Nonhomogeneous Initial Boundary Value Problems for Two-Dimensional Nonlinear Schrödinger Equations

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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March 27, 2014
Blacksburg, Virginia

Keywords: Nonlinear Schrödinger Equations, Nonhomogeneous Boundary Value Problem,
Harmonic Analysis

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(ABSTRACT)

The dissertation focuses on the initial boundary value problems (IBVPs) of a class of nonlinear Schrödinger equations posed on a half plane $\mathbb{R}\times\mathbb{R}^+$ and on a strip domain $\mathbb{R}\times[0,L]$ with Dirichlet nonhomogeneous boundary data in a two-dimensional plane. Compared with pure initial value problems (IVPs), IBVPs over part of entire space with boundaries are more applicable to the reality and can provide more accurate data to physical experiments or practical problems. Although there is less research that has been made for IBVPs than that for IVPs, more attention has been paid for IBVPs recently. In particular, this thesis studies the local well-posedness of the equation for the appropriate initial and boundary data in Sobolev spaces H^s with $s \geq 0$ and investigates the global well-posedness in the H^1-space. The main strategy, especially for the local well-posedness, is to derive an equivalent integral equation (whose solution is called mild solution) from the original equation by semi-group theory and then perform the Banach fixed-point argument. However, along the process, it is essential to select proper auxiliary function spaces and prepare all the corresponding norm estimates to complete the argument. In fact, the IBVP posed on $\mathbb{R}\times\mathbb{R}^+$ and the one posed on $\mathbb{R}\times[0,L]$ are two independent problems because the techniques adopted are different. The first problem is more related to the initial value problem (IVP) posed on the whole plane $\mathbb{R}^2$ and the major ingredients are Strichartz's estimate and its generalized theory. On the other hand, the second problem can be studied as an IVP over a half-line and periodic domain, which is established on the analysis for series inspired by Bourgain's work. Moreover, the corresponding smoothing properties and regularity conditions of the solution are also considered.

Dedication

This dissertation is dedicated to my parents.

Acknowledgments

First of all, I want to thank my parents. Their constant support is the reason that I could devote myself to the academic world.

I want to express my sincerest gratitude to my advisor Dr. Shu-Ming Sun, who led me into this wonderful field of partial differential equations with his knowledge, wisdom and enthusiasm. Without his guidance, this dissertation could not be completed. I also would like to thank Dr. Shu-Ming Sun's family, who greatly helped me while I was in Blacksburg.

I want to thank Dr. Jong U. Kim, Dr. Michael Renardy, Dr. David Russell and Dr. Pengtao Yue for serving on my committee and sharing their insightful advices.

I am also grateful to my friends and my teachers in the Department of Mathematics at Virginia Tech, who helped and supported me along this journey. Thank you!

Contents

Chapter 1

Introduction

The (linear) Schrödinger equation was first formulated in late 1920s by Austrian physicist E. Schrödinger [75] to model the quantum state of a physical system (molecules, atoms and subatomic particles, etc.) in terms of time variable. Over time, the nonlinear Schrödinger (NLS) equations were brought into sight and became an essential part in the field of physics, mechanics and applied mathematics. The NLS equations are one type of nonlinear dispersive evolution equation whose solutions describe the wave propagation spreading out in space as they evolve in time. They have been broadly introduced to explain many physical phenomena. For instance, the NLS equations establish a typical model for the deep water wave modulation in which the water depth or the wave number are assumed to be large, (cf. Chabchoub and Hoffmann [34], Hui and Hamilton [54], Johnson [55], Kato [57], Ma [70], Peregrine [73]), as well as some other dispersive equations like wave equations or Korteweg-de Vries (KdV) equations which are used to model the small-amplitude water wave, i.e. shallow wave (cf. Miles [71]). The NLS equations can be adopted to depict the propagation of light in nonlinear optical fibers (cf. Bergé [6], C. Sulem and P.-L. Sulem [81]). Furthermore, the NLS equation collects attention in the study of quantum field theory (cf. Avron, Herbst and Simon [3, 4, 5], Bialynicki-Birula and Mycielski [8], Combes, Schrader and Seiler [37], Eboli and Marques [39], Gogny and Lions [50], Reed and Simon [74], Simon [77], and C. Sulem and P.-L. Sulem [81]).

The focus in the following context will mostly stand in the perspective of mathematical analysis in contrast to the aspect of mechanical or physical applications. This thesis provides certain essential features of the NLS equation in the form of:

$$\begin{cases} i\partial_t u + \Delta u + g(u) = 0 \qquad (\boldsymbol{x}, t) \in \Omega \times I \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) \\ u|_{\partial\Omega}(\boldsymbol{x}, t) = h(\boldsymbol{x}, t) \end{cases} \tag{1.1}$$

where $g(u)$, only determined by u, represents a nonlinear term which is determined by specific applications. ∂_t and Δ denote the partial derivative with respect to t and the Laplace operator with respect to $\boldsymbol{x} \in \Omega$, respectively, while I is the time interval with $0 \in I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ is the spatial domain with boundary $\partial\Omega$. By properly choosing $\varphi : \Omega \to \mathbb{C}$ as the initial data and $h : \partial\Omega \times I \to \mathbb{C}$ as the boundary data, one can expect a solution of (1.1) that satisfies some properties. The shape of the domain Ω is determined at the beginning of the investigation. Here, we are interested in having results for I as large as possible, and when $I = \mathbb{R}$ we say the result is global.

In this thesis, we only give our attention to the power nonlinear term, i.e. $g(u) = \lambda|u|^{p-2}u$ for $p > 2$ and $\lambda \in \mathbb{R}/\{0\}$, since it is one of the most typical nonlinearities and appears in many practical models that take a great deal of common interests. Note that if $\lambda > 0$, the equation is said to be forced. There are other kinds of nonlinearities, for example (cf. Cazenave [27]):

(i) $g(u) = V \cdot u$, the external potential, where $V : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ and $V \in L^q$ with $q > \max\{1, N/2\}$;

(ii) $g(u)(\boldsymbol{x}) = \frac{u}{|u|} f(\boldsymbol{x}, |u|)$, the local nonlinearity, where $f(\boldsymbol{x}, |u|)$ is measurable in $\boldsymbol{x}$ and continuous in $|u|$ satisfying that $f(\boldsymbol{x}, 0) = 0$ and for a function L mapping from $\mathbb{R}$ to $\mathbb{R}$ $|f(\boldsymbol{x}, u) - f(\boldsymbol{x}, v)| \leq L(M)|u - v|$ requiring $L \in C(\mathbb{R}^+)$ when $N = 1$ and $L(M) \leq M^\sigma$ for $0 \leq \sigma < \frac{4}{N-2}$ when $N \geq 2$;

(iii) $g(u)(\boldsymbol{x}) = (W * |u|^2)u$, the Hartree nonlinearity, for an even function $W : \mathbb{R}^N \to \mathbb{R}$ and $W \in L^q(\mathbb{R}^N)$ with $q > \max\{1, N/4\}$.

There are other types of interesting Cauchy problems with different nonlinearities, for instance, $g(u) = P(u, \nabla_{\boldsymbol{x}} u, \overline{u}, \nabla_{\boldsymbol{x}} \overline{u})$ for $\boldsymbol{x} \in \mathbb{R}^N$ and $P : \mathbb{C}^{2n+2} \to \mathbb{C}$, which shows derivatives in the nonlinear term (cf. Kenig, Ponce and Vega [68]).

Given the NLS equation (1.1) as a mathematical model, it is natural to consider its solution and any corresponding mathematical properties. Questions can arise such as: whether or not there is a topological space (Banach space) in which the solution exists; whether or not the solution we find is the only solution; what the asymptotical behavior of the solution can be observed; what kind of initial and boundary conditions are necessary for the solution to exist in the topological space, and so forth. Some of these questions are aligned together known as the well-posedness for the equation, leading to a great deal of mathematical investigation on the NLS equation. In this thesis, we concentrate on the well-posedness in the fractional Sobolev spaces $H^s(\Omega)$ for some real number s; that is, we consider the solutions continuous in time and belonging to the classical space H^s at any fixed time. Such canonical function space is denoted by $C_t(I; H^s(\Omega))$. To be explicit, we summarize the general content of well-posedness as follows which will be precisely defined later in this thesis:

Definition 1.1 (Well-posedness)**.** *We say that the initial value problem (1.1) is well-posed in a Banach space X^s (e.g. $C_t(I; H^s(\Omega))$ for $s \geq 0$) if the following properties hold:*

(i) (Existence and uniqueness) there exists a unique solution u in X^s for the problem..

(ii) (Blow-up property) for every pair of (φ, h) satisfying certain conditions there exists a solution which is defined on a maximal interval $(-T_{\min}, T_{\max})$ with $T_{\max}, T_{\min} \in (0, \infty]$; $u(t) \to \infty$ in X^s as $t \to T_{\max}$ or $t \to -T_{\min}$. Then, such solution is called the maximal solution in X^s.

(iii) (Continuous dependence) the maximal solution $u \in X^s$ depends continuously on the initial data φ and boundary data h.

Equation (1.1) can be understood in the traditional sense if $u \in C\left(I; H^2(\Omega)\right) \bigcap C^1\left(I; L^2(\Omega)\right)$. However when the regularity s is chosen small, one fails to recognize the equation with a classical solution. Therefore we define the generalized solution of (1.1) in the following way

Definition 1.2 (Generalized Solution)**.** *$u \in C\left(I; H^s(\Omega)\right)$ is said to be a generalized solution to (1.1), if for $\varphi \in H^s(\Omega)$ and h contained in some space $Y^s\left(I \times \partial\Omega\right))$ depending on s, there exists a sequence $u_n \in C\left(I; H^2(\Omega)\right) \bigcap C^1\left(I; L^2(\Omega)\right)$ such that:*

(i) u_n solves the equation of (1.1) in $L^2(\Omega)$ for $0 \le t < T$;

(ii) $u_n \to u$ in $C\left(I; H^s(\Omega)\right)$;

(iii) $\varphi_n = u_n(0)$ converges to φ in $H^s(\Omega)$;

(iv) $h_n = u_n|_{\partial\Omega}$ converges to h in $Y^s\left(I \times \partial\Omega\right))$.

Remark. Let a solution be in the space $X^s = C_t\left(I;\, H^s(\Omega)\right)$ with $s \ge 0$. If it does not cause any confusion, we can simply say the solution is in H^s or this is a H^s-solution.

We say that the IBVP (1.1) is conditionally well-posed on X^s if the problem is well-posed with the only solution in X^s and there is an auxiliary space Y^s such that the solution obtained for the well-posedness must be contained in Y^s as well, which can be indispensable to the study of NLS equations when the low regularity becomes an issue; on the other hand, auxiliary ingredients (function spaces) sometimes may not be required to complete the proceeding argument for well-posedness, and under such circumstance the equation is recognized as unconditionally well-posed. However, whether or not the well-posedness is conditional can be an interest of research by itself. Some conclusions were provided by Bona, Sun and Zhang [12] to answer the questions in this subject, for instance, how to clarify that different auxiliary conditions are equivalent when all of them guarantee the well-posedness, or when one can even remove the auxiliary conditions, etc. To this subject, an important concept introduced in [12] will appear in this thesis regarding unconditional well-posedness, called regularity persistence:

Definition 1.3 (Regularity Persistence)**.** *Let $s_1 < s$. Assume the IBVP (1.1) be conditionally well-posed in the space H^{s_1} with the auxiliary space $Y_T^{s_1}$ for some $T > 0$. The problem* *is said to have the property of regularity persistence if any solution* $u \in C([0,T]; H^{s_1}(\Omega))$ *of (1.1) with initial data* $\varphi \in H^{s_1}$ *belongs to the space* $C([0,T]; H^s(\Omega))$ *as long as* $\varphi \in H^s$.

The NLS equation can be studied with different methodologies as more and more techniques are developed. For example, the inverse scattering transform (IST) is a well-known method for solving nonlinear partial differential equations. It can especially be adopted to solve forced cubic NLS equations with $g(u) = \lambda|u|^2u$ and $\lambda > 0$ in (1.1) of dimension one in the spatial variable (cf. Ablowitz and Segur [1], Bona and Fokas [9], Fokas [42], Fokas and Ablowitz [44], Kamvissis [56], Lenells and Fokas [69], and the references therein). The key step of performing the IST is to find the scattering data (namely, the time evolution of the eigenfunctions, the norm constants, and the reflection coefficient) and then solve for a nonlinear integro-differential equation for the scattering data, which however can be very challenging because of the nonhomogeneous boundary condition(s). Another disadvantage of IST is that one can not apply it to study the existence of the globe solution in Sobolev space (cf. Bu [24]). There are also numerous numerical studies on NLS equations (cf. Kaup [62, 63]).

Mathematically, the method of nonlinear analysis, especially Harmonic analysis, is an efficient machinery to study the NLS equations as well, which is also a fundamental part of this thesis. Ever since the middle of the twentieth century, the idea of applying Fourier analysis and integral theory has provided significant contributions to the research of integrable differential equations, such as NLS equations or KdV equation. The general strategy usually is to transform the original equation into an equivalent integral equation with the nonlinear term and the initial and boundary conditions combined, and then conduct an argument by using Banach fixed point theorem (also known as the Contraction Mapping principle). In the attempt to fulfill the use of this method, it is impossible to neglect the Strichartz's estimates.

This critical technique was put forward as of late 1970s, when Strichartz first presented estimate as a Fourier restriction theorem for wave equation in his far-reaching paper [80], which initiated the exploration to estimates on operators that lead to the solution. It was later nicely generalized and applied to solve the initial value problems (IVPs), or namely the Cauchy problem, considering the entire space as the domain for the solution by Ginibre and Velo with a more concise proof in [49], Yajima in [88], and Cazenave and Weissler in [30]. Additionally, the endpoints case of Strichartz's estimates were provided by Keel and Tao [64] and Tao [83]. Nowadays, analyzing estimates based on the Strichartz's inequalities with the norms of Sobolev spaces $W^{s,r}$, Besov spaces $B^s_{p,q}$ or many other function spaces has been the primary task to the well-posedness problems. There is a key concept along with Strichartz's inequalities

Definition 1.4 (Admissible Pair). *Let* $r \in [2,\infty]$ *if* $N = 1$ *or* $r \in [2,\infty)$ *if* $N = 2$ *or* $r \in [2, 2N/(N-2)]$ *if* $N > 2$. (q,r) *is said to be an admissible pair with respect to* $\mathbb{R}^N$ *if* $\frac{2}{Nq} + \frac{1}{r} = \frac{1}{2}$.

In a pure IVP, one intends to search for the solution over the entire space vanishing at infinity with properly chosen initial data. The problem is as follows,

$$\begin{cases} i\partial_t u + \Delta_{\boldsymbol{x}} u + g(u) = 0 & (\boldsymbol{x}, t) \in \mathbb{R}^N \times I \\ u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}) \, . \end{cases} \tag{1.2}$$

Decades of efforts were devoted to the IVP and a number of results are established, among of which there is a basic result for well-posedness from the angle of Semi-group theory (cf. Pazy [72]). Given a complex Hilbert space X and a complex-valued self-joint negative operator A on X with domain $D(A)$, let $\overline{A}$ be the extension of A to the dual space of $D(A)$, i.e. $(D(A))^*$. Suppose $\varphi \in X$ and g is a Lipschitz continuous operator on bounded sets of X. Then there exists a unique solution $u \in C(\mathbb{R}; X) \bigcap C^1(\mathbb{R}; (D(A))^*)$ of the equation

$$i\partial_t u + \overline{A} u + g(u) = 0 \, , \quad u(0) = \varphi \, .$$

If choosing a stronger initial condition $\varphi \in D(A)$, one can expect a nicer solution in $C(\mathbb{R}; D(A)) \bigcap C^1(\mathbb{R}; X)$. Note that conservation laws are also discussed for considering the global behavior of the solution (cf. Cazenave and Haraux [29], Segal [76]). Nevertheless, this result is not quite applicable in general for the local Cauchy problems when the nonlinearities are not necessarily Lipschitz continuous.

From above, one can see that the Semi-group theory gives the well-posedness in an abstract way. With further details of the equation, more powerful results will be obtained and a long list of contributions to the Cauchy problem can be given by many mathematicians. Ginibre and Velo verified the global existence and uniqueness for the solution of problem (1.2) with certain general nonlinear $g(u)$ in the space $C(\mathbb{R}; H^1)$ [46] and obtained asymptotical behavior in [47] for any $N \geq 2$. Moreover, if $g(u) = \lambda |u|^{p-2} u$, then the conclusion holds for $2 \leq p < 2N/(N-2)$ if $\lambda > 0$ and $2 \leq p < 2(N+2)/N$ if $\lambda < 0$. Kato explicitly applied the Strichartz's estimate for his work on the NLS equation to build the contraction mapping, which is known as Kato's method. He used $L^q(L^r)$-type of Lebesgue spaces as the auxiliary spaces and showed the well-posedness of the IVP for solution in $C(\mathbb{R}; H^1)$ and weaker assumptions (cf. [58, 59]). The results for the well-posedness of a solution in $C(\mathbb{R}; L^2(\mathbb{R}^N))$ (L^2-solution) or in $C(\mathbb{R}; H^2(\mathbb{R}^N)$ (H^2-solution) were added to the project as well (cf. Kato [59] and Y. Tsutrumi [86, 87]). Thereafter, the general results of well-posedness for solution included in the space $C(\mathbb{R}; H^s(\mathbb{R}^N))$ were presented with $s < N/2$ (cf. Cazenave and Weissler [32], Fang and Han [40], Fang and Zhong [41], and Kato [60]) and with $s > N/2$ (cf. [48, 49]).

Note that to have a smoother solution (with higher order derivatives), the discussion is likely to be restricted by Sobolev embedding theorem which causes some strict conditions on the solution's regularity (the index s as in the L^r-based Sobolev space $W^{s,r}$) and on the nonlinearity regarding to the auxiliary function spaces. In particular, for solutions that are expected in the space $C(\mathbb{R}; H^s(\mathbb{R}^N))$ with $s < N/2$ and $g(u) = \lambda |u|^{p-2} u$, we say that the problem is subcritical if $p - 2 < 4/(N - 2s)$ and is critical if $p - 2 = 4/(N - 2s)$. The critical cases are usually studied separately (cf. Cazenave and Weissler [30, 31, 32], and Cazenave, Fang and Han [28]).

Eventually one can get the sufficient conclusion on the IVP: let $[s]$ be the largest integer smaller or equal to s and let (q,r) be an admissible pair.

(i) If $0 < s < N/2$ with $N \leq 4$ and $2 < p \leq 2 + \frac{4}{N-2s}$, or $2 \leq s < \min\{4, N/2\}$ with $4 < N \leq 8$ and $s < p \leq 2+\frac{4}{N-2s}$, or $4 \leq s < N/2$ with $N > 8$ and $s-1 < p \leq 2+\frac{4}{N-2s}$, there is an admissible pair (q,r) such that given $\varphi \in H^s(\mathbb{R}^N)$, the equation is conditionally well-posed with the unique maximal solution $u \in C\left((-T_{\min}, T_{\max});\ H^s(\mathbb{R}^N)\right)$ satisfying an extra condition that

$$u \in L^q_{\text{loc}}\left((-T_{\min}, T_{\max});\ B^s_{r,2}(\mathbb{R}^N)\right)\ .$$

(ii) If $s > N/2$ and $\max\{2, [s]+1\} < p < \infty$ ($[s] < p < \infty$ if $s \in \mathbb{Z}$), or $2 < p < \infty$ when p is an even integer, for every $\varphi \in H^s(\mathbb{R}^N)$ the equation is unconditionally well-posed and has a unique maximal solution $u \in C\left((-T_{\min}, T_{\max});\ H^m(\mathbb{R}^N)\right)$.

The global well-posedness is obtained for the H^1-solution in higher dimensional spaces $\mathbb{R}^N$.

For every $\varphi \in H^1(\mathbb{R}^N)$, the maximal H^1-solution to (1.2) ($g(u) = \lambda|u|^{p-2}u$) given by the conclusion of local well-posedness for the IVP is global and $\sup_t\{\|u(t)\|_{H^1} : t \in \mathbb{R}\} < \infty$, i.e., the choices of $T_{\max}$ and $T_{\min}$ are independent of the initial data.

However, with more general nonlinearity, H^1 initial data require further constraints. Note that the assumption on the initial data φ can be altered. For example, if given the oscillating data, i.e. $\varphi_\beta = \varphi \exp\left(i\frac{\beta|\boldsymbol{x}|^2}{4}\right)$ where $\varphi \in H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |\boldsymbol{x}\,\varphi(\boldsymbol{x})|^2\, d\boldsymbol{x} < \infty$, then a global solution can be found (cf. Cazenave and Weissler [33]). Some of the results listed above are organized in Bourgain [18] and Cazenave [27].

Furthermore, let the initial data be periodic in space for IVP of the NLS equation where the equation is posed on a torus $\mathbb{T}^N \times I$ (the period equals 1). The problem is given of the form:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{p-2}u = 0 & (\boldsymbol{x}, t) \in \mathbb{T}^N \times I\,, \\ u(0) = \varphi\,. \end{cases} \tag{1.3}$$

Bourgain in his foundational work [14, 16] developed a method with harmonic analysis analogous to Strichartz's estimates for solving periodic (in spacial variables) NLS equations and obtained the global well-posedness using an invariant measure, Gibbs measure [17]. Fokas in [43] studied one dimensional periodic problem with IST method. We present some Bourgain's important results here.

When $u = u(\boldsymbol{x}, t)$ is 1-periodic in each coordinate of $\boldsymbol{x}$; i.e. $\boldsymbol{x} \in \mathbb{T}^N$

(i) ($N = 1$) The NLS equation (1.3) is locally well-posed for $\varphi \in H^s(\mathbb{T})$ given $2 < p < \frac{6-4s}{1-2s}$.

(ii) ($N = 2$) (1.3) is globally well-posed for $p = 4$ and $\varphi \in H^1(\mathbb{T}^2)$ with $\|\varphi\|_{L^2}$ sufficiently small and globally well-posed for $p \geq 4$ and $\|\varphi\|_{H^1}$ sufficiently small.

(iii) ($N = 3$) (1.3) is globally well-posed for $4 \leq p < 6$ and $\varphi \in H^1(\mathbb{T}^3)$ with $\|\varphi\|_{H^1}$ is sufficiently small.

(iv) ($N \geq 4$) (1.3) is locally well-posed for $4 \leq p < \frac{2N-4s+4}{N-2s}$ and $s > \frac{3N}{N+4}$.

It is worth mentioning that Takaoka and Tzvetkov [82] continued the study on a two dimensional problem considering the solution on $\boldsymbol{x} \in \mathbb{R} \times \mathbb{T}$ and verified that the equation $i\partial_t u + \Delta u + |u|^{p-2}u = 0$ with $u(0) = \varphi$ is globally well-posed for $2 < p < 4$ on $\mathbb{R} \times \mathbb{T}$ given $\varphi \in L^2(\mathbb{R} \times \mathbb{T})$, and is globally well-posed for $p = 4$ with $\|\varphi\|_{L^2(\mathbb{R}\times\mathbb{T})}$ sufficiently small.

Comparing to an IVP, one may replace $\mathbb{R}^N$ in (1.2) by a subset Ω and prescribe (Dirichlet) boundary conditions $u|_{\partial\Omega} = h(\boldsymbol{x}, t)$ for some h that has certain smoothing condition to have an initial boundary value problem (IBVP). Although for IBVPs, there are relatively fewer results than these for IVPs in the past, they are gaining more attentions these days as more techniques are developed. The studies on IBVPs with homogeneous boundary data, i.e. $u|_{\partial\Omega} = 0$, follow behind the work of Brezis and Gallouet [19], M. Tsutsumi [84, 85], Y. Tsutsumi [86]. In the study of the nonhomogeneous IBVP, one can see references by Holmer [52], Bona, Sun and Zhang [10]. Moreover, the global well-posedness for the H^1-solution posed on a subset Ω of $\mathbb{R}^N$ was provided by Bu [20, 21, 22, 23], Bu, Shull, Wang and Chu [25], Bu, Tsutaya and Zhang [26], Strauss and Bu [79].

For one-dimensional IBVP posed on the half line

$$\begin{cases} iu_t + u_{xx} + \lambda|u|^{p-2}u = 0\,, & (x,t) \in \mathbb{R}^+ \times (0,T)\,, \\ u(x,0) = \varphi(x)\,, \quad u(0,t) = h(t)\,, \end{cases} \tag{1.4}$$

the low regularity feature was studied by Holmer based on Riemann-Liouville fractional integral theory. Bona, Sun and Zhang [10] provided a more general result using the boundary integral method. The main results are following

(i) If $0 \leq s < \frac{1}{2}$ with $3 \leq p < \frac{6-4s}{1-2s}$, (1.4) is conditionally locally well-posed in $H^s(\mathbb{R}^+)$; i.e. there exists a $T > 0$ depending on s, $\varphi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(0,T)$ small, (1.4) admits a unique solution $u \in C\left([0,T];\, H^s(\mathbb{R}^+)\right)$ with $\|u\|_{L^q([0,T];\, L^r(\mathbb{R}^+))} < \infty$ satisfying the blow-up and continuous dependence properties.

(ii) If $s > \frac{1}{2}$ and $3 \leq p < \infty$ for $\varphi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(0,T)$, (1.4) is unconditionally locally well-posed where $u \in C\left([0,T];\, H^s(\mathbb{R}^+)\right)$.

(iii) (1.4) is globally well-posed for $s \geq 1$, $\varphi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{s+1}{2}}_{\mathrm{loc}}(\mathbb{R}^+)$, if either $p \geq 3$ and $\lambda < 0$ or $3 \leq p \leq 4$ and $\lambda > 0$.

Meanwhile, Bona, Sun and Zhang [10] studied the IBVP on a finite interval $(0, L)$,

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2} u = 0 \,, & (x,t) \in [0,1] \times (0,T) \\ u(x,0) = \varphi(x) \,, \quad u(0,t) = h_1(t) \,, \quad u(1,t) = h_2(t) \,. \end{cases} \tag{1.5}$$

They showed the following

(i) If $0 \le s < \frac{1}{2}$ with $3 \le p < \frac{6-4s}{1-2s}$, (1.5) is conditionally locally well-posed in $H^s(0,1)$ for $h_j \in H^{\frac{s+1}{2}}(0,T)$ with $\|u\|_{L^4[0,L]\times[0,T]} < \infty$ as the extra space, $j = 1, 2$.

(ii) If $s > \frac{1}{2}$ and $3 \le p < \infty$ for $\varphi \in H^s(0,L)$ and $h_1,\ h_2 \in H^{\frac{s+1}{2}}(0,T)$, (1.5) is unconditionally locally well-posed where $u \in C\left([0,T];\, H^s(0,L)\right)$.

(iii) (1.5) is globally well-posed for $s \ge 1$, $\varphi \in H^s[0,L]$ and $h_1,\ h_2 \in H_{\text{loc}}^{\frac{s+1}{2}}(\mathbb{R}^+)$, if either $p \ge 3$ and $\lambda < 0$ or $3 \le p \le \frac{10}{3}$ and $\lambda > 0$.

Following these works, the main purpose of this thesis is to discuss the well-posedness for IBVP of the NLS equation posed on a half plane $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ (or a half space $(x, y) \in \mathbb{R}^n \times \mathbb{R}^+$) and on a strip $(x, y) \in \mathbb{R} \times [0, L]$; i.e.

$$\begin{cases} iu_t + u_{xx} + u_{yy} + \lambda |u|^{p-2} u = 0 \,, & (x,y,t) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \\ u(x,y,0) = \varphi(x,y) \,, & (x,y) \in \mathbb{R} \times \mathbb{R}^+ \\ u(x,0,t) = h(x,t) \,, & (x,t) \in \mathbb{R} \times \mathbb{R} \end{cases} \tag{1.6}$$

and

$$\begin{cases} iu_t + u_{xx} + u_{yy} + \lambda |u|^{p-2} u = 0 \,, & (x,y,t) \in \mathbb{R} \times [0,1] \times \mathbb{R} \\ u(x,y,0) = \varphi(x,y) \,, & (x,y) \in \mathbb{R} \times [0,1] \\ u(x,0,t) = h_1(x,t) \,, \; u(x,1,t) = h_2(x,t) \,, & (x,t) \in \mathbb{R} \times \mathbb{R} \end{cases} \tag{1.7}$$

where $p \ge 3$ is a constant (note that we only study the case for $p \ge 3$, although many of the results in the paper still hold for $p > 2$; also in $\mathbb{R}^n \times \mathbb{R}^+$, u_{xx} is replaced by the Laplacian $\Delta_x u$ in $\mathbb{R}^n$). The initial data φ and the boundary data h, h_1 and h_2 satisfy certain smoothing conditions and compatibility conditions.

The basic plot of the proof especially for local well-posedness is to derive an equivalent integral equation from the NLS equation above by semi-group theory and perform Banach fixed point argument to obtain the existence and uniqueness, and then obtain the result globally using the energy estimates. Moreover along the whole process, it is critical to select suitable auxiliary function spaces and prepare necessary estimates using Strichartz's inequality and Bourgain's method, etc. Although two IBVPs share the general strategy, they are independent problems because the techniques adopted are different.

The idea of proof of the local well-posedness for (1.6) is inspired by the method introduced for the KdV equation by Bona, Sun and Zhang in [11]. First, by properly extending the initial condition and nonhomogeneous term (i.e., the nonlinear term) to the half plane $y < 0$ so that they are defined in $\mathbb{R}^2$, the original equation is decomposed into three parts: one in $\mathbb{R}^2$ with initial condition and linear Schrödinger equation, one in $\mathbb{R}^2$ with homogeneous initial condition and nonhomogeneous equation, and the last one (the most important one) with a nonhomogeneous boundary condition, homogeneous initial condition and linear equation. The first two equations have been studied comprehensively and the solutions can be obtained by classical semi-group theory. The solution formula for the last member from decomposition and its corresponding estimates are obtained in this thesis. We apply the Laplace transform to the linear equation in (1.6) with respect to t and derive an integral formula of an operator in terms of boundary condition, called boundary integral operator. Then, the Strichartz's estimates are proved for the boundary integral operator (over the half plane and on its boundary $y = 0$), which gives us the necessary tools for Banach fixed point theorem.

At the end of the discussion, conclusion for the global well-posedness is provided using Energy estimates and corresponding conservation laws in $H^1(\mathbb{R} \times \mathbb{R}^+)$. Then we establish the results below:

Theorem 1.5. *(i) If $0 \leq s < 1$ with $3 \leq p \leq \frac{4-2s}{1-s}$, (1.6) is locally well-posed in $C_t(\mathbb{R}; H^s(\mathbb{R}\times\mathbb{R}^+))$ for $\varphi \in H^s(\mathbb{R}\times\mathbb{R}^+)$ and $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R};\ L_x^2(\mathbb{R})) \cap L_{loc}^2\left(\mathbb{R};\ H_x^{s+\frac{1}{2}}(\mathbb{R})\right)$ and additionally*

$$u \in L_t^q\left([0,T];\ W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right) \cap L_t^\gamma\left([0,T];\ W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)$$

(ii) If $s = 1$ and $3 \leq p < \infty$, (1.6) is locally well-posed on $C_t(\mathbb{R}; H^s(\mathbb{R}\times\mathbb{R}^+))$ for φ and h properly chosen from the spaces above with an extra condition

$$u \in L_t^{\tilde{q}}\left([0,T];\ W_{xy}^{s,\tilde{r}}(\mathbb{R}\times\mathbb{R}^+)\right) \text{ for any } \tilde{r} > 2$$

(iii) If $s > 1$ and $3 \leq p < \infty$ (assume $s \leq p-1$ for $s \in \mathbb{Z}$ or $[s] \leq p-2$ for $s \notin \mathbb{Z}$ when p is not an even integer), (1.6) is unconditionally locally well-posed on $C_t(\mathbb{R}; H^s(\mathbb{R}\times\mathbb{R}^+))$.

(iv) If $0 \leq s \leq 1$ is given, then the condition can be removed, and therefore the well-posedness is unconditional.

(v) If either $3 \leq p < \infty$ and $\lambda < 0$ or $3 \leq p \leq \frac{10}{3}$ and $\lambda > 0$, (1.6) is globally well-posed in H^1 for $\varphi \in H^1(\mathbb{R}\times\mathbb{R}^+)$ and $h \in H^1\left(\mathbb{R};\ L_x^2(\mathbb{R})\right) \cap L^2\left(\mathbb{R};\ H_x^{3/2}(\mathbb{R})\right)$.

To consider the local well-posedness for (1.7), we use the ideas in [82]. For this part of investigation, we set the same decomposition as that for (1.6), except that the homogeneous boundary conditions will be included in the first two equations. We use Fourier series to formulate the solution, which limits the use of Strichartz's estimates. Based on the work in [16] and [82] and many others, we can eventually show the local well-posedness. As a part of the whole argument, the global well-posedness in $H^1(\mathbb{R}\times[0,1])$ is also completed. It follows the main point of this problem:

Theorem 1.6. *(i) If $0 \le s < \frac{1}{2}$ with $3 \le p \le 4$ or $\frac{1}{2} \le s < 1$ with $3 \le p \le \frac{3-2s}{1-s}$ or $s = 1$ with $3 \le p < \infty$, (1.7) is conditionally locally well-posed in $C\left(\mathbb{R}; H^s(\mathbb{R} \times (0,1))\right)$ for $\varphi \in H^s(\mathbb{R} \times (0,1))$ and $h_j \in H_{loc}^{\frac{s+1}{2}}\left(\mathbb{R}; L^2(\mathbb{R})\right) \cap L_{loc}^2\left(\mathbb{R}; H^{s+1}(\mathbb{R})\right)$, $j = 1, 2$ and*

$$u \in L_t^r\left([0,T]; W_{xy}^{s,r}(\mathbb{R} \times [0,1])\right) \text{ for } r \in [2,4]$$

(ii) If $s > 1$ and $3 \le p < \infty$ (assume $s \le p-1$ for $s \in \mathbb{Z}$ or $[s] \le p-2$ for $s \notin \mathbb{Z}$ when p is not an even integer), (1.7) is unconditionally locally well-posed for small φ and h_1, h_2 chosen from the same spaces.

(iii) If $0 \le s \le 1$ is given, then the condition can be removed, and therefore the well-posedness is unconditional.

(iv) (1.7) is globally well-posed in H^1 for $\varphi \in H^1(\mathbb{R} \times [0,L])$ and h_1, $h_2 \in H_{loc}^1\left(\mathbb{R}; L^2(\mathbb{R})\right) \cap L_{loc}^2\left(\mathbb{R}; H^2(\mathbb{R})\right)$ with any $\gamma > 0$, if either $p \ge 3$ and $\lambda < 0$ or $p = 3$ and $\lambda > 0$.

Many classic techniques for the study of NLS equations are developed from the works contributed for the KdV equation (cf. [13, 15, 36, 53, 66]). For the general materials in functional analysis, interpolation theory, Sobolev Embedding theorem, etc., one can also see references provided in [2, 7, 72, 78, 89]. Other studies on smoothing properties and regularity that will be applied in this thesis can be found in papers by Constantin and Saut [38], Kenig, Ponce and Vega [67], etc.

This thesis is organized as follows. Chapter 2 lists a collection of notations and well-known preliminary lemmas on integration theory, Fourier analysis, Sobolev spaces and some useful inequalities as preparations for the later chapters. In Chapter 3, we establish the local and global well-posedness results for the NLS equation posed over the half plane $\mathbb{R} \times \mathbb{R}^+$. In Section 3.1 we begin the investigation by decomposing the initial boundary problem into three simpler equations: a linear equation with nonhomogeneous initial condition (the Cauchy problem), a linear equation with nonhomogeneous boundary condition, and a non-linear equation with both homogeneous initial and boundary conditions, which forms the Duhamel forcing boundary operator, the semi-group and Duhamel inhomogeneous solution operator respectively. Thus we can obtain the equivalent integral form of the original equation. Some smooth conditions and compatibility conditions are also given. Section 3.2 provides a detailed work of deriving the formulation and estimates for the corresponding operators of the solution to each problem. In Section 3.3 we perform the Contraction mapping theorem for the well-posedness, which also contains three subsections. In Section 3.4, we discuss the regularity of the result in 3.3 and establish the unconditional well-posedness. Section 3.5 provides the global well-posedness result for the half plane problem. Chapter 4 is devoted to the discussion on the IBVP over a strip domain $\mathbb{R} \times [0,1]$ in space with a parallel structure to Chapter 3. In Chapter 5, we briefly talk about the IBVP posed on the square domain $[0,L] \times [0,L]$ with formulation of each decomposed equation. In Chapter 6 the future research plan is presented.

Chapter 2

Preliminary Tools

In this Chapter, we introduce some notations, definitions, and fundamental lemmas in Analysis.

2.1 Definitions and Notations

1). For two real-valued terms A and B, write: (a) $A \approx B$ if there is a positive constant c so that $A = cB$; (b) $A \sim B$ if there exist two independent positive numbers c_1 and c_2 so that $c_1 A \leq B \leq c_2 A$;(c) $A \lesssim B$ (or $A \gtrsim B$) if there is a positive constant c so that $A \leq cB$ (or $A \geq cB$).

2). $\mathcal{D}(\mathbb{R}^N) = C_0^\infty(\mathbb{R}^N)$ denotes the space of test functions and $\mathcal{D}'(\mathbb{R}^N)$ denotes the space of distributions.

3). Define a function $\psi(x)$ as the cut-off function if $\psi \in \mathcal{D}(\mathbb{R}^N)$:

$$\psi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases} \tag{2.1}$$

Moreover we can choose ψ to be non-increasing when $1 \leq |x| \leq 2$ if necessary.

4). $\mathcal{S}(\mathbb{R}^N)$ denotes the space of all complex-valued functions u on $\mathbb{R}^N$ which are of class C^∞ so that $|\boldsymbol{x}|^k |D^\alpha u(\boldsymbol{x})|$ is bounded for every $k \in \mathbb{N}$ and every multi-index α. $\mathcal{S}'(\mathbb{R}^N)$ denotes the space of tempered distributions.

5). For $u \in \mathcal{S}'(\mathbb{R}^N)$, $\mathcal{F}[u]$ denotes the Fourier transform of u, i.e.,

$$\mathcal{F}[u](\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} u(\boldsymbol{x})\, d\boldsymbol{x}\,.$$

Let $\boldsymbol{x} = (x_1, \cdots, x_j, \cdots, x_N)$. $\mathcal{F}_{x_j}[u]$ denotes the Fourier transform of u with respect to only x_j; i.e.,

$$\mathcal{F}_{x_j}[u](x_1, \cdots, \xi_j, \cdots, x_N) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix_j\xi_j} u(\cdots, x_j, \cdots)\, dx_j\,.$$

Besides, $\widehat{u} = \mathcal{F}[u]$ and $\widehat{u^{x_j}} = \mathcal{F}_{x_j}[u]$. In addition, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Moreover, $\widehat{u}$ also denotes the coefficient for Fourier series; i.e. for $\boldsymbol{n} \in \mathbb{Z}^N$

$$\widehat{u}(\boldsymbol{n}) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{T}^N} e^{-i\boldsymbol{x}\cdot\boldsymbol{n}} u(\boldsymbol{x})\, d\boldsymbol{x}$$

Then $u(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} e^{i\boldsymbol{x}\cdot\boldsymbol{n}}\widehat{u}(\boldsymbol{n})$, etc. Note that for any $a \in \mathbb{R}/\{0\}$,

$$\widehat{u}(\boldsymbol{\xi}) \eqsim \int_{\mathbb{R}^N} e^{-ia\boldsymbol{x}\cdot\boldsymbol{\xi}} u(\boldsymbol{x})\, d\boldsymbol{x}\,.$$

6). For Banach spaces X and Y, write $X \hookrightarrow Y$ if X is continuously embedded in Y; i.e. $X \subset Y$ with a continuous injection.

7). X^s denotes a Banach space such that if $s_1 < s_2$ then $X^{s_2} \hookrightarrow X^{s_1}$.

8). $L^r(\Omega)$ denotes the Lebesgue space with $r \in [1, \infty]$; we denote the conjugate index of r by r', that is $\frac{1}{r} + \frac{1}{r'} = 1$.

9). $W^{s,r}(\mathbb{R}^N)$ denotes the fractional Sobolev space for $r \in [1, \infty]$ and $s \in \mathbb{R}$ with the norm

$$\|u\|_{W^{s,r}(\mathbb{R}^N)} = \left\| \mathcal{F}^{-1} \left[(1 + |\boldsymbol{\xi}|)^{\frac{s}{2}}\, \widehat{u}(\boldsymbol{\xi}) \right] \right\|_{L^r(\mathbb{R}^N)} \text{ for every } u \in W^{s,r}(\mathbb{R}^N)$$

10). $H^s(\mathbb{R}^N)$ denotes the L^2-based Sobolev space, i.e. $H^s = W^{s,2}(\mathbb{R}^N)$.

11). $B^s_{r,q}(\mathbb{R}^N)$ denotes a Besov space for $s \in \mathbb{R}$ and $1 \le r,\, q \le \infty$.

12). $\dot{W}^{s,r}$, $\dot{H}^s$ and $\dot{B}^s_{r,q}$ denote the homogeneous version of $W^{s,r}$, H^s and $B^s_{r,q}$, respectively.

13). Let $X = L^r$, $W^{s,r}$, H^s or $B^s_{r,q}$. Let $u : \Omega \subset \mathbb{R}^N \to \mathbb{C}$ and $u = u(\boldsymbol{x}) \in X(\Omega)$. Furthermore, let u^* be an extension of u to $X(\mathbb{R}^N)$, i.e. $u^* : \mathbb{R}^N \to \mathbb{C}$ and $u^*|_\Omega = u$. Then define $\|u\|_{X(\Omega)} = \inf_{u^*\in X(\mathbb{R}^N)} \|u\|_{X(\mathbb{R}^N)}$.

We also write $X_{\boldsymbol{x}}$ to specify the independent variable $\boldsymbol{x}$.

14). Let $u \in H^s(\mathbb{R}^N)$ with $s \in \mathbb{R}$ and let α be the multi-index with $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $|\alpha| = s$. We denote the fractional derivative of order α by

$$D^\alpha u(\boldsymbol{x}) = \mathcal{F}^{-1} \left[|\boldsymbol{\xi}|^\alpha \widehat{u}(\boldsymbol{\xi}) \right]$$

where $|\boldsymbol{\xi}|^\alpha = |\xi_1|^{\alpha_1} \cdots |\xi_N|^{\alpha_N}$.

15). $W^{\sigma,q}(I,X)$ denotes the Banach space of measurable functions $u: I \to X$ such that $\|\partial_t^\sigma u(t)\|_X \in L^q(I)$ with the norm

$$\|u\|_{W^{\sigma,q}(I,X)} = \|\partial_t^\sigma u\|_{L^q(I,X)}$$

16). $X^{\sigma,s}$ denotes a Bourgain space over $\mathbb{R} \times \mathbb{T}$ by

$$X^{\sigma,s} = \left\{ u \,\middle|\, \|u\|_{X^{\sigma,s}} = \|\mathcal{U}(-t)u\|_{H_t^\sigma(H_x^s)} < \infty \right\}, \tag{2.2}$$

where $\mathbb{T}$ stands for the quotient $\mathbb{R}/\mathbb{Z}$ and $\mathcal{U}(t)$ is given in [82] by

$$\mathcal{U}(t)\phi = \int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} e^{-i\left(\xi^2+n^2\right)t+i(x\xi+yn)} \widehat{\phi}(\xi, n)\, d\xi\,. \tag{2.3}$$

Therefore

$$\|u\|_{X^{\sigma,s}} = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} (1+|\xi|+|n|)^{2s} \left(1+\left|\lambda+\xi^2+n^2\right|\right)^{2\sigma} |\widehat{u}(\xi,n,\lambda)|^2 \, d\xi\, d\lambda \right\}^{\frac{1}{2}} \tag{2.4}$$

17). $e^{it\Delta}\varphi$ denotes the group of isometries generated by the skew-adjoint operator $i\Delta$ and also the solution of

$$\begin{cases} i\partial_t u + \Delta u = 0 & \text{in } \mathbb{R}^N \times I \ni 0\,, \\ u(0) = \varphi\,, \end{cases}$$

with the representation given in [27]; i.e. we write, interchangeably, for $\boldsymbol{x}$, $\boldsymbol{\xi} \in \mathbb{R}^N$

$$e^{it\Delta}\varphi = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i|\boldsymbol{\xi}|^2 t + i\boldsymbol{x}\cdot\boldsymbol{\xi}} \widehat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

The notation $W_{\mathbb{R}^N}(t)\varphi$ is also in use.

2.2 Lemmas

Here is a miscellaneous collection of basic propositions of functional analysis and integration theory presented as lemmas in support of the whole argument of this thesis.

Lemma 2.1. *Some inequalities*

(i) For a, $b \geq 0$,

$$ab \leq \frac{a^q}{q} + \frac{b^{q'}}{q'} \tag{2.5}$$

where the equality holds when $a^q = b^{q'}$.

(ii) *With $0 \le r \le n$ and $r \in \mathbb{R}$*

$$a^{n-r} \cdot b^r \le a^n + b^n \tag{2.6}$$

(iii) *For each $\alpha \ge 1$,*

$$(a^\alpha + b^\alpha) \le (a+b)^\alpha \le 2^{\alpha-1}(a^\alpha + b^\alpha) \tag{2.7}$$

i.e. $(a+b)^\alpha \sim a^\alpha + b^\alpha$.

The result is reversed if $0 < \alpha \le 1$.

Proof. (Outlines): In (2.5) letting $p = \frac{n}{n-r}$, then

$$a^{n-r} \cdot b^r \le \frac{(a^{n-r})^{\frac{n}{n-r}}}{\frac{n}{n-r}} + \frac{(b^r)^{\frac{n}{r}}}{\frac{n}{r}} = a^n \cdot \left(1 - \frac{r}{n}\right) + b^n \cdot \frac{r}{n} \le a^n + b^n\,.$$

Note that the left half inequality of (2.7) can be verified by showing that

$$f(x) = (1+x)^\alpha - x$$

is an increasing function on $[0, \infty)$ $(f' \ge 0)$ and $f(0) = 1$; the right half holds because $f(x) = x^\alpha$ which is convex on $(0, \infty)$. $\square$

Lemma 2.2 (Gagliardo-Nirenberg Inequality)**.** *We have:*

(i) *(cf. [27]) Suppose $1 \le p, q, r \le \infty$, $j, m \in \mathbb{N}$, with $0 \le j \le m$. If $\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-a}{q}$ for $a \in \left[\frac{j}{m}, 1\right]$, $\exists\ C = C(N, m, j, a, q, r)$ such that*

$$\sum_{|\alpha|=j} \|D^\alpha u\|_{L^p} \le C \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{L^r} \right)^a \cdot \|u\|_{L^q}^{1-a} \qquad \text{for } u \in \mathcal{D}\left(\mathbb{R}^N\right) \tag{2.8}$$

(ii) *See Corollary 1.5 in [51]. Let $1 < p, q, r < \infty$, $s, s_1 \in \mathbb{R}$, $0 \le \theta \le 1$. Then the fractional Gagliardo-Nirenberg inequality of the form*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|_{\dot{H}^{s_1,r}}^\theta \cdot \|u\|_{L^q}^{1-\theta} \tag{2.9}$$

holds if and only if $\frac{N}{p} - s = \theta\left(\frac{N}{r} - s_1\right) + (1-\theta)\frac{N}{q}$ with $s \le \theta s_1$.

Lemma 2.3 (Sobolev Embedding Theorem - cf. [27] and [2])**.** *If Ω has Lipschitz continuous boundary in $\mathbb{R}^N$, then*

(i) *for $s > 0$, $1 \le r < N/s$ and every $q \in \left[r, \frac{Nr}{N-sr}\right]$, we have*

$$W^{s,r}(\Omega) \hookrightarrow L^q(\Omega); \tag{2.10}$$

moreover for $u \in W^{s,r}(\Omega)$ and $|\alpha| = s$

$$\|u\|_{L^r(\Omega)} \lesssim \|D^\alpha u\|_{L^q(\Omega)}. \tag{2.11}$$

(ii) when $r = N/s$, for every $q \in [r, \infty)$ (2.10) and (2.11) hold.

(iii) for $s > N/2$,

$$H^s(\Omega) \hookrightarrow C_b(\Omega) \tag{2.12}$$

and

$$\|u\|_{L^\infty(\Omega)} \lesssim \|D^\alpha u\|_{L^2(\Omega)} \tag{2.13}$$

Lemma 2.4 (Strichartz's Estimates - cf. [27, 78])**.** *We have*

(i) For every $\varphi \in H^s(\mathbb{R}^N)$, the function $t \mapsto e^{it\Delta}\varphi$ belongs to

$$L^q(\mathbb{R}; W^{s,r}(\mathbb{R}^N)) \bigcap C(\mathbb{R}; H^s(\mathbb{R}^N))$$

for every admissible pair (q, r). Moreover,

$$\left\| e^{i\cdot\Delta}\varphi \right\|_{L^q(\mathbb{R}; W^{s,r}(\mathbb{R}^N))} \lesssim \|\varphi\|_{H^s(\mathbb{R}^N)} \tag{2.14}$$

(ii) Let I be an interval of $\mathbb{R}$ (bounded or not), $J = \overline{I} \ni 0$. If (γ, ρ) is an admissible pair and $f \in L^{\gamma'}(I; H^{s,\rho'}(\mathbb{R}^N))$, then for every admissible pair (q, r), the function

$$t \mapsto \Phi_f(t) = \int_0^t e^{i(t-\tau)\Delta} f(\tau)\, d\tau \text{ for } t \in I$$

belongs to $L^q(I; W^{s,r}(\mathbb{R}^N)) \bigcap C(J; H^s(\mathbb{R}^N))$. Additionally,

$$\|\Phi_f\|_{L^q(I; W^{s,r}(\mathbb{R}^N))} \lesssim \|f\|_{L^{\gamma'}(I; H^{s,\rho'}(\mathbb{R}^N))} \tag{2.15}$$

Lemma 2.5. *Another L^2-based estimate - Lemma 3.1 in [11]*

$$\left\| \int_0^\infty e^{-y\eta} f(\eta)\, d\eta \right\|_{L^2(\mathbb{R}^+)} \leq \|f\|_{L^2(\mathbb{R}^+)} \tag{2.16}$$

Lemma 2.6 (Representation Lemma)**.** *Suppose $g \in L^{q'}(\Omega)$. Define a linear functional $\mathcal{G}$ over $L^q(\Omega)$ such that $\mathcal{G}(f) := \int_\Omega f(\boldsymbol{x}) \cdot \overline{g}(\boldsymbol{x})\, d\boldsymbol{x}$ for any $f \in L^q$. Then*

$$\|g\|_{L^{q'}} = \|\mathcal{G}\|_{(L^q)^*} \tag{2.17}$$

The following two lemmas are used to study differentiation of fractional order:

Lemma 2.7 (The Chain Rule for Fractional Derivatives - *Proposition 3.1* in [35])**.** *Let $0 < s < 1$ and α be a nonnegative multi-index with $|\alpha| = s$. Let $u : \mathbb{R}^2 \to \mathbb{C}$ and $f \in C^1(\mathbb{C})$. Then*

$$\|D^\alpha f(u)\|_{L^{r_2}} \lesssim \|f'(u)\|_{L^{r_1}} \|D^s u\|_{L^r} \tag{2.18}$$

for $\frac{1}{r_2} = \frac{1}{r_1} + \frac{1}{r}$ with $1 < r$, r_1, $r_2 < \infty$. In particular, if $|f'(u)|$ is uniformly bounded, then

$$\|D^\alpha f(u)\|_{L^r} \lesssim \|f'(u)\|_{L^\infty} \|D^\alpha u\|_{L^r} \tag{2.19}$$

Lemma 2.8 (The Product Rule (Leibnitz Rule) for Fractional Derivatives - *Proposition 3.3* in [35] or *Lemma 2.6* in [65] for high-dimensional case)**.** *Let $0 < s < 1$, u, $v : \mathbb{R}^2 \to \mathbb{C}$ and $f \in C^1(\mathbb{C})$. Then*

$$\|D^\alpha(uv)\|_{L^{r_2}} \lesssim \|v\|_{L^{r_1}} \|D^\alpha u\|_{L^r} + \|u\|_{L^{\tilde{r}_1}} \|D^\alpha v\|_{L^{\tilde{r}}} \tag{2.20}$$

for $\frac{1}{r_2} = \frac{1}{r_1} + \frac{1}{r} = \frac{1}{\tilde{r}_1} + \frac{1}{\tilde{r}}$ with $1 < r,\ r_1,\ \tilde{r},\ \tilde{r}_1,\ r_2 < \infty$.

The proof follows the idea in appendix of [61].

Lemma 2.9 (Inequalities involving the norm of the Bourgain's space)**.** *Recall that*

$$\|u\|_{X^{\sigma,s}} = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} (1 + |\xi| + |n|)^{2s} \left(1 + \left|\lambda + \xi^2 + n^2\right|\right)^{2\sigma} |\widehat{u}(\xi, n, \lambda)|^2 \, d\xi \, d\lambda \right\}^{\frac{1}{2}}$$

(i) ([82]) *Let $(x, y, t) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$. For any $\sigma > \frac{1}{2}$*

$$\|u\|_{L^r(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \tag{2.21}$$

(ii)

$$\|u\|_{L_t^\infty\left(\mathbb{R};\, L^2_{xy}(\mathbb{R}\times\mathbb{T})\right)} \lesssim \|u\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \tag{2.22}$$

(iii) (Lemma 3.2 in [45]) Let $-\frac{1}{2} < \sigma' \leq 0 \leq \sigma \leq \sigma' + 1$. Then for $\mathcal{U}(t)$ by (2.3) and $\psi_T(t)$ a C_0^∞-function compactly supported in $[0, T]$,

$$\left\| \psi_T(t) \int_0^t \mathcal{U}(t-\tau) f(\tau)\, \tau \right\|_{X^{\sigma,s}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \lesssim T^{1-(\sigma-\sigma')} \|f\|_{X^{\sigma',s}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \tag{2.23}$$

Note that the operator $\mathcal{U}(t)$ is the same as $W_0(t)$ in Chapter 4.

Proof. (i) Theorem 2.3 of [82] gives the argument of

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})}.$$

Meanwhile we know that

$$\|u\|_{L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} = \|u\|_{X^{0,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \leq \|u\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})}$$

Then by using interpolation theorem, we land on the result we expected.

(ii) Let $\widehat{u_k} = \widehat{u_k}(\xi, n, \lambda)$ be such that for $|\lambda + \xi^2 + n^2| \in \left[2^{k-1}, 2^k\right)$;.

$$\|u\|_{L_t^\infty\left(\mathbb{R};\, L^2_{xy}(\mathbb{R}\times\mathbb{T})\right)} \lesssim \left\|\widehat{u^t}\right\|_{L^1_\lambda\left(\mathbb{R};\, L^2_{xy}(\mathbb{R}\times\mathbb{T})\right)} \eqsim \|\widehat{u}\|_{L^1_\lambda\left(\mathbb{R};\, L^2_{\xi n}\right)}$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}} \|\widehat{u_k}\|_{L^1_\lambda\left(\mathbb{R};\,L^2_{\xi n}\right)} \approx \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} |\widehat{u_k}(\xi,n,\lambda)|^2 \, d\xi\right)^{\frac{1}{2}} d\lambda \\
&\le \sum_{k\in\mathbb{Z}} 2^{k-1} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} |\widehat{u_k}(\xi,n,\lambda)|^2 \, d\lambda\, d\xi\right)^{\frac{1}{2}} \\
&\le \sum_{k\in\mathbb{Z}} 2^{\frac{k-1}{2}-(k-1)\sigma} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} \left(1+\left|\lambda+\xi^2+n^2\right|\right)^{2\sigma} |\widehat{u_k}(\xi,n,\lambda)|^2 \, d\lambda\, d\xi\right)^{\frac{1}{2}} \\
&\le \sum_{k\in\mathbb{Z}} \|u_k\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})} \\
&\lesssim \|u\|_{X^{\sigma,0}(\mathbb{R}\times\mathbb{T}\times\mathbb{R})}
\end{aligned}
$$

$\square$

Lemma 2.10 (Lemma A-1 in [10]). *Let ψ be a standard cut-off function by (2.1) supported in $(-2,2)$. Then*

$$
\sum_{n=1}^{\infty} \left|\int_0^{\infty} \frac{f(\mu)}{\mu-n}\left(1-\psi\left(n^2-\mu^2\right)\right) d\mu\right|^2 \lesssim \int_0^{\infty} (1+\mu)|f(\mu)|^2 \, d\mu \tag{2.24}
$$

Lemma 2.11 (Section 1.1 in [27]). *Let $X \hookrightarrow Y$ be Banach spaces. Assume X is reflexive and consider $y \in Y$ and a uniformly bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$. If $x_n \rightharpoonup y$ in Y as $n\to\infty$, then $y\in X$ and $x_n \rightharpoonup y$ in X as $n\to\infty$.*

Lemma 2.12 (The Generalized Gronwell's inequality). *Let the real-valued function $\phi(t)$ be continuous in $J=[0,T]$ and let*

$$
\phi(t) \le \alpha + \int_0^t h(s)\phi(s)\, ds \quad in \quad J
$$

where $\alpha\in\mathbb{R}$ and $h(t)$ is nonnegative and continuous (or Lebesgue integrable) in J. Then

$$
\phi(t) \le \alpha \exp\left(\int_0^t h(s)\, ds\right)
$$

Chapter 3

2D Nonlinear Schrödinger Equations in a Half Plane

In this chapter, we consider the IBVP for $(x, y) \in \mathbb{R} \times \mathbb{R}^+$. Let $T \in (0, \infty]$. In the study of the "half plane" problem, the main ingredients are Fourier transform, Strichartz's estimate and Sobolev Embedding theorems.

