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The thin-layer effect and interfacial stability in a two-layer Couette flow with similar liquids

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The linear stability of Couette flow composed of two layers of immiscible fluids, one lying on top of the other, is considered for the special case when the two fluids have similar mechanical properties. The interfacial eigenvalue is found in closed form by considering the two-fluid problem as a perturbation of the one-fluid problem. The importance of the role played by the viscosity difference, when one of the fluids is in a thin layer, is illustrated.

I. INTRODUCTION

There is a large number of examples of flows of two immiscible fluids in industry in which viscosity stratification has been observed to play an important role in determining the position of the interface.[1–9] In some of these practical examples, the densities of the fluids are different so that we need to examine the effects of both viscosity and density stratification. Theoretically, steady parallel shear flows often have many possible solutions. On the other hand, the arrangement with a thin layer of the less viscous fluid, say next to a wall, tends to be observed in experiments. We need to examine why such an arrangement is preferred. In earlier analyses of the linear stability of the interface in two-layer shearing flows, Yih[10] examined long waves for the case of viscosity stratification. His results show that the arrangement where the less viscous fluid lies in a relatively thin layer is stable to long waves. This result was, however, not emphasized at the time. Later, Hooper[11] examined the linear stability of a Couette flow with the upper fluid occupying a semi-infinite half-plane and the lower fluid occupying a finite strip. This analysis was also for long waves and showed stability if the lower fluid is the less viscous and instability otherwise, and this phenomenon was termed the "thin-layer" effect. Attention was not yet focused on the effect of density stratification. In addition, since an instability need not necessarily involve long waves, it became essential that all wavelengths be examined if we are to show that the thin-layer arrangement can be observed. Hence, numerical investigations of the linear stability of a number of two-layer shearing flows[12–14] were undertaken for all wavelengths. Results showed that the thin-layer arrangement can be stable not only to long waves but to order 1 waves as well. Since short waves at the interface are stabilized by surface tension,[15] it became plausible that there may be situations where the thin-layer arrangement is linearly stable at all wavelengths. We found one such situation numerically.[12] The Reynolds number for that situation is low and we believe that the linear stability of the thin-layer arrangement is a low Reynolds number phenomenon. On top of this, numerical work on the effect of density stratification[12,13] revealed that there are instances where the linear stability of a thin-layer arrangement is robust to an adverse density stratification. Hence, this raised the possibility that we may observe viscosity stratification counteracting what we expect density stratification to do. This result was met with a variety of reactions: at one extreme, that it is obvious, and at the other extreme, that it is unbelievable. It is actually not obvious because a heuristic reasoning for interfacial instability or stability could not be arrived at *a priori*. *A posteriori*, we may argue that when the density difference is small, gravity is a relatively small force, and although it tends to destabilize an arrangement with the heavier fluid on top, this destabilizing effect can be offset by choosing the viscosities so that stresses at the interface counter the effect of gravity. However, such a statement still does not give us a precise criterion for instability. Clearly, the study of such arrangements requires further analysis and further numerical work. In particular, why does the viscosity stratification play such a crucial role in the linear stability of thin layers?

In this article, we answer this for one type of shearing flow by obtaining the interfacial eigenvalue in closed form and investigating the thin-layer asymptotics of that eigenvalue. Along the way, we must compute some coefficients that involve integrals of products of Airy functions with exponential functions in our expansion. This does not detract from the fact that the work is basically an analytical and not a numerical investigation. We show that in the thin-layer limit, the effect of viscosity stratification appears at leading order. The main assumptions and steps of this article are sketched in more detail below.

We assume that the two fluids have mechanical properties that differ only by a small amount, say of $O(\epsilon)$, and the surface tension is also of $O(\epsilon)$. We choose this situation in order to take advantage of the fact that the eigenvector and eigenvalue of the basic unperturbed problem (the linear stability analysis for the case of identical fluids) and its adjoint are available in closed form. We note that for two-fluid flows in general, those quantities are not known in closed form but must be computed numerically. The case of "similar liquids" is an exception (for other examples, see Refs. 16 and 17). The calculation of the interfacial eigenvalue for the case of similar liquids for two-layer Couette flow involves a straightforward regular perturbation expansion of a boundary-value problem (operator plus boundary conditions). Hence, one can either assume an ansatz for the eigenvalue and eigenvector in a series in ϵ and use the Fredholm alternative theorem to obtain a solvability condition or, alternatively, to expand the resolvent of the differential operator as done in Refs. 16 and 17 and also in this article. Both methods are essentially equivalent and lead to the same amount of algebra.

After calculating the interfacial eigenvalue for the case of similar liquids, we take a "thin-layer limit" of it. This is different from the usual long-wave approximation. In a long-wave approximation, the disturbance wavelength needs to be long with respect to the plate separation. In our thin-layer approximation, the thickness of the thin layer must be small compared with other length scales. We have checked that in the long-wave limit, our interfacial eigenvalue is identical to Yih's[10] when his two fluids have similar properties. It is shown that the leading term in the growth rate (the real part of the eigenvalue) is proportional to the viscosity difference: it is negative (stabilizing) when the fluid in the thin layer is the less viscous, and positive otherwise. Terms containing the density difference and surface tension are found to be an order of magnitude less. This highlights the importance of the viscosity difference in the linear stability of thin layers (to disturbance wavelengths that are not too short).

II. GOVERNING EQUATIONS

Two fluids of viscosity μ_i, kinematic viscosity ν_i, and density ρ_i $(i = 1,2)$ lie between two rigid parallel boundaries of infinite extent in the (x^*,z^*) plane. Subscripts 1 or 2 on the physical properties of the fluids denote fluid 1 or 2, respectively. Fluid 1 occupies $0 \leqslant z^* \leqslant l_1^*$ and fluid 2 occupies $l_1^* \leqslant z \leqslant l^*$. Asterisks denote dimensional variables. The upper boundary at $z^* = l^*$ moves with velocity $(U^*,0)$. The lower boundary is fixed. The fluids are incompressible and satisfy the Navier–Stokes equations. At the interface, the velocity and shear stress must be continuous, the jump in the normal stress is balanced by surface tension and curvature, and the kinematic free surface condition must hold. We introduce the following dimensionless variables (without asterisks):

$$(x,z) = (x^*,z^*)/l^*, \quad \mathbf{u} = \mathbf{u}^*/U^*, \quad t = t^*U^*/l^*,$$
$$p = p^*/\rho_1 U^{*2}, \tag{1}$$

where $\mathbf{u}^*$ is the velocity and p^* is the pressure. There are six dimensionless parameters: $m = \mu_1/\mu_2$; $r = \rho_1/\rho_2$; $l_1 = l_1^*/l^*$; a Froude number F given by $F^2 = U^{*2}/gl^*$, where g denotes the gravitational acceleration constant; a surface tension parameter $S = S^*/(\mu_1 U^*)$, where S^* is the surface tension coefficient; and a Reynolds number $R = U^*l^*/\nu_1^*$. We denote $l_2 = 1 - l_1$. A steady shearing flow solution to the two-layer problem is given by a velocity $[U_1(z),0]$, a pressure P, and a flat interface at $z = l_1$, where

$$\begin{aligned} U_1(z) &= z/(l_1 + ml_2) \quad \text{for } 0 \leqslant z \leqslant l_1, \\ &= m(z-1)/(l_1 + ml_2) + 1 \quad \text{for } l_1 \leqslant z \leqslant 1, \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{\partial P}{\partial z} &= -\frac{1}{F^2} \quad \text{for } 0 \leqslant z \leqslant l_1, \\ &= -1/rF^2 \quad \text{for } l_1 \leqslant z \leqslant 1. \end{aligned} \tag{3}$$

We add small disturbances, $\mathbf{u} = (u,v)$ to the velocity and h to the interface position, that are taken to be proportional to $\exp(i\alpha x + \sigma t)$. The resulting equations governing linear stability are in fluid i,

$$\frac{1}{R_i}\Delta u - \frac{\rho_1}{\rho_i}\frac{\partial p}{\partial x} - vU_{1z}(z) - U_1(z)i\alpha u = \sigma u, \tag{4}$$

$$\frac{1}{R_i}\Delta v - \frac{\rho_1}{\rho_i}\frac{\partial p}{\partial z} - i\alpha v U_1(z) = \sigma v, \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{6}$$

where $R_i = U^*l^*/\nu_i$. The Froude number does not enter into these equations since the pressure gradient in the basic flow field cancels the effect of gravity.

The interface conditions, linearized at $z = l_1$, yield

kinematic free surface condition: $v - hi\alpha U_1(l_1) = h\sigma$, (7)

continuity of velocity: $[\![v]\!] = 0$, (8)

$$[\![u]\!] + h[\![U_{1z}(l_1)]\!] = 0, \tag{9}$$

continuity of shear stress: $\left[\!\!\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right)\right]\!\!\right] = 0,$ (10)

balance of normal stress:

$$\left[\!\!\left[-p + \frac{\mu}{\mu_1}\frac{2}{R}\frac{\partial v}{\partial z}\right]\!\!\right] = h\left(-\frac{\alpha^2 S}{R} + \frac{1}{F^2}\left(\frac{1}{r} - 1\right)\right). \tag{11}$$

The boundary conditions are

$$u = 0, \quad v = 0 \quad \text{at } z = 0,1. \tag{12}$$

Here, $[\![\circ]\!]$ denotes the jump of $\circ$ across the interface, or $\circ_1 - \circ_2$. For the one-fluid problem ($r = 1$, $m = 1$, $S = 0$) with no interface at $z = l_1$, Romanov[18] has proved that the real parts of all eigenvalues σ are negative at any Reynolds number and wavenumber α. The addition of an interface at $z = l_1$ to the linear stability problem gives rise to an eigenvalue that was referred to as the "interfacial mode" by Yih.[10] This follows from the observation that if the two fluids have identical viscosity and density, and if there is no surface tension, then

$$u = 0, \quad v = 0, \quad h = \exp(i\alpha x + \sigma t), \quad \sigma = -i\alpha l_1 \tag{13}$$

satisfy Eqs. (4)–(12).

