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DRAW RESONANCE REVISITED*

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Abstract. We consider the problem of isothermal fiber spinning in a Newtonian fluid with no inertia. In particular, we focus on the effect of the downstream boundary condition. For prescribed velocity, it is well known that an instability known as draw resonance occurs at draw ratios in excess of about 20.2. We shall revisit this problem. Using the closed form solution of the differential equation, we shall show that an infinite family of eigenvalues exists and discuss its asymptotics. We also discuss other boundary conditions. If the force in the filament is prescribed, no eigenvalues exist, and the problem is stable at all draw ratios. If the area of the cross section is prescribed downstream, on the other hand, the problem is unstable at any draw ratio. Finally, we discuss the stability when the drawing speed is controlled in response to changes in cross section or force.

Key words. extensional flow, fiber spinning, draw resonance

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1. Formulation of the problem.

Fiber spinning is a manufacturing process used in making textile or glass fibers. A highly viscous fluid is extruded vertically from a nozzle. It is then cooled by the ambient air and solidifies. The solidified fiber is then wound on a spool at the end of the spinline.

Many physical effects are potentially significant in the study of this problem: viscosity, inertia, gravity, surface tension, cooling, elasticity, and air drag may all be relevant. In this paper, we focus on the simplest model and study the influence of varying boundary conditions. We assume that the force in the fiber is purely due to viscous effects, and we ignore temperature dependence. We use a one-dimensional model based on slender geometry and cross-sectional averaging. Let $u(x,t)$ denote the axial speed and $A(x,t)$ the area of the cross section. The spinneret is located at $x=0$ and the spool is at $x=L$. The conservation of mass implies that

$$A_t + (uA)_x = 0. \tag{1}$$

If only viscous forces contribute, the tension in the fiber is given by $3\eta A u_x$, where η is the viscosity. The requirement of constant tension in the fiber leads to

$$(Au_x)_x = 0. \tag{2}$$

Boundary conditions in industrial processes are notoriously ill defined. It is customary to assume that A and u are given at the spinneret: $A(0,t)=A_0$, $u(0,t)=u_0$. This of course, is an idealization; in reality there is a transition to an upstream flow, which cannot be described by the one-dimensional model. At the spool, it is sensible to prescribe either the speed or the force with which the fiber is wound. One might also consider control strategies where the flow is monitored and the speed of the spool adjusted to achieve a given objective. Since the goal of the manufacturing process is

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a fiber of uniform cross section, a control strategy might aim to keep the cross section constant. We shall consider what happens in the case of perfect success of such a control, i.e., when constant area is imposed as a boundary condition. We shall thus focus on the following three boundary conditions:

1. Prescribed speed: $u(L,t) = u_1$.
2. Prescribed force: $A(L,t)u_x(L,t) = F$, where F denotes the force divided by the elongational viscosity 3η.
3. Prescribed cross section: $A(L,t) = A_1$.

It is easy to see that the problem admits the steady solution

$$u_s(x) = u_0 e^{kx}, \quad A_s(x) = A_0 e^{-kx}, \tag{3}$$

where, respectively,

$$e^{kL} = u_1/u_0, \quad k = F/(A_0 u_0), \quad e^{kL} = A_0/A_1 \tag{4}$$

for the three choices of boundary conditions. The dimensionless quantity $q = e^{kL}$ is called the draw ratio.

2. Linear stability. The stability of the steady solution was first analyzed by Kase, Matsuo, and Yoshimoto [5] and Pearson and Matovich [6]. For subsequent reviews and textbook chapters, see also [2, 7, 9, 10]. We note that much of the literature on draw resonance is concerned with the effect of additional physical mechanism, which are not included in our analysis. Inertia, elasticity, and cooling generally have a stabilizing effect, while surface tension and shear thinning are destabilizing. This paper, on the other hand, will focus purely on the case of Stokes flow and investigate the effect of varying the downstream boundary condition. In the case of prescribed speed, an instability known as draw resonance is found for draw ratios in excess of about 20.2, while no such instability is found for prescribed force. In [8], the case of a linear combination of speed and force is also investigated; as expected, the stability threshold increases from 20.2 to infinity as the relevant coefficient is varied. The case of prescribed cross section does not seem to have been analyzed in the literature. We shall see that this boundary condition leads to instability at all draw ratios.

