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Citation: Physics of Fluids (1958-1988) **27**, 821 (1984); doi: 10.1063/1.864710
View online: http://dx.doi.org/10.1063/1.864710
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof1/27/4?ver=pdfcov
Published by the AIP Publishing

The evolution of long-wave solutions to the nonlinear Schrödinger equation

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(Received 31 May 1983; accepted 7 October 1983)

In water of moderate depth, the behavior of small perturbations superimposed on Stokes wave trains is described by the nonlinear (cubic) Schrödinger equation. In the present study wave-like solutions to this equation are examined, and it is shown that when these perturbations are neutrally stable and sufficiently long, solutions to the Schrödinger equation may be approximated by the well-known Korteweg–deVries equation. As a result, sufficiently long perturbations to Stokes wave trains may be regarded as mathematically analogous to those imposed on a free surface separating two fluids of different densities. This result is established independently by singular perturbation techniques, numerical computation, and comparison of exact stationary wave solutions.

I. INTRODUCTION

In the study of the propagation of nonlinear dispersive waves both the Korteweg-deVries equation and the nonlinear Schrödinger equation have played important roles. The first was derived by Korteweg and deVries[1] in their analysis of shallow water waves in a uniform channel. This equation typically describes weakly nonlinear, weakly dispersive waves which may either be pulse-like or periodic. Much of the recent interest in this equation is due to the soliton properties of its solutions as demonstrated by Zabusky and Kruskal,[2] the inverse-scattering techniques developed by Gardner, Greene, Kruskal, and Muira[3,4] and the wide variety of applications of this equation. For a review of this equation, its solutions and applications, we refer the reader to Miles[5] and Whitham.[6] The nonlinear Schrödinger equation typically describes the behavior of relatively long modulations imposed on fully dispersive wave trains. One of the first applications in fluid mechanics is due to Hasimoto and Ono[7] who, in their study of surface gravity waves, have shown that this equation is capable of describing the Benjamin–Feir[8] or modulational stability of Stokes wave trains. Hasimoto and Ono[7] have also shown that this equation can describe the behavior of isolated wave packets. It was later shown by Zakharov and Shabat[9] that these wave packets were also soliton-like in the sense that two packets could collide, interact nonlinearly, and emerge unchanged except for a phase shift. Other solutions of interest include the simple and complex recurrence phenomena described numerically by Yuen and Ferguson[10] and analytically by Stiassnie and Korszynski.[11] One of the primary motivations for these studies has been to describe the behavior of water waves. However, a number of investigations have since shown that the nonlinear Schrödinger equation can be expected to arise in a wide variety of applications which are also of interest to the fluid mechanist. In addition to surface gravity waves these include internal gravity waves,[12] rotating flows and vortex dynamics,[13] nonlinear duct acoustics,[14] the dynamics of liquid sheets,[15] and the stability of plane Poiseuille flow.[16] For futher references and a review of solutions, extensions, and applications of this equation, we refer the reader to Yuen and Lake[17] and Lamb.[18]

As described in more detail in the next section, the nonlinear Schrödinger equation frequently is used to describe the evolution of small perturbations to constant-amplitude wave trains, e.g., Stokes wave trains in the theory of water waves. We have found that when such perturbations are neutrally stable and sufficiently long, the solutions to the Schrödinger equation may be approximated by those of the Korteweg–deVries equation. The work presented here establishes this connection between these two key equations independently through use of singular perturbation techniques and numerical computation. Thus, under the conditions stated, perturbations to a Stokes wave train in water of moderate depth can be described by the Korteweg–deVries equation; because of the generality of the result, similar remarks hold for other dispersive and weakly nonlinear systems, such as rotating flows, in which the Schrödinger equation governs the wave-train envelope. Furthermore, numerical solutions can be simplified due to the reduction of the number of partial differential equations to be solved from two to one.