In the following analysis, we let the two fluids have mechanical properties that differ only slightly. We keep R and α fixed but arbitrary. We introduce a small parameter ϵ and regard $1 - m$, $1 - r$, and S as small quantities proportional to ϵ; that is, we set

$$1 - m = \overline{m}\epsilon, \quad 1 - r = \overline{r}\epsilon, \quad S = \overline{S}\epsilon.$$

Hence, at $\epsilon = 0$, there is an algebraically onefold and geometrically simple eigenvalue $\sigma = -i\alpha l_1$ arising from the presence of the interface. We can make an ansatz for the eigenvalue and for the velocity as power series in ϵ. The purpose of the following analysis is to find the coefficient of ϵ in the expansion for σ.

III. PERTURBATION OF NEUTRAL EIGENVALUE

The method of perturbing a simple eigenvalue[19] involves evaluating the eigenfunction belonging to the eigenvalue $\sigma = -i\alpha l_1$ for the unperturbed problem and its adjoint. The method will be explained in detail below in the present context to avoid confusion.

The relevant results from Ref. 19 are quoted below in the context of the perturbation of a finite-dimensional matrix equation. Some modifications will then be made to apply the method to our differential operator. Suppose σ_0 is an algebraically simple eigenvalue of a matrix L_0. Let A be an eigenfunction of L_0 with eigenvalue σ_0, and let C be an eigenfunction of L_0^* (the adjoint of L_0) with eigenvalue $\bar{\sigma}_0$ (the overbar here denotes the complex conjugate). Let L_0 be perturbed into $L(\epsilon) = L_0 + \epsilon L_1 + L_2(\epsilon)$ with $L_2 = O(\epsilon^2)$ and $\epsilon L_1 + L_2$ depending smoothly on ϵ. Then the perturbed eigenvalue σ is analytic in ϵ and is given by setting the following expression $\Psi(\epsilon,\sigma)$, which represents to $O(\epsilon)$ the projection of $L(\epsilon) - \sigma$, first onto the eigenspace of the unperturbed problem and then onto the adjoint eigenspace, to zero:

$$\Psi(\epsilon,\sigma) = \langle C,(L_0 + \epsilon L_1 - \sigma)A \rangle + O(\epsilon^2) . \tag{14}$$

Care must be taken when this result is applied to unbounded operators in infinite-dimensional spaces, for example, differential operators. Such an operator has a "domain" that is specified not only by smoothness requirements on the function but also by the boundary conditions. If the domain of the operator that is being perturbed depends on ϵ, we cannot apply (14); the domains of $L(\epsilon)$ and L_0 may be different, and their combination would not make sense. We can, however, circumvent this problem by not looking at the differential operator itself, but at its resolvent $[L(\epsilon) - \lambda \mathbf{I}]^{-1}$, where λ is not an eigenvalue of $L(\epsilon)$. The domain of this does not depend on ϵ and we will need to redefine Ψ accordingly, i.e., replace $L_0 + \epsilon L_1$ in (14) by the expansion for the resolvent. As will be seen from the following, the resolvent itself does not ever need to be computed.

Let X denote the set of functions (u,v,h). We introduce an inner product by

$$\langle X_1,X_2\rangle = \int_0^{2\pi/\alpha} \int_{z=0}^{l_1} \bar{u}_1 u_2 + \bar{v}_1 v_2 \, dz\, dx + \int_0^{2\pi/\alpha} \int_{z=l_1}^{1} \bar{u}_1 u_2 + \bar{v}_1 v_2 \, dz\, dx + \int_0^{2\pi/\alpha} \bar{h}_1 h_2 \, dx \tag{15}$$

to generate a Hilbert space. In this Hilbert space, we consider the subspace determined by the "Hodge projection" (see space H in Theorem 1.4, Ref. 20), that is, by the conditions that the velocity field be divergence-free, that the vertical velocity vanish at the walls, and be continuous across the interface. By $L(\epsilon)X$ we denote the left-hand sides of Eqs. (4), (5), and (7). We regard $L(\epsilon)$ as an operator in the subspace so that the conditions on v in (8) and (12) and the normal stress balance in (11) are an integral part of the definition of $L(\epsilon)$. The domain of definition of $L(\epsilon)$ is determined by the rest of the boundary conditions in (9), (10), and (12), which we write in the form $B(\epsilon)X = 0$. The range of the operator $L(\epsilon)$ must satisfy the following conditions in order for the pressure p occurring on the left sides of (4) and (5) to be determined as a function of X: The "velocity part" of $L(\epsilon)X$ must be divergence-free, the vertical velocity must vanish on the walls and be continuous across the interface, and the jump in p across the interface must be given by the normal stress balance. Thus, the problem we wish to solve is as follows: for small ϵ, find σ satisfying

$$L(\epsilon)X = \sigma X, \quad B(\epsilon)X = 0, \tag{16}$$
$$L(\epsilon) = L_0 + \epsilon L_1 + O(\epsilon^2), \quad B(\epsilon) = B_0 + \epsilon B_1 + O(\epsilon^2).$$

Explicitly,

$$L_0 X = \begin{pmatrix} \frac{1}{R}\Delta u - \frac{\partial p}{\partial x} - v - i\alpha u z \\ \frac{1}{R}\Delta v - \frac{\partial p}{\partial z} - i\alpha v z \\ v - hi\alpha l_1 \end{pmatrix} \text{ in fluids 1 and 2,}$$

and

$$L_1 X = \begin{pmatrix} -v\bar{m}l_2 - i\alpha u z \bar{m} l_2 - \frac{\partial \tilde{p}}{\partial x} \\ -i\alpha v \bar{m} l_2 z - \frac{\partial \tilde{p}}{\partial z} \\ -hi\alpha \bar{m} l_1 l_2 \end{pmatrix} \text{ in fluid 1,}$$

and

$$\begin{pmatrix} \frac{1}{R}(\bar{m} - \bar{r})\Delta u - \frac{\partial \tilde{p}}{\partial x} + v l_1 \bar{m} + i\alpha u l_1 (z-1)\bar{m} + \bar{r}\frac{\partial p}{\partial x} \\ \frac{1}{R}(\bar{m} - \bar{r})\Delta v - \frac{\partial \tilde{p}}{\partial z} + i\alpha v l_1 (z-1)\bar{m} + \bar{r}\frac{\partial p}{\partial z} \\ -hi\alpha l_1 l_2 \bar{m} \end{pmatrix} \text{ in fluid 2,}$$

where $\tilde{p}$ denotes the $O(\epsilon)$ perturbation to the pressure; i.e., at $\epsilon = 0$,

$$\left[\!\!\left[-p + \frac{2}{R}\frac{\partial v}{\partial z} \right]\!\!\right] = 0, \tag{17}$$

and for $\epsilon \neq 0$,

$$[\![-\tilde{p}]\!] - \frac{2\bar{m}}{R}\frac{\partial v_2}{\partial z} = h\left(-\alpha^2 \frac{\bar{S}}{R} + \frac{\bar{r}}{F^2} \right),$$

$$B_0 X = \begin{pmatrix} u_1 - u_2 \text{ at } z = l_1 \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z} \text{ at } z = l_1 \\ u \text{ at } z = 0,1 \end{pmatrix},$$

and

$$B_1 X = \begin{pmatrix} \bar{m}h \text{ at } z = l_1 \\ -\bar{m}\left(\frac{\partial u_1}{\partial z} + \frac{\partial v_1}{\partial x} \right) \text{ at } z = l_1 \\ 0 \text{ at } z = 0,1 \end{pmatrix}.$$

With the above definitions, we are now ready to look at the resolvent of $L(\epsilon)$ and then to redefine Ψ given in (14). Since ϵ is small, the eigenvalues of $L(\epsilon)$ are close to those of L_0, so that the λ in the resolvent should be chosen away from its eigenvalues, such as $-i\alpha l_1$. We choose $\lambda = 0$. Hence, instead of looking at (16) directly, we study the equivalent problem

$$L(\epsilon)^{-1} X = \sigma^{-1} X = : \hat{\sigma} X$$

and perturb around $\hat{\sigma} = -1/(i\alpha l_1)$. We note that the definition of $L(\epsilon)^{-1}$ already incorporates the boundary conditions. The expression (14) is applied to this problem; thus

$$\Psi(\epsilon,\hat{\sigma}) = \langle C,[L(\epsilon)^{-1} - \hat{\sigma}]A \rangle + O(\epsilon^2), \tag{18}$$

where the C and A are as before, is set equal to zero. We will require an expansion of the resolvent in powers of ϵ in order to carry out the calculation of Ψ. We note that the inverse of L_0 is defined. Parts of the calculation are organized into appendices of this paper.