We linearize at the steady solution and consider exponentially varying perturbations:

$$u(x,t) = u_s(x) + \tilde{u}(x)e^{\lambda t}, \quad A(x,t) = A_s(x) + \tilde{a}(x)e^{\lambda t}. \tag{5}$$

The linearized equations are

$$\lambda \tilde{a} + (u_s \tilde{a} + A_s \tilde{u})_x = 0, \quad (A_s \tilde{u}_x + \tilde{a}(u_s)_x)_x = 0. \tag{6}$$

It is advantageous to make the transformation $z = e^{kx}$. The steady solution then takes the form $u_s(z) = u_0 z$, $A_s(z) = A_0/z$. The linearized equations (6) are transformed to

$$\lambda \tilde{a} + kz(zu_0\tilde{a} + A_0\tilde{u}/z)_z = 0, \quad (A_0\tilde{u}_z + zu_0\tilde{a})_z = 0. \tag{7}$$

We can rewrite these equations in the form

$$\begin{aligned} \frac{\lambda}{ku_0}\frac{\tilde{a}}{A_0} + z\frac{\tilde{a}}{A_0} + z^2\frac{\tilde{a}_z}{A_0} - \frac{\tilde{u}}{u_0 z} + \frac{\tilde{u}_z}{u_0} &= 0, \\ \frac{\tilde{u}_z}{u_0} + z\frac{\tilde{a}}{A_0} &= C_1. \end{aligned} \tag{8}$$

We simplify by setting $\lambda/(ku_0) = \mu$, $\tilde{a}/A_0 = a$, $\tilde{u}/u_0 = u$. This is equivalent to nondimensionalizing the equations by scaling the velocity and area with their steady state values at the spinneret, length along the filament with $1/k$, and time with $1/(ku_0)$. The resulting dimensionless equations are

$$\begin{aligned} \mu a + za + z^2 a_z - \frac{u}{z} + u_z &= 0, \\ u_z + za &= C_1. \end{aligned} \tag{9}$$

We can now solve the second equation for a:

$$a = \frac{C_1 - u_z}{z}. \tag{10}$$

After inserting this into the first equation, we obtain

$$-u + (-\mu + z)u_z + \mu C_1 - z^2 u_{zz} = 0. \tag{11}$$

Clearly, $u = \mu C_1$ is a particular solution, and $u = z - \mu$ is a particular solution of the homogeneous equation. We can then obtain the full solution using the reduction of order method:

$$u(z) = \mu C_1 + (z - \mu)C_2 + C_3(-ze^{\mu/z} + (\mu - z)\mathrm{Ei}(\mu/z)); \tag{12}$$

see also [8]. Here Ei is the exponential integral defined for $z > 0$ by

$$\mathrm{Ei}(x) = \int_{-\infty}^{z} \frac{e^t}{t}\, dt, \tag{13}$$

where the integral is understood in the principal value sense (see Ch. 5 of [1]).

We have the boundary conditions $u(1) = a(1) = 0$ at the spinneret and one of the following three at the take-up point:

1. Fixed speed: $u(q) = 0$.
2. Fixed force: $u_z(q) + qa(q) = 0$.
3. Fixed cross section: $a(q) = 0$.

The requirement of a nontrivial solution leads to the following characteristic equations. For fixed speed,

$$(e^{\mu} - e^{\mu/q})q + (q - \mu)(\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q)) = 0. \tag{14}$$

For fixed force,

$$e^{\mu} = 0. \tag{15}$$

For fixed cross section,

$$\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q) = 0. \tag{16}$$

3. Remarks on well-posedness. For the case of fixed force, no eigenvalues exist, and indeed it can be shown for this case that any initial disturbance will decay to zero in finite time. To see this, we note that the linearized equations (9) (without the assumption of exponential time dependence) are

$$\begin{aligned} a_t + z^2 a_z - \frac{u}{z} &= 0, \\ u_z + za &= 0. \end{aligned} \tag{17}$$

We can integrate the second of these equations to find

$$u = -\int_1^z ya\,dy. \tag{18}$$

For any given initial data $a(z,0) = a_0(z)$, we can now solve the equation using the iterative procedure

$$\begin{aligned} u^1 &= 0,\\ a_t^n + z^2 a_z^n - \frac{u^n}{z} &= 0, \quad a^n(1,t) = 0, \quad a_n(z,0) = a_0(z),\\ u^{n+1} &= -\int_0^x ya^n\,dy. \end{aligned} \tag{19}$$

It follows by induction that

$$a(z,t) = 0 \tag{20}$$

for $1 < z < Z(t)$, where

$$Z'(t) = Z(t)^2, \quad Z(0) = 1. \tag{21}$$

The solution will thus become identically zero as soon as $Z(t)$ reaches the value q.

