

MESH INDEPENDENCE OF KLEINMAN-NEWTON ITERATIONS FOR RICCATI EQUATIONS IN HILBERT SPACE*

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Abstract. In this paper we consider the convergence of the infinite dimensional version of the Kleinman-Newton algorithm for solving the algebraic Riccati operator equation associated with the linear quadratic regulator problem in a Hilbert space. We establish mesh independence for this algorithm and apply the result to systems governed by delay equations. Numerical examples are presented to illustrate the results.

Key words. Riccati, Kleinman-Newton, mesh independence

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1. Introduction. The problem of constructing numerical schemes for optimization-based design and control of infinite dimensional systems leads to technical and practical issues that are not present if one is interested only in simulation. For example, if one uses finite elements or the method of lines to simulate a system of partial differential equations (PDEs), then the resulting finite dimensional approximate system is often very large and can have millions of state variables. The corresponding approximating Riccati equations are immense, and special numerical techniques are required to solve such equations. Many of these large-scale Riccati solvers are based on iterative algorithms (see [9], [10], and [30]) and take advantage of the mathematical structure of the approximating system (symmetry, sparseness, etc.).

There are two basic issues that need to be addressed in developing practical numerical approximations for control. First, it is essential that the approximation scheme leads to finite dimensional approximating Riccati equations that converge (under mesh refinement) to the solution of the infinite dimensional Riccati equation. This is a well-studied problem (see [7], [14], [26], [33], and [43]). It is now well known that to obtain norm convergence for the Riccati equation, the approximation scheme must satisfy some form of convergence, dual convergence, and uniform preservation of stabilizability and detectability under mesh refinement (see [7] and [33]). These concepts will be made more precise in section 7.1. The important point here is that many "standard" convergent approximation schemes do not satisfy all the conditions necessary for norm convergence of the Riccati operators (see [16]). If this issue is ignored when one develops an approximation scheme for control design and optimization, then the resulting numerical algorithm can fail to produce accurate and useful results (see the numerical examples in section 9). In this paper we show that these properties are also key ingredients in establishing mesh independence of Newton-type algorithms.

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The second important issue is concerned with the development of an effective algorithm for the numerical solution of the (large-scale) finite dimensional Riccati equations that arise once the problem has been discretized. During the past five years considerable attention has been devoted to the problem of developing accurate and fast numerical methods for control of large-scale systems. In 2004, the first issue of the *IEEE Control Systems Magazine* (Volume 24, Issue 1) was devoted to this topic. Much of the motivation for this emphasis comes from the fact that such systems often arise as discretizations of control problems with PDEs as the governing system. The observation that these large-scale finite dimensional Riccati equations come from discretizations of PDE control systems makes it possible to exploit special algorithms such as multigrid techniques (see [42] and [45]) and parallel iterative solvers (see [30]). Considerable progress has been made at this level by Benner and Saak (see [9] and [10]). Also, Grasedyck, Hackbusch, and Khoromskij (see [30] and the references therein) have developed impressive computational algorithms for Riccati and Lyapunov equations that arise in these cases. Many of these large-scale Riccati solvers are based on iterative algorithms.

It is impossible to address all the potential problems in constructing approximation schemes for optimal control of infinite dimensional systems in a single paper, so we limit our discussion to the well-studied linear quadratic optimal control problem and show how specific approximation assumptions are needed to address convergence and efficiency of an algorithm. In particular, we focus on convergence and mesh independence of the Kleinman–Newton algorithm for solving the operator Riccati equation defined by the linear quadratic regulator (LQR) problem. This problem is simple enough to allow for a rather complete analysis of convergence and mesh independence and yet complex enough to illustrate how both convergence and mesh independence might fail for perfectly good “standard” (convergent) numerical approximations.

2. A short review of the mesh independence principle. There are two basic aspects of the mesh independence principle (MIP) for Newton-type methods (see [1] and [2]). Roughly speaking, the MIP may be broken down into convergence under mesh refinement of the Newton iteration counts on a given mesh.

Let $\mathcal{F} : D(\mathcal{F}) \subseteq E \rightarrow E$ be a nonlinear operator on an infinite dimensional Hilbert space E , and consider the equation

$$(2.1) \quad \mathcal{F}(x) = 0.$$

Let $E^N \subseteq E$ be a sequence of finite dimensional approximating spaces, and consider the sequence of discretized equations

$$(2.2) \quad \mathcal{F}^N(x^N) = 0,$$

where $\mathcal{F}^N : D(\mathcal{F}^N) \subseteq E^N \rightarrow E^N$. In this paper we are interested in the problem of solving the Riccati operator equation associated with LQR feedback control of systems governed by delay and PDEs. In this setting, (2.1) is an infinite dimensional Riccati equation defined by a PDE control system, and (2.2) is an approximating Riccati equation obtained by some type of finite element or finite difference scheme applied to the PDE system. Here N is related to the size of the mesh used to define the discretized equations on a grid. Assume that (2.1) and (2.2) have unique solutions $x_\infty \in D(\mathcal{F})$ and $x_\infty^N \in D(\mathcal{F}^N)$, respectively. We say that the *approximation scheme converges* if

$$(2.3) \quad \lim_{N \rightarrow +\infty} \|x_\infty^N - P^N x_\infty\|_{E^N} = 0,$$

where $P^N : E \rightarrow E^N$ is the orthogonal projection of E onto E^N .

Now assume that one applies a Newton-type algorithm to the infinite dimensional problem (2.1) and that the same algorithm is also applied to the corresponding finite dimensional approximate problem (2.2). For the moment assume both schemes produce quadratically convergent iterations x_k and x_k^N , $k = 1, 2, \dots$. For a given $\varepsilon > 0$, $x_0 \in D(\mathcal{F})$ and $x_0^N \in D(\mathcal{F}^N)$ define the numbers $M(\varepsilon, x_0)$ and $M^N(\varepsilon, x_0^N)$ by

$$M(\varepsilon, x_0) \triangleq \inf\{k : \|x_k - x_\infty\| < \varepsilon\} \quad \text{and} \quad M^N(\varepsilon, x_0^N) \triangleq \inf\{k : \|x_k^N - x_\infty^N\|_{E^N} < \varepsilon\},$$

respectively. Here, x_0 and x_0^N are the starting values for the iterations. The (strong) MIP (see Theorem 2.1 in [2]) takes the form

$$(2.4) \quad M(\varepsilon, x_0) = M^N(\varepsilon, P^N x_0) + \tau(N),$$

where $\tau(N) \rightarrow 0$ as $N \rightarrow +\infty$. Also, assume there are constants c and c^N such that

$$(2.5) \quad \|x_{k+1} - x_\infty\| \leq c \|x_k - x_\infty\|^2$$

and

$$(2.6) \quad \|x_{k+1}^N - x_\infty^N\|_{E^N} \leq c^N \|x_k^N - x_\infty^N\|_{E^N}^2,$$

respectively. Let \hat{c} and \hat{c}^N be the minimal values of c and c^N that satisfy (2.5) and (2.6), where $x_0^N = P^N x_0$. As noted in [2], since $P^N : E \rightarrow E^N$ is the orthogonal projection of E onto E^N , in some cases one can show that another form of the strong MIP is given by

$$(2.7) \quad \hat{c}^N = \hat{c} + \gamma(N),$$

where $\gamma(N) \rightarrow 0$ as $N \rightarrow +\infty$. The basic idea behind these strong versions of mesh independence is that the number of iterations required to achieve a given error tolerance is independent of the mesh size and asymptotically converges to the number of infinite dimensional iterations (theoretically) required to attain the same tolerance. A weaker form of the MIP would require only that, if one has the estimates (2.5) and (2.6), then

$$(2.8) \quad c^N = c + \delta(N),$$

where $\delta(N) \rightarrow 0$ as $N \rightarrow +\infty$. Although the constants c and c^N are not the minimal values, it follows that the number of iterations required to solve the discretized equations $\mathcal{F}^N(x^N) = 0$ is essentially independent of the mesh size.