II. LINEAR THEORY AND PROBLEM STATEMENT

In order to illustrate the generality of our result, we consider a Schrödinger equation of the form

$$A_t = (i\omega''/2)A_{XX} + iF(|A|)A\,, \tag{1}$$

where ω'' is a real constant, $|A| \equiv (A\bar{A})^{1/2}$, and $\bar{A}$ is the complex conjugate of A. In order that constant amplitude solutions exist, we will assume that F is a real function of its argument and in addition has a simple Taylor series of the form

$$F(\eta) = F(\eta_0) + (\eta - \eta_0)F'(\eta_0) + [(\eta - \eta_0)^2/2]F''(\eta_0) + o[(\eta - \eta_0)^2],$$

 0031-9171/84/040821-06$01.90

where primes denote differentiation. Equation (1) is recognized to be a generalization of the usual cubic Schrödinger equation used in many studies of wave-train stability; this is obtained by setting $F(\eta) \equiv \text{const} \times \eta^2$. In the most frequently encountered applications and, in terms of suitable nondimensional variables, the physical quantity of interest (e.g., in the theory of surface gravity waves, the disturbance elevation of the free surface) is given by

$$u = A(X,t)\exp i(k\bar{x} - \omega\bar{t}),$$

where k and ω are the wavenumber and frequency of the carrier wave. In the usual way ω is related to k through a dispersion relation and the quantity ω'' appearing in (1) is typically the second derivative of ω with respect to k. The variables X and t are so-called slow variables; in particular, t is typically a scaled version of the time $\bar{t}$ and X is a scaled version of $\bar{x} - \omega'\bar{t}$, where ω' is the group velocity associated with the carrier wave. For a review and derivation of equations of the form (1) in the context of specific physical systems, we refer the reader to Hasimoto and Ono,[7] Yuen and Lake,[17] and Lamb.[18]

In what follows it will be convenient to work with the set of real equations obtained by setting $A = (a/2)\exp i\beta$, where a and β are real functions of X and t; these equations read

$$\begin{aligned} &a_t + (\omega''/2)(a\beta_{XX} + 2\beta_X a_X) = 0, \\ &\beta_t = (\omega''/2a)(a_{XX} - a\beta_X^2) + F(a/2). \end{aligned} \tag{2}$$

A well-known solution to these equations is the constant-amplitude solution given by

$$a = a_0 \quad \text{and} \quad \beta = F(a_0/2)t, \tag{3}$$

where a_0 is a positive constant; this is recognized as the analog of the Stokes wave solution in the theory of surface gravity waves. The main objective of the present study is to describe the behavior of small disturbances superimposed on this constant-amplitude solution. If we assume that a and β are given by

$$a = a_0 + \tilde{a}e^{i(KX - \Omega t)},$$
$$\beta = F(a_0/2)t + \tilde{\beta}e^{i(KX - \Omega t)},$$

where $\tilde{a}$ and $\tilde{\beta}$ are small constants and K and Ω are the modulation wavenumber and frequency, the usual linearized analysis yields the following dispersion relation:

$$\left(\frac{\Omega}{K}\right)^2 = -\frac{\omega'' a_0}{4}F'\left(\frac{a_0}{2}\right)\left(1 - \frac{\omega'' K^2}{a_0 F'(a_0/2)}\right),$$

provided $F'(a_0/2) \neq 0$. It is clear that the modulation can become unstable if

$$\omega'' F'(a_0/2) > 0 \quad \text{and} \quad K^2 < a_0 F'(a_0/2)/\omega''.$$

These, of course, are essentially the modulational stability conditions derived by Hasimoto and Ono.[7] However, our interest is in the behavior of long waves propagating on the base state (3). In the limit as $K \to 0$, the wave speed becomes

$$\Omega/K \sim \pm C\,[1 + (\omega''^2/8C^2)K^2 + O(K^4)],$$

where

$$C \equiv [-(\omega'' a_0/4)F'(a_0/2)]^{1/2}. \tag{4}$$

In order that the modulations be wavelike, we will restrict our attention to the case where $\omega'' F'(a_0/2) < 0$; i.e., when the constant-amplitude solution (3) is stable to all modulations. Clearly, in this limit the waves are only weakly dispersive and we expect that the effects of dispersion will only be noticeable over relatively long times of the order K^{-2}. Inspection of the above dispersion relation indicates that these effects can be described by

$$a_t \pm C a_X = \pm(\omega''^2/8C)a_{XXX},$$

which is recognized as an Airy or linearized Korteweg–deVries equation.