3.1 Formulation of the Problem

To consider the well-posedness of the following IBVP for $(x, y) \in \mathbb{R} \times \mathbb{R}^+$ and $T \in (0, \infty]$,

$$\begin{cases} iu_t + u_{xx} + u_{yy} + g = 0\,, & (x,y,t) \in \mathbb{R} \times \mathbb{R}^+ \times (0,T); \\ u(x,y,0) = \varphi(x,y)\,, & (x,y) \in \mathbb{R} \times \mathbb{R}^+\,; \\ u(x,0,t) = h(x,t)\,, & (x,t) \in \mathbb{R} \times (0,T)\,, \end{cases} \tag{3.1}$$

where $g(x, y, t) := \lambda |u(x, y, t)|^{p-2} u(x, y, t)$ for $p \geq 3$, we will reformulate it into an integral equation. If $T > 0$ is small, the wellposedness problem of (3.1) is said to be local and the local well-posedness in $C([0, T]; H^s(\mathbb{R} \times \mathbb{R}^+))$ will be discussed.

First, we decompose (3.1) into three relatively simple problems. Let $\phi : \mathbb{R}^2 \to \mathbb{C}$ be an extension of φ such that $\phi(x, y) = \varphi(x, y)$ for $y \geq 0$ preserving the function spaces used later and the relative norms. Let the solution of the linear initial value problem

$$\begin{cases} iv_t + v_{xx} + v_{yy} = 0\,, & (x,y,t) \in \mathbb{R}^2 \times (0,T)\,; \\ v(x,y,0) = \phi(x,y)\,, & (x,y) \in \mathbb{R}^2\,, \end{cases} \tag{3.2}$$

be $v = W_{\mathbb{R}^2}(t)\phi$. Here, $W_{\mathbb{R}^2}(t)$ is a C_0-semigroup for the infinitesimal generator $i\Delta$ in $\mathbb{R}^2$.

Also, let $f : \mathbb{R}^2 \times [0, T] \to \mathbb{C}$ be an extension of g such that $f(x, y, t) = g(x, y, t)$ for $y \geq 0$ preserving the function spaces and the relative norms. Then, the solution of the initial value problem

$$\begin{cases} iw_t + w_{xx} + w_{yy} + f = 0\,, & (x,y,t) \in \mathbb{R}^2 \times (0,T)\,; \\ w(x,y,0) = 0\,, & (x,y) \in \mathbb{R}^2\,, \end{cases} \tag{3.3}$$

can be found by $w = \Phi_f := i\int_0^t W_{\mathbb{R}^2}(t-\tau)f(\tau)\,d\tau$ as in [27].

Finally, the solution of the following homogeneous IBVP

$$\begin{cases} iz_t + z_{xx} + z_{yy} = 0\,, \qquad \text{for } (x,y,t) \in \mathbb{R} \times \mathbb{R}^+ \times (0,T)\,, \\ z(x,y,0) = 0\,, \\ z(x,0,t) = h(x,t) - \left[W_{\mathbb{R}^2}(t)\phi + i\int_0^t W_{\mathbb{R}^2}(t-\tau)f(\tau)\,d\tau\right]\Big|_{y=0} \end{cases} \tag{3.4}$$

is denoted by

$$z := W_b\left[h - \left[W_{\mathbb{R}^2}(\cdot)\phi + i\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot-\tau)f(\tau)\,d\tau\right]\Big|_{y=0}\right](x,y,t)\,, \tag{3.5}$$

where the term $\left[W_{\mathbb{R}^2}(\cdot)\phi + i\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot-\tau)f(\tau)\,d\tau\right]\Big|_{y=0}$ is the trace on the boundary $y = 0$. Therefore, if the solution of (3.1) is smooth enough, it is then transformed to the solution of the following integral equation,

$$\begin{aligned} u(x,y,t) \;\; &= \;\; W_b\left[h - W_{\mathbb{R}^2}(\cdot)\phi\Big|_{y=0} - i\left(\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot-\tau)f(\tau)\,d\tau\right)\Big|_{y=0}\right](x,y,t) \\ &\qquad + W_{\mathbb{R}^2}(t)\phi(x,y) + \, i\left(\int_0^t W_{\mathbb{R}^2}(t-\tau)f(\tau)\,d\tau\right)(x,y)\,. \end{aligned} \tag{3.6}$$

From now on, a solution of (3.6) is called a mild solution of (3.1) and we will only focus on the solutions of (3.6). In general, the solution of (3.6) is a solution of (3.1) in the sense of distribution. However, under some conditions on $f(x,y,t)$, we have the following lemma.

Lemma 3.1. *For $s \geq 0$ and every fixed $t \in [0,T]$ with $f(t) : H^s \to H^\sigma$, $u \in C([0,T];\, H^s) \bigcap C^1([0,T];\, H^{\min\{s-2,\sigma\}})$ is a solution of (3.1) if and only if (3.6) holds for u.*

The proof of the Lemma is analogous to *Lemma 4.2.8* in [27] using the semigroup theory.

We note that the existence of trace in (3.6) depends on the function space used to define $W_{\mathbb{R}^2}$. In the Sobolev space $u \in H^s(\mathbb{R}^2)$ with $s \geq 0$, the trace of u at $y = 0$ is not defined if $s < 1/2$. In this case, the trace in (3.6) is just taken as zero, i.e., (3.6) is

$$u(x,y,t) = W_b[h](x,y,t) + W_{\mathbb{R}^2}(t)\phi(x,y) + \, i\left(\int_0^t W_{\mathbb{R}^2}(t-\tau)f(\tau)\,d\tau\right)(x,y)\,. \tag{3.7}$$

If $s > 1/2$, the trace of u is well defined and some compatibility conditions on φ and h in (3.1) have to be imposed, i.e., $\varphi|_{y=0} = h|_{t=0} \in H_x^{s-\frac{1}{2}}$ if $h|_{t=0}$ is also defined in $H_x^{s-\frac{1}{2}}$. If $s > 5/2$, more compatibility conditions on φ and h from the equation in (3.1) must be imposed. The details on the compatibility conditions are referred to the similar situations for the IBVP of KdV equation [11].

3.2 Representations and Estimates of Solution Operators

In this section, we will derive the formulas of the solutions in (3.2)-(3.4) and obtain the corresponding solution operators.

3.2.1 Equation with the Boundary Condition

First, consider the following equation with nonhomogeneous boundary condition:

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0\,, & (x,y,t) \in \mathbb{R}\times\mathbb{R}^+\times(0,T)\,; \\ u(x,y,0) = 0\,, & (x,y) \in \mathbb{R}\times\mathbb{R}^+\,, \\ u(x,0,t) = h(x,t) & (x,t) \in \mathbb{R}\times(0,T)\,. \end{cases} \tag{3.8}$$

The boundary condition $h(x,t)$ is in $H_t^{\frac{2s+1}{4}}\left([0,T];\, L_x^2(\mathbb{R})\right)$. It will be shown that the solution $W_b[h]$ is in $L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)\bigcap L_t^\infty\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right)$, where (q,r) is an admissible pair, i.e., $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$. The solution map $W_b[h]$ is continuous with respect to the spaces of h and $W_b[h]$. Recall that, we use $\lesssim$ or $\approx$ to denote $\le$ or $=$ with a difference of a factor of a generic constant.

First, we derive the explicit formula of the solution operator $W_b[h]$.

Proposition 3.2. *The solution of (3.8) can be expressed by*

$$\begin{aligned} u(x,y,t) = W_b[h](x,y,t) &= \frac{1}{\pi^2}\Big(\int_{-\infty}^{\infty} e^{i\xi x}\Big(\int_0^\infty e^{-i(\xi^2-\eta^2)t-y\eta}\eta\tilde{h}(\xi,-i(\xi^2-\eta^2))\,d\eta \\ &\quad + \int_0^\infty e^{-i(\xi^2+\eta^2)t+iy\eta}\eta\tilde{h}(\xi,-i(\xi^2+\eta^2))\,d\eta\Big)\,d\xi\Big) \\ &\approx W_{b_1}[h](x,y,t) + W_{b_2}[h](x,y,t)\,, \end{aligned} \tag{3.9}$$

where

$$W_{b_1}[h](x,y,t) = \int_{-\infty}^{\infty}\int_0^\infty e^{-i(\xi^2+\eta^2)t+i(y\eta+x\xi)}\eta\tilde{h}(\xi,-i(\xi^2+\eta^2))\,d\eta\,d\xi\,, \tag{3.10}$$

$$W_{b_2}[h](x,y,t) = \int_{-\infty}^{\infty}\int_0^\infty e^{-i(\xi^2-\eta^2)t+ix\xi-y\eta}\eta\tilde{h}(\xi,-i(\xi^2-\eta^2))\,d\eta\,d\xi\,. \tag{3.11}$$

Proof. For any fixed $t > 0$, we apply Fourier transform on both sides of the equation in (3.8) with respect to $x \in \mathbb{R}$ so that the equation of $u(x, y, t)$ is converted to the following equation of $\widehat{u^x}(\xi, y, t)$,

$$\begin{cases} i\widehat{u^x}_t - \xi^2 \widehat{u^x} + \widehat{u^x}_{yy} = 0\,, & (\xi, y, t) \in \mathbb{R} \times \mathbb{R}^+ \times (0, \infty)\,; \\ \widehat{u^x}(\xi, y, 0) = 0\,, & \\ \widehat{u^x}(\xi, 0, t) = \widehat{h^x}(\xi, t)\,. & \end{cases} \tag{3.12}$$

Then, apply Laplace transform on both sides of (3.12) with respect to $t > 0$ (here $\widehat{u^x}(\xi, y, 0) = 0$) with $\tilde{u}(\xi, y, \omega) = \mathcal{L}_t[\widehat{u^x}](\xi, y, \omega)$ and $\tilde{h}(\xi, \omega) = \mathcal{L}_t[\widehat{h^x}](\xi, \omega)$, which gives

$$\begin{cases} (i\omega - \xi^2)\tilde{u} + \tilde{u}_{yy} = 0\,, & (\xi, y, \omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{C}\,; \\ \tilde{u}(\xi, 0, \omega) = \tilde{h}(\xi, \omega)\,. & \end{cases} \tag{3.13}$$

The characteristic equation of (3.13) is $z^2 + \omega i - \xi^2 = 0$ or $z^2 = \xi^2 - \eta - i\gamma$, where $\omega = \gamma - \eta i$ with $\gamma > 0$ that is required in the form of inverse Laplace transform. If $z = re^{i\theta}$ with $\theta = \arg z$, then $r^2 \sin(2\theta) = -\gamma < 0$, which implies $\frac{\pi}{2} < \theta < \pi$ or $\frac{3\pi}{2} < \theta < 2\pi$. Therefore, two roots of the characteristic equation are $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+\pi)}$ with $\frac{\pi}{2} < \theta < \pi$. Since $\tilde{u} \to 0$ as $y \to \infty$, $\tilde{u} = c(\xi, \omega)e^{re^{i\theta}y} = c(\xi, \omega)e^{ry(\cos\theta + i\sin\theta)}$ where

$$r = \sqrt[4]{(\xi^2 - \eta)^2 + \gamma^2}\,, \quad \cos 2\theta = \frac{\xi^2 - \eta}{\sqrt{(\xi^2 - \eta)^2 + \gamma^2}}\,, \quad \sin 2\theta = \frac{-\gamma}{\sqrt{(\xi^2 - \eta)^2 + \gamma^2}}.$$

If $\gamma \to 0^+$, then $r \to \sqrt{|\xi^2 - \eta|}$. We divide it into two cases. If $\eta < \xi^2$, then $\cos 2\theta \to 1$ and $\sin 2\theta \to 0^-$ with $\frac{\pi}{2} < \theta < \pi$, which yields $\cos\theta \to -1\,, \sin\theta \to 0^+$ or $z \to -\sqrt{\xi^2 - \eta}$. Thus, $\tilde{u}(\xi, y, \omega) = \tilde{h}(\xi, \omega)e^{-y\sqrt{\xi^2 - \eta}}$. If $\eta > \xi^2$, then $\cos 2\theta \to -1$ and $\sin 2\theta \to 0^-$, which implies that $\cos\theta \to 0^-$, $\sin\theta \to 1$ or $z \to i\sqrt{\eta - \xi^2}$. Thus, $\tilde{u}(\xi, y, \omega) = \tilde{h}(\xi, \omega)e^{iy\sqrt{\eta - \xi^2}}$.

Perform the inverse Laplace transform on $\widetilde{u^x}$. If $\gamma \to 0$ with $\omega = \gamma - i\eta$, then

$$\begin{aligned} \widehat{u^x}(\xi, y, t) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\omega t} e^{zy} \tilde{h}(\xi, \omega)\, d\omega \\ &\overset{\gamma \to 0}{=} \frac{-1}{2\pi} \left(\int_{\infty}^{\xi^2} e^{-i\eta t} e^{iy\sqrt{\eta - \xi^2}} \tilde{h}(\xi, -i\eta)\, d\eta + \int_{\xi^2}^{-\infty} e^{-i\eta t} e^{-y\sqrt{\xi^2 - \eta}} \tilde{h}(\xi, -i\eta)\, d\eta \right) \\ &= \frac{1}{\pi} \left(\int_0^\infty e^{-i(\xi^2 + \eta^2)t + iy\eta} \eta \tilde{h}(\xi, -i(\xi^2 + \eta^2))\, d\eta + \int_0^\infty e^{-i(\xi^2 - \eta^2)t - y\eta} \eta \tilde{h}(\xi, -i(\xi^2 - \eta^2))\, d\eta \right)\,, \end{aligned}$$

where we have replaced η by $(\xi^2 + \eta^2)$ in the first integral and η by $(\xi^2 - \eta^2)$ in the second integral. Finally, we can take the inverse Fourier transform over ξ to find

$$\begin{aligned} u(x, y, t) = \frac{1}{\pi^2} \Bigg(\int_{-\infty}^\infty e^{i\xi x} \Bigg(&\int_0^\infty e^{-i(\xi^2 - \eta^2)t - y\eta} \eta \tilde{h}(\xi, -i(\xi^2 - \eta^2))\, d\eta \\ &+ \int_0^\infty e^{-i(\xi^2 + \eta^2)t + iy\eta} \eta \tilde{h}(\xi, -i(\xi^2 + \eta^2))\, d\eta \Bigg)\, d\xi \Bigg) \\ = W_{b_1}[h](x, y, t) &+ W_{b_2}[h](x, y, t)\,, \end{aligned}$$

or $W_b[h](x, y, t) = W_{b_1}[h](x, y, t) + W_{b_2}[h](x, y, t)$ as the solution of (3.8). $\square$

To study the operators $W_{b_1}[h](x,y,t)$ and $W_{b_2}[h](x,y,t)$, we first rewrite them into more convenient forms. For $W_{b_1}[h](x,y,t)$, we let $\boldsymbol{x}=(x,y)$ and $\boldsymbol{\xi}=(\xi,\eta)\in\mathbb{R}^2$ and define

$$\widehat{\Phi_h}(\boldsymbol{\xi})=\nu_1(\xi,\eta)=\begin{cases}\eta\tilde{h}(\xi,-i(\xi^2+\eta^2)) & , \quad \eta\geq 0\,,\\ 0 & , \quad \eta<0\,.\end{cases}$$

Then,

$$W_{b_1}[h](x,y,t)=\int_{-\infty}^{\infty}\int_0^{\infty}e^{-i(\xi^2+\eta^2)t+i(y\eta+x\xi)}\widehat{\Phi_h}(\xi,\eta)\,d\eta\,d\xi=\int_{\mathbb{R}^2}e^{-i|\boldsymbol{\xi}|^2t+i\boldsymbol{\xi}\cdot\boldsymbol{x}}\widehat{\Phi_h}(\boldsymbol{\xi})\,d\boldsymbol{\xi}\,, \quad (3.14)$$

which is exactly the integral solution formula of initial value problem (3.2) of the linear Schrödinger equation over the whole plane $\mathbb{R}^2$, i.e., $W_{b_1}[h](x,y,t)=W_{\mathbb{R}^2}(t)\Phi_h(x,y)$. Similarly, if

$$\widehat{\Psi_h}(\xi,\eta)=\nu_2(\xi,\eta)=\begin{cases}\eta\tilde{h}(\xi,-i(\xi^2-\eta^2)) & , \quad \eta\geq 0\,,\\ 0 & , \quad \eta<0\,,\end{cases}$$

then

$$W_{b_2}[h](x,y,t)=\int_{\mathbb{R}^2}e^{-i(\xi^2-\eta^2)t+ix\xi-y\eta}\widehat{\Psi_h}(\xi,\eta)\,d\eta\,d\xi\,. \quad (3.15)$$

Note that $\widehat{\Psi_h}(\xi,\eta)$ is continuous in η at $\eta=0$.

Next, we prove the estimates for the operator $W_b[h]$.

Proposition 3.3. *For $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ with $r\in[2,\infty)$, the following estimates hold.*

$$\|W_b[h]\|^2_{L^q_t([0,T];W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+))}\lesssim\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\beta|+\xi^2)^s\sqrt{|\beta|}\left(\int_0^{\infty}e^{i(\beta+\xi^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right)^2d\beta\,d\xi\,, \quad (3.16)$$

$$\|W_b[h]\|_{L^q_t\left([0,T];W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\lesssim\|h\|_{H_t^{\frac{2s+1}{4}}([0,T];L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];H_x^{s+\frac{1}{2}}(\mathbb{R})\right)}\,. \quad (3.17)$$

Proof. We need derive the estimates for both $W_{b_1}[h]$ and $W_{b_2}[h]$. We first prove (3.16) for $W_{b_1}[h]$. From the form of $W_{b_1}[h]$ in (3.10) and the definition of Φ_h in (3.14), we have

$$\begin{aligned}\|\Phi_h\|^2_{H^s}&=\|\left(1+\xi^2+\eta^2\right)^{\frac{s}{2}}\widehat{\Phi_h}(\xi,\eta)\|^2_{L^2}\\&=\int_{-\infty}^{\infty}\int_0^{\infty}\left(1+\xi^2+\eta^2\right)^s\eta^2\left|\tilde{h}(\xi,-i(\xi^2+\eta^2))\right|^2d\eta\,d\xi\\&=\int_{-\infty}^{\infty}\int_0^{\infty}\left(1+\xi^2+\eta^2\right)^s\eta^2\left|\int_0^{\infty}e^{i(\xi^2+\eta^2)t}\widehat{h^x}(\xi,t)\,dt\right|^2d\eta\,d\xi\\&\simeq\int_{-\infty}^{\infty}\int_0^{\infty}\left(1+\xi^2+\eta^2\right)^s\eta^2\left|\int_0^{\infty}e^{i(\xi^2+\eta^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2d\eta\,d\xi\\&=\frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty}(1+\beta+\xi^2)^s\frac{\beta}{\sqrt{\beta}}\left|\int_0^{\infty}e^{i(\beta+\xi^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2d\beta\,d\xi\end{aligned}$$

$$= C \int_{-\infty}^{\infty} \int_{0}^{\infty} (1+\beta+\xi^2)^s \sqrt{\beta} \left| \int_0^\infty \int_{-\infty}^{\infty} e^{i\beta t} e^{i\xi^2 t} e^{-i\xi x} h(x,t)\, dx\, dt \right|^2 d\beta\, d\xi \tag{3.18}$$

(replacing η by $\sqrt{\beta}$), and

$$\|W_{b_1}[h]\|_{L_t^q\left(\mathbb{R}^+;\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \|\Phi_h\|_{H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)}$$

for any admissible pair (q,r) with $r \in [2,\infty)$, by the fact that $W_{b_1}[h] = W_{\mathbb{R}^2}(t)\Phi_h$ solves (3.2) and the Strichartz's estimate in *Remark 2.3.8* in [27]. After combining these two estimates, it is deduced that

$$\begin{aligned} &\|W_{b_1}[h]\|_{L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\ &\quad \lesssim \left\{ \int_{-\infty}^{\infty} \int_{0}^{\infty} (1+|\beta|+\xi^2)^s \sqrt{|\beta|} \left(\int_0^\infty \int_{-\infty}^{\infty} e^{i\left(\beta+\xi^2\right)t} e^{-i\xi x} h(x,t)\, dx\, dt \right)^2 d\beta\, d\xi \right\}^{\frac{1}{2}} . \end{aligned} \tag{3.19}$$

For $W_{b_2}[h]$, we first consider the case $s=0$, i.e., in the space L^2. Next several steps are devoted to prove that

$$\|W_{b_2}[h]\|_{L_t^q\left([0,T];\, L_{xy}^r(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \|\Psi_h\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} . \tag{3.20}$$

We rephrase $W_{b_2}[h]$ from (3.15) by

$$\begin{aligned} &W_{b_2}[h](x,y,t) \\ &\quad = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y\eta} \widehat{\Psi}_h(\xi,\eta)\, d\eta\, d\xi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y|\eta|} \widehat{\Psi}_h(\xi,\eta)\, d\eta\, d\xi \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y|\eta|} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_h(\rho,\tau) \cdot e^{-i(\xi\rho+\eta\tau)}\, d\rho\, d\tau \right] d\eta\, d\xi \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_h(\rho,\tau) \cdot \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y|\eta|-i(\xi\rho+\eta\tau)}\, d\eta\, d\xi \right] d\rho\, d\tau\, . \end{aligned}$$

After denoting

$$\begin{aligned} K_t(x,y,\rho,\tau) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y|\eta|-i(\xi\rho+\eta\tau)}\, d\eta\, d\xi \\ &= \left(\int_{-\infty}^{\infty} e^{-i\xi^2 t+ix\xi-i\xi\rho}\, d\xi \right) \cdot \left(\int_{-\infty}^{\infty} e^{i\eta^2 t-y|\eta|-i\eta\tau}\, d\eta \right) , \end{aligned} \tag{3.21}$$

we can rewrite $W_{b_2}[h]$ as

$$W_{b_2}[h](x,y,t) := \mathcal{K}(t)\Psi_h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_h(\rho,\tau) \cdot K_t(x,y,\rho,\tau)\, d\rho\, d\tau\, , \tag{3.22}$$

where $\mathcal{K}(t)$ is an operator acting on each reasonable choice of $\Psi_h \in L^2(\mathbb{R}\times\mathbb{R}^+)$ with an integral kernel K_t.

The next task is to prove that

$$\int_0^T \langle \mathcal{K}(t)\psi, \phi(t)\rangle_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\, dt \lesssim \|\psi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} \cdot \|\phi\|_{L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \tag{3.23}$$

for $\phi \in L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)$ and $\psi \in L^2_{xy}$, where q' and r' are conjugate indices of q and r.

The reason of studying (3.23) is that the left hand side $\int_0^T \langle \mathcal{K}(t)\psi, \phi(t)\rangle_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\, dt$, given by the inner product of $\mathcal{K}(\cdot)\psi$ and ϕ over $L^2_{xyt}(\mathbb{R}\times\mathbb{R}^+\times[0,T])$, corresponds to a linear functional on $L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)$. Since $\|\mathcal{K}(\cdot)\psi\|_{L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)}$ is the norm of this functional and if (3.23) can be proven, then (3.20) is proved according to the definition of functional norm. Thus, it is necessary to define another operator $\mathcal{K}_T$ that maps each $\phi \in L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)$ to a function without t, that is,

$$\mathcal{K}_T\phi(\rho,\tau) := \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} K_t(x,y,\rho,\tau)\overline{\phi(x,y,t)}\, dy\, dx\, dt\,. \tag{3.24}$$

Moreover, we have

$$\begin{aligned}
\int_0^T < \mathcal{K}(t)\psi,&\phi(\cdot,\cdot,t) >_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\, dt = \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \mathcal{K}(t)\psi\,(x,y) \cdot \overline{\phi(x,y,t)}\, dx\, dy\, dt \\
&= \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\rho,\tau)\cdot K_t(x,y,\rho,\tau)\, d\rho\, d\tau \cdot \overline{\phi(x,y,t)}\, dx\, dy\, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\rho,\tau) \underbrace{\int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} K_t(x,y,\rho,\tau)\overline{\phi(x,y,t)}\, dy\, dx\, dt}_{\mathcal{K}_T\phi(\rho,\tau)}\, d\rho\, d\tau \\
&\le C\|\psi\|_{L^2_{\rho,\tau}(\mathbb{R}\times\mathbb{R}^+)} \cdot \|\mathcal{K}_T\phi\|_{L^2_{\rho\tau}(\mathbb{R}\times\mathbb{R}^+)}\,,
\end{aligned}$$

which yields

$$\int_0^T \langle \mathcal{K}(t)\psi, \phi(t)\rangle_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\, dt \lesssim \|\psi\|_{L^2_{\rho\tau}(\mathbb{R}\times\mathbb{R}^+)} \cdot \|\mathcal{K}_T\phi\|_{L^2_{\rho\tau}(\mathbb{R}\times\mathbb{R}^+)}\,. \tag{3.25}$$

Now we need one more step from (3.25) to (3.23) to achieve the final result. The following arguments will mainly focus on the connection between $\|\phi\|_{L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)}$ and $\|\mathcal{K}_T\phi\|_{L^2_{\rho\tau}(\mathbb{R}\times\mathbb{R}^+)}$. Recall K_t in (3.21) and let

$$K_{s,\sigma}(x,y,z,w) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_s(x,y,\rho,\tau)\overline{K_\sigma(z,w,\rho,\tau)}\, d\rho\, d\tau\,. \tag{3.26}$$

Then, it is obtained that

$$\|\mathcal{K}_T\phi\|^2_{L^2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} K_s(x,y,\rho,\tau)\overline{\phi(x,y,s)}\, dy\, dx\, ds \right) \cdot$$

$$
\begin{aligned}
&\qquad \left(\int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \overline{K_\sigma(z,w,\rho,\tau)} \phi(z,w,\sigma)\, dw\, dz\, d\sigma\right) d\rho\, d\tau \\
&= \int_0^T \int_0^T \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} \overline{\phi(x,y,s)}\phi(z,w,\sigma) \\
&\qquad K_s(x,y,\rho,\tau)\overline{K_\sigma(z,w,\rho,\tau)} dx\, dy\, dz\, dw\, d\rho\, d\tau\Big]\, ds\, d\sigma \\
&= \int_0^T \int_0^T \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \overline{\phi(x,y,s)}\phi(z,w,\sigma)\cdot \\
&\qquad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_s(x,y,\rho,\tau)\overline{K_\sigma(z,w,\rho,\tau)}\, d\rho\, d\tau\right) dw\, dy\, dz\, dx\Big]\, ds\, d\sigma \\
&= \int_0^T \int_0^T \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \overline{\phi(x,y,s)}\phi(z,w,\sigma) K_{s,\sigma}(x,y,z,w) dw\, dy\, dz\, dx\right) ds\, d\sigma\,.
\end{aligned}
$$

For $y,\, w \geq 0$,

$$
\begin{aligned}
K_{s,\sigma} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_s(x,y,\rho,\tau)\overline{K_\sigma(z,w,\rho,\tau)}\, d\rho\, d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)s+ix\xi-y|\eta|-i(\xi\rho+\eta\tau)}\, d\eta\, d\xi\right)\cdot \\
&\qquad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde\xi^2-\tilde\eta^2)\sigma-iz\tilde\xi-w|\tilde\eta|+i(\tilde\xi\rho+\tilde\eta\tau)}\, d\tilde\eta\, d\tilde\xi\right) d\rho\, d\tau \\
&= (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)s+ix\xi-y|\eta|} \Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi\rho+\eta\tau)} \\
&\qquad \left(\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tilde\xi^2-\tilde\eta^2)\sigma-iz\tilde\xi-w|\tilde\eta|} e^{i(\tilde\xi\rho+\tilde\eta\tau)}\, d\tilde\eta\, d\tilde\xi\right) d\rho\, d\tau\Big]\, d\eta\, d\xi \\
&= (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(-\xi^2-\eta^2)s+ix\xi-y|\eta|}\mathcal{F}_{\rho,\tau}\circ\mathcal{F}^{-1}_{\tilde\xi,\tilde\eta}\left[e^{i(\tilde\xi^2-\tilde\eta^2)\sigma-iz\tilde\xi-w\tilde\eta}\right] d\eta\, d\xi \\
&= (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)(s-\sigma)+i\xi(x-z)-|\eta|(y+w)}\, d\eta\, d\xi \\
&= C\left(\int_{-\infty}^{\infty} e^{-i\xi^2(s-\sigma)+i\xi(x-z)}\, d\xi\right)\cdot\left(\int_{-\infty}^{\infty} e^{i\eta^2(s-\sigma)-|\eta|(y+w)}\, d\eta\right)\,.
\end{aligned}
$$

If we define

$$
\widetilde{K}_t(x,y,z,w) = \left(\int_{-\infty}^{\infty} e^{-i\xi^2 t+i\xi(x-z)}\, d\xi\right)\cdot\left(\int_{-\infty}^{\infty} e^{i\eta^2 t-|\eta|(y+w)}\, d\eta\right) \qquad \text{for } y>0 \text{ and } w>0\,, \tag{3.27}
$$

then $K_{s,\sigma}(x,y,z,w) = \widetilde{K}_{s-\sigma}(x,y,z,w)$. Now, we use $\widetilde{K}_{s-\sigma}$ to study the operator

$$
\widetilde{\mathcal{K}}(t)\varphi(x,y) = \int_{-\infty}^{\infty}\int_0^{\infty} \varphi(z,w)\widetilde{K}_t(x,y,z,w)\, dz\, dw \tag{3.28}
$$

for $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$. To estimate the kernel $\widetilde{K}_t$, for $(x,y,z,w)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^+$, it is derived that

$$
\left|\int_{-\infty}^{\infty} e^{-i\xi^2 t+i(x-z)\xi}\, d\xi\right| = \frac{1}{\sqrt{t}}\left|\int_{-\infty}^{\infty} e^{-i\xi_0^2+i\frac{(x-z)}{\sqrt{t}}\xi_0}\, d\xi_0\right| \quad \text{where } \xi_0 = \xi\sqrt{t}
$$

$$= \frac{1}{\sqrt{t}} \left| \int_{-\infty}^{\infty} e^{-i\xi_1^2} \, d\xi \right| = \frac{C}{\sqrt{t}} \,, \quad \text{where } \xi_1 = \left(\xi_0 - \frac{x-z}{2\sqrt{t}} \right).$$

An estimate from [10] yields

$$\left| \int_0^{\infty} e^{i\eta^2 t - a\eta - i\eta b} \, d\eta \right| \le \frac{C}{\sqrt{t}} \quad \text{for any } a \in \mathbb{R}^+ \,, b \in \mathbb{R} \,.$$

By substituting $y+w$ for a and 0 for b, we obtain

$$\left| \int_{-\infty}^{\infty} e^{i\eta^2 t - |\eta|(y+w)} \, d\eta \right| \le \frac{C}{\sqrt{t}} \,.$$

Hence

$$\left| \widetilde{K}_t(x,y,z,w) \right| = \left| \int_{-\infty}^{\infty} e^{-i\xi^2 t + ix\xi - i\xi z} \, d\xi \right| \cdot \left| \int_{-\infty}^{\infty} e^{i\eta^2 t - |\eta|(y+w)} \, d\eta \right| \le \frac{C}{\sqrt{t}} \cdot \frac{C}{\sqrt{t}} \le \frac{C}{t} \tag{3.29}$$

For some fixed $t > 0$, (3.28) and (3.29) imply

$$\begin{aligned} \left\| \widetilde{\mathcal{K}}(t)\varphi \right\|_{L^\infty_{xy}} &\le \int_{-\infty}^{\infty} \int_0^{\infty} \left| \varphi(z,w) \widetilde{K}_t(x,y,z,w) \right| \, dw \, dz \\ &\le \frac{C}{|t|} \int_{-\infty}^{\infty} \int_0^{\infty} |\varphi(z,w)| \, dw \, dz = C \cdot |t|^{-1} \|\varphi\|_{L^1_{xy}} \end{aligned} \tag{3.30}$$

for each fixed $t > 0$.

On the other hand, for $t > 0$, we can also prove that $\|\widetilde{\mathcal{K}}(t)\varphi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} \le \|\varphi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}$ by an estimate $\left\| \int_0^{\infty} e^{-y\eta} f(\eta) \, d\eta \right\|_{L^2(\mathbb{R}^+)} \le \|f\|_{L^2(\mathbb{R}^+)}$ as follows. From

$$\begin{aligned} \widetilde{\mathcal{K}}(t)\varphi(x,y) &= \int_{-\infty}^{\infty} \int_0^{\infty} \varphi(z,w) \widetilde{K}_t(x,y,z,w) \, dz \, dw \quad y > 0 \, w > 0 \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \varphi(z,w) \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\xi^2-\eta^2)t + ix\xi - y|\eta| - i\xi z - |\eta| w} \, d\eta \, d\xi \right] dw \, dz \\ &= 4 \int_0^{\infty} e^{-y\eta} \int_{-\infty}^{\infty} e^{ix\xi} \left[\int_{-\infty}^{\infty} \int_0^{\infty} e^{-i\xi z - \eta w} e^{-i(\xi^2-\eta^2)t} \varphi(z,w) \, dw \, dz \right] d\xi \, d\eta \\ &= C \int_0^{\infty} e^{-y\eta} \, \mathcal{F}_\xi^{-1} \left[\int_{-\infty}^{\infty} \int_0^{\infty} e^{-i\xi z - \eta w} e^{-i(\xi^2-\eta^2)t} \varphi(z,w) \, dw \, dz \right] (x) \, d\eta \\ &= C \int_0^{\infty} e^{-y\eta} \, \mathcal{F}_\xi^{-1} \left[\mathcal{F}_z \left(\int_0^{\infty} e^{-w\eta} e^{-i(\xi^2-\eta^2)t} \varphi(z,w) \, dw \right) (\xi) \right] (x) \, d\eta \,, \end{aligned}$$

we have

$$\left\| \widetilde{\mathcal{K}}(t)\varphi \right\|^2_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)} = \int_{-\infty}^{\infty} \left\| \widetilde{\mathcal{K}}(t)\varphi \right\|^2_{L^2_y(\mathbb{R}^+)} \, dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left\| \int_0^{\infty} e^{-y\eta}\, \mathcal{F}_\xi^{-1} \left[e^{-i(\xi^2-\eta^2)t} \int_0^{\infty} e^{-w\eta} \mathcal{F}_z \left(\varphi(z,w)\right)(\xi)\, dw \right](x)\, d\eta \right\|_{L^2_y(\mathbb{R}^+)}^2 dx \\
&\le \int_{-\infty}^{\infty} \left\| \mathcal{F}_\xi^{-1} \left[e^{-i(\xi^2-\eta^2)t} \int_0^{\infty} e^{-w\eta} \mathcal{F}_z \left(\varphi(z,w)\right)(\xi)\, dw \right](x) \right\|_{L^2_\eta(\mathbb{R}^+)}^2 dx \\
&= \int_0^{\infty} \left\| \mathcal{F}_\xi^{-1} \left[e^{-i(\xi^2-\eta^2)t} \int_0^{\infty} e^{-w\eta} \mathcal{F}_z \left(\varphi(z,w)\right)(\xi)\, dw \right](\cdot) \right\|_{L^2_x(\mathbb{R})}^2 d\eta \\
&= \int_0^{\infty} \left\| \int_0^{\infty} e^{-w\eta} \mathcal{F}_z \left(\varphi(z,w)\right)(\cdot)\, dw \right\|_{L^2_\xi(\mathbb{R})}^2 d\eta \\
&= \int_{-\infty}^{\infty} \left\| \int_0^{\infty} e^{-w\eta} \mathcal{F}_z \left(\varphi(z,w)\right)(\xi)\, dw \right\|_{L^2_\eta(\mathbb{R}^+)}^2 d\xi \\
&\le \left\| \mathcal{F}_z \left(\varphi(z,w)\right) \right\|_{L^2_{\xi w}(\mathbb{R}\times\mathbb{R}^+)}^2 = \|\varphi\|_{L^2_{z,w}(\mathbb{R}\times\mathbb{R}^+)}^2 .
\end{aligned} \tag{3.31}
$$

By (3.30), (3.31) and interpolation, we obtain

$$
\left\| \widetilde{\mathcal{K}}(t)\varphi \right\|_{L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)} \lesssim t^{-2\left(\frac{1}{2}-\frac{1}{r}\right)} \|\varphi\|_{L^{r'}(\mathbb{R}\times\mathbb{R}^+)} . \tag{3.32}
$$

Then, for $\phi \in C_c\left([0,T];\, L^{r'}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)$, a classical density argument leads to

$$
\left\| \widetilde{\mathcal{K}}(t)\phi \right\|_{L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)} \lesssim t^{-2\left(\frac{1}{2}-\frac{1}{r}\right)} \|\phi(t)\|_{L^{r'}(\mathbb{R}\times\mathbb{R}^+)} \tag{3.33}
$$

if $t > 0$ is fixed. Thus,

$$
\left\| \int_0^t \widetilde{\mathcal{K}}(t-s)\phi(s)\, ds \right\|_{L^r_{xy}} \le C \int_0^T |t-s|^{-2\left(\frac{1}{2}-\frac{1}{r}\right)} \|\phi(s)\|_{L^{r'}_{xy}}\, ds = C \int_0^T |t-s|^{\frac{-2}{q}} \|\phi(s)\|_{L^{r'}_{xy}}\, ds
$$

By Riesz potential inequality,

$$
\begin{aligned}
\left\| \int_0^{(\cdot)} \widetilde{\mathcal{K}}(\cdot - s)\phi(s)\, ds \right\|_{L^q_t\left([0,T];\, L^r_{xy}\right)} &\le \left\| \int_0^{T} \widetilde{\mathcal{K}}(\cdot - s)\phi(s)\, ds \right\|_{L^q_t\left([0,T];\, L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\lesssim \|\phi\|_{L^{q'}_t\left([0,T];\, L^{r'}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}
\end{aligned}
$$

or

$$
\left\| \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \phi(z,w,s) \widetilde{K}_{t-s}(x,y,z,w)\, dz\, dw\, ds \right\|_{L^q_t\left([0,T];\, L^r_{xy}\right)} \lesssim \|\phi\|_{L^{q'}_t\left([0,T];\, L^{r'}_{xy}\right)} . \tag{3.34}
$$

To find the estimate of $\|\mathcal{K}_T\phi\|_{L^2}^2$, by (3.34) we have

$$
\begin{aligned}
&\|\mathcal{K}_T\phi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}^2 = \\
&\quad \int_0^T \int_0^T \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \overline{\phi(x,y,s)} \phi(z,w,\sigma) \widetilde{K}_{s-\sigma}(x,y,z,w) dw\, dy\, dz\, dx \right) ds\, d\sigma
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
&= \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \overline{\phi(x,y,s)} \left[\int_0^T \left(\int_{-\infty}^{\infty} \int_0^{\infty} \phi(z,w,\sigma) \widetilde{K}_{s-\sigma}(x,y,z,w)\, dw\, dz \right) d\sigma \right] dy\, dx\, ds \\
&\leq C \|\phi\|_{L^{q'}([0,T];L^{r'})} \left\| \int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \phi(z,w,\sigma) \widetilde{K}_{s-\sigma}(x,y,z,w)\, dz\, dw\, d\sigma \right\|_{L^q([0,T];L^r)} \\
&\leq C \|\phi\|_{L^{q'}([0,T];L^{r'}(\mathbb{R}\times\mathbb{R}^+))} \|\phi\|_{L^{q'}([0,T];L^{r'}(\mathbb{R}\times\mathbb{R}^+))} = C\|\phi\|^2_{L^{q'}([0,T];L^{r'}(\mathbb{R}\times\mathbb{R}^+))} . \qquad (3.36)
\end{aligned}
$$

Finally, from (3.23) and (3.25), it is deduced that

$$
\int_0^T \langle \mathcal{K}(t)\psi, \phi(t) \rangle_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\, dt \lesssim \|\psi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} \cdot \|\phi\|_{L_t^{q'}\left([0,T];\, L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)}
$$

or

$$
\|\mathcal{K}(t)\psi\|_{L^q([0,T];L^r(\mathbb{R}\times\mathbb{R}^+))} \leq C\|\psi\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} ,
$$

which yields

$$
\|W_{b_2}[h]\|_{L_t^q\left(\mathbb{R}^+;\, L_{xy}^r(\mathbb{R}\times\mathbb{R}^+)\right)} = \|\mathcal{K}(t)\Psi_h\|_{L_t^q\left(\mathbb{R}^+;\, L_{xy}^r(\mathbb{R}\times\mathbb{R}^+)\right)} \leq C \cdot \|\Psi_h\|_{L^2_{(\xi,\eta)}}
$$

The proof of (3.20) is then completed.

Similar to (3.18), we have

$$
\begin{aligned}
\|\Psi_h\|^2_{L^2} = \|\widehat{\Psi_h}(\xi,\eta)\|^2_{L^2} &\eqsim \int_{-\infty}^{\infty} \int_0^{\infty} \eta^2 \left| \widetilde{h}(\xi, -i(\xi^2-\eta^2)) \right|^2 d\eta\, d\xi \\
&\eqsim \int_{-\infty}^{\infty} \int_0^{\infty} \eta^2 \left| \int_0^{\infty} e^{i(\xi^2-\eta^2)t} \int_{-\infty}^{\infty} e^{-i\xi x} h(x,t)\, dx\, dt \right|^2 d\eta\, d\xi \\
&= C \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{\beta} \left| \int_0^{\infty} \int_{-\infty}^{\infty} e^{-i\beta t} e^{i\xi^2 t} e^{-i\xi x} h(x,t)\, dx\, dt \right|^2 d\beta\, d\xi ,
\end{aligned}
$$

which implies

$$
\|W_{b_2}[h]\|_{L_t^q\left([0,T];\, L_{xy}^r(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \left\{ \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{\beta} \left| \int_0^{\infty} \int_{-\infty}^{\infty} e^{-i\beta t} e^{i\xi^2 t} e^{-i\xi x} h(x,t)\, dx\, dt \right|^2 d\beta\, d\xi \right\}^{\frac{1}{2}} . \qquad (3.37)
$$

Now, we study the case with $s > 0$, i.e.,

$$
\begin{aligned}
&\|W_{b_2}[h]\|_{L_t^q\left([0,T];\, W_{xy}^{s;r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\quad \lesssim \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{0} (1+|\beta|+\xi^2)^s \sqrt{|\beta|} \left| \int_0^{\infty} \int_{-\infty}^{\infty} e^{i\left(\beta+\xi^2\right)t} e^{-i\xi x} h(x,t)\, dx\, dt \right|^2 d\beta\, d\xi \right\}^{\frac{1}{2}} . \qquad (3.38)
\end{aligned}
$$

Write the multi-index $\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{Z}^2$; i.e. $(\xi\eta)^\alpha = \xi^{\alpha_1}\eta^{\alpha_2}$, and let $\alpha_1, \alpha_2 \geq 0$ with $|\alpha| = \alpha_1 + \alpha_2 = m$. From $|(\xi\eta)^\alpha| \leq |\xi|^m + |\eta|^m$,

$$
D^\alpha_{xy} W_{b_2}[h] \eqsim \int_{-\infty}^{\infty} \int_0^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y\eta} \cdot [(i\xi)(-\eta)]^\alpha \cdot \eta \widetilde{h}(\xi, -i(\xi^2-\eta^2))\, d\eta\, d\xi
$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-i(\xi^2-\eta^2)t+ix\xi-y\eta}\Psi_{h,\alpha}(\xi,\eta)\,d\eta\,d\xi\,,$$

where $\Psi_{h,\alpha}(\xi,\eta) = [(i\xi)(-\eta)]^\alpha \cdot \eta\tilde{h}(\xi,-i(\xi^2-\eta^2))$ for $\eta \geq 0$ and otherwise $\Psi_{h,\alpha} = 0$. Thus,

$$\begin{aligned}
&\sum_{|\alpha|=m}\left\|D^\alpha_{xy}W_{b_2}[h]\right\|^2_{L^q_t([0,T];\,L^r_{xy}(\mathbb{R}\times\mathbb{R}^+))} \lesssim \sum_{|\alpha|=m}\|\Psi_{h,\alpha}\|^2_{L^2}\\
&\quad\leq C(m)\int_{-\infty}^{\infty}\int_0^{\infty}(\beta+\xi^2)^m\sqrt{\beta}\left|\int_0^{\infty}e^{-i(\beta-\xi^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\beta\,d\xi\\
&\quad= C(m)\int_{-\infty}^{\infty}\int_{-\infty}^{0}(|\beta|+\xi^2)^m\sqrt{|\beta|}\left|\int_0^{\infty}e^{i(\beta+\xi^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\beta\,d\xi\,.
\end{aligned}$$

A simple interpolation argument leads to (3.38). (3.19) and (3.38) finish the proof of (3.16).

To prove (3.17), if we let $\lambda = -\left(\beta+\xi^2\right)$, which gives $\beta = -\left(\lambda+\xi^2\right)$, then

$$\begin{aligned}
&\|W_b[h]\|^2_{L^q_t([0,T];\,W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+))}\\
&\quad\leq C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\beta|+\xi^2)^s\sqrt{|\beta|}\left|\int_0^{\infty}e^{i(\beta+\xi^2)t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\beta\,d\xi\\
&\quad= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda+\xi^2|+\xi^2\right)^s\sqrt{|\lambda+\xi^2|}\left|\int_0^{\infty}e^{-i\lambda t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\lambda\,d\xi\\
&\quad\leq C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda|+\xi^2\right)^s\sqrt{|\lambda+\xi^2|}\left|\int_0^{\infty}e^{-i\lambda t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\lambda\,d\xi\\
&\quad\leq C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda|+\xi^2\right)^s\sqrt{|\lambda|+\xi^2}\left|\int_{-\infty}^{\infty}\int_0^{\infty}e^{-i(\xi x+\lambda t)}h(x,t)\,dt\,dx\right|^2 d\lambda\,d\xi\\
&\quad\leq C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda|+\xi^2\right)^{s+\frac{1}{2}}\left|\int_{-\infty}^{\infty}\int_0^{\infty}e^{-i(\xi x+\lambda t)}h(x,t)\,dt\,dx\right|^2 d\lambda\,d\xi\\
&\quad= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda|+\xi^2\right)^{s+\frac{1}{2}}\left|\widehat{h}(\xi,\lambda)\right|^2 d\lambda\,d\xi\\
&\quad= C\|h\|^2_{H_t^{\frac{2s+1}{4}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)}\,,
\end{aligned}$$

where h can be first chosen as a smooth function of compactly supported with respect to t in $[0,T]$ and then a density argument can be applied. Thus, we finish the proof of (3.17). This completes the proof of the proposition. □

Remark 3.4. *(i) From (3.16), we can see that the weaker condition on $h(x,t)$ is*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+|\lambda|+\xi^2\right)^s\sqrt{|\lambda+\xi^2|}\left|\int_0^{\infty}e^{-i\lambda t}\int_{-\infty}^{\infty}e^{-i\xi x}h(x,t)\,dx\,dt\right|^2 d\lambda\,d\xi < \infty\,,$$

which is also consistent with estimates on the trace of $W_{\mathbb{R}^2}\phi$ with $\phi \in H^s(\mathbb{R}^2)$. However, the condition is not easy to be expressed in a simple and clear form. Thus, we use a slightly stronger condition that $h \in H_t^{\frac{2s+1}{4}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)$.

(ii) By using a similar argument, for $s,\ \sigma \geq 0$, we can show the following two estimates:

$$\|W_b[h]\|_{W_t^{s,q}\left([0,T];\, L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|\lambda|)^{2s}\sqrt{|\lambda|+\xi^2}\left|\widehat{h}(\xi,\lambda)\right|^2 d\beta\, d\xi\right\}^{\frac{1}{2}} ,$$
$$\sup_{y\in\mathbb{R}^+} \|W_b[h](y)\|_{H_t^\sigma([0,T];\, L^2_x(\mathbb{R}))} \leq C\|h\|_{H_t^\sigma([0,T];\, L^2_x(\mathbb{R}))} .$$

3.2.2 Equation with the Initial Condition

In this section, we derive the estimates for the solution $v(x,y,t) = W_{\mathbb{R}^2}(t)\phi(x,y)$ in (3.2).

Proposition 3.5. *It is known that*

$$W_{\mathbb{R}^2}(t)\phi(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2+\eta^2)t+i(\xi x+\eta y)}\widehat{\phi}(\xi,\eta)\, d\xi\, d\eta\,. \tag{3.39}$$

Proof. Detailed derivation can be found in *Section 2.2* of [27]. □

The following is the estimates that are needed later,

Proposition 3.6. *Let $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ where $r\in[2,\infty)$.*

$$\|W_{\mathbb{R}^2}\phi\|_{L^q_t\left([0,T];\, W^{s,r}_{xy}(\mathbb{R}^2)\right)} \lesssim \|\phi\|_{H^s_{xy}(\mathbb{R}^2)}\ , \tag{3.40}$$
$$\left\|W_b\left[\left(W_{\mathbb{R}^2}\phi\right)\Big|_{y=0}\right]\right\|_{L^q_t\left([0,T];\, W^{s,r}_{xy}(\mathbb{R}^2)\right)} \lesssim \|\phi\|_{H^s(\mathbb{R}^2)}\,. \tag{3.41}$$

Proof. (3.40) is proved in *Section 2.3* of [27] as the classic Strichartz's estimate. For (3.41), we evaluate $W_{\mathbb{R}^2}(t)\phi$ at $y=0$

$$\left(W_{\mathbb{R}^2}(t)\phi\right)\Big|_{y=0} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2+\eta^2)t+i\xi x}\widehat{\phi}(\xi,\eta)\, d\xi\, d\eta\,. \tag{3.42}$$

It should be noted that when $W_b[h]$ is defined in (3.9), $h(x,0)=0$ if $h(x,t)$ has some regularity on t. Since we only consider t in a finite interval, we multiply $W_{\mathbb{R}^2}(t)\phi(x,0)$ by any smoothly compacted supported even function $\chi(t)$ satisfying $\chi(t)=1$ for $|t|\in[0,T]$ and $\chi(t)=0$ for $|t|>2T$.

First, we prove the case for $s=0$. From (3.16), we have

$$\begin{aligned}
&\|W_b\left[\chi(\cdot)W_{\mathbb{R}^2}(\cdot)\varphi(\cdot,0)\right]\|^2_{L^q_t\left([0,T];\, L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{|\beta|}\left|\int_0^\infty e^{i(\beta+\xi^2)t}\chi(t)\int_{-\infty}^{\infty}e^{-i\xi x}\left(W_{\mathbb{R}^2}(t)\varphi(x,0)\right)\, dx\, dt\right|^2 d\beta\, d\xi \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{|\beta|}\cdot|\mathcal{E}(\xi,\beta)|^2\, d\beta\, d\xi\,,
\end{aligned} \tag{3.43}$$

where

$$
\begin{aligned}
\mathcal{E} &= \int_0^\infty e^{i\beta t}\chi(t)e^{i\xi^2 t}\int_{-\infty}^\infty e^{-i\xi x}\left(W_{\mathbb{R}^2}(t)\phi(x,0)\right)\,dx\,dt\\
&= \int_0^\infty e^{i\beta t}e^{i\xi^2 t}\chi(t)\int_{-\infty}^\infty e^{-i\xi x}\left(\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-i(\xi_0^2+\eta_0^2)t+i\xi x}\widehat{\phi}(\xi_0,\eta_0)\,d\eta_0\,d\xi_0\right)\,dx\,dt\\
&= \int_0^\infty e^{i\beta t}e^{i\xi^2 t}\chi(t)\left\{\int_{-\infty}^\infty e^{-i\xi x}\left[\int_{-\infty}^\infty e^{i\xi_0 x}\left(\int_{-\infty}^\infty e^{-i\xi_0^2 t}e^{-i\eta_0^2 t}\widehat{\phi}(\xi_0,\eta_0)\,d\eta_0\right)\,d\xi_0\right]\,dx\right\}\,dt\\
&= \int_0^\infty e^{i\beta t}e^{i\xi^2 t}\chi(t)\left(\int_{-\infty}^\infty e^{-i\xi^2 t}e^{-i\eta_0^2 t}\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right)\,dt = \int_0^\infty \chi(t)\int_{-\infty}^\infty e^{i\beta t}e^{-i\eta_0^2 t}\widehat{\phi}(\xi,\eta_0)\,d\eta_0\,dt\\
&= \int_{-\infty}^\infty\int_0^\infty \chi(t)e^{i(\beta-\eta_0^2)t}\,dt\,\widehat{\phi}(\xi,\eta_0)\,d\eta_0\\
&= \int_{-\infty}^\infty \frac{1}{i(\beta-\eta_0^2)}\left(1-\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\,.
\end{aligned}
$$

Here, we first deal with the estimate for small η_0. Let

$$
\widehat{\phi}(\xi,\eta_0) = \chi(\eta_0/T)\widehat{\phi}(\xi,\eta_0) + (1-\chi(\eta_0/T))\widehat{\phi}(\xi,\eta_0) = \widehat{\phi}_0(\xi,\eta_0) + \widehat{\phi}_1(\xi,\eta_0)\,.
$$

Then, by (3.43), we obtain

$$
\begin{aligned}
&\|W_b\left[W_{\mathbb{R}^2}(\cdot)\phi_0(\cdot,0)\right]\|^2_{L_t^q\left([0,T];\,W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}\\
&\qquad\le \int_{-\infty}^\infty\int_{-\infty}^\infty\left|\beta\int_{-2}^2\int_0^\infty \chi(t)e^{i(\beta^2-\eta_0^2)t}\,dt\,\widehat{\phi}_0(\xi,\eta_0)\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\le \int_{-\infty}^\infty\int_{-4}^4\left|\beta\int_{-2}^2\int_0^\infty \chi(t)e^{i(\beta^2-\eta_0^2)t}\,dt\,\widehat{\phi}_0(\xi,\eta_0)\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\qquad + \int_{-\infty}^\infty\int_{|\beta|\ge 4}\left|\beta\int_{-2}^2\int_0^\infty \chi(t)e^{i(\beta^2-\eta_0^2)t}\,dt\,\widehat{\phi}_0(\xi,\eta_0)\,d\eta_0\right|^2\,d\beta\,d\xi\,.
\end{aligned}
$$

The first integral is bounded by $C\|\widehat{\phi}_0\|^2_{L^2(\mathbb{R}^2)}$. In the second integral, we have $\beta^2-\eta_0^2\ge 10$. Thus,

$$
\begin{aligned}
&\int_{-\infty}^\infty\int_{|\beta|\ge 4}\left|\beta\int_{-2}^2\int_0^\infty \chi(t)e^{i(\beta^2-\eta_0^2)t}\,dt\,\widehat{\phi}_0(\xi,\eta_0)\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\le \int_{-\infty}^\infty\int_{|\beta|\ge 4}\left|\int_{-2}^2\frac{\beta}{(\beta^2-\eta_0^2)}\,|\widehat{\phi}_0(\xi,\eta_0)|\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\le \int_{-\infty}^\infty\int_{|\beta|\ge 4}\left|\int_{-2}^2\frac{1}{(\beta-\eta_0)}\,|\widehat{\phi}_0(\xi,\eta_0)|\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\qquad + \int_{-\infty}^\infty\int_{|\beta|\ge 4}\left|\int_{-2}^2\frac{1}{(\beta+\eta_0)}\,|\widehat{\phi}_0(\xi,\eta_0)|\,d\eta_0\right|^2\,d\beta\,d\xi\\
&\qquad\le \int_{-\infty}^\infty\int_{-\infty}^\infty\left|\int_{-\infty}^\infty\frac{\zeta_{[-2,2]}(\eta_0)}{(\beta-\eta_0)}\,|\widehat{\phi}_0(\xi,\eta_0)|\,d\eta_0\right|^2\,d\beta\,d\xi
\end{aligned}
$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\zeta_{[-2,2]}(\eta_0)}{(\beta+\eta_0)}\,|\widehat{\phi}_0(\xi,\eta_0)|\,d\eta_0\right|^2 d\beta\,d\xi \le C\|\widehat{\phi}_0\|^2_{L^2(\mathbb{R}^2)}\,,$$

where $\zeta(\eta_0)$ is a characteristic function.

