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TRANSFER FUNCTION APPROACH TO OUTPUT SPECIFICATION IN CERTAIN LINEAR DISTRIBUTED PARAMETER SYSTEMS

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Abstract. In this paper we study, using Laplace transform methods, some questions of *output determination* for the wave equation and the Euler–Bernoulli beam equation. Specifically we study the problem of determining, over a specified time interval, the displacement at a particular spatial point, via control exercised by means of an external force applied at another spatial point. Some indications of a more general theory are given.

1. **Output Specification.** Most of this paper is concerned with quite specific systems, the wave equation and the Euler–Bernoulli beam equation, but we will introduce our subject in a somewhat more abstract setting. We consider a linear input–output system in a Banach (often Hilbert) space $\mathcal{W}$ with scalar input $u(t)$ and scalar output $y(t)$:

$$\dot{w} = A\,w + b\,u, \quad y = \langle w, c\rangle \equiv c^*\,w, \quad w \in \mathcal{W}, \; u \in \mathcal{U}. \tag{1.1}$$

We suppose that the (generally unbounded) linear operator A generates a strongly continuous semigroup $S(t), \; t \geq 0$, in $\mathcal{W}$. The simplest situation arises when b and c are elements of $\mathcal{W}$ and its dual $\mathcal{W}^{'}$, respectively, but, to include the most important and interesting cases, we allow b to lie in a larger space than $\mathcal{W}$ in general and we allow c to be an unbounded linear functional on $\mathcal{W}$ with a domain smaller than $\mathcal{W}$ itself. In any case, $\langle w, c\rangle$ denotes the value of the linear functional c at $w \in \mathcal{W}$ which, for convience, we also write as c^*w. We do require that (1.1) should be an *admissible* input–output system in the sense developed in [4], [17] and numerous other papers and it may be necessary to interpret c^*w in the integrated sense discussed there. Similar considerations apply to b when it does not lie in $\mathcal{W}$. The *input space* $\mathcal{U}$ will usually, but not invariably, be taken to be $L^2[0, \infty)$.

With zero initial state $w(0) = 0$ in $\mathcal{W}$ we have, for almost all $t \geq 0$,

$$y(t) = c^* \int_0^t S(t-\tau) b\, u(\tau)\, d\tau. \tag{1.2}$$

The admissibility condition implies that on each finite interval $[0, T]$ the mapping so defined carries $\mathcal{U}$ into a designated *output space* $\mathcal{Y}$ and is a bounded operator. The *output specification problem* is to find a particular $T > 0$ and a Hilbert space

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$\mathcal{Z} \subset \mathcal{Y}$ of functions $z(t)$ defined on $[0, \infty)$, with an appropriate norm $\| \, \|_{\mathcal{Z}}$, at least as strong as $\| \, \|_{\mathcal{Y}}$, such that, given any $z \in \mathcal{Z}$ we can find an input $u \in \mathcal{U}$ such that the corresponding output $y(t)$ agrees with $z(t-T)$ on the interval $[T, \infty)$. If this is the case we say that (1.1) is *output specifiable* in $\mathcal{Z}$. It will be recognized at once that this is a *control problem* for (1.1). We should take particular pains to point out that, as such, it is quite distinct from the classical *tracking problem* (cf., e.g., [2]). It has been discussed, as in [7], [10], under the subject heading of *motion planning*. However, the literature in that area is primarily concerned with geometric problems involved in robot kinematics so we have elected to use the terms "output specification" or "output determination". In addition to the above references, the work of the present article is also related to that of [6] and [8]. In our work the analysis is carried out in the "frequency domain" via estimation and manipulation of the transfer function and solution of exponential moment problems rather than via delay systems as in the last two articles or via construction of motion planning operators as in the first two cited articles.

Using $\hat{f}(s)$ to denote the Laplace transform of an exponentially bounded, locally square integrable function $f(t)$ defined on $[0, \infty)$,

$$\hat{y}(s) = \mathcal{T}(s)\,\hat{u}(s); \quad \mathcal{T}(s) = c^*\,(s\,I - A)^{-1}\,b \tag{1.3}$$

being the so-called *transfer function* of the system. In many cases where the abstract system (1.1) is realized in terms of a partial differential equation it is rather difficult to get a closed form expression for $\mathcal{T}(s)$ directly from this definition. Fortunately, the transfer function can be computed by an alternate procedure. Choosing s with $Re(s)$ large enough so that $e^{-st}\,S(t) = e^{(A-sI)t}$ has uniform exponential decay, we consider exponential inputs $u(t,s) = e^{st}$ and denote the outputs by $y(s,t)$. Replacing the initial time $t = 0$ by $t = -\tau_0,\ \tau_0 > 0$ and defining $w(-\tau_0) = 0$ we can write

$$y(s,t) = c^* \int_{-\tau_0}^{t} S(t-\tau)\,b\,e^{s\tau}\,d\tau = e^{st}\,c^* \int_{-\tau_0}^{t} e^{(A-sI)(t-\tau)}\,d\tau\,b.$$

Letting $t - \tau = \sigma$, we have

$$y(s,t) = e^{st}\,c^* \int_0^{t+\tau_0} e^{(A-sI)\sigma}\,d\sigma\,b = e^{st}\,c^*(sI-A)^{-1}\left(I - e^{(A-sI)(t+\tau_0)}\right)b.$$

Letting $\tau_0 \to \infty$ we obtain, in the limit,

$$y(s,t) = e^{st}\,c^*(sI-A)^{-1}\,b = e^{st}\,\mathcal{T}(s). \tag{1.4}$$

In other words, to obtain the transfer function we seek, we use an exponential input $u(t) = e^{st}$ and look for the corresponding timewise exponential solution for which $lim_{\tau_0 \to \infty} w(-\tau_0) = 0$. Then, dividing the resulting output by e^{st} we obtain $\mathcal{T}(s)$. This is, of course, simply a variation on the familiar representation of the resolvent of a semigroup generator in terms of an operator-valued integral of the semigroup.