The above linear solution is only valid for relatively short times. Over a longer time, the effects of dispersion and nonlinearity will need to be taken into account even at lowest order. Thus, in the next section, we will use the method of multiple scales to derive the equation governing long time behavior of long, small-amplitude waves on the constant-amplitude solution (3). In what follows, it will be useful to deal with the equations obtained by extracting the speed shift (3) from β. These are

$$\begin{aligned} &a_t + (\omega''/2)(ab_{XX} + 2b_X a_X) = 0, \\ &b_t = (\omega''/2a)(a_{XX} - ab_X^2) + F(a/2) - F(a_0/2), \end{aligned} \tag{5}$$

where $b \equiv \beta - F(a_0/2)t$.

III. DERIVATION OF THE KORTEWEG–DEVRIES EQUATION

As indicated in the linear theory, Eq. (5) admits both left- and right-moving waves. In order to focus on right-moving waves, we will transform to a coordinate system moving with the speed (4) and therefore replace X and t by

$$\chi \equiv (X - Ct)/L \quad \text{and} \quad \tau \equiv (C/L)t,$$

where L gives a measure of the length of the modulations. Furthermore, the dependent variables a and b will be replaced by

$$a \equiv a_0(1 + \Delta\hat{a}) \quad \text{and} \quad b \equiv (\Delta\, 4CL/\omega'')\hat{b},$$

where Δ gives a measure of the wave amplitude. Here, the scaling for b was suggested by a more detailed examination of the linear theory. Equations (5) now read

$$\begin{aligned} &\hat{a}_\tau - \hat{a}_\chi + 2\hat{b}_{\chi\chi} + 2\Delta(\hat{a}\hat{b}_{\chi\chi} + 2\hat{a}_\chi\hat{b}_\chi) = 0, \\ &\hat{b}_\tau - \hat{b}_\chi = -\frac{F[(a_0/2)(1 + \Delta\hat{a})] - F(a_0/2)}{a_0\Delta F'(a_0/2)} \\ &\qquad + 2\mu^2[\hat{a}_{\chi\chi}/(1 + \Delta\hat{a})] - 2\Delta\hat{b}_\chi^2, \end{aligned} \tag{6}$$

where, for the sake of convenience, we have defined $\mu = \omega''/4CL$.

The Korteweg–deVries equation is obtained by requiring that the waves be long, i.e., $L \to \infty$ or $\mu \to 0$, and of small amplitude, i.e., $\Delta \to 0$. In order that the effects of nonlinearity and dispersion be noticeable over the same scales, it is also necessary to require that $\Delta = O(\mu^2) = o(1)$; thus, we will set $\Delta = \gamma\mu^2$ in (6) and look for small μ solutions which are valid at times $\tau = O(\mu^{-2})$. The procedure to be employed is equivalent to the multiple-scale approach described in Chap. IV of Leibovich and Seebass.[19] Here we simply assume that, in the coordinate system chosen, $\hat{a}$ and $\hat{b}$ vary only slowly with respect to τ and therefore take small μ expansions in the form

$$\hat{a} \sim a_1(\chi, \hat{\tau}) + \mu^2 a_2(\chi, \hat{\tau}) + o(\mu^2),$$
$$\hat{b} \sim b_1(\chi, \hat{\tau}) + \mu^2 b_2(\chi, \hat{\tau}) + o(\mu^2),$$

where $\hat{\tau} \equiv \mu^2 \tau$. When these are substituted in (6) and coefficients of like powers of μ are equated, we find that a_1 and b_1 satisfy

$$a_{1\chi} = 2b_{1\chi\chi}, \qquad b_{1\chi} = \tfrac{1}{2} a_1, \tag{7}$$

and that a_2 and b_2 satisfy

$$2b_{2\chi\chi} - a_{2\chi} = -a_{1\hat{\tau}} - 2\gamma(a_1 b_{1\chi\chi} + 2a_{1\chi} b_{1\chi}),$$
$$a_2 - 2b_{2\chi} = -2b_{1\hat{\tau}} + 4a_{1\chi\chi} - 4\gamma b_{1\chi}^2 - (\gamma\Lambda/2)a_1^2,$$

where

$$\Lambda \equiv a_0 F''(a_0/2)/2F'(a_0/2).$$

Equation (7) is seen to provide a relatively simple relation between the amplitude and phase perturbations. When (7) is used to replace b_1 by a_1 in the equations for a_2 and b_2 we can show through straightforward manipulation that a_1 must satisfy

$$a_{1\hat{\tau}} - 2a_{1\chi\chi\chi} + [(5+\Lambda)/2]\gamma a_1 a_{1\chi} = 0. \tag{8}$$

