

SUFFICIENT CONDITIONS AND PERFECT DUALITY IN NONCONVEX MINIMIZATION WITH INEQUALITY CONSTRAINTS

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Abstract. This paper presents a duality theory for solving concave minimization problem and nonconvex quadratic programming problem subjected to nonlinear inequality constraints. By use of the *canonical dual transformation* developed recently, two canonical dual problems are formulated, respectively. These two dual problems are perfectly dual to the primal problems with zero duality gap. It is proved that the sufficient conditions for global minimizers and local extrema (both minima and maxima) are controlled by the triality theory discovered recently [5]. This triality theory can be used to develop certain useful primal-dual methods for solving difficult nonconvex minimization problems. Results shown that the difficult quadratic minimization problem with quadratic constraint can be converted into a one-dimensional dual problem, which can be solved completely to obtain all KKT points and global minimizer.

1. Concave Minimization Problem and Parametrization. The concave minimization problem to be discussed in this paper is denoted as the primal problem (\mathcal{P}) in short)

$$(\mathcal{P}) : \quad \min P(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{X}_f, \quad (1.1)$$

where $P(\mathbf{x})$ is a real-valued concave function defined on a suitable convex set $\mathcal{X}_a \subset \mathbb{R}^n$, and $\mathcal{X}_f \subset \mathbb{R}^n$ is the feasible space, defined by

$$\mathcal{X}_f = \{\mathbf{x} \in \mathcal{X}_a \subset \mathbb{R}^n \mid B\mathbf{x} \leq \mathbf{b}\}, \quad (1.2)$$

in which, $B \in \mathbb{R}^{m \times n}$ is given matrix such that $\text{rank} B = m$, and $\mathbf{b} \in \mathbb{R}^m$ is a given vector. The goal in this problem is to find both global and local minimum values that P can achieve in the feasible space and, if this value is not $-\infty$, to find, if it exists, at least one vector $\bar{\mathbf{x}} \in \mathcal{X}_f$ that achieves this value. The concave minimization problem (\mathcal{P}) appears in many applications. Methods and solutions to this very difficult problem are fundamentally important in both mathematics and engineering science.

Mathematically speaking, if the function P is continuous on its domain \mathcal{X}_a and the feasible set \mathcal{X}_f is compact, then by the well-known Weierstrass Theorem, the global minimum value is finite, and at least one point in \mathcal{X}_f exists which attains this value. From point view of convex analysis, if the convex subset $\mathcal{X}_f \subset \mathbb{R}^n$

2000 *Mathematics Subject Classification.* 49N15, 49M37, 90C26, 90C20.

Key words and phrases. global optimization, duality, concave minimization, quadratic programming, NP-hard problems, optimality condition.

is compact, it must be bounded and closed, i.e. the radius r_0 of \mathcal{X}_f , defined by $|\mathbf{x}| \leq r_0 \quad \forall \mathbf{x} \in \mathcal{X}_f$, is finite. Thus, the problem (\mathcal{P}) has at least one solution. However, due to the concavity of the total cost $P(\mathbf{x})$, the problem (\mathcal{P}) generally will possess many solutions that are local, but not global, minimizers. The application of standard algorithms designed for solving convex programming problems will generally fail to solve this nonconvex global minimization problem [11]. A detailed discussion and comprehensive review on this topic are given by Benson (1995).

Physically speaking, the primal problem (\mathcal{P}) is called *realizable* if there exists a vector $\bar{\mathbf{x}} \in \mathcal{X}_f$ such that $P(\bar{\mathbf{x}}) = \min\{P(\mathbf{x}) \mid \forall \mathbf{x} \in \mathcal{X}_f\} > -\infty$. Thus, if (\mathcal{P}) is realizable, the total cost P must be bounded below on \mathcal{X}_f , i.e. there exists a parameter $\mu > -\infty$ such that $P(\mathbf{x}) \geq \mu \quad \forall \mathbf{x} \in \mathcal{X}_f$. From point view of applications, if a problem is not realizable, then the mathematical modelling of this problem might be not well proposed. By this philosophy, we consider only the realizable primal problem (\mathcal{P}) so that the primal problem can be written in the parametrization form (\mathcal{P}_μ)

$$(\mathcal{P}_\mu) : \quad \min P(\mathbf{x}) \quad \text{s.t.} \quad B\mathbf{x} \leq \mathbf{b}, \quad P(\mathbf{x}) \geq \mu. \quad (1.3)$$

Lemma 1.1 (Parametrization). *Suppose that $P : \mathcal{X}_a \rightarrow \mathbb{R}$ is a canonical function. For an given parameter $\mu \in \mathbb{R}$, the parametrized problem (\mathcal{P}_μ) has at least one global minimizer $\bar{\mathbf{x}}$ in the parametric feasible set $\mathcal{X}_\mu \subset \mathbb{R}^n$ defined by*

$$\mathcal{X}_\mu = \{\mathbf{x} \in \mathbb{R}^n \mid B\mathbf{x} \leq \mathbf{b}, \quad P(\mathbf{x}) \geq \mu\}. \quad (1.4)$$

Moreover, if the problem (\mathcal{P}) is realizable, then there exists a constant $\mu > -\infty$ such that the solution of the parametrized problem (\mathcal{P}_μ) solves also the primal problem (\mathcal{P}) .