3. Mesh independence for the infinite dimensional Riccati equation. In this paper we focus on the case where the nonlinear function $\mathcal{F} = \mathcal{F}(\Pi)$ is defined by an infinite dimensional Riccati operator equation of the form

$$(3.1) \quad \mathcal{F}(\Pi) = A^* \Pi + \Pi A - \Pi B B^* \Pi + C^* C = 0,$$

where A generates a strongly continuous semigroup on a Hilbert space H . Here $\mathcal{F} : D(\mathcal{F}) \subseteq E \rightarrow E$, where E is the space of bounded linear operators on H .

Remark 3.1. It is important to note that in most applications the operator A is unbounded, and even if the B and C operators are bounded, the nonlinear operator

\mathcal{F} will not be continuous on its domain. Therefore, \mathcal{F} will not have a Lipschitz continuous Fréchet derivative, and the analysis used in [1] and [2] is not directly applicable. In particular, convergence proofs for the infinite dimensional Newton algorithm that depend on the existence of the Fréchet derivative cannot be used in this setting. As noted by Damm and Hinrichsen in [23], the existence of the Fréchet derivative can be relaxed if one works in ordered Banach spaces. Indeed, they provide a general convergence result under the blanket assumptions that E is ordered by a closed, solid, regular convex cone and that \mathcal{F} is continuous on its domain (see page 50 in [23]). Moreover, even when using the ordered space approach, we see that Fréchet differentiability was needed to obtain quadratic convergence of the Newton method (see page 56 in [23]). However, for the delay systems below (and other PDE control systems) these assumptions do not hold.

In the finite dimensional case, the “natural” space of operators is the set $E = \mathcal{H}^n$ of $n \times n$ Hermitian matrices with (Frobenius) trace norm. As noted in [23], if one uses the cone $\mathcal{C} = \mathcal{H}_+^n = \{\Pi \in \mathcal{H}^n : \Pi \geq 0\}$, then \mathcal{C} satisfies the blanket assumptions above. In an infinite dimensional setting, verifying these assumptions is nontrivial or impossible, depending on the choice of E . One might be tempted to use the infinite dimensional analogue and set $E = \mathcal{H}$ to be the set of all trace class operators on the Hilbert space H . If the solution Π to (3.1) is not of trace class (see Example 1), then this is not a reasonable choice for E . Even if the solution is of trace class, one might still need to work in a larger space to develop practical approximation schemes for numerical solutions. In this setting, if $E = \mathcal{L}(H, H)$ is the space of bounded linear operators on H and one sets $\mathcal{C} = \mathcal{H}_+ = \{\Pi \in \mathcal{H} : \Pi \geq 0\}$ to be the cone of nonnegative definite trace operators, then \mathcal{C} is not solid. Hence, a direct application of the results in [23] is not possible.

In the case when the nonlinear equation (3.1) is a Riccati equation defining an LQR controller, it is possible to extend the finite dimensional proof of Mehrmann in [40] to a rather general class of infinite dimensional problems. We take this approach and present a complete convergence proof for the infinite dimensional Kleinman–Newton algorithm in the space $E = \mathcal{L}(H, H)$. Although this proof is similar in spirit to the results in [40], there are some technical details that require attention. Moreover, this approach provides explicit bounds and estimates that we later use to establish mesh independence. This is another reason we do not use the approach in [23] based on ordered spaces. The following example illustrates that even if A , B , and C are bounded, the solution to the operator Riccati equation (3.1) does not have to be of trace class. Later we shall use this example to illustrate the importance of the compactness assumptions.

Example 1. Let $H = \mathbb{R} \times L^2$, and define the operators A , B , and C on H by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -I \end{bmatrix}, \quad B = [1 \ 0]^T, \quad \text{and} \quad C = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2}I \end{bmatrix},$$

respectively. Here I is the identity on L^2 . By direct computation it follows that

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$$

is the solution to the Riccati equation (3.1). Since the identity operator is not compact, Π is not of trace class. On the other hand, for this simple example, \mathcal{F} is continuously Fréchet differentiable, and convergence of the infinite dimensional Kleinman–Newton algorithm follows directly from [1] and [2].

It is well known that under very mild conditions on the LQR problem, and assuming that the discretization scheme preserves the basic control system properties, both (2.1) and (2.2) have unique solutions in the set of nonnegative self-adjoint bounded linear operators. The issues to be resolved are as follows:

- (i) What conditions must be placed on the discretization scheme to guarantee that the solutions to the approximating equations (2.2) converge *in norm* to the solution of the infinite dimensional problem (2.1)? In some sense this is a classic numerical analysis problem. However, for the Riccati equation (3.1), the conditions for norm convergence are nontrivial and the best results are stated in terms of control system properties.
- (ii) Does the infinite dimensional Kleinman–Newton algorithm converge quadratically when applied to the infinite dimensional operator Riccati equation? As noted above, there are several approaches to this question. For applications to delay and PDE systems, our approach offers explicit bounds which are helpful in establishing mesh independence, and this approach does not require continuity of \mathcal{F} .
- (iii) Finally, what conditions must be placed on the discretization scheme to guarantee that the Kleinman–Newton algorithm satisfies MIP estimates of the form (2.4), (2.7), or (2.8)?

In this paper we focus on these issues. First, we give a brief review of what is known about convergence of discretization schemes for the infinite dimensional LQR control problem.

4. A summary of approximation results for LQR control. Most of the numerical schemes for approximating systems governed by PDEs developed during the past 50 years focused on methods that provided convergent and efficient simulations. However, the LQR problem is an optimal control problem on an infinite time interval, and it is possible to clearly identify two additional requirements that need to be placed on an approximation scheme to ensure convergence of the control design. Moreover, we shall show that these requirements also play a role in determining mesh independence of the Kleinman–Newton algorithm. In particular, dual convergence and preservation of exponential stability (POES) play central roles in both convergence and mesh independence. The POES condition was first introduced by Banks and Kunisch in [7] as a technical assumption needed to establish strong convergence of the Riccati operators for parabolic PDE control problems. This condition is equivalent to the uniform stabilizability defined in Assumption 7.3 below. In some cases one can relax the POES assumption and still obtain strong (and even norm) convergence of the Riccati operators. For example, the spline scheme developed for delay systems by Kappel and Salamon in [36] produced convergent Riccati operators even though POES was not satisfied (see [35] and [37]). Kappel and Salamon replaced the POES assumption with a uniform output and input-output stability condition and proved strong convergence of the Riccati operators. Ito [33], [34] used the version given in Assumption 7.3 to establish norm convergence. We will say more about this when we discuss the numerical results below.

In 1969 Sasai and Shimemura [47] was among the first researchers to recognize the importance of dual convergence for infinite dimensional LQR problems (also see [46] and [48]). Gibson (see [27], [28], [29]) established a general framework for developing approximation schemes for LQR problems and applied his results to control systems governed by delay and hyperbolic PDEs. If one is interested only in weak convergence of the functional gains, then dual convergence may not be essential (see [24] and

[25]). However, as observed in [8] and [13], weak convergence may not be sufficient for practical design, and, as shown in [16], not all standard schemes yield dual convergent algorithms. In particular, the finite element scheme developed in [5] is not dual convergent and does not produce strongly convergent functional gains.

At this point, there is no general method that can address the issue of dual convergence. However, for delay systems, two approaches have emerged. The first method is based on constructing numerical schemes that are dual convergent. The excellent survey by Kappel [35] focuses on this approach. A second approach is based on constructing separate numerical schemes for the forward problem and the dual problem. This is the approach carried out for delay equations by Germani, Manes, and Pepe in the paper [26]. It is important to note that extending any of these methods to other types of PDE-based systems is not a trivial exercise.