Once (8) is solved for a_1, (7) may be integrated to find b_1. Thus, in this limit, wavelike solutions to (2) on (3) are governed by the Korteweg–deVries equation (8). Because of the sign of the dispersive term, any solitons must have negative amplitudes. A special case of interest is when

$$F(\eta) = \text{const} \times \eta^{2N},$$

where $N > 0$. In this case $\Lambda = 2N - 1$ and the coefficient of $\gamma a_1 a_{1\chi}$ becomes $2 + N$. By setting $N = 1$, we obtain the result for the cubic Schrödinger equation.

We can make a simple check on this solution by noting that, in the case of the cubic Schrödinger equation, an exact solution to (6) corresponding to a stationary wave is

$$\hat{a} = [G(\xi) - 1]/\gamma\mu^2,$$
$$\hat{b} = \pm(1/\gamma\mu)\arccos[G_0/G(\xi)], \quad \text{for } \xi \lessgtr 0,$$

where $\xi \equiv \chi - \sigma\mu^2\tau$, $G_0 \equiv 1 + \sigma\mu^2$, and

$$G(\xi) \equiv \left\{1 + (G_0^2 - 1)\operatorname{sech}^2\left[\left(\frac{1 - G_0^2}{16\mu^2}\right)^{1/2}\xi\right]\right\}^{1/2};$$

the parameter $\sigma < 0$ simply determines the speed of the wave. In the limit $\mu \to 0$ with all other parameters fixed this becomes

$$\hat{a} \sim (\sigma/\gamma)\operatorname{sech}^2[(-\sigma/8)^{1/2}\xi] + o(1),$$
$$\hat{b} \sim -[(-2\sigma)^{1/2}/\gamma]\tanh[(-\sigma/8)^{1/2}\xi] + o(1),$$

which is just the stationary wave solution to (7) and (8).

IV. NUMERICAL RESULTS

In this section, numerical solutions to (6) and (8) will be obtained. The nonlinear term was taken to be that of the cubic Schrödinger equation. As a result, (6) was written

$$\hat{a}_\tau = \hat{a}_\chi(1 - 4\gamma\mu^2\hat{b}_\chi) - 2(1 + \gamma\mu^2\hat{a})b_{\chi\chi},$$
$$\hat{b}_\tau = \hat{b}_\chi(1 - 2\gamma\mu^2\hat{b}_\chi) + \frac{2\mu^2}{1 + \gamma\mu^2\hat{a}}\hat{a}_{\chi\chi} - \frac{\hat{a}}{2}\left(1 + \frac{\gamma\mu^2\hat{a}}{2}\right). \tag{9}$$

Because $\Lambda = 1$ for this case, the Korteweg–deVries equation was written

$$\hat{a}_\tau - 2\mu^2\hat{a}_{\chi\chi\chi} + 3\gamma\mu^2\hat{a}\hat{a}_\chi = 0, \tag{10}$$

where the fact that $\hat{\tau} = \mu^2\tau$ and $a_1 \approx \hat{a}$ has been used. The initial conditions used were

$$\hat{a}(\chi,0) \equiv [G(\chi) - 1]/\gamma\mu^2,$$
$$\hat{b}(\chi,0) \equiv \pm(1/\gamma\mu\alpha)\arccos[G_0/G(\chi)],$$

where the upper and lower signs on $\hat{b}(\chi,0)$ are to be chosen when χ is less than and greater than zero, respectively. Here α is a positive constant and G_0 and $G(\chi)$ are given by

$$G_0 \equiv 1 + \gamma\mu^2,$$
$$G(\chi) \equiv \left\{1 + (G_0^2 - 1)\operatorname{sech}^2\left[\left(\frac{1 - G_0^2}{16\mu^2}\right)^{1/2}\alpha\chi\right]\right\}^{1/2}.$$

In addition, we seek solutions corresponding to isolated disturbances and the solutions are therefore required to satisfy

$$\hat{a} \to 0 \quad \text{and} \quad \hat{b} \to \mp(1/\gamma\mu\alpha)\arccos G_0, \tag{11}$$

as $\chi \to \pm\infty$ for all values of $\tau \geqslant 0$. The parameter α allows us to vary the width of the wave relative to the amplitude; when $\alpha = 1$ the wave is just the stationary wave solution discussed in the previous section.