Proof. Since for any given $\mu \in \mathbb{R}$ the feasible space \mathcal{X}_μ is a convex subset of \mathbb{R}^n and the cost P is bounded below on \mathcal{X}_μ , the parametric optimization problem (\mathcal{P}_μ) has at least one global minimizer $\bar{\mathbf{x}}_\mu$ such that

$$P(\bar{\mathbf{x}}_\mu) = \inf\{P(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}_\mu\} \geq \mu > -\infty.$$

On the other hand, if (\mathcal{P}) is realizable, then the global minimizer $\bar{\mathbf{x}} \in \mathcal{X}_f$ exists such that $P(\bar{\mathbf{x}}) = \min\{P(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}_f\} > -\infty$. By choosing $\mu = P(\bar{\mathbf{x}}) < \infty$, then the global minimizer $\bar{\mathbf{x}}_\mu$ of the parametrized problem (\mathcal{P}_μ) solves also the primal problem (\mathcal{P}) . \square

In this paper, we will find the canonical dual formulation of the parametric problem (\mathcal{P}_μ) .

2. Canonical Dual Problem. By the definition introduced in [5], a Gâteaux differentiable function $P : \mathcal{X}_a \rightarrow \mathbb{R}$ is called a *canonical function* on $\mathcal{X}_a \subset \mathbb{R}^n$ if and only if its Gâteaux derivative $\delta P : \mathcal{X}_a \rightarrow \mathcal{X}_a^* \subset \mathbb{R}^n$ is a one-to-one mapping from \mathcal{X}_a onto its range \mathcal{X}_a^* (in particular, if $P(\mathbf{x})$ is differentiable, then $\delta P(\mathbf{x}) = \partial P / \partial \mathbf{x}$). Thus, if P is a canonical function on \mathcal{X}_a , then the duality relation $\mathbf{x}^* = \delta P(\mathbf{x})$ is invertible for all $\mathbf{x}^* \in \mathcal{X}_a^*$, and the Legendre conjugate $P^* : \mathcal{X}_a^* \rightarrow \mathbb{R}$ of P can be uniquely defined by

$$P^*(\mathbf{x}^*) = \{\mathbf{x}^T \mathbf{x}^* - P(\mathbf{x}) \mid \mathbf{x}^* = \delta P(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{X}_a\}. \quad (2.5)$$

Clearly, $P(\mathbf{x})$ is a canonical function on \mathcal{X}_a , if and only if the equivalent relations

$$\mathbf{x}^* = \delta P(\mathbf{x}) \quad \Leftrightarrow \quad \mathbf{x} = \delta P^*(\mathbf{x}^*) \quad \Leftrightarrow \quad P(\mathbf{x}) + P^*(\mathbf{x}^*) = \mathbf{x}^T \mathbf{x}^* \quad (2.6)$$

hold on $\mathcal{X}_a \times \mathcal{X}_a^*$. Thus, by the standard procedure of the canonical dual transformation, a dual problem (\mathcal{P}_μ^d) in short) can be formulated as the following:

$$(\mathcal{P}_\mu^d) : \max P^d(\boldsymbol{\epsilon}^*, \rho^*) \quad \forall (\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_\mu^*, \quad (2.7)$$

where $P^d : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$P^d(\boldsymbol{\epsilon}^*, \rho^*) = (\rho^* - 1)P^*(B^T \boldsymbol{\epsilon}^*/(\rho^* - 1)) + \mu \rho^* - \mathbf{b}^T \boldsymbol{\epsilon}^*, \quad (2.8)$$

in which, $P^*(\mathbf{x}^*)$ is the Legendre conjugate function of $P(\mathbf{x})$ on \mathcal{X}_a^* , and the dual feasible space \mathcal{Y}_μ^* is defined by

$$\mathcal{Y}_\mu^* = \{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R} \mid \boldsymbol{\epsilon}^* \geq 0 \in \mathbb{R}^m, \rho^* \geq 0, B^T \boldsymbol{\epsilon}^*/(\rho^* - 1) \in \mathcal{X}_a^*\}. \quad (2.9)$$

Since $P(\mathbf{x})$ is a canonical function over \mathcal{X}_f , the canonical dual function P^d is well defined on \mathcal{Y}_μ^* .

Theorem 2.1 (Canonical Duality Theorem). *Suppose that $P(\mathbf{x})$ is a canonical function on \mathcal{X}_a , then for a given parameter $\mu > -\infty$ and an m -vector \mathbf{b} , the problem (\mathcal{P}_μ^d) is canonically (or perfectly) dual to the primal problem (\mathcal{P}_μ) in the sense that if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) \in \mathcal{Y}_\mu^*$ is a KKT point of (\mathcal{P}_μ^d) such that $\bar{\mathbf{x}}^* = B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1)$, then the vector $\bar{\mathbf{x}}$ determined by*

$$\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*) \quad (2.10)$$

is a KKT point of (\mathcal{P}_μ), and

$$P(\bar{\mathbf{x}}) = P^d(\bar{\mathbf{y}}^*). \quad (2.11)$$

Moreover, if the primal problem (\mathcal{P}) is realizable, then there exists a finite parameter $\mu > -\infty$ such that the vector $\bar{\mathbf{x}}$ is also a KKT point of the problem (\mathcal{P}).