If in (1.2), (1.3) we choose a desired output $y(t)$ with transform $\hat{y}(s)$ we might (naively) hope to find an input $u(t)$ realizing this output via the inverse relationship

$$\hat{u}(s) = \mathcal{T}(s)^{-1}\,\hat{y}(s). \tag{1.5}$$

In general this will not be immediately feasible for a variety of reasons. The transfer function $\mathcal{T}(s)$ may have decay properties in the right half complex plane which would preclude $\hat{u}(s)$, determined via (1.5), from being the Laplace transform of an input $u \in \mathcal{U}$. Equally well, any zeros of $\mathcal{T}(s)$ in the closed right half complex plane will, in general, produce poles of $\hat{u}(s)$ inconsistent with the requirement $u \in \mathcal{U}$.

These problems are addressed by the use of two distinct modifications of the actual output as compared with the target output. As anticipated in our definition of output specifiability, the desired *output profile* z may be right-shifted (postponed, as it were) by T, replacing $\hat{z}(s)$ by $e^{-sT}\,\hat{z}(s)$ and a "bridging" *transitional output* $\eta(t)$ may be defined with support restricted to $[0,T]$, or part of that interval. The overall output $y(t)$ then consists of $\eta(t)$ on $[0,T]$ and $z(t-T)$ on $[0,\infty)$. This allows (1.5) to be replaced by

$$\hat{u}(s) \,=\, \mathcal{T}(s)^{-1}\hat{y}(s) \,=\, \mathcal{T}(s)^{-1}\left(\hat{\eta}(s) \,+\, e^{-sT}\hat{z}(s)\right). \tag{1.6}$$

Examples to follow indicate that with judicious choice of T and the *output profile space* $\mathcal{Z}$, followed by selection of $\eta(t)$ in such a way that $\hat{\eta}(s) + e^{-sT}\hat{z}(s)$ has zeros appropriately cancelling the poles of $\mathcal{T}(s)^{-1}$, it often becomes possible to solve the output specification problem. The process of selecting the transitional output $\eta(t)$ typically involves the solution of *exponential moment problems* similar to those discussed, e.g., in [1] and [11].

2. **Remote Endpoint Position Specification for the Wave Equation with Boundary Control.** Our first example can be studied quite directly by the "transverse integration" methods described in [13] for the Euler-Bernoulli beam equation but we elect to treat it here as a simple application of the method described in §1 . We consider the wave equation, corresponding to motion of a stretched string,

$$\frac{\partial^2 w}{\partial t^2} \,-\, \kappa^2\,\frac{\partial^2 w}{\partial x^2} \,=\, 0, \quad 0 \,\le\, x \,\le\, L. \tag{2.1}$$

The control force is active at $x = 0$ and at $x = 0$ we have a free endpoint, corresponding to boundary conditions

$$\kappa^2\,\frac{\partial w}{\partial x}(0,t) \,=\, u(t), \quad \frac{\partial w}{\partial x}(L,t) \,=\, 0. \tag{2.2}$$

The output $y(t)$ is defined by

$$y(t) \,=\, w(L,t). \tag{2.3}$$

Using the method described in conjunction with (1.4) we set $u(t) = e^{st}$ in (2.2) and seek a solution of (2.1), (2.2) of the form $w(x,t) = e^{st}\hat{w}(x,s)$; for $\hat{w}(x,s)$ (2.1), (2.2) are replaced by

$$\frac{d^2\hat{w}}{dx^2}(x,s) \,-\, \frac{s^2}{\kappa^2}\hat{w}(x,s) \,=\, 0, \quad \frac{d\hat{w}}{dx}(0,s) \,=\, 1, \quad \frac{d\hat{w}}{dx}(L,s) \,=\, 0. \tag{2.4}$$

Computing the solution of (2.4) we have

$$\hat{w}(x,s) \,=\, \frac{1}{\kappa\, s\, \sinh\frac{s\,L}{\kappa}}\left(\sinh\frac{s\,x}{\kappa}\,\sinh\frac{s\,L}{\kappa} \,-\, \cosh\frac{s\,x}{\kappa}\,\cosh\frac{s\,L}{\kappa}\right), \tag{2.5}$$

so that

$$\mathcal{T}(s) \,=\, \hat{w}(L,s) \,=\, \frac{\sinh^2\frac{s\,L}{\kappa} \,-\, \cosh^2\frac{s\,L}{\kappa}}{\kappa\, s\, \sinh\frac{s\,L}{\kappa}} \,=\, \frac{-1}{\kappa\, s\, \sinh\frac{s\,L}{\kappa}} \tag{2.6}$$