For the case of prescribed speed, a rigorous proof of well-posedness and spectrally determined growth can be given along similar lines as [4]. The case of prescribed cross section is somewhat different and will be discussed now. In this case, the linear problem, again without the assumption of exponential time dependence, is

$$\begin{aligned} a_t + za + z^2 a_z - \frac{u}{z} + u_z &= 0,\\ u_z + za &= \phi(t), \end{aligned} \tag{22}$$

where $\phi(t)$ is a function to be determined after the boundary conditions are imposed. We can integrate the second equation to find

$$u(z,t) = -\int_1^z ya(y,t)\,dy + (z-1)\phi(t) \tag{23}$$

and insert this result into the first equation. This yields

$$a_t + z^2 a_z + \frac{1}{z}\int_1^z ya\,dy + \frac{\phi(t)}{z} = 0. \tag{24}$$

We next set $a = b + \gamma/z$, where γ is independent of z, and b satisfies

$$\int_1^q b(z,t)\chi(z)\,dz, \tag{25}$$

with χ to be determined. We shall denote the projections of a onto b and γ/z by P and Q.

We want χ to be such that we also have

$$\int_1^q z^2 b_z \chi(z)\,dz = 0. \tag{26}$$

We note that the boundary condition $a(1,t) = a(q,t) = 0$ leads to $b(1,t) = qb(q,t)$. We now integrate by parts to find

$$\int_1^q z^2 b_z \chi\, dz = -\int_1^q b(z^2\chi)'\, dz + q^2 b(q,t)\chi(q) - b(1,t)\chi(1). \tag{27}$$

We achieve our objective if

$$\frac{d}{dz}(z^2\chi) = K\chi, \quad \chi(1) = q\chi(q) \tag{28}$$

for some constant K. This leads to

$$\chi(z) = \frac{1}{z^2} e^{-K/z}, \quad e^{-K} = \frac{1}{q} e^{-K/q}. \tag{29}$$

With χ thus determined, (24) can be decomposed as follows:

$$\begin{gathered} \gamma_t - \gamma + zQ\left(\frac{1}{z}\int_1^z yb\, dy\right) + \phi(t) = 0, \\ b_t + z^2 b_z + P\left(\frac{1}{z}\int_1^z yb\, dy\right) = 0. \end{gathered} \tag{30}$$

The solution procedure is now obvious. We solve the second equation for b with the boundary condition $b(1,t) = qb(q,t)$, and after b is determined, the first equation, and the condition $\gamma(t) = -b(1,t)$, determine $\phi(t)$. Well-posedness and spectral growth are obvious from this reformulation of the equations. We note that the boundary condition for b is a two-point condition rather than an upstream condition. This is reflected in the nature of the eigenspectrum below; the limit of the real part of large eigenvalues will be a finite number rather than $-\infty$.

We note that a general discussion of boundary conditions for the hyperbolic systems arising in viscoelastic flows is given in [3]; in the context of that discussion the Newtonian case is degenerate, even if inertia is included.

4. Asymptotics of large eigenvalues. In this section, we focus on the asymptotic behavior of large eigenvalues. It is instructive to look at this case for a number of reasons. As we shall see, some instabilities can be predicted from the analysis of this limit. The asymptotic formula also gives insights into the qualitative nature of the eigenspectrum; it shows that there are infinitely many eigenvalues and that they line up along a curve and shows what the approximate spacing is.

We begin with the simpler case of fixed cross section.

