{\partial s}{\partial p} \right|_{T, w^2} = - \left. \frac{\partial}{\partial T} \left( \frac{1}{\rho} \right) \right|_{p, w^2}, \quad (\text{A5})$$

and an equation obtained by combining the definition of the thermal expansion coefficient (10) and the Maxwell relation (A5),

$$\beta = - \rho \left. \frac{\partial s}{\partial p} \right|_{T, w^2}. \quad (\text{A6})$$

Furthermore equation (11) and the definition (10) can be combined to yield

$$a^2 = \frac{c_p}{c_p (\partial \rho / \partial p)|_{T, w^2} - \beta^2 T} \quad (\text{A7})$$

and

$$\gamma = a^2 \left. \frac{\partial \rho}{\partial p} \right|_{T, w^2}. \quad (\text{A8})$$

Finally we use the definition of the specific heat ratio to obtain an equation for  $c_v$ ,

$$c_v = \frac{c_p}{\gamma}, \quad (\text{A9})$$

and the definition of the  $\beta = 0$  version of the linear wave speed of second sound (23),

$$u_T^2 := \frac{s^2 T \alpha}{c_v (1 - \alpha)}. \quad (\text{A10})$$

Taking the singularity law for  $c_p$ , (A2) and definition (A4), one obtains

$$s(p, \varepsilon) \sim s_\lambda(p) + \mathcal{A} \varepsilon \ln \varepsilon - (\mathcal{A} + \mathcal{B}) \varepsilon + O(\varepsilon^2 \ln \varepsilon) \rightarrow s_\lambda, \quad (\text{A11})$$

for the entropy by integration. Integrating (A5) after insertion of (A11) leads to the expansion for the density,

$$\rho(p, \varepsilon) \sim \rho_\lambda(p) + \mathcal{S}_1 \varepsilon \ln \varepsilon + \mathcal{S}_2 \varepsilon + O(\varepsilon^2 \ln^2 \varepsilon) \rightarrow \rho_\lambda. \quad (\text{A12})$$

The definition of the pressure dependent coefficients is given in the next section, here we simply note that  $\mathcal{S}_1$  is equal to  $R$ , the corresponding coefficient in the paper of Maynard.<sup>9</sup> Computation of (A6) under consideration of the expression for  $s$  and  $\rho$  yields

$$\beta(p, \varepsilon) \sim \mathcal{C}_1 \ln \varepsilon + \mathcal{C}_2 + \mathcal{C}_3 \varepsilon \ln^2 \varepsilon + \mathcal{C}_4 \varepsilon \ln \varepsilon + \mathcal{C}_5 \varepsilon + O(\varepsilon^2 \ln^3 \varepsilon) \rightarrow -\infty, \quad (\text{A13})$$

for the thermal expansion coefficient. Application of (A2), (A13), and (A12) to the equation for the linear wave speed for first sound (A7) leads to

$$a^2(p, \varepsilon) \sim (C_1^\lambda)^2 + \frac{k_1}{\ln \varepsilon} + \frac{k_2}{\ln^2 \varepsilon} + O\left(\frac{1}{\ln^3 \varepsilon}\right) \rightarrow (C_1^\lambda)^2 \quad (\text{A14})$$

and

$$a(p, \varepsilon) \sim C_1^\lambda + \frac{j_1}{\ln \varepsilon} + \frac{j_2}{\ln^2 \varepsilon} + O\left(\frac{1}{\ln^3 \varepsilon}\right) \rightarrow C_1^\lambda. \quad (\text{A15})$$

The specific heat ratio, calculated from (A8) is given by

$$\gamma(p, \varepsilon) \sim m_1 \ln \varepsilon + m_2 + \frac{m_3}{\ln \varepsilon} + O\left(\frac{1}{\ln^2 \varepsilon}\right) \rightarrow \infty. \quad (\text{A16})$$

Further evaluation of (A9) yields

$$c_v(p, \varepsilon) \sim c_v^\lambda + \frac{n_1}{\ln \varepsilon} + \frac{n_2}{\ln^2 \varepsilon} + O\left(\frac{1}{\ln^3 \varepsilon}\right) \rightarrow c_v^\lambda. \quad (\text{A17})$$