We first have to find the boundary value problem adjoint to (16). This is done in Appendix A. Then we determine the eigenvector at $\epsilon = 0$ for both (16) and the adjoint problem. These are denoted by A and C, respectively, and satisfy

$$\begin{aligned} &L_0 A = -i\alpha U_1(l_1)A, \quad B_0 A = 0,\\ &L_0^* C = i\alpha U_1(l_1)C, \quad B_0^* C = 0, \end{aligned} \tag{19}$$

where $U_1(l_1) = l_1$ for $\epsilon = 0$. These eigenfunctions are determined in Appendix B. In order to apply formula (18), we must determine the expressions

$$\langle C,L(\epsilon)^{-1}A \rangle \tag{20}$$

to first order in ϵ. To facilitate this calculation, we introduce x^0 and x^1 defined by

$$L(\epsilon)^{-1}A = x^0 + \epsilon x^1 + O(\epsilon^2). \tag{21}$$

Equating the coefficients of equal powers of ϵ, we find the equations governing x^0 and x^1

$$L_0 x^0 = A, \quad B_0 x^0 = 0, \tag{22}$$

and

$$L_1 x^0 + L_0 x^1 = 0, \quad B_1 x^0 + B_0 x^1 = 0. \tag{23}$$

From (19), we find $x^0 = -A/i\alpha l_1$. We will not need the solutions x^1 to the perturbation problem (23) but only the inner product $\langle C,x^1\rangle$. This is seen from (20) and (21):

$$\Psi(\epsilon,\hat{\sigma}) = \langle C,x^0\rangle + \epsilon\langle C,x^1\rangle - \hat{\sigma}\langle C,A \rangle + O(\epsilon^2). \tag{24}$$

We calculate $\langle C,x^1\rangle$ from (23) and an integration by parts:

$$\begin{aligned} -\langle C,L_1 x^0\rangle &= \langle C,L_0 x^1\rangle\\ &= \langle L_0^* C,x^1\rangle + \text{boundary integrals}, \end{aligned} \tag{25}$$

where the boundary integrals are evaluated using the second part of (23), and $\langle L_0^* C,x^1\rangle = \langle i\alpha l_1 C,x^1\rangle = -i\alpha l_1\langle C,x^1\rangle$. (The boundary integrals would vanish if $B_0 x^1$ were zero.) Details of these calculations are in Appendix C.

IV. DISCUSSION OF RESULTS

Using Eqs. (C1)–(C7) of Appendix C, we have

$$\begin{aligned} \sigma \sim &-i\alpha l_1 - i\epsilon\alpha l_1 l_2 \overline{m}\\ &+ \frac{\epsilon}{e^{-\alpha l_1}(\overline{c}_4 - \overline{d}_4)}\Bigg\{ -\frac{1}{2}\, i\overline{m}(e^{\alpha l_1}\overline{d}_3 + e^{-\alpha l_1}\overline{d}_4)\\ &+ \left[\frac{\alpha^{1/3}R^{1/3}}{2}\left(\frac{\alpha^2\overline{S}}{R} - \frac{\overline{r}}{F^2}\right)\right.\\ &\left.+ \frac{i\overline{m}\alpha^{4/3}}{R^{2/3}}\right](\overline{d}_3\overline{V}_3 + \overline{d}_4\overline{V}_4)\Bigg\} + O(\epsilon^2), \end{aligned} \tag{26}$$

where the coefficients $\overline{d}_3$, $\overline{d}_4$, and $\overline{c}_4$ are complex conjugates of those defined by (B22)–(B24) of Appendix B, and $\overline{V}_3$ and $\overline{V}_4$ are complex conjugates of $V_3(s_L)$ and $V_4(s_L)$, defined by (B14) and (B15).

We first show that $\overline{c}_4 - \overline{d}_4$, occurring in the denominator of (26), is not zero. From (B22) and (B23),

$$\begin{aligned} c_4 - d_4 &= (V_4 - V_2)(V_1' - V_3') + (V_2' - V_4')(V_1 - V_3)\\ &= \pi\bigg(\int_{s_0}^{s_1} \mathrm{Bi}(t)\exp(i\alpha^{2/3}R^{-1/3}t)dt\\ &\quad\times\int_{s_0}^{s_1} \mathrm{Ai}(t)\exp(-i\alpha^{2/3}R^{-1/3}t)dt\\ &\quad-\int_{s_0}^{s_1} \mathrm{Bi}(t)\exp(-i\alpha^{2/3}R^{-1/3}t)dt\\ &\quad\times\int_{s_0}^{s_1} \mathrm{Ai}(t)\exp(i\alpha^{2/3}R^{-1/3}t)dt\bigg). \end{aligned}$$

We change variables from t to θ:

$$t = s_L\{1 + i(R/\alpha)[-l_2 + \tfrac{1}{2}(\theta + 1)]\},$$

so that

$$c_4 - d_4 = -\tfrac{1}{4}\alpha^{2/3}R^{2/3}\pi\Lambda,$$

where

$$\begin{aligned} \Lambda = &\int_{-1}^{1} \mathrm{Bi}(t(\theta))e^{\alpha\theta/2}\,d\theta\int_{-1}^{1} \mathrm{Ai}(t(\theta))e^{-\alpha\theta/2}\,d\theta\\ &-\int_{-1}^{1} \mathrm{Bi}(t(\theta))e^{-\alpha\theta/2}\,d\theta\int_{-1}^{1} \mathrm{Ai}(t(\theta))e^{\alpha\theta/2}\,d\theta. \end{aligned} \tag{27}$$

We define $\alpha^* = \alpha/2$, $t = e^{i\pi/6}z$, and use the identity[21]

$$\mathrm{Bi}(t) = 2e^{i\pi/6}\mathrm{Ai}(te^{2\pi i/3}) - e^{i\pi/2}\,\mathrm{Ai}(t)$$

to obtain

$$\begin{aligned} \Lambda &= 2e^{i\pi/6}\bigg(\int_{-1}^{1} \mathrm{Ai}(e^{5\pi i/6}z(\theta))e^{\alpha^*\theta}\,d\theta\\ &\quad\times\int_{-1}^{1} \mathrm{Ai}(e^{i\pi/6}z(\theta))e^{-\alpha^*\theta}\,d\theta\\ &\quad-\int_{-1}^{1} \mathrm{Ai}(e^{5\pi i/6}z(\theta))e^{-\alpha^*\theta}\,d\theta\\ &\quad\times\int_{-1}^{1} \mathrm{Ai}(e^{i\pi/6}z(\theta))e^{\alpha^*\theta}\,d\theta\bigg)\\ &= 2i\bigg(\int_{-1}^{1} \mathrm{Ai}(e^{5\pi i/6}z(\theta))e^{\alpha^*\theta}\,d\theta\\ &\quad\times\int_{-1}^{1} \mathrm{Ai}(-iz(\theta))e^{-\alpha^*\theta}\,d\theta\\ &\quad-\int_{-1}^{1} \mathrm{Ai}(e^{5\pi i/6}z(\theta))e^{-\alpha^*\theta}\,d\theta\\ &\quad\times\int_{-1}^{1} \mathrm{Ai}(-iz(\theta))e^{\alpha^*\theta}\,d\theta\bigg) \end{aligned} \tag{28}$$

by using formula 10.4.7 in Ref. 21. The above is equal to $-2i$ multiplied by Romanov's Δ (see Eq. 1.15 in Ref. 18), where his $e^{i\pi/6}[(x - id)/\epsilon]$ is our $e^{5\pi i/6}z$. Romanov's α, x, c, $-id$, and ϵ correspond to our α^*, θ, $2l_2 - 1$, $1 - 2l_2 - 4i\alpha^*/R$, and $2^{2/3}(\alpha^* R)^{-1/3}$, respectively. Romanov showed (see Lemma 4 in Ref. 18) that $\Delta \neq 0$ if c is real. Therefore, $c_4 - d_4 \neq 0$.

We now discuss the thin-layer effect. We examine the asymptotic behavior of σ given by (26) in the limit of $l_1 \to 0$. We show below that in (26), the coefficient of $\overline{m}/(\overline{c}_4 - \overline{d}_4)$ is $O(l_1^2)$ and the coefficient of $\overline{r}/(\overline{c}_4 - \overline{d}_4)$ is $O(l_1^3)$.

In the following, we evaluate various terms at $l_1 = 0$. In Eq. (B9),

$$s_0 = s_L = -\alpha^{4/3} R^{-2/3}, \quad s_1 = s_L + i\alpha^{1/3} R^{1/3}, \qquad (29)$$

and in (B17) and (B18),

$$V_1(s_L) = V_2(s_L) = V_1'(s_L) = V_2'(s_L) = 0, \qquad (30)$$

where $'$ denotes $\partial/\partial s$, so that

$$d_3 = d_4 = 0 \qquad (31)$$

in Eqs. (B23) and (B24). Hence, we find that the $O(\epsilon)$ terms in σ vanish when $l_1 = 0$. Next,

$$\frac{\partial V_1}{\partial l_1} = \frac{\partial V_2}{\partial l_1} = 0, \qquad (32)$$

$$\frac{\partial V_1'}{\partial l_1} = \frac{\partial V_2'}{\partial l_1} = i\alpha^{1/3} R^{1/3}, \qquad (33)$$

so that

$$\frac{\partial d_3}{\partial l_1} = \frac{\partial d_4}{\partial l_1} = 0 \qquad (34)$$

so that the $O(\epsilon)$ terms vanish also at $O(l_1)$.

We find that at $l_1 = 0$,

$$\frac{\partial^2 \sigma}{\partial l_1^2} = 2i\epsilon\alpha\overline{m} + \frac{\epsilon}{(\overline{c} - \overline{d}_4)} \left\{ -\frac{1}{2} i\overline{m} \left(\frac{\partial^2 \overline{d}_3}{\partial l_1^2} + \frac{\partial^2 \overline{d}_4}{\partial l_1^2} \right) + \left[\frac{\alpha^{1/3} R^{1/3}}{2} \left(\frac{\alpha^2 \overline{S}}{R} - \frac{\overline{r}}{F^2} \right) + \frac{i\overline{m}\alpha^{4/3}}{R^{2/3}} \right] \times \left(\overline{V}_3 \frac{\partial^2 \overline{d}_3}{\partial l_1^2} + \overline{V}_4 \frac{\partial^2 \overline{d}_4}{\partial l_1^2} \right) \right\} + O(\epsilon^2). \qquad (35)$$

Since

$$\frac{\partial^2 V_1}{\partial l_1^2} = \frac{\partial^2 V_2}{\partial l_1^2} = -\alpha^{2/3} R^{2/3} \qquad (36)$$

and

$$\frac{\partial^2 V_1'}{\partial l_1^2} = i\alpha^{4/3} R^{1/3} = -\frac{\partial^2 V_2'}{\partial l_1^2}, \qquad (37)$$

at $l_1 = 0$, we find that

$$\frac{\partial^2 d_3}{\partial l_1^2} = -2i\alpha^{4/3} R^{1/3} V_4, \quad \frac{\partial^2 d_4}{\partial l_1^2} = 2i\alpha^{4/3} R^{1/3} V_3. \qquad (38)$$

Hence in Eq. (35),

$$V_3 \frac{\partial^2 d_3}{\partial l_1^2} + V_4 \frac{\partial^2 d_4}{\partial l_1^2} = 0$$

and

$$\frac{\partial^2 d_3}{\partial l_1^2} + \frac{\partial^2 d_4}{\partial l_1^2} = 2i\alpha^{4/3} R^{1/3} [V_3(l_1 = 0) - V_4(l_1 = 0)]. \qquad (39)$$

Therefore, the coefficient of $\overline{m}/(\overline{c}_4 - \overline{d}_4)$ in σ is $O(l_1^2)$.