We use the asymptotic expansion of the exponential integral for large argument [1]:

$$\mathrm{Ei}(\mu) = \pi i\,\mathrm{sgn}(\mathrm{Im}\,\mu) + \frac{e^\mu}{\mu}\left(1 + O\left(\frac{1}{\mu}\right)\right). \tag{31}$$

Using this, we can approximate the characteristic equation by

$$e^\mu = q e^{\mu/q}, \tag{32}$$

which leads to

$$\mu = \frac{q}{q-1}(2n\pi i + \ln q). \tag{33}$$

Since $\ln q/(q-1)$ is positive, we find an infinite family of unstable eigenvalues for any value of q. For $q = 2$ the following table compares the eigenvalues found from the asymptotic formula (33) with exact roots of the characteristic equation found by Newton's method:

n	Result from (33)	Exact eigenvalue
1	$1.38629 + 12.5664i$	1.35405 + 12.42i
2	$1.38629 + 25.1327i$	1.37705 + 25.0552i
3	$1.38629 + 37.6991i$	1.38204 + 37.6467i
4	$1.38629 + 50.2655i$	1.38387 + 50.226i
5	$1.38629 + 62.8319i$	1.38473 + 62.8002i

For the case of fixed speed, we need to carry the approximation of the exponential integral a little further:

$$\mathrm{Ei}(\mu) = \pi i \,\mathrm{sgn}(\mathrm{Im}\,\mu) + \frac{e^\mu}{\mu}\left(1 + \frac{1}{\mu} + \frac{2}{\mu^2} + O\left(\frac{1}{\mu^3}\right)\right). \tag{34}$$

Using this, we obtain the approximate characteristic equation

$$e^{\mu - \mu/q} = -\frac{q^3}{(q-1)\mu^2}. \tag{35}$$

For large $|\mu|$, we obtain the following asymptotic formula for the eigenvalues

$$\mu_n = \frac{q}{q-1}\left(2n\pi i + \ln\left(\frac{q^3}{q-1}\right) - 2\ln\left(2n\pi\frac{q}{q-1}\right)\right). \tag{36}$$

Here n is any integer. For $n \to \infty$, the real parts of these eigenvalues tend to $-\infty$ logarithmically, i.e., they are stable.

Since the asymptotic approximation depends on $|\mu/q|$ being large in addition to $|\mu|$, the first few eigenvalues are predicted poorly if q is large. The following table illustrates this behavior for $q = 20.218$, the value at which onset of draw resonance occurs:

n	μ_n given by (36)	Exact eigenvalue
1	$2.40565 + 6.61013i$	4.66015i
2	$0.947223 + 13.2203i$	−0.738622 + 11.4532i
3	$0.094096 + 19.8304i$	−1.20379 + 18.2453i
4	$-0.511207 + 26.4405i$	−1.55854 + 25.0014i
5	$-0.980716 + 33.0506i$	−1.85118 + 31.7307i
10	$-2.43915 + 66.1013i$	−2.8734 + 65.1607i
20	$-3.89758 + 132.203i$	−4.0729 + 131.605i
50	$-5.82551 + 330.506i$	−5.86414 + 330.225i

Another limit which can be approached by asymptotics is that of large draw ratio. If we consider the case $\mu \to \infty$, $q \to \infty$ in (14) with the expectation that $\mu/q \to 0$, the balance of leading order terms yields

$$e^\mu + \ln q = 0, \tag{37}$$

i.e.,

$$\mu_n = (2n-1)i\pi + \ln\ln q. \tag{38}$$

Since q must be really large for $\ln\ln q$ to be considered "large," this approximation is not useful in practice. For $q = 5 * 10^8$, a totally unrealistic value of course, we have

$$i\pi + \ln\ln q = 2.99724 + 3.14159i, \quad 3i\pi + \ln\ln q = 2.99724 + 9.42478i, \tag{39}$$

compared to exact eigenvalues of $2.72203 + 3.471i$ and $2.76567 + 9.63623i$, respectively.

5. Control strategies. In this section, we consider how the onset of draw resonance is affected if we add a control which adjusts the drawing speed in response to observed fluctuations. Since the goal of the manufacturing process is a uniform thread, it seems natural to change the speed in response to fluctuations in the cross-sectional area. This leads to a downstream boundary condition

$$u(q) - \epsilon a(q) \tag{40}$$

to be imposed on (9). Intuitively, we would be tempted to increase the drawing speed when the cross section becomes larger, i.e., $\epsilon > 0$.