The key point here is that in order to develop numerical schemes for control of infinite dimensional systems, one must first ensure that certain control system properties are preserved under the approximation. Once this issue is resolved, it is important to consider the problem of numerically solving the finite dimensional problem. In particular, it is possible to construct several numerical schemes that preserve the required control system properties (stabilizability, detectability, etc.), but the resulting finite dimensional control problems may differ dramatically in conditioning and computational complexity.

In this paper we focus on an iterative method for solving the Riccati equations associated with LQR problems. We show that the infinite dimensional Kleinman–Newton iterations converge to the Riccati operator for the infinite dimensional problem, and we investigate mesh independence. Although the basic ideas used in the proof are similar to those found in papers on the finite dimensional problem, there are certain estimates that provide insight into general connections between preservation of control system properties, convergence, and mesh independence. These results provide a framework that can be employed to analyze specific finite dimensional approximations. We close with an application of these results to delay systems and a discussion of the averaging schemes found in [3] and [16] and the spline/finite element schemes given in [4], [5], and [8].

As noted above, the MIP does not make sense unless one has *norm convergence* of the discretized Riccati operators. It is known (see [16]) that the spline/finite element scheme for delay systems given in [5] does not produce norm convergent Riccati operators. This problem can also be seen in the much simpler Example 2 given in section 7.1 below.

5. Problem setting and basics. We consider the following LQR problem in an abstract Hilbert space setting. Let U , H , and Y be Hilbert spaces over the reals. If Z and W are any two Hilbert spaces, then we denote by $\mathcal{L}(Z, W)$ the linear space of linear bounded operators from Z into W . In the special case where $W = Z$, we set $\mathcal{L}(Z) = \mathcal{L}(Z, Z)$. The system equation is given in state space form by

$$(5.1) \quad \dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0 \in H,$$

where A generates a strongly continuous semigroup on H and $B \in \mathcal{L}(U, H)$.

The assumption that $B \in \mathcal{L}(U, H)$ implies we are considering only bounded input operators. Let $C \in \mathcal{L}(H, Y)$, and define the quadratic cost function $J(u)$ by

$$(5.2) \quad J(u) = \int_0^\infty (\|Cz(s)\|^2 + \|u(s)\|^2) ds,$$

where $z(s)$ is the solution to (5.1) for a given control $u \in L^2(0, \infty; U)$. The LQR control problem is to minimize the quadratic cost $J(u)$ over all controls $u \in L^2(0, \infty; U)$.

It is well known (see [11], [12]) that under certain assumptions, the optimal control is given by state feedback $u_{opt} = -Kz(t)$, where

$$(5.3) \quad K = B^*X,$$

and $X \in \mathcal{L}(H)$ is a solution of an abstract algebraic Riccati operator equation of the form

$$(5.4) \quad A^*X + XA - XBB^*X + C^*C = 0.$$

In order to formulate an abstract Newton method for solving this nonlinear operator equation in $\mathcal{L}(H)$, we first have to specify the appropriate mappings. Let $\Sigma(H) = \{\Pi \in \mathcal{L}(H) : \Pi \text{ self-adjoint}\}$ be the space of self-adjoint bounded linear operators on H , and let $\Sigma^+(H) = \{\Pi \in \Sigma(H) : (\Pi x, x) \geq 0 \text{ for all } x \in H\}$ denote the subspace of nonnegative operators in $\Sigma(H)$.

Since A is an unbounded operator in (5.4), we define a map \mathcal{A} which can be formally written as $\mathcal{A}(\Pi) = A^*\Pi + \Pi A$ and can be defined rigorously as in [12, page 151]. In particular, for a given $\Pi \in \Sigma(H)$, set

$$\phi_\Pi(x, y) = (\Pi x, Ay) + (Ax, \Pi y), \quad x, y \in D(A),$$

and define

$$D(\mathcal{A}) = \{\Pi \in \Sigma(H) : \phi_\Pi \text{ can be extended to a continuous sesquilinear operator on } H \times H\}.$$

This unique extension of ϕ_Π as a continuous sesquilinear form on $H \times H$ will also be denoted by ϕ_Π . For each $\Pi \in D(\mathcal{A})$, one can define a linear operator, denoted by $\mathcal{A}(\Pi) \in \Sigma(H)$, by the identity

$$(5.5) \quad (\mathcal{A}(\Pi)x, y) = \phi_\Pi(x, y) = (\Pi x, Ay) + (Ax, \Pi y), \quad x, y \in D(A), \Pi \in D(\mathcal{A}).$$

Therefore, we have defined a linear operator $\mathcal{A} : D(\mathcal{A}) \subset \Sigma(H) \rightarrow \Sigma(H)$, and in [12, page 152], it is shown that for $\Pi \in D(\mathcal{A})$ and $x \in D(A)$ one has $\Pi x \in D(A^*)$ and

$$(5.6) \quad \mathcal{A}(\Pi)x = A^*\Pi x + \Pi Ax.$$

The previous notation allows us to precisely define a solution of the abstract Riccati operator equation.

DEFINITION 5.1. *The bounded linear operator X is called a strict solution of the Riccati equation (5.4) if $X \in D(\mathcal{A})$ and*

$$(5.7) \quad \mathcal{A}(X) - XBB^*X + C^*C = 0.$$

The bounded linear operator X is called a weak solution of the Riccati equation (5.4) if $X \in D(\mathcal{A})$ and

$$(5.8) \quad (Xx, Ay) + (Ax, Xy) - (B^*Xx, B^*Xy) + (Cx, Cy) = 0, \quad x, y \in D(A).$$

It is shown on page 262 in [12] that for $X \in \Sigma^+(H)$ a strict solution is equivalent to a weak solution of (5.4). Although we might use the notation of the operator equation, in this paper we deal with weak solutions. The existence and uniqueness of

solutions to the Riccati operator equation are not necessarily guaranteed. We follow the definitions and notation in [12].

DEFINITION 5.2. (i) *The system (A, B) is called stabilizable if there exists a bounded linear operator $K : H \rightarrow U$ such that $(A - BK)$ generates an exponentially stable C_0 -semigroup on H .*

(ii) *The pair (A, C) is called detectable if there exists a bounded linear operator $F : Y \rightarrow X$ such that $(A + FC)$ generates an exponentially stable C_0 -semigroup on H .*

Since B and C are bounded, the following theorem follows from [11, Part III, Prop. 2.3, Prop. 3.2, and Cor. 4.2].

THEOREM 5.3. *If (A, B) is stabilizable, then there exists a minimal solution $X_{min} \in \Sigma^+(H)$ of the Riccati equation (5.8). If, in addition, (A, C) is detectable, then $A - BB^*X_{min}$ generates an exponentially stable semigroup and X_{min} is the unique solution of the Riccati equation (5.8) in $\Sigma^+(H)$.*

In the Newton iteration, we will see that the Newton steps are defined by the solutions of a generalized Lyapunov equation. Therefore, we recall a few facts about Lyapunov equations in Hilbert spaces. The following result is found on pages 19–28 in [11].

THEOREM 5.4. *Let $S(\cdot)$ denote a strongly continuous semigroup on a Hilbert space H with an infinitesimal generator A . Then the following statements are equivalent.*

(i) *The semigroup $S(\cdot)$ is exponentially stable; i.e., there exist $\omega > 0$ and $M \geq 1$ such that*

$$(5.9) \quad \|S(t)x\| \leq Me^{-\omega t}\|x\| \quad \text{for all } x \in H, \quad t \geq 0.$$

(ii) *There exists a positive $P \in \Sigma^+(H)$ such that*

$$(5.10) \quad (Px, Ay) + (Ax, Py) + (x, y) = 0, \quad x, y \in D(A).$$

When applying Newton's method, we obtain Lyapunov equations that are more general where the identity term (x, y) is replaced by a more general term (x, Qy) with possible nonnegativity or positivity properties. However, from the representation formula for solutions of Lyapunov equations, one can establish the following result (see [22, page 252]).