Let $f(l_1) = d_3 V_3 + d_4 V_4$. Then

$$\frac{\partial^3 f}{\partial l_1^3}(0) = 6i\alpha^{4/3} R^{1/3} \left(V_3 \frac{\partial V_4}{\partial l_1} - V_4 \frac{\partial V_3}{\partial l_1} \right) + V_3 \frac{\partial^3 d_3}{\partial l_1^3} + V_4 \frac{\partial^3 d_4}{\partial l_1^3}, \qquad (40)$$

where

$$\frac{\partial^3 d_3}{\partial l_1^3}(l_1 = 0) = V_4' \frac{\partial^3 V_2}{\partial l_1^3} - V_4 \frac{\partial^3 V_2'}{\partial l_1^3} - 6\alpha^{5/3} R^{2/3} V_4' + 6i\alpha^{7/3} R^{1/3} V_4 - V_4' \frac{\partial^3 V_1}{\partial l_1^3} + V_4 \frac{\partial^3 V_1'}{\partial l_1^3} - 6i\alpha^{4/3} R^{1/3} \frac{\partial V_4}{\partial l_1}, \qquad (41)$$

$$\frac{\partial^3 d_4}{\partial l_1^3}(l_1 = 0) = V_3' \frac{\partial^3 V_1}{\partial l_1^3} - V_3 \frac{\partial^3 V_1'}{\partial l_1^3} + 6\alpha^{5/3} R^{2/3} V_3' + 6i\alpha^{4/3} R^{1/3} \frac{\partial V_3}{\partial l_1} - 6i\alpha^{7/3} R^{1/3} V_3 + V_3 \frac{\partial^3 V_2'}{\partial l_1^3} - V_3' \frac{\partial^3 V_2}{\partial l_1^3}, \qquad (42)$$

$$\frac{\partial^3 V_1}{\partial l_1^3} = -\alpha^{5/3} R^{2/3} = -\frac{\partial^3 V_2}{\partial l_1^3}. \qquad (43)$$

Hence, Eq. (40) becomes

$$\frac{\partial^3 f}{\partial l_1^3}(l_1 = 0) = 4\alpha^{5/3} R^{2/3} (V_4 V_3' - V_3 V_4'). \qquad (44)$$

We note that when $l_1 = 0$,

$$c_4 - d_4 = V_4' V_3 - V_4 V_3' \neq 0. \qquad (45)$$

Therefore, the coefficient of $\overline{r}/(\overline{c}_4 - \overline{d}_4)$ is $O(l_1^3)$. As $l_1 \to 0$, (26) becomes

$$\sigma \sim -i\alpha l_1 - i\epsilon\alpha l_1 l_2 \overline{m} + \epsilon \left[\overline{m}\alpha^{4/3} R^{1/3} \frac{(\overline{V}_4 - \overline{V}_3)}{2(\overline{c}_4 - \overline{d}_4)} \times [l_1^2 + O(l_1^3)] - \frac{\alpha^2 R}{3} \left(\frac{\alpha^2 \overline{S}}{R} - \frac{\overline{r}}{F^2} \right) \times [l_1^3 + O(l_1^4)] \right] + O(\epsilon^2), \qquad (46)$$

where

$$V_4 - V_3 = i\pi\alpha^{1/3} R^{1/3} \times \int_{-1}^{1} (\mathrm{Ai}(s_L)\mathrm{Bi}(t(\theta)) - \mathrm{Bi}(s_L)\mathrm{Ai}(t(\theta))) \times \sinh \frac{\alpha(\theta - 1)}{2} \, d\theta,$$

$$t(\theta) = -\alpha^{4/3} R^{-2/3} - i\alpha^{1/3} R^{1/3} (\theta - 1)/2,$$

$$c_4 - d_4 = \frac{-\pi\alpha^{2/3} R^{2/3}}{4} \left(\int_{-1}^{1} \mathrm{Bi}(t(\theta)) e^{\alpha\theta/2} \, d\theta \times \int_{-1}^{1} \mathrm{Ai}(t(\theta)) e^{-\alpha\theta/2} \, d\theta - \int_{-1}^{1} \mathrm{Ai}(t(\theta)) e^{\alpha\theta/2} \, d\theta \times \int_{-1}^{1} \mathrm{Bi}(t(\theta)) e^{-\alpha\theta/2} \, d\theta \right).$$

We have checked that for long waves ($\alpha \to 0$), the coefficient of $\overline{m}$ in σ at $O(l_1^2)$ and the coefficient of $\overline{r}$ in σ at $O(l_1^3)$ agree with the limit $\epsilon \to 0$, $l_1 \to 0$ of the long-wave analysis of Yih[10] (see Appendix D).

Equation (46) presents the main result of our thin-layer analysis. It shows that the dominant term in the real part of σ is $O(l_1^2)$ and is proportional to the viscosity difference. We need to find out the sign of this dominant term in order to make conclusions about stability. This is done below.

The Airy functions of complex arguments were computed using the method of Schulten, Anderson, and Gordon.[22] Our computer program was checked against the thin-layer limit of Yih's long-wave formula [see Eq. (D2) of Appendix D] and also against the exact formulation of σ for the two-layer Couette flow without the thin-layer, small-ϵ, or long-wave approximations, for which a code is available.[12] For example, at $\alpha = 0.1$, $m = 0.99$, $l_1 = 0.05$, $R = 1.0$, $\overline{S} = \overline{r} = 0$, (46) yields σ as $(-0.4168E-8, -0.5043E-2)$, (D2) yields $(-0.4167E-8, -0.5042E-2)$, and using 25 Chebyshev polynomials in each fluid yields $(-0.4357E-8, 0.5043E-2)$.

Figure 1 displays the coefficient of $\epsilon \overline{m} l_1^2$ in $re(\sigma)$ of (46) versus α for $\overline{S} = \overline{r} = 0$ for a variety of Reynolds numbers. This shows that when the less viscous fluid is in the thin layer ($\overline{m} > 0$), the flow is linearly stable to wavelengths that are not too short; otherwise ($\overline{m} < 0$) the flow is unstable.

In the following two paragraphs, we present some features of Eq. (46), which, at first glance, may appear inconsistent, but are not. In Appendix E, it is shown that the growth rate asymptotes to a constant as $\alpha \to \infty$. However, in the problem without the thin-layer approximation and without the restriction that the fluids be similar, it is known that[12,15]

$$\sigma \sim -i\alpha U_1(l_1) + \frac{Rm(1-m)(1-m^2/r)}{2(l_2+ml_2)^2\alpha^2(1+m)^2} - \frac{\alpha S}{2(1+m)} - \frac{mR(1-1/r)}{2(1+m)\alpha F^2} \quad \text{as } \alpha \to \infty, \quad (47)$$

so that the growth rate in the viscosity term is $O(1/\alpha^2)$ and proportional to ϵ^2. This difference is explained by noting that the large α limit of the thin-layer approximation is an intermediate limit in the sense that $1 \ll \alpha \ll l_1^{-1}$. Thus the two limits, $\alpha \to \infty$ and $l_1 \to 0$, need not be interchangeable. We remark that in the two-layer Bénard problem,[16] the eigenvalue tends to zero in either limit but at different rates. In a previous article on thin-layer effects,[12] the author showed from numerical studies that although there is a critical Reynolds number beyond which short-wave asymptotics for the unbounded problem of Ref. 15 approximates the bounded problem, there is another unstable regime that is missed out by the short-wave asymptotics. The limit analyzed in Appendix E is linearly stable and therefore not related to this band of instability. This band has recently been analyzed by Hooper[23] who considered the problem of Ref. 11 for all wavelengths. She identified three distinct forms of instability depending on the magnitude of two dimensionless parameters, β and $(\alpha R)^{1/3}$, where β is a wavenumber measured on a viscous length scale, α is a wavenumber measured on the

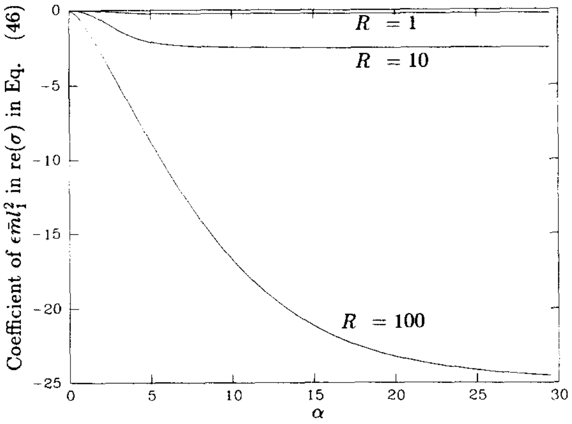

FIG. 1. The coefficient of $\epsilon \overline{m} l_1^2$ in the growth rate $re(\sigma)$ in the "thin-layer" limit given by Eq. (46) versus α. Surface tension and density difference are zero. Reynolds numbers are 1, 10, and 100. The growth rates asymptote to constants for short waves.

scale of the depth of the lower fluid, and R is the Reynolds number of the lower fluid. At small β and large $(\alpha R)^{1/3}$, she found a new type of instability that arose at the viscous boundary layer at the wall and this is an explanation for the band of instability that was numerically found in Ref. 12.