Proof. Suppose that $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) \in \mathcal{Y}_\mu^*$ is a KKT point of (\mathcal{P}_μ^d) such that $\bar{\mathbf{x}}^* = B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1)$, then the complementarity conditions

$$0 \leq \bar{\boldsymbol{\epsilon}}^* \perp B \delta P^*(\bar{\mathbf{x}}^*) - \mathbf{b} \leq 0, \quad (2.12)$$

$$0 \leq \bar{\rho}^* \perp (\delta P^*(\bar{\mathbf{x}}^*))^T \bar{\mathbf{x}}^* - P^*(\bar{\mathbf{x}}^*) - \mu \geq 0, \quad (2.13)$$

hold. Since $(\bar{\mathbf{x}}, \bar{\mathbf{x}}^*)$ is a canonical dual pair on $\mathcal{X}_a \times \mathcal{X}_a^*$, we have

$$\begin{aligned} \bar{\mathbf{x}}^* &= \frac{B^T \bar{\boldsymbol{\epsilon}}^*}{(\bar{\rho}^* - 1)} = \delta P(\bar{\mathbf{x}}) \Leftrightarrow \\ \bar{\mathbf{x}} &= \delta P^*(\bar{\mathbf{x}}^*) \Leftrightarrow \\ P(\bar{\mathbf{x}}) &= \bar{\mathbf{x}}^T \bar{\mathbf{x}}^* - P^*(\bar{\mathbf{x}}^*). \end{aligned}$$

Thus in terms of $\bar{\mathbf{x}}^* = B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1)$ and $\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*)$, the complementarity conditions (2.12) and (2.13) can be written as

$$0 \leq \bar{\boldsymbol{\epsilon}}^* \perp B \bar{\mathbf{x}} - \mathbf{b} \leq 0, \quad (2.14)$$

$$0 \leq \bar{\rho}^* \perp P(\bar{\mathbf{x}}) - \mu \geq 0. \quad (2.15)$$

This shows that $\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*)$ is, indeed, a KKT point of the parametric problem (\mathcal{P}_μ).

By the complementarity conditions (2.14) and (2.15), we have $\mathbf{b}^T \bar{\boldsymbol{\epsilon}}^* = (B \bar{\mathbf{x}})^T \bar{\boldsymbol{\epsilon}}^*$ and $\bar{\rho}^* \mu = \bar{\rho}^* P(\bar{\mathbf{x}})$, respectively. Thus, in term of $\bar{\mathbf{x}} = \delta P^*(B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1))$, we have

$$P^d(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) = (\bar{\rho}^* - 1)P^*(B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1)) - \bar{\mathbf{x}}^T B^T \bar{\boldsymbol{\epsilon}}^* + \bar{\rho}^* P(\bar{\mathbf{x}}) = P(\bar{\mathbf{x}})$$

due to the fact that $\bar{\mathbf{x}}^T \bar{\mathbf{x}}^* - P^*(\bar{\mathbf{x}}^*) = P(\bar{\mathbf{x}})$. This proves the theorem. \square

Remark. In Theorem 2.1, it is required that $\rho^* \neq 1$. Actually, if $\rho^* = 1$, the complementarity condition (2.15) leads to $P(\mathbf{x}) = \mu$, and from the condition $B^T \boldsymbol{\epsilon}^*/(\rho^* - 1) \in \mathcal{X}_a^*$ in the dual feasible space \mathcal{Y}_μ^* , we have $\boldsymbol{\epsilon}^* = 0$. This means that the inequality constraint $B\mathbf{x} \leq \mathbf{b}$ is inactive. In this case, the primal problem is a concave minimization with equality constraint $P(\mathbf{x}) = \mu$. If $P(\mathbf{x})$ is a quadratic function, a complete set of solutions has been obtained recently in [8].

Theorem 2.1 shows that there is no duality gap between the problems (\mathcal{P}_μ) and (\mathcal{P}_μ^d) . As we know that the KKT stationary conditions are only necessary for the nonconvex optimization problem (\mathcal{P}_μ) . The sufficient condition will be given by a triality theorem in the next section.

3. Triality Theory: Sufficient Conditions for Local and Global Extrema.

Let

$$\mathcal{Y}_+^* = \{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_\mu^* \mid \rho^* > 1\}, \quad (3.16)$$

$$\mathcal{Y}_-^* = \{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_\mu^* \mid 0 \leq \rho^* < 1\}, \quad (3.17)$$

and

$$\mathcal{X}_{\mu b} = \{\mathbf{x} \in \mathcal{X}_f \mid P(\mathbf{x}) = \mu\}. \quad (3.18)$$

Theorem 3.1 (Triality Theorem). *Suppose that for a given finite parameter $\mu > -\infty$, the vector $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a KKT point of the problem (\mathcal{P}_μ^d) , and $\bar{\mathbf{x}}^* = B^T \bar{\boldsymbol{\epsilon}}^*/(\bar{\rho}^* - 1)$.*

If $\bar{\rho}^ > 1$, then $\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*)$ is a global minimizer of $P(\mathbf{x})$ over $\mathcal{X}_{\mu b}$ if and only if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a global maximizer of P^d over \mathcal{Y}_+^* , and*

$$P(\bar{\mathbf{x}}) = \min_{\mathbf{x} \in \mathcal{X}_{\mu b}} P(\mathbf{x}) = \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_+^*} P^d(\boldsymbol{\epsilon}^*, \rho^*) = P^d(\bar{\mathbf{y}}^*). \quad (3.19)$$