Thus, from now on, we assume that $\widehat{\phi}(\xi,\eta_0)=0$ for $|\eta_0|\le 1$. By (3.43) again, we obtain

$$\begin{aligned}
&\|W_b\left[W_{\mathbb{R}^2}(\cdot)\phi(\cdot,0)\right]\|^2_{L^q_t\left([0,T];\,W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\\
&\quad\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{|\beta|^{1/4}}{i(\beta-\eta_0^2)}\left(1-\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\quad\le \int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\beta}{i(\beta^2-\eta_0^2)}\left(1-\int_T^{2T}\chi'(t)e^{i(\beta^2-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\qquad+\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\beta}{i(\beta^2+\eta_0^2)}\left(1-\int_T^{2T}\chi'(t)e^{i(-\beta^2-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\quad= I_1+I_2\,.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
I_2 &\le C\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\beta}{\beta^2+\eta_0^2}\left|\widehat{\phi}(\xi,\eta_0)\right|\,d\eta_0\right|^2 d\beta\,d\xi\\
&\le C\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{1}{\beta+|\eta_0|}\left|\widehat{\phi}(\xi,\eta_0)\right|\,d\eta_0\right|^2 d\beta\,d\xi \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\widehat{\phi}(\xi,\beta)\right|^2 d\beta\,d\xi\,.
\end{aligned}$$

For I_1, we first consider

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\beta}{i(\beta^2-\eta_0^2)}\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\qquad\le \int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{0}^{\infty}\frac{1}{(\beta-\eta_0)}\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\qquad\qquad+\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{0}^{\infty}\frac{1}{(\beta-\eta_0)}\widehat{\phi}(\xi,-\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\qquad\qquad+\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{1}{\beta+|\eta_0|}\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\widehat{\phi}(\xi,\beta)\right|^2 d\beta\,d\xi\,.
\end{aligned}$$

The estimate for next term in I_1 is

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{-\infty}^{\infty}\frac{\beta}{i(\beta^2-\eta_0^2)}\left(\int_T^{2T}\chi'(t)e^{i(\beta^2-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\le \int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\int_{0}^{\infty}\frac{\beta}{(\beta^2-\eta_0^2)}\left(\int_T^{2T}\chi'(t)e^{i(\beta^2-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{\beta}{(\beta^2-\eta_0^2)}\left(\int_T^{2T}\chi'(t)e^{i(\beta^2-\eta_0^2)t}\,dt\right)\widehat{\phi}(\xi,-\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&\le \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{\beta^{1/4}}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\eta_0^{-1/2}\widehat{\phi}(\xi,\sqrt{\eta_0})\,d\eta_0\right|^2 d\beta\,d\xi\\
&\quad + \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{\beta^{1/4}}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\eta_0^{-1/2}\widehat{\phi}(\xi,-\sqrt{\eta_0})\,d\eta_0\right|^2 d\beta\,d\xi\\
&= I_{11}+I_{12}\,.
\end{aligned}
$$

Since it is similar to estimate I_{11} and I_{12}, we only study I_{11}. Write I_{11} as

$$
\begin{aligned}
I_{11} &\le \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{\beta^{1/4}-\eta_0^{1/4}}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\eta_0^{-1/2}\widehat{\phi}(\xi,\sqrt{\eta_0})\,d\eta_0\right|^2 d\beta\,d\xi\\
&\quad + \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{1}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\eta_0^{-1/4}\widehat{\phi}(\xi,\sqrt{\eta_0})\,d\eta_0\right|^2 d\beta\,d\xi\\
&= I_{111}+I_{112}\,.
\end{aligned}
$$

I_{112} can be estimated as follows. By Parseval's theorem, if we let $\Phi_0 = \eta_0^{-1/4}\widehat{\phi}(\xi,\sqrt{\eta_0})$ for $\eta_0 \ge 0$ and $\Phi_0 = 0$ for $\eta_0 < 0$,

$$
\begin{aligned}
I_{112} &= \int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{1}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\eta_0^{-1/4}\widehat{\phi}(\xi,\sqrt{\eta_0})\,d\eta_0\right|^2 d\beta\,d\xi\\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{1}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\Phi_0(\xi,\eta_0)\,d\eta_0\right|^2 d\beta\,d\xi\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{F}_{\eta_0}\left[\frac{1}{\eta_0}\int_T^{2T}\chi'(t)e^{i\eta_0 t}\,dt\right](\beta)\mathcal{F}_{\eta_0}[\Phi_0(\xi,\eta_0)](\beta)\,d\eta_0\right|^2 d\beta\,d\xi\,.
\end{aligned}
$$

It is straightforward to check that $\mathcal{F}_{\eta_0}\left[\frac{1}{\eta_0}\int_T^{2T}\chi'(t)e^{i\eta_0 t}\,dt\right](\beta)$ is uniformly bounded for any $\beta \in \mathbb{R}$. Thus,

$$
\begin{aligned}
I_{112} &\le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathcal{F}_{\eta_0}[\Phi_0(\xi,\eta_0)](\beta)\,d\eta_0\right|^2 d\beta\,d\xi \le C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Phi_0(\xi,\eta_0)\right|^2 d\eta_0\,d\xi\\
&\le C\int_{-\infty}^{\infty}\int_0^{\infty}\left|\eta_0^{-1/4}\widehat{\phi}(\xi,\sqrt{\eta_0})\right|^2 d\eta_0\,d\xi \le C\int_{-\infty}^{\infty}\int_0^{\infty}\left|\widehat{\phi}(\xi,\eta_0)\right|^2 d\eta_0\,d\xi\,.
\end{aligned}
$$

For I_{111}, we note that

$$
\left|\frac{\beta^{1/4}-\eta_0^{1/4}}{(\beta-\eta_0)}\left(\int_T^{2T}\chi'(t)e^{i(\beta-\eta_0)t}\,dt\right)\right| \le C\left((\beta^{3/4}+\eta_0^{3/4})(1+(\beta-\eta_0)^2)\right)^{-1}
$$

and

$$
\sup_{\eta_0\in(0,\infty)}\int_0^{\infty}\left((\beta^{3/4}+\eta_0^{3/4})(1+(\beta-\eta_0)^2)\right)^{-1} d\beta \le C < \infty\,,
$$

where $(\beta^{3/4}+\eta_0^{3/4})(1+(\beta-\eta_0)^2)$ is symmetric with respect to β and η_0. Hence, by theory of integral operators, we have

$$\begin{aligned}
I_{111} &\le C\int_{-\infty}^{\infty}\int_0^{\infty}\left|\int_0^{\infty}\frac{1}{(\beta^{3/4}+\eta_0^{3/4})(1+(\beta-\eta_0)^2)}\left|\eta_0^{-1/2}\widehat{\phi}(\xi,\sqrt{\eta_0})\right|\,d\eta_0\right|^2\,d\beta\,d\xi \\
&\le C\int_{-\infty}^{\infty}\int_0^{\infty}\left|\eta_0^{-1/2}\widehat{\phi}(\xi,\sqrt{\eta_0})\right|^2\,d\eta_0\,d\xi \le C\int_{-\infty}^{\infty}\int_0^{\infty}\eta_0^{-1}\left|\widehat{\phi}(\xi,\eta_0)\right|^2\,d\eta_0\,d\xi \\
&\le C\int_{-\infty}^{\infty}\int_0^{\infty}\left|\widehat{\phi}(\xi,\eta_0)\right|^2\,d\eta_0\,d\xi\,,
\end{aligned}$$

since $\widehat{\phi}(\xi,\eta_0)=0$ for $|\eta_0|\le 1$. Thus, we finish the proof of (3.41) for $s=0$.

For $s>0$, we use interpolation. First, let $s=2$. Then, denote $u=W_b[\chi(t)W_{\mathbb{R}^2}\phi](t)$ and let $v=u_t$. The equations for v are

$$\begin{aligned}
&iv_t+v_{xx}+v_{yy}=0,\quad v\Big|_{t=0}=i(u_{xx}+u_{yy})\Big|_{t=0}=0\,,\\
&\qquad\qquad\text{and } v\Big|_{y=0}=(W_{\mathbb{R}^2}\phi)_t\Big|_{y=0}=(W_{\mathbb{R}^2}(i\phi_{xx}+i\phi_{yy}))\Big|_{y=0}
\end{aligned}$$

for $t\in[0,T]$. By (3.41) with $s=0$, we have

$$\|v\|^2_{L^q_t\left([0,T];\,L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\le C\,\|\phi_{xx}+\phi_{yy}\|^2_{L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\le C\,\|\phi\|^2_{H^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\,.$$

Then, by u satisfying

$$u_{xx}+u_{yy}=-iv\,,\quad\text{for}\quad(x,y)\in\mathbb{R}\times\mathbb{R}^+\,,\qquad u\Big|_{y=0}=(W_{\mathbb{R}^2}\phi)\Big|_{y=0}$$

and the classical theory of elliptic equations for Dirichlet boundary conditions, we obtain that

$$\|u\|^2_{H^{2,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)}\le C\left(\|v\|^2_{L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)}+\|W_{\mathbb{R}^2}\phi\|^2_{H^{2,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)}\right)\,,$$

which yields

$$\begin{aligned}
&\|u\|^2_{L^q_t\left([0,T];\,H^{2,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\\
&\quad\le C\left(\|v\|^2_{L^q_t\left([0,T];\,L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}+\|W_{\mathbb{R}^2}\phi\|^2_{L^q_t\left([0,T];\,H^{2,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\le C\,\|\phi\|^2_{H^2_{xy}(\mathbb{R}\times\mathbb{R}^+)}\,.
\end{aligned}$$

Thus, the estimate (3.41) for $s=2$ is proved. The case for $0<s<2$ is obtained by interpolation. The proof for $s>2$ is similar. Therefore, we have

$$\|W_b\left[W_{\mathbb{R}^2}(\cdot)\phi(\cdot,0)\right]\|^2_{L^q_t\left([0,T];\,W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\le C\,\|\phi\|^2_{H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)}\,.$$

This finishes the proof. $\square$

3.2.3 Equation with the Nonlinearity

Finally, we derive the estimates for the nonhomogeneous terms in the equation. Let $f : \mathbb{R}^2 \times [0,T] \to \mathbb{C}$ be an extension of g, i.e. $f(x,y,t) = g(x,y,t)$ for $y \geq 0$. We will show that the solution $u = \Phi_f$ of (3.3) satisfies some estimates needed later, where it is well-known that

Proposition 3.7.

$$\begin{aligned}\Phi_f(x,y,t) &= i\left(\int_0^t W_{\mathbb{R}^2}(t-\tau)f(\tau)\,d\tau\right)(x,y)\\ &= i\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi^2+\eta^2)(t-\tau)+i(\xi x+\eta y)}\widehat{f^{xy}}(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau\,.\end{aligned} \tag{3.44}$$

Proof. This is a direct result from the semigroup theory since $W_{\mathbb{R}^2}(t)\phi$ is a semigroup with respect to $t > 0$ so that $W_{\mathbb{R}^2}(t)\phi$ solves the extended linear Cauchy problem. $\square$

Thus the following estimates hold.

Proposition 3.8. *Let $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ and $\frac{1}{\gamma}+\frac{1}{\rho}=\frac{1}{2}$ where $r,\ \rho \in [2,\infty)$. Then,*

$$\|\Phi_f\|_{L_t^q\left([0,T];\,W_{xy}^{s,r}(\mathbb{R}^2)\right)} \lesssim \|f\|_{L_t^{\gamma'}\left([0,T];\,H_{xy}^{s,\rho'}(\mathbb{R}^2)\right)}\,, \tag{3.45}$$

$$\|W_b\left[\Phi_f|_{y=0}\right]\|_{L_t^q\left([0,T];\,W_{xy}^{s,r}(\mathbb{R}^2)\right)} \lesssim \|f\|_{L_t^1\left([0,T];\,H_{xy}^{s}(\mathbb{R}^2)\right)}\,. \tag{3.46}$$

Proof. The proof of (3.45) can be found as the Strichartz's estimate in *Section 2.3* of [27]. For (3.46), we first show that

$$W_b\left[\Phi_f|_{y=0}\right] = i\left(\int_0^t W_b\left[\left[W_{\mathbb{R}^2}(t)f(\tau)\right]\Big|_{y=0}\right](t-\tau,x,y)\,d\tau\right)\,. \tag{3.47}$$

Let v be the right side of (3.47). Then, v satisfies

$$\begin{aligned}&iv_t + v_{xx} + v_{yy}\\ &= -W_b\left[W_{\mathbb{R}^2}(t)f(\tau)\Big|_{y=0}\right]\Big|_{\tau=t}(0,x,y) + i\int_0^t\left(i\left(W_b\left[\left[W_{\mathbb{R}^2}(t)f(\tau)\right]\Big|_{y=0}\right](t-\tau,x,y)\right)_t +\right.\\ &\qquad\left.\left(W_b\left[\left[W_{\mathbb{R}^2}(t)f(\tau)\right]\Big|_{y=0}\right](t-\tau,x,y)\right)_{xx} + \left(W_b\left[\left[W_{\mathbb{R}^2}(t)f(\tau)\right]\Big|_{y=0}\right](t-\tau,x,y)\right)_{yy}\right)d\tau\\ &= -W_b\left[W_{\mathbb{R}^2}(t)f(\tau)\Big|_{y=0}\right]\Big|_{\tau=t}(0,x,y) = 0\,,\\ &\text{with } \ v|_{t=0} = 0\,,\quad v|_{y=0} = i\int_0^t\left[W_{\mathbb{R}^2}(t-\tau)f(\tau)\right]\Big|_{y=0}d\tau\,.\end{aligned}$$

Thus,

$$v = W_b \left[i \int_0^t \left[W_{\mathbb{R}^2}(t-\tau) f(\tau) \right] \, d\tau \Big|_{y=0} \right] = W_b \left[\Phi_f|_{y=0} \right] .$$

Now, by the Minkowski inequality and *Proposition 3.5*, we have

$$\begin{aligned}
&\| W_b \left[\Phi_f|_{y=0} \right] \|_{L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\qquad = \left\| \int_0^T \zeta_{[0,t]}(\tau) W_b \left[\left[W_{\mathbb{R}^2}(t) f(\tau) \right] \Big|_{y=0} \right] (t-\tau, x, y) \, d\tau \right\|_{L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\qquad \le \int_0^T \left\| \zeta_{[\tau,\infty)}(t) W_b \left[\left[W_{\mathbb{R}^2}(t) f(\tau) \right] \Big|_{y=0} \right] (t-\tau, x, y) \right\|_{L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} d\tau \\
&\qquad = \int_0^T \left\| W_b \left[\left[W_{\mathbb{R}^2}(t) f(\tau) \right] \Big|_{y=0} \right] (t, x, y) \right\|_{L_t^q\left([0,T-\tau];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} d\tau \\
&\qquad \le C \int_0^T \| f(\tau, x, y) \|_{H_{xy}^s(\mathbb{R}^2)} \, d\tau = \| f \|_{L_t^1\left([0,T];\, H_{xy}^s(\mathbb{R}^2)\right)} ,
\end{aligned}$$

where ζ is a characteristic function. Thus, the proof of the estimates with nonhomogeneous term in the equation is completed. □

3.3 Local Well-posedness

In the following, we will show that the integral equation

$$\begin{aligned}
u(x,y,t) = & W_b \left[h - W_{\mathbb{R}^2}(\cdot)\phi \Big|_{y=0} - i \left(\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot - \tau) f(\tau) \, d\tau \right) \Big|_{y=0} \right] (x,y,t) \\
& + W_{\mathbb{R}^2}(t)\phi(x,y) + i \left(\int_0^t W_{\mathbb{R}^2}(t-\tau) f(\tau) \, d\tau \right) (x,y) = \mathcal{A}[u](x,y,t)
\end{aligned} \tag{3.48}$$

has a solution in some function spaces with $H^s(\mathbb{R}\times\mathbb{R}^+)$, i.e., $\mathcal{A}[u](x,y,t)$ has a fixed point, where $f(u) = \lambda |u|^{p-2} u$ for $p \ge 3$and $(x,y,t) \in \mathbb{R}\times\mathbb{R}^+\times(0,T)$. Also note that as long as $0 \le s < 1/2$, the trace operator is not defined and we may take the trace as zero and have a simpler form for $\mathcal{A}[u](x,y,t)$. Suppose (q,r) and (γ,ρ) two independent admissible pairs and define

$$\mathcal{X}_T^s := C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L_t^\gamma\left([0,T];\, W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)$$

with $\|u\|_{\mathcal{X}_T^s} = \sup_{t\in[0,T]} \|u(t)\|_{H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)} + \|u\|_{L_t^q\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|u\|_{L_t^\gamma\left([0,T];\, W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)}$. Let $(\tilde{q},\tilde{r})$ be a third pair of admissible pair such that $r_1 > 2$ and define

$$\mathcal{Y}_T^s := C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L_t^{\tilde{q}}\left([0,T];\, W_{xy}^{s,\tilde{r}}(\mathbb{R}\times\mathbb{R}^+)\right)$$

with $\|u\|_{\mathcal{Y}_T^s} = \sup_{t\in[0,T]} \|u(t)\|_{H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)} + \|u\|_{L_t^{\tilde{q}}\left([0,T];\, W_{xy}^{s,\tilde{r}}(\mathbb{R}\times\mathbb{R}^+)\right)}$. At last denote

$$\mathcal{Z}_T^s := C_t\left([0,T];\, H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)$$

with the regular sup-norm in the time variable, i.e. $\|u\|_{\mathcal{Z}_T^s} = \sup_{t\in[0,T]} \|u(t)\|_{H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)}$.

For some $M > 0$, define the closed balls in $\mathcal{X}_T^s, \mathcal{Y}_T^s$ and $\mathcal{Z}_T^s$ as

$$B_M^{\mathcal{X}^s} := \left\{u : \|u\|_{\mathcal{X}_T^s} \le M\right\}, \quad B_M^{\mathcal{Y}_T^s} := \left\{u : \|u\|_{\mathcal{Y}_T^s} \le M\right\} \quad \text{and} \quad B_M^{\mathcal{Z}^s} := \left\{u : \|u\|_{\mathcal{Z}_T^s} \le M\right\}.$$

3.3.1 Existence and Uniqueness

In this section, we concentrate on the discussion of the existence and uniqueness of the solution using the contraction mapping argument. It is necessary to define some function spaces as follows. If $0 \le s < 1$, let two admissible pairs (q,r) and (γ,ρ) be independent of each other such that

$$r = \frac{2p}{2+s(p-2)}, \qquad q = \frac{2p}{(1-s)(p-2)}, \tag{3.49}$$

$$\rho = \frac{2(p-1)}{1+s(p-2)}, \qquad \gamma = \frac{2(p-1)}{(1-s)(p-2)}, \tag{3.50}$$

and the main goal is following.

Theorem 3.9. *Let $\mu > 0$ be a constant so that*

$$\|\varphi\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} + \|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} \le \mu$$

and φ, h satisfy some compatibility conditions.

(a) Let (q,r), (γ,ρ) be defined by (3.49) and (3.50). For $0 \le s < 1$ and $3 \le p < (4-2s)/(1-s)$, there exists a $T > 0$ such that there is a unique solution $u \in \mathcal{X}_T^s$ of (3.48). In particular, we have

$$\|\mathcal{A}[u] - \mathcal{A}[v]\|_{\mathcal{X}_T^s} \le (1/2)\, \|u - v\|_{\mathcal{X}_T^s}, \tag{3.51}$$

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \le M \tag{3.52}$$

for any u and $v \in B_M^{\mathcal{X}^s}$. Moreover, if $p = (4-2s)/(1-s)$, then μ must be small.

(b) Let (q,r) be any admissible pair with $r > 2$. For $s = 1$, there exists a $T > 0$ so that the integral equation (3.48) has a unique solution

$$u \in \mathcal{Y}_T^1 = C_t\left([0,T];\, H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L_t^q\left([0,T];\, W_{xy}^{1,r}(\mathbb{R}\times\mathbb{R}^+)\right).$$

In particular, we have

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Y}_T^1} \leq (1/2)\,\|u-v\|_{\mathcal{Y}_T^1}\,, \tag{3.53}$$

$$\|\mathcal{A}[u]\|_{\mathcal{Y}_T^1} \leq M \tag{3.54}$$

for any u and $v \in B_M^{\mathcal{Y}^1}$.

(c) *For $s>1$, we assume that $p \geq s+1$ if $s \in \mathbb{Z}$ or $p \geq [s]+2$ if $s \notin \mathbb{Z}$ only when p is not an even integer. There exists a $T>0$ such that the integral equation (3.48) has a unique solution $u \in \mathcal{Z}_T^s = C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right)$. Also, for u and $v \in B_M^{\mathcal{Z}^s}$*

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}_T^0} \leq (1/2)\,\|u-v\|_{\mathcal{Z}_T^0}\,, \tag{3.55}$$

$$\|\mathcal{A}[u]\|_{\mathcal{Z}_T^s} \leq M \tag{3.56}$$

In addition if we further assume $p \geq s+2$ with $s \in \mathbb{Z}$ or $p \geq [s]+3$ with $s \notin \mathbb{Z}$ or p is an even integer, then (3.55) can be improved by

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}_T^s} \leq (1/2)\,\|u-v\|_{\mathcal{Z}_T^s}\,, \tag{3.57}$$

Proof. For $0 \leq s < 1$, assume that

$$3 \leq p \leq \frac{4-2s}{1-s}\,. \tag{3.58}$$

It is easy to verify that $r>2$ and (q,r) is an admissible pair. We will use the contraction mapping theorem. First, let $3 \leq p < \frac{4-2s}{1-s}$, the subcritical case. In [27] (the proof of *Theorem 4.6.1*), we find the estimate

$$\|f(u)\|_{L_t^{q'}\left([0,T];L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim T^{\frac{q-p}{q}}\,\|u\|^{p-1}_{L_t^{q}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)}\,. \tag{3.59}$$

Let $\alpha \in \mathbb{R}^{+,2}$ be a multi-index such that $|\alpha|=s$. We know that $u:\mathbb{R}^N \to \mathbb{R}^2$ and $f \in C^1(\mathbb{C})$. For $0 \leq s < 1$, the chain rule for the fractional derivative here gives

$$\|D^\alpha f(u)\|_{L^{r_2}} \lesssim \|f'(u)\|_{L^{r_1}}\|D^\alpha u\|_{L^r}$$

for $\frac{1}{r_2}=\frac{1}{r_1}+\frac{1}{r}$ with $1<r,\, r_1,\, r_2<\infty$. Let u_R and u_I be the real and imaginary parts of u, respectively, with which $f(u)=\lambda|u|^{p-2}u=\lambda u_R\left(u_R^2+u_I^2\right)^{\frac{p}{2}-1}+i\lambda u_I\left(u_R^2+u_I^2\right)^{\frac{p}{2}-1}$. Therefore

$$f'(u)=\lambda\begin{pmatrix} (p-2)|u|^{p-4}u_R^2+|u|^{p-2} & (p-2)|u|^{p-4}u_Ru_I \\ (p-2)|u|^{p-4}u_Ru_I & (p-2)|u|^{p-4}u_I^2+|u|^{p-2}\end{pmatrix}.$$

which yields that $|f'(u)| \lesssim |u|^{p-2} < \infty$ for $p \geq 2$. In fact, let $k \in \mathbb{Z}^+$ and we continue differentiating to have higher order of derivatives of f. If $p \geq k+1$, it is observed that $(k-1)$-th derivative $f^{(k-1)}$ can be composed of terms in the form $|u|^{a_1}u_R^{a_2}u_I^{a_3}$ where $a_1+a_2+a_3=p-k$. Through a simple calculation,

$$\partial_{u_R}\left(|u|^{a_1}u_R^{a_2}u_I^{a_3}\right)=a_1|u|^{a_1-2}u_R^{a_2+1}u_I^{a_3}+a_2|u|^{a_1}u_R^{a_2-1}u_I^{a_3}.$$

Thus $f^{(k)}$ is term of $|u|^{b_1} u_R^{b_2} u_I^{b_3}$ where $b_1+b_2+b_3=p-k-1$. Moreover,

$$|f^{(k)}(u)| \lesssim |u|^{p-k-1}\,, \tag{3.60}$$

for $p \geq k+1$.

For $u \in B_M^{\mathcal{X}^s}$

$$\begin{aligned}\|D^\alpha f(u)(t)\|_{L^{r'}} &\lesssim \|f'(u)(t)\|_{L^{r_1}} \|D^\alpha u(t)\|_{L^r} \\ &\leq \left\||u(t)|^{p-2}\right\|_{L^{r_1}} \|D^\alpha u\|_{L^r} = \|u(t)\|^{p-2}_{L^{r_1(p-2)}} \|D^\alpha u(t)\|_{L^r}\end{aligned}$$

where $\frac{1}{r'} = 1 - \frac{1}{r}$ and $\frac{1}{r_1} = 1 - \frac{2}{r}$ for $r > 2$. According to Gagliardo-Nirenburg inequality and (3.49), $\frac{1}{r_1(p-2)} = \frac{1}{r} - \frac{s}{2}$ which implies $\|u(t)\|_{L^{r_1(p-2)}} \leq \|D^\alpha u(t)\|_{L^r}$. Thus,

$$\|D^\alpha f(u)(t)\|_{L^{r'}} \lesssim \|D^\alpha u(t)\|^{p-2}_{L^r} \cdot \|D^\alpha u(t)\|_{L^r}\,.$$

Next, we give the norm in t over the above inequality. That is, with $\frac{1}{q'} = 1 - \frac{1}{q}$ and $\frac{1}{q_1} = 1 - \frac{2}{q}$

$$\|D^\alpha f(u)\|_{L_t^{q'}\left([0,T];L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \|D^\alpha u\|^{p-2}_{L_t^{q_1(p-2)}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)} \cdot \|D^\alpha u\|_{L_t^{q}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)}\,.$$

On account of (3.58), it can be shown that $q \geq q_1(p-2)$. By an imbedding theorem for L^p space,

$$\|D^\alpha u\|_{L_t^{q_1(p-2)}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim T^{\frac{1}{q_1} - \frac{p-2}{q}} \|D^\alpha u\|_{L_t^{q}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)}\,,$$

which means

$$\|D^\alpha f(u)\|_{L_t^{q'}\left([0,T];L_{xy}^{r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim T^{\frac{q-p}{q}} \|D^\alpha u\|^{p-1}_{L_t^{q}\left([0,T];L_{xy}^{r}(\mathbb{R}\times\mathbb{R}^+)\right)}\,. \tag{3.61}$$

Thus, with (3.59) and (3.61) it is not difficult to see that

$$\begin{aligned}&\|f(u)\|_{L_t^{q'}\left([0,T];H_{xy}^{s,r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \\ &\quad\lesssim T^{\frac{q-p}{q}} \|u\|^{p-1}_{L_t^{q}\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} = T^{1-\frac{(1-s)(p-2)}{2}} \|u\|^{p-1}_{L_t^{q}\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}\,.\end{aligned} \tag{3.62}$$

Similarly, let $u \in B_M^{\mathcal{X}^s}$.

$$\begin{aligned}\|D^\alpha f(u)(t)\|_{L^2} &\lesssim \|f'(u)(t)\|_{L^{\frac{2\rho}{\rho-2}}} \|D^\alpha u(t)\|_{L^\rho} \\ &\lesssim \left\||u(t)|^{p-2}\right\|_{L^{\frac{2\rho}{\rho-2}}} \|D^\alpha u\|_{L^\rho} = \|u(t)\|^{p-2}_{L^{\frac{2\rho(p-2)}{\rho-2}}} \|D^\alpha u(t)\|_{L^\rho}\,.\end{aligned}$$

Since $\rho = \frac{2(p-1)}{1+s(p-2)}$ by (3.50), then $\frac{\rho-2}{2\rho(p-2)} = \frac{1}{\rho} - \frac{s}{2}$. Thus again using Gagliardo-Nirenberg inequality, one can obtain that

$$\|D^\alpha f(u)(t)\|_{L^2} \lesssim \|D^\alpha u(t)\|^{p-2}_{L^\rho} \cdot \|D^\alpha u(t)\|_{L^\rho}\,.$$

Because (3.58) exactly leads to $\gamma \geq \frac{\gamma(p-2)}{\gamma-1}$, it follows that by taking norm with respect to t,

$$\begin{aligned}\|D^\alpha f(u)\|_{L_t^1\left([0,T];L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} &\lesssim \|D^\alpha u\|^{p-2}_{L_t^{\frac{\gamma(p-2)}{\gamma-1}}\left([0,T];L^\rho_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \cdot \|D^\alpha u\|_{L_t^\gamma\left([0,T];L^\rho_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\ &\lesssim T^{\frac{\gamma-1}{\gamma}-\frac{p-2}{\gamma}} \|D^\alpha u\|^{p-1}_{L_t^\gamma\left([0,T];L^\rho_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} = T^{\frac{\gamma-p+1}{\gamma}} \|D^\alpha u\|^{p-1}_{L_t^\gamma\left([0,T];L^\rho_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} ,\end{aligned} \tag{3.63}$$

Also it is straightforward to see that

$$\|f(u)\|_{L_t^1\left([0,T];L^2_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim T^{\frac{\gamma-p+1}{\gamma}} \|u\|^{p-1}_{L_t^\gamma\left([0,T];L^\rho_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} . \tag{3.64}$$

Hence, (3.63) and (3.64) imply

$$\begin{aligned}\|f(u)\|_{L_t^1\left([0,T];H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim& T^{\frac{\gamma-p+1}{\gamma}} \|u\|^{p-1}_{L_t^\gamma\left([0,T];W^{s,\rho}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\ =& T^{1-\frac{(1-s)(p-2)}{2}} \|u\|^{p-1}_{L_t^\gamma\left([0,T];W^{s,\rho}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} .\end{aligned} \tag{3.65}$$

Let $u, v \in \mathcal{X}^s_T$ and $w = u\cdot\theta + v\cdot(1-\theta)$ for $\theta \in [0,1]$. Then

$$|u|^{p-2}u - |v|^{p-2}v = \int_0^1 \left[\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right]\cdot(u-v)\,d\theta\,.$$

By (2.20) in *Lemma 2.8*,

$$\begin{aligned}&\left\|D^\alpha\left[|u|^{p-2}u(t) - |v|^{p-2}v(t)\right]\right\|_{L^{r'}} \\ &\quad= \left\|\int_0^1 D^\alpha\left[\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right]\cdot(u-v)\,d\theta\right\|_{L^{r'}} \\ &\quad\leq \sup_{\theta\in[0,1]}\left\|D^\alpha\left[\left(\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right)\cdot(u-v)\right]\right\|_{L^{r'}} \\ &\quad\leq \sup_{\theta\in[0,1]}\left\|D^\alpha\left[\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right]\right\|_{L^{r_1}}\cdot\|u-v\|_{L^{r_2}} \\ &\qquad+ \sup_{\theta\in[0,1]}\left\|\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right\|_{L^{\frac{r}{r-2}}}\cdot\|D^\alpha[u-v]\|_{L^r} .\end{aligned}$$

Note that if we choose

$$r_2 = \frac{2r}{2-sr} = \frac{p}{1-s}\,,$$

then the Sobolev embedding theorem implies $W^{s,r} \hookrightarrow L^{r_2}$. Therefore

$$r_1 = \frac{r'r_2}{r_2-r'} = \frac{2p}{2p-sp-4+4s} \quad \text{and} \quad \frac{r(p-2)}{r-2} = \frac{p}{1-s}\,.$$

For $p > 3$, we can see that the derivative of $\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2} - 1\right)|w|^{p-4}w^2$ is absolutely bounded by $|w|^{p-3}$. By *Lemma 2.7*,

$$\begin{aligned}
&\left\| D^\alpha\left[|u|^{p-2}u(t) - |v|^{p-2}v(t)\right]\right\|_{L^{r'}} \\
&\quad \lesssim \sup_{\theta\in[0,1]} \left\||w(t)|^{p-3}\right\|_{L^{\frac{r_1 r}{r-r_1}}} \cdot \|D^\alpha w(t)\|_{L^r} \cdot \|u(t) - v(t)\|_{L^{\frac{p}{1-s}}} \\
&\qquad + \sup_{\theta\in[0,1]} \|w(t)\|^{p-2}_{L^{\frac{r(p-2)}{r-2}}} \cdot \|D^\alpha(u(t) - v(t))\|_{L^r} \\
&\quad = \sup_{\theta\in[0,1]} \|w(t)\|^{p-3}_{L^{\frac{p}{1-s}}} \cdot \|D^\alpha w(t)\|_{L^r} \cdot \|u(t) - v(t)\|_{L^{\frac{p}{1-s}}} \\
&\qquad + \sup_{\theta\in[0,1]} \|w(t)\|^{p-2}_{L^{\frac{p}{1-s}}} \cdot \|D^\alpha(u(t) - v(t))\|_{L^r} \\
&\quad \lesssim \left(\|u(t)\|^{p-2}_{W^{s,r}} + \|v(t)\|^{p-2}_{W^{s,r}}\right) \cdot \|(u(t) - v(t))\|_{W^{s,r}}.
\end{aligned}$$

As for $p = 3$, by (2.19) in *Lemma 2.7* it can be shown that

$$\begin{aligned}
&\left\| D^\alpha\left[|u|^{p-2}u(t) - |v|^{p-2}v(t)\right]\right\|_{L^{r'}} \\
&\lesssim \sup_{\theta\in[0,1]} \|D^\alpha w(t)\|_{L^r} \cdot \|u(t) - v(t)\|_{L^{\frac{3}{1-s}}} + \sup_{\theta\in[0,1]} \|w(t)\|_{L^{\frac{3}{1-s}}} \cdot \|D^\alpha(u(t) - v(t))\|_{L^r} \\
&\lesssim (\|u(t)\|_{W^{s,r}} + \|v(t)\|_{W^{s,r}}) \cdot \|(u(t) - v(t))\|_{W^{s,r}}.
\end{aligned}$$

Since $q \geq \frac{q(p-2)}{q-2}$, we obtain

$$\begin{aligned}
&\|f(u) - f(v)\|_{L_t^{q'}\left([0,T];W_{xy}^{s,r'}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\leq \left(\|u\|^{p-2}_{L_t^{\frac{q(p-2)}{q-2}}\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^{\frac{q(p-2)}{q-2}}\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\|u - v\|_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\leq T^{\frac{q-p}{q}}\left(\|u\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\|u - v\|_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\lesssim T^{1-\frac{(1-s)(p-2)}{2}}\left(\|u\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\|u - v\|_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)}.
\end{aligned}$$

For each $t \in [0, T]$ and the admissible pair (γ, ρ) in (3.50), if replacing r' by 2 and r by ρ in the above argument, we have $r_2 = \frac{2(p-1)}{1-s}, r_1 = \frac{2r_2}{r_2-2} = \frac{2(p-1)}{p+s-2}$ and $W^{s,\rho} \hookrightarrow L^{r_2}$. Moreover, it is clear that $\frac{2\rho(p-2)}{\rho-2} = \frac{2(p-1)}{1-s}$. Thus for any $p \geq 3$, a similar discussion shows that

$$\begin{aligned}
&\left\| D^\alpha\left[|u|^{p-2}u(t) - |v|^{p-2}v(t)\right]\right\|_{L^2} \\
&\quad \leq \sup_{\theta\in[0,1]} \|w(t)\|^{p-3}_{L^{\frac{2(p-1)}{1-s}}} \cdot \|D^\alpha w(t)\|_{L^\rho} \cdot \|u(t) - v(t)\|_{L^{\frac{2(p-1)}{1-s}}} \\
&\qquad + \sup_{\theta\in[0,1]} \|w(t)\|^{p-2}_{L^{\frac{2(p-1)}{1-s}}} \cdot \|D^\alpha(u(t) - v(t))\|_{L^\rho}
\end{aligned}$$

$$\lesssim \left(\|u(t)\|_{W^{s,\rho}}^{p-2} + \|v(t)\|_{W^{s,\rho}}^{p-2}\right) \cdot \|(u(t) - v(t))\|_{W^{s,\rho}} .$$

Because of $1 = \frac{\gamma-1}{\gamma} + \frac{1}{\gamma}$ with $\gamma \geq \frac{\gamma(p-2)}{\gamma-1}$, we have

$$\begin{aligned}
&\|D^\alpha(f(u) - f(v))\|_{L_t^1\left([0,T];L_{xy}^2(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\lesssim \left(\|u\|^{p-2}_{L_t^{\frac{\gamma(p-2)}{\gamma-1}}\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^{\frac{\gamma(p-2)}{\gamma-1}}\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \right) \cdot \|u - v\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\leq T^{\frac{\gamma-p+1}{\gamma}} \left(\|u\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \right) \cdot \|u - v\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\lesssim T^{1-\frac{(1-s)(p-2)}{2}} \left(\|u\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \right) \|u - v\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} .
\end{aligned}$$

Now, we choose u in $B_M^{\mathcal{X}^s}$, that is, $\|u\|_{\mathcal{X}_T^s} \leq M$. We study the operator $\mathcal{A}(u)$ in (3.48) with the estimate

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} = \|\mathcal{A}[u]\|_{L_t^\infty\left([0,T];H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right)} + \|\mathcal{A}[u]\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|\mathcal{A}[u]\|_{L_t^q\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times\mathbb{R}^+)\right)} .$$

By applying *Proposition 3.3*, *3.6*, and *3.8*, we obtain

$$\begin{aligned}
&\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \\
&\quad \leq \|W_b[h]\|_{\mathcal{X}_T^s} + \|W_{\mathbb{R}^2}\phi\|_{\mathcal{X}_T^s} + \|\Phi_f\|_{\mathcal{X}_T^s} + \left\| W_b\left[(W_{\mathbb{R}^2}\phi)\Big|_{y=0} \right] \right\|_{\mathcal{X}_T^s} + \left\| W_b\left[(\Phi_f)\Big|_{y=0} \right] \right\|_{\mathcal{X}_T^s} \\
&\quad \lesssim \|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + \|f\|_{L_t^{q'}\left([0,T];H_{xy}^{s,r'}\right)} + \|f\|_{L_t^1\left([0,T];H_{xy}^s\right)} \\
&\quad \leq \mu + \|f\|_{L_t^{q'}\left([0,T];H_{xy}^{s,r'}\right)} + \|f\|_{L_t^1\left([0,T];H_{xy}^s\right)} \\
&\quad \lesssim \mu + T^{1-\frac{(1-s)(p-2)}{2}} \left(\|u\|^{p-1}_{L_t^q\left([0,T];W_{xy}^{s,r}\right)} + \|u\|^{p-1}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \right) \\
&\quad \sim \mu + T^{1-\frac{(1-s)(p-2)}{2}} \left(\|u\|_{L_t^q\left([0,T];W_{xy}^{s,r}\right)} + \|u\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}(\mathbb{R}\times\mathbb{R}^+)\right)} \right)^{p-1} \\
&\quad \leq \mu + T^{1-\frac{(1-s)(p-2)}{2}} \|u\|^{p-1}_{\mathcal{X}_T^s} .
\end{aligned}$$

where (3.62) has been applied. Since $u \in B_M^{\mathcal{X}^s}$, for some $C_0 > 0$,

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \leq C_0 \left(\mu + T^{1-\frac{(1-s)(p-2)}{2}} M^{p-1} \right) . \tag{3.66}$$

Then, select M sufficiently large such that $M > C_0\mu$ and let T be small enough with

$$0 < T \leq \left(\frac{M - C_0\mu}{C_0 M^{p-1}} \right)^{\frac{2}{2-(1-s)(p-2)}} . \tag{3.67}$$

This implies that for $\|u\|_{\mathcal{X}_T^s} \leq M$, $\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \leq M$ which restates (3.52).

Moreover, for $\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{X}_T^s}$ and the estimate (3.51), let $\Phi_f(u)$ denote the operator on the nonlinearity f given by u, where $f \in L_t^{q'}\left([0,T];H_{xy}^{s,r'}\right) \bigcap L_t^1\left([0,T];H_{xy}^s\right)$. If we choose u and v in $B_M^{\mathcal{X}^s}$, then

$$\begin{aligned}
&\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{X}_T^s}\\
&\quad\le \|\Phi_f(u)-\Phi_f(v)\|_{\mathcal{X}_T^s}+\left\|W_b\left[(\Phi_f(u)-\Phi_f(v))\Big|_{y=0}\right]\right\|_{\mathcal{X}_T^s}\\
&\quad\lesssim \|f(u)-f(v)\|_{L_t^{q'}\left([0,T];H_{xy}^{s,r'}\right)}+\|f(u)-f(v)\|_{L_t^1\left([0,T];H_{xy}^s\right)}\\
&\quad\lesssim T^{1-\frac{(1-s)(p-2)}{2}}\left(\|u\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}+\|v\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}\right)\|u-v\|_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}\\
&\qquad\quad+\|f(u)-f(v)\|_{L_t^1\left([0,T];H_{xy}^s\right)}\\
&\quad\lesssim T^{1-\frac{(1-s)(p-2)}{2}}\left(\|u\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}+\|v\|^{p-2}_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}\right)\|u-v\|_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}\\
&\qquad\quad+T^{1-\frac{(1-s)(p-2)}{2}}\left(\|u(t)\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}\right)}+\|v(t)\|^{p-2}_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}\right)}\right)\|u-v\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}\right)}\\
&\quad\lesssim T^{1-\frac{(1-s)(p-2)}{2}}M^{p-2}\left(\|u-v\|_{L_t^q\left([0,T];W_{xy}^{s,r}\right)}+\|u-v\|_{L_t^\gamma\left([0,T];W_{xy}^{s,\rho}\right)}\right)\\
&\quad\le T^{1-\frac{(1-s)(p-2)}{2}}M^{p-2}\|u-v\|_{\mathcal{X}_T^s},
\end{aligned}$$

i.e., for some constant C_1,

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{X}_T^s}\le C_1T^{1-\frac{(1-s)(p-2)}{2}}M^{p-2}\|u-v\|_{\mathcal{X}_T^s}. \tag{3.68}$$

Then, choose T and M satisfying (3.67) and $C_1T^{1-\frac{(1-s)(p-2)}{2}}M^{p-2}\le(1/2)$, which yields (3.51), i.e.

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{X}_T^s}\le(1/2)\|u-v\|_{\mathcal{X}_T^s}.$$

Now we can prove the existence and uniqueness. Using Contraction Mapping theorem, we consider the problem (3.2) (or (3.3) if $0\le s<1/2$) has a unique solution u in $\mathcal{X}_T^s$.

For the critical case with $p=\frac{4-2s}{1-s}$, we let $r=\frac{2p}{p-2}$ and $q=p$. Note that in (3.66), we need that

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s}\le C_0\left(\mu+M^{p-1}\right).$$

Thus, instead of (3.66) and (3.68), for $u,\,v\in B_{2C_0\mu}^{\mathcal{X}^s}:=\left\{u:\ \|u\|_{\mathcal{X}_T^s}\le 2C_0\mu\right\}$, we have

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s}\le C_0\left(\mu+C_1\mu^{p-1}\right) \tag{3.69}$$

and

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{X}_T^s}\le\widetilde{C_1}\mu^{p-2}\|u-v\|_{\mathcal{X}_T^s}. \tag{3.70}$$

As long as $T>0$ finite, we can choose μ small enough so that $C_0\left(\mu+C_1\mu^{p-1}\right)\le 2C_0\mu$ and $\widetilde{C_1}\mu^{p-2}\le(1/2)$. Thus, (3.51) and (3.52) also hold for this case. That leads to a conclusion that there is a fixed point of (3.6) (or (3.7)) in $B^{\mathcal{X}^s}_{2C_0\mu}$. By contraction mapping theorem, the existence and uniqueness of this problem is also guaranteed. Here, we note that the initial and boundary conditions are small.

For $s = 1$, pick u in $B^{\mathcal{Y}^s}_{M}$. Let (q, r) be any admissible pair with $r > 2$.

$$\begin{aligned}\|\nabla f(u)(t)\|_{L^2} &\lesssim \|f'(u)(t)\|_{L^{\frac{2r}{r-2}}} \|\nabla u(t)\|_{L^r} \lesssim \left\||u(t)|^{p-2}\right\|_{L^{\frac{2r}{r-2}}} \|\nabla u\|_{L^r} \\ &= \|u(t)\|^{p-2}_{L^{\frac{2r(p-2)}{r-2}}} \|\nabla u(t)\|_{L^r} \lesssim \|u(t)\|^{p-2}_{H^1} \|\nabla u(t)\|_{L^r}\end{aligned}$$

by the Sobolev embedding theorem (here, note that we can let $p > 2$ by choosing r near 2). Note that $\nabla = \nabla_{xy}$. Thus,

$$\|f(u)(t)\|_{H^1} \lesssim \|u(t)\|^{p-2}_{H^1} \|u(t)\|_{W^{1,r}} .$$

Consequently, adding the norm and applying the Holder's inequality with respect to t show that

$$\begin{aligned}\|f(u)\|_{L^1_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} &\lesssim \|u\|^{p-2}_{L_t^{\frac{q(p-2)}{q-1}}\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \cdot \|u\|_{L^q_t\left([0,T];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\ &\lesssim T^{\frac{q-1}{q}} \|u\|^{p-2}_{L^\infty_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \|u\|_{L^q_t\left([0,T];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} .\end{aligned} \tag{3.71}$$

Moreover, Sobolev embedding theorem gives

$$\begin{aligned}&\left\|\nabla\left[|u|^{p-2}u(t) - |v|^{p-2}v(t)\right]\right\|_{L^2} \\ &\quad= \left\|\int_0^1 \nabla\left[\frac{p}{2}|w(t)|^{p-2} + \left(\frac{p}{2}-1\right)|w(t)|^{p-4}w^2(t)\right] \cdot (u(t)-v(t))\,d\theta\right\|_{L^2} \\ &\quad\le \sup_{\theta\in[0,1]} \left\|\nabla\left[\left(\frac{p}{2}|w(t)|^{p-2} + \left(\frac{p}{2}-1\right)|w(t)|^{p-4}w^2(t)\right) \cdot (u(t)-v(t))\right]\right\|_{L^2} \\ &\quad\le \sup_{\theta\in[0,1]} \left\|\nabla\left[\frac{p}{2}|w(t)|^{p-2} + \left(\frac{p}{2}-1\right)|w(t)|^{p-4}w^2(t)\right]\right\|_{L^{r_1}} \cdot \|u(t)-v(t)\|_{L^{\frac{2r_1}{r_1-2}}} \\ &\qquad+ \sup_{\theta\in[0,1]} \left\|\frac{p}{2}|w(t)|^{p-2} + \left(\frac{p}{2}-1\right)|w(t)|^{p-4}w^2(t)\right\|_{L^{\frac{2r}{r-2}}} \cdot \|(u(t)-v(t))\|_{W^{1,r}}\end{aligned}$$

where $r_1 > 2$ is to be chosen later. Notice that

$$\begin{aligned}&\left\|\nabla\left[\frac{p}{2}|w(t)|^{p-2} + \left(\frac{p}{2}-1\right)|w(t)|^{p-4}w^2(t)\right]\right\|_{L^{r_1}} \\ &\quad\le \left|\frac{p}{2}\right| \cdot \left\|\nabla|w(t)|^{p-2}\right\|_{L^{r_1}} + \left|\frac{p}{2}-1\right| \cdot \left\|\nabla|w(t)|^{p-4}w^2(t)\right\|_{L^{r_1}} \\ &\quad\lesssim \left\||w(t)|^{p-4}|w(t)+\overline{w}(t)| \cdot |\nabla w(t)|\right\|_{L^{r_1}} + \left\||w(t)|^{p-3} \cdot |\nabla w(t)|\right\|_{L^{r_1}} .\end{aligned}$$

Hence

$$\left\|\nabla\left[|u(t)|^{p-2}u(t) - |v(t)|^{p-2}v(t)\right]\right\|_{L^2} \lesssim \sup_{\theta\in[0,1]} \left\||w(t)|^{p-3} \cdot |\nabla w(t)|\right\|_{L^{r_1}} \cdot \|u(t)-v(t)\|_{L^{\frac{2r_1}{r_1-2}}}$$

$$+\sup_{\theta\in[0,1]}\|w(t)\|^{p-2}_{L^{\frac{2r(p-2)}{r-2}}}\cdot\|(u(t)-v(t))\|_{W^{1,r}}.$$

If $p=3$, then we let $r_1=r$ which gives

$$\begin{aligned}
&\Big\|\nabla\big[|u(t)|^{p-2}u(t)-|v(t)|^{p-2}v(t)\big]\Big\|_{L^2}\\
&\quad\lesssim\sup_{\theta\in[0,1]}\|w(t)\|_{W^{1,r}}\cdot\|u(t)-v(t)\|_{W^{1,r}}+\sup_{\theta\in[0,1]}\|\nabla w(t)\|_{L^r}\cdot\|\nabla(u(t)-v(t))\|_{L^r}\\
&\quad\le(\|\nabla u(t)\|_{L^r}+\|\nabla v(t)\|_{L^r})\cdot\|u(t)-v(t)\|_{W^{1,r}}.
\end{aligned}$$

When $p>3$, continuing the discussion above, we can let $2<r_1<r$ and get

$$\begin{aligned}
&\Big\|\nabla\big[|u(t)|^{p-2}u(t)-|v(t)|^{p-2}v(t)\big]\Big\|_{L^2}\\
&\quad\lesssim\sup_{\theta\in[0,1]}\Big\||w(t)|^{p-3}\cdot|\nabla w(t)|\Big\|_{L^{r_1}}\cdot\|u(t)-v(t)\|_{L^{\frac{2r_1}{r_1-2}}}\\
&\qquad+\sup_{\theta\in[0,1]}\|w(t)\|^{p-2}_{L^{\frac{2r(p-2)}{r-2}}}\cdot\|u(t)-v(t)\|_{W^{1,r}}\\
&\quad\le\sup_{\theta\in[0,1]}\|w(t)\|^{p-3}_{L^{\frac{rr_1(p-3)}{r-r_1}}}\cdot\|\nabla w(t)\|_{L^r}\cdot\|u(t)-v(t)\|_{L^{\frac{2r_1}{r_1-2}}}\\
&\qquad+\sup_{\theta\in[0,1]}\|w(t)\|^{p-2}_{L^{\frac{2r(p-2)}{r-2}}}\cdot\|u(t)-v(t)\|_{W^{1,r}}\\
&\quad\le\left(\|u(t)\|^{p-2}_{W^{1,r}}+\|v(t)\|^{p-2}_{W^{1,r}}\right)\cdot\|u(t)-v(t)\|_{W^{1,2}}\\
&\qquad+\left(\|u(t)\|^{p-2}_{W^{1,2}}+\|v(t)\|^{p-2}_{W^{1,2}}\right)\cdot\|u(t)-v(t)\|_{W^{1,r}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Big\||u(t)|^{p-2}u(t)-|v(t)|^{p-2}v(t)\Big\|_{H^1}&\lesssim\left(\|u(t)\|^{p-2}_{W^{1,r}}+\|\nabla v(t)\|^{p-2}_{W^{1,r}}\right)\cdot\|u(t)-v(t)\|_{H^1}\\
&\quad+\left(\|u(t)\|^{p-2}_{H^1}+\|\nabla v(t)\|^{p-2}_{H^1}\right)\cdot\|u(t)-v(t)\|_{W^{1,r}}.
\end{aligned}$$

Integrating with respect to t implies

$$\begin{aligned}
&\|f(u)-f(v)\|_{L^1_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\\
&=T^{\frac{q-1}{q}}\left(\|u\|^{p-2}_{L^q_t\left([0,T];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}+\|v\|^{p-2}_{L^q_t\left([0,T];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\|u-v\|_{L^\infty_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\\
&+T^{\frac{q-1}{q}}\left(\|u\|^{p-2}_{L^\infty_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}+\|v\|^{p-2}_{L^\infty_t\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\right)\|u-v\|_{L^q_t\left([0,T];L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}.
\end{aligned}\tag{3.72}$$

Therefore, by (3.71),

$$\begin{aligned}
\|\mathcal{A}[u]\|_{\mathcal{Y}^1_T}&\le\|W_b[h]\|_{\mathcal{Y}^1_T}+\|W_{\mathbb{R}^2}\phi\|_{\mathcal{Y}^1_T}+\|\Phi_f\|_{\mathcal{Y}^1_T}+\|W_b\left[(W_{\mathbb{R}^2}\phi)_{y=0}\right]\|_{\mathcal{Y}^1_T}+\|W_b\left[(\Phi_f)_{y=0}\right]\|_{\mathcal{Y}^1_T}\\
&\lesssim\|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];\,H_x^{s+\frac12}(\mathbb{R})\right)}+\|\phi\|_{H^s_{xy}}+\|f\|_{L^1_t\left([0,T];H^s_{xy}\right)}
\end{aligned}$$

$$\begin{aligned}
&\le \mu + \|f\|_{L_t^1\left([0,T];H^s_{xy}\right)} \lesssim \mu + T^{\frac{q-1}{q}} \|u\|^{p-2}_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \|u\|_{L_t^q\left([0,T];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\le C_0 \left(\mu + T^{\frac{q-1}{q}} \|u\|^{p-1}_{\mathcal{Y}^1_T}\right) \qquad (3.73)
\end{aligned}$$

for some $C_0 > 0$. Furthermore, by (3.72),

$$\begin{aligned}
&\|\mathcal{A}[u] - \mathcal{A}[v]\|_{\mathcal{Y}^1_T} \qquad (3.74) \\
&\le \|\Phi_f(u) - \Phi_f(v)\|_{\mathcal{Y}^1_T} + \|W_b\left[(\Phi_f(u) - \Phi_f(v))_{y=0}\right]\|_{\mathcal{Y}^1_T} \lesssim \|f(u) - f(v)\|_{L_t^1\left([0,T];L^2_{xy}\right)} \\
&\lesssim T^{\frac{q-1}{q}} \left(\|u\|^{p-2}_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} + \|v\|^{p-2}_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\right) \|u - v\|_{L_t^q\left([0,T];L^r_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\le T^{\frac{q-1}{q}} M^{p-2} \|u - v\|_{\mathcal{Y}^1_T}\,. \qquad (3.75)
\end{aligned}$$

In order to apply the contraction mapping theorem, we need

$$C_1 T^{\frac{q-1}{q}} M^{p-2} \le 1/2\,, \qquad (3.76)$$

and

$$0 < T \le \left(\frac{M - C_0\mu}{C_0 M^{p-1}}\right)^{\frac{q}{q-1}}\,. \qquad (3.77)$$

Therefore, by above estimates, we again show that there is a unique solution u to (3.6) in $\mathcal{Y}^1_T$.

For $s > 1$, first assume p is not an even integer. Let $p \ge s + 1$ with $s \in \mathbb{Z}$ or $p \ge [s] + 2$ with $s \notin \mathbb{Z}$. Then in the next few steps we will show that the mapping $u \mapsto f(u)$ is continuous and bounded $H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$. More precisely, given any $M > 0$ and $t > 0$, there exists $C(M)$ such that

$$\|f(u)(t)\|_{H^s} \lesssim \|u(t)\|^{p-1}_{H^s} \qquad (3.78)$$

$$\|f(u)(t) - f(v)(t)\|_{H^s} \lesssim M^{p-2}\|u(t) - v(t)\|_{H^s} + \varepsilon\{\|u(t) - v(t)\|_{L^2}\} \qquad (3.79)$$

for all u, $v \in C_t(\mathbb{R}^+; H^s(\mathbb{R}^2))$ such that $\|u\|_{C_t(\mathbb{R}^+;H^s(\mathbb{R}^2))}$, $\|v\|_{C_t(\mathbb{R}^+;H^s(\mathbb{R}^2))} \le M$. $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function so that $\varepsilon(t) \to 0^+$ as $t \to 0^+$. We can agree on a stronger condition on p such that for $p \ge s + 2$ with $s \in \mathbb{Z}$ or $p \ge [s] + 3$ with $s \notin \mathbb{Z}$ the Lipschitz continuity holds for f over the H^s norm:

$$\|f(u)(t) - f(v)(t)\|_{H^s} \lesssim M^{p-2}\|u(t) - v(t)\|_{H^s} \qquad (3.80)$$

Now we start the proof. Let α be a nonnegative multi-index with $|\alpha| = s$. Let α_m be any nonnegative multi-index with integer components where $|\alpha_m| = m = [s]$ denotes the integer part of s. Denote $\alpha - \alpha_m$ by $\widetilde{\alpha}$. Since $s > 1$, $H^s(\mathbb{R}^2) \hookrightarrow C_b(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for $q \in [2, \infty]$. More specifically $\|u\|_{L^\infty_{xyt}}$, $\|v\|_{L^\infty_{xyt}} \le M$.