And, finally expression (A10) leads to the expansion for the linear wave speed for second sound:

$$u_T^2(p, \varepsilon) \sim h_1 \varepsilon^{2/3} + \frac{h_2 \varepsilon^{2/3}}{\ln \varepsilon} + \frac{h_3 \varepsilon^{2/3}}{\ln^2 \varepsilon} + O\left(\frac{\varepsilon^{2/3}}{\ln^3 \varepsilon}\right) \rightarrow 0. \quad (\text{A18})$$

The derivatives of  $a$ ,  $c_v$  and  $\alpha$  with respect to  $p$  and  $T$  can be written as

$$\left. \frac{\partial a}{\partial p} \right|_{T, w^2} (p, \varepsilon) \sim -\frac{j_1}{T_\lambda \varepsilon \ln^2 \varepsilon} \frac{dT_\lambda}{dp} - \frac{2j_2}{T_\lambda \varepsilon \ln^3 \varepsilon} \frac{dT_\lambda}{dp} + O\left(\frac{1}{\varepsilon \ln^4 \varepsilon}\right) \rightarrow -\infty, \quad (\text{A19})$$

$$\left. \frac{\partial a}{\partial T} \right|_{p, w^2} (p, \varepsilon) \sim \frac{j_1}{T_\lambda \varepsilon \ln^2 \varepsilon} + \frac{2j_2}{T_\lambda \varepsilon \ln^3 \varepsilon} + O\left(\frac{1}{\varepsilon \ln^4 \varepsilon}\right) \rightarrow -\infty, \quad (\text{A20})$$

$$\left. \frac{\partial c_v}{\partial p} \right|_{T, w^2} (p, \varepsilon) \sim -\frac{n_1}{T_\lambda \varepsilon \ln^2 \varepsilon} \frac{dT_\lambda}{dp} - \frac{2n_2}{T_\lambda \varepsilon \ln^3 \varepsilon} \frac{dT_\lambda}{dp} + O\left(\frac{1}{\varepsilon \ln^4 \varepsilon}\right) \rightarrow \infty, \quad (\text{A21})$$

$$\left. \frac{\partial c_v}{\partial T} \right|_{p, w^2} (p, \varepsilon) \sim \frac{n_1}{T_\lambda \varepsilon \ln^2 \varepsilon} + \frac{2n_2}{T_\lambda \varepsilon \ln^3 \varepsilon} + O\left(\frac{1}{\varepsilon \ln^4 \varepsilon}\right) \rightarrow \infty, \quad (\text{A22})$$

$$\left. \frac{\partial \alpha}{\partial p} \right|_{T, w^2} (p, \varepsilon) \sim \frac{2k}{3T_\lambda \varepsilon^{1/3}} \frac{dT_\lambda}{dp} + O(\varepsilon^{1/6}) \rightarrow -\infty, \quad (\text{A23})$$

$$\left. \frac{\partial \alpha}{\partial T} \right|_{p, w^2} (p, \varepsilon) \sim -\frac{2k}{3T_\lambda \varepsilon^{1/3}} + O(\varepsilon^{1/6}) \rightarrow -\infty. \quad (\text{A24})$$

To complete the description of the behaviour at the  $\lambda$ -line, the following expressions for the sound speed ratio, the eigenvalues (22) and the abbreviations (46) are useful:

$$\left(\frac{u_T}{a_o}\right)^2 (p, \varepsilon) \sim \frac{h_1 \varepsilon^{2/3}}{(C_1^\lambda)^2} + O\left(\frac{\varepsilon^{2/3}}{\ln \varepsilon}\right) \rightarrow 0, \quad (\text{A25})$$

$$(\lambda^+)^2 (p, \varepsilon) \sim (C_1^\lambda)^2 + O(\varepsilon^{2/3}) \rightarrow (C_1^\lambda)^2, \quad (\text{A26})$$

$$(\lambda^-)^2 (p, \varepsilon) \sim O(\varepsilon^{2/3}) \rightarrow 0, \quad (\text{A27})$$

$$A(p, \varepsilon) \sim O(\varepsilon^{2/3}) \rightarrow 0, \quad (\text{A28})$$

$$B(p, \varepsilon) \sim 1 + O(\varepsilon^{2/3}) \rightarrow 1. \quad (\text{A29})$$