The scheme chosen to solve (9) and (10) was an implicit finite difference scheme. In χ, second-order accurate central differences were used and, in time, a second-order accurate fully implicit Crank–Nicolson scheme was employed, see, e.g., Richtmyer and Morton.[20] This results in a relatively large system of nonlinear equations which have been solved by a least change secant update quasi-Newton algorithm based on a model trust region; in particular, the subroutine HYBRD from the Argonne Laboratory's MINPACK package was used. The Jacobian matrix corresponding to this system is banded and was approximated by the Curtis–Powell–Reid algorithm; see Moré, Garbow, and Hillstrom.[21] Because the nonlinear system was solved at small time intervals, good starting points for the quasi-Newton algorithm were easily obtained. If this had not been the case, a homotopy method as in Watson[22–24] would have had to be used.

For the cases described below, the χ and τ steps were taken to be 0.1 and 0.125, respectively. The χ interval was taken to be 10 which resulted in a system of equations of dimension 198; to compute the solution to $\tau = 10$, this required roughly five hours of CPU time on a VAX 11/780. To incorporate the boundary conditions necessary as $\chi \to \pm\infty$, we have required that (11) be satisfied at $\chi = \pm 5$. Although test runs for larger intervals were made to assess the effects of the finite interval, it was felt that the above was a reasonable balance between accuracy and computation time.

In Figs. 1 and 2 solutions corresponding to $\mu = 0.16$ and $\gamma = 16.8$ have been plotted. In Fig. 1, the parameter α was taken to be 1.0. The resultant exact solution to (9) is therefore a stationary wave which propagates without

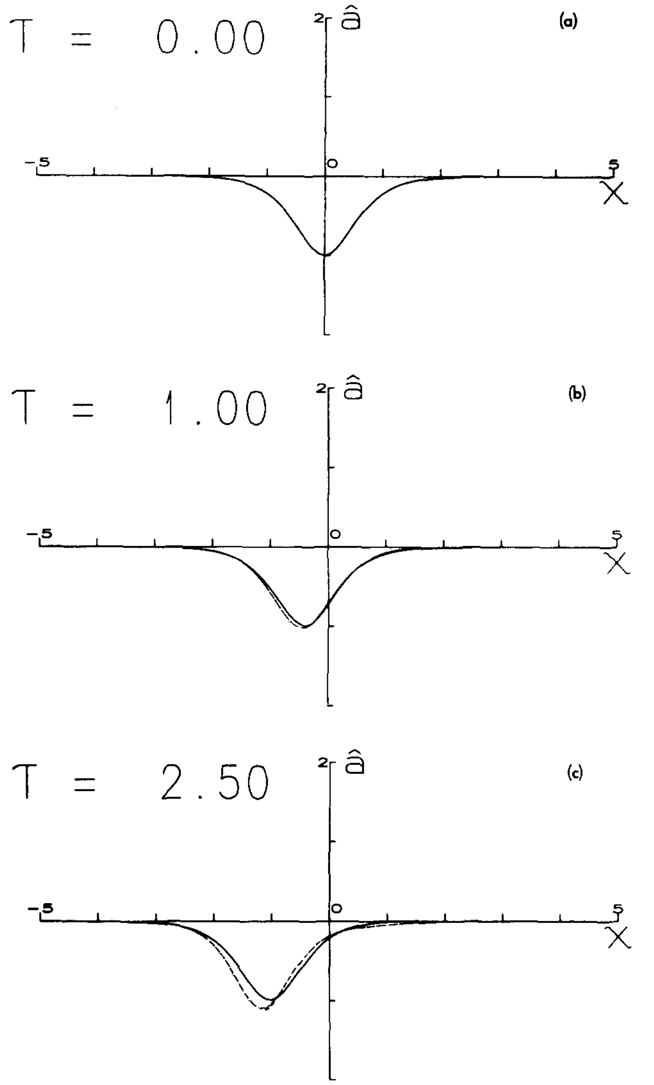

FIG. 1. Computed waveforms for $\mu = 0.16$, $\gamma = 16.8$, and $\alpha = 1.0$. Dark solid lines denote solutions to the Schrödinger equation (9) and dotted lines denote those of the Korteweg–deVries equation (10).