Let $z(t)$, posed first as a function in $\mathcal{Y} = L^2[0,\infty)$, be the desired output with Laplace transform $\hat{z}(s)$ and let $\hat{u}(s)$ be the Laplace transform of the input $u \in \mathcal{U} = L^2[0,\infty)$. We set

$$\hat{y}(s) \,=\, e^{-\frac{sL}{\kappa}}\,\hat{z}(s) \,=\, e^{-\frac{sL}{\kappa}}\,\frac{1}{s}\hat{z}_1(s).$$

Then $\hat{z}_1(s)$ is the Laplace transform of a function $z_1(t) \in L^2[0,\infty)$ and we have

$$\hat{u}(s) = -\kappa\, s \sinh \frac{s\,L}{\kappa}\, \frac{1}{s} e^{-\frac{sL}{\kappa}}\, \hat{z}_1(s) = -\frac{\kappa}{2}\left(1 - e^{-2\frac{s\,L}{\kappa}}\right) \hat{z}_1(s). \qquad (2.7)$$

Thus the overall output $y(t)$ is obtained from $z_1(t)$ via integration and a right shift of magnitude $\frac{L}{\kappa}$, which implies that the restriction of $y(t)$ to $[\frac{L}{\kappa}, \infty)$ has the form $y(t) = z(t - \frac{L}{\kappa})$, where $z(t)$ has the Laplace transform $\frac{-1}{s}\hat{z}_1(s)$, corresponding to $z(t) = -\int_0^t z_1(\tau)\, d\tau$. Thus $z(t)$ is restricted to $H_0^1[0,\infty) \equiv \mathcal{Z}$. With $z(t)$ so restricted, the right half plane boundedness of $1 - e^{-2\frac{s\,L}{\kappa}}$ shows that (2.7) yields an input $u(t)$ in $L^2[0,\infty)$ producing the shifted output $z(t - \frac{L}{\kappa})$ on $[\frac{L}{\kappa}, \infty)$. In this example the only transitional output introduced is the zero output on $0 \le t \le \frac{L}{\kappa}$, required because of the finite speed of propagation of signals from $x = 0$ to $x = L$; the transfer function (2.6) has no zeros requiring the introduction of a nontrivial $\eta(t)$ – in marked contrast to the example to follow.

3. **An Interior, Point to Point, Input–Output Problem for the Wave Equation.** Here we consider a system modeled by an inhomogeneous variant of (2.1):

$$\frac{\partial^2 w}{\partial t^2} - \kappa^2 \frac{\partial^2 w}{\partial x^2} = \delta_\xi\, u(t), \quad 0 \le x \le L, \qquad (3.1)$$

now with boundary conditions

$$w(0,t) \equiv w(L,t) \equiv 0. \qquad (3.2)$$

The input element $b = \delta_\xi$ corresponds to the Dirac distribution with support $\{\xi\}$, $\xi \in (0,L)$. We take as the output

$$y(t) = w(\zeta,t), \ \zeta \in (\xi, L); \qquad (3.3)$$

corresponding to $c = \delta_\zeta$; the assumption $\zeta > \xi$ could obviously be reversed without changing any of the essentials of the problem.

As in the previous section, setting $u(t) = e^{st}$, $w(x,t) = \hat{w}(x,s)\, e^{st}$, we have

$$s^2 \hat{w}(x,s) - \kappa^2 \frac{d^2 \hat{w}}{dx^2}(x,s) = \delta_\xi, \quad \hat{w}(0,s) = \hat{w}(L,s) = 0. \qquad (3.4)$$

Taking $u \in L^2[0,\infty) \equiv \mathcal{U}$, $y \in L^2[0,\infty) \equiv \mathcal{Y}$, we have

$$\hat{y}(s) = \hat{w}(\zeta,s)\, \hat{u}(s) \equiv \mathcal{T}(s)\, \hat{u}(s) = g(\zeta,\xi,s)\, \hat{u}(s), \qquad (3.5)$$

where $g(x,\xi,s)$ is Green's function for the boundary value problem

$$s^2 \hat{w}(x,s) - \kappa^2 \frac{d^2 \hat{w}}{dx^2}(x,s) = f(x), \quad \hat{w}(0,s) = \hat{w}(L,s) = 0,$$

as extensively developed in, e.g., [9] and [15]. From those references we immediately have

$$g(x,\xi,s) = \frac{\sinh \frac{s(\xi - L)}{\kappa} \sinh \frac{s\,x}{\kappa}}{\kappa^2 \sinh \frac{s\,L}{\kappa}}, \quad x \le \xi, \quad \frac{\sinh \frac{s\,\xi}{\kappa} \sinh \frac{s(x-L)}{\kappa}}{\kappa^2 \sinh \frac{s\,L}{\kappa}}, \quad x > \xi. \qquad (3.6)$$

Then, since we have assumed $\zeta > \xi$, we have

$$\mathcal{T}(s) = \frac{\sinh \frac{s\,\xi}{\kappa} \sinh \frac{s(\zeta - L)}{\kappa}}{\kappa^2 \sinh \frac{s\,L}{\kappa}} \qquad (3.7)$$

and, to achieve an output $y(t)$ with Laplace transform $\hat{y}(s)$ with zero initial state and input $u(t)$, with Laplace transform $\hat{u}(s)$, we need to have

$$\hat{u}(s) \,=\, \mathcal{T}(s)^{-1}\,\hat{y}(s) \,=\, \frac{\kappa^2 \sinh \frac{s\,L}{\kappa}}{\sinh \frac{s\,\xi}{\kappa}\, \sinh \frac{s(\zeta - L)}{\kappa}}\,\hat{y}(s). \tag{3.8}$$