## 2. Coefficients

The pressure dependent coefficients  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $k$ ,  $b$  have to be determined from experimental data, values can be found in Refs. 12, 13, and 14, respectively. Furthermore, the integration constant  $s_\lambda(p)$  is obtained from a relation given by Ahlers<sup>12</sup> and  $\rho_\lambda(p)$  is evaluated using an expression from Kierstead.<sup>15</sup> The  $\lambda$ -line is represented through a polynomial fit for  $p_\lambda(T_\lambda)$ , also given by Kierstead. All other coefficients are links between  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $k$ ,  $b$ ,  $s_\lambda$ ,  $\rho_\lambda$ ,  $T_\lambda(p)$ , and derivatives of them.

$$\mathcal{G}_1(p) = -\mathcal{A} \rho_\lambda^2 \frac{dT_\lambda}{dp},$$

$$\mathcal{G}_2(p) = -T_\lambda \rho_\lambda^2 \left( \frac{ds_\lambda}{dp} - \frac{(\mathcal{A} + \mathcal{B})}{T_\lambda} \frac{dT_\lambda}{dp} \right),$$

$$\mathcal{E}_1(p) = -\frac{\mathcal{A} \rho_\lambda}{T_\lambda} \frac{dT_\lambda}{dp},$$

$$\mathcal{E}_2(p) = \frac{\mathcal{B} \rho_\lambda}{T_\lambda} \frac{dT_\lambda}{dp} - \rho_\lambda \frac{ds_\lambda}{dp},$$

$$\mathcal{E}_3(p) = -\frac{\mathcal{A} \mathcal{G}_1}{T_\lambda} \frac{dT_\lambda}{dp},$$

$$\mathcal{E}_4(p) = \mathcal{G}_1 \frac{ds_\lambda}{dp} - \rho_\lambda \frac{d\mathcal{A}}{dp} + \frac{1}{T_\lambda} \frac{dT_\lambda}{dp} (\mathcal{A} \rho_\lambda + \mathcal{B} \mathcal{G}_1 - \mathcal{A} \mathcal{E}_2),$$

$$\mathcal{E}_5(p) = -\mathcal{E}_2 \frac{ds_\lambda}{dp} + \rho_\lambda \frac{d(\mathcal{A} + \mathcal{B})}{dp} + \frac{\mathcal{B}}{T_\lambda} \frac{dT_\lambda}{dp} (\mathcal{E}_2 - \rho_\lambda),$$

$$C_1^\lambda(p) = \left( \frac{d\rho_\lambda}{dp} + \frac{ds_\lambda}{dp} \frac{dT_\lambda}{dp} \rho_\lambda^2 \right)^{-1/2},$$

$$k_1(p) = -\frac{(C_1^\lambda)^4}{\mathcal{A}} \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda,$$

$$k_2(p) = \frac{(C_1^\lambda)^4}{\mathcal{A}^2} \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda \left[ (C_1^\lambda)^2 \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda - \mathcal{B} \right],$$

$$j_1(p) = \frac{k_1}{2C_1^\lambda} = -\frac{(C_1^\lambda)^3 \rho_\lambda^2 T_\lambda}{2\mathcal{A}} \left( \frac{ds_\lambda}{dp} \right)^2,$$