THEOREM 5.5. *Let $S(\cdot)$ denote a strongly continuous semigroup on a Hilbert space H with an infinitesimal generator A . If $S(\cdot)$ is exponentially stable and $Q \in \Sigma(H)$, then there exists a unique solution $X \in \Sigma(H)$ of*

$$(5.11) \quad (Xx, Ay) + (Ax, Xy) + (x, Qy) = 0, \quad x, y \in D(A).$$

Moreover, X has the representation

$$(5.12) \quad X = \int_0^\infty S^*(t)QS(t)dt,$$

and if $Q \in \Sigma^+(H)$, then $X \in \Sigma^+(H)$.

6. The Kleinman–Newton method in Hilbert space. In this section we define the Kleinman–Newton algorithm and establish convergence. Throughout this section we make the following assumption.

ASSUMPTION 6.1. *Let A be the infinitesimal generator of a semigroup on H , $B \in \mathcal{L}(U, H)$, and $C \in \mathcal{L}(H, Y)$. We assume that*

- (i) the system (A, B) is stabilizable and the pair (A, C) is detectable, and
- (ii) an operator $X_0 \in \Sigma^+(H)$ is given such that $A - BB^*X_0$ generates an exponentially stable semigroup on H .

We solve the abstract algebraic equation (5.7) by Newton’s method. We seek a bounded linear operator $X \in D(\mathcal{A})$, which provides a solution to the Riccati equation $\mathcal{F}(X) = 0$ where the nonlinear mapping $\mathcal{F} : D(\mathcal{A}) \subset \Sigma(H) \rightarrow \Sigma(H)$ is defined by

$$(6.1) \quad \mathcal{F}(X) = \mathcal{A}(X) - XBB^*X + C^*C.$$

In particular, we look for weak solutions to the equation $\mathcal{F}(X) = 0$ as defined by (5.8).

Formally applying Newton’s method to $\mathcal{F}(X) = 0$ leads to the iteration

$$(6.2) \quad \begin{aligned} &\mathcal{A}(X_{k+1} - X_k) - X_kBB^*(X_{k+1} - X_k) - (X_{k+1} - X_k)BB^*X_k \\ &+ \mathcal{A}(X_k) - X_kBB^*X_k + C^*C = 0, \end{aligned}$$

or equivalently,

$$(6.3) \quad \mathcal{A}(X_{k+1}) - X_kBB^*X_{k+1} - X_{k+1}BB^*X_k + X_kBB^*X_k + C^*C = 0.$$

After rearranging some terms, it follows that the weak formulation of this scheme has the form

$$(6.4) \quad \begin{aligned} &(X_{k+1}x, (A - BB^*X_k)y) + ((A - BB^*X_k)x, X_{k+1}y) \\ &= -(B^*X_kx, B^*X_ky) - (Cx, Cy), \quad x, y, \in D(A). \end{aligned}$$

In what follows we will establish the following convergence theorems for Newton’s method. We split up the statements into three parts according to the tools being used in the proof. We follow the structure of the proof for the finite dimensional case given in [40] (see pages 91–94).

THEOREM 6.2. *If Assumption 6.1 holds, then*

- (i) the Newton iteration (6.4) is well defined and has a unique solution $X_k \in \Sigma^+(H)$, $k = 1, 2, \dots$, and
- (ii) the closed-loop operators $A - BB^*X_k$, $k = 1, 2, \dots$, generate exponentially stable semigroups $S_k(t)$.

Proof. Let us assume, by induction, that $X_k \in \Sigma^+(H)$ is such that $A - BB^*X_k$ generates an exponentially stable semigroup $S_k(t)$. In particular, there exist $M_k \geq 1$ and $\omega_k > 0$ such that

$$(6.5) \quad \|S_k(t)\| \leq M_k e^{-\omega_k t}.$$

Theorem 5.5 applied to equation (6.4) yields the existence of the next iterate, $X_{k+1} \in \Sigma^+(H)$. By adding and subtracting terms involving X_{k+1} and reordering, (6.4) can now be rewritten as

$$(6.6) \quad \begin{aligned} &(X_{k+1}x, (A - BB^*X_{k+1})y) + ((A - BB^*X_{k+1})x, X_{k+1}y) \\ &= -(B^*X_{k+1}x, B^*X_{k+1}y) - (Cx, Cy) \\ &\quad - (B^*(X_{k+1} - X_k)x, B^*(X_{k+1} - X_k)y), \quad x, y \in D(A). \end{aligned}$$

From (6.6) the operator X_{k+1} can be viewed as a solution $X_{k+1} \in \Sigma^+(H)$ of a Lyapunov equation of the form (5.11) with the infinitesimal generator $A - BB^*X_{k+1}$ of a semigroup denoted by $S_{k+1}(t)$. Since X_{k+1} exists, we define

$$V(t, z) := (X_{k+1}S_{k+1}(t)z, S_{k+1}(t)z), \quad z \in D(A - BB^*X_{k+1}).$$

It follows from (6.6) that $\frac{dV}{dt}(t, z) = -\Phi(t)$, where

$$\Phi(t) = \|B^* X_{k+1} S_{k+1}(t)z\|^2 + \|C S_{k+1}(t)z\|^2 + \|B^*(X_{k+1} - X_k) S_{k+1}(t)z\|^2.$$

Since $X_{k+1} \in \Sigma^+(H)$, an integration of the previous equation yields

$$\int_0^t \Phi(s) ds = V(0, z) - V(t, z) \leq V(0, z)$$

for all $t > 0$ and $z \in D(A - BB^* X_{k+1})$. The domain $D(A - BB^* X_{k+1})$ is dense in H , which implies that for all $z \in H$ there is a constant c_z such that

$$(6.7) \quad \int_0^\infty \|B^*(X_{k+1} - X_k) S_{k+1}(t)z\|^2 dt \leq \int_0^\infty \Phi(t) dt \leq c_z.$$

Set $A_k = A - BB^* X_k$ so that $A_{k+1} = A - BB^* X_{k+1} = A_k + BB^*(X_k - X_{k+1})$. Observe that $z_{k+1}(t) = S_{k+1}(t)z$ is the solution of

$$\dot{z}_{k+1}(t) = A_{k+1} z_{k+1}(t) = A_k z_{k+1}(t) + BB^*(X_k - X_{k+1}) z_{k+1}(t), \quad z_{k+1}(0) = z,$$

and is given by

$$z_{k+1}(t) = S_k(t)z + \int_0^t S_k(t-s) BB^*(X_k - X_{k+1}) z_{k+1}(s) ds.$$

Using the assumption that $S_k(t)$ is exponentially stable so that (6.5) holds, we have

$$\begin{aligned} \|z_{k+1}(t)\| &\leq \|S_k(t)z\| + \int_0^t \|S_k(t-s)\| \|BB^*(X_k - X_{k+1}) z_{k+1}(s)\| ds \\ &\leq M_k e^{-\omega_k t} \|z\| + M_k \|B\| \int_0^t e^{-\omega_k(t-s)} \|B^*(X_k - X_{k+1})\| \|z_{k+1}(s)\| ds. \end{aligned}$$

Gronwall's inequality, together with (6.7), yields

$$\int_0^\infty \|S_{k+1}(t)z\|^2 dt = \int_0^\infty \|z_{k+1}(t)\|^2 dt \leq c_z, \quad z \in D(A - BB^* X_{k+1}).$$

By density this holds for all $z \in H$, and by Theorem 5.1.2 in [22] we obtain that S_{k+1} is an exponentially stable semigroup. This concludes the induction step and the proof. \square

THEOREM 6.3. *If Assumption 6.1 holds, then*

- (i) *the sequence X_k converges in $\Sigma^+(H)$ and $\lim_{k \rightarrow \infty} X_k = X_\infty \in \Sigma^+(H)$. Moreover, X_∞ is a weak solution to $\mathcal{F}(X_\infty) = 0$;*
- (ii) *the closed-loop operator $A - BB^* X_\infty$ generates an exponentially stable semigroup $S_\infty(t)$ satisfying*

$$(6.8) \quad \|S_\infty(t)\| \leq M e^{-\omega t}$$

for constants $M \geq 1$ and $\omega > 0$;