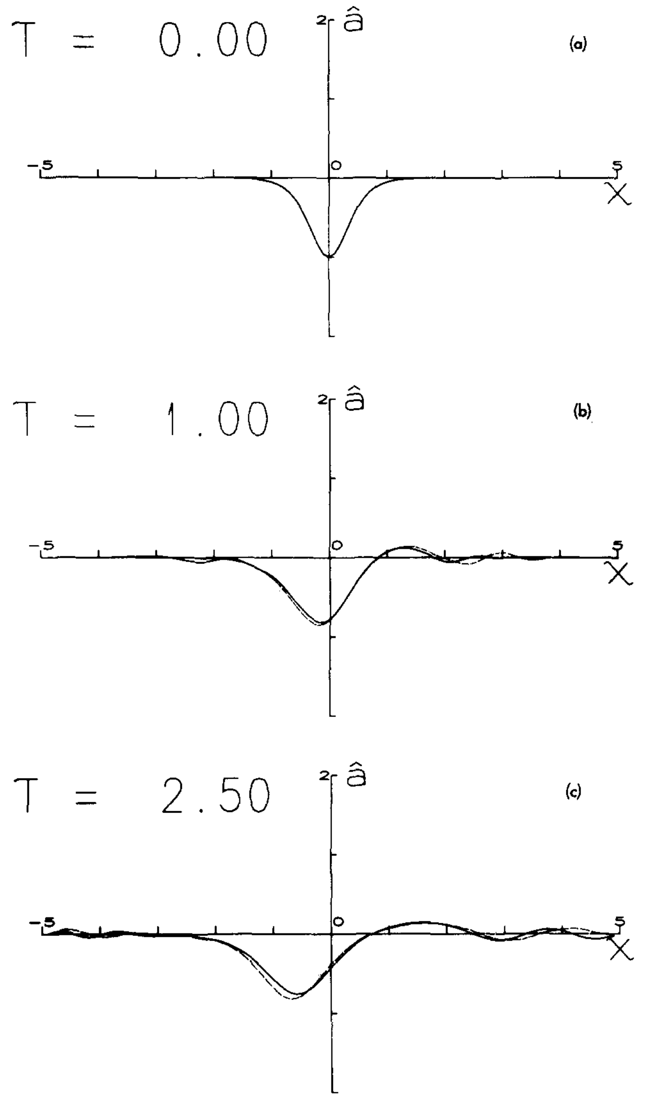

FIG. 2. Computed waveforms for $\mu = 0.16$, $\gamma = 16.8$, and $\alpha = 1.5$. Dark solid lines denote solutions to the Schrödinger equation (9) and dotted lines denote those of the Korteweg–deVries equation (10).

change in form and at a speed $\gamma\mu^2 \approx 0.43$; this is verified by the numerical solution. The main difference between the two solutions is expected to be due to the errors inherent in the asymptotic approximation. In Fig. 2, the value of α was increased to 1.5. The initial waveform then breaks up in the oscillatory manner depicted. By $\tau = 2.50$ the influence of the finite χ interval is felt over most of the wave. Even so, it is encouraging that the agreement is still reasonable.

In Figs. 3 and 4 we have plotted the computed solutions for the same value of γ (i.e., $\gamma = 16.8$) and $\mu = 0.08$. That is, we have halved μ while leaving all other parameters the same. To compare these results to those of Figs. 1 and 2, it is important to note that halving μ will slow down the evolution of the waveform by a factor of about 4. This is exactly true for the stationary wave solution to (9) corresponding to $\alpha = 1.0$, i.e., the wave speed $\gamma\mu^2$ decreases exactly by a factor of 4. For sufficiently small μ, the asymptotic theory along with Eq. (8) or (10) indicates this will also be approximately true for arbitrary initial conditions.