We introduce a time delay of length $\frac{\zeta-\xi}{\kappa}$, corresponding to the signal propagation time from $x = \xi$ to $x = \zeta$, by setting $\hat{y}(s) = e^{\frac{-s(\zeta-\xi)}{\kappa}}\,\hat{y}_0(s)$ so that now

$$\hat{u}(s) \,=\, \frac{\kappa^2 \sinh \frac{s\,L}{\kappa}}{\sinh \frac{s\,\xi}{\kappa}\, \sinh \frac{s(\zeta - L)}{\kappa}}\, e^{\frac{-s(\zeta-\xi)}{\kappa}}\,\hat{y}_0(s) \,\equiv\, \mathcal{U}(s)\,\hat{y}_0(s). \tag{3.9}$$

We then have the problem of dealing with the poles of $\mathcal{U}(s)$ arising from the zeros of $\mathcal{T}(s)$ as given by (3.7). To this end we set

$$\hat{y}_0(s) \,=\, \hat{\eta}(s) + e^{\frac{-s(2L-(\zeta-\xi))}{\kappa}}\,\hat{z}(s) \tag{3.10}$$

and obtain, with $L_0 \,\equiv\, \frac{2L-(\zeta-\xi)}{\kappa}$, $\;L_1 \,=\, \frac{2(L-(\zeta-\xi))}{\kappa}$,

$$\hat{u}(s) \,=\mathcal{U}(s)\left(\hat{\eta}(s) + e^{-L_1 s}\hat{z}(s)\right) \,=\, \frac{\kappa^2 \sinh \frac{s\,L}{\kappa}}{\sinh \frac{s\,\xi}{\kappa} \sinh \frac{s(\zeta - L)}{\kappa}} \left(e^{\frac{-s(\zeta-\xi)}{\kappa}}\,\hat{\eta}(s) + e^{-L_0 s}\hat{z}(s)\right). \tag{3.11}$$

In (3.11) $\mathcal{U}(s)$ is uniformly bounded in each right half plane $\mathrm{Re}\, s \,\geq\, \sigma$ for any positive σ but not for $\sigma \,=\, 0$ because of the zeros of the denominator on the imaginary axis. Our attention turns to selecting the function η, assumed square integrable with support in $[0, L_1]$, so that $\hat{\eta}(s) + e^{-L_1 s}\hat{z}(s)$ will also have zeros at those same points, allowing (3.11) to be the transform of a function in $L^2[0,\infty)$ while maintaining the desired output profile z, in the form $z(t - L_0)$, delayed by L_0, on the interval $[L_0, \infty)$.

We denote the zeros of $\sinh \frac{s\,\xi}{\kappa}$, $\;\sinh \frac{s(\zeta-L)}{\kappa}$, all on the imaginary axis, by $i\,\mu_k,\; i\,\nu_j,\; -\infty \,<\, k, j \,<\, \infty$, respectively. Each of these, listed so that their imaginary parts are increasing with respect to k, $\;j$, respectively, form equally spaced sets with respective uniform gaps $\frac{\pi\,\kappa}{\xi}$, $\;\frac{\pi\,\kappa}{L-\zeta}$. If these two gaps are equal then the points $i\,\mu_k$ coincide with the points $i\,\nu_j$ and the function

$$\chi(s) \,\equiv\, \sinh \frac{s\,\xi}{\kappa}\,\sinh \frac{s(\zeta - L)}{\kappa} \tag{3.12}$$

agrees with $\sinh^2\left(s\,\xi/\kappa\right)$ and thus has zeros of multiplicity 2 at each of the points in the two coinciding sets. This is clearly a very special case. In the more general case one of the gaps will be smaller than the other; we may, without loss of generality, suppose the smaller gap to be $\gamma \,=\, \frac{\pi\,\kappa}{\xi}$ since the treatment of the other case is entirely similar. Forming intervals of length γ along the positive imaginary axis with boundary points $i\,\mu_k$, $\;k \,=\, 0, 1, 2, ...,$, careful inspection shows that such intervals contain zeros of $\chi(s)$ which we can organize into sets $\mathcal{S}_k$; the set $\mathcal{S}_0$ consists of the double zero $s \,=\, 0$. Each $\mathcal{S}_k$ contains a simple zero of $\chi(s)$, which we rename as $i\,\sigma_k$, or a single zero with multiplicity 2 (true for all k in the special case described in the preceding paragraph) , which we will redesignate as $i\,\hat{\sigma}_k$, or two zeros, each of single multiplicity, which we will designate as σ_k and ρ_k. We also define conjugate sets $\mathcal{S}_{-k}$, $\;k \,>\, 0$, on the negative imaginary axis. The sets $\mathcal{S}_k$, $\;-\infty \,<\, k \,<\, \infty$ can be, and are assumed to be, constructed so that the minimum gap between successive sets is at least $\frac{\pi\,\kappa}{2\,\xi}$. Corresponding to the three mutually exclusive cases described we consider sets $\mathcal{E}_k$ of "generalized exponentials"