The proof of (3.78) with $s \in \mathbb{Z}$ can be directly found from *Lemma 4.10.2* in [27]. However, if $s \notin \mathbb{Z}$ (hence $s > m$), according to the proof of the same lemma, we can say that

$$\begin{aligned}\|D^\alpha[f(u)]\|_{L^2} &= \left\|D^{\widetilde{\alpha}} D^{\alpha_m}[f(u)]\right\|_{L^2} \\ &= \text{a collection of terms in the form } \left\|D^{\widetilde{\alpha}} f^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u\right\|_{L^2},\end{aligned}$$

where $k \in \mathbb{Z}$ and $k \in \{1, \cdots, m\}$ and β_j's are multi-indices with $\alpha = \beta_1 + \cdots + \beta_k$ and $|\beta_j| \geq 1$. Let $r = \frac{2}{1-s+m}$ and $q_j = \frac{mr}{|\beta_j|}$. Then, $\frac{2r}{r-2} = \frac{2}{s-m}$. Then by Leibnitz rule (2.20)

$$\begin{aligned}&\left\|D^{\widetilde{\alpha}} f^{(k)}(u) \prod_{j=1}^k D^{\beta_j} u\right\|_{L^2} \\ &\leq \left\|D^{\widetilde{\alpha}}\left[f^{(k)}(u)\right]\right\|_{L^{\frac{2r}{r-2}}} \left\|\prod_{j=1}^k D^{\beta_j} u\right\|_{L^r} + \left\|f^{(k)}(u)\right\|_{L^\infty} \left\|D^{\widetilde{\alpha}} \prod_{j=1}^k D^{\beta_j} u\right\|_{L^2}.\end{aligned}$$

We can firstly see that, since $m \geq 1$, then over $\mathbb{R}^2$ we can have $H^m(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{s-m}}(\mathbb{R}^2)$.

$$\begin{aligned}\left\|D^{\widetilde{\alpha}}\left[f^{(k)}(u)\right]\right\|_{L^{\frac{2r}{r-2}}} &\leq \|f^{(k+1)}(u)\|_{L^\infty} \|D^{\widetilde{\alpha}} u\|_{L^{\frac{2}{s-m}}} \\ &\lesssim \|u\|_{L^\infty}^{p-2-k} \|D^{\widetilde{\alpha}} u\|_{H^m} = \|u\|_{H^s}^{p-1-k}.\end{aligned}$$

Secondly, let $\frac{|\beta_j|}{mr} = \frac{1}{q_j} = \frac{|\beta_j|}{2} + a_j\left(\frac{1}{2} - \frac{s}{2}\right)$, i.e. $a_j = \frac{|\beta_j|(1-\frac{2}{mr})}{s-1}$. Thus with Gagliardo-Nirenberg inequality, it leads to

$$\begin{aligned}\left\|\prod_{j=1}^k D^{\beta_j} u\right\|_{L^r} &\leq \prod_{j=1}^k \left\|D^{\beta_j} u\right\|_{L^{q_j}} \leq \prod_{j=1}^k \|D^\alpha u\|_{L^2}^{a_j} \|u\|_{L^\infty}^{1-a_j} \\ &\leq \|D^\alpha u\|_{L^2}^{\frac{m-\frac{2}{r}}{s-1}} \|u\|_{L^\infty}^{k-\frac{m-\frac{2}{r}}{s-1}} \\ &\leq \|D^\alpha u\|_{L^2} \|u\|_{L^\infty}^{k-1} \leq \|u\|_{H^s}^k\end{aligned}$$

Thirdly, it can be checked that $\left\|f^{(k)}(u)\right\|_{L^\infty} \lesssim \|u\|_{L^\infty}^{p-1-k} \leq \|u\|_{H^s}^{p-1-k}$ by simply differentiating f with respect to u. Finally, let $a_j = \frac{|\beta_j|}{m}$. Let q_j's be such that $\frac{1}{q_j} = \frac{|\beta_j|}{2} + a_j\left(\frac{1}{2} - \frac{s}{2}\right)$ when $j \neq l$ and $\frac{1}{q_l} = \frac{|\beta_l|+s-m}{2} + a_l\left(\frac{1}{2} - \frac{s}{2}\right)$.

$$\begin{aligned}\left\|D^{\widetilde{\alpha}} \prod_{j=1}^k D^{\beta_j} u\right\|_{L^2} &\leq \sum_{l=1}^k \left\|D^{\widetilde{\alpha}+\beta_l}\right\|_{L^{q_l}} \left\|\prod_{\substack{j=1 \\ j\neq l}}^k D^{\beta_j} u\right\|_{L^{\frac{2q_l}{q_l-2}}} \\ &\leq \sum_{l=1}^k \left\|D^{\widetilde{\alpha}+\beta_l}\right\|_{L^{q_l}} \prod_{\substack{j=1 \\ j\neq l}}^k \left\|D^{\beta_j} u\right\|_{L^{q_j}}\end{aligned}$$

$$\leq \|D^\alpha u\|_{L^2}\|u\|_{L^\infty}^{k-1} \leq \|u\|_{H^s}^k$$

Thus (3.78) is verified.

On the other hand, for (3.80), in regard of *Lemma 4.10.2* in [27], $D^\alpha[f(u)-f(v)]$ consists of terms in the form

$$D^{\widetilde{\alpha}}\left[f^{(k)}(u)\prod_{j=1}^{k} D^{\beta_j}u - f^{(k)}(v)\prod_{j=1}^{k} D^{\beta_j}v\right]$$

where $k \in \mathbb{Z}$ and β_j's are defined the same as above. Moreover, each of these terms can be set apart into

$$D^{\widetilde{\alpha}}\left[f^{(k)}(v)\prod_{j=1}^{k} D^{\beta_j}w_j\right] \quad \text{and} \quad D^{\widetilde{\alpha}}\left[\left(f^{(k)}(u)-f^{(k)}(v)\right)\prod_{j=1}^{k} D^{\beta_j}u\right]$$

where $w_j = u$ or v except for one that equals $u - v$. Now applying the same steps for the first part of the proof, we can let $q_j = \frac{2m}{|\beta_j|}$. Thus Sobolev embedding theorem gives

$$D^{\widetilde{\alpha}}\left[f^{(k)}(v)\prod_{j=1}^{k} D^{\beta_j}w_j\right] \leq \prod_{j=1}^{k}\|w_j\|_{H^s} \lesssim M^{p-2}\|u-v\|_{H^s}$$

We now show that if $p \geq m+3$ with $s \notin \mathbb{Z}$ or $p \geq m+2$ with $s \in \mathbb{Z}$, $f^{(k)}$ is Lipschitz on a bounded domain for $0 \leq k \leq m+1$. Recall from an earlier claim, we know that $f^{(k)}$ is term of $|u|^{a_1}u_R^{a_2}u_I^{a_3}$ where $a_1+a_2+a_3 = p-k-1$. Thus $\partial_u^k\left(|u|^{p-2}u\right) - \partial_v^k\left(|v|^{p-2}v\right)$ is the sum of terms in the form $|u|^{a_1}u_R^{a_2}u_I^{a_3} - |v|^{a_1}v_R^{a_2}v_I^{a_3}$ where $a_1+a_2+a_3 = p-k-1$. With the mean value theorem in Calculus, we have

$$\begin{aligned}
&\left||u|^{a_1}u_R^{a_2}u_I^{a_3} - |v|^{a_1}v_R^{a_2}v_I^{a_3}\right| \\
&\leq \left||u|^{a_1}-|v|^{a_1}\right||u_R|^{a_2}|u_I|^{a_3} + |v|^{a_1}\left|u_R^{a_2}-v_R^{a_2}\right||u_I|^{a_3} + |v|^{a_1}v_R^{a_2}\left|u_I^{a_3}-v_I^{a_3}\right| \\
&\lesssim (\max\{|u|,|v|\})^{a_1+a_2+a_3-1}|u-v| \lesssim (|u|+|v|)^{p-k-2}|u-v|
\end{aligned}$$

Therefore we obtain

$$\left\|\partial_u^k\left(|u|^{p-2}u\right) - \partial_v^k\left(|v|^{p-2}v\right)\right\|_{L^\infty} \lesssim M^{p-2-k}\|u-v\|_{L^\infty}$$

for $0 \leq k \leq m+1$. If $s \in \mathbb{Z}$, the result can be revealed straightforwardly according to *Lemma 4.10.2* in [27]. For $s \notin \mathbb{Z}$, $s > m$. Then

$$\begin{aligned}
&\left\|D^{\widetilde{\alpha}}\left[\left(f^{(k)}(u)-f^{(k)}(v)\right)\prod_{j=1}^{k} D^{\beta_j}u\right]\right\|_{L^2} \\
&\leq \left\|D^{\widetilde{\alpha}}\left[f^{(k)}(u)-f^{(k)}(v)\right]\right\|_{L^{\frac{2}{s-m}}}\left\|\prod_{j=1}^{k} D^{\beta_j}u\right\|_{L^{\frac{2}{1-s+m}}} + \left\|f^{(k)}(u)-f^{(k)}(v)\right\|_{L^\infty}\left\|D^{\widetilde{\alpha}}\prod_{j=1}^{k} D^{\beta_j}u\right\|_{L^2}.
\end{aligned}$$

We have discussed the estimate on each factor here except $\left\| D^{\widetilde{\alpha}} \left[f^{(k)}(u) - f^{(k)}(v) \right] \right\|_{L^{\frac{2}{s-m}}}$. Applying Sobolev embedding theorem with $H^{1-s+m} \hookrightarrow L^{\frac{2}{s-m}}$, we have

$$
\begin{aligned}
&\left\| D^{\widetilde{\alpha}} \left[f^{(k)}(u) - f^{(k)}(v) \right] \right\|_{L^{\frac{2}{s-m}}} \lesssim \left\| \nabla \left[f^{(k)}(u) - f^{(k)}(v) \right] \right\|_{L^2} \\
&\leq \left\| \left[f^{(k+1)}(u) - f^{(k+1)}(v) \right] \nabla u \right\|_{L^2} + \left\| f^{(k+1)}(v)(\nabla u - \nabla v) \right\|_{L^2} \\
&\leq \left\| f^{(k+1)}(u) - f^{(k+1)}(v) \right\|_{L^\infty} \|\nabla u\|_{L^2} + \left\| f^{(k+1)}(v) \right\|_{L^\infty} \|\nabla (u-v)\|_{L^2} \\
&\lesssim M^{p-3-k} \|u-v\|_{L^\infty} \|u\|^{H^s} + \|u\|_{L^\infty}^{p-2-k} \|u-v\|_{H^s} \\
&\lesssim M^{p-2-k} \|u-v\|_{H^s} ,
\end{aligned}
$$

which completes the discussion on (3.80). When $m+2 \leq p < m+3$ for $s \notin \mathbb{Z}$ or $m+1 \leq p < m+2$ for $s \in \mathbb{Z}$, one can conclude that $f^{(k)}$ is Lipschitz continuous for $0 \leq k \leq m$ and only continuous for $k = m+1$. In that, $\left\| f^{(m+1)}(u) - f^{(m+1)}(v) \right\|_{L^\infty} \leq \delta\{\|u-v\|_{L^\infty}\}$ where $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ represents a continuous function. By (2.8), we have $\|u-v\|_{L^\infty} \lesssim \|u-v\|_{H^m}^{\frac{1}{m}} \|u-v\|_{L^2}^{1-\frac{1}{m}} \leq \|u-v\|_{H^s}^{\frac{1}{m}} \|u-v\|_{L^2}^{1-\frac{1}{m}}$, which implies

$$
\left\| f^{(m+1)}(u) - f^{(m+1)}(v) \right\|_{L^\infty} \leq \delta\{\|u-v\|_{L^2}\} .
$$

Then (3.79) is proved.

From (3.78) we get

$$
\|f(u)\|_{L^1((0,T);\, H^s)} \lesssim T \|u\|_{L^\infty((0,T);\, H^s)}^{p-1} , \tag{3.81}
$$

and likewise, from (3.80) it is true that for $p \geq s+2$ with $s \in \mathbb{Z}$ or $p \geq [s]+3$ with $s \notin \mathbb{Z}$

$$
\|f(u) - f(v)\|_{L^1((0,T);\, H^s)} \lesssim T M^{p-2} \|u-v\|_{L^\infty((0,T);\, H^s)} , \tag{3.82}
$$

while with (3.79) for $p \geq s+1$ if p is an odd integer or $p \geq [s]+2$ if $s \notin \mathbb{Z}$

$$
\|f(u) - f(v)\|_{L^1((0,T);\, L^2)} \lesssim T M^{p-2} \|u-v\|_{L^\infty((0,T);\, L^2)} \tag{3.83}
$$

$$
\|f(u) - f(v)\|_{L^1((0,T);\, H^s)} \lesssim T M^{p-2} \|u-v\|_{L^\infty((0,T);\, H^s)} + T\varepsilon\{\|u-v\|_{L^\infty((0,T);\, L^2)}\} . \tag{3.84}
$$

Note that we apply (3.82) and (3.83) here for the proof of existence and uniqueness; however (3.84) will be used for the continuous dependence argument later.

If we assume u and v are from $B_M^{\mathcal{Z}^s}$, (3.81) yields

$$
\begin{aligned}
&\|\mathcal{A}[u]\|_{\mathcal{Z}_T^s} \leq \|W_b[h]\|_{\mathcal{Z}_T^s} + \|W_{\mathbb{R}^2}\phi\|_{\mathcal{Z}_T^s} + \|\Phi_f\|_{\mathcal{Z}_T^s} + \|W_b\left[(W_{\mathbb{R}^2}\phi)_{y=0}\right]\|_{\mathcal{Z}_T^s} + \|W_b\left[(\Phi_f)_{y=0}\right]\|_{\mathcal{Z}_T^s} \\
&\lesssim \|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + \|f\|_{L_t^1\left([0,T]; H_{xy}^s\right)} \leq \mu + \|f\|_{L_t^1\left([0,T]; H_{xy}^s\right)} \\
&\lesssim \ \mu + T\|u\|_{L^\infty((0,T);\, H^s)}^{p-1} \leq C_0 \left(\mu + T\|u\|_{\mathcal{Z}_T^s}^{p-1} \right) ,
\end{aligned} \tag{3.85}
$$

for some $C_0 > 0$. Furthermore, use (3.82) for $p \geq s+2$ with $s \in \mathbb{Z}$ or $p \geq [s]+3$ with $s \notin \mathbb{Z}$ to obtain

$$\begin{aligned}\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}_T^s} &\leq \|\Phi_f(u)-\Phi_f(v)\|_{\mathcal{Z}_T^s} + \|W_b\left[(\Phi_f(u)-\Phi_f(v))_{y=0}\right]\|_{\mathcal{Z}_T^s} \\ &\lesssim \|f(u)-f(v)\|_{L_t^1\left([0,T];L_{xy}^2\right)} \lesssim TM^{p-2}\|u-v\|_{L^\infty((0,T);\,H^s)} \leq TM^{p-2}\|u-v\|_{\mathcal{Z}_T^s}. \quad (3.86)\end{aligned}$$

Additionally, if $s+1 \leq p < s+2$ for $s \in \mathbb{Z}$ or $[s]+2 \leq p < [s]+3$ for $s \notin \mathbb{Z}$, then (3.83) gives

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}_T^0} \leq C_1 TM^{p-2}\|u-v\|_{\mathcal{Z}_T^0}. \quad (3.87)$$

Hence, (3.55), (3.56) and (3.57) are proved if M and T satisfy

$$C_1 TM^{p-2} \leq 1/2 \quad (3.88)$$

and

$$0 < T \leq \left(\frac{M-C_0\mu}{C_0 M^{p-1}}\right). \quad (3.89)$$

Thus, with the contraction mapping principle we can find a fixed point $u \in \mathcal{Z}_T^s$ when p is an even integer, or p is not an even integer but $p \geq s+2$ for $s \in \mathbb{Z}$ or $p \geq [s]+3$ for $s \notin \mathbb{Z}$. Nevertheless, if $s+1 \leq p < s+2$ for $s \in \mathbb{Z}$ or $[s]+2 \leq p < [s]+3$ for $s \notin \mathbb{Z}$, a fixed point $u \in \mathcal{Z}_T^0$ can be found which has a sequence $\{u_n\}_{n\geq 1} \subset B_M^{\mathcal{Z}^s} \subset \mathcal{Z}_T^s$ approaching u in $\mathcal{Z}_T^0$; i.e., choose $u_1 \in B_M^{\mathcal{Z}^s}$. Then let $u_2 = \mathcal{A}[u_1]$ and hence $u_2 \in B_M^{\mathcal{Z}^s}$ because of (3.56). We define $u_3 = \mathcal{A}[u_2]$ and again $u_3 \in B_M^{\mathcal{Z}^s}$; moreover $\|u_3-u_2\|_{\mathcal{Z}_T^s} = \|\mathcal{A}[u_2]-\mathcal{A}[u_1]\|_{\mathcal{Z}_T^0} \leq \frac{1}{2}\|u_2-u_1\|_{\mathcal{Z}_T^0}$ by (3.55). If repeating this procedure, we will generate a Cauchy sequence $\{u_n\}_{n\geq 1} \subset B_M^{\mathcal{Z}^s} \subset \mathcal{Z}_T^s$ and $u_n \to u$ in $\mathcal{Z}_T^s$. This means the problem has a unique solution in $\mathcal{Z}_T^s$. Since $\mathcal{Z}_T^s$ is also a reflexive Banach space, we conclude that $u \in \mathcal{Z}_T^s$ is the unique solution of (3.6) for $s > 1$.

Note that for p an even integer, $f \in C^\infty(\mathbb{C})$, i.e., there is a unique solution $u \in C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right)$ for any $s > 1$ if the initial and boundary conditions are in appropriate spaces discussed above. However, for general $s > 1$ we can only allow $[s] < p-1$ so that the existence and uniqueness hold for $u \in C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times\mathbb{R}^+)\right)$.

Finally, *Lemma 3.1* guarantees the existence and uniqueness for problem (3.1). □

3.3.2 The Maximal Solution

Now, we discuss the maximum existence interval of solutions found in *Theorem 3.9*.

Theorem 3.10. *Let $p \geq 3$ and: (a) $0 \leq s < \infty$ if p is an even integer; or (b) $0 \leq s \leq p-1$ if p is not an even integer but $s \in \mathbb{Z}$; or (c) $0 \leq [s] \leq p-2$ p is not an even integer and $s \notin \mathbb{Z}$. Assume that a unique solution u to the problem (3.1) exists in $\mathcal{X}_T^s$ if $0 \leq s < 1$ and $3 \leq p < \frac{4-2s}{1-s}$, or $\mathcal{Y}_T^s$ if $s = 1$, or $\mathcal{Z}_T^s$ if $s > 1$, for $t \in [0,T]$ with $\varphi \in H^s(\mathbb{R}\times\mathbb{R}^+)$ and $h \in H_t^{\frac{2s+1}{4}}([0,T_0];L_x^2(\mathbb{R})) \cap L_t^2([0,T_0];H_x^{s+1/2}(\mathbb{R}))$ for any $T_0 > 0$. Let $T_{\max} = \sup T$ and* *suppose* $T_{\max} < \infty$. *Also Define* u^* *on* $[0, T_{\max})$ *as the solution of (3.1) in* $\mathcal{X}^s_{T_{\max}}$ *if* $0 \leq s < 1$ *and* $3 \leq p < \frac{4-2s}{1-s}$, *or* $\mathcal{Y}^s_{T_{\max}}$ *if* $s = 1$, *or* $\mathcal{Z}^s_{T_{\max}}$ *if* $s > 1$, *with* $u^*(t) = u(t)$ *on* $[0, T]$ *whose existence and uniqueness have been proved in Theorem 3.9. Then* $\lim\limits_{t\uparrow T_{\max}} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} = \infty$.

Proof. The proof can be easily obtained using classical extension procedure. For any $T < T_{\max}$ suppose $\|u(t)\|_{H^s} \leq \mathcal{C}$ for some $\mathcal{C} > 0$ on $[0, T]$. Choose small $\delta > 0$ and $T = t_n$ large enough for some $n \geq 3$ so that $|T_{\max} - T| < \frac{T}{n}$. Let u_1 be the unique solution guaranteed in $\mathcal{X}_T$ for $t \in [0, T]$. Note that by the uniqueness, $u^*(t) = u_1(t)$ for any $t \in [0, T]$. By (3.6)

$$\begin{aligned} u_1(t) = W_b &\left(h - W_{\mathbb{R}^2}(\cdot)\phi|_{y=0} - i\lambda \left(\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot - \tau)\, |u_1(\tau)|^{p-2}\, u_1(\tau)\, d\tau \right)_{y=0} \right)(t) \\ &+ W_{\mathbb{R}^2}(t)\phi + i\lambda \int_0^t W_{\mathbb{R}^2}(t-\tau)\, |u_1(\tau)|^{p-2}\, u_1(\tau)\, d\tau \end{aligned} \tag{3.90}$$

Furthermore, we want to extend u_1 forward in time. Suppose $u_2(t) = u(t+T)$ solves

$$\begin{cases} i(u_2)_t + (u_2)_{xx} + (u_2)_{yy} + i\lambda\, |u_2|^{p-2}\, u_2 = 0 & (x,y,t) \in \mathbb{R}\times\mathbb{R}^+\times(0,\delta) \\ u_2(x,y,0) = u_1(x,y,T) & (x,y) \in \mathbb{R}\times\mathbb{R}^+ \\ u_2(x,0,t) = h(x,t+T) & (x,t) \in \mathbb{R}\times(0,\delta) \end{cases} \tag{3.91}$$

in

$$\mathcal{X}^s_\delta := C_t\left([0,\delta];\, H^s_{xy}(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L^q_t\left([0,\delta];\, W^{s,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L^\gamma_t\left([0,\delta];\, W^{s,\rho}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)$$

which is equivalent to

$$\begin{aligned} u_2(t) = W_b &\left(h(\cdot + T) - W_{\mathbb{R}^2}(\cdot)u_1(T)|_{y=0} - i\lambda \left(\int_0^{\cdot} W_{\mathbb{R}^2}(\cdot - \tau)\, |u_2(\tau)|^{p-2}\, u_2(\tau)\, d\tau \right)_{y=0} \right)(t) \\ &+ W_{\mathbb{R}^2}(t)u_1(T) + i\lambda \int_0^t W_{\mathbb{R}^2}(t-\tau)\, |u_2(\tau)|^{p-2}\, u_2(\tau)\, d\tau \end{aligned} \tag{3.92}$$

for $t \in [0, \delta]$. Since the H^s-norm of $\phi = u_1(t_0)$ and $u_1(T) = u_1(t_n)$ are both bounded by $\mathcal{C}$, then it can be ensured that there is a $\mu > 0$ such that

$$\begin{aligned} &\|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H^s_{xy}} \leq \mu\,, \\ &\|h(\cdot + T)\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|u_1(T)\|_{H^s_{xy}} \leq \mu\,. \end{aligned}$$

According to (3.67) and (3.68), we realize that the measure of T is genuinely determined by μ. Thus, we can choose any value for $\delta \leq T$; say $\delta = \frac{2T}{n} < T$. Thus we define that

$$u(t) = \begin{cases} u_1(t) & t \in (0,T) \\ u_2(t-T) & t \in (T, T+\delta) \end{cases}\,. \tag{3.93}$$

Then clearly

$$\|u\|_{\mathcal{X}^s_{T+\delta}} \le \|u_1\|_{\mathcal{X}^s_T} + \|u_2\|_{\mathcal{X}^s_\delta} < \infty \quad \text{and} \quad \lim_{t\uparrow T}\|u(t)\|_{H^s} = \lim_{t\downarrow T}\|u(t)\|_{H^s} = u_1(T)\,.$$

Thus, $u \in \mathcal{X}^s_{T+\delta}$ while $T+\delta \ge T + \frac{2T}{n} > T_{\max}$ which means $T_{\max} \neq \sup S$ and causes a contradiction with the assumption. Thus it must be true that $\lim\limits_{t\uparrow T_{\max}} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} = \infty$ when $0 \le s < 1$ and $3 \le p < \frac{4-2s}{1-s}$. Here, h has to be defined for any large time $t > 0$. Note that we are not able to construct the blow-up statement with the critical case when $p = \frac{4-2s}{1-s}$ because based on (3.69) and (3.70) the existence and uniqueness results are only for small initial and boundary data.

Moreover, replacing $\mathcal{X}^s_T$ by $\mathcal{Y}^s_T$ if $s=1$, and by $\mathcal{Z}^s_T$ if $s>1$ for $p \ge s+1$ with $s \in \mathbb{Z}$ or $p \ge [s]+2$ with $s \notin \mathbb{Z}$ when p is not an even integer, and repeating a similar argument as above, one can obtain the analogous conclusions as stated in the theorem. $\square$

3.3.3 Continuous Dependence

The continuous dependence property of solutions on the initial condition and the boundary condition can be proved as follows.

Theorem 3.11. *Let $p \ge 3$ and: (a) $0 \le s < \infty$ if p is an even integer; or (b) $0 \le s \le p-1$ if p is not an even integer but $s \in \mathbb{Z}$; or (c) $0 \le [s] \le p-2$ p is not an even integer and $s \notin \mathbb{Z}$. Assume $\{\varphi_n\}$ be a sequence of functions in $H^s(\mathbb{R}\times\mathbb{R}^+)$ and $\varphi \in H^s(\mathbb{R}\times\mathbb{R}^+)$ so that $\varphi_n \to \varphi$ as $n\to\infty$ in $H^s(\mathbb{R}\times\mathbb{R}^+)$. Let h be a function and $\{h_n\}$ be a sequence of functions such that*

$$h, h_n \in H_t^{\frac{2s+1}{4}}\left([0,T];\, L^2_x(\mathbb{R})\right) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right) \quad \textit{and} \quad h_n \to h \quad \textit{as } n\to\infty$$

in $H_t^{\frac{2s+1}{4}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)$ for any $T>0$. Let u_n be the solutions to the equation (3.1) with $u_n(x,y,0) = \varphi_n(x,y)$ and $u_n(x,0,t) = h_n(x,t)$ and u be the solution with $u(x,y,0) = \varphi(x,y)$ and $u(x,0,t) = h(x,t)$, respectively. Then $u_n \to u$ as $n\to\infty$ in X_T with $\|u_n\|_{X_T} \le M$ where $X_T = \mathcal{X}^s_T$ for $0 \le s < 1$ and $3 \le p < \frac{4-2s}{1-s}$ or $\mathcal{Y}^1_T$ for $s=1$ or $\mathcal{Z}^s_T$ for $s>1$, respectively.

Proof. First, we consider $0 \le s < 1$ and $3 \le p < \frac{4-2s}{1-s}$. *Theorem 3.9* and *3.10* guarantee the existence of a common existence interval $[0,T_c]$ for u_n and u because of the choice of $T_{\max}$ only dependent on the initial and boundary conditions. Furthermore, from the proof of (3.51) we can obtain

$$\|u-u_n\|_{\mathcal{X}^s_T} = \|\mathcal{A}[u] - \mathcal{A}[u_n]\|_{\mathcal{X}^s_T} \le C\Big(\|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times\mathbb{R}^+)}$$

$$+\|h-h_n\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + T^{1-\frac{(1-s)(p-2)}{2}} M^{p-2}\|u-u_n\|_{\mathcal{X}_T^s}\Bigg).$$

Let T be sufficiently small so that $T^{1-\frac{(1-s)(p-2)}{2}} M^{p-2} < 1/2$. Then

$$\|u-u_n\|_{\mathcal{X}_T^s} \le 2C\Bigg(\|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} + \|h-h_n\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} \Bigg).$$

Since T only depends on the uniform bounds for $u, u_n, \varphi, \varphi_n, h, h_n$ in their respective norms for $t \in [0, T_c]$, the above inequality holds for $T, 2T, \ldots$ until reaching T_c. The continuous dependence is proved for $0 \le s < 1$.

The case for $s = 1$ can be obtained similarly if $\mathcal{X}_T^s$ is changed to $\mathcal{Y}_T^1$.

For $s > 1$, we first notice, by (3.55) or (3.87), that

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}_T^0} \le C_1 T M^{p-2}\,\|u-v\|_{\mathcal{Z}_T^0}.$$

Hence

$$\|u-u_n\|_{\mathcal{Z}_T^0} = \|\mathcal{A}[u]-\mathcal{A}[u_n]\|_{\mathcal{Z}_T^0} \le C\Bigg(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}$$

$$+\|h-h_n\|_{H_t^{\frac{1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{\frac{1}{2}}(\mathbb{R})\right)} + T M^{p-2}\|u-u_n\|_{\mathcal{Z}_T^0}\Bigg).$$

Thus choosing $T \le 1$ small enough so that $TM^{p-2} < 1/2$,

$$\|u-u_n\|_{\mathcal{Z}_T^0} \le 2C\Bigg(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} + \|h-h_n\|_{H_t^{\frac{1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{\frac{1}{2}}(\mathbb{R})\right)} \Bigg),$$

which shows that $\|u-u_n\|_{\mathcal{Z}_T^0} \to 0$ as $n \to 0$. The proof of continuous dependence in $\mathcal{Z}_T^s$ has been ensured by (3.84) where $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function so that $\varepsilon(t) \to 0^+$ as $t \to 0^+$; that is

$$\|u-u_n\|_{\mathcal{Z}_T^s} = \|\mathcal{A}[u]-\mathcal{A}[u_n]\|_{\mathcal{Z}_T^s} \le C\Bigg(\|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times\mathbb{R}^+)}$$

$$+\|h-h_n\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + TM^{p-2}\|u-u_n\|_{\mathcal{Z}_T^s} + T\varepsilon\{\|u-u_n\|_{\mathcal{Z}_T^0}\}\Bigg).$$

Rewrite this as

$$\|u-u_n\|_{\mathcal{Z}_T^s}$$
$$\le 2C\left(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} + \|h-h_n\|_{H_t^{\frac{1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{\frac{1}{2}}(\mathbb{R})\right)} + \varepsilon\{\|u-u_n\|_{\mathcal{Z}_T^0}\}\right).$$

Since the right hand side of the inequality approaches 0 as $n \to 0$, so does the left hand side. Thus the proof of the continuous dependence is completed. □

Eventually, the statements *(i)-(iii)* in *Theorem 1.5* are proved and we move to discuss the possibility of removing the auxiliary space from the well-posedness for $0 \le s \le 1$.

3.4 Regularity and Unconditional Well-posedness

In this section, we discuss the regularity property and unconditional well-posedness of equation (3.1) for $0 \le s \le 1$. Using *(i)-(iii)* in *Theorem 1.5*, the proof of Theorem part *(iv)* follows directly from the argument given in [12]. Here, we note that the regularity theorems discussed in Sections 5.1-5.5 of [27] hold for the IBVP (3.1), i.e., the persistence of regularity defined in [12] is valid and unconditional well-posedness theorems in [12] can be applied for (3.1) (also see Section 4 of [12]).

Proposition 3.12. *For $0 \le s \le 1$ the IBVP (3.1) has the property of regularity persistence (see Definition 1.3); i.e., let $0 \le s_1 < s$. Also let u on $[0, T_{\max})$ be the unique maximal solution of (3.1) in $\mathcal{X}^{s_1}_{T_{\max}}$ if $0 \le s_1 < 1$ and $3 \le p < \frac{4-2s_1}{1-s_1}$, or $\mathcal{Y}^{s_1}_{T_{\max}}$ if $s_1 = 1$, $[0, T_{\max}]$ provided in Theorem 3.9 and 3.10 with $\varphi \in H^{s_1}(\mathbb{R} \times \mathbb{R}^+)$ and $h \in H_t^{\frac{2s_1+1}{4}}([0,T]; L^2_x(\mathbb{R})) \cap L^2_t([0,T]; H_x^{s_1+1/2}(\mathbb{R}))$. If $\varphi \in H^s(\mathbb{R}\times\mathbb{R}^+)$ and $h \in H_t^{\frac{2s+1}{4}}([0,T]; L^2_x(\mathbb{R})) \cap L^2_t([0,T]; H_x^{s+\frac{1}{2}}(\mathbb{R}))$, then $u \in \mathcal{Y}^s_{T_{\max}}$ for $s = 1$, or $u \in \mathcal{Z}^s_{T_{\max}}$ for $s > 1$, respectively.*

Proof. We first consider whether the regularity persistence holds for (3.1) in H^1. Let $s_1 = 1$. We can also let u be a maximal H^1-solution on $[0, T_{\max})$ according to *Theorems 3.9* and *3.10*. We know that for any $r > 2$ $u \in \mathcal{Y}^1_T = C_t\left([0,T];\, H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right) \bigcap L^q_t\left([0,T];\, W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)$ is an H^s-solution on $[0,T)$ with $T \le T_{\max}$ and want to show that $T = T_{\max}$ by contradiction: suppose $T < T_{\max}$ and $\lim\limits_{t\uparrow T} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} = \infty$.

Since $r > 2$, by Sobolev embedding theorem $W^{1,r} \hookrightarrow L^\infty$ and therefore $u(t) \in L^\infty$ for any $t \in [0, T_{\max})$. Moreover from the proof of *Theorem 3.9* for $s > 1$, we have

$$\|f(u)(t)\|_{H^s} \lesssim \|u(t)\|^{p-2}_{L^\infty} \|u(t)\|_{H^s}\,, \text{ for all } \ t \in [0, T_{\max}).$$

Then use the equation (3.48) and *Proposition 3.3*, *3.6*, and *3.8* to obtain

$$\begin{aligned}
\|u(t)\|_{H^s} &= \|\mathcal{A}[u](t)\|_{H^s} \\
&\lesssim \|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H^s_{xy}} + \|f\|_{L^1_t\left([0,T]; H^s_{xy}\right)} \\
&\lesssim \|h\|_{H_t^{\frac{2s+1}{4}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H^s_{xy}} + \int_0^t \|u(s)\|^{p-2}_{L^\infty} \|u(s)\|_{H^s}\, ds\,.
\end{aligned}$$

for every $t \in [0, T)$. Using Gronwall's inequality *Lemma 2.12*, we deduce that

$$\|u(t)\|_{H^s} \lesssim \exp\left(C \int_0^t \|u(s)\|^{p-2}_{L^\infty}\, ds \right), \text{ for } \ t \in [0, T).$$

If $T < T_{\max}$, then $\lim_{t \uparrow T} \|u(t)\|_{H^s(\mathbb{R}\times\mathbb{R}^+)} < \infty$ which contradicts the blowup alternative in H^s.

Next we take our attention for the H^{s_1}-solution when $0 \leq s_1 < 1$ and let $s = 1$. We again let u be the unique maximal solution of (3.1) in $\mathcal{X}^{s_1}_{T_{\max}}$ on $[0, T_{\max})$ if $0 \leq s_1 < 1$ and $3 \leq p < \frac{4-2s_1}{1-s_1}$ and in $\mathcal{Y}^1_T$ on $[0,T)$ with $T \leq T_{\max}$. The constraint on p and s implies that $\frac{2r(p-2)}{r-2} \leq \frac{2}{1-s_1}$ which means by Sobolev embedding theorem $H^s \hookrightarrow L^{\frac{2r(p-2)}{r-2}}$. Similarly suppose $T < T_{\max}$ and $\lim_{t\uparrow T} \|u(t)\|_{H^1(\mathbb{R}\times\mathbb{R}^+)} = \infty$. By the same argument for deriving (3.71), we have for every interval $[0,\tau] \subset [0,T)$,

$$\begin{aligned}
\|f(u)\|_{L^1_t\left([0,\tau];H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} &\lesssim \|u\|^{p-2}_{L^\infty_t\left([0,\tau];L_{xy}^{\frac{2r(p-2)}{r-2}}(\mathbb{R}\times\mathbb{R}^+)\right)} \|u\|_{L^1_t\left([0,\tau];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \\
&\lesssim \|u\|^{p-2}_{L^\infty_t\left([0,\tau];H^{s_1}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \|u\|_{L^1_t\left([0,\tau];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)} \lesssim \|u\|_{L^1_t\left([0,\tau];W^{1,r}_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)}\,.
\end{aligned}$$

For some small $\varepsilon > 0$, note that

$$\|u\|_{L^1_t\left([0,\tau];W^{1,r}_{xy}\right)} = \|u\|_{L^1_t\left([0,\tau-\varepsilon-\varepsilon];W^{1,r}_{xy}\right)} + \|u\|_{L^1_t\left([\tau-\varepsilon,\tau];W^{1,r}_{xy}\right)} \leq C_\varepsilon + \varepsilon^{\frac{q-1}{q}} \|u\|_{L^q_t\left([0,T];W^{1,r}_{xy}\right)}.$$

Then

$$\begin{aligned}
\|u\|_{\mathcal{Y}^1_\tau} = \|\mathcal{A}[u]\|_{\mathcal{Y}^1_\tau} &\lesssim \|h\|_{H_t^{\frac{2s+1}{4}}([0,\tau];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,\tau];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H^s_{xy}} + \|f\|_{L^1_t\left([0,\tau];H^s_{xy}\right)} \\
&\lesssim C + C_\varepsilon + \varepsilon^{\frac{q-1}{q}} \|u\|_{L^q_t\left([0,\tau];W^{1,r}_{xy}\right)}\,.
\end{aligned}$$

We can fix ε sufficiently small so that $\varepsilon^{\frac{q-1}{q}} \leq (1/2)$ which leads to

$$\|u\|_{L^\infty\left([0,T];\,H^1_{xy}\right)} + \|u\|_{L^q([0,T];W^{1,r})} \leq C\,.$$

where C is independent of τ. If we let $\tau \to T^-$, then a contradiction arises. Therefore $T = T_{\max}$. □

Now we can move onto the conclusion for unconditional well-posedness.

Theorem 3.13. *For $0 \leq s \leq 1$, the problem (3.1) is unconditionally well-posed.*

Proof. Since (3.1) is unconditionally well-posed for some $s > 1$ according to part *(c)* in *Theorem 3.9* and the equation has the property of regularity persistence when $s = 1$, which is verified in *Proposition 3.12*, we can conclude that (3.1) is unconditionally well-posed for $s = 1$ by *Theorem2.6* in [12].

Furthermore, because (3.1) is unconditionally well-posed for $s = 1$ and has the property of regularity persistence when $0 \leq s < 1$ as well, we find the equation is also unconditionally well-posed for $0 \leq s < 1$. Then proof is finished. □

3.5 Global Well-posedness

In this section, we will discuss the global existence of the solutions for (3.1) for any $T \in (0, \infty]$. To prove the global existence, we first derive several identities.

Lemma 3.14. *If the solution u of (3.1) exists for any $t > 0$ and is smooth enough, then u satisfies the following identities.*

$$(|u|^2)_t = -2Im\left[(u_x\overline{u})_x + (u_y\overline{u})_y\right] , \tag{3.94}$$

$$\left(|u_x|^2 + |u_y|^2 - \frac{2\lambda}{p}|u|^p\right)_t = 2Re\left[(\overline{u_t}u_x)_x + (\overline{u_t}u_y)_y\right] , \tag{3.95}$$

$$\left(|u_y|^2 - |u_x|^2 + \frac{2\lambda}{p}|u|^p\right)_y = -2Re(\overline{u}_x u_y)_x - i(u\overline{u_y})_t + i(u\overline{u_t})_y\,. \tag{3.96}$$

Proof. The first identity (3.94) can be obtained by multiplying both sides of the equation (3.1) by $\overline{u_t}$ and then retaining the imaginary parts. For the proof of (3.95), first we know that

$$\begin{aligned}
(|u_x|^2)_t &= (u_x\overline{u}_x)_t = 2\mathrm{Re}(u_{xt}\overline{u}_x) = 2\mathrm{Re}(\overline{u_t}u_x)_x - 2\mathrm{Re}(\overline{u_t}u_{xx})\,,\\
(|u_y|^2)_t &= 2\mathrm{Re}(\overline{u_t}u_y)_y - 2\mathrm{Re}(\overline{u_t}u_{yy})\,.
\end{aligned}$$

Add them together to have

$$\begin{aligned}
(|u_x|^2 + |u_y|^2)_t &= 2\mathrm{Re}\left[(\overline{u_t}u_x)_x + (\overline{u_t}u_y)_y\right] - 2\mathrm{Re}(\overline{u_t}\Delta u)\\
&= 2\mathrm{Re}\left[(\overline{u_t}u_x)_x + (\overline{u_t}u_y)_y\right] + 2\mathrm{Re}\left(i|u_t|^2 + \lambda\overline{u_t}u|u|^{p-2}\right)\\
&= 2\mathrm{Re}\left[(\overline{u_t}u_x)_x + (\overline{u_t}u_y)_y\right] + \frac{2\lambda}{p}\cdot\frac{p}{2}\cdot\left(|u|^2\right)^{\frac{p}{2}-1}\left(|u|^2\right)_t\\
&= 2\mathrm{Re}\left[(\overline{u_t}u_x)_x + (\overline{u_t}u_y)_y\right] + \left(\frac{2\lambda}{p}|u|^p\right)_t ,
\end{aligned}$$

which gives (3.95), where (3.1) has been used. The proof of (3.96) is as follows.

$$\begin{aligned}
-\left(|u_y|^2 - |u_x|^2\right)_y &= (u_y\overline{u}_x)_x + (\overline{u_y}u_x)_x - u_y(\overline{u}_{xx} + \overline{u}_{yy}) - \overline{u_y}(u_{xx} + u_{yy})\\
&= 2\mathrm{Re}(u_y\overline{u}_x)_x + u_y(-i\overline{u_t} + \lambda\overline{u}|u|^{p-2}) + \overline{u_y}(iu_t + \lambda u|u|^{p-2})\\
&= 2\mathrm{Re}(u_y\overline{u}_x)_x + i(u_t\overline{u_y} - \overline{u_t}u_y) + \lambda|u|^{p-2}(u_y\overline{u} + \overline{u_y}u)\\
&= 2\mathrm{Re}(u_y\overline{u}_x)_x + i\left[(u\overline{u_y})_t - (\overline{u_t}u)_y\right] + \frac{2\lambda}{p}\cdot\frac{p}{2}\cdot\left(|u|^2\right)^{\frac{p}{2}-1}\left(|u|^2\right)_y\\
&= 2\mathrm{Re}(u_y\overline{u}_x)_x + i(u\overline{u_y})_t - i(\overline{u_t}u)_y + \left(\frac{2\lambda}{p}|u|^p\right)_y .
\end{aligned}$$

$\square$

Next, we derive an *a-priori* estimate of the solution for (3.1) in $H^1(\mathbb{R}\times\mathbb{R}^+)$.

Proposition 3.15. *Assume that either $3 \le p < \infty$ if $\lambda < 0$ or $3 \le p \le \frac{10}{3}$ if $\lambda > 0$. Let $T > 0$ be given. If u is a solution of (3.1) in $C_t\left([0,T];\, H^1_{xy}(\mathbb{R}\times\mathbb{R}^+)\right)$, then there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $\varphi \in H^1(\mathbb{R}\times\mathbb{R}^+)$ and $h \in H^1(\mathbb{R}\times\mathbb{R}^+)$,*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^1(\mathbb{R}\times\mathbb{R}^+)} \le \phi\left(\|\varphi\|_{H^1(\mathbb{R}\times\mathbb{R}^+)} + \|h\|_{H^1(\mathbb{R}\times\mathbb{R}^+)}\right) . \tag{3.97}$$

Proof. First, we show the estimate of the L^2-norm over variables x and y. By (3.94),

$$\begin{aligned}
\|u(t)\|^2_{L^2_{xy}} &= \int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,t)|^2\, dy\, dx \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,0)|^2\, dy\, dx + \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^t \left(|u(x,y,\tau)|^2\right)_t\, d\tau\, dy\, dx \\
&=\|\varphi\|^2_{L^2_{xy}} - 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^t \left[(u_x(x,y,\tau)\overline{u}(x,y,\tau))_x + (u_y(x,y,\tau)\overline{u}(x,y,\tau))_y\right]\, d\tau\, dy\, dx \\
&= \|\varphi\|^2_{L^2_{xy}} + \int_{-\infty}^{\infty}\int_0^t 2\mathrm{Im}\left(u_y(x,0,\tau)\overline{u}(x,0,\tau)\right)\, d\tau\, dx\, ,
\end{aligned}$$

which gives

$$\|u(t)\|^2_{L^2_{xy}} \le \|\varphi\|^2_{L^2_{xy}} + 2\|h\|_{L^2_{xt}}\left(\int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2\, d\tau\, dx\right)^{\frac12} . \tag{3.98}$$

Then, we derive the estimate on $u_y(x,0,\tau)$. Integrate (3.96) with respect to x, y and t to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2\, d\tau\, dx - \int_{-\infty}^{\infty}\int_0^t |u_x(x,0,\tau)|^2\, d\tau\, dx \\
&\quad = i\int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,t)\overline{u_y}(x,y,t)|\, dy\, dx - i\int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,0)\overline{u_y}(x,y,0)|\, dy\, dx \\
&\qquad + i\int_{-\infty}^{\infty}\int_0^t |u(x,0,\tau)\overline{u_t}(x,0,\tau)|\, d\tau\, dx - \frac{2\lambda}{p}\int_{-\infty}^{\infty}\int_0^t |u(x,0,\tau)|^p\, d\tau\, dx\, ,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2\, d\tau\, dx = \int_{-\infty}^{\infty}\int_0^t |u_x(x,0,\tau)|^2\, d\tau\, dx \\
&\quad = +i\int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,t)\overline{u_y}(x,y,t)|\, dy\, dx - i\int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,0)\overline{u_y}(x,y,0)|\, dy\, dx \\
&\qquad + i\int_{-\infty}^{\infty}\int_0^t |u(x,0,\tau)\overline{u_t}(x,0,\tau)|\, d\tau\, dx - \frac{2\lambda}{p}\int_{-\infty}^{\infty}\int_0^t |u(x,0,\tau)|^p\, d\tau\, dx \\
&\quad \le \|h_x\|^2_{L^2_{xt}} + \left(\int_{-\infty}^{\infty}\int_0^{\infty} |u(x,y,t)|^2\, dy\, dx\right)^{\frac12}\cdot\left(\int_{-\infty}^{\infty}\int_0^{\infty} |u_y(x,y,t)|^2\, dy\, dx\right)^{\frac12} \\
&\qquad + \|\varphi\|_{L^2_{xy}}\cdot\|\varphi_y\|_{L^2_{xy}} + \|h\|_{L^2_{xt}}\cdot\|h_t\|_{L^2_{xt}} + \frac{2|\lambda|}{p}\|h\|^p_{L^p_{xt}}
\end{aligned}$$

$$= \|h_x\|_{L^2_{xt}}^2 + \|u\|_{L^2_{xy}} \cdot \|u_y\|_{L^2_{xy}} + \|\varphi\|_{L^2_{xy}} \cdot \|\varphi_y\|_{L^2_{xy}} + \|h\|_{L^2_{xt}} \cdot \|h_t\|_{L^2_{xt}} + \frac{2|\lambda|}{p}\|h\|_{L^p_{xt}}^p .$$

Then, it is derived from (3.98) that

$$\begin{aligned}
\|u(t)\|_{L^2_{xy}}^2 &\le \|\varphi\|_{L^2_{xy}}^2 + 2\|h\|_{L^2_{xt}} \left(\int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2 \, d\tau \, dx\right)^{\frac{1}{2}} \\
\le& 2\|h\|_{L^2_{xt}} \cdot \left(\|u\|_{L^2_{xy}}\|u_y\|_{L^2_{xy}} + \|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} + \|h\|_{L^2_{xt}}\|h_t\|_{L^2_{xt}} + \frac{2|\lambda|}{p}\|h\|_{L^p_{xt}}^p + \|h_x\|_{L^2_{xt}}^2\right)^{\frac{1}{2}} \\
&\quad + \|\varphi\|_{L^2_{xy}}^2 \\
\lesssim& \|u(t)\|_{L^2_{xy}}^{\frac{1}{2}} \left(2\|h\|_{L^2_{xt}}\|u_y\|_{L^2_{xy}}^{\frac{1}{2}}\right) \\
&+ 2\|h\|_{L^2_{xt}} \left(\|\varphi\|_{L^2_{xy}}^{\frac{1}{2}}\|\varphi_y\|_{L^2_{xy}}^{\frac{1}{2}} + \|h\|_{L^2_{xt}}^{\frac{1}{2}}\|h_t\|_{L^2_{xt}}^{\frac{1}{2}} + \sqrt{\frac{2|\lambda|}{p}}\|h\|_{L^p_{xt}}^{\frac{p}{2}} + \|h_x\|_{L^2_{xt}}\right) + \|\varphi\|_{L^2_{xy}}^2 \\
\le& \frac{1}{4}\|u(t)\|_{L^2_{xy}}^2 + \frac{3}{4}\left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} \|u_y\|_{L^2_{xy}}^{\frac{2}{3}} \\
&+ 2\|h\|_{L^2_{xt}} \left(\|\varphi\|_{L^2_{xy}}^{\frac{1}{2}}\|\varphi_y\|_{L^2_{xy}}^{\frac{1}{2}} + \|h\|_{L^2_{xt}}^{\frac{1}{2}}\|h_t\|_{L^2_{xt}}^{\frac{1}{2}} + \sqrt{\frac{2|\lambda|}{p}}\|h\|_{L^p_{xt}}^{\frac{p}{2}} + \|h_x\|_{L^2_{xt}}\right) + \|\varphi\|_{L^2_{xy}}^2 ,
\end{aligned}$$

which yields

$$\begin{aligned}
\|u(t)\|_{L^2_{xy}}^2 \le& \|u_y\|_{L^2_{xy}}^{\frac{2}{3}} \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} \\
&+ \frac{4}{3}\|h\|_{L^2_{xt}} \left(\|\varphi\|_{L^2_{xy}}^{\frac{1}{2}}\|\varphi_y\|_{L^2_{xy}}^{\frac{1}{2}} + \|h\|_{L^2_{xt}}^{\frac{1}{2}}\|h_t\|_{L^2_{xt}}^{\frac{1}{2}} + \sqrt{\frac{2|\lambda|}{p}}\|h\|_{L^p_{xt}}^{\frac{p}{2}} + \|h_x\|_{L^2_{xt}}\right) \\
&+ \frac{4}{3}\|\varphi\|_{L^2_{xy}}^2 . \qquad (3.99)
\end{aligned}$$

With $L^\infty L^2$-norm of u, we then integrate the identity (3.95) with respect to x, y and t. For the sake of simplicity of notations, we let $C = C\left(\|\varphi\|_{H^1_{xy}}\right)$ be an increasing function depending on the initial data and $C(t) = C\left(\|h\|_{H^1_{xt}}\right)$ denote an increasing function for the boundary value and the time t. Moreover, $C(0) = 0$ and they both equal zero for $\|\varphi\|_{H^1_{xy}} = 0$ and $\|h\|_{H^1_{xt}} = 0$.

First, assume $\lambda < 0$. Apply (3.95) and (3.99) to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_0^{\infty} \left(|u_y(t)|^2 + |u_x(t)|^2\right) dy\, dx &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(|u_y(x,y,0)|^2 + |u_x(x,y,0)|^2\right) dy\, dx \\
&+ \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\lambda}{p}|u|^p \, dy\, dx - \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\lambda}{p}|u(x,y,0)|^p \, dy\, dx \\
&- 2\mathrm{Re}\int_{-\infty}^{\infty}\int_0^t \overline{u_t}(x,0,\tau)u_y(x,0,\tau)\, d\tau\, dx
\end{aligned}$$

$$
\begin{aligned}
&\le \|\varphi\|^2_{H^1_{xy}} + \frac{2|\lambda|}{p}\|\varphi\|^p_{L^p_{xy}} + \int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2 \, d\tau\, dx + \|h_t\|^2_{L^2_{xy}} \\
&\le \|u_y\|_{L^2_{xy}} \cdot \|u\|_{L^2_{xy}} + \Big(\|\varphi\|_{L^2_{xy}} \cdot \|\varphi_y\|_{L^2_{xy}} + \|h\|_{L^2_{xt}} \cdot \|h_t\|_{L^2_{xt}} + \frac{2|\lambda|}{p}\|h\|^p_{L^p_{xt}} + \|h_x\|^2_{L^2_{xt}} \\
&\quad + \|\varphi\|^2_{H^1_{xy}} + \frac{2|\lambda|}{p}\|\varphi\|^p_{L^p_{xy}} + \|h_t\|^2_{L^2_{xy}} \Big) \\
&\le \frac{1}{2}\|u_y\|^{\frac{4}{3}}_{L^2_{xy}} \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} + \|u_y\|_{L^2_{xy}} \left(C + C(t)\right) \\
&\quad + \Big(\|\varphi\|_{L^2_{xy}} \cdot \|\varphi_y\|_{L^2_{xy}} + \|h\|_{L^2_{xt}} \cdot \|h_t\|_{L^2_{xt}} + \frac{2|\lambda|}{p}\|h\|^p_{L^p_{xt}} + \|h_x\|^2_{L^2_{xt}} \\
&\quad + \|\varphi\|^2_{H^1_{xy}} + \frac{2|\lambda|}{p}\|\varphi\|^p_{L^p_{xy}} + \|h_t\|^2_{L^2_{xy}} \Big) \\
&= \frac{1}{2}\|u_y\|^{\frac{4}{3}}_{L^2_{xy}} \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} + \|u_y\|_{L^2_{xy}} \left(C + C(t)\right) + C + C(t) \le \frac{1}{6}\|u_y\|^2_{L^2_{xy}} + C + C(t)\,.
\end{aligned}
$$

Thus, by moving the terms of $\|u_y\|^2_{L^2_{xy}}$ on the right hand side to the left side, we conclude that $\|u(t)\|_{H^1(\mathbb{R}\times\mathbb{R}^+)}$ is uniformly bounded by the initial and boundary data, i.e.,

$$
\|u(t)\|_{H^1(\mathbb{R}\times\mathbb{R}^+)} \le C + C(t) := \phi_1\,.
$$

One may notices that during the derivation of above inequality with C and $C(t)$, there are some terms involving $\|\varphi\|_{L^p_{xy}}$ and $|h\|_{L^p_{xt}}$, which can be bounded by their H^1-norms using Sobolev's embedding theorem since H^1 is embedded in L^p for any $\infty > p \ge 2$ if the underlying domain is 2-dimensional.

Next, let us consider the case for $\lambda > 0$. Here, we need the following Gagliardo-Nirenburg's inequality (2.8),

$$
\|u(t)\|_{L^p_{xy}} \lesssim \left(\|u_x(t)\|_{L^2_{xy}} + \|u_y(t)\|_{L^2_{xy}}\right)^{1-\frac{2}{p}} \cdot \|u(t)\|^{\frac{2}{p}}_{L^2_{xy}}\,,
$$

i.e.,

$$
\int_{\mathbb{R}^2} |u(t)|^p \, dy\, dx \lesssim \left(\|u_x(t)\|_{L^2_{xy}} + \|u_y(t)\|_{L^2_{xy}}\right)^{p-2} \cdot \|u(t)\|^2_{L^2_{xy}}\,. \tag{3.100}
$$

If no confusion arises, again we denote "$\lesssim$" by "$\le$" in the proof. From (3.95) and using the same functions C and $C(t)$ together with 3.99) and (3.100) and the estimate obtained for the L^2-norm of $u_y(x,0,t)$, it is shown that

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_0^{\infty} \left(|u_y(t)|^2 + |u_x(t)|^2\right) dy\, dx &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(|u_y(x,y,0)|^2 + |u_x(x,y,0)|^2\right) dy\, dx \\
&\quad + \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\lambda}{p}|u|^p \, dy\, dx - \int_{-\infty}^{\infty}\int_0^{\infty} \frac{2\lambda}{p}|u(x,y,0)|^p \, dy\, dx \\
&\quad - 2\mathrm{Re}\int_{-\infty}^{\infty}\int_0^{t} \overline{u_t}(x,0,\tau) u_y(x,0,\tau) \, d\tau\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\varphi\|_{H^1_{xy}}^2 + \frac{2|\lambda|}{p}\|\varphi\|_{L^p_{xy}}^p + \int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2 \, d\tau\, dx + \|h_t\|_{L^2_{xy}}^2 + \frac{2\lambda}{p}\int_{-\infty}^{\infty}\int_0^{\infty} |u|^p \, dy\, dx \\
&\leq \frac{2\lambda}{p}\int_{-\infty}^{\infty}\int_0^{\infty} |u|^p \, dy\, dx + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq \left(\|u_x\|_{L^2_{xy}} + \|u_y\|_{L^2_{xy}}\right)^{p-2} \cdot \|u(t)\|_{L^2_{xy}}^2 + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq \left(\|u_x\|_{L^2_{xy}} + \|u_y\|_{L^2_{xy}}\right)^{p-2} \cdot \Big(C(t)\|u_y\|_{L^2_{xy}}^{\frac{2}{3}} + C(t) + C\Big) + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq C(t)\|u_y\|_{L^2_{xy}}^{\frac{2}{3}}\|u_x\|_{L^2_{xy}}^{p-2} + C(t)\|u_y\|_{L^2_{xy}}^{\frac{3p-4}{3}} + \left(\|u_x\|_{L^2_{xy}} + \|u_y\|_{L^2_{xy}}\right)^{p-2} \cdot (C(t)+C) \\
&\qquad + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C(t)\|u_x\|_{L^2_{xy}}^{\frac{3(p-2)}{2}} + C(t)\|u_y\|_{L^2_{xy}}^{\frac{3p-4}{3}} + \frac{1}{2}\|u_x\|_{L^2_{xy}}^2 + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 \\
&\qquad + C + C(t)\,,
\end{aligned}
$$

from which we can see that

$$\|u(t)\|_{H^1(\mathbb{R}\times\mathbb{R}^+)} \leq C + C(t) := \phi_2$$

if and only if $\frac{3(p-2)}{2} < 2$ and $\frac{3p-4}{3} < 2$, i.e., $3 \leq p < \frac{10}{3}$. Therefore, in this case $\sum_{|\alpha|=1} \|D^\alpha u(t)\|_{L^2}$ or $\|u(t)\|_{H^1_{xy}}$ is uniformly bounded by $C + C(t)$.