In Fig. 3, solutions corresponding to $\alpha = 1.0$ have been plotted for time $\tau = 0, 4, 10$. Due to the slowing of the wave speed, these can be compared directly to the results of Fig. 1. As expected, the decrease in μ results in better agreement for the same wave displacement. The computed waveforms for $\alpha = 1.5$ have been plotted in Fig. 4. If we again rescale the time by a factor of 4, a comparison with Fig. 2 further supports our claim that solutions to the Schrödinger equation will approach those of the Korteweg–deVries equation in the limit of vanishing μ.

In conclusion, our numerical computations show good agreement between the solutions to (9) and (10); this is true even when the resultant behavior is relatively complicated. Furthermore, these computations as well as others not presented verify that the agreement becomes better and better as μ becomes smaller and smaller.

Here we note that we have also attempted to compare our solutions when the initial conditions are such that wave fissioning occurs. However, for the examples considered, the

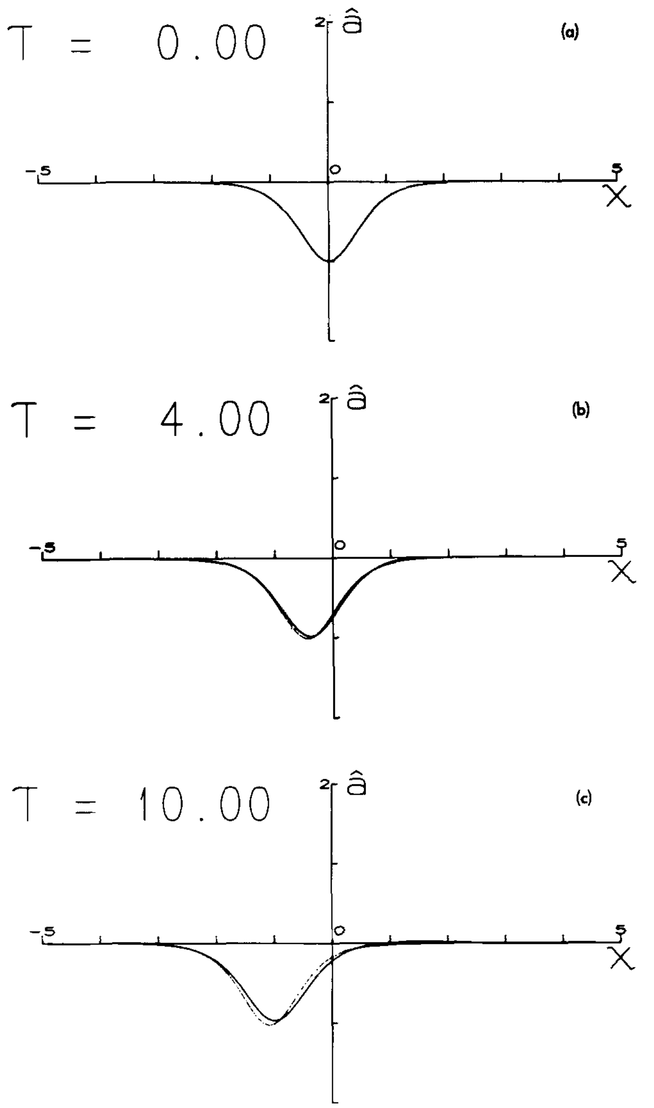

FIG. 3. Computed waveforms for $\mu = 0.08$, $\gamma = 16.8$, and $\alpha = 1.0$. Dark solid lines denote solutions to the Schrödinger equation (9) and dotted lines denote those of the Korteweg–deVries equation (10).