$$\mathcal{E}_k \;=\; \left\{e^{i\,\sigma_k\,t}\right\} \text{ or } \mathcal{E}_k \;=\; \left\{e^{i\,\sigma_k\,t},\, t\,e^{i\,\sigma_k\,t}\right\},\text{ or } \mathcal{E}_k \;=\; \left\{e^{i\,\sigma_k\,t},\; \frac{e^{i\,\sigma_k\,t} - e^{i\,\rho_k\,t}}{i\,\sigma_k - i\,\rho_k}\right\}, \tag{3.13}$$

respectively, according to the membership of $\mathcal{S}_k$. We will indicate these functions generically by $\epsilon_k(t),\ \upsilon_k(t)$, it being understood that the latter is not present when $\mathcal{E}_k$ consists of just one function.

To cancel the zeros in the denominator of $\tilde{\mathcal{T}}(s)$ (or $\mathcal{T}(s)^{-1}$), the function $\eta(t)$, with support in $[0, L_1]$, L_1 as defined just prior to (3.11), should be such that $\eta(t)$ satisfies the *moment problem*, wherein $\tilde{z}(t) \;\equiv\; z(t - L_1)$,

$$\int_0^{L_1} \overline{\epsilon_k(t)}\,\eta(t)\,dt \;=\; -\int_{L_1}^{\infty} \overline{\epsilon_k(t)}\,\tilde{z}(t)\,dt, \equiv\, c_k,\;\; -\infty \,<\, k \,<\, \infty, \tag{3.14}$$

This suffices if $\mathcal{E}_k$ contains a single exponential $e^{i\,\sigma_k\,t}$. Where $\mathcal{E}_k$ contains a second generalized exponential we adjoin a second equation to (3.14) with $\epsilon_k(t)$ replaced by $\upsilon_k(t)$, as described above, and

$$d_k \;=\; \int_{L_1}^{\infty} t\,e^{-i\,\sigma_k\,t}\,\tilde{z}(t)\,dt \;\text{ or }\; \frac{\left(\int_{L_1}^{\infty} e^{-i\,\sigma_k\,t}\,\tilde{z}(t)\,dt \;-\; \int_{L_1}^{\infty} e^{-i\,\rho_k\,t}\,\tilde{z}(t)\,dt\right)}{i\,\sigma_k - i\,\rho_k} \tag{3.15}$$

depending on kind of generalized exponential, as indicated in (3.13).

Theorem 1 *Let the functions $z(t - L_0)$, $z^{'}(t - L_0)$, $t\,z(t - L_0)$ and $t\,z^{'}(t - L_0)$, with support in $[L_0, \infty)$, extended to have zero values on $[0, L_0)$, all lie in $L^1[0, \infty)$ and, in addition, let $z(t - L_0)$ itself also lie in $L^2[0, \infty)$. Then $\eta \,\in\, L^2[0, L_1)$ may be chosen so that $\hat{u}(s)$, as given by (3.11) also lies in $L^2[0, \infty)$ and the corresponding output $y(t)$, with Laplace transform as indicated by (3.5), incorporates the (delayed) output profile z via $y(t) \,=\, z(t - L_0),\;\; L_0 \,\le\, t \,<\, \infty$.*

Sketch of Proof A routine integration in the complex plane shows that, under the conditions of the theorem, the sequences $\{c_k\}$ and $\{d_k\}$ are square summable. Using the uniform separation of the $\mathcal{S}_k$ along the imaginary axis, described earlier, with results in [16], [1], [12] the functions $\epsilon_k(t),\;\upsilon_k(t),\; -\infty \,<\, k \,<\, \infty$, are seen to form a *Riesz basis* (cf. [1], e.g.) for $L^2[0, L_1)$. As a consequence there exists a unique function $\eta(t)$ whose norm in that space is bounded above and below by $\left[\sum_{k=-\infty}^{\infty}\left(|c_k|^2 \,+\, |d_k|^2\right)\right]^{\frac{1}{2}}$, satisfying the moment problem (3.14). With $\eta(t)$ thus chosen on $[0, L_1)$ and extended by 0 to $[L_1, \infty)$, some rather delicate estimates, which will appear elsewhere, are then required to show that (3.9) yields $\hat{u}(s)$ as an analytic function whose restrictions to lines Re $s \,=\, \sigma \,>\, 0$ and whose trace on Re $s \,=\, 0$ lie in, and are uniformly bounded in $L^2(-\infty, \infty)$, i.e., as a function in the Paley-Wiener space of the right half plane. The complexities of the required argument arise from the fact that the poles of $\mathcal{U}(s)$, i.e., the zeros of $\mathcal{T}(s)$, lie on the imaginary axis which, in general, is the boundary of the region of analyticity (i.e., the open right half plane) of $\hat{y}(s)$. If the support of the output $y(t)$ is restricted to a finite closed subinterval of $[0, \infty)$ then $\hat{y}(s)$ becomes an entire function. The desired estimates can then be obtained with the use of a system of contours consisting of lines parallel to and on either side of the imaginary axis together with techniques similar to those to be applied in the case of the Euler–Bernoulli beam system in the section to follow.