The resulting characteristic equation is

$$(e^{\mu} - e^{\mu/q})q + \left(q + \frac{\epsilon}{q} - \mu\right)(\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q)) = 0. \tag{41}$$

The asymptotic behavior of the eigenvalues can be discussed by the same methods as above. We obtain

$$\mu_n \sim \frac{q}{q-1}\left(2\pi n i + \ln\left(\frac{\epsilon}{2n\pi q}\right) - \mathrm{sgn}(\epsilon)\frac{i\pi}{2}\right). \tag{42}$$

For large n, these eigenvalues become stable.

Next, we consider the onset value for draw resonance as a function of ϵ. The results are summarized in the following table:

ϵ	Critical draw ratio
0	20.218
10	18.872
20	17.224
30	14.904

Contrary to intuition, the effect of the control is destabililizing, and the critical draw ratio decreases. For negative ϵ, if we just track the eigenvalue that is responsible for draw resonance at $\epsilon = 0$, the critical draw ratio seems to increase:

ϵ	Critical draw ratio
-20000	200.00
-10000	147.10
-5000	109.13
-1000	57.324
-500	44.843
-100	28.536
-50	25.056
-20	22.420
-10	21.3817

It would be wrong to think, however, that we can achieve stability at any draw ratio by choosing ϵ large and negative. In fact, there are new instabilities at low draw ratios when $|\epsilon|$ is large. We can see this by looking at the asymptotic behavior of eigenvalues assuming that both $|\mu|$ and $|\epsilon|$ are large. The result is

$$\mu_n \sim \frac{q}{q-1}\left(2\pi i n + \ln\left(\frac{\epsilon q}{\epsilon + 2\pi i n q(q-1)}\right)\right). \tag{43}$$

For $q = 4$, $\epsilon = -100$, for instance, this formula yields $\mu_1 = 1.54832 + 9.23897i$, while the actual eigenvalue is $1.21466 + 9.25047i$. The instability resulting from this eigenvalue persists for $q < 8.01516$. We thus have two separate instabilities, one for low draw ratio and another for high draw ratio. The next table shows the first eigenvalue for $\epsilon = -100$ as a function of the draw ratio q (the higher eigenvalues are more stable):

q	First eigenvalue
2	$1.33788 + 12.9402i$
3	$1.40897 + 10.04i$
4	$1.21466 + 9.25047i$
6	$0.579954 + 8.55471i$
8	$0.004134 + 8.08557i$
10	$-0.533251 + 7.66538i$
15	$-1.59346 + 5.79611i$
20	$-0.601555 + 4.77314i$
25	$-0.173004 + 4.58841i$
30	$0.0558642 + 4.49297i$
40	$0.31093 + 4.37887i$

Another control strategy is to monitor the force in the thread and change the drawing speed in response. This leads to the boundary condition

$$u(q) + \epsilon(u'(q) + qa(q)) = 0. \tag{44}$$

This boundary condition was also considered in [8]. The resulting characteristic equation is

$$(e^{\mu} - e^{\mu/q})q + (q - \mu)(\mathrm{Ei}(\mu) - \mathrm{Ei}(\mu/q)) + \epsilon e^{\mu} = 0. \tag{45}$$

The behavior of large eigenvalues becomes

$$\mu_n \sim \frac{q}{q-1}\left(2n\pi i + \ln\left(\frac{q^3}{q+\epsilon-1}\right) - 2\ln\left(2n\pi\frac{q}{q-1}\right)\right). \tag{46}$$

We see from this asymptotic formula that a positive ϵ is stabilizing, as would heuristically be expected. The effect on draw resonance follows the same trend, and the results show no surprises.

ϵ	Critical draw ratio
-5	5.387
-2	16.786
0	20.218
5	26.561
10	31.632
20	40.137
50	60.37
100	87.468

6. Conclusions. We have investigated the simplest model of fiber spinning in a viscous fluid, which includes only viscous forces, neglecting all other effects. In this simple case, the linear stability problem has a closed form solution in terms of an exponential integral, which can be exploited to gain substantial qualitative insight into the behavior of the eigenvalues. The stability of the flow depends crucially on the choice of downstream boundary conditions. If the speed is prescribed, then, as is well known, the flow becomes unstable beyond a critical draw ratio. On the other hand, prescribed force leads to no instabilities, while prescribed cross section leads to instability at all draw ratios. In terms of strategies to control instability, adjustment of the speed in reaction to changes in cross section has an effect opposite of what is intuitively expected. In addition to changing the threshold for high draw ratio instabilities, such a control also produces new instabilities at low draw ratios.

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