$$\begin{aligned}
j_2(p) &= \frac{k_2}{2C_1^\lambda} - \frac{k_1^2}{8(C_1^\lambda)^3} \\
&= \frac{(C_1^\lambda)^3}{8\mathcal{A}^2} \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda \left[ 3(C_1^\lambda)^2 \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda - 4\mathcal{B} \right], \\
m_1(p) &= -\frac{\mathcal{A}}{T_\lambda} (C_1^\lambda)^2 \left( \frac{dT_\lambda}{dp} \right)^2 \rho_\lambda^2, \\
m_2(p) &= (C_1^\lambda)^2 \left[ \frac{\mathcal{B}}{T_\lambda} \left( \frac{dT_\lambda}{dp} \right)^2 \rho_\lambda^2 + (C_1^\lambda)^2 \left( \frac{d\rho_\lambda}{dp} \right)^2 \right], \\
m_3(p) &= -\frac{(C_1^\lambda)^6}{\mathcal{A}} \left( \frac{d\rho_\lambda}{dp} \right)^2 \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda, \\
c_v^\lambda(p) &= \frac{T_\lambda}{(C_1^\lambda)^2 \rho_\lambda^2} \left( \frac{dT_\lambda}{dp} \right)^{-2} = -\frac{\mathcal{A}}{m_1}, \\
n_1(p) &= \frac{T_\lambda^2}{\mathcal{A} \rho_\lambda^4} \left( \frac{d\rho_\lambda}{dp} \right)^2 \left( \frac{dT_\lambda}{dp} \right)^{-4}, \\
n_2(p) &= \frac{T_\lambda^2}{\mathcal{A}^2 \rho_\lambda^6} \left( \frac{d\rho_\lambda}{dp} \right)^2 \left( \frac{dT_\lambda}{dp} \right)^{-6} \left[ \mathcal{B} \left( \frac{dT_\lambda}{dp} \right)^2 \rho_\lambda^2 + \frac{d\rho_\lambda}{dp} T_\lambda \right. \\
&\quad \left. - \frac{ds_\lambda}{dp} \frac{dT_\lambda}{dp} \rho_\lambda^2 T_\lambda \right], \\
h_1(p) &= \frac{ks_\lambda^2 T_\lambda}{c_v^\lambda} = ks_\lambda^2 (C_1^\lambda)^2 \rho_\lambda^2 \left( \frac{dT_\lambda}{dp} \right)^2, \\
h_2(p) &= \frac{k}{\mathcal{A}} (C_1^\lambda)^4 s_\lambda^2 T_\lambda \left( \frac{d\rho_\lambda}{dp} \right)^2, \\
h_3(p) &= \frac{k}{\mathcal{A}^2} (C_1^\lambda)^6 s_\lambda^2 T_\lambda \left( \frac{d\rho_\lambda}{dp} \right)^2 \left[ \left( \frac{ds_\lambda}{dp} \right)^2 \rho_\lambda^2 T_\lambda - \frac{\mathcal{B}}{(C_1^\lambda)^2} \right].
\end{aligned}$$

### 3. Derivation of a set of self-consistent asymptotic formulas for the thermodynamic quantities at absolute zero

From the third law of thermodynamics, which usually is expressed in the form

$$\lim_{T \rightarrow 0} s(p, T, w^2) = \lim_{T \rightarrow 0} s(\rho, T, w^2) = 0, \quad (\text{A30})$$

it is clear, that in the limit  $T \rightarrow 0$  the entropy  $s$  depends on  $T$  only. Therefore we can write

$$\begin{aligned}
\lim_{T \rightarrow 0} \frac{\partial s}{\partial p} \Big|_{T, w^2} &= \lim_{T \rightarrow 0} \frac{\partial s}{\partial \rho} \Big|_{T, w^2} = \lim_{T \rightarrow 0} \frac{\partial s}{\partial w^2} \Big|_{p, T} = \lim_{T \rightarrow 0} \frac{\partial s}{\partial w^2} \Big|_{p, T} \\
&= 0. \quad (\text{A31})
\end{aligned}$$

After some basic manipulations equations (A31) and (3) leads to the well-known relations

$$\lim_{T \rightarrow 0} \frac{\partial \rho}{\partial T} \Big|_{p, w^2} = 0 \quad (\text{A32})$$

and

$$\lim_{T \rightarrow 0} \frac{\partial \alpha}{\partial T} \Big|_{p, w^2} = 0. \quad (\text{A33})$$

Following Landau and Lifschitz,<sup>16</sup> the phonon contribution to the specific heat of He II for  $T \rightarrow 0$  can be written in the form

$$c(p, T) \equiv c_p = c_v \sim \frac{K}{\tilde{a}^3 \tilde{\rho}} T^3 \rightarrow 0, \quad (\text{A34})$$

where  $K$  is a constant and the tilde denotes quantities evaluated at  $T=0$ . For small temperatures the first sound speed and the density depend on the pressure  $p$  only. From  $c = T \partial s / \partial T$  one then obtains

$$s(p, T) \sim \frac{K}{3\tilde{a}^3 \tilde{\rho}} T^3 = \frac{c}{3} \rightarrow 0. \quad (\text{A35})$$

The integration constant  $s(p, T=0)$  was chosen to be zero in order to satisfy the third law of thermodynamics. Equation (A5) leads to

$$\rho(p, T) \sim \tilde{\rho} + \frac{K\tilde{\rho}^2}{12} \frac{d}{dp} \left( \frac{1}{\tilde{a}^3 \tilde{\rho}} \right) T^4 + O(T^8) \rightarrow \tilde{\rho}. \quad (\text{A36})$$