If $0 \leq \bar{\rho}^ < 1$, then $\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*)$ is a local maximizer of $P(\mathbf{x})$ over \mathcal{X}_μ if and only if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a local maximizer of P^d over \mathcal{Y}_-^* and*

$$P(\bar{\mathbf{x}}) = \max_{\mathbf{x} \in \mathcal{X}_\mu} P(\mathbf{x}) = \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} P^d(\boldsymbol{\epsilon}^*, \rho^*) = P^d(\bar{\mathbf{y}}^*). \quad (3.20)$$

If $0 < \bar{\rho}^ < 1$, then $\bar{\mathbf{x}} = \delta P^*(\bar{\mathbf{x}}^*)$ is a local minimizer of $P(\mathbf{x})$ over $\mathcal{X}_{\mu b}$ if and only if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a local minimizer of P^d over $\mathcal{Y}_-^* \cap \{\rho^* \neq 0\}$ and*

$$P(\bar{\mathbf{x}}) = \min_{\mathbf{x} \in \mathcal{X}_{\mu b}} P(\mathbf{x}) = \min_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} P^d(\boldsymbol{\epsilon}^*, \rho^*) = P^d(\bar{\mathbf{y}}^*). \quad (3.21)$$

Proof. By the concavity of the canonical function P on \mathcal{X}_a , the Legendre conjugate

$$P^*(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathcal{X}_a} \{\mathbf{x}^T \mathbf{x}^* - P(\mathbf{x})\}$$

is uniquely defined on \mathcal{X}_a^* . Particularly, for a given $\mathbf{y}^* = (\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_\mu^*$, we let $\mathbf{x}^* = B^T \boldsymbol{\epsilon}^*/(\rho^* - 1)$. Thus, we have

$$P^d(\boldsymbol{\epsilon}^*, \rho^*) = \min_{\mathbf{x} \in \mathcal{X}_a} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \quad (3.22)$$

where

$$\Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) = \mathbf{x}^T B^T \boldsymbol{\epsilon}^* - (\rho^* - 1)P(\mathbf{x}) + \mu\rho^* - \mathbf{b}^T \boldsymbol{\epsilon}^* \quad (3.23)$$

is the so-called *total complementary function*, or the *extended Lagrangian*, which can be obtained by the standard canonical dual transformation (see [5]). Clearly,

for any given $\rho^* > 1$, $\Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*)$ is convex in \mathbf{x} and concave (linear) in $\boldsymbol{\epsilon}^*$ and ρ^* . Thus, by the classical saddle-minimax theory, we have

$$\begin{aligned}
 P^d(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) &= \max_{\rho^* > 1} \max_{\boldsymbol{\epsilon}^* \geq 0} P^d(\boldsymbol{\epsilon}^*, \rho^*) \\
 &= \max_{\rho^* > 1} \max_{\boldsymbol{\epsilon}^* \geq 0} \min_{\mathbf{x} \in \mathcal{X}_a} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
 &= \max_{\rho^* > 1} \min_{\mathbf{x} \in \mathcal{X}_a} \max_{\boldsymbol{\epsilon}^* \geq 0} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
 &= \min_{\mathbf{x} \in \mathcal{X}_a} \max_{\rho^* > 1} \{(1 - \rho^*)P(\mathbf{x}) + \mu\rho^*\} \quad s.t. \quad B\mathbf{x} \leq \mathbf{b} \\
 &= \min_{\mathbf{x} \in \mathcal{X}_f} \left\{ P(\mathbf{x}) - \min_{\rho^* > 1} \rho^*(P(\mathbf{x}) - \mu) \right\} \\
 &= \min_{\mathbf{x} \in \mathcal{X}_f} P(\mathbf{x}) \quad s.t. \quad P(\mathbf{x}) = \mu,
 \end{aligned}$$

since the linear programming

$$\theta_1 = \min_{\rho^* > 1} \rho^*(P(\mathbf{x}) - \mu)$$

has a solution in the open domain $(1, +\infty)$ if and only if $P(\mathbf{x}) = \mu$. This shows that the KKT point $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ maximize P^d on \mathcal{Y}_+^* if and only if $\bar{\mathbf{x}}$ is a global minimizer of $P(\mathbf{x})$ on $\mathcal{X}_{\mu b}$.

In the case that $0 \leq \bar{\rho}^* < 1$, the total complementary function $\Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*)$ is concave in $\mathbf{x} \in \mathbb{R}^n$ and concave in either $\boldsymbol{\epsilon}^* \in \mathbb{R}_+^m$ or $\rho^* \in [0, 1)$. Thus, if $(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a global minimizer of P^d on \mathcal{Y}_-^* , then by the so-called *bi-duality theory* developed in [5], we have either

$$\max_{\mathbf{x} \in \mathbb{R}^n} \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) = \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} \max_{\mathbf{x} \in \mathbb{R}^n} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \quad (3.24)$$

or

$$\min_{\mathbf{x} \in \mathbb{R}^n} \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) = \min_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} \max_{\mathbf{x} \in \mathbb{R}^n} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*). \quad (3.25)$$