The analysis of Eq. (46) has shed further light on why we can obtain the thin-layer effect that was described in the Introduction. It is of interest to compare our analysis with an example of the effect that was found numerically in Ref. 12. The parameters used in the computations were $R_1 = 10$, $m = 0.01$, $r = 0.95$, $S = 0.1$, and $F^2 = 0.1$. We note that the analysis here does not apply directly to these parameters because the viscosity ratio m is close to 0, whereas in our analysis, such ratios are to be close to 1. From comparing the graphs for $l_1 = 0.05$ and $l_1 = 0.1$ in Fig. 1, Ref. 12, we notice that the growth rate $re(\sigma)$ is proportional to l_1 as $l_1 \to 0$ instead of the l_1^2 behavior found in our analysis above. This is reconciled by examining Yih's[10] long-wave analysis of σ. His formula for $re(\sigma)$ for $\alpha \to 0$ contains terms such as $m + (l_1/l_2)$. The order of magnitude of this is $O(1)$ in the limit $l_1 \to 0$, but when m is $O(l_1)$, the magnitude becomes $O(l_1)$. When m is assumed to be $O(l_1)$, it is found that $\sigma = O(l_1)$ as $l_1 \to 0$ in Yih's long-wave analysis.

Hooper[11] recently analyzed the thin-layer effect for two-layer Couette flow when the top fluid is unbounded. We discuss here how her work is related to ours. At first glance, the limit $l_1 \to 0$ might appear identical to putting the upper wall at infinity, but this is not true. Our problem approaches Hooper's not simply when $l_1 \to 0$ but when both $l_1 \to 0$ and $l^* \to \infty$. Hooper uses long-wave asymptotics where her wavenumber, denoted by α_H, is our αl_1 and α_H is small. She shows that the growth rate for the semibounded problem is $O(\alpha^{4/3})$. This appears to be in contrast with what Yih[10] and the present article find, viz. the growth rate for long waves is $O(\alpha^2)$. However, there is no disagreement because long-wave results are sensitive to boundary conditions and Hooper's problem has no upper boundary whereas there is one in Yih's problem and ours. It is natural, then, that in our problem, the growth rates for long waves retrieve Yih's

$O(\alpha^2)$. We note that in Hooper's long-wave asymptotics, the dimensional wavelength is long compared with the width of fluid 1, but short compared with the width of the channel: i.e., $1 \ll \alpha \ll l_1^{-1}$ yields $O(\alpha^{4/3})$ growth rates. In the long-wave asymptotics of the present article, the dimensional wavelength is long compared with the width of the channel. How might we retrieve Hooper's results? We note that her Reynolds number, denoted by $R_{\rm H}$, is $U_0 l_1^*/\nu_1$, where U_0 is equivalent to our basic flow speed at the undisturbed interface, and corresponds to our $U^* U_1(l_1)$. Here, Eq. (2) yields $U_1(l_1) = l_1/(l_1 + m l_2)$, U^* is the upper plate speed, and l_1^* is the dimensional depth of the lower fluid. Our Reynolds number R is $U^* l^*/\nu_1$, where l^* is the plate separation. The Reynolds numbers are related by $R_{\rm H} = R l_1^2/(l_1 + m l_2)$. Thus, if our top plate is kept at a finite distance with speed U^* and the fluid volume ratio l_2/l_1 is made to approach ∞, then Hooper's Reynolds number $R_{\rm H}$ approaches 0. Her results concerning $O(\alpha^{4/3})$ growth rates require that $R_{\rm H}$ be finite and $\alpha_{\rm H} \ll 1$, and do not allow $R_{\rm H}$ to be zero. We retrieve $R_{\rm H} = O(1)$ if our Reynolds number R is made to approach infinity at order $R = O(l_1^{-2})$ as $l_2/l_1 \to \infty$. This, in effect, moves the upper plate to infinity. This limit also accommodates U^* becoming large. Therefore, retrieval of $O(\alpha^{4/3})$ growth rates $[\alpha_{\rm H} \ll 1, R_{\rm H} = O(1)]$ from our bounded Couette flow problem involves three limits: $l_1 \to 0$, $R \to \infty$, and our α should satisfy $1 \ll \alpha \ll l_1^{-1}$. These limits are not necessarily interchangeable. Since the growth rates for Hooper's and Yih's problems are different, even in the limit $l_1 \to 0$, this suggests that the presence of the upper wall is felt even when one layer becomes thin. For short waves, the stability picture is different from that of long waves. In the short-wave limit, the analysis is localized at the interface and the waves are not aware of the walls. However, as noted just after Eq. (47), the short-wave limit is not interchangeable with the limit of one layer becoming thin.

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APPENDIX A: THE ADJOINT PROBLEM FOR $\epsilon = 0$

We denote the domains occupied by the two fluids by

$$\Omega_1 = \{0 \leqslant x \leqslant 2\pi/\alpha,\ 0 \leqslant z \leqslant l_1\}$$

and

$$\Omega_2 = \{0 \leqslant x \leqslant 2\pi/\alpha,\ l_1 \leqslant z \leqslant 1\}.$$

We denote the interface by I, and the lower and upper boundaries by Γ_1 and Γ_2, respectively. Let $X_1 = (u,v,h)$ and $X_2 = (u^*,v^*,h^*)$. Asterisks denote the adjoint. We have

$$\begin{aligned}\langle X_2, L_0 X_1\rangle = &\int_{\Omega_1} \bar{u}^* \frac{1}{R_1} \Delta u - \bar{u}^* \frac{\partial p}{\partial x} - \bar{u}^* v U_1'(z) \\ &- \bar{u}^* i\alpha u z + \bar{v}^* \frac{1}{R_1} \Delta v - \bar{v}^* \frac{\partial p}{\partial z} - i\alpha \bar{v}^* v z \\ &+ \int_{\Omega_2} \bar{u}^* \frac{1}{R_2} \Delta u - \bar{u}^* \frac{\partial p}{\partial x} - \bar{u}^* v U_1'(z) \\ &- \bar{u}^* i\alpha u z + \bar{v}^* \frac{1}{R_2} \Delta v - \bar{v}^* \frac{\partial p}{\partial z} - i\alpha \bar{v}^* v z \\ &+ \int_I \bar{h}^* (v - h i\alpha U_1(l_1)). \end{aligned} \tag{A1}$$

We integrate by parts and obtain, using the divergence condition $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{u}^* = 0$,

$$\begin{aligned}\langle X_2, L_0 X_1\rangle = &\int_{\Omega_1} u\left(\frac{1}{R} \Delta \bar{u}^* - \frac{\partial \bar{p}^*}{\partial x} - \bar{u}^* i\alpha U_1(z)\right) + v\left(\frac{1}{R} \Delta \bar{v}^* - \frac{\partial \bar{p}^*}{\partial z} - \bar{u}^* - i\alpha U_1(z) \bar{v}^*\right) \\ &+ \int_{\Omega_2} u\left(\frac{1}{R} \Delta \bar{u}^* - \frac{\partial \bar{p}^*}{\partial x} - \bar{u}^* i\alpha U_1(z)\right) + v\left(\frac{1}{R} \Delta \bar{v}^* - \frac{\partial \bar{p}^*}{\partial z} - \bar{u}^* - i\alpha U_1(z) \bar{v}^*\right) \\ &+ \int_{\Gamma_1} -\frac{\bar{u}^*}{R} \frac{\partial u}{\partial z} + \frac{u}{R} \frac{\partial \bar{u}^*}{\partial z} - \frac{\bar{v}^*}{R} \frac{\partial v}{\partial z} + \frac{v}{R} \frac{\partial \bar{v}^*}{\partial z} + \bar{v}^* p - v \bar{p}^* \\ &+ \int_{\Gamma_2} \frac{\bar{u}^*}{R} \frac{\partial u}{\partial z} - \frac{u}{R} \frac{\partial \bar{u}^*}{\partial z} + \frac{\bar{v}^*}{R} \frac{\partial v}{\partial z} - \frac{v}{R} \frac{\partial \bar{v}^*}{\partial z} - \bar{v}^* p + v \bar{p}^* \\ &+ \int_I \bar{h}^* v - \bar{h}^* h i\alpha U_1(l_1) + \left[\!\left[\frac{\bar{u}^*}{R} \frac{\partial u}{\partial z} - \frac{u}{R} \frac{\partial \bar{u}^*}{\partial z} + \frac{\bar{v}^*}{R} \frac{\partial v}{\partial z} - \frac{v}{R} \frac{\partial \bar{v}^*}{\partial z} - \bar{v}^* p + v \bar{p}^* \right]\!\right]. \end{aligned} \tag{A2}$$

From this, we read off the adjoint differential operator to be given by

$$L_0^* X_2 = \begin{pmatrix} \dfrac{1}{R} \Delta u^* - \dfrac{\partial p^*}{\partial x} + U_1(z) \dfrac{\partial u^*}{\partial x} \\ \dfrac{1}{R} \Delta v^* - \dfrac{\partial p^*}{\partial z} - U_1'(z) u^* + U_1(z) \dfrac{\partial v^*}{\partial x} \\ U_1(l_1) \dfrac{\partial h^*}{\partial x} \end{pmatrix}. \tag{A3}$$

Moreover, since X_1 satisfies the boundary conditions (12) on Γ_1 and Γ_2, the integrals over these boundaries vanish if

$$\bar{u}^* = \bar{v}^* = 0 \quad \text{on } z = 0,1. \tag{A4}$$

Into the interface term in (A2), we add

$$\frac{\bar{v}^*}{R}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\right) - \frac{v}{R}\left(\frac{\partial \bar{u}^*}{\partial x} + \frac{\partial \bar{v}^*}{\partial z}\right),$$

which is zero. We integrate the x derivative by parts and use periodicity. This yields

$$\int_I \bar{h}^* v - \bar{h}^* h i\alpha U_1(l_1) + \left[\!\!\left[\frac{\bar{u}^*}{R}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) - \frac{u}{R}\left(\frac{\partial \bar{u}^*}{\partial z} + \frac{\partial \bar{v}^*}{\partial x}\right) + \bar{v}^*\left(\frac{2}{R}\frac{\partial v}{\partial z} - p\right) - v\left(\frac{2}{R}\frac{\partial \bar{v}^*}{\partial z} - \bar{p}^*\right)\right]\!\!\right]. \tag{A5}$$