4. **Endpoint Position Specification for an Euler–Bernoulli Beam System.** We turn now to an output determination problem for the prismatic Euler–Bernoulli beam equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \quad t \geq 0, \quad 0 \leq x \leq 1. \tag{4.16}$$

We assume the beam is hinged at $x = 0$ with controlled angle of attachment while the end $x = 1$ is free. Thus the boundary conditions are

$$w(0,t) = 0, \ \frac{\partial w}{\partial x}(0,t) = u(t), \ \frac{\partial^2 w}{\partial x^2}(1,t) = \frac{\partial^3 w}{\partial x^3}(1,t) = 0. \tag{4.17}$$

The control $u(t)$ is used to influence the output

$$y(t) = w(1,t). \tag{4.18}$$

We will assume a zero initial state

$$w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0. \tag{4.19}$$

As developed in §1, the transfer function $\mathcal{T}(s)$ may be obtained by taking $u(t) = u(s,t) = e^{st}$ in (4.17) and constructing a corresponding timewise exponential solution of (4.16) and (4.17) in the form $w(x,t) = e^{st}\,\tilde{w}(x,s)$; then

$$\mathcal{T}(s) = \tilde{w}(1,s). \tag{4.20}$$

So doing, we obtain for $\tilde{w}(x,s)$ the equation

$$s^2\,\tilde{w}(x,s) + \frac{d^4\tilde{w}}{dx^4}(x,s) = 0. \tag{4.21}$$

and the boundary conditions

$$\tilde{w}(0,s) = 0, \ \frac{d\tilde{w}}{dx}(0,s) = 1, \ \frac{d^2\tilde{w}}{dx^2}(1,s) = 0, \ \frac{d^3\tilde{w}}{dx^3}(1,s) = 0. \tag{4.22}$$

Solving (4.21), (4.22), a tedious but routine exercise, we obtain the input to output transform relationship, with $\omega_k = \left((1+i)/\sqrt{2}\right)^k, \quad k = 1,2,3,4,$

$$\hat{u}(s) = \mathcal{T}(s)\hat{y}(s) = 2\,\frac{\left(1 + \frac{1}{4}\left(e^{(2s)^{\frac{1}{2}}} + e^{-(2s)^{\frac{1}{2}}} + e^{i\,(2s)^{\frac{1}{2}}} + e^{-i\,(2s)^{\frac{1}{2}}}\right)\right)\hat{y}(s)}{s^{-\frac{1}{2}}\left(\overline{\omega_1}e^{\omega_1\,s^{\frac{1}{2}}} + \overline{\omega_4}e^{\omega_4\,s^{\frac{1}{2}}} + \overline{\omega_2}e^{\omega_2\,s^{\frac{1}{2}}} + \overline{\omega_3}e^{\omega_3\,s^{\frac{1}{2}}}\right)} \tag{4.23}$$

Since $\mathcal{T}(s)$ is an even function of s it is easy to verify that it is a meromorphic function in the entire s-plane and analytic in a neighborhood of $s = 0$.

It is clear that if the relationship (4.23) is to play a role in constructing inputs resulting in a given output the behavior of $\mathcal{T}(s)$ in the right half plane, and means for its modification, constituting the "pseudo-inversion" process, must be of critical importance. We begin by investigating the growth of $\mathcal{T}(s)$; we describe the closed right half plane as the set of s such that $s = 0$ or else $s = r\,e^{i\theta}, r > 0, -\pi/2 \leq \theta \leq \pi/2$. With $\rho = r^{\frac{1}{2}}, \varphi = \frac{\theta}{2}$, we then set $s^{\frac{1}{2}} = \sigma = \rho\,e^{i\varphi}, \rho > 0, -\pi/4 \leq \varphi \leq \pi/4$. The dominant term in the numerator of $\mathcal{T}(s)$ is then seen to be

$$\frac{1}{4}e^{\sqrt{2}\rho(\cos\varphi + i\,\sin\varphi)}. \tag{4.24}$$

Except for the factor $s^{-\frac{1}{2}}$, the dominant terms in the denominator are

$$\frac{1-i}{\sqrt{2}}e^{\frac{1+i}{\sqrt{2}}\sigma} + \frac{1+i}{\sqrt{2}}e^{\frac{1-i}{\sqrt{2}}\sigma} = e^{i(\rho\sin\phi - \pi/4)}\,e^{\rho\cos\phi} + e^{i(\rho\sin\psi + \pi/4)}\,e^{\rho\cos\psi}, \tag{4.25}$$

wherein we have set $\phi = \varphi + \frac{\pi}{4}$, $\psi = \varphi - \frac{\pi}{4}$. Comparing (4.24) with (4.25) we see that the growth of $\mathcal{T}(s)$ in the right half s–plane is majorized by

$$e^{\left(\sqrt{2}\rho\cos\varphi - \sup\{\rho\cos\phi,\ \rho\cos\psi\}\right)} \equiv e^{\gamma(\varphi)\rho}. \tag{4.26}$$