If $p = \frac{10}{3}$, we have $\frac{3(p-2)}{2} = \frac{3p-4}{3} = 2$. Thus,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\infty} \left(|u_y(t)|^2 + |u_x(t)|^2\right) \, dy\, dx \\
&\leq \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} \|u_y\|_{L^2_{xy}}^{\frac{2}{3}}\|u_x\|_{L^2_{xy}}^{\frac{4}{3}} + \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} \|u_y\|_{L^2_{xy}}^2 + \left(\|u_x\|_{L^2_{xy}} + \|u_y\|_{L^2_{xy}}\right)^{\frac{4}{3}} \cdot (C(t)+C) \\
&\qquad + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + c\|h\|_{L^2_{xt}}^2\|u_x\|_{L^2_{xy}}^2 + \left(2\|h\|_{L^2_{xt}}\right)^{\frac{4}{3}} \|u_y\|_{L^2_{xy}}^2 + \frac{1}{2}\|u_x\|_{L^2_{xy}}^2 + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 \\
&\qquad + \frac{1}{6}\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&\leq \frac{1}{2}\|u_y\|_{L^2_{xy}}^2 + \left(c\|h\|_{L^2_{xt}}^2 + \frac{1}{2}\right)\|h\|_{L^2_{xt}}^2\|u_x\|_{L^2_{xy}}^2 + c\|h\|_{L^2_{xt}}^2\|u_y\|_{L^2_{xy}}^2 + C + C(t) \\
&= \left(c\|h\|_{L^2_{xt}}^2 + \frac{1}{2}\right)\left(\|u_y\|_{L^2_{xy}}^2 + \|u_x\|_{L^2_{xy}}^2\right) + C + C(t)\,,
\end{aligned}
$$

where c is a fixed positive constant. The above inequality is equivalent to

$$\left(\frac{1}{2} - c\|h\|_{L^2_{xt}}^2\right)\left(\|u_y\|_{L^2_{xy}}^2 + \|u_x\|_{L^2_{xy}}^2\right) \leq C + C(t)\,.$$

Since it is assumed that $\|h\|_{L^2_{xt}(\mathbb{R}\times[0,T]} < \infty$, we can partition $[0,T]$ into a finite number of subintervals (t_{j-1}, t_j) for $j = 1, \cdots, m$ with $\sup_j |t_j - t_{j-1}| \le \delta$ such that $\|h\|_{L^2_{xt}(\mathbb{R}\times[t_{j-1},t_j])} < (1/4c)$. Starting with $[0, t_1]$, we can move forward over one subinterval (t_{j-1}, t_j) at a time to obtain a uniform bound for $\|u\|_{H^1}$ and then use $u(t_j)$ as the initial value for the solution on (t_j, t_{j+1}) to have a uniform bound for $\|u\|_{H^1}$. By repeating the process until reaching T, we prove the uniform bound of $\|u\|_{H^1}$ for $t \in [0,T]$. $\square$

Finally, the following global well-posedness of the IBVP (3.1) can be obtained from *Theorems 3.9* and *Propositions 3.10* and *3.15*.

Theorem 3.16. *Assume that either* $3 \le p < \infty$ *if* $\lambda < 0$ *or* $3 \le p \le \frac{10}{3}$ *if* $\lambda > 0$. *Then, (3.1) is globally well-posed in* $H^1(\mathbb{R}\times\mathbb{R}^+)$ *if* $\varphi \in H^1(\mathbb{R}\times\mathbb{R}^+)$ *and* $h \in H^1_{t,loc}(\mathbb{R}; L^2_x(\mathbb{R})) \bigcap L^2_{t,loc}\left(\mathbb{R}; H^{\frac{3}{2}}_x(\mathbb{R})\right)$.

Chapter 4

2D Nonlinear Schrödinger Equations in a Strip Domain

In this chapter we turn our attention to the IBVP over a strip domain $(x, y) \in \mathbb{R} \times [0, L]$ for some $L > 0$. The NLS equation posed on the strip domain possesses similar structures in many aspects with the "half-periodic" problem, so that the study on this topic can apply Bourgain's method as the "discrete" version of the Strichartz's inequalities to establish the basic estimates for a contraction mapping.

4.1 Formulation of the Problem

Without loss of generality, we consider the IBVP for $(x, y) \in \mathbb{R} \times [0, 1]$ and some $T \in (0, \infty]$ the equation as follows:

$$\begin{cases} iu_t + u_{xx} + u_{yy} + g = 0 & (x, y, t) \in \mathbb{R} \times (0, 1) \times (0, T) \\ u(x, y, 0) = \varphi(x, y) & (x, y) \in \mathbb{R} \times (0, 1) \\ u(x, 0, t) = h_1(x, t) & (x, t) \in \mathbb{R} \times (0, T) \\ u(x, 1, t) = h_2(x, t) & (x, t) \in \mathbb{R} \times (0, T) \end{cases} \tag{4.1}$$

where $g(x, y, t) = \lambda |u(x, y, t)|^{p-2} u(x, y, t)$ for $p \geq 3$ and $(x, y, t) \in \mathbb{R} \times (0, 1) \times (0, T)$. Moreover we pose several extra assumptions, $\varphi|_{y=0} = h_j|_{t=0} = 0$ for $j = 1, 2$ and $x \in \mathbb{R}$, as the compatibility on the initial and boundary conditions is concerned.

Once more we continue to search for the solution of (4.1) in $C([0, T]; H^s(\mathbb{R} \times \mathbb{R}^+))$ for some properly chosen $T \in (0, \infty]$. Then we call for a parallel procedure analogous to *Section 3.1* in which the initial data, boundary condition and the nonlinearity are to be studied separately. However, for the problem with one spatial variable picked within an finite interval, there are slight differences along the process of decomposition regarding the boundary conditions.

First there comes the linear problem provided with homogeneous boundary condition,

$$\begin{cases} iv_t + v_{xx} + v_{yy} = 0 & (x, y, t) \in \mathbb{R} \times (0, 1) \times (0, T) \\ v(x, y, 0) = \varphi(x, y) & (x, y) \in \mathbb{R} \times (0, 1) \\ v(x, 0, t) = v(x, 1, t) = 0 & (x, t) \in \mathbb{R} \times (0, T) \end{cases} \tag{4.2}$$

where the solution is denoted by $v = W_0(t)\phi$. Note that $W_{\mathbb{R}^2}(t)$ represents a C_0-semigroup whose infinitesimal generator is $i\Delta$ as well. It follows the nonlinear equation

$$\begin{cases} iw_t + w_{xx} + w_{yy} + g = 0 & (x, y, t) \in \mathbb{R} \times (0, 1) \times (0, T) \\ w(x, y, 0) = 0 & (x, y) \in \mathbb{R} \times (0, 1) \\ w(x, 0, t) = w(x, 1, t) = 0 & (x, t) \in \mathbb{R} \times (0, T) \end{cases} . \tag{4.3}$$

Similar notation for the solution is given as $w := \Phi_{0,f}(t) := i \int_0^t W_0(t - \tau) f(\tau)\, d\tau$ like in the previous chapter. At last, we study linear NLS equation with homogeneous initial condition and nonhomogeneous boundary conditions,

$$\begin{cases} iz_t + z_{xx} + z_{yy} = 0 & (x, y, t) \in \mathbb{R} \times (0, 1) \times (0, T) \\ z(x, y, 0) = 0 & (x, y) \in \mathbb{R} \times (0, 1) \\ z(x, 0, t) = h_1(x, t) & (x, t) \in \mathbb{R} \times (0, T) \\ z(x, 1, t) = h_2(x, t) & (x, t) \in \mathbb{R} \times (0, T) \end{cases} . \tag{4.4}$$

The solution is denoted by $z := W_b(h_1, h_2)$.

Thus combining solutions from (4.2)-(4.4), we shall be able to generate the integral equation equivalent to (4.1) as follows,

$$u(x, y, t) = W_b(h_1, h_2)(x, y, t) + W_0(t)\phi(x, y) + i \left(\int_0^t W_0(t - \tau) f(\tau)\, d\tau \right)(x, y) \tag{4.5}$$

The equivalency lemma is also valid as stated in *Lemma 4.2.8* in [27] and *Lemma 3.1*,

Lemma 4.1. *For $s \geq 0$ and every $f(t) : H^s \to H^\sigma \ \forall\, t \in [0, T]$,*

$$u \in C([0, T];\, H^s) \bigcap C^1([0, T];\, H^{\min\{s-2, \sigma\}})$$

is a solution of (4.1) if and only if (4.5) holds for u.

4.2 Representations and Estimates of Solution Operators

4.2.1 Equation with the Initial Condition

For (4.2) we let $\phi : \mathbb{R} \times [-1,1] \to \mathbb{C}$ be an odd extension of the given initial data φ in the direction of negative y-axis; i.e. $\phi(x,y) = \varphi(x,y)$ for $y \in [0,1]$ to get the following equation,

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0 & (x,y,t) \in \mathbb{R} \times (0,1) \times (0,T) \\ u(x,y,0) = \phi(x,y) & (x,y) \in \mathbb{R} \times (0,1) \\ u(x,0,t) = u(x,1,t) = 0 & (x,t) \in \mathbb{R} \times (0,T) \end{cases} \tag{4.6}$$

where $u = W_0\phi$ on $\mathbb{R} \times [-1,1] \times [0,T]$. We begin with driving the formula of u.

Proposition 4.2. *Let $W_0(t)\phi$ solve (4.6). Then*

$$W_0(t)\phi(x,y) = \int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}} e^{-i\pi^2\left(\xi^2+n^2\right)t + i\pi(x\xi+ny)} \cdot \widehat{\phi}(\xi,n)\, d\xi \tag{4.7}$$

Proof. For equation (4.6) we perform Fourier transform with respect to x so that

$$\begin{cases} i\widehat{u^x}_t - \xi^2\pi^2\widehat{u^x} + \widehat{u^x}_{yy} = 0, & (\xi,y,t) \in \mathbb{R} \times [0,1] \times [0,T] \\ \widehat{u^x}(\xi,y,0) = \widehat{\phi^x}(\xi,y) \\ \widehat{u^x}(\xi,0,t) = \widehat{u^x}(\xi,1,t) = 0 \end{cases} \tag{4.8}$$

Based on method of separation of variables, suppose that $\widehat{u^x}(\xi,y,t) = Y(\xi,y)T(\xi,t)$. Then (4.8) implies that

$$\begin{cases} -\dfrac{Y''}{Y} = \lambda = i\dfrac{T'}{T} - \xi^2\pi^2 \\ Y(0) = Y(1) = 0 \end{cases}$$

When $\lambda \le 0$, $Y \equiv 0$. For $\lambda > 0$, $Y_n(\xi,y) = c_n(\xi)\sin(n\pi y)$ with $\lambda = n^2\pi^2$, $n \in \mathbb{Z}^+$. This gives that $i\frac{T'}{T} - \xi^2\pi^2 - n^2\pi^2 = 0$ and therefore $T = C(\xi) \cdot e^{-i\pi^2\left(\xi^2+n^2\right)t}$. Thus

$$\widehat{u^x}(\xi,y,t) = \sum_{n=1}^{\infty} C_n(\xi) \cdot e^{-i\pi^2\left(\xi^2+n^2\right)t} \cdot \sin(n\pi y)$$

Let $t = 0$, then

$$\widehat{\phi^x}(\xi,y) = \widehat{u^x}(\xi,y,0) = \sum_{n=1}^{\infty} C_n(\xi) \cdot \sin(n\pi y)$$

Since $\int_0^1 \left(\sqrt{2}\sin(n\pi y)\right)^2\, dy = 1$ and $\int_0^1 \sin(n\pi y)\sin(m\pi y)\, dy = 0$ for $m \neq n$, we integrate $\widehat{\phi^x}(\xi, y)\sin(n\pi y)$ to get

$$\int_0^1 \widehat{\phi^x}(\xi, y)\sin(n\pi y)\, dy = C_n(\xi)\int_0^1 \sin^2(n\pi y)\, dy = \frac{C_n(\xi)}{2}$$

Hence $C_n(\xi) = 2\int_0^1 \widehat{\phi^x}(\xi, y)\sin(n\pi y)\, dy$. Given that ϕ is the odd extension of φ for $y \in [-1, 1]$, then $\widehat{\phi^x}$ is also an odd function with respect to y with

$$\widehat{\phi^x}(\xi, y) = \begin{cases} \widehat{\varphi^x}(\xi, y) & y \geq 0 \\ -\widehat{\varphi^x}(\xi, -y) & y < 0 \end{cases}.$$

Therefore

$$\begin{aligned} u(x, y, t) &= \int_{-\infty}^{\infty} e^{i\pi x\xi}\left[\sum_{n=1}^{\infty} C_n(\xi)\cdot e^{-i\pi^2\left(\xi^2+n^2\right)t}\cdot \sin(n\pi y)\right] d\xi \\ &= \frac{1}{2i}\int_{-\infty}^{\infty} e^{i\pi x\xi}\left[\sum_{n\in\mathbb{Z}} \widetilde{C}_n(\xi)\cdot e^{-i\pi^2\left(\xi^2+n^2\right)t+in\pi y}\right] d\xi \\ &= \frac{1}{i}\int_{-\infty}^{\infty}\left[\sum_{n\in\mathbb{Z}}\left(\int_0^1 \widehat{\phi^x}(\xi, \eta)\sin(n\pi\eta)\, d\eta\right)\cdot e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi x\xi+in\pi y}\right] d\xi \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left[\sum_{n\in\mathbb{Z}}\left(\int_{-1}^1 \widehat{\phi^x}(\xi, \eta)e^{-in\pi\eta}\, d\eta\right)\cdot e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\right] d\xi \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}} e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi x\xi+i\pi yn}\cdot\widehat{\phi}(\xi, n)\, d\xi \end{aligned}$$

where

$$\widetilde{C}_n = \begin{cases} 2\int_0^1 \widehat{\phi^x}(\xi, \eta)\sin(n\pi\eta)\, d\eta & n > 0 \\ 0 & n = 0 \\ -2\int_0^1 \widehat{\phi^x}(\xi, \eta)\sin(-n\pi\eta)\, d\eta & n < 0 \end{cases}$$

In summary,

$$W_0(t)\phi(x, y) := \int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}} e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot\widehat{\phi}(\xi, n)\, d\xi \tag{4.9}$$

$\square$

Remark. It is straightforward to show that $W_0(t)\phi(x, y)$ is also an odd function in y if ϕ is an odd function.

We can show that W_0 maps from $H^s(\mathbb{R}^2)$ to $L_t^r\left([0, T];\, W_{xy}^{s,r}(\mathbb{R}^2)\right)\bigcap L_t^\infty\left([0, T];\, H_{xy}^s(\mathbb{R}^2)\right)$ for $r \in [2, 4]$ based on the estimates that follow.

Proposition 4.3. *Let $r \in [2,4]$ and $s \geq 0$. Then for some $\sigma > \frac{1}{2}$,*

$$\|W_0\phi\|_{L_t^r\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)} \lesssim \left(T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right) \|\phi\|_{H_{xy}^s(\mathbb{R}\times[0,1])} \, , \tag{4.10}$$

$$\|W_0\phi\|_{L_t^\infty\left([0,T];\, H_{xy}^{s}(\mathbb{R}\times[0,1])\right)} \lesssim \|\phi\|_{H_{xy}^s(\mathbb{R}\times[0,1])} \, . \tag{4.11}$$

Proof. In fact the estimate for $\|W_0(\cdot)\phi\|_{L_{xyt}^r(\mathbb{R}\times[-1,1]\times[0,T])}$ was studied for $r \in [2,4]$ in *Section 3* of [82]. Because of the symmetry of $W_0(t)\phi(x,y)$ with respect to y we also agree on the estimate restricted to the strip domain $\mathbb{R}\times[0,1]$,

$$\|W_0\phi\|_{L_{xyt}^r(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \left(T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right) \|\phi\|_{L_{xy}^2(\mathbb{R}\times[0,1])} \, . \tag{4.12}$$

However, the representation of $W_0(t)\phi$ in (4.7) shows that $D^\alpha W_0(t)\phi = W_0(t)D^\alpha\phi$ for $|\alpha| = s$. Thus from (4.12) we have the estimate for the norm on $L_t^r\left([0,T], H_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)$. Then (4.10) is obtained.

Now, fix $t \in [0,T]$. From (4.7) we derive

$$\begin{aligned}
\|W_0(t)\phi\|^2{}_{L_{xy}^2(\mathbb{R}\times[0,1])} &= \frac{1}{2}\|W_0(t)\phi\|^2_{L_{xy}^2(\mathbb{R}\times[-1,1])} \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-1}^{1}\left|\int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}} e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot\widehat{\phi}(\xi,n)\,d\xi\right|^2 dy\,dx \\
&\approx \int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}}\left|e^{-i\pi^2\left(\xi^2+n^2\right)t}\cdot\widehat{\phi}(\xi,n)\right|^2 d\xi \\
&= \int_{-\infty}^{\infty}\sum_{n\in\mathbb{Z}}\left|\widehat{\phi}(\xi,n)\right|^2 d\xi = \|\phi\|^2_{L_{xy}^2(\mathbb{R}\times[-1,1])} \approx \|\phi\|^2_{L_{xy}^2(\mathbb{R}\times[0,1])}
\end{aligned}$$

Additionally, for the same reason that $D^\alpha W_0(t)\phi = W_0(t)D^\alpha\phi$, the estimate (4.11) is valid for any $s \geq 0$. □

One may notice a pivotal step during the derivation of (4.12) that follows, which is also mentioned in (2.21).

Remark 4.4. *For $r \in [2,4]$ and any $\sigma > \dfrac{1}{2}$, let $\phi = \phi(x,y)$ and $f = f(x,y,t)$. Then*

$$\|W_0\phi\|_{L_{xyt}^r(\mathbb{R}\times[-1,1]\times[0,T])} \lesssim \|W_0\phi\|_{X^{\sigma,0}(\mathbb{R}\times[-1,1]\times[0,T])} \tag{4.13}$$

and according to the definition of the Bourgain space (2.2),

$$\|W_0 f\|_{L_{xyt}^r(\mathbb{R}\times[-1,1]\times[0,T])} \lesssim \|W_0 f\|_{X^{\sigma,0}(\mathbb{R}\times[-1,1]\times[0,T])} = \|f\|_{H_t^\sigma\left([0,T];\, L_{xy}^2(\mathbb{R}\times[-1,1])\right)} \, . \tag{4.14}$$

4.2.2 Equation with the Nonlinearity

We continue to discuss the nonlinear equation with nonhomogeneous initial and boundary conditions,

$$\begin{cases} iu_t + u_{xx} + u_{yy} + f = 0 & (x, y, t) \in \mathbb{R} \times (0,1) \times (0,T) \\ u(x,y,0) = 0 & (x,y) \in \mathbb{R} \times (0,1) \\ u(x,0,t) = u(x,1,t) = 0 & (x,t) \in \mathbb{R} \times (0,T) \end{cases} \tag{4.15}$$

Then according to the semigroup theory, let $f : \mathbb{R} \times [-1,1] \times [0,T] \to \mathbb{C}$ be an odd extension of g to the direction of negative y-axis; i.e. $f(x,y,t) = g(x,y,t)$ for $y \in [0,1]$. We write $\Phi_{0,f}$ as the solution to (4.15) and with the assistance of *Proposition 4.2* at the end of this section we are able to conclude that

$$\Phi_{0,f} \in L_t^q \left([0,T];\, W_{xy}^{s,r}(\mathbb{R} \times [-1,1])\right) \bigcap L_t^\infty \left([0,T];\, H_{xy}^s(\mathbb{R} \times [-1,1])\right)$$

for every $f \in L^q\left([0,T];\, W^{s,q}(\mathbb{R} \times [0,1])\right)$ with nicely picked q for $r \in [2,4]$. Using semigroup theory and by (4.7), we first derive the formula

Proposition 4.5.

$$\begin{aligned} \Phi_{0,f}(x,y,t) &= i \left(\int_0^t W_0(t-\tau) f(\tau)\, d\tau \right)(x,y) \\ &= i \int_0^t \int_{-\infty}^{\infty} \sum_{n \in \mathbb{Z}} e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)+i\pi(\xi x+ny)} \widehat{f^{xy}}(\xi, n, \tau)\, d\xi\, d\tau \end{aligned} \tag{4.16}$$

Then the associated estimates are listed below.

Proposition 4.6. $\forall\, r \in [2,4]$ *and* $\sigma \in (1/2, 1]$, *there is a* $q \in [4\sigma/(1+\sigma), 2]$ *such that,*

$$\|\Phi_{0,f}\|_{L^r(\mathbb{R}\times[0,1]\times[0,T]) \bigcap L_t^\infty\left([0,T];\, L_{xy}^2(\mathbb{R}\times[0,1])\right)} \lesssim T^{1+\sigma-\frac{4\sigma}{q}} \|f\|_{L^q(\mathbb{R}\times[0,1]\times[0,T])}, \tag{4.17}$$

$$\|\Phi_{0,f}\|_{L^r([0,T];\, W^{s,r}(\mathbb{R}\times[0,1])) \bigcap L_t^\infty\left([0,T];\, H_{xy}^s(\mathbb{R}\times[0,1])\right)} \lesssim T^{1+\sigma-\frac{4\sigma}{q}} \|f\|_{L^q([0,T];\, W^{s,q}(\mathbb{R}\times[0,1]))}, \tag{4.18}$$

$$\|\Phi_{0,f}\|_{L^4([0,T];\, W^{s,4}(\mathbb{R}\times[0,1])) \bigcap L_t^\infty\left([0,T];\, H_{xy}^s(\mathbb{R}\times[0,1])\right)} \lesssim C_T \|f\|_{L^{4/3}\left([0,T];\, W^{s,4/3}(\mathbb{R}\times[0,1])\right)}. \tag{4.19}$$

where C_T *only depends on* T.

Proof. Choose $\sigma \in (\frac{1}{2}, 1]$ as in (4.13) and some $q \in [4\sigma/(1+\sigma), 2]$. Using duality with (2.21) when $r = 4$, we shall be able to say

$$\|f\|_{X^{-\sigma,0}(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|f\|_{L^{\frac{4}{3}}(\mathbb{R}\times[0,1]\times[0,T])}$$

as well as

$$\|f\|_{X^{0,0}(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|f\|_{L^2(\mathbb{R}\times[0,1]\times[0,T])}$$

Essentially we can obtain the following by interpolation

$$\|f\|_{X^{\sigma',0}(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|f\|_{L^q(\mathbb{R}\times[0,1]\times[0,T])} \tag{4.20}$$

with $\frac{1}{q} = \frac{a}{4/3} + \frac{1-a}{2}$ and $\sigma' = a(-\sigma)$ for $0 \leq a \leq 1$. Therefore

$$\sigma' = \frac{2\sigma(q-2)}{q} \tag{4.21}$$

where $\frac{4}{3} < \frac{4\sigma}{1+\sigma} \leq q \leq 2$ for $\sigma < \frac{1}{2}$. It is easy to check that $\sigma' \leq 0$ and

$$-\sigma' = 4\sigma\left(\frac{1}{q} - \frac{1}{2}\right) \leq 4\sigma\left(\frac{1+\sigma}{4\sigma} - \frac{1}{2}\right) = 1 - \sigma < \frac{1}{2}\,.$$

Moreover,

$$1 - (\sigma - \sigma') = 1 + \sigma - \frac{4\sigma}{q} \geq 0\,.$$

Thus by (2.21)-(2.23) with $s = 0$ and (4.21) we have

$$\begin{aligned}
&\|\Phi_{0,f}\|_{L^r(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\
&\quad= \left\|\int_0^t W_0(t-\tau)f(\tau)\,d\tau\right\|_{L^r(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\
&\quad\lesssim \left\|\int_0^t W_0(t-\tau)f(\tau)\,d\tau\right\|_{X^{\sigma,0}(\mathbb{R}\times[0,1]\times[0,T])} \\
&\quad\lesssim T^{1-(\sigma-\sigma')}\|f\|_{X^{\sigma',0}(\mathbb{R}\times[0,1]\times[0,T])} \\
&\quad\lesssim T^{1+\sigma-\frac{4\sigma}{q}}\|f\|_{L^q(\mathbb{R}\times[0,1]\times[0,T])}
\end{aligned}$$

If considering $s > 0$, let α be a 2-dimensional multi-index with $|\alpha| = s$. We can derive the following estimates using (2.21)-(2.23) again, with the same σ and σ' defined above

$$\begin{aligned}
&\left\|D^\alpha_{xy}\Phi_{0,f}\right\|_{L^r(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\
&\quad= \left\|\int_0^t W_0(t-\tau)\left(D^\alpha_{xy}f(\tau)\right)\,d\tau\right\|_{L^r(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\
&\quad\lesssim \left\|\int_0^t W_0(t-\tau)\left(D^\alpha_{xy}f(\tau)\right)\,d\tau\right\|_{X^{\sigma,0}(\mathbb{R}\times[0,1]\times[0,T])} \\
&\quad\lesssim T^{1-(\sigma-\sigma')}\|D^\alpha_{xy}f\|_{X^{\sigma',0}(\mathbb{R}\times[0,1]\times[0,T])} \\
&\quad\leq T^{1+\sigma-\frac{4\sigma}{q}}\|D^\alpha_{xy}f\|_{L^q(\mathbb{R}\times[0,1]\times[0,T])}
\end{aligned}$$

Hence (4.17) and (4.18) are both proved. □

4.2.3 Equation with the Boundary Conditions

Next we study the equation with the nonhomogeneous boundary conditions h_1, h_2.

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0 & (x,y,t) \in \mathbb{R}\times(0,1)\times(0,T) \\ u(x,y,0) = 0 & (x,y) \in \mathbb{R}\times(0,1) \\ u(x,0,t) = h_1(x,t) & (x,t) \in \mathbb{R}\times(0,T) \\ u(x,1,t) = h_2(x,t) & (x,t) \in \mathbb{R}\times(0,T) \end{cases} \tag{4.22}$$

We expect the solution, denoted by $W_b(h_1,h_2)(x,y,t)$, presented in following form.

Proposition 4.7.

$$\begin{aligned} u(x,y,t) &= W_b(h_1,h_2)(x,y,t) \\ &= \pi \int_{-\infty}^{\infty} \sum_n \left[n \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)+i\pi(\xi x+ny)} \left(\widehat{h_1^x} - (-1)^n \widehat{h_2^x}\right)(\xi,\tau)\, d\tau \right] d\xi \\ &\approx W_{b_1}h_1(x,y,t) + W_{b_2}h_2(x,y,t) \end{aligned} \tag{4.23}$$

where

$$W_{b_1}h(x,y,t) = \int_{-\infty}^{\infty} \sum_n \left[n \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)+i\pi(\xi x+ny)} \widehat{h^x}(\xi,\tau)\, d\tau \right] d\xi \tag{4.24}$$

$$W_{b_2}h(x,y,t) = \int_{-\infty}^{\infty} \sum_n \left[n \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)+i\pi(\xi x+ny)} (-1)^{n+1}\widehat{h^x}(\xi,\tau)\, d\tau \right] d\xi \tag{4.25}$$

Proof. Again, we apply the Fourier transform with respect to x to get

$$\begin{cases} i\widehat{u^x}_t - \xi^2\pi^2\widehat{u^x} + \widehat{u^x}_{yy} = 0 & (x,y,t) \in \mathbb{R}\times(0,1)\times(0,T) \\ \widehat{u^x}(\xi,y,0) = 0 & (x,y) \in \mathbb{R}\times(0,1) \\ \widehat{u^x}(\xi,0,t) = \widehat{h_1^x}(\xi,t) & (x,t) \in \mathbb{R}\times(0,T) \\ \widehat{u^x}(\xi,1,t) = \widehat{h_2^x}(\xi,t) & (x,t) \in \mathbb{R}\times(0,T) \end{cases} \tag{4.26}$$

If $h_2 \equiv 0$ and $h_1(x,0) = 0$, then $\widehat{h_2^x} \equiv 0$ and $\widehat{h_1^x}(\xi,0) = 0$. That is

$$\begin{cases} i\widehat{u_1^x}_t - \xi^2\pi^2\widehat{u_1^x} + \widehat{u_1^x}_{yy} = 0 & (x,y,t) \in \mathbb{R}\times(0,1)\times(0,T) \\ \widehat{u_1^x}(\xi,y,0) = 0 & (x,y) \in \mathbb{R}\times(0,1) \\ \widehat{u_1^x}(\xi,0,t) = \widehat{h_1^x}(\xi,t) & (x,t) \in \mathbb{R}\times(0,T) \\ \widehat{u_1^x}(\xi,1,t) = 0 & (x,t) \in \mathbb{R}\times(0,T) \end{cases} \tag{4.27}$$

Let $v(\xi, y, t) = \widehat{u_1^x}(\xi, y, t) - (1-y)\widehat{h_1^x}(\xi, t)$. For v,

$$\begin{cases} iv_t - \xi^2\pi^2 v + v_{yy} = -i(1-y)\dfrac{\partial \widehat{h_1^x}}{\partial t} + \xi^2\pi^2(1-y)\widehat{h_1^x} \\ v(\xi, y, 0) = 0 \\ v(\xi, 0, t) = v(\xi, 1, t) = 0 \end{cases} \tag{4.28}$$

Consider the Fourier Sine series of v, $v(\xi, y, t) = \sum_{k=1}^{\infty} \alpha_k(\xi, t) \cdot \sin(k\pi y)$. Plug this form into (4.28) and have

$$\begin{aligned} i\int_0^1 \sum_{k=1}^{\infty} \frac{\partial \alpha_k}{\partial t} \cdot \sin(k\pi y) \cdot \sin(n\pi y)\, dy - \xi^2\pi^2 \cdot \int_0^1 \sum_{k=1}^{\infty} \alpha_k \cdot \sin(k\pi y) \cdot \sin(n\pi y)\, dy \\ - \int_0^1 \sum_{k=1}^{\infty} \alpha_k \cdot k^2\pi^2 \cdot \sin(k\pi y) \cdot \sin(n\pi y)\, dy \\ = i\frac{\partial \widehat{h_1^x}}{\partial t} \int_0^1 (y-1)\sin(n\pi y)\, dy - \xi^2\pi^2 \widehat{h_1^x} \int_0^1 (y-1)\sin(n\pi y)\, dy \\ = \left(i\frac{\partial \widehat{h_1^x}}{\partial t} - \xi^2\pi^2\widehat{h_1^x} \right) \int_0^1 (y-1)\sin(n\pi y)\, dy \end{aligned}$$

It is the same as

$$\begin{aligned} \frac{i}{2}\frac{\partial \alpha_n}{\partial t} - \frac{\xi^2\pi^2 + n^2\pi^2}{2}\alpha_n &= \left(i\frac{\partial \widehat{h_1^x}}{\partial t} - \xi^2\pi^2\widehat{h_1^x} \right) \int_0^1 (y-1)\sin(n\pi y)\, dy \\ &= \left(i\frac{\partial \widehat{h_1^x}}{\partial t} - \xi^2\pi^2\widehat{h_1^x} \right) \cdot \frac{-1}{n\pi} = \frac{-i\dfrac{\partial \widehat{h_1^x}}{\partial t} + \xi^2\pi^2\widehat{h_1^x}}{n\pi} \end{aligned}$$

or simply

$$\frac{\partial \alpha_n}{\partial t} + i\pi^2\left(\xi^2 + n^2\right) \cdot \alpha_n = \frac{-2}{n\pi}\left(\frac{\partial \widehat{h_1^x}}{\partial t} + i\xi^2\pi^2\widehat{h_1^x} \right)$$

Solve for α_n to get

$$\alpha_n(t) = e^{-i\pi^2\left(\xi^2+n^2\right)t}\left[\int_0^t e^{i\pi^2\left(\xi^2+n^2\right)\tau} \cdot \frac{-2}{n\pi}\left(\frac{\partial \widehat{h_1^x}}{\partial t} + i\xi^2\pi^2\widehat{h_1^x} \right) d\tau + C \right]$$

However, $C = 0$ on account of the boundary conditions. Thus

$$\alpha_n(t) = \frac{-2}{n\pi}\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \cdot \left(\frac{\partial \widehat{h_1^x}}{\partial t} + i\xi^2\pi^2\widehat{h_1^x} \right) d\tau$$

Also we can see that

$$\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \frac{\partial \widehat{h_1^x}}{\partial t}\, d\tau$$

$$= e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \cdot \widehat{h_1^x}(\xi,\tau)\Big|_{\tau=0}^{\tau=t} - i\pi^2\left(\xi^2+n^2\right)\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau$$
$$= \widehat{h_1^x}(\xi,t) - i\pi^2\left(\xi^2+n^2\right)\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau$$

because of which we have

$$\alpha_n = \frac{-2\widehat{h_1^x}}{n\pi} + 2in\pi\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau$$

Since $\int_0^1 (1-y)\cdot\sqrt{2}\sin(n\pi y)\,dy = \frac{\sqrt{2}}{n\pi}$ and $1-y = \sum_{n=1}^{\infty}\frac{\sqrt{2}}{n\pi}\cdot\sqrt{2}\sin(n\pi y)$, then

$$v(\xi,y,t) = -\widehat{h_1^x}\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi y) + \sum_{n=1}^{\infty} 2in\pi\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau\sin(n\pi y)$$
$$= -(1-y)\widehat{h_1^x} + \sum_{n=1}^{\infty} 2in\pi\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau\sin(n\pi y)$$

and

$$\widehat{u_1^x}(\xi,y,t) = \sum_{n=1}^{\infty} 2in\pi\int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)}\widehat{h_1^x}\,d\tau\sin(n\pi y)$$

Likewise, if $h_1 \equiv 0$ and $h_2(x,0)=0$, then $\widehat{h_1^x}\equiv 0$ and $\widehat{h_2^x}(\xi,0)=0$.

$$\begin{cases} i\widehat{u_2^x}_t - \xi^2\pi^2\widehat{u_2^x} + \widehat{u_2^x}_{yy} = 0 & (x,y,t) \in \mathbb{R}\times(0,1)\times(0,T) \\ \widehat{u_2^x}(\xi,y,0) = 0 & (x,y) \in \mathbb{R}\times(0,1) \\ \widehat{u_2^x}(\xi,0,t) = 0 & (x,t) \in \mathbb{R}\times(0,T) \\ \widehat{u_2^x}(\xi,1,t) = \widehat{h_2^x}(\xi,t) & (x,t) \in \mathbb{R}\times(0,T) \end{cases} \tag{4.29}$$

This time we let $v(\xi,y,t) = \widehat{u_2^x}(\xi,y,t) - y\widehat{h_2^x}(\xi,t)$.

$$\begin{cases} iv_t - \xi^2\pi^2 v + v_{yy} = -iy\frac{\partial\widehat{h_2^x}}{\partial t} + \xi^2\pi^2 y\widehat{h_2^x} \\ v(\xi,y,0) = 0 \\ v(\xi,0,t) = v(\xi,1,t) = 0 \end{cases} \tag{4.30}$$

Again write $v(\xi,y,t) = \sum_{k=1}^{\infty}\alpha_k(\xi,t)\cdot\sin(k\pi y)$. From equation (4.30) we can derive

$$\frac{i}{2}\frac{\partial\alpha_n}{\partial t} - \frac{\xi^2\pi^2+n^2\pi^2}{2}\alpha_n = \left(-i\frac{\partial\widehat{h_2^x}}{\partial t} + \xi^2\pi^2\widehat{h_2^x}\right)\int_0^1 y\sin(n\pi y)\,dy$$
$$= \left(-i\frac{\partial\widehat{h_2^x}}{\partial t} + \xi^2\pi^2\widehat{h_2^x}\right)\cdot\frac{-(-1)^n}{n\pi}$$

$$= \frac{(-1)^n \cdot \left(i\dfrac{\partial \widehat{h_2^x}}{\partial t} - \xi^2\pi^2 \widehat{h_2^x} \right)}{n\pi};$$

that is,

$$\frac{\partial \alpha_n}{\partial t} + i\pi^2\left(\xi^2 + n^2\right) \cdot \alpha_n = \frac{2(-1)^n}{n\pi} \left(\frac{\partial \widehat{h_2^x}}{\partial t} + i\xi^2\pi^2 \widehat{h_2^x} \right)$$

Solve this first order differential equation for α_n with respect to t and

$$\begin{aligned}\alpha_n &= \frac{2(-1)^n}{n\pi} \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \cdot \left(\frac{\partial \widehat{h_2^x}}{\partial t} + i\xi^2\pi^2 \widehat{h_2^x} \right) d\tau \\ &= \frac{2(-1)^n \widehat{h_2^x}}{n\pi} - 2i(-1)^n n\pi \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \widehat{h_2^x}\, d\tau\end{aligned}$$

by using $\int_0^1 y \cdot \sqrt{2}\sin(n\pi y)\, dy = \dfrac{-\sqrt{2}(-1)^n}{n\pi}$ and $y = \sum_{n=1}^{\infty} \dfrac{-\sqrt{2}(-1)^n}{n\pi} \cdot \sqrt{2}\sin(n\pi y)$, which lead directly to

$$\begin{aligned}v(\xi, y, t) &= \widehat{h_2^x} \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(n\pi y) - \sum_{n=1}^{\infty} 2i(-1)^n n\pi \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \widehat{h_2^x}\, d\tau \sin(n\pi y) \\ &= -y\widehat{h_2^x} + \sum_{n=1}^{\infty} 2in\pi \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \left(-(-1)^n \widehat{h_2^x} \right) d\tau \sin(n\pi y)\end{aligned}$$

Therefore

$$\widehat{u_2^x}(\xi, y, t) = \sum_{n=1}^{\infty} 2in\pi \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \left(-(-1)^n \widehat{h_2^x} \right) d\tau \sin(n\pi y)$$

In all, the formula of the solution to equation (4.22) is found as

$$\begin{aligned}u(x, y, t) &= \int_{-\infty}^{\infty} e^{i\pi x\xi} \left(\widehat{u_1^x}(\xi, y, t) + \widehat{u_2^x}(\xi, y, t) \right) d\xi \\ &= 2i\pi \int_{-\infty}^{\infty} e^{i\pi x\xi} \sum_{n=1}^{\infty} \left[n \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \left(\widehat{h_1^x} - (-1)^n \widehat{h_2^x} \right) d\tau \sin(n\pi y) \right] d\xi\end{aligned}$$

which is exactly (4.23). Thus (4.24) and (4.25) are obtained. □

Remark 4.8. *One may notice the equivalency between $|W_{b_1}h|$ and $|W_{b_2}h|$, as far as the norm is concerned. Moreover $|W_b(h_1, h_2) \leq |W_{b_1}h_1| + |W_{b_1}h_2|$, which means that we only need work on deriving estimates for $W_{b_1}h$. The following form is also adopted for $W_{b_1}h$ alternatively,*

$$W_{b_1}h(x, y, t) \approx \int_{-\infty}^{\infty} e^{i\pi x\xi} \sum_{n=1}^{\infty} \left[n \int_0^t e^{-i\pi^2\left(\xi^2+n^2\right)(t-\tau)} \left(\widehat{h} \right) d\tau \sin(n\pi y) \right] d\xi \tag{4.31}$$

If no confusion occurs, we use W_bh instead of $W_{b_1}h$ for the most of time in this section.

Then we shall start discussing the estimates on $|W_b(h_1, h_2)$ (or $W_b h$).

Proposition 4.9. *For $r \in [2,4]$ and any $\sigma > \frac{1}{2}$,*

$$\begin{aligned} &\|W_b(h_1,h_2)\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\ &\qquad\lesssim \sum_{j=1,2}\|h_j\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t([0,T];\,H^\sigma_x(\mathbb{R}))} \end{aligned} \tag{4.32}$$

$$\begin{aligned} &\|W_b(h_1,h_2)\|_{L^4_t\left([0,T];\,W^{s,4}_{xy}(\mathbb{R}\times[0,1])\right)\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\ &\qquad\lesssim \sum_{j=1,2}\|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];\,H^{s+1}_x(\mathbb{R})\right)} \end{aligned} \tag{4.33}$$

$$\begin{aligned} &\|W_b(h_1,h_2)\|_{L^r_t\left([0,T];\,W^{s,r}_{xy}(\mathbb{R}\times[0,1])\right)\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\ &\qquad\lesssim \sum_{j=1,2}\|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\,L^2_x(\mathbb{R}))\bigcap L^2_t\left([0,T];\,H^{s+1}_x(\mathbb{R})\right)} \end{aligned} \tag{4.34}$$

Proof. In this proof we substitute W_b for W_{b_1} unless it is indicated otherwise. We first let

$$\widetilde{h}(\xi,\lambda) = \int_0^T e^{i\pi^2\lambda t}\widehat{h^x}(\xi,t)\,dt \approx \widehat{h}(\xi,-\lambda)\,. \tag{4.35}$$

Thus $\widehat{h^x}(\xi,t) \approx \int_{-\infty}^{\infty} e^{-i\pi^2\lambda t}\widetilde{h}(\xi,\lambda)\,d\lambda$. Apply (4.35) to (4.24) to get

$$\begin{aligned} &W_b h(x,y,t) \\ &\approx \int_{-\infty}^{\infty}\sum_n \left[e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^t e^{i\pi^2\left(\xi^2+n^2\right)\tau}\left(\int_{-\infty}^{\infty} e^{-i\pi^2\lambda\tau}\widetilde{h}(\xi,\lambda)\,d\lambda\right)\,d\tau\right]\,d\xi \\ &= \int_{-\infty}^{\infty}\sum_n \left[e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_{-\infty}^{\infty}\left(\int_0^t e^{i\pi^2\left(\xi^2+n^2-\lambda\right)\tau}\,d\tau\right)\widetilde{h}(\xi,\lambda)\,d\lambda\right]\,d\xi \\ &= \int_{-\infty}^{\infty}\sum_n \left[e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_{-\infty}^{\infty}\frac{e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}-1}{i\pi^2\left(\xi^2+n^2-\lambda\right)}\widetilde{h}(\xi,\lambda)\,d\lambda\right]\,d\xi \\ &= \int_{-\infty}^{\infty}\sum_n \left[e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^{\infty}\frac{e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}-1}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\,d\lambda\right]\,d\xi \\ &\qquad + \int_{-\infty}^{\infty}\sum_n \left[e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^{\infty}\frac{e^{i\pi^2(\xi^2+n^2+\lambda)t}-1}{\xi^2+n^2+\lambda}\widetilde{h}(\xi,-\lambda)\,d\lambda\right]\,d\xi \\ &= \frac{1}{i\pi^2}\left(I^+ + I^-\right) \end{aligned}$$

where

$$I^+ = \int_{-\infty}^{\infty}\sum_n \left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^{\infty}\frac{e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}-1}{\xi^2+n^2-\lambda}\cdot\widetilde{h}(\xi,\lambda)\,d\lambda\right)\,d\xi$$

$$I^- = \int_{-\infty}^{\infty}\sum_n \left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^{\infty}\frac{e^{i\pi^2(\xi^2+n^2+\lambda)t}-1}{\xi^2+n^2+\lambda}\cdot\widetilde{h}(\xi,-\lambda)\,d\lambda\right)\,d\xi$$

Then we keep splitting up I^+ and I^- in the following way

$$I^+ =$$
$$\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^\infty \frac{e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}-1}{\xi^2+n^2-\lambda}\cdot\widetilde{h}(\xi,\lambda)\psi\left(\xi^2+n^2-\lambda\right)\,d\lambda\right)\,d\xi$$
$$+\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi$$
$$+\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{-1}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi$$
$$= I_1^+ + I_2^+ + I_3^+$$

with

$$I_1^+ =$$
$$\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^\infty \frac{e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}-1}{\xi^2+n^2-\lambda}\cdot\widetilde{h}(\xi,\lambda)\psi\left(\xi^2+n^2-\lambda\right)\,d\lambda\right)\,d\xi \tag{4.36}$$

$$I_2^+ =$$
$$\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\cdot\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi \tag{4.37}$$

$$I_3^+ =$$
$$\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{1}{\xi^2+n^2-\lambda}\cdot\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi \tag{4.38}$$

Likewise, rewrite

$$\begin{aligned}
I^- &\approx \int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{e^{i\pi^2(\xi^2+n^2+\lambda)t}-1}{\xi^2+n^2+\lambda}\widetilde{h}(\xi,-\lambda)\,d\lambda\right)\,d\xi\\
&=\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{e^{i\pi^2\lambda t}-e^{-i\pi^2\left(\xi^2+n^2\right)t}}{\xi^2+n^2+\lambda}\widetilde{h}(\xi,-\lambda)\,d\lambda\right)\,d\xi\\
&= I_1^- - I_2^-
\end{aligned}$$

with

$$I_1^- = \int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_0^\infty\frac{e^{i\pi^2\lambda t}}{\xi^2+n^2+\lambda}\widetilde{h}(\xi,-\lambda)\,d\lambda\right)\,d\xi \tag{4.39}$$

$$I_2^- = \int_{-\infty}^{\infty} \sum_n \left(e^{i\pi(x\xi+ny)} \cdot n \int_0^\infty \frac{e^{-i\pi^2\left(\xi^2+n^2\right)t}}{\xi^2+n^2+\lambda} \widetilde{h}(\xi,-\lambda)\, d\lambda \right) d\xi \tag{4.40}$$

We first study I_1^+. Rewrite $e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}$ from (4.36) in the Taylor expansion

$$\begin{aligned} I_1^+ &= \sum_{k=0}^{\infty} \frac{\left(i\pi^2\cdot t\right)^{k+1}}{(k+1)!} \cdot \left[\int_{-\infty}^{\infty} \sum_n e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)} \cdot \right. \\ &\qquad \left. \left(n \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \psi\left(\xi^2+n^2-\lambda\right) \right) d\lambda\, d\xi \right] \\ &= \sum_{k=0}^{\infty} I_{1,k}^+ \end{aligned}$$

where

$$\begin{aligned} I_{1,k}^+ = \frac{\left(i\pi^2\cdot t\right)^{k+1}}{(k+1)!} \cdot \left[\int_{-\infty}^{\infty} \sum_n e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)} \cdot \right. \\ \left. \left(n \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \psi\left(\xi^2+n^2-\lambda\right) \right) d\lambda\, d\xi \right] \end{aligned}$$

Recall $\psi(x)$ as a cut-off function from (2.1). Note that $\psi\left(\xi^2+n^2-\lambda\right) \not\equiv 0$ for $-2 \le \xi^2+n^2-\lambda \le 2$, i.e. $-2+\lambda \le n^2+\xi^2 \le 2+\lambda$. Then for each $I_{1,k}^+$, apply (4.11) and (4.12) so that

$$\begin{aligned} &\|I_{1,k}^+\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L_t^\infty\left([0,T];\, L^2_{xy}(\mathbb{R}\times[0,1])\right)} \\ &= \left\| \frac{\left(i\pi^2\cdot t\right)^{k+1}}{(k+1)!} \cdot \int_{-\infty}^{\infty} \sum_n e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)} \right. \\ &\qquad \left. \left(n \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \psi\left(\xi^2+n^2-\lambda\right) \right) d\lambda\, d\xi \right\|_{L^4\bigcap L_t^\infty\left(L^2_{xy}\right)} \\ &\le \frac{\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!} \left\| \int_{-\infty}^{\infty} \sum_n e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)} \cdot \right. \\ &\qquad \left. \left(n \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \psi\left(\xi^2+n^2-\lambda\right) \right) d\lambda\, d\xi \right\|_{L^4\bigcap L_t^\infty\left(L^2_{xy}\right)} \\ &\lesssim \frac{\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!} \left\| n \cdot \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \cdot \psi\left(\xi^2+n^2-\lambda\right) d\lambda \right\|_{L^2_{\xi n}} \\ &= \frac{\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!} \left\{ \int_{-\infty}^{\infty} \sum_n \left| n \cdot \int_0^\infty \left(\xi^2+n^2-\lambda\right)^k \cdot \widetilde{h}(\xi,\lambda) \cdot \psi\left(\xi^2+n^2-\lambda\right) d\lambda \right|^2 d\xi \right\}^{\frac{1}{2}} \\ &\le \frac{\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!} \left\{ \int_{-\infty}^{\infty} \sum_n \left(\int_0^\infty |n| \cdot \left|\xi^2+n^2-\lambda\right|^k \cdot \left|\widetilde{h}(\xi,\lambda)\right| \cdot \left|\psi\left(\xi^2+n^2-\lambda\right)\right| d\lambda \right)^2 d\xi \right\}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{2^k\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!}\left\{\int_{-\infty}^{\infty}\sum_n\left(\int_0^\infty\left(1+\xi^2+|\lambda|\right)\cdot\left|\widetilde{h}(\xi,\lambda)\right|^2\left|\psi\left(\xi^2+n^2-\lambda\right)\right|\,d\lambda\right)\right.\\
&\qquad\left.\left(\int_0^\infty\left|\psi\left(\xi^2+n^2-\lambda\right)\right|\,d\lambda\right)\,d\xi\right\}^{\frac{1}{2}}\\
&\leq \frac{2^k\left(\pi^2\cdot T\right)^{k+1}}{(k+1)!}\left[\int_{-\infty}^{\infty}\sum_n\left(\int_{\xi^2+n^2-2}^{\xi^2+n^2+2}\left(1+\xi^2+|\lambda|\right)\cdot\left|\widetilde{h}(\xi,\lambda)\right|^2\,d\lambda\right)\cdot\left(\int_{\xi^2+n^2-2}^{\xi^2+n^2+2}1\,d\lambda\right)\,d\xi\right]^{\frac{1}{2}}\\
&\lesssim \frac{\left(2\pi^2\cdot T\right)^{k+1}}{(k+1)!}\left[\int_{-\infty}^{\infty}\left(\sum_n\int_{\xi^2+n^2-2}^{\xi^2+n^2+2}\left(1+|\lambda|\right)\cdot\left|\widetilde{h}(\xi,\lambda)\right|^2\,d\lambda\right)\,d\xi\right]^{\frac{1}{2}}
\end{aligned}$$

Since $(n+1)^2-2>n^2+2$ for $n\geq 1$, then the sequence of intervals

$$\left\{\left[\xi^2+n^2-2,\xi^2+n^2+2\right]\right\}_{n\in\mathbb{Z}}$$

is basically disjoint. Thus continuing with the argument above, we find that

$$\begin{aligned}
&\|I_{1,k}^+\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)}\\
&\lesssim \frac{\left(2\pi^2\cdot T\right)^{k+1}}{(k+1)!}\left[\int_{-\infty}^{\infty}\int_0^\infty\left(1+|\lambda|\right)\left|\widetilde{h}(\xi,\lambda)\right|^2\,d\lambda\,d\xi\right]^{\frac{1}{2}}\\
&= \frac{\left(2\pi^2\cdot T\right)^{k+1}}{(k+1)!}\|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x(\mathbb{R}))}
\end{aligned}$$

Hence

$$\begin{aligned}
&\|I_1^+\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)}\\
&\leq \sum_{k=0}^{\infty}\|I_{1,k}^+\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)}\\
&\lesssim \sum_{k=0}^{\infty}\frac{\left(2\pi^2\cdot T\right)^{k+1}}{(k+1)!}\|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x(\mathbb{R}))}\leq \|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x(\mathbb{R}))}\,.
\end{aligned}\tag{4.41}$$

Next we consider the term I_2^+ in (4.37). First, fix a $t\in(0,T]$ and see

$$\begin{aligned}
\|I_2^+(t)\|^2_{L^2_{xy}} &\approx \int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left|\left(\int_0^{\xi^2}+\int_{\xi^2}^{\infty}\right)n\cdot\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right|^2\,d\xi\\
&\leq \int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left|\int_0^{\xi^2}n\cdot\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right|^2\,d\xi\\
&\qquad+\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left|\int_{\xi^2}^{\infty}n\cdot\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right|^2\,d\xi\\
&= S_1+S_2
\end{aligned}$$

where we define

$$S_1 = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{\xi^2} n \cdot \frac{e^{-i\pi^2 \lambda t}}{\xi^2 + n^2 - \lambda} \widetilde{h}(\xi, \lambda) \left(1 - \psi\left(\xi^2 + n^2 - \lambda\right)\right) d\lambda \right|^2 d\xi \tag{4.42}$$

$$S_2 = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{\xi^2}^{\infty} n \cdot \frac{e^{-i\pi^2 \lambda t}}{\xi^2 + n^2 - \lambda} \widetilde{h}(\xi, \lambda) \left(1 - \psi\left(\xi^2 + n^2 - \lambda\right)\right) d\lambda \right|^2 d\xi \tag{4.43}$$

In S_1, we can substitute $\xi^2 - \mu$ for λ, and use Holder's inequality with

$$\begin{aligned}
S_1 &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{\xi^2} n \cdot \frac{e^{-i\pi^2 (\xi^2 - \mu) t}}{n^2 + \mu} \widetilde{h}\left(\xi, \xi^2 - \mu\right) \left(1 - \psi(n^2 + \mu)\right) d\mu \right|^2 d\xi \\
&\le \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_0^{\xi^2} \left(1 + \xi^2 - \mu\right) \left|\widetilde{h}\left(\xi, \xi^2 - \mu\right)\right|^2 d\mu \right) \cdot \left(\int_0^{\xi^2} \frac{n^2 \cdot (1 - \psi(n^2 + \mu))}{(n^2 + \mu)^2 \left(1 + \xi^2 - \mu\right)} d\mu \right) d\xi \\
&= \int_{-\infty}^{\infty} \left(\int_0^{\xi^2} \left(1 + \xi^2 - \mu\right) \left|\widetilde{h}(\xi, \xi^2 - \mu)\right|^2 d\mu \right) \cdot \left(\sum_{n=1}^{\infty} \int_0^{\xi^2} \frac{n^2 \cdot (1 - \psi(n^2 + \mu))}{(n^2 + \mu)^2 \left(1 + \xi^2 - \mu\right)} d\mu \right) d\xi \\
&\le \int_{-\infty}^{\infty} \left(\int_0^{\xi^2} \left(1 + \xi^2 - \mu\right) \left|\widetilde{h}(\xi, \xi^2 - \mu)\right|^2 d\mu \right) \cdot \left(\int_0^{\xi^2} \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \mu)^2 \left(1 + \xi^2 - \mu\right)} d\mu \right) d\xi
\end{aligned}$$

Let us take a moment to investigate the second integral factor. Since $\frac{1}{n^2+\mu}$ is strictly decreasing in n, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + \mu} \le \int_0^{\infty} \frac{d\eta}{\eta^2 + \mu}$. Thus

$$\begin{aligned}
\int_0^{\xi^2} \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \mu)^2 \left(1 + \xi^2 - \mu\right)} d\mu &\le \int_0^{\xi^2} \sum_{n=1}^{\infty} \frac{1}{(n^2 + \mu) \left(1 + \xi^2 - \mu\right)} d\mu \\
&\le \int_0^{\xi^2} \frac{1}{1 + \xi^2 - \mu} \left(\int_0^{\infty} \frac{d\eta}{\eta^2 + \mu} \right) d\mu \overset{\eta = t\sqrt{\mu}}{\le} \int_0^{\xi^2} \frac{1}{1 + \xi^2 - \mu} \left(\frac{1}{\sqrt{\mu}} \int_0^{\infty} \frac{1}{t^2 + 1} dt \right) d\mu \\
&\lesssim \int_0^{\xi^2} \frac{1}{\sqrt{\mu}\left(1 + \xi^2 - \mu\right)} d\mu \overset{\sigma = \sqrt{\mu}}{\lesssim} \int_0^{|\xi|} \frac{1}{1 + \xi^2 - \sigma^2} d\sigma \\
&\approx \frac{1}{\sqrt{1 + \xi^2}} \int_0^{|\xi|} \left(\frac{1}{\sqrt{1 + \xi^2} - \sigma} + \frac{1}{\sqrt{1 + \xi^2} + \sigma} \right) d\sigma \\
&\approx \frac{1}{\sqrt{1 + \xi^2}} \ln \left(\frac{\sqrt{1 + \xi^2} + \sigma}{\sqrt{1 + \xi^2} - \sigma} \right) \Bigg|_0^{|\xi|} = \frac{1}{\sqrt{1 + \xi^2}} \ln \left(\frac{\sqrt{1 + \xi^2} + |\xi|}{\sqrt{1 + \xi^2} - |\xi|} \right) \\
&= \frac{2}{\sqrt{1 + \xi^2}} \ln \left(\sqrt{1 + \xi^2} + |\xi| \right) \lesssim \frac{1}{\sqrt{1 + \xi^2}} \left(\sqrt{1 + \xi^2} + |\xi| \right) \le 2
\end{aligned}$$

Since we have showed that $\displaystyle\int_0^{\xi^2} \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \mu)^2 \left(1 + \xi^2 - \mu\right)} d\mu$ is uniformly bounded, then

$$S_1 \lesssim \int_{-\infty}^{\infty} \int_0^{\xi^2} \left(1 + \xi^2 - \mu\right) \left|\widetilde{h}(\xi, \xi^2 - \mu)\right|^2 d\mu \, d\xi$$

$$\approx \int_{-\infty}^{\infty} \int_{0}^{\xi^2} (1+\lambda) \left| \widetilde{h}(\xi,\lambda) \right|^2 \, d\lambda \, d\xi \lesssim \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\,L_x^2)}$$