From this we find the adjoint interface conditions:

$$\begin{aligned} &[\![u^*]\!] = 0, \quad [\![v^*]\!] = 0, \\ &\left[\!\!\left[\frac{\partial u^*}{\partial z} + \frac{\partial v^*}{\partial x}\right]\!\!\right] = 0, \\ &\left[\!\!\left[p^* - \frac{2}{R}\frac{\partial v^*}{\partial z}\right]\!\!\right] + h^* = 0. \end{aligned} \tag{A6}$$

APPENDIX B: EIGENFUNCTION OF THE UNPERTURBED PROBLEM

If $\epsilon = 0$, (13) is a solution of Eqs. (4)–(9) and we have the eigenfunction

$$A = e^{i\alpha x}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{B1}$$

The eigenvector C of the adjoint satisfies

$$L_0^* C = i\alpha l_1 C, \quad B_0^* C = 0.$$

We denote C by (u,v,h) and drop the asterisks used in Appendix A. This leads to the equations

$$\begin{aligned} &\frac{1}{R}\Delta u - \frac{\partial p}{\partial x} + u i\alpha U_1(z) = i\alpha l_1 u, \\ &\frac{1}{R}\Delta v - \frac{\partial p}{\partial z} - u + i\alpha U_1(z) v = i\alpha l_1 v, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0. \end{aligned} \tag{B2}$$

We set $v = v_0 e^{i\alpha x}$, etc., and obtain, by combining the equations,

$$\frac{i}{\alpha R}\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)^2 v + (l_1 - z)\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right) v - 2\frac{\partial v}{\partial z} = 0.$$

This equation factorizes in a manner that is analogous to the factorization of the Orr–Sommerfeld equations governing stability of the one-fluid plane Couette flow (see Sec. 31.1, p. 212, Ref. 24):

$$\Delta\left(\frac{1}{R}\Delta v - (l_1 - z)\frac{\partial v}{\partial x}\right) = 0. \tag{B3}$$

The general solution of this equation is constructed from the two solutions of the equation

$$\frac{1}{R}\Delta v - (l_1 - z)\frac{\partial v}{\partial x} = 0 \tag{B4}$$

and the two particular solutions of the equations

$$\frac{1}{R}\Delta v - (l_1 - z)\frac{\partial v}{\partial x} = -R^{1/3}\alpha^{-2/3}e^{i\alpha x}e^{\pm\alpha z}. \tag{B5}$$

Equation (B4) is $v_{0zz} - (\alpha^2 + i\alpha R l_1 - i\alpha R z)v_0 = 0$ which transforms into Airy's equation $V_{ss} - sV = 0$ via

$$v_0(z) = V(s), \quad s = \frac{\alpha^2 + i\alpha R l_1 - i\alpha R z}{-\alpha^{2/3}R^{2/3}}. \tag{B6}$$

The solutions of Airy's equation are $\mathrm{Ai}(s)$ and $\mathrm{Bi}(s)$. Equations (B5) become

$$V_{ss} - sV = \exp[\pm(-i\alpha^2/R + \alpha l_1)] \times \exp(\mp i\alpha^{2/3}R^{-1/3}s), \tag{B7}$$

for which the particular solutions are

$$V(s) = -\mathrm{Ai}(s)\pi\int^s \mathrm{Bi}(t)W(t)dt + \mathrm{Bi}(s)\pi\int^s \mathrm{Ai}(t)W(t)dt, \tag{B8}$$

where $W(t)$ is one of

$$\begin{aligned} &W_1(t) = \exp(-i\alpha^2/R + \alpha l_1)\exp(-i\alpha^{2/3}R^{-1/3}t), \\ &W_2(t) = \exp(i\alpha^2/R - \alpha l_1)\exp(i\alpha^{2/3}R^{-1/3}t). \end{aligned}$$

We have used the result[21] that the Wronskian of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ is $1/\pi$. We denote s_0 to be the value of s at $z = 0$, s_1 to be the value of s at $z = 1$, and s_L to be the value of s at $z = l_1$:

$$\begin{aligned} &s_L = -\alpha^{4/3}R^{-2/3}, \\ &s_0 = \frac{\alpha^{4/3} + i\alpha^{1/3}Rl_1}{-R^{2/3}} = s_L - i\alpha^{1/3}l_1R^{1/3}, \\ &s_1 = \frac{\alpha^2 - i\alpha R l_2}{-\alpha^{2/3}R^{2/3}} = s_L + i\alpha^{1/3}l_2R^{1/3}. \end{aligned} \tag{B9}$$

The general solution is

$$V(s) = c_1\,\mathrm{Ai}(s) + c_2\,\mathrm{Bi}(s) + c_3V_1(s) + c_4V_2(s) \tag{B10}$$

in fluid 1, and

$$V(s) = d_1\,\mathrm{Ai}(s) + d_2\,\mathrm{Bi}(s) + d_3V_3(s) + d_4V_4(s) \tag{B11}$$

in fluid 2, where

$$V_1(s) = -\mathrm{Ai}(s)\pi\int_{s_0}^s \mathrm{Bi}(t)W_1(t)dt + \mathrm{Bi}(s)\pi\int_{s_0}^s \mathrm{Ai}(t)W_1(t)dt, \tag{B12}$$

$$V_2(s) = -\mathrm{Ai}(s)\pi\int_{s_0}^s \mathrm{Bi}(t)W_2(t)dt + \mathrm{Bi}(s)\pi\int_{s_0}^s \mathrm{Ai}(t)W_2(t)dt, \tag{B13}$$

$$V_3(s) = -\mathrm{Ai}(s)\pi\int_{s_1}^s \mathrm{Bi}(t)W_1(t)dt + \mathrm{Bi}(s)\pi\int_{s_1}^s \mathrm{Ai}(t)W_1(t)dt, \tag{B14}$$

$$V_4(s) = -\mathrm{Ai}(s)\pi\int_{s_1}^s \mathrm{Bi}(t)W_2(t)dt + \mathrm{Bi}(s)\pi\int_{s_1}^s \mathrm{Ai}(t)W_2(t)dt. \tag{B15}$$

At the solid boundaries, V and $\partial V/\partial s$ are 0:

$$\begin{aligned} &d_2\,\mathrm{Bi}(s_1) = -d_1\,\mathrm{Ai}(s_1), \\ &c_2\,\mathrm{Bi}(s_0) = -c_1\,\mathrm{Ai}(s_0), \end{aligned}$$

$$c_1\,\mathrm{Ai}'(s_0)+c_2\,\mathrm{Bi}'(s_0)+c_3V_1'(s_0)+c_4V_2'(s_0)=0,$$
$$d_1\,\mathrm{Ai}'(s_1)+d_2\,\mathrm{Bi}'(s_1)+d_3V_3'(s_1)+d_4V_4'(s_1)=0.$$

Since $V_1'(s_0)$, $V_2'(s_0)$, $V_3'(s_1)$, and $V_4'(s_1)$ are zero,

$$c_1=0,\quad c_2=0,\quad d_1=0,\quad d_2=0. \tag{B16}$$

The coefficients c_3, c_4, d_3, and d_4 must be determined such that the jump conditions across the interface given by Eqs. (A6) at $z=l_1$ are satisfied. These are equivalent to the continuity of v,

$$c_3V_1(s_L)+c_4V_2(s_L)-d_3V_3(s_L)-d_4V_4(s_L)=0; \tag{B17}$$

the continuity of v_z,

$$c_3V_1'(s_L)+c_4V_2'(s_L)-d_3V_3'(s_L)-d_4V_4'(s_L)=0; \tag{B18}$$

and the continuity of v_{zz},

$$c_3V_1''(s_L)+c_4V_2''(s_L)-d_3V_3''(s_L)-d_4V_4''(s_L)=0, \tag{B19}$$

and

$$h=(-1/\alpha^2R)[\![v_{zzz}]\!]. \tag{B20}$$

Using Eqs. (B7), Eqs. (B17) and (B19) imply that

$$c_3W_1(s_L)+c_4W_2(s_L)-d_3W_1(s_L)-d_4W_2(s_L)=0, \tag{B21}$$

where

$$W_1(s_L)=\exp(\alpha l_1),\quad W_2(s_L)=\exp(-\alpha l_1).$$

From (B18) and (B21), we have at $s=s_L$,

$$\begin{aligned}c_4[&V_2V_1'-V_2'V_1+V_2'V_3-V_2V_3'\\&+e^{-2\alpha l_1}(V_1V_3'-V_3V_1')]\\=d_4[&V_4V_1'-V_4'V_1+V_4'V_3-V_4V_3'\\&+e^{-2\alpha l_1}(V_1V_3'-V_1'V_3)].\end{aligned}$$

We choose

$$\begin{aligned}c_4=&V_4(V_1'-V_3')-V_4'(V_1-V_3)\\&+e^{-2\alpha l_1}(V_1V_3'-V_1'V_3),\end{aligned} \tag{B22}$$

$$\begin{aligned}d_4=&V_2(V_1'-V_3')-V_2'(V_1-V_3)\\&+e^{-2\alpha l_1}(V_1V_3'-V_3V_1'),\end{aligned} \tag{B23}$$

$$\begin{aligned}d_3=&e^{-2\alpha l_1}[V_1'(V_4-V_2)+V_1(V_2'-V_4')]\\&+V_2V_4'-V_4V_2'.\end{aligned} \tag{B24}$$

We will not need to calculate c_3.

APPENDIX C: EVALUATION OF INNER PRODUCTS

We calculate the terms in Ψ defined in (24).