- (iii) *the Newton iterates satisfy $0 \leq X_\infty \leq \dots \leq X_{k+1} \leq X_k \leq \dots \leq X_1$.*

Proof. If we increase the index in (6.4) by one, we obtain

$$\begin{aligned} (X_{k+2}x, (A - BB^* X_{k+1}y)) + ((A - BB^* X_{k+1})x, X_{k+2}y) \\ = -(B^* X_{k+1}x, B^* X_{k+1}y) - (Cx, Cy), \quad x, y \in D(A). \end{aligned}$$

Subtracting this from (6.6) yields

$$(6.9) \quad \begin{aligned} & ((X_{k+1} - X_{k+2})x, (A - BB^*X_{k+1}y)) + ((A - BB^*X_{k+1})x, (X_{k+1} - X_{k+2})y) \\ & = -(B^*(X_{k+1} - X_k)x, (B^*(X_{k+1} - X_k)y)), \quad x, y \in D(A). \end{aligned}$$

We can infer from Theorem 5.5 that $X_{k+1} - X_{k+2} \geq 0$, $k = 0, 1, 2, \dots$

Therefore $X_k \in \Sigma^+(H)$ is a sequence of operators which is decreasing and bounded from below by 0. By [39, page 282], there exists an $X_\infty \in \Sigma^+(H)$ with

$$\lim_{k \rightarrow +\infty} X_k x = X_\infty x \quad \text{for all } x \in H.$$

Passing to the limit in (6.4), we deduce that X_∞ satisfies the Riccati equation (5.8) in its weak form, and hence $\mathcal{F}(X_\infty) = 0$. Theorem 5.3 implies that the solution X_∞ is unique, and by Assumption 6.1, $A - BB^*X_\infty$ generates an exponentially stable semigroup. \square

THEOREM 6.4. *If Assumption 6.1 holds, then for all $k = 0, 1, 2, \dots$,*

$$\|X_{k+1} - X_\infty\| \leq c \|X_k - X_\infty\|^2,$$

where

$$c = \int_0^\infty \|S_\infty^*(t)\| \|BB^*\| \|S_\infty(t)\| dt \leq \frac{M^2}{2\omega} \|BB^*\|$$

and the constants $M \geq 1$ and $\omega > 0$ are given by (6.8).

Proof. By (6.3), the limit operator X_∞ satisfies the equation

$$(6.10) \quad \mathcal{A}(X_\infty) - X_\infty BB^* X_\infty + C^* C = 0.$$

To shorten notation in the rest of the proof, all the equations are to be understood in the weak sense. Equation (6.10) can be rewritten as

$$(6.11) \quad \begin{aligned} & (A - BB^*X_{k+1})^* X_\infty + X_\infty (A - BB^*X_{k+1}) \\ & = -X_\infty BB^* X_\infty - C^* C + (X_\infty - X_{k+1}) BB^* X_\infty + X_\infty BB^* (X_\infty - X_{k+1}). \end{aligned}$$

If we subtract (6.6) from (6.11), we obtain

$$(6.12) \quad \begin{aligned} & (A - BB^*X_{k+1})^* (X_\infty - X_{k+1}) + (X_\infty - X_{k+1}) (A - BB^*X_{k+1}) \\ & = X_\infty BB^* X_\infty - X_{k+1} BB^* X_\infty - X_\infty BB^* X_{k+1} + X_{k+1} BB^* X_{k+1} \\ & \quad + (X_{k+1} - X_k) BB^* (X_{k+1} - X_k) \\ & = (X_\infty - X_{k+1}) BB^* (X_\infty - X_{k+1}) + (X_{k+1} - X_k) BB^* (X_{k+1} - X_k). \end{aligned}$$

This implies that

$$(6.13) \quad \begin{aligned} & (A - BB^*X_\infty)^* (X_\infty - X_{k+1}) + (X_\infty - X_{k+1}) (A - BB^*X_\infty) \\ & = -(X_\infty - X_{k+1}) BB^* (X_\infty - X_{k+1}) + (X_{k+1} - X_k) BB^* (X_{k+1} - X_k). \end{aligned}$$

Note that $\Delta = X_\infty - X_{k+1}$ is the solution to the Lyapunov equation

$$(A_\infty)^* \Delta + \Delta(A_\infty) = -\hat{Q},$$

where

$$\hat{Q} = (X_\infty - X_{k+1})BB^*(X_\infty - X_{k+1}) - (X_{k+1} - X_k)BB^*(X_{k+1} - X_k)$$

and $A_\infty = A - BB^*X_\infty$ generates the exponentially stable semigroup $S_\infty(t)$. It follows from (5.12) in Theorem 5.5 above that the representation formula in Corollary 4.2 of [28] can be used to derive the following:

(6.14)

$$\begin{aligned} 0 \leq X_{k+1} - X_\infty &= \int_0^\infty S_\infty^*(t) \{-(X_\infty - X_{k+1})BB^*(X_\infty - X_{k+1}) \\ &\quad + (X_{k+1} - X_k)BB^*(X_{k+1} - X_k)\} S_\infty(t) dt \\ &\leq \int_0^\infty S_\infty^*(t) ((X_{k+1} - X_k)BB^*(X_{k+1} - X_k)) S_\infty(t) dt. \end{aligned}$$

Taking norms and using the fact that, for self-adjoint operators, $\|S\| = \sup_{\|x\| \leq 1} (x, Sx)$, we obtain

(6.15)

$$\|X_{k+1} - X_\infty\| \leq \|X_{k+1} - X_k\|^2 \int_0^\infty \|S_\infty^*(t)\| \|BB^*\| \|S_\infty(t)\| dt = c \|X_{k+1} - X_k\|^2,$$

where

(6.16)

$$c = \int_0^\infty \|S_\infty^*(t)\| \|BB^*\| \|S_\infty(t)\| dt = \|BB^*\| \int_0^\infty \|S_\infty^*(t)\| \|S_\infty(t)\| dt \leq \frac{M^2}{2\omega} \|BB^*\|$$

follows from (6.8). Since all the operators are self-adjoint, we have

$$0 \leq X_k - X_{k+1} \leq X_k - X_\infty \quad \Rightarrow \quad \|X_k - X_{k+1}\| \leq \|X_k - X_\infty\|,$$

which implies the quadratic rate of convergence

$$\|X_{k+1} - X_\infty\| \leq c \|X_k - X_\infty\|^2. \quad \square$$

7. Approximation and mesh independence results. In this section we focus on the problem of developing numerical schemes that yield convergent and mesh-independent approximations of the infinite dimensional Riccati equation

$$\mathcal{F}(X) = \mathcal{A}(X) - XBB^*X + C^*C = 0,$$

where $BB^*, CC^* \in \mathcal{L}(H)$ and $\mathcal{A} : D(\mathcal{A}) \subset \Sigma(H) \rightarrow \Sigma(H)$ is defined as in section 5 by $\mathcal{A}(\Pi)x = A^*\Pi x + \Pi Ax$. Although it is possible to work in a very abstract setting, we use the approximation setup found in Ito's paper [33]. The resulting framework is general enough to handle a large class of problems and helps us keep the technical discussion to a minimum.

We consider a sequence of approximating problems defined by (H^N, A^N, B^N, C^N) , where $H^N \subset H$ is a sequence of finite dimensional subspaces of H , and $A^N \in$

$\mathcal{L}(H^N, H^N)$, $B^N \in \mathcal{L}(U, H^N)$, and $C^N \in \mathcal{L}(H^N, Y)$ are bounded linear operators. Let $P^N : H \rightarrow H^N$ denote the orthogonal projection of H onto H^N satisfying $\|P^N\| \leq 1$, and as $N \rightarrow \infty$ we have $\|P^N x - x\| \rightarrow 0$ for all $x \in H$. Note that if T^N is any bounded linear operator on H^N , i.e., $T^N \in \mathcal{L}(H^N, H^N)$, then $T^N P^N$ belongs to $\mathcal{L}(H, H^N)$ and the operator norms satisfy

$$\|T^N\|_{\mathcal{L}(H^N, H^N)} = \|T^N P^N\|_{\mathcal{L}(H, H^N)}.$$

Therefore, we can use the notation $\|T^N\| = \|T^N P^N\|$ without referring to the specific spaces.