Thus, if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a maximizer of P^d over \mathcal{Y}_-^* then we have

$$\begin{aligned}
 P^d(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) &= \max_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} P^d(\boldsymbol{\epsilon}^*, \rho^*) \\
 &= \max_{\boldsymbol{\epsilon}^* \geq 0} \max_{\rho^* \in [0, 1)} \max_{\mathbf{x} \in \mathbb{R}^n} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
 &= \max_{\rho^* \in [0, 1)} \max_{\mathbf{x} \in \mathbb{R}^n} \max_{\boldsymbol{\epsilon}^* \geq 0} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
 &= \max_{\rho^* \in [0, 1)} \max_{\mathbf{x} \in \mathbb{R}^n} \{(1 - \rho^*)P(\mathbf{x}) + \mu\rho^*\} \quad s.t. \quad B\mathbf{x} \leq \mathbf{b} \\
 &= \max_{\mathbf{x} \in \mathcal{X}_f} \left\{ P(\mathbf{x}) - \min_{\rho^* \in [0, 1)} \rho^*(P(\mathbf{x}) - \mu) \right\} \\
 &= \max_{\mathbf{x} \in \mathcal{X}_f} P(\mathbf{x}) \quad s.t. \quad P(\mathbf{x}) - \mu \geq 0,
 \end{aligned}$$

by the fact that the domain $[0, 1)$ is closed on the lower bound and open on the upper bound, the problem

$$\theta_2 = \min_{\rho^* \in [0, 1)} \{\rho^*(P(\mathbf{x}) - \mu)\} \quad (3.26)$$

has a solution if and only if $P(\mathbf{x}) \geq \mu$, and for this solution, $\theta_2 = 0$.

On the other hand, if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*)$ is a minimizer of P^d over $\mathcal{Y}_-^* \cap \{\rho^* \neq 0\}$ then we have

$$\begin{aligned}
P^d(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) &= \min_{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_-^*} P^d(\boldsymbol{\epsilon}^*, \rho^*) \\
&= \min_{\rho^* \in (0,1)} \min_{\boldsymbol{\epsilon}^* \geq 0} \max_{\mathbf{x} \in \mathbb{R}^n} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
&= \min_{\rho^* \in (0,1)} \min_{\mathbf{x} \in \mathbb{R}^n} \max_{\boldsymbol{\epsilon}^* \geq 0} \Xi(\mathbf{x}, \boldsymbol{\epsilon}^*, \rho^*) \\
&= \min_{\rho^* \in (0,1)} \min_{\mathbf{x} \in \mathbb{R}^n} \{(1 - \rho^*)P(\mathbf{x}) + \mu\rho^*\} \quad s.t. \quad B\mathbf{x} \leq \mathbf{b} \\
&= \min_{\mathbf{x} \in \mathcal{X}_f} \left\{ P(\mathbf{x}) - \max_{\rho^* \in (0,1)} \rho^*(P(\mathbf{x}) - \mu) \right\} \\
&= \min_{\mathbf{x} \in \mathcal{X}_f} P(\mathbf{x}) \quad s.t. \quad P(\mathbf{x}) \geq \mu,
\end{aligned}$$

since the linear minimization

$$\theta_3 = \max_{\rho^* \in (0,1)} \rho^*(P(\mathbf{x}) - \mu)$$

has a solution on the open domain $(0, 1)$ if and only if $P(\mathbf{x}) - \mu = 0$. By the fact that $P^d(\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) = P(\bar{\mathbf{x}})$ for all KKT points of (\mathcal{P}_μ) , the theorem is proved. \square

Theorem 3.1 shows that if the KKT point $\bar{\rho}^* \neq 0$, then $\bar{\mathbf{x}}$ is a minimizer of P only if $\bar{\mathbf{x}}$ is located on the boundary of the feasible set \mathcal{X}_μ , i.e. $P(\bar{\mathbf{x}}) = \mu$, which can not be located, generally speaking, by standard algorithms designed for convex minimization problems. This is the main reason that why the primal problem (\mathcal{P}) is NP-hard. However, the dual problem in this case may provide an alternative approach. The triality theorem provides sufficient conditions for both global and local minima, which can be used to develop certain potentially useful primal-dual algorithms for solving this concave minimization problem with inequality constraints.

4. Quadratic Minimization with Quadratic Constraint. The canonical dual problem (\mathcal{P}_μ^d) of the nonconvex problem (\mathcal{P}) can be formulated in different ways, depending on the inequality constraint $P(\mathbf{x}) \geq \mu$ in the parametrical feasible set \mathcal{X}_μ . In the case that $P(\mathbf{x})$ is a quadratic function, the canonical dual problem and optimality conditions have been studied recently by the use of a normality parametrization $|\mathbf{x}|^2 \leq \mu$ [8]. In this section, we will consider the following quadratic minimization problem with quadratic constraint $((\mathcal{P}_q)$ in short):

$$(\mathcal{P}_q) : \quad \begin{aligned} \min \quad & P_q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{f}, \\ s.t. \quad & B\mathbf{x} \leq \mathbf{b}, \quad \frac{1}{2}\mathbf{x}^T C\mathbf{x} \leq \mu, \end{aligned} \quad (4.27)$$

where A and C are two given symmetrical matrices. As it was pointed by Floudas and Visweswaran[4], due to the nonlinear (quadratic) constraint, even finding a feasible solution for this problem can be a formidable task. It is also known that the quadratic minimization problem with only linear equality constraint is NP-hard [4, 14]. In this section, we will present a canonical dual approach for solving this problem.