Similarly, for the second term S_2, use $\nu^2 + \xi^2$ to substitute for λ. As a result,

$$\begin{aligned}
S_2 &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{\infty} \frac{2n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{n-1} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n-1}^{n+1} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n+1}^{\infty} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&= S_{2,1} + S_{2,2} + S_{2,3}
\end{aligned}$$

while we label term in the following way that

$$S_{2,1} = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{n-1} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \tag{4.44}$$

$$S_{2,2} = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n-1}^{n+1} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \tag{4.45}$$

$$S_{2,3} = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n+1}^{\infty} \frac{n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \tag{4.46}$$

Regarding (4.44), we obtain

$$\begin{aligned}
S_{2,1} &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{n-1} \frac{2n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\approx \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{n-1} \left[\frac{1}{n-\nu} + \frac{1}{n+\nu} \right] e^{-i\pi^2(\nu^2+\xi^2)t} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_0^{n-1} \frac{e^{-i\pi^2(\nu^2+\xi^2)t}}{n-\nu} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1-\psi(n^2-\nu^2)\right) \, d\nu \right|^2 \, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_0^{n-1} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \frac{\nu^{2\alpha+2}}{(n-\nu)^{2-2\beta}} \, d\nu \right) \left(\int_0^{n-1} \frac{d\nu}{\nu^{2\alpha}(n-\nu)^{2\beta}} \right) \, d\xi
\end{aligned}$$

We now take a look at the second integral factor and choose α and β such that $0 < \alpha, \beta < \frac{1}{2}$ and $1 - 2\alpha - 2\beta < 0$.

$$\int_0^{n-1} \frac{d\nu}{\nu^{2\alpha}(n-\nu)^{2\beta}} = \int_0^{n/2} \frac{d\nu}{\nu^{2\alpha}(n-\nu)^{2\beta}} + \int_{n/2}^{n-1} \frac{d\nu}{\nu^{2\alpha}(n-\nu)^{2\beta}}$$

$$
\begin{aligned}
&\leq \left(\frac{2}{n}\right)^{2\beta} \int_0^{n/2} \frac{d\nu}{\nu^{2\alpha}} + \left(\frac{2}{n}\right)^{2\alpha} \int_0^{n/2} \frac{d\nu}{(n-\nu)^{2\beta}} \\
&= \left(\frac{2}{n}\right)^{2\beta} \frac{\nu^{1-2\alpha}}{1-2\alpha}\Bigg|_0^{n/2} + \left(\frac{2}{n}\right)^{2\alpha} \frac{(n-\nu)^{1-2\beta}}{1-2\beta}\Bigg|_{n/2}^{n-1} \\
&= \left(\frac{n}{2}\right)^{1-2\alpha-2\beta} \left(\frac{1}{1-2\alpha} + \frac{1}{1-2\beta}\right) + \left(\frac{2}{n}\right)^{2\alpha} \frac{1}{2\beta-1} \leq C
\end{aligned}
$$

Therefore, with $0 < \alpha,\ \beta < \frac{1}{2}$ and $1 - 2\alpha - 2\beta < 0$

$$
\begin{aligned}
S_{2,1} &\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \int_0^{n-1} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \frac{\nu^{2\alpha+2}}{(n-\nu)^{2-2\beta}}\, d\nu\, d\xi \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^{2\alpha+2} \left(\sum_{n=[\nu+2]}^{\infty} \frac{1}{(n-\nu)^{2-2\beta}}\right) d\nu\, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^{2\alpha+2} \left(1 + \sum_{n=[\nu+3]}^{\infty} \frac{1}{(n-\nu)^{2-2\beta}}\right) d\nu\, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^{2\alpha+2} \left(1 + \int_{\nu+1}^{\infty} \frac{d\eta}{(\eta-\nu)^{2-2\beta}}\right) d\nu\, d\xi \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^{2\alpha+2} \left(1 + \frac{(\eta-\nu)^{2\beta-1}}{2\beta-1}\Bigg|_{\nu+1}^{\infty}\right) d\nu\, d\xi \\
&= \int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^{2\alpha+2}\, d\nu\, d\xi \lesssim \int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left(\lambda - \xi^2\right)^{\frac{1}{2}+\alpha} \left|\widetilde{h}(\xi, \lambda)\right|^2 d\lambda\, d\xi \\
&\leq \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}.
\end{aligned}
$$

The symbol $[\cdot]$ represents the largest integer which is smaller or equal to the number inside. It is clear that when $n = 1$ $S_{2,1} = 0$ and at the same time the estimate on $S_{2,2}$ is given by the steps above. Hence for $S_{2,2}$ we only need to consider for $n \geq 2$. Also note that $\nu \geq n-1$ implies $\frac{1}{\nu} \leq \frac{1}{n-1}$. Thus

$$
\begin{aligned}
S_{2,2} &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n-1}^{n+1} \frac{2n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1 - \psi(n^2-\nu^2)\right) d\nu \right|^2 d\xi \\
&= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n-1}^{n+1} \left[\frac{1}{n-\nu} + \frac{1}{n+\nu}\right] e^{-i\pi^2(\nu^2+\xi^2)t} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1 - \psi(n^2-\nu^2)\right) d\nu \right|^2 d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \left(\int_{n-1}^{\sqrt{n^2-1}} + \int_{\sqrt{n^2+1}}^{n+1}\right) \frac{e^{-i\pi^2(\nu^2+\xi^2)t}}{n-\nu} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1 - \psi(n^2-\nu^2)\right) d\nu \right|^2 d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{n-1}^{\sqrt{n^2-1}} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^3\, d\nu\right) \left(\int_{n-1}^{\sqrt{n^2-1}} \frac{d\nu}{\nu(n-\nu)^2}\right) d\xi \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{\sqrt{n^2+1}}^{n+1} \left|\widetilde{h}\left(\xi, \nu^2+\xi^2\right)\right|^2 \nu^3\, d\nu\right) \left(\int_{\sqrt{n^2+1}}^{n+1} \frac{d\nu}{\nu(n-\nu)^2}\right) d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{n-1}^{\sqrt{n^2-1}} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^3 \, d\nu \right) \left. \frac{1}{(n-1)(n-\nu)} \right|_{n-1}^{\sqrt{n^2-1}} d\xi \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{\sqrt{n^2+1}}^{n+1} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^3 \, d\nu \right) \left. \frac{1}{(n-1)(n-\nu)} \right|_{\sqrt{n^2+1}}^{n+1} d\xi \\
&= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{n-1}^{\sqrt{n^2-1}} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^3 \, d\nu \right) \left(\frac{n+\sqrt{n^2-1}}{n-1} - 1 \right) d\xi \\
&\quad + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{\sqrt{n^2+1}}^{n+1} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^3 \, d\nu \right) \left(\frac{\sqrt{n^2+1}+n}{n-1} - 1 \right) d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{n-1}^{\sqrt{n^2-1}} + \int_{\sqrt{n^2+1}}^{n+1} \right) \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^3 \, d\nu \, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \int_{0}^{\infty} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^2 \, d\nu^2 \, d\xi \le \int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left| \widetilde{h}(\xi, \lambda) \right|^2 (\lambda - \xi^2) \, d\lambda \, d\xi \le \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}
\end{aligned}
$$

Finally the only part left to show is the estimate for $S_{2,3}$ in (4.46).

$$
\begin{aligned}
S_{2,3} &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n+1}^{\infty} \frac{2n \cdot e^{-i\pi^2(\nu^2+\xi^2)t}}{n^2-\nu^2} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1 - \psi(n^2-\nu^2)\right) d\nu \right|^2 d\xi \\
&= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left| \int_{n+1}^{\infty} \left[\frac{1}{\nu-n} - \frac{1}{\nu+n} \right] e^{-i\pi^2(\nu^2+\xi^2)t} \cdot \nu \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \left(1 - \psi(n^2-\nu^2)\right) d\nu \right|^2 d\xi \\
&\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(\int_{n+1}^{\infty} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \frac{\nu^{2\alpha+2}}{(\nu-n)^{2-2\beta}} \, d\nu \right) \left(\int_{n+1}^{\infty} \frac{d\nu}{\nu^{2\alpha}(\nu-n)^{2\beta}} \right) d\xi
\end{aligned}
$$

Analogously, we still need to work on the second factor. Let $0 < \alpha,\, \beta < \frac{1}{2}$ and $1-2\alpha-2\beta < 0$. Also, $\nu > 2n$ guarantees that $2(\nu - n) > \nu$. Then

$$
\begin{aligned}
\int_{n+1}^{\infty} \frac{d\nu}{\nu^{2\alpha}(\nu-n)^{2\beta}} &= \int_{n+1}^{2n} \frac{d\nu}{\nu^{2\alpha}(\nu-n)^{2\beta}} + \int_{2n}^{\infty} \frac{d\nu}{\nu^{2\alpha}(\nu-n)^{2\beta}} \\
&\le \frac{1}{(2n)^{2\alpha}} \int_{n+1}^{2n} \frac{d\nu}{(\nu-n)^{2\beta}} + 2^{2\beta} \int_{2n}^{\infty} \frac{d\nu}{\nu^{2\alpha+2\beta}} \\
&\le \frac{1}{n^{2\alpha}} \left. \frac{(\nu-n)^{1-2\beta}}{1-2\beta} \right|_{n+1}^{2n} + 2^{2\beta} \left. \frac{\nu^{1-2\alpha-2\beta}}{2\alpha+2\beta-1} \right|_{\infty}^{2n} \lesssim n^{1-2\alpha-2\beta} \le C
\end{aligned}
$$

Therefore (note that $\nu \ge 2$),

$$
\begin{aligned}
S_{2,3} &\lesssim \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \int_{n+1}^{\infty} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \frac{\nu^{2\alpha+2}}{(\nu-n)^{2-2\beta}} \, d\nu \, d\xi \\
&= \int_{-\infty}^{\infty} \int_{2}^{\infty} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^{2\alpha+2} \left(\sum_{n=1}^{[\nu-1]} \frac{1}{(\nu-n)^{2-2\beta}} \right) d\nu \, d\xi \\
&\le \int_{-\infty}^{\infty} \int_{2}^{\infty} \left| \widetilde{h}\left(\xi, \nu^2+\xi^2\right) \right|^2 \nu^{2\alpha+2} \left(1 + \sum_{n=1}^{[\nu-2]} \frac{1}{(\nu-n)^{2-2\beta}} \right) d\nu \, d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{\infty}\int_{2}^{\infty}\left|\widetilde{h}\left(\xi,\nu^2+\xi^2\right)\right|^2\nu^{2\alpha+2}\left(1+\int_1^{\nu-1}\frac{d\eta}{(\nu-\eta)^{2-2\beta}}\right)d\nu\,d\xi\\
&=\int_{-\infty}^{\infty}\int_{2}^{\infty}\left|\widetilde{h}\left(\xi,\nu^2+\xi^2\right)\right|^2\nu^{2\alpha+2}\left(1+\frac{1}{(1-2\beta)(\nu-\eta)^{1-2\beta}}\Bigg|_{\nu-1}^{1}\right)d\nu\,d\xi\\
&\eqsim \int_{-\infty}^{\infty}\int_{2}^{\infty}\left|\widetilde{h}\left(\xi,\nu^2+\xi^2\right)\right|^2\nu^{2\alpha+1}\,d\nu^2\,d\xi=\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\widetilde{h}(\xi,\lambda)\right|^2\left(\lambda-\xi^2\right)^{\alpha+\frac{1}{2}}\,d\lambda\,d\xi\\
&\leq \int_{-\infty}^{\infty}\int_{0}^{\infty}\lambda\left|\widetilde{h}(\xi,\lambda)\right|^2\,d\lambda\,d\xi\leq\|h\|^2_{H_t^{\frac{1}{2}}([0,T];\,L_x^2)}
\end{aligned}
$$

Eventually by adding together from $S_{2,1}$ to $S_{2,3}$, we get $S_2\lesssim\|h\|^2_{H_t^{\frac{1}{2}}([0,T];\,L_x^2)}$. Hence

$$
\sup_{0\leq t\leq T}\|I_2^+(t)\|_{L^2_{xy}}\lesssim\|h\|_{H_t^{\frac{1}{2}}([0,T];\,L_x^2)} \tag{4.47}
$$

On the other hand, to deal with the estimates for L^4-norm, we split I_2^+ as follows:

$$
\begin{aligned}
I_2^+=&\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{\xi^2+n^2}^{\infty}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi\\
&+\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{\xi^2+\frac{n^2}{2}}^{\xi^2+n^2}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi\\
&+\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{0}^{\xi^2+\frac{n^2}{2}}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi\\
=&\,I_{2,1}^++I_{2,2}^++I_{2,3}^+
\end{aligned}
$$

while letting

$$
I_{2,1}^+=\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{\xi^2+n^2}^{\infty}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi \tag{4.48}
$$

$$
I_{2,2}^+=\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{\xi^2+\frac{n^2}{2}}^{\xi^2+n^2}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi \tag{4.49}
$$

$$
I_{2,3}^+=\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{0}^{\xi^2+\frac{n^2}{2}}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)d\lambda\right)d\xi \tag{4.50}
$$

We first write from (4.48)

$$
I_{2,1}^+=
$$

$$\int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_{\xi^2+n^2}^{\infty}\frac{e^{-i\pi^2\lambda t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi$$
$$=\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\right.$$
$$\left.\int_{\xi^2+n^2}^{\infty}\frac{ne^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}}{\left(\xi^2+n^2-\lambda\right)}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right)\,d\xi$$

Then choose $\mu=\xi^2+n^2-\lambda$ and $s=\sqrt{\lambda-\xi^2}$ for the step below. By (4.14) we can get for $\frac{1}{2}<\sigma<1$

$$\begin{aligned}
&\|I_{2,1}^{+}\|^2_{L^4(\mathbb{R}\times[0,1]\times[0,T])}\\
&\lesssim\left\|\int_{\xi^2+n^2}^{\infty}\frac{ne^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right\|^2_{H_t^{\sigma}\left([0,T];\,L^2_{\xi n}\right)}\\
&=\left\|\int_{-\infty}^{\infty}\frac{ne^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}}{\xi^2+n^2-\lambda}\chi_{[\xi^2+n^2,\infty)}(\lambda)\cdot\widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)\,d\lambda\right\|^2_{H_t^{\sigma}\left([0,T];\,L^2_{\xi n}\right)}\\
&=\left\|\int_{-\infty}^{\infty}\frac{ne^{i\pi^2\mu t}}{\mu}\chi_{(-\infty,0]}(\mu)\cdot\widetilde{h}\left(\xi,\xi^2+n^2-\mu\right)\left(1-\psi(\mu)\right)\,d\mu\right\|^2_{H_t^{\sigma}\left([0,T];\,L^2_{\xi n}\right)}
\end{aligned}$$

Since the term only with the cut-off function ψ can be easily shown to be controlled by $\|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x)}$ up to a coefficient as in the study on (4.41) for I_1^{+}, then we may focus just on

$$\begin{aligned}
&\left\|\int_{-\infty}^{\infty}\frac{ne^{i\pi^2\mu t}}{\mu}\chi_{(-\infty,-1]}(\mu)\cdot\widetilde{h}\left(\xi,\xi^2+n^2-\mu\right)\,d\mu\right\|^2_{H_t^{\sigma}\left([0,T];\,L^2_{\xi n}\right)}\\
&\approx\int_{-\infty}^{\infty}\sum_n\int_{-\infty}^{\infty}\left|\frac{n(1+|\mu|)^{\sigma}}{|\mu|}\chi_{(-\infty,-1]}(\mu)\cdot\widetilde{h}\left(\xi,\xi^2+n^2-\mu\right)\right|^2\,d\mu\,d\xi\\
&\leq\int_{-\infty}^{\infty}\sum_n\int_{-\infty}^{\infty}\left|\frac{n}{(1+|\mu|)^{1-\sigma}}\chi_{(-\infty,-1]}(\mu)\cdot\widetilde{h}\left(\xi,\xi^2+n^2-\mu\right)\right|^2\,d\mu\,d\xi\\
&=\int_{-\infty}^{\infty}\sum_n\int_{-\infty}^{\infty}\left|\frac{n}{(1+|\lambda-\xi^2-n^2|)^{1-\sigma}}\chi_{[\xi^2+n^2+1,\infty)}(\lambda)\cdot\widetilde{h}(\xi,\lambda)\right|^2\,d\lambda\,d\xi\\
&\approx\int_{-\infty}^{\infty}\sum_n\int_{-\infty}^{\infty}\left|\frac{n}{(1+s^2-n^2)^{1-\sigma}}\chi_{[\sqrt{n^2+1},\infty)}(s)\cdot\widetilde{h}\left(\xi,s^2+\xi^2\right)\right|^2\,s\,ds\,d\xi\\
&=\int_{-\infty}^{\infty}\sum_n\left(\int_{\sqrt{n^2+1}}^{n+1}+\int_{n+1}^{\infty}\right)\frac{n^2}{(1+s^2-n^2)^{2-2\sigma}}\cdot\left|\widetilde{h}\left(\xi,s^2+\xi^2\right)\right|^2\,s\,ds\,d\xi
\end{aligned}$$

The first integral can be estimated from

$$\int_{-\infty}^{\infty}\sum_n\int_{\sqrt{n^2+1}}^{n+1}\frac{n^2}{(1+s^2-n^2)^{2-2\sigma}}\cdot\left|\widetilde{h}\left(\xi,s^2+\xi^2\right)\right|^2\,s\,ds\,d\xi$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{\infty} \sum_n \int_{\sqrt{n^2+1}}^{n+1} n^2 \cdot \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 s\, ds\, d\xi \\
&\leq \int_{-\infty}^{\infty} \sum_n \int_{\sqrt{n^2+1}}^{n+1} s^3 \cdot \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 ds\, d\xi \leq \int_{-\infty}^{\infty} \int_1^{\infty} s^3 \cdot \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 ds\, d\xi \\
&\lesssim \int_{-\infty}^{\infty} \int_{1+\xi^2}^{\infty} \left(\lambda-\xi^2\right) \cdot \left|\widetilde{h}(\xi,\lambda)\right|^2 d\lambda\, d\xi \lesssim \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}
\end{aligned}
$$

and the second integral satisfies

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \sum_n \int_{n+1}^{\infty} \frac{n^2}{(1+s^2-n^2)^{2-2\sigma}} \cdot \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 s\, ds\, d\xi \\
&\quad\lesssim \int_{-\infty}^{\infty} \sum_n \int_{n+1}^{\infty} \left(\frac{n}{s^2-n^2}\right)^{2-2\sigma} n^{2\sigma} \cdot \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 s\, ds\, d\xi \\
&\quad\lesssim \int_{-\infty}^{\infty} \sum_{n\neq 0} \int_{n+1}^{\infty} \frac{1}{(s-n)^{2-2\sigma}} \cdot s^{2\sigma+1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 ds\, d\xi \\
&\quad= \int_{-\infty}^{\infty} \int_2^{\infty} s^{2\sigma+1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 \left(\sum_{n=1}^{\lfloor s-1\rfloor} \frac{1}{(s-n)^{2-2\sigma}}\right) ds\, d\xi \\
&\quad\leq \int_{-\infty}^{\infty} \int_2^{\infty} s^{2\sigma+1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 \left(\int_0^{s-1} \frac{d\eta}{(s-\eta)^{2-2\sigma}}\right) ds\, d\xi \\
&\quad= \int_{-\infty}^{\infty} \int_2^{\infty} s^{2\sigma+1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 \left(\left.\frac{(s-\eta)^{2\sigma-1}}{1-2\sigma}\right|_0^{s-1}\right) ds\, d\xi \\
&\quad\lesssim \int_{-\infty}^{\infty} \int_2^{\infty} s^{2\sigma+1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 s^{2\sigma-1}\, ds\, d\xi \\
&\quad= \int_{-\infty}^{\infty} \int_2^{\infty} s^{4\sigma} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 ds\, d\xi \approx \int_{-\infty}^{\infty} \int_2^{\infty} s^{4\sigma-1} \left|\widetilde{h}\left(\xi, s^2+\xi^2\right)\right|^2 ds^2\, d\xi \\
&\quad= \int_{-\infty}^{\infty} \int_{4+\xi^2}^{\infty} \left(\lambda-\xi^2\right)^{2\sigma-\frac{1}{2}} \left|\widetilde{h}(\xi,\lambda)\right|^2 d\lambda\, d\xi\,.
\end{aligned}
$$

With $\sigma = \frac{3}{4} \in \left(\frac{1}{2}, 1\right)$ we reach to the critical point

$$
\|I_{2,1}^+\|_{L^4(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|h\|_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \tag{4.51}
$$

Similarly, we may also manage to control the L^4-norm by the boundary data in $H^\alpha(L^2)$-norm for $I_{2,2}^+$ in (4.49) with $\frac{1}{2} < \sigma \leq \frac{3}{4}$. Note that $I_{2,2}^+$ equals a nonzero number only when $n \geq 2$. Let $s = \sqrt{\lambda - \xi^2}$. Then as we tried with $I_{2,1}^+$, use (4.14) to get

$$
\begin{aligned}
&\|I_{2,2}^+\|^2_{L^4(\mathbb{R}\times[0,1]\times[0,T])} \\
&\quad\lesssim \left\| \int_{\xi^2+\frac{n^2}{2}}^{\xi^2+n^2} \frac{n e^{i\pi^2\left(\xi^2+n^2-\lambda\right)t}}{\xi^2+n^2-\lambda} \widetilde{h}(\xi,\lambda)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right) d\lambda \right\|^2_{H_t^\sigma\left([0,T];\, L^2_{\xi n}\right)} \\
&\quad\lesssim \int_{-\infty}^{\infty} \sum_n \int_{-\infty}^{\infty} \left| \frac{n}{(1+|\xi^2+n^2-\lambda|)^{1-\alpha}} \chi_{\left[\xi^2+\frac{n^2}{2},\, \xi^2+n^2-1\right)}(\lambda) \cdot \widetilde{h}(\xi,\lambda)\right|^2 d\lambda\, d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\eqsim \int_{-\infty}^{\infty} \sum_n \int_{-\infty}^{\infty} \left| \frac{n}{(1+n^2-s^2)^{1-\alpha}} \chi_{\left[\frac{n}{\sqrt{2}}, \sqrt{n^2-1}\right)}(s) \cdot \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi \\
&= \int_{-\infty}^{\infty} \sum_n \left(\int_{n-1}^{\sqrt{n^2-1}} + \int_{\frac{n}{\sqrt{2}}}^{n-1} \right) \frac{n^2}{(1+n^2-s^2)^{2-2\alpha}} \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi\,.
\end{aligned}
$$

We want to show that in the last line, the integral with respect to s over the interval $\left[\frac{n}{\sqrt{2}}, n-1\right]$ exists only for $n \geq 4$. First we see that any two elements in the sequence $\left\{\left[n-1, \sqrt{n^2-1}\right]\right\}_{n\in\mathbb{Z}}$ are disjoint. Thus

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{n\geq 2} \int_{n-1}^{\sqrt{n^2-1}} \frac{n^2}{(1+n^2-s^2)^{2-2\sigma}} \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi \\
&\qquad \leq \int_{-\infty}^{\infty} \sum_{n\geq 2} \int_{n-1}^{\sqrt{n^2-1}} n^2 \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi \\
&\qquad \leq \int_{-\infty}^{\infty} \sum_{n\geq 2} \int_{n-1}^{\sqrt{n^2-1}} s(1+s)^2 \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 ds\, d\xi \\
&\qquad \leq \int_{-\infty}^{\infty} \int_{1}^{\infty} s(1+s)^2 \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 ds\, d\xi \\
&\qquad \lesssim \int_{-\infty}^{\infty} \int_{1+\xi^2}^{\infty} \left(1+\lambda-\xi^2\right) \cdot \left| \widetilde{h}(\xi,\lambda) \right|^2 d\lambda\, d\xi \leq \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}\,.
\end{aligned}
$$

For $n \geq 4$,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{n\geq 4} \int_{\frac{n}{\sqrt{2}}}^{n-1} \frac{n^2}{(1+n^2-s^2)^{2-2\sigma}} \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi \\
&\qquad \leq \int_{-\infty}^{\infty} \sum_{n\geq 4} \int_{\frac{n}{\sqrt{2}}}^{n-1} \left(\frac{n}{n^2-s^2} \right)^{2-2\sigma} n^{2\sigma} \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 s\, ds\, d\xi \\
&\qquad \lesssim \int_{-\infty}^{\infty} \sum_{n\geq 4} \int_{\frac{n}{\sqrt{2}}}^{n-1} \frac{s^{2\sigma+1}}{(n-s)^{2-2\sigma}} \cdot \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 ds\, d\xi \\
&\qquad = \int_{-\infty}^{\infty} \int_{2\sqrt{2}}^{\infty} s^{2\sigma+1} \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 \left(\sum_{n=\lceil s+1 \rceil}^{\lfloor \sqrt{2}s \rfloor} \frac{1}{(n-s)^{2-2\sigma}} \right) ds\, d\xi \\
&\qquad \leq \int_{-\infty}^{\infty} \int_{2\sqrt{2}}^{\infty} s^{2\sigma+1} \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 \left(\int_{s+1}^{\sqrt{2}s} \frac{d\eta}{(\eta-s)^{2-2\sigma}} \right) ds\, d\xi \\
&\qquad = \int_{-\infty}^{\infty} \int_{2\sqrt{2}}^{\infty} s^{2\sigma+1} \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 \left(\left. \frac{(\eta-s)^{2\sigma-1}}{2\sigma-1} \right|_{s+1}^{\sqrt{2}s} \right) ds\, d\xi \\
&\qquad \lesssim \int_{-\infty}^{\infty} \int_{2\sqrt{2}}^{\infty} s^{4\sigma} \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 ds\, d\xi \\
&\qquad \eqsim \int_{-\infty}^{\infty} \int_{2\sqrt{2}}^{\infty} (s^2)^{2\sigma-\frac{1}{2}} \left| \widetilde{h}\left(\xi, s^2+\xi^2\right) \right|^2 ds^2\, d\xi
\end{aligned}
$$

$$= \int_{-\infty}^{\infty} \int_{2\sqrt{2}+\xi^2}^{\infty} \left(\lambda - \xi^2\right)^{2\sigma - \frac{1}{2}} \left|\widetilde{h}(\xi, \lambda)\right|^2 \, d\lambda \, d\xi \,.$$

Again by letting $\sigma = \frac{3}{4}$ we obtain the result

$$\|I^+_{2,2}\|_{L^4(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|h\|_{H_t^{\frac{1}{2}}([0,T];\, L^2_x)} \tag{4.52}$$

At last, we need to study $I^+_{2,3}$ in (4.50).

$$\begin{aligned} I^+_{2,3} &\approx \int_{-\infty}^{\infty} \int_0^{\infty} \widetilde{h}(\xi, \lambda) e^{-i\pi^2\lambda t + i\pi x\xi} \sum_{n=1}^{\infty} \left[n \sin(n\pi y) \cdot \chi_{\left[0, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] d\lambda \, d\xi \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \widetilde{h}(\xi, \lambda) e^{-i\pi^2\lambda t + i\pi x\xi} \left[\sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[\xi^2, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] d\lambda \, d\xi \\ &\quad + \int_{-\infty}^{\infty} \int_0^{\infty} \widetilde{h}(\xi, \lambda) e^{-i\pi^2\lambda t + i\pi x\xi} \chi_{[0,\xi^2]}(\lambda) \left[\sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] d\lambda \, d\xi \end{aligned}$$

Let

$$K_1 = \int_{-\infty}^{\infty} \int_0^{\infty} \widetilde{h}(\xi, \lambda) e^{-i\pi^2\lambda t + i\pi x\xi} \left[\sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[\xi^2, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] d\lambda \, d\xi \tag{4.53}$$

$$K_2 = \int_{-\infty}^{\infty} \int_0^{\infty} \widetilde{h}(\xi, \lambda) e^{-i\pi^2\lambda t + i\pi x\xi} \chi_{[0,\xi^2]}(\lambda) \left[\sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] d\lambda \, d\xi \tag{4.54}$$

We pick $\sigma \in \left(\frac{3}{4}, 1\right)$ again. Referring to the *Proposition 3.6* in [10], we have the inequality

$$\left| \sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[0, \frac{n^2}{2}\right]}(\mu) \frac{1 - \psi(n^2 - \mu)}{n^2 - \mu} \right| \lesssim \frac{|y|^{\sigma - 1}}{(1 + \sqrt{\mu})^{1 - \sigma}} \qquad 0 \le \sigma \le 1 \,.$$

Replace μ by $\lambda - \xi^2$ and then obtain that

$$\left| \sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[\xi^2, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right| \lesssim \frac{|y|^{\sigma - 1}}{\left(1 + \sqrt{\lambda - \xi^2}\right)^{1 - \sigma}} \qquad 0 \le \sigma \le 1 \tag{4.55}$$

Recall the Housedorff-Young inequality $\|\mathcal{F}[f]\|_{L^p} \lesssim \|f\|_{L^{p'}}$ for each $r \in [2, \infty]$ and $f \in L^{r'}$ where $r' = \frac{r}{r-1}$. Take $r = 4$. Then for (4.53),

$$\|K_1\|^4_{L^4_{xyt}} \lesssim$$

$$\int_0^1 \left\| \widetilde{h}(\xi, \lambda) \cdot \left[\sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[\xi^2, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1 - \psi\left(\xi^2 + n^2 - \lambda\right)}{\xi^2 + n^2 - \lambda} \right] \right\|^4_{L^{\frac{4}{3}}_{\xi\lambda}} dy$$

$$
\begin{aligned}
&\lesssim \int_0^1 \left\{ \int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left|\widetilde{h}(\xi,\lambda)\right|^{\frac{4}{3}} \cdot \left| \sum_{n=1}^{\infty} n \sin(n\pi y) \cdot \chi_{\left[\xi^2, \xi^2 + \frac{n^2}{2}\right]}(\lambda) \frac{1-\psi\left(\xi^2+n^2-\lambda\right)}{\xi^2+n^2-\lambda} \right|^{\frac{4}{3}} d\lambda\, d\xi \right\}^3 dy \\
&\overset{(4.55)}{\lesssim} \int_0^1 \left\{ \int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left|\widetilde{h}(\xi,\lambda)\right|^{\frac{4}{3}} \cdot \left(\frac{|y|^{\sigma-1}}{\left(1+\sqrt{\lambda-\xi^2}\right)^{1-\sigma}} \right)^{\frac{4}{3}} d\lambda\, d\xi \right\}^3 dy \\
&\le \int_0^1 \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left|\widetilde{h}(\xi,\lambda)\right|^2 (1+\lambda)\, d\lambda\, d\xi \right]^2 \cdot \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} (1+\lambda)^{-2} \frac{|y|^{4\sigma-4}}{\left(1+\sqrt{\lambda-\xi^2}\right)^{4-4\sigma}} d\lambda\, d\xi \right] dy \\
&\lesssim \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \left|\widetilde{h}(\xi,\lambda)\right|^2 (1+\lambda)\, d\lambda\, d\xi \right]^2 \cdot \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \frac{(1+\lambda)^{-2}}{\left(1+\sqrt{\lambda-\xi^2}\right)^{4-4\sigma}} d\lambda\, d\xi \right] \\
&\lesssim \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \cdot \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{\infty} \frac{1}{(1+\lambda-\xi^2)^{2-2\sigma}(1+\lambda)^2} d\lambda\, d\xi \right] \\
&\le \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \cdot \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{2\xi^2} \frac{1}{(1+\lambda-\xi^2)^{2-2\sigma}(1+\lambda)^2} d\lambda\, d\xi \right. \\
&\qquad \left. + \int_{-\infty}^{\infty} \int_{2\xi^2}^{\infty} \frac{1}{(1+\lambda-\xi^2)^{2-2\sigma}(1+\lambda)^2} d\lambda\, d\xi \right] \\
&\le \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \cdot \left[\int_{-\infty}^{\infty} \int_{\xi^2}^{2\xi^2} \frac{1}{(1+\xi^2)^2} d\lambda\, d\xi + \int_{-\infty}^{\infty} \int_{2\xi^2}^{\infty} \frac{1}{(1+\xi^2)^{2-2\sigma}(1+\lambda)^2} d\lambda\, d\xi \right] \\
&\le \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \cdot \left[\tan^{-1}(\xi)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{2-2\sigma}} \int_{2\xi^2}^{\infty} \frac{1}{(1+\lambda)^2} d\lambda\, d\xi \right] \\
&\lesssim \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)} \cdot \left[1 + \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{3-2\sigma}} d\xi \right] \lesssim \|h\|^4_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}.
\end{aligned}
$$

Here we adopt the idea in the proof of *Proposition 3.6* in [10], again. Let $S_k = \sum_{n=1}^{k} \sin(n\pi y)$. Thus, $|S_k| \le |y|^{\gamma-1} k^{\gamma}$ for $0 \le \gamma < 1$. Then consider $0 < \lambda \le \xi^2$ and $\mu = \xi^2 - \lambda > 0$. Thus

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n \sin(n\pi y)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda} \\
&= \sum_{n=1}^{\infty} \frac{n \sin(n\pi y)\left(1-\psi(n^2+\mu)\right)}{n^2+\mu} \\
&= \lim_{k\to\infty} \sum_{n=2}^{k} \sin(n\pi y) \cdot \frac{n}{n^2+\mu} + \sin(\pi y) \frac{1-\psi(1+\mu)}{1+\mu} \\
&= \lim_{k\to\infty} \left[\sum_{n=2}^{k} \frac{n}{n^2+\mu} (S_n - S_{n-1}) \right] + \sin(\pi y) \frac{1-\psi(1+\mu)}{1+\mu} \\
&= \lim_{k\to\infty} \left[\sum_{n=2}^{k} \frac{n}{n^2+\mu} S_n - \sum_{n=2}^{k} \frac{n}{n^2+\mu} S_{n-1} \right] + \sin(\pi y) \frac{1-\psi(1+\mu)}{1+\mu}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\left[\sum_{n=2}^{k}\frac{n}{n^2+\mu}S_n - \sum_{n=1}^{k-1}\frac{n+1}{(n+1)^2+\mu}S_n\right] + \sin(\pi y)\frac{1-\psi(1+\mu)}{1+\mu} \\
&= \lim_{k\to\infty}\left[\sum_{n=2}^{k-1}\left(\frac{n}{n^2+\mu} - \frac{n+1}{(n+1)^2+\mu}\right)S_n + \frac{k}{k^2+\mu}S_k - \frac{2}{4+\mu}S_1\right] + \sin(\pi y)\frac{1-\psi(1+\mu)}{1+\mu}\,.
\end{aligned}
$$

Therefore the absolute value satisfies

$$
\begin{aligned}
&\left|\sum_{n=1}^{\infty}\frac{n\sin(n\pi y)\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\right| \\
&\lesssim |y|^{\gamma-1}\left(\sum_{n=2}^{\infty}\frac{|n^2+n-\mu|n^\gamma}{(n^2+\mu)^2}\right) + \frac{1}{1+\mu} \\
&\lesssim |y|^{\gamma-1}\left(\sum_{n=2}^{\infty}\frac{1}{(n^2+\mu)^{1-\frac{\gamma}{2}}}\right) + \frac{1}{1+\mu} \\
&\lesssim |y|^{\gamma-1}\left(\sum_{n=2}^{\infty}\frac{1}{(n+\sqrt{\mu})^{2-\gamma}}\right) + \frac{1}{1+\mu} \\
&\le |y|^{\gamma-1}\left(\int_1^{\infty}\frac{d\eta}{(\eta+\sqrt{\mu})^{2-\gamma}}\right) + \frac{1}{1+\mu} \\
&\eqsim |y|^{\gamma-1}\left(\frac{1}{(1+\sqrt{\mu})^{1-\gamma}}\right) + \frac{1}{1+\mu} \\
&\lesssim \frac{|y|^{\gamma-1}}{(1+\sqrt{\mu})^{1-\gamma}} = \frac{|y|^{\gamma-1}}{\left(1+\sqrt{\xi^2-\lambda}\right)^{1-\gamma}}\,.
\end{aligned}
\tag{4.56}
$$

With (4.54) and (4.56) we know that for each fixed γ as long as $\frac{3}{4} < \gamma < 1$,

$$
\begin{aligned}
\|K_2\|_{L^4_{xyt}}^4 &= \int_0^1 \|K_2\|_{L^4_{xt}}^4\,dy \\
&\eqsim \int_0^1\left(\int_{-\infty}^{\infty}\int_0^{\xi^2}\left|\sum_{n=1}^{\infty}\frac{n\sin(n\pi y)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\widetilde{h}(\xi,\lambda)\right|^{\frac{4}{3}}d\lambda\,d\xi\right)^{\frac{3}{4}\cdot 4} dy \\
&\lesssim \int_0^1\left(\int_{-\infty}^{\infty}\int_0^{\xi^2}\left|\frac{|y|^{\gamma-1}}{\left(1+\sqrt{\xi^2-\lambda}\right)^{1-\gamma}}\widetilde{h}(\xi,\lambda)\right|^{\frac{4}{3}}d\lambda\,d\xi\right)^{3} dy \\
&\lesssim \left(\int_{-\infty}^{\infty}\int_0^{\xi^2}\left|\widetilde{h}(\xi,\lambda)\right|^2\left(1+\xi^2\right)^{\sigma}\,d\lambda\,d\xi\right)^2\cdot\int_{-\infty}^{\infty}\int_0^{\xi^2}\frac{1}{\left(1+\xi^2\right)^{2\sigma}\left(1+\xi^2-\lambda\right)^{2-2\gamma}}\,d\lambda\,d\xi \\
&\eqsim \|h\|_{L_t^2([0,T];\,H_x^\sigma)}^4\cdot\int_{-\infty}^{\infty}\frac{1}{\left(1+\xi^2\right)^{2\sigma}}\int_0^{\xi^2}\frac{1}{\left(1+\xi^2-\lambda\right)^{2-2\gamma}}\,d\lambda\,d\xi \\
&= \|h\|_{L_t^2([0,T];\,H_x^\sigma)}^4\cdot\int_{-\infty}^{\infty}\frac{1}{\left(1+\xi^2\right)^{2\sigma}}\int_1^{\xi^2+1}\frac{1}{\mu^{2-2\gamma}}\,d\mu\,d\xi
\end{aligned}
$$

$$\lesssim \|h\|^4_{L^2_t([0,T];\,H^\sigma_x)} \cdot \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{2\sigma}} \cdot \frac{1}{(1+\xi^2)^{1-2\gamma}(2\gamma-1)}\, d\xi$$
$$\lesssim \|h\|^4_{L^2_t([0,T];\,H^\sigma_x)} \cdot \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{2\sigma-2\gamma+1}}\, d\xi \lesssim \|h\|^4_{L^2_t([0,T];\,H^\sigma_x)}$$

if and only if $2\sigma - 2\gamma + 1 > \frac{1}{2}$, i.e. $\sigma > \gamma - \frac{1}{4} > \frac{1}{2}$. Hence,

$$\|I^+_{2,3}\|_{L^4(\mathbb{R}\times[0,1]\times[0,T])} \lesssim \|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x)\bigcap L^2_t([0,T];\,H^\sigma_x)} \tag{4.57}$$

Adding (4.47), (4.51), (4.52) and (4.57), the estimate for the $L^4 \bigcap L^\infty(L^2)$-norm of $I^+_{2,3}$ is obtained:

$$\|I^+_2\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim \|h\|_{H_t^{\frac{1}{2}}([0,T];\,L^2_x)\bigcap L^2_t([0,T];\,H_x^{\frac{1}{2}+\epsilon})} \tag{4.58}$$

for any $\epsilon > 0$. Now we can study I^+_3 in (4.38). By *Proposition 4.3*

$$\|I^+_3\|^2_{L^4(\mathbb{R}\times[0,1]\times[0,T])\,\bigcap\, L^\infty_t([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1]))}$$
$$\lesssim \int_{-\infty}^{\infty} \sum_n \left| n \int_0^\infty \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi$$
$$\le \int_{-\infty}^{\infty} \sum_n \left| n \int_0^{\xi^2} \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi$$
$$+ \int_{-\infty}^{\infty} \sum_n \left| n \int_{\xi^2}^\infty \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi = L_1 + L_2$$

that is

$$L_1 = \int_{-\infty}^{\infty} \sum_n \left| n \int_0^{\xi^2} \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi \tag{4.59}$$
$$L_2 = \int_{-\infty}^{\infty} \sum_n \left| n \int_{\xi^2}^\infty \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi \tag{4.60}$$

For (4.59), we let $\mu = \xi^2 - \lambda$ so that

$$L_1 = \int_{-\infty}^{\infty} \sum_n \left| n \int_0^{\xi^2} \frac{\widetilde{h}(\xi,\lambda)\cdot\left(1-\psi\left(\xi^2+n^2-\lambda\right)\right)}{\xi^2+n^2-\lambda}\, d\lambda \right|^2 d\xi$$
$$\eqsim \int_{-\infty}^{\infty} \sum_n \left| n \int_0^{\xi^2} \frac{\widetilde{h}\left(\xi,\xi^2-\mu\right)\cdot\left(1-\psi\left(n^2+\mu\right)\right)}{n^2+\mu}\, d\mu \right|^2 d\xi$$
$$\le \int_{-\infty}^{\infty} \sum_n \left[\int_0^{\xi^2} \left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2 d\mu \cdot \int_0^{\xi^2} \frac{n^2}{(n^2+\mu)^2}\, d\mu \right] d\xi$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{\infty}\left[\int_0^{\xi^2}\left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2 d\mu\cdot\int_0^{\xi^2}\sum_n\frac{n^2}{\left(n^2+\mu\right)^2}\,d\mu\right]d\xi\\
&\leq \int_{-\infty}^{\infty}\left[\int_0^{\xi^2}\left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2 d\mu\cdot\int_0^{\xi^2}\sum_n\frac{1}{n^2+\mu}\,d\mu\right]d\xi\\
&\lesssim \int_{-\infty}^{\infty}\left[\int_0^{\xi^2}\left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2 \mu\,d\mu\cdot\int_0^{\xi^2}\int_0^{\infty}\frac{1}{\eta^2+\mu}\,d\eta\,d\mu\right]d\xi\\
&= \int_{-\infty}^{\infty}\left[\int_0^{\xi^2}\left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2 \mu\,d\mu\cdot\int_0^{\xi^2}\frac{1}{\sqrt{\mu}}\int_0^{\infty}\frac{1}{\eta^2+1}\,d\eta\,d\mu\right]d\xi\\
&= \int_{-\infty}^{\infty}\int_0^{\xi^2}|\xi|\left|\widetilde{h}\left(\xi,\xi^2-\mu\right)\right|^2\mu\,d\mu\,d\xi \approx \int_{-\infty}^{\infty}\int_0^{\xi^2}|\xi|\left|\widetilde{h}\left(\xi,\lambda\right)\right|^2 d\lambda\,d\xi \leq \|h\|^2_{L^2_t\left([0,T];\,\dot{H}^{\frac12}_x\right)}.
\end{aligned}
$$

Also let $\nu=\sqrt{\lambda-\xi^2}$ so that we apply *Lemma 2.10* to this argument.

$$
\begin{aligned}
L_2 &= 2\int_{-\infty}^{\infty}\sum_n\left|\int_0^{\infty}\frac{n\nu\widetilde{h}\left(\xi,\xi^2+\nu^2\right)\cdot\left(1-\psi\left(n^2-\nu^2\right)\right)}{n^2-\nu^2}\,d\nu\right|^2 d\xi\\
&\leq 2\int_{-\infty}^{\infty}\sum_n\left|\int_0^{\infty}\frac{\nu\widetilde{h}\left(\xi,\xi^2+\nu^2\right)\cdot\left(1-\psi\left(n^2-\nu^2\right)\right)}{n-\nu}\,d\nu\right|^2 d\xi\\
&\lesssim \int_{-\infty}^{\infty}\int_0^{\infty}(1+\nu)\nu^2\left|\widetilde{h}\left(\xi,\xi^2+\nu^2\right)\right|^2 d\nu\,d\xi\\
&\lesssim \int_{-\infty}^{\infty}\int_{\xi^2}^{\infty}\left(1+\sqrt{\lambda}\right)\sqrt{\lambda}\left|\widetilde{h}(\xi,\lambda)\right|^2 d\lambda\,d\xi \lesssim \|h\|^2_{H_t^{\frac12}([0,T];\,L^2_x)}.
\end{aligned}
$$

Hence

$$
\|I_3^+\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L^\infty_t\left([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim \|h\|_{H_t^{\frac12}([0,T];\,L^2_x)\bigcap L^2_t\left([0,T];\,H_x^{\frac12}\right)} \tag{4.61}
$$

The next part to work on is the estimate for I^-. First we look at I_1^- in (4.39).

$$
\begin{aligned}
I_1^- &= \int_{-\infty}^{\infty}\sum_n\left(e^{i\pi(x\xi+ny)}\cdot n\int_0^{\infty}\frac{e^{i\pi^2\lambda t}}{\xi^2+n^2+\lambda}\widetilde{h}(\xi,-\lambda)\,d\lambda\right)d\xi\\
&\eqsim \int_{-\infty}^{\infty}\int_0^{\infty}e^{i\pi x\xi+i\pi^2\lambda t}\widetilde{h}(\xi,-\lambda)\sum_{n=1}^{\infty}\frac{n\sin(n\pi y)}{\xi^2+n^2+\lambda}\,d\lambda\,d\xi
\end{aligned}
$$

Recall (4.56) and it can be realized that if we replace $-\lambda$ by λ then

$$
\left|\sum_{n=1}^{\infty}\frac{n\sin(n\pi y)}{\xi^2+n^2+\lambda}\right| \lesssim \frac{|y|^{\sigma-1}}{\left(1+\sqrt{\xi^2+\lambda}\right)^{1-\sigma}} \tag{4.62}
$$

which works for any $\lambda>0$ and $\sigma\in[1/2,1]$. Thus

$$
\|I_1^-\|^2_{L^\infty_t([0,T];\,L^2_{xy}(\mathbb{R}\times[0,1]))} = \sup_{t\in[0,T]}\left\|I_1^-(t)\right\|^2_{L^2_{xy}}
$$

$$
\begin{aligned}
&\lesssim \sup_{t\in[0,T]} \int_0^1 \int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{i\pi^2\lambda t} \widetilde{h}(\xi,-\lambda) \sum_{n=1}^{\infty} \frac{n\sin(n\pi y)}{\xi^2+n^2+\lambda}\, d\lambda \right|^2 d\xi\, dy \\
&\le \int_0^1 \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right| \cdot \left| \sum_{n=1}^{\infty} \frac{n\sin(n\pi y)}{\xi^2+n^2+\lambda} \right| d\lambda \right)^2 d\xi\, dy \\
&\lesssim \int_0^1 \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right| \cdot \left| \frac{|y|^{\sigma-1}}{\left(1+\sqrt{\xi^2+\lambda}\right)^{1-\sigma}} \right| d\lambda \right)^2 d\xi\, dy \\
&\eqsim \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right| \cdot \left| \frac{1}{\left(1+\sqrt{\xi^2+\lambda}\right)^{1-\sigma}} \right| d\lambda \right)^2 d\xi \\
&\le \int_{-\infty}^{\infty} \left(\int_0^{\infty} (1+|\lambda|) \left|\widetilde{h}(\xi,-\lambda)\right|^2 d\lambda \right) \cdot \left(\int_0^{\infty} \frac{1}{(1+|\lambda|)\left(1+\xi^2+\lambda\right)^{1-\sigma}}\, d\lambda \right) d\xi \\
&\le \int_{-\infty}^{\infty} \left(\int_0^{\infty} (1+|\lambda|) \left|\widetilde{h}(\xi,-\lambda)\right|^2 d\lambda \right) \cdot \left(\int_0^{\infty} \frac{1}{(1+\lambda)^{2-\sigma}}\, d\lambda \right) d\xi \\
&\lesssim \int_{-\infty}^{\infty} \left(\int_0^{\infty} (1+|\lambda|) \left|\widetilde{h}(\xi,-\lambda)\right|^2 d\lambda \right) d\xi \le \|h\|^2_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}
\end{aligned}
$$

We continue to work on the L^4-norm by using the same technique for K_2 with the assistance from (4.62). Again we only choose $\sigma \in \left(\frac{3}{4}, 1\right)$.

$$
\begin{aligned}
\|I_1^-\|^4_{L^4(\mathbb{R}\times[0,1]\times[0,T])} &= \int_0^1 \|I_1^-\|^4_{L^4_{xt}}\, dy \\
&\lesssim \int_0^1 \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right|^{\frac{4}{3}} \cdot \left| \sum_{n=1}^{\infty} \frac{n\sin(n\pi y)}{\xi^2+n^2+\lambda} \right|^{\frac{4}{3}} d\lambda\, d\xi \right)^3 dy \\
&\lesssim \int_0^1 \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right|^{\frac{4}{3}} \cdot \left| \frac{|y|^{\sigma-1}}{\left(1+\sqrt{\xi^2+\lambda}\right)^{1-\sigma}} \right|^{\frac{4}{3}} d\lambda\, d\xi \right)^3 dy \\
&\lesssim \left(\int_{-\infty}^{\infty} \int_0^{\infty} \left|\widetilde{h}(\xi,-\lambda)\right|^{\frac{4}{3}} \cdot \left| \frac{1}{\left(1+\sqrt{\xi^2+\lambda}\right)^{1-\sigma}} \right|^{\frac{4}{3}} d\lambda\, d\xi \right)^3 \\
&\le \int_{-\infty}^{\infty} \left(\int_0^{\infty} (1+|\lambda|)^{\frac{1}{2}} \left|\widetilde{h}(\xi,-\lambda)\right|^2 d\lambda \right)^2 \cdot \left(\int_0^{\infty} \frac{1}{(1+|\lambda|)\left(1+\xi^2+\lambda\right)^{2-2\sigma}}\, d\lambda \right) d\xi \\
&\le \int_{-\infty}^{\infty} \left(\int_0^{\infty} (1+|\lambda|)^{\frac{1}{2}} \left|\widetilde{h}(\xi,-\lambda)\right|^2 d\lambda \right)^2 \cdot \left(\int_0^{\infty} \frac{1}{(1+\lambda)^{3-2\sigma}}\, d\lambda \right) d\xi \lesssim \|h\|^4_{H_t^{\frac{1}{4}}([0,T];\, L_x^2)}
\end{aligned}
$$

which completes the conclusion that

$$
\|I_1^-\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T]) \bigcap L_t^{\infty}\left([0,T];\, L^2_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim \|h\|_{H_t^{\frac{1}{2}}([0,T];\, L_x^2)}\,. \tag{4.63}
$$

Now we can put our energy on studying I_2^- in (4.40) and then the whole argument can be finished. By *Proposition 4.3*

$$
\begin{aligned}
&\|I_2^-\|^2_{L^4(\mathbb{R}\times[0,1]\times[0,T])\bigcap L_t^\infty([0,T];L^2_{xy}(\mathbb{R}\times[0,1]))} \\
&\quad\leq \left\|\int_{-\infty}^{\infty}\sum_n\left(e^{-i\pi^2\left(\xi^2+n^2\right)t+i\pi(x\xi+ny)}\int_0^\infty \frac{n\widetilde{h}(\xi,-\lambda)}{\xi^2+n^2+\lambda}\,d\lambda\right)\,d\xi\right\|^2_{L^4\bigcap L_t^\infty\left([0,T];L^2_{xy}\right)} \\
&\quad\lesssim \int_{-\infty}^{\infty}\sum_n\left|\int_0^\infty \frac{n\widetilde{h}(\xi,-\lambda)}{\xi^2+n^2+\lambda}\,d\lambda\right|^2\,d\xi \\
&\quad\leq \int_{-\infty}^{\infty}\sum_n\left(\int_0^\infty(1+|\lambda|)^{2\gamma}\left|\widetilde{h}(\xi,-\lambda)\right|^2\,d\lambda\right)\cdot\left(\int_0^\infty\frac{n^2}{(1+|\lambda|)^{2\gamma}(\xi^2+n^2+\lambda)^2}\,d\lambda\right)\,d\xi \\
&\quad\leq \int_{-\infty}^{\infty}\sum_n\left(\int_0^\infty(1+|\lambda|)^{2\gamma}\left|\widetilde{h}(\xi,-\lambda)\right|^2\,d\lambda\right)\cdot\left(\int_0^\infty\frac{n^2}{(1+|\lambda|)^{2\gamma}(n^2+\lambda)^2}\,d\lambda\right)\,d\xi \\
&\quad\leq \|h\|_{H_t^\gamma([0,T];\,L^2_x)}\cdot\left(\sum_n\int_0^\infty\frac{n^2}{(1+|\lambda|)^{2\gamma}(n^2+\lambda)^2}\,d\lambda\right)
\end{aligned}
$$

We let $\beta>0$ and $\gamma>\frac{\beta}{2}+\frac{1}{4}>\frac{1}{4}$ and look into the second factor.

$$
\begin{aligned}
\sum_n\int_0^\infty\frac{n^2}{(1+|\lambda|)^{2\gamma}(n^2+\lambda)^2}\,d\lambda &\approx \int_0^\infty\sum_{n=1}^\infty\frac{n^2}{(1+|\lambda|)^{2\gamma}(n^2+\lambda)^2}\,d\lambda \\
&\leq \int_0^\infty\sum_{n=1}^\infty\frac{n^2}{(n^2+\lambda)^{\frac{3}{2}+\beta}}\cdot\frac{1}{(1+|\lambda|)^{2\gamma}(n^2+\lambda)^{\frac{1}{2}-\beta}}\,d\lambda \\
&\leq \int_0^\infty\sum_{n=1}^\infty\frac{1}{n^{1+2\beta}}\cdot\frac{1}{(1+|\lambda|)^{2\gamma+\frac{1}{2}-\beta}}\,d\lambda \\
&\leq \sum_{n=1}^\infty\frac{1}{n^{1+2\beta}}\cdot\int_0^\infty\frac{1}{(1+|\lambda|)^{2\gamma+\frac{1}{2}-\beta}}\,d\lambda\leq C
\end{aligned}
$$

Therefore

$$
\|I_2^-\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L_t^\infty\left([0,T];L^2_{xy}(\mathbb{R}\times[0,1])\right)}\lesssim\|h\|_{H_t^\gamma([0,T];\,L^2_x)}\quad\forall\,\gamma>\frac{1}{4}\tag{4.64}
$$

Finally combine (4.41), (4.58), (4.61), (4.63) and (4.64) to attain the expected estimate (4.32)

$$
\|W_bh\|_{L^4_{xyt}(\mathbb{R}\times[0,1]\times[0,T])\bigcap L_t^\infty\left([0,T];L^2_{xy}(\mathbb{R}\times[0,1])\right)}\lesssim\|h\|_{H_t^{\frac{1}{2}}([0,T];L^2_x(\mathbb{R}))\bigcap L_t^2\left([0,T];H_x^{\epsilon+\frac{1}{2}}(\mathbb{R})\right)}
$$

for any $\epsilon>0$.

On the next stage, we take derivatives into account. Let $\alpha=(\alpha_1,\alpha_2)$ be a nonnegative two-dimensional multi-index with $|\alpha|=\alpha_1+\alpha_2=s$ again. We also let u solve (4.22) and claim that for $0<\epsilon\leq\frac{1}{2}$,

$$
\left\|\partial_y^{2s}u\right\|_X\lesssim\sum_{j=1,2}\|(\lambda+|\xi|^{\epsilon+\frac{1}{2}})(\lambda+|\xi|^2)^s\widehat{h_j}\|_{L^2_{\xi\lambda}}\tag{4.65}
$$

where $X = L^r \bigcap L_t^\infty\left(L^2_{xy}\right)$ for $r \in [2,4]$. To begin the proof of this claim, we observe by (4.22) that $\partial_t u = i\left(\partial_x^2 + \partial_y^2\right)u$. Then check the equality for $s = 1$, 2,

$$\left\|\partial_y^2 u\right\|_X \le \left\|\left(\partial_x^2 + \partial_y^2\right)u\right\|_X + \left\|\partial_x^2 u\right\|_X = \|\partial_t u\|_{L^q} + \left\|\partial_x^2 u\right\|_X$$

and

$$\begin{aligned}\left\|\partial_y^4 u\right\|_X &= \left\|\partial_y^2\left(\partial_y^2 u\right)\right\|_X \le \left\|\partial_t\left(\partial_y^2 u\right)\right\|_{L^q} + \left\|\partial_x^2\left(\partial_y^2 u\right)\right\|_X \\ &= \left\|\partial_y^2\left(\partial_t u\right)\right\|_{L^q} + \left\|\partial_y^2\left(\partial_x^2 u\right)\right\|_X \\ &\le \left\|\left(\partial_t^2 u\right)\right\|_X + 2\left\|\partial_t\partial_x^2 u\right\|_X + \left\|\left(\partial_x^4 u\right)\right\|_X\end{aligned}$$

Then the pattern of the estimate as s rises can be discovered for $m \in \mathbb{N}$ by induction as follows.

$$\left\|\partial_y^{2m} u\right\|_X \le \sum_{k=0}^{m}\binom{m}{k}\left\|\partial_t^k\partial_x^{2(m-k)}u\right\|_X$$

For $k \ge 1$, we let $v = \partial_t^k\partial_x^{2(m-k)}u$, and then v solves (4.22) with the boundary data replaced as follows.

$$\begin{cases} iv_t + v_{xx} + v_{yy} = 0 \\ v(x,y,0) = (i)^k\left[(\partial_x^2 + \partial_y^2)^k u\right](x,y,0) = 0 \\ v(x,0,t) = \partial_t^k\partial_x^{2(m-k)}h_1(x,t)\,, \qquad v(x,1,t) = \partial_t^k\partial_x^{2(m-k)}h_2(x,t)\end{cases}$$

Thus by (4.32) for any $\epsilon > 0$

$$\|\partial_t^k\partial_x^{2(m-k)}u\|_X = \|v\|_X \lesssim \sum_{j=1,2}\|\partial_t^k\partial_x^{2(m-k)}h_j\|_{H_t^{\frac{1}{2}}([0,T];L_x^2)\bigcap L_t^2\left([0,T];H^{\epsilon+\frac{1}{2}}\right)}$$

On the other hand, with the formula (4.24), it is easy to verify that

$$\|\partial_x^{2m}v\|_X \lesssim \sum_{j=1,2}\|\partial_x^{2m}h_j\|_{H_t^{\frac{1}{2}}([0,T];L_x^2)\bigcap L_t^2\left([0,T];H^{\epsilon+\frac{1}{2}}\right)}$$

Therefore for $\left\|\partial_y^{2m}u\right\|_X$, we obtain

$$\begin{aligned}\|\partial_y^{2m}u\|_X &\lesssim \sum_{j=1,2}\sum_{k=0}^{m}\binom{m}{k}\|\partial_t^k\partial_x^{2(m-k)}h_j\|_{H_t^{\frac{1}{2}}([0,T];L_x^2)\bigcap L_t^2\left([0,T];H^{\epsilon+\frac{1}{2}}\right)} \\ &\eqsim \sum_{j=1,2}\|(|\lambda| + |\xi|^{2\epsilon+1})^{\frac{1}{2}}(|\lambda| + |\xi|^2)^m\widehat{h_j}\|_{L^2_{\xi\lambda}}\end{aligned}$$

By interpolation, we have proved (4.65).