Define the finite dimensional approximations for \mathcal{A} ,

$$\mathcal{A}^N : \Sigma(H^N) \rightarrow \Sigma(H^N), \quad N = 1, 2, \dots,$$

by

$$\mathcal{A}^N(\Pi^N) = [A^N]^* \Pi^N + \Pi^N A^N.$$

The resulting approximating Riccati equation becomes

$$(7.1) \quad \mathcal{F}^N(X^N) = \mathcal{A}^N(X^N) - X^N B^N (B^N)^* X^N + (C^N)^* C^N = 0.$$

In this section we distinguish between two types of sequences. Let $X_k \in \mathcal{L}(H)$ denote the iterates of the Newton method for the infinite dimensional Riccati equation $\mathcal{F}(X) = 0$. Likewise, $X_k^N \in \mathcal{L}(H^N)$ denotes the iterates of the Newton method for the discretized Riccati equation $\mathcal{F}^N(X^N) = 0$.

We first review the conditions on the approximating scheme (H^N, A^N, B^N, C^N) which are sufficient to guarantee that the approximating Riccati equation (7.1) admits a unique nonnegative solution X_∞^N , and $X_\infty^N P^N$ converges to the unique nonnegative solution X_∞ of the operator Riccati equation (6.1). These results can be found in Ito's paper [33]. We then focus on the issue of mesh independence for the Kleinman-Newton algorithm and present convergence rates.

7.1. Convergence of approximating Riccati operators. In order to discuss convergence of the finite dimensional approximating Riccati operators, we need to assume that the numerical scheme preserves the basic stabilizability and detectability conditions needed to guarantee that the LQR problem is well-posed. It is important to note that even standard numerical schemes may not preserve these important control system properties. However, for the delay systems considered below it is known that all of the schemes discussed in Kappel's survey [35] satisfy these conditions (see [15], [18], [19], and [28]). Moreover, as we see below, although these conditions are sufficient for the finite dimensional Newton iterates to converge, they do not guarantee that the limit of the Newton iterates X_∞^N converges to X_∞ as $N \rightarrow +\infty$. We shall need additional properties on the approximating sequence (H^N, A^N, B^N, C^N) . We break these assumptions into three distinct hypotheses concerning the convergence of the operators, convergence of the adjoint operators, and preservation of uniform stabilizability/detectability under the approximation.

ASSUMPTION 7.1 (convergence). *Assume that there is an N_s such that for all $N > N_s$ the following conditions hold:*

- (C-i) *For each $x \in H$, $S^N(t)P^N x \rightarrow S(t)x$ and the convergence is uniform in t on bounded subintervals of $[0, +\infty)$.*
- (C-ii) *For each $u \in U$, $B^N u \rightarrow Bu$, and for each $x \in H$, $C^N P^N x \rightarrow Cx$.*

ASSUMPTION 7.2 (dual convergence). Assume that there is an N_s such that for all $N > N_s$ the following conditions hold:

(C*-i) For each $x \in H$, $[S^N(t)]^* P^N x \rightarrow S^*(t)x$ and the convergence is uniform in t on bounded subintervals of $[0, +\infty)$.

(C*-ii) For each $x \in H$, $[B^N]^* P^N x \rightarrow B^* x$, and for each $y \in Y$, $[C^N]^* y \rightarrow C^* y$.

ASSUMPTION 7.3 (uniformly stabilizable and detectable). Assume that there is an N_s such that for all $N > N_s$ the following conditions hold:

(US) The family of pairs (A^N, B^N) is uniformly stabilizable; i.e., there exist a sequence of operators $K^N \in \mathcal{L}(H^N, U)$ and positive constants $M_1 \geq 1$, $\omega_1 > 0$ such that $\sup \|K^N\| < +\infty$ and the semigroups $e^{(A^N - B^N K^N)t}$ generated by closed-loop operators $A^N - B^N K^N$ satisfy

$$\|e^{(A^N - B^N K^N)t} P^N\| \leq M_1 e^{-\omega_1 t}, \quad t \geq 0.$$

(UD) The family of pairs (A^N, C^N) is uniformly detectable; i.e., there exist a sequence of operators $G^N \in \mathcal{L}(Y, H^N)$ and positive constants $M_2 \geq 1$, $\omega_2 > 0$ such that $\sup \|G^N\| < +\infty$ and the semigroups $e^{(A^N - G^N C^N)t}$ generated by closed-loop operators $A^N - G^N C^N$ satisfy

$$\|e^{(A^N - G^N C^N)t} P^N\| \leq M_2 e^{-\omega_2 t}, \quad t \geq 0.$$

The following results may be found in [33, Theorems 2.1 and 2.2].

THEOREM 7.4. If Assumptions 7.1, 7.2, and 7.3 hold, then for all $N > N_s$ the Riccati equation (7.1) admits a unique nonnegative solution X_∞^N , $\sup \|X_\infty^N\| < +\infty$, and there exist positive constants $M_3 \geq 1$, $\omega_3 > 0$ (independent of N) such that the closed-loop semigroups $S_\infty^N(t)$ generated by operators $(A^N - B^N [B^N]^* X_\infty^N)$ satisfy

$$(7.2) \quad \|S_\infty^N(t)\| = \|S_\infty^N(t) P^N\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0.$$

THEOREM 7.5. If (A, B) is stabilizable, (A, C) is detectable, and Assumptions 7.1, 7.2, and 7.3 hold, then the unique nonnegative solutions X_∞^N to the Riccati equation (7.1) converge strongly to X_∞ . Moreover, the closed-loop semigroups $S_\infty^N(t)$ converge strongly to the closed-loop semigroup $S_\infty(t)$ and

$$(7.3) \quad \|S_\infty(t)\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0.$$

Note that the previous results yield only strong convergence of the solutions of the Riccati equations. However, if B is bounded with finite dimensional range, then strong convergence of X_∞^N to X_∞ implies norm convergence of the feedback gain operators. In particular, if $\text{rank}(B) < +\infty$, then

$$(7.4) \quad \lim_{N \rightarrow +\infty} \|K^N - K\| = 0,$$

where $K^N = [B^N]^* X_\infty^N$ (see Theorems 6.2 and 6.8 in [28]). Moreover, under certain compactness assumptions on B and C , one can establish norm convergence of the Riccati operators. There are a number of results along this line (see [24], [28], [33], [34], [43], and [44]) and some make use of the smoothing property of the semigroup (e.g., analytic, differentiable). The following theorem is not the most general result, but it is directly applicable to delay and parabolic PDE control systems. Also, this result can be used to obtain rates of convergence for the approximating Riccati operators. The proof follows directly from Ito's paper [33, pp. 158–160].