1. The calculation of $\langle C,L_1x^0\rangle$

We denote $C=(u^*,v^*,h^*)$ and $x^0=(u,v,h)$. Below, the interval of integration I extends over one wavelength in x, at $z=l_1$,

$$\begin{aligned}\langle C,L_1x^0\rangle&=\int_{\Omega_1}-\bar{u}^*\cdot\nabla\tilde{p}+\int_{\Omega_2}-\bar{u}^*\cdot\nabla\tilde{p}+\int_I\bar{h}^*\overline{m}l_2e^{i\alpha x}\\&=\int_{\Omega_1}\tilde{p}\nabla\cdot\bar{\mathbf{u}}^*+\int_{\Omega_2}\tilde{p}\nabla\cdot\bar{\mathbf{u}}^*+\int_I\bar{h}^*\overline{m}l_2e^{i\alpha x}\\&\quad-[\![\bar{v}^*\tilde{p}]\!]+\int_{\Gamma_1}\bar{v}^*\tilde{p}-\int_{\Gamma_2}\bar{v}^*\tilde{p},\end{aligned}$$

after an integration by parts. We have $\nabla\cdot\bar{\mathbf{u}}^*=0$. At $z=l_1$, $-[\![\tilde{p}]\!]=h(-\alpha^2(\bar{s}/R)+\bar{r}/F^2)$, and $\partial v_2/\partial z=0$ for h and v_2 belonging to $x^0=-A/i\alpha l_1$. Here, the h belonging to A is $e^{i\alpha x}$; hence, $-[\![\tilde{p}]\!]=-(e^{i\alpha x}/i\alpha l_1)(-\alpha^2\overline{S}/R+\bar{r}/F^2)$. Therefore,

$$\begin{aligned}\langle C,L_1x^0\rangle&=\int_I\bar{h}^*\overline{m}l_2e^{i\alpha x}-\bar{v}^*[\![\tilde{p}]\!]\\&=\int_I\bar{h}^*\overline{m}l_2e^{i\alpha x}-\bar{v}^*\frac{e^{i\alpha x}}{i\alpha l_1}\left(-\frac{\alpha^2\overline{S}}{R}+\frac{\bar{r}}{F^2}\right)\\&=\Gamma_1.\end{aligned} \tag{C1}$$

From (B20),

$$\begin{aligned}h^*&=(ie^{i\alpha x}/\alpha)[\![V_{sss}]\!]\\&=(ie^{i\alpha x}/\alpha)[c_3V_1'''(s_L)+c_4V_2'''(s_L)\\&\quad-d_3V_3'''(s_L)-d_4V_4'''(s_L)]\end{aligned}$$

and using (B7), e.g., $V_1''(s)=sV_1+W_1(s)$ and hence $V_1'''(s)=V_1(s)+sV_1'(s)+W_1'(s)$,

$$\begin{aligned}h^*=(ie^{i\alpha x}/\alpha)[&c_3(V_1+s_LV_1'+W_1')\\&+c_4(V_2+s_LV_2'+W_2')-d_3(V_3+s_LV_3'+W_1')\\&-d_4(V_4+s_LV_4'+W_2')].\end{aligned}$$

We use conditions (B17) and (B18) to find

$$h^*=(ie^{i\alpha x}/\alpha)[W_1'(s_L)(c_3-d_3)+W_2'(s_L)(c_4-d_4)].$$

We use (B21) to express c_3-d_3 in terms of c_4-d_4:

$$\begin{aligned}h^*&=\frac{ie^{i\alpha x}}{\alpha}\left(-\frac{W_1'(s_L)}{W_1(s_L)}W_2(s_L)+W_2'(s_L)\right)(c_4-d_4)\\&=-\frac{2\exp(i\alpha x-\alpha l_1)}{\alpha^{1/3}R^{1/3}}(c_4-d_4).\end{aligned} \tag{C2}$$

From (B11) and (B16), v^* at $z=l_1$ is

$$V(s_L)=d_3V_3(s_L)+d_4V_4(s_L). \tag{C3}$$

2. The calculation of $\langle C,L_0x^1\rangle$

We denote $x^1=(u,v,h)$. The boundary conditions from (23) yield

$$[u]=\frac{\overline{m}e^{i\alpha x}}{i\alpha l_1}\quad\text{at } z=l_1,$$

$$\left[\!\!\left[\frac{\partial u}{\partial z}\right]\!\!\right]=0\quad\text{at } z=l_1,$$

$$u=0\quad\text{at } z=0,1.$$

The normal stress condition (17) for x^1 is

$$\left[\!\!\left[-p+\frac{2}{R}\frac{\partial v}{\partial z}\right]\!\!\right]=0,$$

$$\langle C,L_0x^1\rangle=\langle L_0^*C,x^1\rangle+\Gamma_2,$$

where Γ_2 can be read off from the calculation of the adjoint in Appendix A:

$$\Gamma_2 = \int_I \bar{h}^* v + \left[\!\left[\frac{1}{R}\left(\bar{u}^* \frac{\partial u}{\partial z} + \bar{v}^* \frac{\partial v}{\partial z} - u \frac{\partial \bar{u}^*}{\partial z} - v \frac{\partial \bar{v}^*}{\partial z}\right) - \bar{v}^* p + v \bar{p}^* \right]\!\right]$$

$$= \int_I \frac{\overline{m}}{\alpha^2 R l_1} (\bar{v}^*_{zz} + \alpha^2 \bar{v}^*) e^{i\alpha x}. \tag{C4}$$

From (B6) and (B11), $v_{0zz}(l_1) = -\alpha^{2/3} R^{2/3} V_{ss}(s_L)$, where

$$\begin{aligned} V_{ss}(s_L) &= d_3 V_3''(s_L) + d_4 V_4''(s_L) \\ &= d_3[s_L V_3(s_L) + W_1(s_L)] \\ &\quad + d_4[s_L V_4(s_L) + W_2(s_L)]. \end{aligned}$$

Therefore,

$$\begin{aligned} v^*_{0zz} + \alpha^2 v^*_0 &= d_3(2\alpha^2 V_3 - \alpha^{2/3} R^{2/3} W_1) \\ &\quad + d_4(2\alpha^2 V_4 - \alpha^{2/3} R^{2/3} W_2). \end{aligned} \tag{C5}$$

Equation (25) becomes

$$-i\alpha l_1 \langle C, x^1 \rangle = -\Gamma_1 - \Gamma_2. \tag{C6}$$

3. The calculation of $\langle C, A \rangle$

We have

$$\langle C, A \rangle = \int_I \bar{h}^* e^{i\alpha x}. \tag{C7}$$

APPENDIX D: LONG-WAVE ANALYSIS OF THIN-LAYER LIMIT

Yih[10] has found that

$$\sigma \sim -i\alpha[c_0' + U_1(l_1)] + \alpha^2 l_2^2 R_2 J \quad \text{as } \alpha \to 0, \tag{D1}$$

where $U_1(l_1) = l_1 + \overline{m}\epsilon l_1 l_2$, and c_0' and J are defined in Eqs. (34) and (42), respectively, of Ref. 10. When evaluated in the limit $l_1 \to 0$, (D1) becomes

$$\sigma \sim -i\alpha\left(-2\frac{l_1^2}{l_2^2}\overline{m}\epsilon + l_1 + \overline{m}\epsilon l_1 l_2\right) + \epsilon\alpha^2 R\left(\frac{-l_1^2 \overline{m}}{60} + \frac{\overline{r} l_1^3}{3F^2}\right) + O(\epsilon^2, l_1^4, \alpha^3). \tag{D2}$$

We now show that our formula (46) reduces to (D2) for small α. We evaluate the expression $c_4 - d_4$ defined just above Eq. (27). At $l_1 = 0$, $t = -\alpha^{4/3} R^{-2/3} - i\alpha^{1/3} R^{1/3}(\theta - 1)/2$ in (27). For small α, we use Taylor expansions for the integrands, e.g.,

$$\begin{aligned} \mathrm{Ai}(t) &\sim \mathrm{Ai}(0) + t\,\mathrm{Ai}'(0) + \frac{t^3}{6}\mathrm{Ai}(0) + \frac{t^4}{12}\mathrm{Ai}'(0) \\ &\quad + \frac{4t^6}{6!}\mathrm{Ai}(0) + \frac{10t^7}{7!}\mathrm{Ai}'(0) + \frac{28t^9}{9!}\mathrm{Ai}(0) \\ &\quad + O(\alpha^{10/3}), \end{aligned} \tag{D3}$$

and we find

$$c_4 - d_4 \sim \frac{\alpha^3 R^2}{180} + \frac{i\alpha^2 R}{6} + O(\alpha^4). \tag{D4}$$

Similarly,

$$V_4 - V_3 \sim \alpha^{1/3} R^{1/3}\left(\frac{2}{3}\alpha^{4/3} R^{1/3} - \frac{i\alpha^{7/3} R^{4/3}}{36} + O(\alpha^{10/3})\right), \tag{D5}$$

so that the $O(\alpha l_1^2)$ term in (46) is $2i\epsilon\alpha l_1^2 \overline{m}$, as in (D2). Expressions (46) and (D2) agree to order α^2.