Following the standard procedure of the canonical dual transformation presented in [5, 8], we introduce a so-called canonical geometrical operator $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}$:

$$\mathbf{y} = (\boldsymbol{\epsilon}, \rho) = \Lambda(\mathbf{x}) = (B\mathbf{x}, \frac{1}{2}\mathbf{x}^T C\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R},$$

where $\boldsymbol{\epsilon} = B\mathbf{x}$ is an m -vector, and $\rho = \frac{1}{2}\mathbf{x}^T C\mathbf{x}$ is a scale. Let \mathcal{Y}_μ be a convex subset of $\mathcal{Y} = \mathbb{R}^m \times \mathbb{R}$ defined by

$$\mathcal{Y}_\mu = \{\mathbf{y} = (\boldsymbol{\epsilon}, \rho) \in \mathbb{R}^m \times \mathbb{R} \mid \boldsymbol{\epsilon} \leq \mathbf{b}, \rho \leq \mu\}.$$

Its indicator $\bar{W} : \mathcal{Y} \rightarrow \mathbb{R} \cup \{+\infty\}$, defined by

$$\bar{W}(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} \in \mathcal{Y}_\mu, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex, lower semi-continuous on \mathcal{Y} . Thus, the inequality constraints in the problem (\mathcal{P}_q) can be relaxed by the indicator of \mathcal{Y}_μ and the primal problem (\mathcal{P}_q) takes the unconstrained canonical form

$$(\mathcal{P}_\mu) : \min \left\{ \bar{W}(\Lambda(\mathbf{x})) + \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{f} \right\} \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (4.28)$$

By the general result presented in [6], the canonical dual function for this unconstrained problem is

$$P_q^d(\mathbf{y}^*) = P_q^\Lambda(\mathbf{y}^*) - \bar{W}^\sharp(\mathbf{y}^*),$$

where \bar{W}^\sharp is canonical conjugate of \bar{W} defined by the sup-Fenchel transformation:

$$\begin{aligned} \bar{W}^\sharp(\mathbf{y}^*) &= \sup_{\mathbf{y} \in \mathcal{Y}} \{\mathbf{y}^T \mathbf{y}^* - \bar{W}(\mathbf{y})\} = \sup_{\boldsymbol{\epsilon} \leq \mathbf{b}} \sup_{\rho \leq \mu} \{\boldsymbol{\epsilon}^T \boldsymbol{\epsilon}^* + \rho \rho^*\} \\ &= \begin{cases} \mathbf{b}^T \boldsymbol{\epsilon}^* + \mu \rho^* & \text{if } \boldsymbol{\epsilon}^* \geq 0, \rho^* \geq 0, \\ +\infty & \text{otherwise;} \end{cases} \end{aligned} \quad (4.29)$$

and $P_q^\Lambda(\mathbf{y}^*)$ is the so-called Λ -canonical conjugate of the canonical function $P_q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{f}$ defined by the Λ -canonical dual transformation (cf. [5])

$$\begin{aligned} P_q^\Lambda(\mathbf{y}^*) &= \{(\Lambda(\mathbf{x}))^T \mathbf{y}^* - P_q(\mathbf{x}) \mid \delta P_q(\mathbf{x}) = (\delta \Lambda(\mathbf{x}))^T \mathbf{y}^*\} \\ &= -\frac{1}{2}(\mathbf{f} - B^T \boldsymbol{\epsilon}^*)^T (A + \rho^* C)^{-1} (\mathbf{f} - B^T \boldsymbol{\epsilon}^*). \end{aligned}$$

On the dual feasible space defined by

$$\mathcal{Y}_q^* = \{(\boldsymbol{\epsilon}^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R} \mid \boldsymbol{\epsilon}^* \geq 0, \rho^* \geq 0, \det(A + \rho^* C) \neq 0\}, \quad (4.30)$$

the canonical dual function $P_q^d(\mathbf{y}^*) = P_q^\Lambda(\mathbf{y}^*) - \bar{W}^\sharp(\mathbf{y}^*)$ takes the following form

$$P_q^d(\boldsymbol{\epsilon}^*, \rho^*) = -\frac{1}{2}(\mathbf{f} - B^T \boldsymbol{\epsilon}^*)^T (A + \rho^* C)^{-1} (\mathbf{f} - B^T \boldsymbol{\epsilon}^*) - \mu \rho^* - \mathbf{b}^T \boldsymbol{\epsilon}^*. \quad (4.31)$$

Thus, the canonical dual problem associated with the quadratic minimization problem (\mathcal{P}_q) can be formulated as the following:

$$(\mathcal{P}_q^d) : \max P_q^d(\boldsymbol{\epsilon}^*, \rho^*) \quad \forall (\boldsymbol{\epsilon}^*, \rho^*) \in \mathcal{Y}_q^*. \quad (4.32)$$

Theorem 4.1. *The problem (\mathcal{P}_q^d) is canonically (perfectly) dual to the primal optimization problem (\mathcal{P}_q) in the sense that if $\bar{\mathbf{y}}^* = (\bar{\boldsymbol{\epsilon}}^*, \bar{\rho}^*) \in \mathcal{Y}_q^*$ is a KKT point of (\mathcal{P}_q^d) , then the vector defined by*

$$\bar{\mathbf{x}} = (A + \bar{\rho}^* C)^{-1} (\mathbf{f} - B^T \bar{\boldsymbol{\epsilon}}^*) \quad (4.33)$$

is a KKT point of (\mathcal{P}_q) , and

$$P_q(\bar{\mathbf{x}}) = P_q^d(\bar{\mathbf{y}}^*). \quad (4.34)$$

Moreover, if C and $(A + \bar{\rho}^* C)$ are positive definite, then $\bar{\mathbf{x}} = (A + \bar{\rho}^* C)^{-1} (\mathbf{f} - B^T \bar{\boldsymbol{\epsilon}}^*)$ is a global minimizer of the problem (\mathcal{P}_q) . If $\bar{\rho}^* > 0$, then $\frac{1}{2}\bar{\mathbf{x}}^T C\bar{\mathbf{x}} = \mu$.