THEOREM 7.6. *Suppose Assumptions 7.1, 7.2, and 7.3 hold, B and C are compact, and $X_\infty x \in D(A^*)$ for all $x \in H$. If $B^N = P^N B$ and $C^N = CP^N$, then (A, B) is stabilizable, (A, C) is detectable, and there exists a constant $\hat{\beta} > 0$ such that*

$$(7.5) \quad \|X_\infty^N - P^N X_\infty P^N\| \leq \Delta^N,$$

where Δ^N is given by

$$(7.6) \quad \Delta^N \triangleq \hat{\beta} \{ \|(A^* - [A^N]^* P^N) X_\infty\| + \|B\| \|(B^* - [B^N]^* P^N) X_\infty\| \}.$$

If, in addition, $\lim_{N \rightarrow +\infty} \|(A^* - [A^N]^* P^N) X_\infty\| = 0$, then

$$\lim_{N \rightarrow +\infty} \|X_\infty^N - P^N X_\infty P^N\| = 0.$$

Remark. The assumptions of convergence, dual convergence, and uniform preservation of stability and detectability in Ito’s result are sufficient, but it is not yet clear if they are necessary for operator norm convergence (see [13], [16], and [26]). However, most approximations that yield operator norm convergence satisfy these or even stronger assumptions (see [14]). We note that, especially for nonnormal problems, it may not be easy to check these conditions for a specific approximation scheme. For example, it is not known which numerical algorithms used in computational fluid dynamics are dual convergent when applied to nonnormal control systems typical in this area (see [15] and [20]).

At first glance the compactness assumptions on the B and C operators seem rather strong. This assumption certainly excludes some idealized boundary control problems. On the other hand, if one includes actuator or sensor dynamics at the boundary (a reasonable assumption in many boundary control problems), then the resulting B and C operators are often compact in practical problems. Moreover, the simple example below provides some insight into the importance of this assumption and perhaps a way out of this technical difficulty.

Example 2. Consider Example 1 above and let $H^N = \mathbb{R} \times \mathbb{R}^N$, and define $P^N : H \rightarrow H^N$ to be the natural projection onto H^N . If $A^N = P^N A$, $B^N = P^N B$, and $C^N = CP^N$, then all the conditions in the previous theorem are satisfied except that C is not compact. The solution to the finite dimensional Riccati equation

$$\mathcal{F}^N(X) \triangleq [A^N]^* X^N + X^N [A^N] - X^N [B^N] [B^N]^* X^N + Q^N = 0$$

is $X^N = \begin{bmatrix} 1 & 0 \\ 0 & I^N \end{bmatrix}$, where I^N is the identity on \mathbb{R}^N . Clearly, X^N does not converge to X in the uniform operator norm since I is not compact. It is interesting to note that the feedback gain operators

$$(7.7) \quad K^N = [B^N]^* X^N = \begin{bmatrix} 1 & 0 \end{bmatrix} = K$$

converge uniformly. This situation occurs in many problems and can be exploited to address mesh independence issues. We shall discuss this issue in a future paper. We turn now to the issue of mesh independence.

7.2. Mesh independence of the Kleinman–Newton algorithm. We turn now to the application of the Kleinman–Newton algorithm to the finite dimensional Riccati equation

$$(7.8) \quad \mathcal{F}^N(X^N) = A^N(X^N) - X^N B^N (B^N)^* X^N + (C^N)^* C^N = 0.$$

We solve the abstract algebraic equation (7.8) by Newton's method. In particular, we seek a bounded linear operator $X^N \in D(\mathcal{A}^N)$, which provides a solution to the Riccati equation $\mathcal{F}^N(X^N) = 0$, where the nonlinear mapping $\mathcal{F}^N : D(\mathcal{A}^N) \subset \Sigma(H^N) \rightarrow \Sigma(H^N)$ is as defined by (7.1) above. Just as for the infinite dimensional case, applying Newton's method to $\mathcal{F}^N(X^N) = 0$ leads to the iteration

$$(7.9) \quad \begin{aligned} \mathcal{A}^N(X_{k+1}^N - X_k^N) - X_k^N B^N [B^N]^* (X_{k+1}^N - X_k^N) - (X_{k+1}^N - X_k^N) B^N [B^N]^* X_k^N \\ + \mathcal{A}^N(X_k^N) - X_k^N B^N [B^N]^* X_k^N + [C^N]^* C^N = 0, \end{aligned}$$

or equivalently,

$$(7.10) \quad \mathcal{A}^N(X_{k+1}^N) - X_k^N B^N [B^N]^* X_{k+1}^N - X_{k+1}^N B^N [B^N]^* X_k^N + X_k^N B^N [B^N]^* X_k^N + [C^N]^* C^N = 0.$$

We assume that the approximation scheme preserves the basic Assumption 6.1 for all N sufficiently large. This ensures that the Newton iterations in the finite dimensional spaces converge monotonically as in Theorem 6.3. In particular, we shall use the following hypothesis.

ASSUMPTION 7.7. *Let $S^N(t)$ be the semigroup generated by A^N on H^N , $B^N \in \mathcal{L}(U, H^N)$, and $C^N \in \mathcal{L}(H^N, Y)$. Assume that there is an N_s such that for all $N > N_s$ the following conditions hold:*

- (N-i) *The system (A^N, B^N) is stabilizable and the pair (A^N, C^N) is detectable.*
- (N-ii) *An operator $X_0^N \in \Sigma^+(H^N)$ is given such that $A^N - B^N [B^N]^* X_0^N$ generates an exponentially stable semigroup on H^N .*

The following results are the finite dimensional versions of Theorems 6.3 and 6.4 above. The proofs are almost identical.

THEOREM 7.8. *If Assumption 7.7 holds, then for all $N > N_s$,*

- (i) *the sequence X_k^N converges in $\Sigma^+(H^N)$ and $\lim_{k \rightarrow +\infty} X_k^N = X_\infty^N \in \Sigma^+(H^N)$. Moreover, X_∞^N is a solution to $\mathcal{F}^N(X_\infty^N) = 0$;*
- (ii) *the closed-loop operator $A^N - B^N [B^N]^* X_\infty^N$ generates an exponentially stable semigroup $S_\infty^N(t)$ satisfying*

$$(7.11) \quad \|S_\infty^N(t)\| \leq M_N e^{-\omega_N t}$$

for constants $M_N \geq 1$ and $\omega_N > 0$;

- (iii) *the Newton iterates satisfy $0 \leq X_\infty^N \leq \dots \leq X_{k+1}^N \leq X_k^N \leq \dots \leq X_1^N$.*

THEOREM 7.9. *If Assumption 7.7 holds, then for all $N > N_s$ and $k = 0, 1, 2, \dots$,*

$$\|X_{k+1}^N - X_\infty^N\| \leq c^N \|X_k^N - X_\infty^N\|^2,$$

where

$$(7.12) \quad \begin{aligned} c^N &= \int_0^\infty \|[S_\infty^N(t)]^*\| \|B^N [B^N]^*\| \|S_\infty^N(t)\| dt \\ &= \|B^N [B^N]^*\| \int_0^\infty \|[S_\infty^N(t)]^*\| \|S_\infty^N(t)\| dt \leq (M_N^2 / 2\omega_N) \|B^N [B^N]^*\| \end{aligned}$$

and the constants $M_N \geq 1$ and $\omega_N > 0$ are as given by (7.11) above.

The constant c^N in Theorem 7.9 is not necessarily the minimal value \hat{c}^N . However, it is possible that there exists an α , independent of N , such that the finite dimensional iterates X_k^N satisfy

$$(7.13) \quad \|X_{k+1}^N - X_\infty^N\| \leq \alpha \|X_k^N - X_\infty^N\|^2.$$

Clearly, the bound

$$c^N \leq (M_N^2/2\omega_N)\|B^N[B^N]^*\|$$

is not tight. As we shall see below, there are convergent approximation schemes satisfying Assumption 7.1 and a fixed α satisfying (7.13) with

$$c^N \leq \alpha < \lim_{N \rightarrow +\infty} (M_N^2/2\omega_N)\|B^N[B^N]^*\| = +\infty.$$

Therefore, the rate of convergence dictated by the constant α in (7.13) provides some level of mesh independence for the finite dimensional problems. However, it is important to note that the previous estimates do not imply that the approximating Riccati operators X_∞^N converge in norm to X_∞ . Hence, even the existence of a constant α for which (7.13) holds does not provide true mesh independence.