Since α_1, $\alpha_2 \ge 0$ such that $\alpha_1 + \alpha_2 = s$,

$$\left\|\partial_x^{\alpha_1}\partial_y^{\alpha_2}u\right\|_X \lesssim \sum_{j=1,2}\left\|(|\lambda| + |\xi|^{2\epsilon+1})^{\frac{1}{2}}(|\lambda| + |\xi|^2)^{\frac{\alpha_2}{2}}|\xi|^{\alpha_1}\widehat{h_j}\right\|_{L^2_{\xi\lambda}}$$

$$\lesssim \sum_{j=1,2} \left\| (|\lambda| + |\xi|^{2\epsilon+1})^{\frac{1}{2}} (|\lambda| + |\xi|^2)^{\frac{s}{2}} \widehat{h_j} \right\|_{L^2_{\xi\lambda}}$$
$$\lesssim \sum_{j=1,2} \left\| (|\lambda| + |\xi|^2)^{\frac{s+1}{2}} \widehat{h_j} \right\|_{L^2_{\xi\lambda}} .$$

Hence, it is clear to see that (4.33) is derived.

As the final part of this proof, we attempt to reach a more general conclusion with the function space. In fact by using (4.33) we know that if $T < \infty$,

$$L^2_t \left([0,T];\, H^{s,2}_{xy}(\mathbb{R} \times [0,1]) \right) \hookrightarrow L^\infty_t \left([0,T];\, L^2_{xy}(\mathbb{R} \times [0,1]) \right).$$

Then

$$\|W_b(h_1,h_2)\|_{L^2_t\left([0,T];\, H^{s,2}_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim T^{\frac{1}{2}} \sum_{j=1,2} \|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t\left([0,T];\, H^{s+1}_x(\mathbb{R})\right)} \tag{4.66}$$

By interpolation from L^2 to L^4 we can get (4.34) and therefore the whole proof is complete. □

Remark 4.10. *(i) In fact we have*

$$\|W_b(h_1,h_2)\|_{L^r_t\left([0,T];\, H^{s,r}_{xy}(\mathbb{R}\times[0,1])\right) \bigcap L^\infty_t\left([0,T];\, L^2_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim \sum_{j=1,2} \left\| (\lambda + |\xi|^{2\epsilon+1})^{\frac{1}{2}} (\lambda + |\xi|^2)^{\frac{s}{2}} \widehat{h_j} \right\|_{L^2_{\xi\lambda}}$$

as a weaker condition on $h_j(x,t)$, $j = 1, 2$. However, again, in order to present the result in a succinct expression, we take (4.34) which is slightly stronger.

(ii) The conclusion of the above proposition seems a little weaker for the regularity with respect to t compared to the problem posed on the half plane. However it can be shown that the boundary condition with $h_j \in H_t^{\frac{1}{2}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t([0,T];\, H^1_x(\mathbb{R}))$ is sharp.

4.3 Local Well-posedness

Now we are prepared for the investigation on the local well-posedness for equation (4.1) with estimates established in the previous section. Recall the mild solution of (4.1) given by (4.5) and define the operator $\mathcal{A}[u](x,y,t)$:

$$u(x,y,t) = W_b(h_1,h_2)(x,y,t) + W_0(t)\phi(x,y) + i \left(\int_0^t W_0(t-\tau) f(\tau)\, d\tau \right)(x,y) = \mathcal{A}[u](x,y,t). \tag{4.67}$$

where $f(u) = \lambda |u|^{p-2} u$ for $p \geq 3$ and $(x,y,t) \in \mathbb{R} \times [0,1] \times (0,T)$. We want to show that there is a unique maximal solution in $H^s(\mathbb{R} \times [0,1]$ continuously depending on the suitable initial and boundary conditions; that is, the operator $\mathcal{A}[u](x,y,t)$ has a fixed point. Note that we assume $\varphi|_{y=0} = h_j|_{t=0} = 0$ for $j = 1, 2$ and $\forall\, x \in \mathbb{R}$. Based on the same strategy in *Section 3.3.1*, we study the well-posedness. Let $r \in [2,4]$ and define the following function spaces.

$$\mathcal{X}_T^s := C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times[0,1])\right)\bigcap L_t^r\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)$$

with $\|u\|_{\mathcal{X}_T^s} = \sup\limits_{t\in[0,T]} \|u(t)\|_{H_{xy}^s(\mathbb{R}\times[0,1])} + \|u\|_{L_t^r\left([0,T];\, W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)}$. Let

$$\mathcal{Y}_T^s := C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times[0,1])\right)$$

with $\|u\|_{\mathcal{Y}_T^s} = \sup\limits_{t\in[0,T]} \|u(t)\|_{H_{xy}^s(\mathbb{R}\times[0,1])}$. For some $M>0$, define the closed balls in $\mathcal{X}_T$ and $\mathcal{Y}_T$ of radius M as

$$B_M^{\mathcal{X}^s} := \left\{u:\, \|u\|_{\mathcal{X}_T^s} \le M\right\} \quad \text{and} \quad B_M^{\mathcal{Y}^s} := \left\{u:\, \|u\|_{\mathcal{Y}_T^s} \le M\right\}$$

4.3.1 Existence and Uniqueness

It is aimed to prove the theorem as follows.

Theorem 4.11. *Choose $r \in [2,4]$. Let $\mu > 0$ such that*

$$\|\varphi\|_{H^s(\mathbb{R}\times[0,1])} + \sum_{j=1,2} \|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} \le \mu$$

and φ, h satisfy certain compatibility conditions. Then

(a) *For $0 \le s < 1/2$ and $3 \le p \le 4$, or $1/2 \le s < 1$ and $3 \le p \le \frac{3-2s}{1-s}$, or $s = 1$ and $3 \le p < \infty$, $\exists\, T > 0$ such that with $r \in [2,4]$ there must be a unique generalized solution $u \in \mathcal{X}_T^s$ of the integral equation (4.67). Besides,*

$$\|\mathcal{A}[u] - \mathcal{A}(v)\|_{\mathcal{X}_T^s} \le (1/2)\,\|u-v\|_{\mathcal{X}_T^s} \tag{4.68}$$

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \le M \tag{4.69}$$

for any u and $v \in B_M^{\mathcal{X}^s}$.

(b) *we assume that $p \ge s+1$ if $s \in \mathbb{Z}$ or $p \ge [s]+2$ if $s \notin \mathbb{Z}$ only when p is not an even integer. There exists a $T>0$ such that the integral equation (4.67) has a unique solution $u \in \mathcal{Y}_T^s = C_t\left([0,T];\, H_{xy}^s(\mathbb{R}\times[0,1])\right)$. Also, for u and $v \in B_M^{\mathcal{Y}^s}$*

$$\|\mathcal{A}[u] - \mathcal{A}[v]\|_{\mathcal{Y}_T^0} \le (1/2)\,\|u-v\|_{\mathcal{Y}_T^0}\,, \tag{4.70}$$

$$\|\mathcal{A}[u]\|_{\mathcal{Y}_T^s} \le M \tag{4.71}$$

In addition if we further assume $p \geq s+2$ with $s \in \mathbb{Z}$ or $p \geq [s]+3$ with $s \notin \mathbb{Z}$ or p is an even integer, then (4.70) can be improved by

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Y}_T^s} \leq (1/2)\,\|u-v\|_{\mathcal{Y}_T^s}\,, \tag{4.72}$$

for u and $v \in B_M^{\mathcal{Y}^s}$.

Proof. The case for $0 \leq s < 1/2$ comes to our attention first. We start with assuming

$$3 \leq p \leq r \leq 4\,, \tag{4.73}$$

which implies that

$$1+\frac{s(p-2)}{2} \geq \frac{p}{r} \quad \text{and equivalently} \quad \frac{r(p-2)}{r-2} \leq \frac{2r}{2-rs}\,.$$

In addition,

$$r \geq r'(p-1) = \frac{r(p-1)}{r-1}$$

Then by Sobolev embedding theorem *Lemma 2.3*, $W^{s,r} \hookrightarrow L^{\frac{r(p-2)}{r-2}}$ and $\|u(t)\|_{L^{\frac{r(p-2)}{r-2}}} \leq \|D^\alpha u(t)\|_{L^r}$.

If we choose $q = r' = \frac{r}{r-1}$, then $q \in \left(\frac{4}{3}, 2\right]$. Let $u \in B_M^{\mathcal{X}}$ and α be a multi-index such that $|\alpha| = s$ as usual. We know that $u : \mathbb{R}^N \to \mathbb{C}$ and $f \in C^1(\mathbb{C})$ and $|f'(u)| \lesssim |u|^{p-2} < \infty$ for $p \geq 3$. For $0 \leq s \leq 1/2$ and $r \in (2,4]$, the chain rule (2.18) suggests that

$$\begin{aligned}\|D^\alpha f(u)(t)\|_{L^{r'}} &\lesssim \|f'(u)(t)\|_{L^{\frac{r}{r-2}}}\,\|D^\alpha u(t)\|_{L^r}\\ &\leq \|u(t)\|^{p-2}_{L^{\frac{r(p-2)}{r-2}}}\|D^\alpha u(t)\|_{L^r} \leq \|D^\alpha u(t)\|^{p-1}_{L^r}\,,\end{aligned}$$

Next, we take the $L^{r'}$-norm of both sides of the inequality above in time t to obtain

$$\begin{aligned}\|D^\alpha f(u)\|_{L_t^{r'}\left([0,T];L_{xy}^{r'}(\mathbb{R}\times[0,1])\right)} &\lesssim \|D^\alpha u\|^{p-1}_{L_t^{\frac{r(p-1)}{r-1}}\left([0,T];L_{xy}^{r}(\mathbb{R}\times[0,1])\right)}\\ &\leq T^{1-\frac{p}{r}}\|D^\alpha u\|^{p-1}_{L_t^{r}\left([0,T];L_{xy}^{r}(\mathbb{R}\times[0,1])\right)}\end{aligned}$$

Equivalently,

$$\|f(u)\|_{L_t^{r'}\left([0,T];H_{xy}^{s,r'}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{p}{r}}\|u\|^{p-1}_{L_t^{r}\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)} \leq T^{1-\frac{p}{r}}\|u\|^{p-1}_{\mathcal{X}_T^s}\,. \tag{4.74}$$

(The investigation for $s = 0$ can be found in [82].)

Assume $u, v \in \mathcal{X}_T^s$ and $w = u\cdot\theta + v\cdot(1-\theta)$ for $\theta \in [0,1]$ to have

$$|u|^{p-2}u - |v|^{p-2}v = \int_0^1 \left[\frac{p}{2}|w|^{p-2} + \left(\frac{p}{2}-1\right)|w|^{p-4}w^2\right]\cdot(u-v)\,d\theta\,.$$

Based on the proof in *Theorem 3.9*, if $p > 3$ then we use (2.20) in *Lemma 2.8* to obtain

$$\begin{aligned}
&\left\| D^\alpha \left[|u|^{p-2} u(t) - |v|^{p-2} v(t) \right] \right\|_{L^{r'}} \\
&\quad = \left\| \int_0^1 D^\alpha \left[\frac{p}{2} |w|^{p-2} + \left(\frac{p}{2} - 1 \right) |w|^{p-4} w^2 \right] \cdot (u - v) \, d\theta \right\|_{L^{r'}} \\
&\quad \leq \sup_{\theta \in [0,1]} \left\| D^\alpha \left[\left(\frac{p}{2} |w|^{p-2} + \left(\frac{p}{2} - 1 \right) |w|^{p-4} w^2 \right) \cdot (u - v) \right] \right\|_{L^{r'}} \\
&\quad \lesssim \sup_{\theta \in [0,1]} \left\| |w(t)|^{p-3} \right\|_{L^{\frac{r(p-2)}{(r-2)(p-3)}}} \cdot \| D^\alpha w(t) \|_{L^r} \cdot \| u(t) - v(t) \|_{L^{\frac{r(p-2)}{r-2}}} \\
&\qquad + \sup_{\theta \in [0,1]} \| w(t) \|^{p-2}_{L^{\frac{r(p-2)}{r-2}}} \cdot \| D^\alpha (u(t) - v(t)) \|_{L^r} \\
&\quad = \sup_{\theta \in [0,1]} \| D^\alpha w(t) \|^{p-2}_{L^r} \cdot \| u(t) - v(t) \|_{L^{\frac{p}{1-s}}} + \sup_{\theta \in [0,1]} \| D^\alpha w(t) \|_{L^r} \cdot \| D^\alpha (u(t) - v(t)) \|_{L^r} \\
&\quad \lesssim \left(\| u(t) \|^{p-2}_{W^{s,r}} + \| v(t) \|^{p-2}_{W^{s,r}} \right) \cdot \| (u(t) - v(t)) \|_{W^{s,r}} .
\end{aligned}$$

For $p = 3$, by (2.19) we can show that

$$\begin{aligned}
&\left\| D^\alpha \left[|u|^{p-2} u(t) - |v|^{p-2} v(t) \right] \right\|_{L^{r'}} \\
&\lesssim \sup_{\theta \in [0,1]} \| D^\alpha w(t) \|_{L^r} \cdot \| u(t) - v(t) \|_{L^{\frac{r(p-2)}{r-2}}} + \sup_{\theta \in [0,1]} \| w(t) \|_{L^{\frac{r(p-2)}{r-2}}} \cdot \| D^\alpha (u(t) - v(t)) \|_{L^r} \\
&\lesssim \left(\| u(t) \|_{W^{s,r}} + \| v(t) \|_{W^{s,r}} \right) \cdot \| (u(t) - v(t)) \|_{W^{s,r}}
\end{aligned}$$

Thus by Sobolev embedding theorem,

$$\| f(u)(t) - f(v)(t) \|_{W^{s,r'}} \lesssim \left(\| u(t) \|^{p-2}_{W^{s,r}_{xy}} + \| v(t) \|^{p-2}_{W^{s,r}_{xy}} \right) \cdot \| u(t) - v(t) \|_{W^{s,r}} .$$

In addition, $\dfrac{p}{r} \leq 1$ guarantees $r \geq \dfrac{r(p-2)}{r-2}$ and the following estimate for the difference of the nonlinearity

$$\begin{aligned}
&\| f(u) - f(v) \|_{L^{r'}_t \left([0,T]; W^{s,r'}_{xy} (\mathbb{R} \times [0,1]) \right)} \\
&\leq \left(\| u \|^{p-2}_{L_t^{\frac{r(p-2)}{r-2}} \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} + \| v \|^{p-2}_{L_t^{\frac{r(p-2)}{r-2}} \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} \right) \| u - v \|_{L^r_t \left([0,T]; L^r_{xy} (\mathbb{R} \times [0,1]) \right)} \\
&\leq T^{1 - \frac{p}{r}} \left(\| u \|^{p-2}_{L^r_t \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} + \| v \|^{p-2}_{L^r_t \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} \right) \| u - v \|_{L^r_t \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} .
\end{aligned} \tag{4.75}$$

We now remind ourselves of $\mathcal{A}[u]$ with (4.67). For $u, v \in B^{\mathcal{X}^s}_M$ and some $r \in [p, 4)$ if $0 \leq s < 1/2$ and $3 \leq p \leq 4$,

$$\| \mathcal{A}(u) \|_{\mathcal{X}_T} = \| \mathcal{A}(u) \|_{L^\infty_t \left([0,T]; H^s_{xy} (\mathbb{R} \times [0,1]) \right)} + \| \mathcal{A}(u) \|_{L^r_t \left([0,T]; W^{s,r}_{xy} (\mathbb{R} \times [0,1]) \right)} .$$

If $3 \le p < 4$, we can find $r \in (p, 4)$ and let $\sigma \in (\frac{1}{2}, \frac{r}{3r-4}]$ and therefore $r' \in [\frac{4\sigma}{1+\sigma}, 2]$. Suppose such r has been chosen as a constant. Then we can pick σ from the left half of the interval $(\frac{1}{2}, \frac{r}{3r-4}]$, i.e. $\frac{1}{2} < \sigma \le \frac{5r-4}{4(3r-4)}$. Thus it can be found that

$$\begin{aligned} 1 + \sigma - \frac{4\sigma}{r'} &= 1 - \left(3 - \frac{4}{r}\right)\sigma \\ &\ge 1 - \left(3 - \frac{4}{r}\right) \cdot \frac{5r-4}{4(3r-4)} = \frac{1}{r} - \frac{1}{4} := \theta_r > 0 \end{aligned} \tag{4.76}$$

According to *Proposition 4.3*, and (4.18), (4.34) in *Proposition 4.6 4.9*, and (4.74)

$$\begin{aligned} \|\mathcal{A}(u)\|_{\mathcal{X}_T^s} &\le \|W_b(h_1,h_2)\|_{\mathcal{X}_T^s} + \|W_0\phi\|_{\mathcal{X}_T} + \|\Phi_{0,f}\|_{\mathcal{X}_T^s} \\ &\lesssim \sum_{j=1,2} \|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + T^{\theta_r}\|f\|_{L_t^{r'}\left([0,T];H_{xy}^{s,r'}\right)} \\ &\le \left(1 + T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right)\mu + T^{\theta_r}\|f\|_{L_t^{r'}\left([0,T];H_{xy}^{s,r'}\right)} \\ &\lesssim \left(1 + T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right)\mu + T^{\theta_r+1-\frac{p}{r}}\|u\|^{p-1}_{L_t^r\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)} \\ &\le \left(1 + T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right)\mu + T^{\theta_r+1-\frac{p}{r}}\|u\|^{p-1}_{\mathcal{X}_T^s} \end{aligned}$$

where $\theta_r + 1 - \frac{p}{r} > \theta_r$ is fixed. Considering the distance, by (4.75) we have

$$\begin{aligned} \|\mathcal{A}(u) - \mathcal{A}(v)\|_{\mathcal{X}_T^s} &= \|\Phi_{0,f}(u) - \Phi_{0,f}(v)\|_{\mathcal{X}_T^s} \\ &\lesssim T^{\theta_r}\|f(u) - f(v)\|_{L_t^{r'}\left([0,T];L_{xy}^{r'}\right)} \\ &\lesssim T^{\theta_r+1-\frac{p}{r}}\left(\|u\|^{p-2}_{\mathcal{X}_T^s} + \|v\|^{p-2}_{\mathcal{X}_T^s}\right)\|u-v\|_{\mathcal{X}_T^s} \\ &\lesssim T^{\theta_r+1-\frac{p}{r}}M^{p-2}\|u-v\|_{\mathcal{X}_T^s} \end{aligned}$$

Thus we obtain with some constants C_0, C_1,

$$\|\mathcal{A}[u]\|_{\mathcal{X}_T^s} \le C_0\left(\mu + T^{\frac{1}{2}}\mu + T^{\frac{1}{2}-\sigma}\mu + T^{\theta_r+1-\frac{p}{r}}M^{p-1}\right), \tag{4.77}$$

and

$$\|\mathcal{A}[u] - \mathcal{A}[v]\|_{\mathcal{X}_T^s} \le C_1 T^{\theta_r+1-\frac{p}{r}}M^{p-2}\|u-v\|_{\mathcal{X}_T^s}. \tag{4.78}$$

Choose T sufficiently small so that $T \le 1$ and $T^{\theta_r+1-\frac{p}{r}}M^{p-1} \le T^{\frac{1}{2}-\sigma}\mu$. This indicates that we need select $T \le \min\left\{1, \left(\frac{\mu}{M^{p-1}}\right)^{1/(\theta_r+1-\frac{p}{r}-(\frac{1}{2}-\sigma))}, \left(\frac{1}{2C_1M^{p-2}}\right)^{1/(\theta_r+1-\frac{p}{r})}\right\}$. Then we observe that the right hand side of (4.77) implies

$$C_0\left(\mu + T^{\frac{1}{2}}\mu + T^{\frac{1}{2}-\sigma}\mu + T^{\theta_r+1-\frac{p}{r}}M^{p-1}\right) \le 4C_0\mu T^{\frac{1}{2}-\sigma} \le 4C_0\mu\left(\frac{\mu}{M^{p-1}}\right)^{(\frac{1}{2}-\sigma)/(\theta_r+1-\frac{p}{r}-(\frac{1}{2}-\sigma))}.$$

It can be achieved by any $M \ge \max\{4C_0\mu,\, \mu^{1/(p-1)}\}$. With the choice of T above we further obtain that $C_1T^{\theta_r+1-\frac{p}{r}}M^{p-2} \le \frac{1}{2}$ for (4.78).

Then we consider the critical case (based on the technique applied here) when $p = 4$. Let $r = 4$. By (4.19) we claim that

$$\|\mathcal{A}(u)\|_{\mathcal{X}_T^s} \lesssim C_T(\mu + \|u\|_{\mathcal{X}_T^s}^{p-1})$$
$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{\mathcal{X}_T^s} \lesssim C_T M^{p-2} \|u - v\|_{\mathcal{X}_T^s}$$

where $C_T > 0$ depends on T only. Thus for a $T > 0$ finite, let $M = 2C_T\mu$. For u, $v \in B_{2C_T\mu}^{\mathcal{X}^s}$ with μ small enough so that

$$\|\mathcal{A}(u)\|_{\mathcal{X}_T^s} \leq C_T(\mu + M^{p-1}) \leq 2C_T\mu$$

and

$$\|\mathcal{A}(u) - \mathcal{A}(v)\|_{\mathcal{X}_T^s} \lesssim \widetilde{C_T}\mu^{p-2}\|u - v\|_{\mathcal{X}_T^s}$$

when $\widetilde{C_T}\mu^{p-2} \leq (1/2)$, we can establish (4.68) and (4.69).

However for $1/2 \leq s < 1$ and by Sobolev embedding theorem, if $\frac{r(p-2)}{r-2} = \frac{2}{1-s}$, $H^s \hookrightarrow L^{\frac{r(p-2)}{r-2}}$; that is $\|u(t)\|_{L^{\frac{r(p-2)}{r-2}}} \leq \|D^\alpha u(t)\|_{L^2}$. This holds if and only if

$$\frac{4}{2 - (p-2)(1-s)} = r < 4 \quad \text{or equivalently} \quad 3 \leq p < \frac{3 - 2s}{1 - s}.$$

We repeat the argument above to get

$$\begin{aligned} \|D^\alpha f(u)(t)\|_{L^{r'}} &\lesssim \|u(t)\|_{L^{\frac{r(p-2)}{r-2}}}^{p-2} \|D^\alpha u(t)\|_{L^r} \\ &\leq \|u(t)\|_{H^s}^{p-2} \|D^\alpha u(t)\|_{L^r}. \end{aligned}$$

Then taking account of norm with respect to t, we have

$$\|D^\alpha f(u)\|_{L_t^{r'}\left([0,T];L_{xy}^{r'}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u(t)\|_{L^\infty([0,T];H^s)}^{p-2} \|D^\alpha u\|_{L_t^r\left([0,T];L_{xy}^r(\mathbb{R}\times[0,1])\right)},$$

and

$$\|f(u)\|_{L_t^{r'}\left([0,T];W_{xy}^{s,r'}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u(t)\|_{L^\infty([0,T];H^s)}^{p-2} \|D^\alpha u\|_{L_t^r\left([0,T];L_{xy}^r(\mathbb{R}\times[0,1])\right)},$$

which gives

$$\|f(u)\|_{L_t^{r'}\left([0,T];H_{xy}^{s,r'}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u\|_{\mathcal{X}_T^s}^{p-1}. \tag{4.79}$$

Similarly, for $p \geq 3$,

$$\begin{aligned} &\|f(u) - f(v)\|_{L_t^{r'}\left([0,T];W_{xy}^{s,r'}(\mathbb{R}\times[0,1])\right)} \\ &\leq T^{1-\frac{2}{r}} \left(\|u\|_{L_t^r\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)}^{p-2} + \|v\|_{L_t^r\left([0,T];W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)}^{p-2} \right) \|u - v\|_{L_t^\infty\left([0,T];H_{xy}^s(\mathbb{R}\times[0,1])\right)} \end{aligned}$$

$$+ T^{1-\frac{2}{r}} \left(\|u\|^{p-2}_{L^\infty_t\left([0,T];H^s_{xy}(\mathbb{R}\times[0,1])\right)} + \|v\|^{p-2}_{L^\infty_t\left([0,T];H^s_{xy}(\mathbb{R}\times[0,1])\right)} \right) \|u-v\|_{L^r_t\left([0,T];W^{s,r}_{xy}(\mathbb{R}\times[0,1])\right)} .$$

Then, we can reach the similar result as

$$\begin{aligned} \|\mathcal{A}(u)\|_{\mathcal{X}^s_T} &\lesssim \left(1 + T^{\frac{1}{2}} + T^{\frac{1}{2}-\sigma}\right)\mu + T^{\theta_r + \frac{(p-2)(1-s)}{2}} \|u\|^{p-1}_{\mathcal{X}^s_T} \\ \|\mathcal{A}(u) - \mathcal{A}(v)\|_{\mathcal{X}^s_T} &\lesssim T^{\theta_r + \frac{(p-2)(1-s)}{2}} M^{p-2} \|u-v\|_{\mathcal{X}^s_T} , \end{aligned}$$

which lead to

$$\|\mathcal{A}[u]\|_{\mathcal{X}^s_T} \le C_0 \left(\mu + T^{\frac{1}{2}}\mu + T^{\frac{1}{2}-\sigma}\mu + T^{\theta_r + \frac{(p-2)(1-s)}{2}} M^{p-1}\right) , \tag{4.80}$$

and

$$\|\mathcal{A}[u] - \mathcal{A}[v]\|_{\mathcal{X}^s_T} \le C_1 T^{\theta_r + \frac{(p-2)(1-s)}{2}} M^{p-2} \|u-v\|_{\mathcal{X}^s_T} . \tag{4.81}$$

Hence we let $M \ge \max\{4C_0\mu,\, \mu^{1/(p-1)}\}$ and

$$T \le \min\left\{1,\, \left(\frac{\mu}{M^{p-1}}\right)^{1/(\theta_r + \frac{(p-2)(1-s)}{2} - (\frac{1}{2}-\sigma))} ,\, \left(\frac{1}{2C_1 M^{p-2}}\right)^{1/(\theta_r + \frac{(p-2)(1-s)}{2})}\right\}$$

so that the verification of (4.68) and (4.69) is fulfilled.

When $p = \frac{3-2s}{1-s}$, $r = 4$. We meet a critical situation again. Then if choose the measure of the initial and boundary data sufficiently small for some fixed $T > 0$, then we can show (4.68) and (4.69).

Using Contraction Mapping theorem, we can conclude that the problem (4.1) (or (4.5)) has a unique solution u in $\mathcal{X}^s_T$ for $0 \le s < \frac{1}{2}$ with $3 \le p \le 4$ or $\frac{1}{2} \le s < 1$ with $3 \le p \le \frac{3-2s}{1-s}$.

Now consider $s = 1$. Let r be arbitrarily chosen from $(2,4)$. We know that $H^1 \hookrightarrow L^r$ for any $r < \infty$. We use the chain rule on the first derivative of $f(u)$ with respect to (x,y).

$$\begin{aligned} \|\nabla f(u)(t)\|_{L^{r'}} &\lesssim \|u(t)\|^{p-2}_{L^{\frac{r(p-2)}{r-2}}} \|\nabla u(t)\|_{L^r} \\ &\le \|u(t)\|^{p-2}_{H^1} \|\nabla u(t)\|_{L^r} , \end{aligned}$$

Then with the norm in t,

$$\|\nabla f(u)\|_{L^{r'}_t\left([0,T];L^{r'}_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u(t)\|^{p-2}_{L^\infty([0,T];H^1)} \|\nabla u\|_{L^r_t\left([0,T];L^r_{xy}(\mathbb{R}\times[0,1])\right)} ,$$

and

$$\|f(u)\|_{L^{r'}_t\left([0,T];W^{1,r'}_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u(t)\|^{p-2}_{L^\infty([0,T];H^1)} \|\nabla u\|_{L^r_t\left([0,T];L^r_{xy}(\mathbb{R}\times[0,1])\right)} ,$$

which implies

$$\|f(u)\|_{L^{r'}_t\left([0,T];W^{1,r'}_{xy}(\mathbb{R}\times[0,1])\right)} \lesssim T^{1-\frac{2}{r}} \|u\|^{p-1}_{\mathcal{X}^1_T} . \tag{4.82}$$

For the distance estimate with $p \geq 3$,

$$\begin{aligned}
&\|f(u) - f(v)\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r'}(\mathbb{R}\times[0,1])\right)} \\
&\leq T^{1-\frac{2}{r}} \left(\|u\|^{p-2}_{L_t^r\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)} + \|v\|^{p-2}_{L_t^r\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)}\right) \|u-v\|_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times[0,1])\right)} \\
&\quad + T^{1-\frac{2}{r}} \left(\|u\|^{p-2}_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times[0,1])\right)} + \|v\|^{p-2}_{L_t^\infty\left([0,T];H^1_{xy}(\mathbb{R}\times[0,1])\right)}\right) \|u-v\|_{L_t^r\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)}
\end{aligned}$$

Thus for any $r \in (2,4)$ and constants C_0, C_1,

$$\begin{aligned}
\|\mathcal{A}(u)\|_{\mathcal{X}_T^1} &\leq C_0 \left(\mu + T^{\frac{1}{2}}\mu + T^{\frac{1}{2}-\sigma}\mu + T^{\theta_r+1-\frac{2}{r}} M^{p-1}\right) \\
\|\mathcal{A}(u) - \mathcal{A}(v)\|_{\mathcal{X}_T^1} &\leq C_1 T^{\theta_r+1-\frac{2}{r}} M^{p-2} \|u-v\|_{\mathcal{X}_T^1}\,.
\end{aligned}$$

If repeating the same argument as that for the case $0 \leq s < 1$, we choose M and T satisfying

$$M \geq \max\{4C_0\mu,\, \mu^{1/(p-1)}\}$$

and

$$T \leq \min\left\{1,\, \left(\frac{\mu}{M^{p-1}}\right)^{1/(\theta_r+1-\frac{2}{r}-(\frac{1}{2}-\sigma))},\, \left(\frac{1}{2C_1M^{p-2}}\right)^{1/(\theta_r+1-\frac{2}{r})}\right\}$$

so that (4.68) and (4.69) are valid again. Thus by the Contraction mapping theorem we can see that the equation has a unique solution in $\mathcal{X}_T^s$. Since $\theta_r + 1 - \frac{2}{r} > 0$, $\forall\, r \in (2,4)$, then we do not need to force any restriction on the measure of the initial and boundary data (to be small).

After finish the discussion for lower regularity, we now study the existence and uniqueness of the solution with $s > 1$. We first recall (3.81), (3.82) and (3.83) and derive a set of similar estimates over the strip domain $\mathbb{R} \times [0,1]$. That is, for u, $v \in B_M^{\mathcal{Y}^s}$, if p is an even integer or p is not an even integer and $p \geq s+1$ with $s \in \mathbb{Z}$, or p is not an even integer and $p \geq [s]+2$ with $s \notin \mathbb{Z}$, given any $M > 0$ and u, $v \in C_t(\mathbb{R}^+; H^s(\mathbb{R}\times[0,1]))$ such that $\|u\|_{C_t(\mathbb{R}^+;H^s(\mathbb{R}\times[0,1]))}$, $\|v\|_{C_t(\mathbb{R}^+;H^s(\mathbb{R}\times[0,1]))} \leq M$,

$$\|f(u)\|_{L^2((0,T);\,H^s)} \lesssim T\|u\|^{p-1}_{L^\infty((0,T);\,H^s)} \tag{4.83}$$

$$\|f(u)-f(v)\|_{L^2((0,T);\,L^2)} \lesssim TM^{p-2}\|u-v\|_{L^\infty((0,T);\,L^2)}\,. \tag{4.84}$$

We also have that for p being an even integer or p not an even integer and $p \geq s+2$ with $s \in \mathbb{Z}$ or p not an even integer and $p \geq [s]+3$ with $s \notin \mathbb{Z}$ the Lipschitz continuity holds for H^s norm:

$$\|f(u)-f(v)\|_{L^1((0,T);\,H^s)} \lesssim TM^{p-2}\|u-v\|_{L^\infty((0,T);\,H^s)}\,. \tag{4.85}$$

Thus we let the stronger assumption hold, i.e., let p be an even integer, or p not be an even integer and $p \geq s+2$ with $s \in \mathbb{Z}$, or p not be an even integer but $p \geq [s]+3$ with $s \notin \mathbb{Z}$. Then

$$\|\mathcal{A}(u)\|_{\mathcal{Y}_T^s} \leq \|W_b(h_1,h_2)\|_{\mathcal{Y}_T^s} + \|W_0\phi\|_{\mathcal{X}_T} + \|\Phi_{0,f}\|_{\mathcal{Y}_T^s}$$

$$
\begin{aligned}
&\lesssim \left(1+T^{\frac{1}{2}}+T^{\frac{1}{2}-\sigma}\right)\mu+T^{\theta_r}\,\|f\|_{L_t^2\left([0,T];H^s_{xy}\right)}\\
&\lesssim \left(1+T^{\frac{1}{2}}+T^{\frac{1}{2}-\sigma}\right)\mu+T^{\theta_r+1}T\|u\|^{p-1}_{L^\infty((0,T);\,H^s)}\\
&\le \left(1+T^{\frac{1}{2}}+T^{\frac{1}{2}-\sigma}\right)\mu+T^{\theta_r+1}M^{p-1}\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
\|\mathcal{A}(u)-\mathcal{A}(v)\|_{\mathcal{Y}^s_T}&=\|\Phi_{0,f}(u)-\Phi_{0,f}(v)\|_{\mathcal{Y}^s_T}\\
&\lesssim T^{\theta_r}\,\|f(u)-f(v)\|_{L_t^2\left([0,T];H^s_{xy}\right)}\\
&\lesssim T^{\theta_r+1}M^{p-2}\|u-v\|_{\mathcal{Y}^s_T}\,.
\end{aligned}
$$

Rewrite them as

$$
\begin{aligned}
&\|\mathcal{A}(u)\|_{\mathcal{Y}^s_T}\le C_0\left(\mu+T^{\frac{1}{2}}\mu+T^{\frac{1}{2}-\sigma}\mu+T^{\theta_r+1}M^{p-1}\right)\\
&\|\mathcal{A}(u)-\mathcal{A}(v)\|_{\mathcal{Y}^s_T}\le C_1T^{\theta_r+1}M^{p-2}\|u-v\|_{\mathcal{Y}^s_T}
\end{aligned}
$$

for C_0, $C_1>0$. We pick σ close enough to $\frac{1}{2}$ from the right hand side so that $\theta_r>0$ again. Then, let M and T be such that

$$
M\ge\max\{4C_0\mu,\,\mu^{1/(p-1)}\}
$$

and

$$
T\le\min\left\{1,\,\left(\frac{\mu}{M^{p-1}}\right)^{1/(\theta_r+\sigma))},\,\left(\frac{1}{2C_1M^{p-2}}\right)^{1/(\theta_r+\frac{1}{2})}\right\}
$$

which yield (4.71) and (4.72). If $s+1\le p<s+2$ for $s\in\mathbb{Z}$ or $[s]+2\le p<[s]+3$ for $s\notin\mathbb{Z}$ when p is not an even number, then (4.84) gives

$$
\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Z}^0_T}\le C_1T^{\theta_r+1}M^{p-2}\|u-v\|_{\mathcal{Z}^0_T}\,. \tag{4.86}
$$

Hence, (4.70) is assured for the same choices of M and T. With the Contraction Mapping principle we can find a fixed point $u\in\mathcal{Y}^s_T$ if p is an even integer, or p is not an even integer but $p\ge s+2$ for $s\in Z$ or $p\ge[s]+3$ for $s\notin\mathbb{Z}$. Nevertheless, if $s+1\le p<s+2$ for $s\in\mathbb{Z}$ or $[s]+2\le p<[s]+3$ for $s\notin\mathbb{Z}$, a fixed point $u\in\mathcal{Y}^0_T$ can also be found which has a sequence $\{u_n\}_{n\ge1}\subset B_M^{\mathcal{Y}^s}\subset\mathcal{Y}^s_T$ converging to u in $\mathcal{Y}^0_T$; i.e., choose $u_1\in B_M^{\mathcal{Y}^s}$. Then let $u_2=\mathcal{A}[u_1]$ and hence $u_2\in B_M^{\mathcal{Y}^s}$ because of (4.71). We keep define $u_{n+1}=\mathcal{A}[u_n]$ and again $u_3\in B_M^{\mathcal{Y}^s}$; moreover $\|u_{n+1}-u_n\|_{\mathcal{Y}^s_T}=\|\mathcal{A}[u_n]-\mathcal{A}[u_{n-1}]\|_{\mathcal{Y}^0_T}\le\frac{1}{2}\|u_n-u_{n-1}\|_{\mathcal{Y}^0_T}$ by (4.70). Then a Cauchy sequence $\{u_n\}_{n\ge1}\subset B_M^{\mathcal{Y}^s}\subset\mathcal{Y}^s_T$ with $u_n\to u$ in $\mathcal{Y}^s_T$ is created. We may see that the problem has a unique solution in $\mathcal{Y}^s_T$. Since $\mathcal{Y}^s_T$ is also a reflexive Banach space, we conclude that $u\in\mathcal{Y}^s_T$ is the unique solution of (4.5) for $s>1$. The proof is finished. □

Then we consider the maximal solution over $[0,T_{\max}]$.

4.3.2 The Maximal Solution

We are interested in the maximum existence interval of solutions found in *Theorem 4.11.*

Theorem 4.12. *Let $p \geq 3$ and: (a) $0 \leq s < \infty$ if p is an even integer; or (b) $0 \leq s \leq p-1$ if p is not an even integer but $s \in \mathbb{Z}$; or (c) $0 \leq [s] \leq p-2$ when p is not an even integer and $s \notin \mathbb{Z}$. Assume that a unique solution u to the problem (3.1) exists in $\mathcal{X}_T^s$ if $0 \leq s < \frac{1}{2}$ with $3 \leq p \leq 4$ or $\frac{1}{2} \leq s < 1$ with $3 \leq p < \frac{3-2s}{1-s}$ or $s = 1$ with $3 \leq p < \infty$, or $\mathcal{Y}_T^s$ if $s > 1$, for $t \in [0,T]$ with $\varphi \in H^s(\mathbb{R} \times [0,1])$ and $h_j \in H_t^{\frac{s+\frac{1}{2}}{2}}([0,T_0]; L_x^2(\mathbb{R})) \cap L_t^2([0,T_0]; H_x^{s+1}(\mathbb{R}))$ for any $T_0 > 0$, $j = 1, 2$. Let $T_{\max} = \sup T$ and suppose $T_{\max} < \infty$. Also, define u^* on $[0, T_{\max})$ as the solution of (4.1) in $\mathcal{X}_{T_{\max}}^s$ if $0 \leq s < \frac{1}{2}$ with $3 \leq p \leq 4$ or $\frac{1}{2} \leq s < 1$ with $3 \leq p < \frac{3-2s}{1-s}$, or $\mathcal{X}_{T_{\max}}^s$ if $s = 1$, or $\mathcal{Y}_{T_{\max}}^s$ if $s > 1$, with $u^*(t) = u(t)$ on $[0,T]$ whose existence and uniqueness have been proved in Theorem 4.11. Then $\lim\limits_{t \uparrow T_{\max}} \|u(t)\|_{H^s(\mathbb{R}\times[0,1])} = \infty$.*

Proof. The proof can be presented using classical extension procedure as an analogue of the proof for *Theorem 3.10* except that the discussion is performed on the domain $\mathbb{R} \times [0,1]$. Suppose that $\|u(t)\|_{H^s} \leq \mathcal{C}$ for some $\mathcal{C} > 0$ on $[0,T]$ and for any $T < T_{\max}$. Choose small $\delta > 0$ and $T = t_n$ such that $|T_{\max} - T| < \frac{T}{n}$ for some $n \geq 3$. Let u_1 be the unique solution guaranteed in $\mathcal{X}_T^s$ for $t \in [0,T]$. Note that by the uniqueness, $u^*(t) = u_1(t)$ for any $t \in [0,T]$. By (4.5)

$$u_1(t) = W_b(h_1,h_2)(t) + W_0(t)\phi + i\lambda \int_0^t W_0(t-\tau)\,|u_1(\tau)|^{p-2}\,u_1(\tau)\,d\tau \tag{4.87}$$

Next we extend u_1 forward in time. Suppose $u_2(t) = u(t+T)$ solves

$$\begin{cases} i(u_2)_t + (u_2)_{xx} + (u_2)_{yy} + i\lambda\,|u_2|^{p-2}\,u_2 = 0 & (x,y,t) \in \mathbb{R} \times [0,1] \times (0,\delta) \\ u_2(x,y,0) = u_1(x,y,T) & (x,y) \in \mathbb{R} \times [0,1] \\ u_2(x,0,t) = h_1(x,t+T)\,, \quad u_2(x,1,t) = h_2(x,t+T) & (x,t) \in \mathbb{R} \times (0,\delta) \end{cases} \tag{4.88}$$

in

$$\mathcal{X}_\delta^s := C_t\left([0,\delta];\, H_{xy}^s(\mathbb{R}\times[0,1])\right) \bigcap L_t^r\left([0,\delta];\, W_{xy}^{s,r}(\mathbb{R}\times[0,1])\right)\,,$$

which is equivalent to

$$u_2(t) = W_b(h_1(\cdot+T),h_2(\cdot+T))(t) + W_0(t)u_1(T) + i\lambda \int_0^t W_0(t-\tau)\,|u_2(\tau)|^{p-2}\,u_2(\tau)\,d\tau \tag{4.89}$$

for $t \in [0,\delta]$. Since H^s-norms of both $\phi = u_1(t_0)$ and $u_1(T) = u_1(t_n)$ are less than $\mathcal{C}$, then there is a $\mu > 0$ such that,

$$\sum_{j=1,2} \|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} \leq \mu\,,$$
$$\sum_{j=1,2} \|h_j(\cdot+T)\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R})) \bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} + \|u_1(T)\|_{H_{xy}^s} \leq \mu\,.$$

Because of the contraction mapping argument, we see that T is only determined by μ. Thus, we can choose any value for $\delta \le T$, say $\delta = \frac{2T}{n} < T$, so that u_2 is uniquely defined. Then let

$$u(t) = \begin{cases} u_1(t) & t \in (0,T) \\ u_2(t-T) & t \in (T, T+\delta) \end{cases},$$

satisfying

$$\|u\|_{\mathcal{X}^s_{T+\delta}} \le \|u_1\|_{\mathcal{X}^s_T} + \|u_2\|_{\mathcal{X}^s_\delta} < \infty \quad \text{and} \quad \lim_{t\uparrow T} \|u(t)\|_{H^s} = \lim_{t\downarrow T} \|u(t)\|_{H^s} = u_1(T)\,.$$

Hence, $u \in \mathcal{X}^s_{T+\delta}$ while $T+\delta \ge T + \frac{2T}{n} > T_{\max}$ leading to the conclusion that $T_{\max} \ne \sup S$. A contradiction arises, which means $\lim\limits_{t\uparrow T_{\max}} \|u(t)\|_{H^s(\mathbb{R}\times[0,1])} = \infty$. Here, h_j's are defined for any $t > 0$. Note that we are not able to construct the blow-up statement with the critical case when $p = 4$ for $0 \le s < \frac{1}{2}$ or $p = \frac{3-2s}{1-s}$ for $\frac{1}{2} \le s < 1$ because the existence and uniqueness results are only for small initial and boundary data.

Moreover, replacing $\mathcal{X}^s_T$ by $\mathcal{Y}^s_T$ if $s > 1$ for $p \ge s+1$ with $s \in \mathbb{Z}$ or $p \ge [s]+2$ with $s \notin \mathbb{Z}$ when p is not an even integer, and repeating a similar argument as above, one can obtain the analogous conclusions as stated in the theorem. □

4.3.3 Continuous Dependence

The continuous dependence property of solutions on the initial condition and the boundary condition can be proved as follows.

Theorem 4.13. *Let $p \ge 3$ and: (a) $0 \le s < \infty$ if p is an even integer; or (b) $0 \le s \le p-1$ if p is not an even integer but $s \in \mathbb{Z}$; or (c) $0 \le [s] \le p-2$ if p is not an even integer and $s \notin \mathbb{Z}$. Assume $\{\varphi_n\}$ be a sequence of functions in $H^s(\mathbb{R}\times[0,1])$ and $\varphi \in H^s(\mathbb{R}\times[0,1])$ so that $\varphi_n \to \varphi$ as $n \to \infty$ in $H^s(\mathbb{R}\times[0,1])$. Let h be a function and $\{h_n\}$ be a sequence of functions such that*

$$h, h_{1,n}, h_{2,n} \in H_t^{\frac{s+1}{2}}\left([0,T];\, L^2_x(\mathbb{R})\right) \bigcap L^2_t\left([0,T];\, H^{s+1}_x(\mathbb{R})\right) \text{ and } h_{1,n} \to h_1,\ h_{2,n} \to h_2$$

as $n \to \infty$ in $H_t^{\frac{s+1}{2}}([0,T];\, L^2_x(\mathbb{R})) \bigcap L^2_t([0,T];\, H^{s+1}_x(\mathbb{R}))$ for any $T > 0$. Let u_n be the solutions to the equation (4.1) with $u_n(x,y,0) = \varphi_n(x,y)$ and $u_n(x,0,t) = h_n(x,t)$ and u be the solution with $u(x,y,0) = \varphi(x,y)$ and $u(x,0,t) = h(x,t)$, respectively. Then $u_n \to u$ as $n \to \infty$ in X_T with $\|u_n\|_{X_T} \le M$ where $X_T = \mathcal{X}^s_T$ if $0 \le s < \frac{1}{2}$ with $3 \le p \le 4$ or $\frac{1}{2} \le s < 1$ with $3 \le p < \frac{3-2s}{1-s}$ or $s = 1$ with $3 \le p < \infty$, or $\mathcal{Y}^s_T$ if $s > 1$, respectively.

Proof. First, we consider $0 \le s \le 1$ with some assumptions on p and r given in *Theorems 4.11* and *4.12* which guarantee the existence of a common interval $[0, T_c]$ for u_n and u because of the choice of $T_{\max}$ only dependent on the initial and boundary conditions. Furthermore, from the proof of (4.68), for θ_r defined by (4.76) and $\sigma > \frac{1}{2}$, we can obtain

$$\|u-u_n\|_{\mathcal{X}_T^s} = \|\mathcal{A}[u]-\mathcal{A}[u_n]\|_{\mathcal{X}_T^s} \le C\Big(\big(T^{\frac{1}{2}}+T^{\frac{1}{2}-\sigma}\big) \|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times[0,1])} + \sum_{j=1,2} \|h_j-h_{j,n}\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\big([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\big)} + T^{\theta_r}M^{p-2}\|u-u_n\|_{\mathcal{X}_T^s}\Big).$$

Let T be sufficiently small so that $CT^{\theta_r}M^{p-2} < 1/2$. Then

$$\|u-u_n\|_{\mathcal{X}_T^s} \le 2\widetilde{C}\Big(\|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times[0,1])} + \sum_{j=1,2} \|h_j-h_{j,n}\|_{H_t^{\frac{2s+1}{4}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\big([0,T];\,H_x^{s+\frac{1}{2}}(\mathbb{R})\big)} \Big).$$

Since T only depends on the uniform bounds for $u, u_n, \varphi, \varphi_n, h, h_n$ in their respective norms with $t \in [0, T_c]$, the above inequality holds for $T, 2T, \ldots$ until reaching T_c. The continuous dependence is proved for $0 \le s \le 1$.

For $s > 1$, we first notice, by (4.86), that

$$\|\mathcal{A}[u]-\mathcal{A}[v]\|_{\mathcal{Y}_T^0} \le C_1 T^{\theta_r+1} M^{p-2}\,\|u-v\|_{\mathcal{Y}_T^0}.$$

Hence

$$\|u-u_n\|_{\mathcal{Y}_T^0} = \|\mathcal{A}[u]-\mathcal{A}[u_n]\|_{\mathcal{Y}_T^0} \le C\Big(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times[0,1])} + \sum_{j=1,2} \|h_j-h_{j,n}\|_{H_t^{\frac{1}{2}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2([0,T];\,H_x^1(\mathbb{R}))} + T^{\theta_r+1}M^{p-2}\|u-u_n\|_{\mathcal{Y}_T^0}\Big).$$

Thus, choosing $T \le 1$ small enough so that $CT^{\theta_r+1}M^{p-2} < 1/2$, we have

$$\|u-u_n\|_{\mathcal{Y}_T^0} \le 2\widetilde{C}\Big(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times[0,1])} + \sum_{j=1,2} \|h_j-h_{j,n}\|_{H_t^{\frac{1}{2}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2([0,T];\,H_x^1(\mathbb{R}))} \Big) \to 0$$

as $n \to 0$. Then we move to the continuous dependence in $\mathcal{Y}_T^s$. Note that (3.84) can be verified on the strip domain $\mathbb{R}\times[0,1]$ by the same proof, i.e.,

$$\|f(u)-f(v)\|_{L^1((0,T);\,H^s(\mathbb{R}\times[0,1]))} \lesssim TM^{p-2}\|u-v\|_{L^\infty((0,T);\,H^s(\mathbb{R}\times[0,1]))} + T^{\theta_r+1}\varepsilon\{\|u-v\|_{L^\infty((0,T);\,L^2(\mathbb{R}\times[0,1]))}\}. \tag{4.90}$$

where $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function so that $\varepsilon(t) \to 0^+$ as $t \to 0^+$. Then it is straightforward to derive

$$\|u-u_n\|_{\mathcal{Y}_T^s} = \|\mathcal{A}[u]-\mathcal{A}[u_n]\|_{\mathcal{Y}_T^s} \le C\Big(\big(T^{\frac{1}{2}}+T^{\frac{1}{2}-\sigma}\big) \|\varphi-\varphi_n\|_{H^s(\mathbb{R}\times[0,1])}$$

$$+\sum_{j=1,2}\|h_j-h_{j,n}\|_{H_t^{\frac{s+1}{2}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{s+1}(\mathbb{R})\right)}$$
$$+T^{\theta_r+1}M^{p-2}\|u-u_n\|_{\mathcal{Y}_T^s}+T^{\theta_r+1}\varepsilon\{\|u-u_n\|_{\mathcal{Y}_T^0}\}\Bigg).$$

Rewrite this as

$$\|u-u_n\|_{\mathcal{Y}_T^s}\leq 2C\Bigg(\|\varphi-\varphi_n\|_{L^2(\mathbb{R}\times[0,1])}+\sum_{j=1,2}\|h_j-h_{j,n}\|_{H_t^{\frac{s+1}{2}}([0,T];\,L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\,H_x^{s+1}(\mathbb{R})\right)}$$
$$+\varepsilon\{\|u-u_n\|_{\mathcal{Y}_T^0}\}\Bigg).$$

Since the right hand side of the inequality approaches 0 as $n\to 0$, so does the left hand side. Thus the proof of the continuous dependent is completed. $\square$

Eventually, the statements *(i)-(iii)* in *Theorem 1.6* are proved and we move to discuss the possibility of removing the auxiliary space from the well-posedness for $0\leq s\leq 1$.

4.4 Regularity and Unconditional Well-posedness

In this section, the regularity property and unconditional well-posedness of equation (4.1) are discussed for $0\leq s\leq 1$. By *(i) and (ii)* of *Theorem 1.6*, the proof of *(iii)* is also motivated by Section 4 of [12] and Sections 5.1-5.5 in [27]. Unconditional well-posedness theorems in [12] can be applied for (4.1) after the persistence of regularity is verified.

Proposition 4.14. *For $0\leq s\leq 1$ the IBVP (4.1) has the property of regularity persistence (see Definition 1.3). Precisely, let $0\leq s_1<s$ and let u on $[0,T_{\max})$ be the unique maximal solution of (4.1) in $\mathcal{X}^{s_1}_{T_{\max}}$ for $0\leq s<\frac{1}{2}$ with $3\leq p\leq 4$ or $\frac{1}{2}\leq s<1$ with $3\leq p<\frac{3-2s}{1-s}$ or $s=1$ with $3\leq p<\infty$, let $[0,T_{\max}]$ be provided in Theorem 4.11 with $\varphi\in H^{s_1}(\mathbb{R}\times[0,1])$ and $h_j\in H_t^{\frac{s_1+1}{2}}([0,T];L_x^2(\mathbb{R}))\cap L_t^2([0,T];H_x^{s_1+1}(\mathbb{R}))$, $j=1,\,2$. If $\varphi\in H^s(\mathbb{R}\times[0,1])$ and $h_j\in H_t^{\frac{s+1}{2}}([0,T];L_x^2(\mathbb{R}))\cap L_t^2([0,T];H_x^{s+1}(\mathbb{R}))$, then $u\in\mathcal{Y}^s_{T_{\max}}$ for $s>1$.*

Proof. We first consider whether the regularity persistence holds for (4.1) in H^1, i.e., $s_1=1$. Let u be a maximal H^1-solution on $[0,T_{\max})$ according to *Theorems 4.11* and *4.12*. We know that for any $r>2$, $u\in\mathcal{X}_T^1=C_t\left([0,T];\,H^1_{xy}(\mathbb{R}\times[0,1])\right)\bigcap L_t^r\left([0,T];\,W^{1,r}_{xy}(\mathbb{R}\times[0,1])\right)$ is an H^s-solution on $[0,T)$ with $T\leq T_{\max}$. We want to show that $T=T_{\max}$ by contradiction: suppose $T<T_{\max}$ and $\lim\limits_{t\uparrow T}\|u(t)\|_{H^s(\mathbb{R}\times[0,1])}=\infty$.

Since $r\in(2,4)$, by Sobolev embedding theorem $W^{1,r}\hookrightarrow L^\infty$, we have $u(t)\in L^\infty$ for any $t\in[0,T_{\max})$. Moreover, from the proof of *Theorem 4.11* for $s>1$, it is obtained that

$$\|f(u)(t)\|_{H^s}\lesssim\|u(t)\|_{L^\infty}^{p-2}\|u(t)\|_{H^s}\,,\text{ for all }\;t\in[0,T_{\max}).$$

Then use the equation (4.67), *Proposition 4.3*, and *4.6*, and *4.9* to obtain

$$\begin{aligned}
\|u(t)\|_{H^s} &= \|\mathcal{A}[u](t)\|_{H^s} \\
&\lesssim \sum_{j=1,2} \|h_j\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + \|f\|_{L_t^1\left([0,T];H_{xy}^s\right)} \\
&\lesssim \sum_{j=1,2} \|h\|_{H_t^{\frac{s+1}{2}}([0,T];\, L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,T];\, H_x^{s+1}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + \int_0^t \|u(s)\|_{L^\infty}^{p-2}\|u(s)\|_{H^s}\, ds
\end{aligned}$$

for every $t \in [0,T)$. Using Gronwall's inequality *Lemma 2.12*, we deduce that

$$\|u(t)\|_{H^s} \lesssim \exp\left(C\int_0^t \|u(s)\|_{L^\infty}^{p-2}\, ds\right),\ \text{for}\ \ t\in[0,T).$$

If $T < T_{\max}$, then $\lim_{t\uparrow T}\|u(t)\|_{H^s(\mathbb{R}\times[0,1])} < \infty$ which contradicts the blowup alternative in H^s.