This theorem can be proved by a similar procedure given in [8]. In the case that there is no linear inequality constraint $B\mathbf{x} \leq \mathbf{b}$ in the problem (\mathcal{P}_q) , the canonical dual problem (\mathcal{P}_q^d) can be simply given as

$$\begin{aligned} \max \quad & -\frac{1}{2}\mathbf{f}^T(A + \rho^*C)^{-1}\mathbf{f} - \mu\rho^* \\ \text{s.t.} \quad & \rho^* \geq 0, \quad \det(A + \rho^*C) \neq 0. \end{aligned} \quad (4.35)$$

This is a one-dimensional maximization problem, which can be solved completely to obtain all KKT points $\rho_i^* \geq 0$. If $(A + \rho_i^*C)$ is positive definite, then the vector $\mathbf{x}_i = (A + \rho_i^*C)^{-1}\mathbf{f}$ is a global minimizer of the quadratic function $\frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{f}$ over the surface $\frac{1}{2}\mathbf{x}^T C\mathbf{x} = \mu$. Particularly, if $C = I$, the primal problem (\mathcal{P}_q) is the so-called quadratic programming over a sphere $\frac{1}{2}\mathbf{x}^T \mathbf{x} \leq \mu$. In this case, the dual problem (4.35) has a unique solution $\bar{\rho}^*$ on the open domain $(-a_1, \infty)$, where a_1 is the smallest eigenvalue of the matrix A , and the vector $\bar{\mathbf{x}} = (A + \bar{\rho}^*I)^{-1}\mathbf{f}$ is a global minimizer of the quadratic function $P_q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T \mathbf{f}$ over the sphere $\frac{1}{2}\mathbf{x}^T \mathbf{x} = \mu$ (see [8]).

5. Applications. In this section we shall give a special application of the theory and method presented in this paper to the quadratic minimization problem (\mathcal{P}_q) with only the quadratic constraint, and both A and C are diagonal matrices. Thus, the problem (\mathcal{P}_q) can be written as:

$$\min \quad \sum_{i=1}^n \left[\frac{1}{2}a_i x_i^2 - f_i x_i \right] \quad (5.36)$$

$$\text{s.t.} \quad \sum_{i=1}^n \frac{1}{2}c_i x_i^2 \leq \mu. \quad (5.37)$$

The coefficients a_i, c_i and f_i are arbitrarily given real numbers. The canonical dual problem is to find ρ^* such that

$$\max \quad -\frac{1}{2} \sum_{i=1}^n \frac{f_i^2}{a_i + \rho^* c_i} - \mu\rho^*, \quad (5.38)$$

$$\text{s.t.} \quad \rho^* \geq 0, \quad \rho^* \neq -\frac{a_i}{c_i}. \quad (5.39)$$

This dual problem has a unique solution over the domain $\rho_1^* > \max\{-a_i/c_i\} \geq 0$. Let $\lambda_i = -a_i/c_i$, $i = 1, 2, \dots, n$, and i_d be the total number of distinct positive λ_i . By a simple analysis (see [8]) we know that the dual problem has at most $2i_d$ KKT points: $\rho_1^* > \rho_2^* \geq \rho_3^* > \dots > \rho_{2i_d}^*$. For each ρ_k^* , the vector $\mathbf{x}_k = \{f_i/(a_i + \rho_k^* c_i)\}$ is a critical point of $P_q(x_i) = \sum_{i=1}^n (\frac{1}{2}a_i x_i^2 - f_i x_i)$, and \mathbf{x}_1 is a global minimizer of P_q .

Example 1. In 2-D space, if we let $a_1 = -0.5, a_2 = -1.0$, and $f_1 = 0.3, f_2 = -0.3$, the quadratic function $P(x_1, x_2)$ is a concave surface (see Fig.1). Let $\mu = 2$. First we consider an indefinite matrix C by choosing $c_1 = -1, c_2 = 0.5$. In this case, $\lambda = \{-0.5, 2\}$ and $i_d = 1$, the dual problem has total $2i_d = 2$ KKT points (see Fig.2a):

$$\rho_1^* = 2.21 > \rho_2^* = 1.79.$$

By the triality theory we known that $\mathbf{x}_1 = \{-0.11, -2.83\}$ is a global minimizer, while $\mathbf{x}_2 = \{-0.13, 2.83\}$ is a local minimizer, and

$$P_q(\mathbf{x}_1) = -4.83 = P_q^d(\rho_1^*) < P_q(\mathbf{x}_2) = -3.13 = P_q^d(\rho_2^*).$$

If we choose $c_1 = 1, c_2 = 0.5$, then C is a positive definite matrix. In this case, $\lambda = \{0.5, 2\}$ and $i_d = 2$, and the dual problem has $2i_d = 4$ KKT points (see Fig.2b):

$$\rho_1^* = 2.213 > \rho_2^* = 1.786 > \rho_3^* = 0.652 > \rho_4^* = 0.349.$$

The vector $\mathbf{x}_1 = \{0.175, -2.818\}$ is a global minimizer, $P_q(\mathbf{x}_1) = P_q^d(\rho_1^*) = -4.875$; $\mathbf{x}_2 = \{0.233, 2.809\}$ and $\mathbf{x}_4 = \{-1.983, 0.363\}$ are local minimizers; while $\mathbf{x}_3 = \{1.975, 0.445\}$ is a local maximizer.