Evaluation of (A6) yields

$$\beta(p, T) \sim -\frac{K\tilde{\rho}}{3} \frac{d}{dp} \left( \frac{1}{\tilde{a}^3 \tilde{\rho}} \right) T^3 \rightarrow 0, \quad (\text{A37})$$

in agreement with equation (A32). From (A7) and (A8) it follows that

$$a^2(p, T) \sim \frac{dp}{d\tilde{\rho}} + O(T^4) = \tilde{a}^2 + O(T^4) \rightarrow \tilde{a}^2 \quad (\text{A38})$$

and

$$\gamma(p, T) \sim 1 + O(T^4) \rightarrow 1. \quad (\text{A39})$$

Taking the density of the normal component in the form given in Ref. 16,

$$\rho_n(p, T) \sim \frac{K}{3\tilde{a}^5} T^4 \rightarrow 0, \quad (\text{A40})$$

we obtain

$$\rho_s(p, T) \sim \tilde{\rho} + \frac{K}{3} \left[ \frac{\tilde{\rho}^2}{4} \frac{d}{dp} \left( \frac{1}{\tilde{a}^3 \tilde{\rho}} \right) - \frac{1}{\tilde{a}^5} \right] T^4 \rightarrow \tilde{\rho} \quad (\text{A41})$$

and

$$\alpha(p, T) \sim 1 - \frac{K}{3\tilde{a}^5 \tilde{\rho}} T^4 + O(T^8) \rightarrow 1. \quad (\text{A42})$$

The second sound speed, calculated from (A10), is given by

$$u_{II}^2(p, T) \sim \frac{\tilde{a}^2}{3} \left( 1 - \frac{K}{3\tilde{a}^5 \tilde{\rho}} T^4 \right) \rightarrow \frac{\tilde{a}^2}{3}. \quad (\text{A43})$$

For a thorough discussion of the result (A43) and its experimental verification the reader is referred to Atkins.<sup>17</sup> Using equations (A34), (A38) and (A42) the derivatives of  $a$ ,  $c$  and  $\alpha$  with respect to  $p$  and  $T$  can be written in the form

$$\frac{\partial a}{\partial p} \Big|_{T, w^2} (p, T) \sim \frac{d\tilde{a}}{dp} + O(T^4) \rightarrow \frac{d\tilde{a}}{dp}, \quad (\text{A44})$$

$$\left. \frac{\partial a}{\partial T} \right|_{p,w^2} (p,T) \sim O(T^3) \rightarrow 0, \quad (\text{A45})$$

$$\left. \frac{\partial c}{\partial p} \right|_{T,w^2} (p,T) \sim K \frac{d}{dp} \left( \frac{1}{\tilde{a}^3 \tilde{\rho}} \right) T^3 \rightarrow 0, \quad (\text{A46})$$

$$\left. \frac{\partial c}{\partial T} \right|_{p,w^2} (p,T) \sim \frac{3K}{\tilde{a}^3 \tilde{\rho}} T^2 \rightarrow 0, \quad (\text{A47})$$

$$\left. \frac{\partial \alpha}{\partial p} \right|_{T,w^2} (p,T) \sim -\frac{K}{3} \frac{d}{dp} \left( \frac{1}{\tilde{a}^5 \tilde{\rho}} \right) T^4 \rightarrow 0, \quad (\text{A48})$$

$$\left. \frac{\partial \alpha}{\partial T} \right|_{p,w^2} (p,T) \sim -\frac{4K}{3\tilde{a}^5 \tilde{\rho}} T^3 \rightarrow 0. \quad (\text{A49})$$

Equation (A49) satisfies the prediction of equation (A33). Finally, the abbreviations (46) take the form

$$A \sim \frac{K\tilde{a}^5 \tilde{\rho}^3}{12} \left[ \frac{d}{dp} \left( \frac{1}{\tilde{a}^3 \tilde{\rho}} \right) \right]^2 T^4 \rightarrow 0, \quad (\text{A50})$$

$$B \sim \frac{2}{3} + O(T^4) \rightarrow \frac{2}{3}. \quad (\text{A51})$$

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