Taking the indicated ranges of the arguments into account we see that the exponent in (4.26) ranges between its maximum value of $\sqrt{2} - \frac{1}{\sqrt{2}}$ when $\varphi = 0$ and its minimum value of 0 at $\varphi = \pm\frac{\pi}{4}$. Then citing (4.23) we conclude that $\mathcal{T}(s)$ grows like $|s|^{\frac{1}{2}}$ as $|s| \to \infty$ on the imaginary axis and, in general, grows like $|s|^{\frac{1}{2}}\ e^{\gamma(\frac{\theta}{2})|s|^{\frac{1}{2}}}$ as $|s| \to \infty$ along the ray $\arg s = \theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

The poles of $\mathcal{T}(s)$ in the right half s–plane arise from the zeros of its denominator. It is easy to see that when $\varphi = 0$ (4.25) takes the form $2\,e^{\frac{\rho}{\sqrt{2}}}\cos\left(\sqrt{2}\rho - \frac{\pi}{4}\right)$ from which an application of Rouché's theorem shows that these zeros take the form

$$s_k = \frac{1}{2}\left((2k-1)\frac{\pi}{2} + \frac{\pi}{4}\right)^2 + o(1),\ \ k \to \infty. \tag{4.27}$$

A more detailed analysis, which we will not pursue here, establishes that these are the only zeros of the denominator in the right half s–plane.

To overcome the growth of $\mathcal{T}(s)$ in the right half plane, as just detailed, we let $\hat{y}(s) = e^{-\epsilon s}\hat{y}_0(s)$ for some $\epsilon > 0$, so that we now have

$$\hat{u}(s) = \mathcal{T}(s)e^{-\epsilon s}\hat{y}_0(s). \tag{4.28}$$

We will assume that $y_0(t)$, with transform $\hat{y}_0(s)$, lies in the Sobolev space $H^2[0,\infty)$ and satisfies

$$y_0(0) = y_0^{'}(0) = 0. \tag{4.29}$$

Theorem 2 *For $\delta > 0$ let $y_0(t)$ satisfy (4.29) and $y_0(t) \equiv \zeta(t) = z(t-\delta)$, $t \ge \delta$, $\zeta(t)$ being the restriction to $[\delta,\infty)$ of a function in $H^2[0,\infty)$. Let the restriction of $y_0(t)$ to $[0,\delta)$ be $\eta(t) \in H^2[0,\delta)$, satisfying the continuity requirements*

$$\eta(\delta) = \zeta(\delta),\ \ \eta^{'}(\delta) = \zeta^{'}(\delta). \tag{4.30}$$

Then for any $\epsilon > 0$ in (4.28) the function $\eta(t) \in H^2[0,\delta)$ can be determined so that there is an input $u(t) \in L^2[0,\infty)$ for which the corresponding output $y(t) \equiv 0$ for $t \in [0,\epsilon]$ and

$$y(t) = y_0(t-\epsilon),\ \ t \in [\epsilon,\infty).$$

Thus the output profile $z(t)$ is achieved in the sense

$$y(t) = z(t-T),\ \ t \in [T,\infty);\ \ T = \epsilon + \delta.$$

Proof. Combining (4.29) with (4.30) we obtain

$$\int_0^\delta \eta^{''}(\tau)\,d\tau = \zeta^{'}(\delta),\quad \int_0^\delta (\tau-\delta)\,\eta^{''}(\tau)\,d\tau = \zeta(\delta). \tag{4.31}$$

Clearly we have

$$\hat{y}_0(s) = (\mathcal{L}y_0)(s) = \frac{1}{s^2}\left(\mathcal{L}y_0^{''}\right)(s). \tag{4.32}$$

The primary purpose served by $\eta(t)$, beyond the smooth connection condition (4.30), is to create zeros in the transform $\hat{y}_0(s)$ at the points s_k cancelling the corresponding zeros in the denominator of $\mathcal{T}(s)$. This is the case if for all s_k as indicated by (4.27) we have

$$\int_0^{\delta} e^{-s_k\tau}\eta(\tau)\,d\tau + \int_{\delta}^{\infty} e^{-s_k\tau}\zeta(\tau)\,d\tau = 0.$$

Integrating by parts twice with use of (4.29) and (4.30), we have, for $k = 1, 2, 3, \ldots$,

$$\int_0^{\delta} e^{-s_k\tau}\eta''(\tau)\,d\tau = -\int_{\delta}^{\infty} e^{-s_k\tau}\zeta''(\tau)\,d\tau = -e^{-s_k\delta}\int_0^{\infty} e^{-s_k\tau}z''(\tau)\,d\tau. \quad (4.33)$$

Equations (4.31) and (4.33) constitute an *exponential moment problem* for $\eta(t)$. Since the sequence $\left\{s_k^{-1}\right\}$ is summable, it is a moment problem of a type which has been extensively studied in the literature (see, e.g., [5], [14], [3]). Using the easy estimate

$$\left|e^{-s_k\delta}\int_0^{\infty} e^{-s_k\tau}z''(\tau)\,d\tau\right| \le \frac{e^{-s_k\delta}}{\sqrt{2\,s_k}}\left\|z''\right\|_{L^2[0,\infty)} \quad (4.34)$$

these results establish, for any given $\delta > 0$, the existence of a solution $\eta''(t)$ of (4.30), (4.33), supported in the interval $[0, \delta]$, with $\left\|\eta''\right\|_{L^2[0,\delta]} \le B \left\|z''\right\|_{L^2[0,\infty)}$, B depending only on δ and the sequence $\{s_k\}$. The conditions (4.29) together with satisfaction of (4.31) imply that the conditions (4.30) are satisfied. Then $y_0(t)$, as described in the theorem statement, lies in $H^2[0,\infty)$ and satisfies (4.29). From (4.32) and the assumption that $z \in H^2[0,\infty) \subset L^2[0,\infty)$ we conclude that