For $\frac{1}{2} \le s_1 < 1$, let $s = 1$ and u be the unique maximal solution of (4.1) in $\mathcal{X}_{T_{\max}}^{s_1}$ on $[0,T_{\max})$ if $3 \le p < \frac{3-2s_1}{1-s_1}$ and in $\mathcal{X}_T^1$ on $[0,T)$ with $T \le T_{\max}$. The constraint on p and s implies that $\frac{r(p-2)}{r-2} \le \frac{2}{1-s_1}$ which means that $H^{s_1} \hookrightarrow L^{\frac{r(p-2)}{r-2}}$ by Sobolev embedding theorem. Similarly, suppose $T < T_{\max}$ and $\lim_{t\uparrow T}\|u(t)\|_{H^1(\mathbb{R}\times[0,1])} = \infty$. By the same argument for deriving (4.82), we have for every interval $[0,\tau]\subset[0,T)$,

$$\begin{aligned}
\|f(u)\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r'}(\mathbb{R}\times[0,1])\right)} &\lesssim \|u(t)\|^{p-2}_{L^\infty\left([0,T];L^{\frac{r(p-2)}{r-2}}\right)}\|u\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)} \\
&\lesssim \|u(t)\|^{p-2}_{L^\infty([0,T];H^{s_1})}\|u\|_{L_t^{r}\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)} \lesssim \|u\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)},
\end{aligned}$$

For some small $\varepsilon > 0$, note that

$$\|u\|_{L^r([0,T];W^{1,r})} = \|u\|_{L^{r'}([0,\tau-\varepsilon];W^{1,r})} + \|u\|_{L^{r'}([\tau-\varepsilon,\tau];W^{1,r})} \le C_\varepsilon + \varepsilon^{1-\frac{2}{r}}\|u\|_{L^r([0,T];W^{1,r})}.$$

Then

$$\begin{aligned}
\|u\|_{\mathcal{X}_\tau^1} = \|\mathcal{A}[u]\|_{\mathcal{X}_\tau^1} &\lesssim \sum_{j=1,2}\|h_j\|_{H_t^{\frac{2s+1}{4}}([0,\tau];\, L_x^2(\mathbb{R}))\bigcap L_t^2\left([0,\tau];\, H_x^{s+\frac{1}{2}}(\mathbb{R})\right)} + \|\phi\|_{H_{xy}^s} + \|f\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r'}\right)} \\
&\lesssim C + C_\varepsilon + \varepsilon^{1-\frac{2}{r}}\|u\|_{L^r([0,T];W^{1,r})}\,.
\end{aligned}$$

We can fix ε sufficiently small so that $\varepsilon^{\frac{q-1}{q}} \le (1/2)$ which leads to

$$\|u\|_{L^\infty\left([0,T];\, H_{xy}^1\right)} + \|u\|_{L^r([0,T];W^{1,r})} \le C\,.$$

where C is independent of τ. If we let $\tau \to T^-$, then a contradiction arises. Therefore $T = T_{\max}$.

At last, let $0 \leq s_1 < \frac{1}{2}$ and let $s = 1$. u is the unique maximal solution of (4.1) in $\mathcal{X}^{s_1}_{T_{\max}}$ on $[0, T_{\max})$ if $3 \leq p < 4$ and in $\mathcal{X}^1_T$ on $[0, T)$ with $T \leq T_{\max}$ again. Remember that we let $r \geq p$ so that $\frac{r(p-2)}{r-2} \leq \frac{2r}{2-s_1 r}$ and $W^{s_1,r} \hookrightarrow L^{\frac{r(p-2)}{r-2}}$. Thus

$$\begin{aligned}\|f(u)\|_{L_t^{r'}\left([0,T];W_{xy}^{1,r'}(\mathbb{R}\times[0,1])\right)} &\lesssim \|u(t)\|^{p-2}_{L^r\left([0,T];L^{\frac{r(p-2)}{r-2}}\right)}\|u\|_{L_t^{fracrr-2}\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)} \\ &\lesssim \|u(t)\|^{p-2}_{L^r([0,T];W^{s_1,r})}\|u\|_{L_t^r\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)} \lesssim \|u\|_{L_t^{fracrr-2}\left([0,T];W_{xy}^{1,r}(\mathbb{R}\times[0,1])\right)},\end{aligned}$$

and

$$\|u\|_{L^r([0,T];W^{1,r})} \leq C_\varepsilon + \varepsilon^{1-\frac{3}{r}}\|u\|_{L^r([0,T];W^{1,r})}$$

for $\varepsilon > 0$. Next, through a routine procedure, as long as fixing ε small enough, we are able to have a contradiction with the assumption $\lim\limits_{t\uparrow T}\|u(t)\|_{H^s(\mathbb{R}\times[0,1])} = \infty$. Hence, the proof is complete. □

Then, the result for unconditional well-posedness is proved.

Theorem 4.15. *For $0 \leq s \leq 1$, the problem (4.1) is unconditionally well-posed.*

Proof. The claim of this theorem is a result directly from *Theorem 2.6* in [12] and *Proposition 4.14*. □

4.5 Global Well-posedness

In this section, we investigate the global existence of the solution for equation (4.1) with $T \in (0, \infty]$. We prepare the following identities.

Lemma 4.16. *If the solution of (4.1) exists for any $t > 0$ and is sufficiently smooth, then for $\eta = \eta(y)$*

$$\begin{aligned}-\eta\left(|u_y|^2 - |u_x|^2 + \frac{2\lambda}{p}|u|^p\right)_y &= 2Re(\eta u_y\overline{u}_x)_x + i(\eta u\overline{u_y})_t - i(\eta\overline{u_t}u)_y \\ &\quad + \eta_y\left(u\overline{u}_x\right)_x - \eta_y|u_x|^2 + \eta_y\left(u\overline{u_y}\right)_y - \eta_y|u_y|^2 + \lambda\eta_y|u|^p \end{aligned}\tag{4.91}$$

and

$$\begin{aligned}&\left[\left(y - \frac{1}{2}\right)\left(|u_y|^2 - |u_x|^2\right)\right]_y \\ &\quad = 2|u_y|^2 - 2Re\left[\left(y-\frac{1}{2}\right)u_y\overline{u}_x\right]_x - i\left[\left(y-\frac{1}{2}\right)u\overline{u_y}\right]_t + i\left[\left(y-\frac{1}{2}\right)\overline{u_t}u\right]_y \\ &\quad\quad - \left(u\overline{u}_x\right)_x - \left(u\overline{u_y}\right)_y - \lambda\left(1-\frac{2}{p}\right)|u|^p - \frac{2\lambda}{p}\left[\left(y-\frac{1}{2}\right)|u|^p\right]_y\end{aligned}\tag{4.92}$$

Proof. For (4.91), we multiply (3.96) by η to obtain

$$
\begin{aligned}
-\eta(|u_y|^2 - |u_x|^2)_y &= 2\mathrm{Re}(\eta u_y \overline{u}_x)_x + i(\eta u\overline{u_y})_t - i\eta(\overline{u_t}u)_y + \eta\left(\frac{2\lambda}{p}|u|^p\right)_y \\
&= 2\mathrm{Re}(\eta u_y \overline{u}_x)_x + i(\eta u\overline{u_y})_t - i(\eta\overline{u_t}u)_y + i\eta_y\overline{u_t}u + \eta\left(\frac{2\lambda}{p}|u|^p\right)_y \\
&= 2\mathrm{Re}(\eta u_y \overline{u}_x)_x + i(\eta u\overline{u_y})_t - i(\eta\overline{u_t}u)_y + \eta\left(\frac{2\lambda}{p}|u|^p\right)_y + \eta_y u\left(\overline{u}_{xx} + \overline{u}_{yy} + \lambda\overline{u}|u|^{p-2}\right) \\
&= 2\mathrm{Re}(\eta u_y \overline{u}_x)_x + i(\eta u\overline{u_y})_t - i(\eta\overline{u_t}u)_y + \eta\left(\frac{2\lambda}{p}|u|^p\right)_y + \eta_y u\overline{u}_{xx} + \eta_y u\overline{u}_{yy} + \lambda\eta_y|u|^p \\
&= 2\mathrm{Re}(\eta u_y \overline{u}_x)_x + i(\eta u\overline{u_y})_t - i(\eta\overline{u_t}u)_y + \eta\left(\frac{2\lambda}{p}|u|^p\right)_y \\
&\quad + \eta_y\left(u\overline{u}_x\right)_x - \eta_y|u_x|^2 + \eta_y\left(u\overline{u_y}\right)_y - \eta_y|u_y|^2 + \lambda\eta_y|u|^p
\end{aligned}
$$

To prove (4.92), we replace each η in (4.91) by $y - \frac{1}{2}$.

$$
\begin{aligned}
-\Big[\left(y-\frac{1}{2}\right)\left(|u_y|^2 - |u_x|^2\right)\Big]_y + \left(|u_y|^2 - |u_x|^2\right) &= -\left(y-\frac{1}{2}\right)\left(|u_y|^2 - |u_x|^2\right)_y \\
&= 2\mathrm{Re}\left[\left(y-\frac{1}{2}\right)u_y\overline{u}_x\right]_x + i\left[\left(y-\frac{1}{2}\right)u\overline{u_y}\right]_t - i\left[\left(y-\frac{1}{2}\right)\overline{u_t}u\right]_y \\
&\quad + \left(u\overline{u}_x\right)_x - |u_x|^2 + \left(u\overline{u_y}\right)_y - |u_y|^2 + \lambda|u|^p + \left(y-\frac{1}{2}\right)\left(\frac{2\lambda}{p}|u|^p\right)_y \\
&= 2\mathrm{Re}\left[\left(y-\frac{1}{2}\right)u_y\overline{u}_x\right]_x + i\left[\left(y-\frac{1}{2}\right)u\overline{u_y}\right]_t - i\left[\left(y-\frac{1}{2}\right)\overline{u_t}u\right]_y \\
&\quad + \left(u\overline{u}_x\right)_x - |u_x|^2 + \left(u\overline{u_y}\right)_y - |u_y|^2 + \lambda|u|^p + \left(\left(y-\frac{1}{2}\right)\frac{2\lambda}{p}|u|^p\right)_y - \frac{2\lambda}{p}|u|^p
\end{aligned}
$$

□

Then we present an *a-priori* estimate of the solution to (4.1) in $H^1(\mathbb{R}\times[0,1])$.

Proposition 4.17. *We assume that either $p \geq 3$ and $\lambda < 0$ or $p = 3$ and $\lambda > 0$. Let $T > 0$ be given. If u is a solution of (4.1) in $C_t\left([0,T]; H^1_{xy}(\mathbb{R}\times[0,1])\right)$, then there exists a $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as a nondecreasing function of $\varphi \in H^1(\mathbb{R}\times[0,1])$ and $h \in H^1(\mathbb{R}\times[0,1])$ such that*

$$
\sup_{t\in[0,T]} \|u(t)\|_{H^1(\mathbb{R}\times[0,1]))} \leq \psi\left(\|\varphi\|_{H^1(\mathbb{R}\times[0,1])} + \|h\|_{H^1(\mathbb{R}\times[0,1])}\right) \tag{4.93}
$$

Proof. We first integrate (4.92) with respect to x, y and t.

$$
\frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \left\{\left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right) - \left(|u_x(x,1,\tau)|^2 + |u_x(x,0,\tau)|^2\right)\right\} d\tau\, dx
$$

$$
\begin{aligned}
&= 2\int_{-\infty}^{\infty}\int_0^1\int_0^t |u_y|^2\,d\tau\,dy\,dx - i\int_{-\infty}^{\infty}\int_0^1 \left(y-\frac{1}{2}\right) u(t)\overline{u_y}(t)\,dy\,dx \\
&\quad + i\int_{-\infty}^{\infty}\int_0^1 \left(y-\frac{1}{2}\right)\varphi\overline{\varphi_y}\,dy\,dx + \frac{i}{2}\int_{-\infty}^{\infty}\int_0^t \left(\left(\overline{h_2}\right)_t h_2 + \left(\overline{h_1}\right)_t h_1\right)\,d\tau\,dx \\
&\quad - \int_{-\infty}^{\infty}\int_0^t h_2\overline{u_y}(x,1,\tau)\,d\tau\,dx + \int_{-\infty}^{\infty}\int_0^t h_1\overline{u_y}(x,0,\tau)\,d\tau\,dx - \frac{\lambda}{p}\int_{-\infty}^{\infty}\int_0^t |h_2|^p\,d\tau\,dx \\
&\quad - \frac{\lambda}{p}\int_{-\infty}^{\infty}\int_0^t |h_1|^p\,d\tau\,dx - \lambda\left(1-\frac{2}{p}\right)\int_{-\infty}^{\infty}\int_0^1\int_0^t |u|^p\,d\tau\,dy\,dx\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx \\
&= 4\|u_y\|_{L^2_{xyt}}^2 - 2i\int_{-\infty}^{\infty}\int_0^1 \left(y-\frac{1}{2}\right) u(t)\overline{u_y}(t)\,dy\,dx + 2i\int_{-\infty}^{\infty}\int_0^1 \left(y-\frac{1}{2}\right)\varphi\overline{\varphi_y}\,dy\,dx \\
&\quad + i\int_{-\infty}^{\infty}\int_0^t \left(\left(\overline{h_2}\right)_t h_2 + \left(\overline{h_1}\right)_t h_1\right)\,d\tau\,dx - 2\int_{-\infty}^{\infty}\int_0^t h_2\overline{u_y}(x,1,\tau)\,d\tau\,dx \\
&\quad + 2\int_{-\infty}^{\infty}\int_0^t h_1\overline{u_y}(x,0,\tau)\,d\tau\,dx - \frac{2\lambda}{p}\|h_2\|_{L^p_{xt}}^p - \frac{2\lambda}{p}\|h_1\|_{L^p_{xt}}^p - 2\lambda\left(1-\frac{2}{p}\right)\|u\|_{L^p_{xyt}}^p \\
&\quad + \|(h_1)_x\|_{L^2_{xt}}^2 + \|(h_2)_x\|_{L^2_{xt}}^2
\end{aligned}
$$

Consider the first case that $\lambda < 0$ and recall the inequality (2.5) $ab \le \frac{(\vartheta a)^q}{q} + \frac{(\frac{1}{\vartheta}a)^{q'}}{q'}$ for ϑ, a, $b \ge 0$.

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx \\
&\le 4\|u_y\|_{L^2_{xyt}}^2 + \|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} + \|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} + \|(h_1)_x\|_{L^2_{xt}}^2 + \|(h_2)_x\|_{L^2_{xt}}^2 \\
&+ \|(h_1)_t\|_{L^2_{xt}}\|h_1\|_{L^2_{xt}} + \|(h_2)_t\|_{L^2_{xt}}\|h_2\|_{L^2_{xt}} - \frac{2\lambda}{p}\|h_2\|_{L^p_{xt}}^p - \frac{2\lambda}{p}\|h_1\|_{L^p_{xt}}^p - 2\lambda\left(1-\frac{2}{p}\right)\|u\|_{L^p_{xyt}}^p \\
&+ 2\|h_1\|_{L^2_{xt}}\left(\int_{-\infty}^{\infty}\int_0^t |u_y(x,0,\tau)|^2\,d\tau\,dx\right)^{\frac{1}{2}} + 2\|h_2\|_{L^2_{xt}}\left(\int_{-\infty}^{\infty}\int_0^t |u_y(x,1,\tau)|^2\,d\tau\,dx\right)^{\frac{1}{2}} \\
&\le 4\|u_y\|_{L^2_{xyt}}^2 + \|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} + \|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} + \|(h_1)_x\|_{L^2_{xt}}^2 + \|(h_2)_x\|_{L^2_{xt}}^2 \\
&+ \|(h_1)_t\|_{L^2_{xt}}\|h_1\|_{L^2_{xt}} + \|(h_2)_t\|_{L^2_{xt}}\|h_2\|_{L^2_{xt}} - \frac{2\lambda}{p}\|h_2\|_{L^p_{xt}}^p - \frac{2\lambda}{p}\|h_1\|_{L^p_{xt}}^p - 2\lambda\left(1-\frac{2}{p}\right)\|u\|_{L^p_{xyt}}^p \\
&+ 2\|h_1\|_{L^2_{xt}}^2 + 2\|h_2\|_{L^2_{xt}}^2 + \frac{1}{2}\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx
\end{aligned}
$$

which provides the L^2-norm of u_y with respect to x, t on the boundary $y = 0$ and $y = 1$:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx \\
&\le 8\|u_y\|_{L^2_{xyt}}^2 + 2\|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} - 4\lambda\left(1-\frac{2}{p}\right)\|u\|_{L^p_{xyt}}^p
\end{aligned}
$$

$$+ 2\left\|(h_1)_t\right\|_{L^2_{xt}}\left\|h_1\right\|_{L^2_{xt}} + 2\left\|(h_2)_t\right\|_{L^2_{xt}}\left\|h_2\right\|_{L^2_{xt}} - \frac{4\lambda}{p}\|h_2\|^p_{L^p_{xt}} - \frac{4\lambda}{p}\|h_1\|^p_{L^p_{xt}}$$
$$+ 4\|h_1\|^2_{L^2_{xt}} + 4\|h_2\|^2_{L^2_{xt}} + 2\|(h_1)_x\|^2_{L^2_{xt}} + 2\|(h_2)_x\|^2_{L^2_{xt}} + 2\|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} \tag{4.94}$$

Then, for each $t \in [0,T]$, apply (3.94),

$$\begin{aligned}
&\|u(t)\|^2_{L^2_{xy}} = \int_{-\infty}^{\infty}\int_0^1 |u(x,y,t)|^2\,dy\,dx \\
&= \int_{-\infty}^{\infty}\int_0^1 |u(x,y,0)|^2\,dy\,dx + \int_{-\infty}^{\infty}\int_0^1\int_0^t \left(|u(x,y,\tau)|^2\right)_t\,d\tau\,dy\,dx \\
&= \|\varphi\|^2_{L^2_{xy}} - 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^1\int_0^t \left[(u_x(x,y,\tau)\overline{u}(x,y,\tau))_x + (u_y(x,y,\tau)\overline{u}(x,y,\tau))_y\right]\,d\tau\,dy\,dx \\
&= \|\varphi\|^2_{L^2_{xy}} + 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^t u_y(x,0,\tau)\overline{u}(x,0,\tau)\,d\tau\,dx - 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^t u_y(x,1,\tau)\overline{u}(x,1,\tau)\,d\tau\,dx \\
&\le \int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx + \|h_1\|^2_{L^2_{xt}} + \|h_2\|^2_{L^2_{xt}} + \|\varphi\|^2_{L^2_{xy}}
\end{aligned}$$

Now use the inequality (4.94) to replace $\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx$ in the result above.

$$\begin{aligned}
\|u(t)\|^2_{L^2_{xy}} \le\ & 8\|u_y\|^2_{L^2_{xyt}} + 2\|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} - 4\lambda\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} \\
&+\ 2\left\|(h_1)_t\right\|_{L^2_{xt}}\left\|h_1\right\|_{L^2_{xt}} + 2\left\|(h_2)_t\right\|_{L^2_{xt}}\left\|h_2\right\|_{L^2_{xt}} - \frac{4\lambda}{p}\|h_2\|^p_{L^p_{xt}} - \frac{4\lambda}{p}\|h_1\|^p_{L^p_{xt}} \\
&+\ 4\|h_1\|^2_{L^2_{xt}} + 4\|h_2\|^2_{L^2_{xt}} + 2\|(h_1)_x\|^2_{L^2_{xt}} + 2\|(h_2)_x\|^2_{L^2_{xt}} + 2\|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} \\
&+\ \|h_1\|^2_{L^2_{xt}} + \|h_2\|^2_{L^2_{xt}} + \|\varphi\|^2_{L^2_{xy}} \\
\le\ & 8\|u_y\|^2_{L^2_{xyt}} + \left(\frac{1}{2}\|u(t)\|^2_{L^2_{xy}} + 2\|u_y(t)\|^2_{L^2_{xy}}\right) - 4\lambda\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} \\
&+\ 2\left\|(h_1)_t\right\|_{L^2_{xt}}\left\|h_1\right\|_{L^2_{xt}} + 2\left\|(h_2)_t\right\|_{L^2_{xt}}\left\|h_2\right\|_{L^2_{xt}} - \frac{4\lambda}{p}\|h_2\|^p_{L^p_{xt}} - \frac{4\lambda}{p}\|h_1\|^p_{L^p_{xt}} \\
&+\ 5\|h_1\|^2_{L^2_{xt}} + 5\|h_2\|^2_{L^2_{xt}} + 2\|(h_1)_x\|^2_{L^2_{xt}} + 2\|(h_2)_x\|^2_{L^2_{xt}} + 2\|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} + \|\varphi\|^2_{L^2_{xy}}
\end{aligned}$$

After combining the same terms, we get

$$\begin{aligned}
\|u(t)\|^2_{L^2_{xy}} \le\ & 16\|u_y\|^2_{L^2_{xyt}} + 4\|u_y(t)\|^2_{L^2_{xy}} - 8\lambda\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} \\
&+\ 4\left\|(h_1)_t\right\|_{L^2_{xt}}\left\|h_1\right\|_{L^2_{xt}} + 4\left\|(h_2)_t\right\|_{L^2_{xt}}\left\|h_2\right\|_{L^2_{xt}} - \frac{8\lambda}{p}\|h_2\|^p_{L^p_{xt}} - \frac{8\lambda}{p}\|h_1\|^p_{L^p_{xt}} \\
&+\ 10\|h_1\|^2_{L^2_{xt}} + 10\|h_2\|^2_{L^2_{xt}} + 4\|(h_1)_x\|^2_{L^2_{xt}} + 2\|(h_2)_x\|^2_{L^2_{xt}} \\
&+\ 4\|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} + 2\|\varphi\|^2_{L^2_{xy}}
\end{aligned} \tag{4.95}$$

Applying the strategy used in *Proposition 3.97*, we integrate the identity (3.95) with respect to x, y and t so that with (4.94)

$$\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|u(t)\|^p_{L^p_{xy}} \\
&= \|\varphi_x\|^2_{L^2_{xy}} + \|\varphi_y\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|\varphi\|^p_{L^p_{xy}} \\
&- 2\mathrm{Re}\int_{-\infty}^{\infty}\int_0^t \left(\overline{h_1}\right)_t u_y(x,0,\tau)\,d\tau\,dx + 2\mathrm{Re}\int_{-\infty}^{\infty}\int_0^t \left(\overline{h_2}\right)_t u_y(x,1,\tau)\,d\tau\,dx \\
&\leq \frac{1}{128}\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,0,\tau)|^2 + |u_y(x,1,\tau)|^2\right)\,d\tau\,dx + 128\,\|(h_1)_t\|^2_{L^2_{xt}} + 128\,\|(h_2)_t\|^2_{L^2_{xt}} \\
&+ \|\varphi_x\|^2_{L^2_{xy}} + \|\varphi_y\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|\varphi\|^p_{L^p_{xy}} \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{64}\|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} \\
&+ c\Bigg(\|(h_1)_t\|_{L^2_{xt}}\,\|h_1\|_{L^2_{xt}} + \|(h_2)_t\|_{L^2_{xt}}\,\|h_2\|_{L^2_{xt}} - \frac{\lambda}{4p}\|h_2\|^p_{L^p_{xt}} - \frac{\lambda}{4p}\|h_1\|^p_{L^p_{xt}} \\
&+ \|h_1\|^2_{L^2_{xt}} + \|h_2\|^2_{L^2_{xt}} + \|(h_1)_x\|^2_{L^2_{xt}} + \|(h_2)_x\|^2_{L^2_{xt}} + \|\varphi\|_{L^2_{xy}}\|\varphi_y\|_{L^2_{xy}} \\
&+ \|(h_1)_t\|^2_{L^2_{xt}} + \|(h_2)_t\|^2_{L^2_{xt}} + \|\varphi_x\|^2_{L^2_{xy}} + \|\varphi_y\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|\varphi\|^p_{L^p_{xy}}\Bigg)
\end{aligned}$$

for some $c > 0$. We continue to use notations, $C = C(\|\varphi\|_{H^1_{xy}})$ and $C(t) = C(\|h_j\|_{H^1_{xt}})$, from *Section 3.5* as functions depending on $\|\varphi\|_{H^1_{xy}}$ and $\|h_j\|_{H^1_{xt}}$ ($j = 1, 2$), respectively; in particular, $C = 0$ if $\|\varphi\|_{H^1_{xy}} = 0$ and $C(t) = 0$ if $\|h_j\|_{H^1_{xt}} = 0$. Then using (4.95)

$$\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|u(t)\|^p_{L^p_{xy}} \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{64}\|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C \\
&+ \frac{1}{64}\|u_y(t)\|_{L^2_{xy}}\left\{16\|u_y\|^2_{L^2_{xyt}} + 4\|u_y(t)\|^2_{L^2_{xy}} - 8\lambda\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C\right\}^{\frac{1}{2}} \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C \\
&+ \frac{1}{64}\|u_y(t)\|_{L^2_{xy}}\left\{4\|u_y\|_{L^2_{xyt}} + 2\|u_y(t)\|_{L^2_{xy}} + \sqrt{8|\lambda|\left(1-\frac{2}{p}\right)}\|u\|^{\frac{p}{2}}_{L^p_{xyt}} + C(t) + C\right\} \\
&= \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + \frac{1}{16}\|u_y(t)\|_{L^2_{xy}}\|u_y\|_{L^2_{xyt}} + C(t) + C \\
&+ \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \sqrt{\frac{|\lambda|}{8^3}\left(1-\frac{2}{p}\right)}\|u_y(t)\|_{L^2_{xy}}\|u\|^{\frac{p}{2}}_{L^p_{xyt}} + \|u_y(t)\|_{L^2_{xy}}(C(t) + C)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{32}\|u_y\|^2_{L^2_{xyt}} + C(t) + C \\
&+ \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{64}\|u_y(t)\|^2_{L^2_{xy}} + \frac{|\lambda|}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} \\
&\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{16}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C
\end{aligned}
$$

Therefore

$$
\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{8\lambda}{3p}\|u(t)\|^p_{L^p_{xy}} \leq \frac{1}{6}\|u_y\|^2_{L^2_{xyt}} - \frac{\lambda}{12}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C\,.
$$

Moreover since $\lambda < 0$ here, we can get

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{8\lambda}{3p}\|u(t)\|^p_{L^p_{xy}} \leq \frac{1}{6}\left(\|u_x\|^2_{L^2_{xyt}} + \|u_y\|^2_{L^2_{xyt}} - \frac{8\lambda}{3p}\|u\|^p_{L^p_{xyt}}\right) + C(t) + C \\
&= \frac{1}{6}\int_0^t \left(\|u_x(\tau)\|^2_{L^2_{xy}} + \|u_y(\tau)\|^2_{L^2_{xy}} - \frac{8\lambda}{3p}\|u(\tau)\|^p_{L^p_{xy}}\right) d\tau + C(t) + C
\end{aligned}
$$

By Gronwell's inequality, we obtain

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{8\lambda}{3p}\|u(t)\|^p_{L^p_{xy}} \\
&\leq (C(t)+C)\cdot \exp\left(\int_0^t \frac{1}{6}\,d\tau\right) = (C(t)+C)\exp\left(\frac{t}{6}\right) := \psi_1
\end{aligned}
$$

where ψ_1 is an increasing function of $\|\varphi\|_{H^1_{xy}}$ and $\|h_j\|_{H^1_{xt}}, j = 1, 2$ and additionally $\psi_1 = 0$ if and only if $\|\varphi\|_{H^1_{xy}}$ $\|h_j\|_{H^1_{xt}}, j = 1, 2$ are zero. Thus

$$
\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} \leq \psi_1
$$

Note that for all the terms with $\|\cdot\|_{L^p}$, one can bound them with H^1-norms according to Sobolev embedding theorem on the domain of dimension 2.

Thus it is clear that $\|u(t)\|_{H^1_{xy}}$ is uniformly bounded for any given T if $\lambda < 0$.

On the other hand, we take $\lambda > 0$ on account. Analogous to the previous argument, (4.95) implies

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} - \frac{2\lambda}{p}\|u(t)\|^p_{L^p_{xy}} \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{64}\|u(t)\|_{L^2_{xy}}\|u_y(t)\|_{L^2_{xy}} - \frac{\lambda}{32}\left(1-\frac{2}{p}\right)\|u\|^p_{L^p_{xyt}} + C(t) + C \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{64}\|u_y(t)\|_{L^2_{xy}}\left\{16\|u_y\|^2_{L^2_{xyt}} + 4\|u_y(t)\|^2_{L^2_{xy}} + C(t) + C\right\}^{\frac{1}{2}} + C(t) + C \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{64}\|u_y(t)\|_{L^2_{xy}}\left\{4\|u_y\|_{L^2_{xyt}} + 2\|u_y(t)\|_{L^2_{xy}} + C(t) + C\right\} + C(t) + C
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{16}\|u_y(t)\|_{L^2_{xy}}\|u_y\|_{L^2_{xyt}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \|u_y(t)\|_{L^2_{xy}}(C(t)+C) + C(t) + C \\
&\leq \frac{1}{16}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{32}\|u_y\|^2_{L^2_{xyt}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{32}\|u_y(t)\|^2_{L^2_{xy}} + C(t) + C \\
&\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + C(t) + C
\end{aligned}
$$

By using Gagliardo-Nirenberg inequality (2.8) and Hölder's inequality,

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} \\
&\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + \frac{2\lambda}{p}\|u(t)\|^p_{L^p_{xy}} + C(t) + C \\
&\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + \frac{2\lambda}{p}\left(\|u_x(t)\|_{L^2_{xy}} + \|u_y(t)\|_{L^2_{xy}}\right)^{p-2} \cdot \|u(t)\|^2_{L^2_{xy}} + C(t) + C
\end{aligned}
$$

Recall an earlier result that with (3.94)

$$
\begin{aligned}
&\|u(t)\|^2_{L^2_{xy}} = \int_{-\infty}^{\infty}\int_0^1 |u(x,y,t)|^2\,dy\,dx \\
&= \int_{-\infty}^{\infty}\int_0^1 |u(x,y,0)|^2\,dy\,dx + \int_{-\infty}^{\infty}\int_0^1\int_0^t \left(|u(x,y,\tau)|^2\right)_t\,d\tau\,dy\,dx \\
&= \|\varphi\|^2_{L^2_{xy}} - 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^1\int_0^t [(u_x(x,y,\tau)\overline{u}(x,y,\tau))_x + (u_y(x,y,\tau)\overline{u}(x,y,\tau))_y]\,d\tau\,dy\,dx \\
&= \|\varphi\|^2_{L^2_{xy}} + 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^t u_y(x,0,\tau)\overline{u}(x,0,\tau)\,d\tau\,dx - 2\mathrm{Im}\int_{-\infty}^{\infty}\int_0^t u_y(x,1,\tau)\overline{u}(x,1,\tau)\,d\tau\,dx \\
&\leq \left(\int_{-\infty}^{\infty}\int_0^t \left(|u_y(x,1,\tau)|^2 + |u_y(x,0,\tau)|^2\right)\,d\tau\,dx\right)^{\frac{1}{2}} \cdot K(t) + C
\end{aligned}
$$

where we let $K(t) = c\max\{\|h_1\|_{L^2_{xt}}, \|h_2\|_{L^2_{xt}}\}$ for any possible constant $c > 0$. Then plug (4.94) into the above inequality to obtain

$$
\begin{aligned}
\|u(t)\|^2_{L^2_{xy}} &\leq \|u_y\|_{L^2_{xyt}} \cdot K(t) + \|u(t)\|^{\frac{1}{2}}_{L^2_{xy}}\|u_y(t)\|^{\frac{1}{2}}_{L^2_{xy}} \cdot K(t) + C(t) + C \\
&\leq \|u_y\|_{L^2_{xyt}} \cdot K(t) + \frac{1}{4}\|u(t)\|^2_{L^2_{xy}} + \frac{3}{4}\|u_y(t)\|^{\frac{2}{3}}_{L^2_{xy}} \cdot K(t)^{\frac{4}{3}} + C(t) + C,
\end{aligned}
$$

i.e.,

$$
\|u(t)\|^2_{L^2_{xy}} \leq \|u_y\|_{L^2_{xyt}} \cdot K(t) + \|u_y(t)\|^{\frac{2}{3}}_{L^2_{xy}} \cdot K(t)^{\frac{4}{3}} + C(t) + C \tag{4.96}
$$

We substitute the revised estimate on the $L^\infty L^2$-norm of u into the inequality for derivatives of u.

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} \\
&\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + \left(\|u_x(t)\|_{L^2_{xy}} + \|u_y(t)\|_{L^2_{xy}}\right)^{p-2} \cdot \|u(t)\|^2_{L^2_{xy}} + C(t) + C
\end{aligned}
$$

$$\leq \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + C(t) + C$$
$$+ \frac{2\lambda}{p}\left(\|u_x(t)\|^{p-2}_{L^2_{xy}} + \|u_y(t)\|^{p-2}_{L^2_{xy}}\right)\cdot\left(\|u_y\|_{L^2_{xyt}}\cdot K(t) + \|u_y(t)\|^{\frac{2}{3}}_{L^2_{xy}}\cdot K(t)^{\frac{4}{3}} + C(t) + C\right)$$
$$= \frac{1}{8}\|u_y(t)\|^2_{L^2_{xy}} + \frac{1}{8}\|u_y\|^2_{L^2_{xyt}} + C(t) + C + \|u_x(t)\|^{p-2}_{L^2_{xy}}\|u_y(t)\|^{\frac{2}{3}}_{L^2_{xy}}\cdot K(t)^{\frac{4}{3}}$$
$$+ \|u_y(t)\|^{\frac{3p-4}{3}}_{L^2_{xy}}\cdot K(t)^{\frac{4}{3}} + \left(\|u_x(t)\|^{p-2}_{L^2_{xy}} + \|u_y(t)\|^{p-2}_{L^2_{xy}}\right)\|u_y\|_{L^2_{xyt}}\cdot K(t)$$
$$+ \left(\|u_x(t)\|^{p-2}_{L^2_{xy}} + \|u_y(t)\|^{p-2}_{L^2_{xy}}\right)(C(t) + C)$$

It turns out that the uniform boundedness can be derived only when $p \leq 3$ in this case. If we allow $p < 3$, then we are able to derive the result:

$$\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} \leq c\int_0^t \left(\|u_x(\tau)\|^2_{L^2_{xy}} + \|u_y(\tau)\|^2_{L^2_{xy}}\right)\, d\tau + C(t) + C$$

Using Gronwell's inequality

$$\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}} \leq (C(t) + C)e^{ct}\,.$$

If we let $p = 3$, then

$$(1 - K(t))\left(\|u_x(t)\|^2_{L^2_{xy}} + \|u_y(t)\|^2_{L^2_{xy}}\right) \leq c\int_0^t \left(\|u_x(\tau)\|^2_{L^2_{xy}} + \|u_y(\tau)\|^2_{L^2_{xy}}\right)\, d\tau + C(t) + C$$

Since $K(t) = c\max\{\|h_1\|_{L^2_{xt}}(\mathbb{R}\times[0,T]), \|h_2\|_{L^2_{xt}}(\mathbb{R}\times[0,T])\} < \infty$. We can again partition [0,T] into a finite number of subintervals (t_{j-1}, t_j) for $j = 1, \cdots, m$ for $m \in N$ with $\sup_j |t_j - t_{j-1}| \sim \delta$ so that, on each interval, $\|h_1\|_{L^2_{xt}(\mathbb{R}\times[t_{j-1},t_j]}$ and $\|h_2\|_{L^2_{xt}(\mathbb{R}\times[t_{j-1},t_j]} < \frac{1}{2c}$. Then starting from $[0,\delta]$, we move over one subinterval and use $u(t_j)$ as the initial value for the argument on (t_j, t_{j+1}). At the end $\sup_{t\in[0,T]}\|u(t)\|_{H^1(\mathbb{R}\times[0,1]))}$ is uniformly bounded by a function depending on the initial and boundary data only. Especially, we let $\psi_2 = (C + C(t))\left(\frac{1}{2} - K(t)\right)^{-1}$. Hence, as a final conclusion, if we let $\psi = \psi_1$ when $p \geq 3$ for $\lambda < 0$ and $\psi = \psi_2$ when $p = 3$ as $\lambda > 0$, $\sup_{t\in[0,T]}\|u(t)\|_{H^1(\mathbb{R}\times[0,1]))}$ is uniformly bounded by ψ. Then the proof is done. □

Eventually, we finish the discussion for global well-posedness of the IBVP (4.1) from *Theorem 4.11*, *Proposition 4.12* and *4.17* as follows.

Theorem 4.18. *Assume that either $p \geq 3$ and $\lambda < 0$ or $p = 3$ and $\lambda > 0$. Then (4.1) is globally well-posed in $H^1(\mathbb{R}\times[0,1])$ if $\varphi \in H^1(\mathbb{R}\times[0,1])$ and $h_j \in H^1_{t\text{-}loc}(\mathbb{R}; L^2_x(\mathbb{R})) \cap L^2_{t\text{-}loc}(\mathbb{R}; H^2_x(\mathbb{R}))$ for $j = 1, 2$.*

Chapter 5

2D Nonlinear Schrödinger Equations in a Square Domain

In this chapter, we briefly discuss the formulation of the NLS equation posed over a square domain $(x, y) \in [0, L] \times [0, L]$ for some $L > 0$. The structure of this IBVP is alike the two-dimensional periodic problem. The study on this topic will even more rely on the Fourier series and Bourgain's method, the "discrete" version of the Strichartz's estimates. However, we only provide the formulation of this problem and leave the work on the estimates and contraction mapping argument in the future.

5.1 Formulation of the Problem

Let $L = 1$. For some $T \in (0, \infty]$, the IBVP is posed on the square domain $[0, 1] \times [0, 1]$. Recall that for $(x, y) \in [0, 1]^2$ and some $T \in (0, \infty]$, the problem is presented as follows:

$$\begin{cases} iu_t + u_{xx} + u_{yy} + g = 0 & (x, y, t) \in (0, 1)^2 \times (0, T) \\ u(x, y, 0) = \varphi(x, y) & (x, y) \in (0, 1)^2 \\ u(0, y, t) = h_1(y, t) & (y, t) \in (0, 1) \times (0, T) \\ u(1, y, t) = h_2(y, t) & (y, t) \in (0, 1) \times (0, T) \\ u(x, 0, t) = h_3(x, t) & (x, t) \in (0, 1) \times (0, T) \\ u(x, 1, t) = h_4(x, t) & (x, t) \in (0, 1) \times (0, T) \end{cases} \tag{5.1}$$

with a group of compatibility conditions listed below:

$$\begin{cases} t = 0 & h_1(y, 0) = h_2(y, 0) = h_3(x, 0) = h_4(x, 0) = 0 \\ (x, y) = (0, 0) & h_1(0, t) = h_3(0, t) \\ (x, y) = (0, 1) & h_1(1, t) = h_4(0, t) \\ (x, y) = (1, 0) & h_2(0, t) = h_3(1, t) \\ (x, y) = (1, 1) & h_2(1, t) = h_4(1, t) \end{cases} \tag{5.2}$$

where

$$g(x,y,t) := \lambda|u(x,y,t)|^{p-2}u(x,y,t) \qquad \text{for } p \geq 3 \text{ and } (x,y,t) \in (0,1)^2 \times (0,T)$$

We are interested in the solution of (5.1) in $C([0,T]; H^s([0,1]\times[0,1]))$ for some properly chosen φ and h_j's for $j = 1, \cdots, 4$. Following the previous chapters, we perform a procedure of decomposition so that we can separately study the linear equations with only nonhomogeneous initial data, the nonlinear equation with homogeneous initial and boundary conditions, and the linear equations with nonhomogeneous boundary conditions, respectively.

Firstly, there is a linear problem provided with homogeneous boundary information

$$\begin{cases} iv_t + v_{xx} + v_{yy} = 0 & (x,y,t) \in (0,1)\times(0,1)\times(0,T) \\ v(x,y,0) = \varphi(x,y) \\ v(0,y,t) = v(1,y,t) = v(x,0,t) = v(x,1,t) = 0 \end{cases} \tag{5.3}$$

where we denote the solution by $W_0(t)\phi$ as before. Note that $W_0(t)\phi$ is still a C_0-semigroup whose infinitesimal generator is $i\Delta$.

Next, we have the nonlinear equation

$$\begin{cases} iw_t + w_{xx} + w_{yy} + g(w) = 0 & (x,y,t) \in (0,1)\times(0,1)\times(0,T) \\ w(x,y,0) = 0 & . \\ w(0,y,t) = w(1,y,t) = w(x,0,t) = w(x,1,t) = 0 \end{cases} \tag{5.4}$$

Use the semigroup theory and denote the solution by $w := \Phi_{0,g}(t) := i\int_0^t W_0(t-\tau)f(\tau)\,d\tau$.

Thirdly, consider the linear equation with nonhomogeneous boundary conditions which requires the major attention for this IBVP.

$$\begin{cases} iz_t + z_{xx} + z_{yy} = 0 & (x,y,t) & \in (0,1)\times(0,1)\times(0,T) \\ z(x,y,0) = 0 & (x,y) & \in \mathbb{R}\times(0,1) \\ z(0,y,t) = h_1(y,t) & (y,t) & \in (0,1)\times(0,T) \\ z(1,y,t) = h_2(y,t) & (y,t) & \in (0,1)\times(0,T) \\ z(x,0,t) = h_3(x,t) & (x,t) & \in (0,1)\times(0,T) \\ z(x,1,t) = h_4(x,t) & (x,t) & \in (0,1)\times(0,T) \end{cases} \tag{5.5}$$

We here write the solution as $z := W_b(h_1,h_2,h_3,h_4)$.

In summary, we shall be able to generate the integral equation equivalent to (5.1) as

$$u(x,y,t) = W_b(h_1,h_2,h_3,h_4)(x,y,t) + W_0(t)\varphi(x,y) + i\left(\int_0^t W_0(t-\tau)|u(\tau)|^{p-2}u(\tau)\,d\tau\right)(x,y) \tag{5.6}$$

Moreover, the following lemma guarantees the equivalency.

Lemma 5.1. *For $s \geq 0$ and every $f(t) : H^s \to H^\sigma \ \forall t \in [0,T]$, $u \in C([0,T]; H^s) \bigcap C^1([0,T]; H^{\min\{s-2,\sigma\}})$ is a solution of (5.1) if and only if (5.6) holds for u.*

5.2 Representations of Solution Operators

Let $\phi : [-1,1] \times [-1,1] \to \mathbb{C}$ be the odd extension of φ in both of negative x-axis and y-axis; i.e. $\phi(x,y) = \varphi(x,y)$ on $[0,1] \times [0,1]$. Then $W_0\phi$ on $[-1,1] \times [-1,1] \times [0,T]$ solves (5.3) whose solution can be formulated in the following form.

Proposition 5.2.

$$W_0(t)\phi(x,y) = \sum_{m,n\in\mathbb{Z}} e^{-i\pi^2\left(m^2+n^2\right)t+i\pi(mx+ny)} \cdot \widehat{\phi}(m,n) \tag{5.7}$$

We let $f : [-1,1] \times [-1,1] \times [0,T] \to \mathbb{C}$ be an odd extension of g in both negative x-axis and negative y-axis; i.e. $f(x,y,t) = g(x,y,t)$ for x, $y \in [0,1]$. Thus the extended solution to (5.4), denoted by $\Phi_{0,f}$, is given in the next proposition.

Proposition 5.3.

$$\begin{aligned} \Phi_{0,f}(x,y,t) &= i\left(\int_0^t W_0(t-\tau)f(\tau)\,d\tau\right)(x,y) \\ &= \int_0^t \sum_{m,n\in\mathbb{Z}} e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)+i\pi(mx+ny)}\widehat{f^{xy}}(m,n,\tau)\,d\tau \end{aligned} \tag{5.8}$$

for x, $y \in [0,1]$ and $t \geq 0$.

Now consider the linear problem with the nonhomogeneous boundary conditions (5.5):

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0 & (x,y,t) \in (0,1)\times(0,1)\times(0,T) \\ u(x,y,0) = 0 & (x,y) \in (0,1)\times(0,1) \\ u(0,y,t) = h_1(y,t) & (y,t) \in (0,1)\times(0,T) \\ u(1,y,t) = h_2(y,t) & (y,t) \in (0,1)\times(0,T) \\ u(x,0,t) = h_3(x,t) & (x,t) \in (0,1)\times(0,T) \\ u(x,1,t) = h_4(x,t) & (x,t) \in (0,1)\times(0,T) \end{cases}$$

equipped with some compatibility conditions as (5.2):

$$\begin{cases} t = 0 & h_1(y,0) = h_2(y,0) = h_3(x,0) = h_4(x,0) = 0 \\ (x,y) = (0,0) & h_1(0,t) = h_3(0,t) \\ (x,y) = (0,1) & h_1(1,t) = h_4(0,t) \\ (x,y) = (1,0) & h_2(0,t) = h_3(1,t) \\ (x,y) = (1,1) & h_2(1,t) = h_4(1,t) \end{cases}$$

To analyze this problem, we need to further decompose it to obtain firstly

$$\begin{cases} iv_t + v_{xx} + v_{yy} = 0 & (x,y,t) \in \mathbb{R} \times (0,1) \times (0,T) \\ v(x,y,0) = 0 & (x,y) \in \mathbb{R} \times (0,1) \\ v(x,0,t) = h_3(x,t) & (x,t) \in \mathbb{R} \times (0,T) \\ v(x,1,t) = h_4(x,t) & (x,t) \in \mathbb{R} \times (0,T) \end{cases} \tag{5.9}$$

where each h_j is considered as an extension of the original boundary function to $x \in \mathbb{R}$ for $j = 3$, 4 and the solution is denoted by $W_{b,y}[h_3, h_4]$ stand for the solution. We also have

$$\begin{cases} iw_t + w_{xx} + w_{yy} = 0 & (x,y,t) \in (0,1) \times (0,1) \times (0,T) \\ w(x,y,0) = 0 & (x,y) \in (0,1) \times (0,1) \\ w(0,y,t) = h_1(y,t) - W_{b,y}[h_4,h_4]\big|_{x=0} & (y,t) \in (0,1) \times (0,T) \\ w(1,y,t) = h_2(y,t) - W_{b,y}[h_4,h_4]\big|_{x=1} & (y,t) \in (0,1) \times (0,T) \\ w(x,0,t) = 0 & (x,t) \in (0,1) \times (0,T) \\ w(x,1,t) = 0 & (x,t) \in (0,1) \times (0,T) \end{cases} \tag{5.10}$$

Obviously for smooth solutions we get the following conditions

$$\begin{cases} w(0,0,t) = w(0,1,t) = 0 \\ w(1,0,t) = w(1,1,t) = 0 \end{cases}$$

We can obtain the necessary results (estimates) of (5.9) from the study of the "strip domain" problem; however for (5.10) we have to start from the very beginning.

Now we rewrite (5.10) as

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0 & (x,y,t) \in (0,1) \times (0,1) \times (0,T) \\ u(x,y,0) = 0 & (x,y) \in (0,1) \times (0,1) \\ u(0,y,t) = h_1(y,t) & (y,t) \in (0,1) \times (0,T) \\ u(1,y,t) = h_2(y,t) & (y,t) \in (0,1) \times (0,T) \\ u(x,0,t) = u(x,1,t) = 0 & (x,t) \in (0,1) \times (0,T) \end{cases} \tag{5.11}$$

where

$$h_1(y,0) = h_2(y,0) = h_1(0,t) = h_2(1,t) = h_1(1,t) = h_2(0,t) = 0$$

We denote the solution by $W_{b,x}[h_1, h_2]$. Thus we can expect to have the following formulation.

Proposition 5.4.

$$\begin{aligned} u(x,y,t) &= W_{b,x}[h_1,h_2](x,y,t) \\ &= \pi \sum_{m,n} \left[n \int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)+i\pi(mx+ny)} \left(\widehat{h_1} - (-1)^n \widehat{h_2}\right)(m,\tau)\, d\tau \right]. \end{aligned} \tag{5.12}$$

Proof. In order to reduce the difficulty of the investigation, we assume $h_2 \equiv 0$, i.e.,

$$\begin{cases} iu_t + u_{xx} + u_{yy} = 0 & (x,y,t) \in (0,1)\times(0,1)\times(0,T) \\ u(x,y,0) = 0 & (x,y) \in (0,1)\times(0,1) \\ u(0,y,t) = h_1(y,t) & (y,t) \in (0,1)\times(0,T) \\ u(1,y,t) = u(x,0,t) = u(x,1,t) = 0 & (x,t) \in (0,1)\times(0,T) \end{cases}$$

Let $z = u - (1-x)h_1(y,t)$. Therefore,

$$\begin{aligned} iz_t &= iu_t - i(1-x)\partial_t h_1 \\ z_{xx} &= u_{xx} \\ z_{yy} &= u_{yy} - (1-x)\partial_y^2 h_1 \end{aligned}$$

which yield

$$\begin{cases} iz_t + z_{xx} + z_{yy} = (x-1)\left(i\partial_t + \partial_y^2\right) h_1 \\ z(x,y,0) = u(x,y,0) - (1-x)h_1(y,0) = 0 \\ z(x,0,t) = u(x,0,t) - (1-x)h_1(0,t) = 0 \\ z(x,1,t) = u(x,1,t) - (1-x)h_1(1,t) = 0 \\ z(0,y,t) = h_1(y,t) - h_1(y,t) = 0 \\ z(1,y,t) = 0 \end{cases}$$

Write $z(x,y,t) = \sum_{m,n=1}^{\infty} \alpha_{m,n}(t)\cdot\sin(m\pi x)\cdot\sin(n\pi y)$ as a convergent series. Thus, we can derive the following equality,

$$\begin{aligned} i\int_0^1\int_0^1 \sum_{k,l}^{\infty} \alpha_{k,l}'(t)\cdot\sin(k\pi x)\cdot\sin(l\pi y)\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dy\,dx \\ -\int_0^1\int_0^1 \sum_{k,l}^{\infty} m^2\pi^2\cdot\alpha_{k,l}(t)\sin(k\pi x)\cdot\sin(l\pi y)\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dy\,dx \\ -\int_0^1\int_0^1 \sum_{k,l}^{\infty} n^2\pi^2\cdot\alpha_{k,l}(t)\sin(k\pi x)\cdot\sin(l\pi y)\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dy\,dx \\ = \int_0^1\int_0^1 (x-1)\left(\partial_t + \partial_y^2\right)\cdot h_1(y,t)\cdot\sin(m\pi x)\cdot\sin(n\pi y)\,dy\,dx \end{aligned}$$

which yields

$$\begin{aligned} &\frac{i\alpha_{m,n}'(t) - \left(m^2+n^2\right)\pi^2\alpha_{m,n}(t)}{4} \\ &= \int_0^1 (x-1)\sin(m\pi x)\,dx\cdot\int_0^1 \left(\partial_t + \partial_y^2\right) h_1(y,t)\sin(n\pi y)\,dy \end{aligned}$$

$$= -\frac{1}{m\pi}\int_0^1 \left(i\partial_t h_1 - n^2\pi^2 h_1\right)\sin(n\pi y)\,dy$$

Note that

$$\begin{aligned}
&\int_0^1 \partial_y^2 h_1 \sin(n\pi y)\,dy \\
&\quad= \left(\partial_y h_1 \sin(n\pi y)\right)\Big|_0^1 - n\pi\int_0^1 \partial_y h_1 \cos(n\pi y)\,dy = -n\pi\int_0^1 \partial_y h_1\cos(n\pi y)\,dy \\
&\quad= -n\pi\; h_1(y,t)\cos(n\pi y)\Big|_0^1 - n^2\pi^2\int_0^1 h_1(y,t)\sin(n\pi y)\,dy \\
&\quad= -n^2\pi^2\int_0^1 h_1(y,t)\sin(n\pi y)\,dy
\end{aligned}$$

Therefore

$$\alpha'_{m,n}(t) + i\left(m^2+n^2\right)\pi^2\alpha_{m,n}(t) = -\frac{4}{m\pi}\int_0^1\left(\partial_t h_1 + in^2\pi^2 h_1\right)\sin(n\pi y)\,dy$$

which implies

$$\begin{aligned}
&\alpha_{m,n}(t) \\
&= \frac{-4}{m\pi}\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}\cdot\int_0^1\left(\partial_t h_1 + in^2\pi^2 h_1\right)\sin(n\pi y)\,dy\,d\tau \\
&= \frac{-4}{m\pi}\int_0^1\left[\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}\partial_t h_1(y,\tau)\,d\tau\right. \\
&\qquad \left. + in^2\pi^2\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)} h_1(y,\tau)\,d\tau\right]\sin(n\pi y)\,dy \\
&= \frac{-4}{m\pi}\int_0^1\left[h_1(y,t) - i(m^2+n^2)\pi^2\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}h_1(y,\tau)\,d\tau\right. \\
&\qquad \left. + in^2\pi^2\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)} h_1(y,\tau)\,d\tau\right]\sin(n\pi y)\,dy \\
&= \frac{-4}{m\pi}\int_0^1 h_1(y,t)\sin(n\pi y)\,dy + 4im\pi\int_0^1\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}h_1(y,\tau)\sin(n\pi y)\,d\tau\,dy\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&z(x,y,t) \\
&= \sum_{m,n=1}^{\infty}\frac{-4}{m\pi}\left(\int_0^1 h_1(\eta,t)\sin(n\pi\eta)\,d\eta\right)\sin(m\pi x)\sin(n\pi y) \\
&\qquad + \sum_{m,n=1}^{\infty} 4im\pi\left(\int_0^1\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}h_1(\eta,\tau)\sin(n\pi\eta)\,d\tau\,d\eta\right)\sin(m\pi x)\sin(n\pi y) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{-4}{m\pi}\left(\int_0^1 h_1(\eta,t)\sin(n\pi\eta)\,d\eta\right)\sin(m\pi x)\sin(n\pi y)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 4im\pi \sin(m\pi x)\left(\int_0^1\int_0^t e^{-i\pi^2\left(m^2+n^2\right)(t-\tau)} h_1(\eta,\tau)\sin(n\pi\eta)\,d\tau\,d\eta\right)\sin(n\pi y)\\
&\approx (x-1)h_1(y,t) + 4\pi i \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t me^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}\\
&\quad \cdot \sin(m\pi x)\sin(n\pi y)\left(\int_0^1 h_1(\eta,t)\sin(n\pi\eta)\,d\eta\right)\,d\tau
\end{aligned}
$$

Thus

$$
\begin{aligned}
&u(x,y,t)\\
&\approx 4\pi i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t me^{-i\pi^2\left(m^2+n^2\right)(t-\tau)}\cdot\sin(m\pi x)\sin(n\pi y)\left(\int_0^1 h_1(\eta,t)\sin(n\pi\eta)\,d\eta\right)\,d\tau\\
&\approx \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t me^{-i\pi^2\left(m^2+n^2\right)(t-\tau)+i\pi(xm+yn)}\widehat{h_1^y}(n,\tau)\,d\tau
\end{aligned}
$$

$\square$

Remark. The estimates and the well-posedness argument will be completed in the future.

Chapter 6

Future Work

Here is a list of topics that I am interested in and plan to start the research in the future.

(i) In this thesis, we discussed well-posedness of the initial boundary value problem for the nonlinear Schrödinger equation (1.1) posed on the half plane $\mathbb{R}\times\mathbb{R}^+$ (which can be generalized to the half space $\mathbb{R}^n \times \mathbb{R}^+$) and the strip domain $\mathbb{R} \times [0, 1]$. For the IBVP on the strip domain, I plan to extend the discussion by considering for $p \geq 4$ when $0 \leq s < 1$. Also, I have started the study of the IBVP posed on the square domain $[0, 1] \times [0, 1]$ by formulating the solution if it exists and then I will complete the study on well-posedness (local and global).

(ii) Moreover, I am interested in expanding my research in regard to well-posedness for the NLS equation (1.1) posed on a domain with a more general shape, for example, inside and outside a region enclosed by certain smooth boundary.

(iii) I am amazed by the stochastic NLS equation as well. Therefore, I plan to begin the study of the following problem in the future: for $\Omega \subset \mathbb{R}^N$ with $N = 2$ or 3 and some interval $I \ni 0$, let $\lambda \in \mathbb{R}$,

$$\begin{cases} id\mathcal{U} + (\Delta\mathcal{U} + \lambda|\mathcal{U}|^{p-2}\mathcal{U})\,dt = \sigma(\mathcal{U})dW_t & (\boldsymbol{\omega}, t) \in \Omega \times I \\ \mathcal{U}(\boldsymbol{\omega}, 0) = \varphi(\boldsymbol{\omega}) \\ \mathcal{U}(\boldsymbol{\omega}, t)|_{\partial\Omega} = h(\boldsymbol{\omega}, t) \end{cases}$$

where $\mathcal{U}(t)$ denotes a random variable in the probability space $(\Omega, \mathcal{F}, P)$. In the stochastic term $\sigma(\mathcal{U})dW_t$, W_t represents the standard Brownian motion.

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