Example 2. For $n = 4$, the graph of the quadratic function $P_q(\mathbf{x})$ can not be viewed. For $\{a_i\} = \{-.5, -1., .4, -1.5\}$, $\{c_i\} = \{1, .5, -1, 1.5\}$, $\{f_i\} = \{.3, -.3, 0, .5\}$, and $\mu = 2$, the graph of its dual function is a plane curve as shown in Fig. 3. The dual problem in this case has total four KKT points, and $\rho_1^* = 2.21604 > \max\{-a_i/c_i\}$. The vector $\mathbf{x}_1 = \{0.175, -2.777, 0., 0.274\}$ is a global minimizer of $P_q(\mathbf{x})$ and $P_q(\mathbf{x}_1) = -4.943 = P_q^d(\rho_1^*)$.

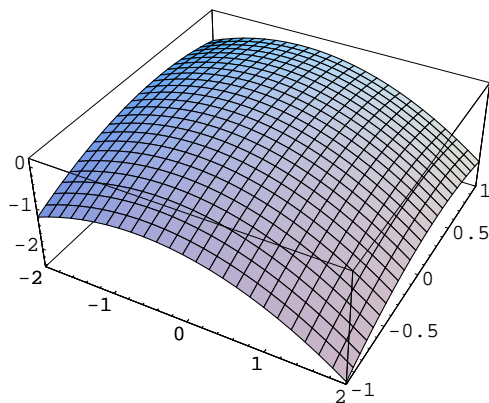


FIGURE 1. Graph of $P_q(x_1, x_2)$

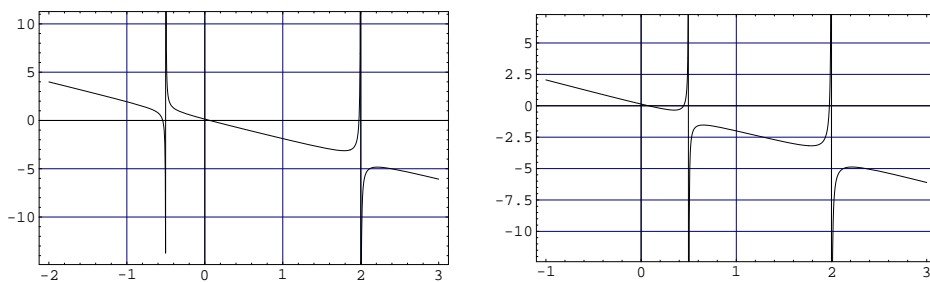


FIGURE 2. Graphs of P^d : (a) Indefinite C ; (b) Positive definite C .

6. Concluding Remarks. Duality theory plays an important role in both analysis and mathematical programming. In convex systems, the saddle-Lagrangian type duality theory has been well studied (cf. [3, 10, 13]). However, these well-developed duality theories usually lead to certain duality gap when the primal problems are

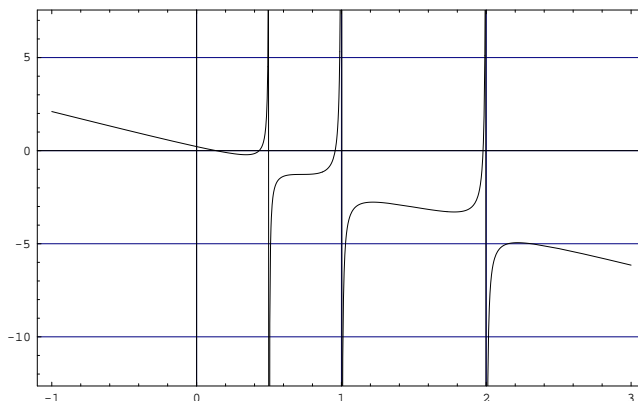


FIGURE 3. Graph of $P^d(\rho^*)$ in 4-D quadratic programming.

nonconvex. In nonconvex analysis, how to use the traditional Legendre transformation to formulate perfect dual problem was listed as one of two open problems in the recent paper by I. Ekeland [2]. The canonical dual transformation method and triality theory were originally developed from nonsmooth and nonconvex mechanics (see Gao, 2000). The key idea of this method is to choose certain geometrically admissible measure $\Lambda : \mathcal{X}_a \rightarrow \mathcal{Y}_a$ such that the Legendre-Fenchel-Moreau transformation holds on the canonical dual spaces $\mathcal{Y}_a \times \mathcal{Y}_a^*$ (see Gao, 2000, 2003, 2004). In the present paper, since the concave function P is a canonical function, this geometrical measure is simply chosen to be $\Lambda(\mathbf{x}) = P(\mathbf{x})$. If P is a quadratic function, the constraint $P(\mathbf{x}) \leq \mu$ can be replaced by $\frac{1}{2}\mathbf{x}^T C \mathbf{x} \leq \mu$. In this case, the parametrized primal problem is a quadratic minimization with quadratic constraint. By the canonical dual transformation, this problem can be easily solved. Particularly, if $C = I$, then problem (P_q) is the well-known quadratic minimization over a sphere, which has been solved completely by the author in a very recent paper (see Gao, 2004). The results presented in this paper show again that the canonical dual transformation and associated triality theory may possess important computational impacts on global optimization.

Acknowledgments: The author would like to offer his sincere thanks to Professor K.L. Teo for a careful reading of the article and helpful suggestions. Referee's useful comments are also gratefully acknowledged.

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Received June 2004; revised December 2004.

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