$$\hat{y}_0(s) \in \mathcal{H}^2_{\mathcal{C}+},\ \ \upsilon_1(s) \equiv s\,\hat{y}_0(s) \in \mathcal{H}^2_{\mathcal{C}+},\ \ \upsilon_2(s) \equiv s^2\hat{y}_0(s) \in \mathcal{H}^2_{\mathcal{C}+}, \quad (4.35)$$

and, with B_0, B_1, B_2 are positive constants independent of z,

$$\|\hat{y}_0(s)\|_{\mathcal{H}^2_{\mathcal{C}+}} \le B_0\,\|z\|_{H^2[0,\infty)}\,,\quad \|\upsilon_k(s)\|_{\mathcal{H}^2_{\mathcal{C}+}} \le B_k\,\|z\|_{H^2[0,\infty)}\,,\ \ k = 1, 2,$$

where $\mathcal{H}^2_{\mathcal{C}+}$ denotes the Hardy space in the right half complex plane $\mathcal{C}^+$.

From (4.28) we now have, for $k = 1, 2$,

$$\hat{u}(s) = \frac{e^{-\epsilon s}\mathcal{T}(s)}{s^k}\upsilon_k(s) \equiv \frac{e^{-\epsilon s}\mathcal{T}(s)}{(s+1)^{\frac{1}{2}}}\left(\frac{(s+1)^{\frac{1}{2}}}{s^k}\upsilon_k(s)\right), \quad (4.36)$$

analytic in $\mathcal{C}^+$ since the poles of $\mathcal{T}(s)$ are cancelled as a consequence of satisfaction of the equations (4.33).

There remains the task of showing that $\hat{u}(s) \in \mathcal{H}^2_{\mathcal{C}+}$ so that, in turn, $u(t) \in L^2[0,\infty)$. Fixing a with (cf. (4.27)) $0 < a < s_1$, (4.36) with $k = 1$ represents $\hat{u}(s)$ as the product of the bounded function $\frac{e^{-\epsilon s}\mathcal{T}(s)}{(s+1)^{\frac{1}{2}}}$ with the H^2 function $\frac{(s+1)^{\frac{1}{2}}}{s}\upsilon_1(s)$ in $0 < \sigma = Re(s) < a$. From this we conclude that $\hat{u}(s)$ is analytic with a uniform bound on $\int_{-\infty}^{\infty}|\hat{u}(\sigma + i\tau)|^2\,d\tau$ in this strip. Since $\upsilon_2(s) \in \mathcal{H}^2_{\mathcal{C}+}$, using an easy estimate on the Laplace integral, we have

$$|\upsilon_2(s)| \le B_a,\ \ Re(s) \ge a, \quad (4.37)$$

B_a being a positive constant. On the other hand if we let (cf. (4.27)) $a_k = \frac{1}{2}(s_k + s_{k+1})$ and define $\mathcal{C}_k,\ k = 1, 2, 3, \ldots$ to be the closed contour consisting of the circular arcs of radii a_k and a_{k+1}, centered at 0, lying in $Re(s) \ge a$, together with the two segments of the line $Re(s) = a$ lying between these arcs, the whole oriented in the positive direction, it is straightforward to establish the existence of a uniform bound $\left|\left(e^{-\epsilon s}\mathcal{T}(s)\right)/(s+1)^{\frac{1}{2}}\right| \le U_a$, $Re(s) \ge a$ for $s \in \mathcal{C}_k$, $k = 1, 2, 3, \ldots$.

Then, using (4.37) and the zero cancellation properties described previously, we see that $|\hat{u}(s)|$ is analytic in $Re(s) \geq a$ and satisfies

$$|\hat{u}(s)| \leq \left(\frac{a+1}{a}\right)^{\frac{1}{2}} \frac{U_a\, B_a}{|s|^{\frac{3}{2}}},\ \ Re(s) \geq a,\ \ s \in \mathcal{C}_k,\ \ k = 1, 2, 3, ...$$

Applying the maximum principle we conclude this inequality holds for all s with $Re(s) \geq a$ and thus $\hat{u}(s)$ is analytic in $Re(s) \geq a$ with

$$\int_{-\infty}^{\infty} |\hat{u}(\sigma + i\tau)|^2\, d\tau \leq \left(\frac{a+1}{a}\right)^{\frac{1}{2}} U_a\, B_a \int_{-\infty}^{\infty} |a + i\tau|^{-3}\, d\tau,\ \ \sigma = Re(s) \geq a.$$

Combined with the earlier estimate in the strip $0 < Re(s) < a$, we conclude that $\hat{u}(s)$ is analytic in $\mathcal{C}^+$ with a uniform bound on $\int_{-\infty}^{\infty} |\hat{u}(\sigma + i\tau)|^2\, d\tau$ for $\sigma = Re(s) > 0$. It follows that $\hat{u}(s) \in \mathcal{H}^2_{\mathcal{C}^+}$ and $u(t) \in L^2[0, \infty)$ and the proof is complete. □

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