APPENDIX E: SHORT-WAVE ANALYSIS OF THIN-LAYER LIMIT

We show that the growth rate of the eigenvalue given by (46) asymptotes to a constant in the short wavelength limit, in agreement with the computed graphs of Fig. 1. In the absence of surface tension and density difference, the growth rate is $\epsilon \overline{m} l_1^2 \alpha^{4/3} R^{4/3}/2$ multiplied by the complex conjugate of $(V_4 - V_3)/(c_4 - d_4)$, where

$$\frac{(V_4 - V_3)}{(c_4 - d_4)} = -\frac{4i}{\alpha^{1/3} R^{1/3}} \frac{\int_{-1}^1 (\mathrm{Ai}(s_L)\mathrm{Bi}(t) - \mathrm{Bi}(s_L)\mathrm{Ai}(t)) \sinh \alpha(\theta - 1)/2\, d\theta}{\int_{-1}^1 \mathrm{Bi}(t) e^{\alpha\theta/2} d\theta \int_{-1}^1 \mathrm{Ai}(t) e^{-\alpha\theta/2} d\theta - \int_{-1}^1 \mathrm{Ai}(t) e^{\alpha\theta/2} d\theta \int_{-1}^1 \mathrm{Bi}(t) e^{-\alpha\theta/2} d\theta},$$
$$t = -\alpha^{4/3} R^{-2/3} - i\alpha^{1/3} R^{1/3}(\theta - 1)/2, \quad s_L = -\alpha^{4/3} R^{-2/3}. \tag{E1}$$

We use asymptotic expansions for large arguments of the Airy function[21]:

$$\begin{aligned} \mathrm{Ai}(-z) &\sim \pi^{-1/2} z^{-1/4}[\sin(\zeta + \pi/4) \\ &\quad - \zeta^{-1} c_1 \cos(\zeta + \pi/4) + O(\zeta^{-2})], \\ \mathrm{Bi}(-z) &\sim \pi^{-1/2} z^{-1/4}[\cos(\zeta + \pi/4) \\ &\quad + \zeta^{-1} c_1 \sin(\zeta + \pi/4) + O(\zeta^{-2})], \end{aligned}$$
$$\zeta = \tfrac{2}{3} z^{3/2}, \quad |\arg(z)| < 2\pi/3, \quad |z| \text{ large}, \quad c_1 = \tfrac{5}{72}.$$

We evaluate the numerator first. For large α,

$$\begin{aligned} &\mathrm{Ai}(s_L)\mathrm{Bi}(t) - \mathrm{Bi}(s_L)\mathrm{Ai}(t) \\ &\quad \sim \pi^{-1} z_1^{-1/4} z_2^{-1/4}[\sin(\zeta_1 - \zeta_2) \\ &\quad + c_1(\zeta_2^{-1} - \zeta_1^{-1})\cos(\zeta_1 - \zeta_2) + O(\zeta^{-2})], \end{aligned} \tag{E2}$$
$$z_1 = \alpha^{4/3} R^{-2/3}, \quad z_2 = \alpha^{4/3} R^{-2/3} + i\alpha^{1/3} R^{1/3}(\theta - 1)/2,$$
$$\zeta_1 = \tfrac{2}{3}\alpha^2/R, \quad \zeta_2 = \tfrac{2}{3} z_2^{3/2}.$$

Therefore,

$$\int_{-1}^1 [\mathrm{Ai}(s_L)\mathrm{Bi}(t) - \mathrm{Bi}(s_L)\mathrm{Ai}(t)] \sinh \frac{\alpha(\theta - 1)}{2} d\theta$$

$$\sim \pi^{-1} z_1^{-1/4} \left(\int_{-1}^{1} z_2^{-1/4} \left(\frac{e^{i(\zeta_1-\zeta_2)} - e^{-i(\zeta_1-\zeta_2)}}{2i} \right) \times \sinh \frac{\alpha(\theta-1)}{2} + c_1 (\zeta_2^{-1} - \zeta_1^{-1}) z_2^{-1/4} \times \left(\frac{e^{i(\zeta_1-\zeta_2)} + e^{-i(\zeta_1-\zeta_2)}}{2} \right) \sinh \frac{\alpha(\theta-1)}{2}\, d\theta \right). \quad \text{(E3)}$$

Since

$$\zeta_1 - \zeta_2 \sim -i\alpha(\theta-1)/2 + R(\theta-1)^2/16 + O(1/\alpha),$$

the resulting integrals are asymptotically expanded using

$$\int_0^1 e^{-\alpha\theta} f(\theta)\, d\theta \sim \frac{f(0)}{\alpha} + \frac{f_\theta(0)}{\alpha^2} + \frac{f_{\theta\theta}(0)}{\alpha^3} + O\left(\frac{1}{\alpha^4}\right), \quad \alpha \to \infty. \quad \text{(E4)}$$

The dominant integral in (E3) is

$$\int_{-1}^{1} z_2^{-1/4} \exp\left(-i(\zeta_1-\zeta_2) - \frac{\alpha(\theta-1)}{2} \right) d\theta = O(e^{2\alpha}\alpha^{-4/3}).$$

Since $\zeta_2^{-1} - \zeta_1^{-1}$ is $O(1/\alpha^3)$, the second term in (E2) will be neglected. The leading term in (E3) is therefore

$$\frac{z_1^{-1/4}}{4\pi i} \int_{-1}^{1} z_2^{-1/4} e^{-i(\zeta_1-\zeta_2) - \alpha(\theta-1)/2}\, d\theta. \quad \text{(E5)}$$

In the denominator of (E1) for large α, we have the expression

$$\int_{-1}^{1} z_2^{-1/4} e^{\alpha\theta/2} \cos\left(\zeta_2 + \frac{\pi}{4}\right) d\theta \times \int_{-1}^{1} z_2^{-1/4} e^{-\alpha\theta/2} \sin\left(\zeta_2 + \frac{\pi}{4}\right) d\theta - \int_{-1}^{1} z_2^{-1/4} e^{-\alpha\theta/2} \cos\left(\zeta_2 + \frac{\pi}{4}\right) d\theta \times \int_{-1}^{1} z_2^{-1/4} e^{\alpha\theta/2} \sin\left(\zeta_2 + \frac{\pi}{4}\right) d\theta. \quad \text{(E6)}$$

Replacing $\zeta_2 + \pi/4$ above by $\pi/4 + 2\alpha^2/3R - (\zeta_1 - \zeta_2)$, and using trigonometric identities, the leading term in (E6) is

$$\frac{1}{2i} \int_{-1}^{1} z_2^{-1/4} e^{i(\zeta_1-\zeta_2) + \alpha(\theta-1)/2}\, d\theta \times \int_{-1}^{1} z_2^{-1/4} e^{-i(\zeta_1-\zeta_2) - \alpha(\theta-1)/2}\, d\theta. \quad \text{(E7)}$$

Using (E5) and (E7), (E1) asymptotes to

$$\frac{-2i}{\alpha^{2/3} R^{1/6} \int_{-1}^{1} z_2^{-1/4} e^{i(\zeta_1-\zeta_2) + \alpha(\theta-1)/2}\, d\theta}, \quad \text{(E8)}$$

where

$$\int_{-1}^{1} z_2^{-1/4} e^{i(\zeta_1-\zeta_2) + \alpha(\theta-1)/2}\, d\theta \sim \int_0^1 z_2^{-1/4} e^{\alpha(\theta-1) + iR(\theta-1)^2/16}\, d\theta.$$

Substituting $\theta - 1 = -\tilde{\theta}$ and using (E4), the above asymptotes to

$$z_1^{-1/4}/\alpha + i\alpha^{1/3} R^{1/3} z_1^{-5/4}/8\alpha^2 + (iR z_1^{-1/4}/8 - 5\alpha^{2/3} R^{2/3} z_1^{-9/4}/64)/\alpha^3.$$

Therefore, (E8) becomes

$$-2i\alpha/\alpha^{1/3} R^{1/3} (1 + iR/4\alpha^2).$$

Hence, the coefficient of $\epsilon \overline{m} l_1^2$ in (46) asymptotes to $-R/4 + i\alpha^2$.

[1] M. E. Charles and P. J. Redberger, Can. J. Chem. Eng. **40**, 70 (1962).
[2] A. E. Everage, Jr., Trans. Soc. Rheol. **17**, 629 (1973).
[3] D. Hasson, U. Mann, and A. Nir, Can. J. Chem. Eng. **48**, 514 (1970).
[4] C. H. Hertz and B. Hermanrud, J. Fluid Mech. **131**, 271 (1983).
[5] D. D. Joseph, K. Nguyen, and G. S. Beavers, J. Fluid Mech. **141**, 319 (1984).
[6] D. L. MacLean, Trans. Soc. Rheol. **17**, 385 (1983).
[7] J. H. Southern and R. L. Ballman, Appl. Polymer Symp. **20**, 175 (1973).
[8] K. Walters, in *Advances in Rheology*, Proceedings of the IX International Congress on Rheology, Mexico, edited by B. Mena, A. Garcia-Rejon, and C. Rangel-Nafaile (Universidad Nacional Autonamía de Mexico, Mexico City, 1984), p. 31.
[9] J. L. White and B.-L. Lee, Trans. Soc. Rheol. **19**, 457 (1975).
[10] C. S. Yih, J. Fluid Mech. **27**, 337 (1967).
[11] A. P. Hooper, Phys. Fluids **28**, 1613 (1985).
[12] Y. Renardy, Phys. Fluids **28**, 3441 (1985).
[13] Y. Renardy and D. D. Joseph, J. Fluid Mech. **150**, 381 (1985).
[14] D. D. Joseph, M. Renardy, and Y. Renardy, J. Fluid Mech. **141**, 309 (1984).
[15] A. P. Hooper and W. G. C. Boyd, J. Fluid Mech. **128**, 507 (1983).
[16] Y. Renardy, Phys. Fluids **29**, 356 (1986).
[17] Y. Renardy and M. Renardy, Phys. Fluids **28**, 2699 (1985).
[18] V. A. Romanov, Funct. Anal. Appl. **7**, 137 (1973).
[19] V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients 1* (Halsted, Jerusalem, 1975), Chap. 4, Sec. 1.
[20] R. Temam, *Navier–Stokes Equations* (North-Holland, Amsterdam, 1979).
[21] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972), Chap. 10.
[22] Z. Schulten, D. G. M. Anderson, and R. G. Gordon, J. Comp. Phys. **31**, 60 (1979).
[23] A. P. Hooper and W. G. C. Boyd, J. Fluid Mech. (in press).
[24] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge U. P., Cambridge, 1982), 1st paperback ed.