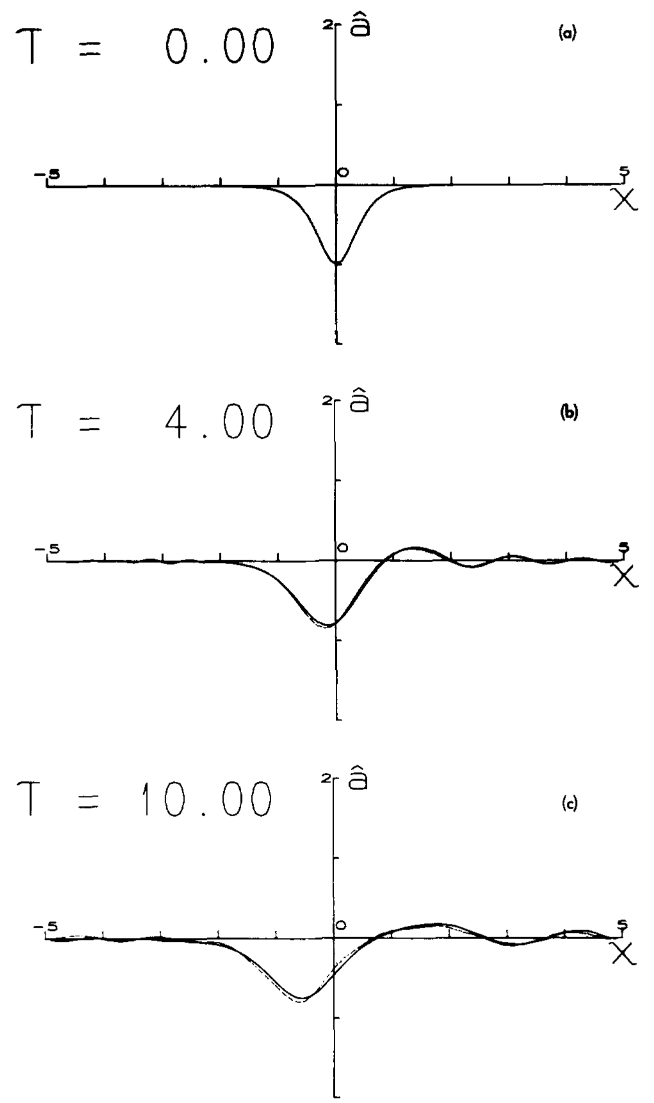

FIG. 4. Computed waveforms for $\mu = 0.08$, $\gamma = 16.8$, and $\alpha = 1.5$. Dark solid lines denote solutions to the Schrödinger equation (9) and dotted lines denote those of the Korteweg–deVries equation (10).

evolution process was relatively slow (numerically, not asymptotically) and it was found that smaller step sizes in the x direction were needed. Because of the high dimension of the system, the computation times necessary for accurate solutions became prohibitively high. Both collocation and Galerkin methods, although more complicated to implement on nonlinear problems, would probably have been more efficient for the present purposes.

V. DISCUSSION

The nonlinear Schrödinger equation (2) has been analyzed, and it has been shown through both analysis and numerical calculation that long, small-amplitude modulations to the constant-amplitude solution (3) are governed by the Korteweg–deVries equation (8). This result is only valid when ω'' and F are such that the phase speed (4) is real, i.e., when $\omega'' F'(a_0/2) < 0$. Thus, this must be distinguished from the studies described by Yuen and Lake,[17] where (3) is unstable to sufficiently long disturbances. Essentially, our results describe the evolution of modulations on unconditionally stable wave trains. Furthermore, our work must also be distinguished from studies which show that the modulational stability of wave-train solutions to the Korteweg–deVries equation can be described by a cubic Schrödinger equation. This fact was first pointed out by Hasimoto and Ono[7] and a detailed derivation is presented by Lamb.[18] In these studies the Korteweg–deVries equation governs the carrier wave and the Schrödinger equation governs the modulations, i.e., the envelope function A. This clearly contrasts with the present work in that we take our starting point to be the Schrödinger equation (1) and then show that certain solutions of (1) can be approximated by the Korteweg–deVries equation (8). Thus, the envelope function A rather than the complete wave train is governed by the Korteweg–deVries equation. This point becomes even more obvious if we note that highly dispersive systems such as surface waves in moderate depth can never be described by the Korteweg–deVries equation although wave-train modulations are frequently governed by (1). One of the contributions of our work is to show that, even in these cases, the envelope function is governed by (8).

In this paper our main interest is in describing the behavior of certain solutions to (1). However, as pointed out in Sec. I, the Schrödinger equation (1) is typically derived by approximating a given physical system and it is of interest to determine relevance of our solutions to such physical systems. The main question is whether higher-order corrections to (1) become necessary before the effects delineated here are seen. Examination of the time scales for the cubic Schrödinger equation indicates that the most severe constraint for the validity of our results is that

$$\epsilon^{1/3} \ll \mu \ll 1,$$

where ϵ gives a measure of the amplitude of the carrier wave. Thus, we expect that our results will also have application to such physical systems for a relatively wide range of values of μ.

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