Although the previous theorem is well known, the explicit value of the constant c^N in (7.12) provides some insight into those approximation properties that might be important in establishing an MIP. There are several factors that influence this constant, but clearly the choice of the approximation scheme (H^N, A^N, B^N, C^N) plays a fundamental role in determining c^N and its value as the mesh is refined. We shall illustrate this dependency with the numerical examples below. However, using Ito's theorem, Theorem 7.4 above, we have the following mesh independence result.

THEOREM 7.10. *If Assumptions 7.1, 7.2, and 7.3 hold and B is compact, then there exist α^N and α such that*

$$(7.14) \quad \|X_{k+1}^N - X_\infty^N\| \leq \alpha^N \|X_k^N - X_\infty^N\|^2,$$

$$(7.15) \quad \|X_{k+1} - X_\infty\| \leq \alpha \|X_k - X_\infty\|^2,$$

and $\alpha^N = \alpha + \delta(N)$, where $\delta(N) \rightarrow 0$ as $N \rightarrow +\infty$.

Proof. First note that Assumption 7.3 implies that Assumption 7.7 holds so that the Kleinman-Newton iterates X_k^N exist and Theorem 7.9 is valid. From Assumption 7.1 we have convergence $B^N u \rightarrow Bu$ for $u \in U$, and Assumption 7.2 yields the dual convergence $[B^N]^* P^N x \rightarrow B^* x$ for $x \in H$. Since B is compact, it follows from Theorem 3.2 in [21] that $\|B^N - B\| \rightarrow 0$, and $\|[B^N]^* P^N - B^*\| \rightarrow 0$ so that $\|B^N [B^N]^* P^N - BB^*\| \rightarrow 0$. Let

$$(7.16) \quad \beta(N) = \|B^N [B^N]^* P^N\| - \|BB^*\|.$$

Theorem 7.4 yields the existence of positive constants $M_3 \geq 1, \omega_3 > 0$ (independent of N) such that the closed-loop semigroups $S_\infty^N(t)$ generated by the operators $(A^N - B^N [B^N]^* X_\infty^N)$ satisfy

$$\|[S_\infty^N(t)]^*\| = \|S_\infty^N(t)\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0,$$

and

$$\|S_\infty^*(t)\| = \|S_\infty(t)\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0.$$

Hence, the estimate for c^N in (7.12) is bounded by

$$c^N = \int_0^\infty \|[S_\infty^N(t)]^*\| \|B^N [B^N]^*\| \|S_\infty^N(t)\| dt \leq \frac{M_3^2}{2\omega_3} \|B^N [B^N]^* P^N\|.$$

Let

$$\alpha^N = \frac{M_3^2}{2\omega_3} \|B^N [B^N]^* P^N\| \geq c^N \text{ and } \alpha = \frac{M_3^2}{2\omega_3} \|BB^*\| \geq c,$$

where c is given by (6.16). It follows that α^N and α satisfy (7.14) and (7.15), respectively. Moreover,

$$\alpha^N = \frac{M_3^2}{2\omega_3} \|B^N [B^N]^* P^N\| = \frac{M_3^2}{2\omega_3} \{\|BB^*\| + \beta(N)\} = \alpha + \frac{M_3^2}{2\omega_3} \beta(N) = \alpha + \delta(N),$$

where

$$\delta(N) = \frac{M_3^2}{2\omega_3} \beta(N) \longrightarrow 0 \text{ as } N \longrightarrow +\infty,$$

and this completes the proof. \square

All that one can imply from Theorem 7.10 is that the finite dimensional iterates X_k^N satisfy

$$\|X_{k+1}^N - X_\infty^N\| \leq (\alpha + \delta(N)) \|X_k^N - X_\infty^N\|^2.$$

If, in addition, one has norm convergence $\lim_{N \rightarrow +\infty} \|P^N X_\infty P^N - X_\infty^N\| = 0$, then the inequality

$$\begin{aligned} \|X_{k+1}^N - P^N X_\infty P^N\| &= \|X_{k+1}^N - X_\infty^N + X_\infty^N - P^N X_\infty P^N\| \\ &\leq \|X_{k+1}^N - X_\infty^N\| + \|X_\infty^N - P^N X_\infty P^N\| \\ &\leq (\alpha + \delta(N)) \|X_k^N - X_\infty^N\|^2 + \|X_\infty^N - P^N X_\infty P^N\| \end{aligned}$$

provides a useful overall convergence rate of

$$(7.17) \quad \|X_{k+1}^N - P^N X_\infty P^N\| \leq (\alpha + \delta(N)) \|X_k^N - X_\infty^N\|^2 + \|X_\infty^N - P^N X_\infty P^N\|$$

in terms of the Newton iterates and the finite dimensional approximations. Applying Theorems 7.5 and 7.6 above yields the following mesh independence result.

THEOREM 7.11. *Suppose Assumptions 7.1, 7.2, and 7.3 hold, B and C are compact, and $X_\infty x \in D(A^*)$ for all $x \in H$. If $B^N = P^N B$ and $C^N = CP^N$, then there exist $\delta(N) \longrightarrow 0$ as $N \longrightarrow +\infty$ and $\hat{\beta}$ such that*

$$(7.18) \quad \|X_{k+1}^N - P^N X_\infty P^N\| \leq (\alpha + \delta(N)) \|X_k^N - X_\infty^N\|^2 + \Delta^N,$$

where α is as given by (7.15) in Theorem 7.10 and $\Delta^N = \hat{\beta} \{ \|(A^* - [A^N]^* P^N) X_\infty\| + \|B\| \|(B^* - [B^N]^* P^N) X_\infty\| \}$. If, in addition, for some $p > 0$ we have

$$\Delta^N = O(1/N^p),$$

then the MIP holds with a rate of $O(1/N^p)$.

Observe that the rate determined by

$$\Delta^N = \hat{\beta} \{ \|(A^* - [A^N]^* P^N) X_\infty\| + \|B\| \|(B^* - [B^N]^* P^N) X_\infty\| \}$$

depends on the order of the approximating scheme (H^N, A^N, B^N, C^N) . In particular, the rate of convergence for the adjoint approximations $[A^N]^*$ plays a key role. For

the delay systems discussed below, it follows that $B^* = [B^N]^*$ for all $N > 1$ so that the rate is essentially determined by how well one can approximate A^* , i.e., if one can obtain an estimate of the form

$$\|(A^* - [A^N]^* P^N)X_\infty\| = O(1/N^p).$$

We shall apply this estimate to the delay systems considered in the next section. In general, the convergence results depend on the regularity of the semigroups and the type of approximations. Obtaining these rates depends on each individual problem. It is considered both delay systems and parabolic PDE control systems. He established convergence rates for the standard finite element scheme applied to the parabolic PDE problem. He also gave convergence rates for the two schemes we will discuss below involving delay systems (see [33] and [34]). In both cases he made heavy use of the regularity of the semigroups generated by A .

8. Control of delay systems. In this section we consider the LQR problem for delay differential equations. In particular, the system is defined by

$$(8.1) \quad \dot{x}(t) = A_0x(t) + A_1x(t - r) + B_0u(t), \quad t > 0,$$

with initial data

$$(8.2) \quad x(0) = \eta, \quad x(s) = \varphi(s), \quad -r < s < 0,$$

where $\eta \in \mathbb{R}^n$ and $\varphi(\cdot) \in L^2(-r, 0; \mathbb{R}^n)$. Here, A_0 and A_1 are $n \times n$ constant real matrices and B_0 is an $n \times m$ matrix.

Let $C_0 = [C_0]^T \geq 0$ be a symmetric real-valued matrix and define the cost function

$$(8.3) \quad J(u) = \int_0^{+\infty} \{(C_0x(s))^T C_0x(s) + \|u(s)\|^2\} ds.$$

The corresponding LQR problem is to minimize the quadratic cost (8.3) over all controls $u \in L^2(0, +\infty; \mathbb{R}^m)$.

In order to present this problem in an infinite dimensional setting we use the Hilbert space $H = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$. Define the operator A with domain

$$D(A) = \{(\eta, \phi(\cdot)) \in H : \phi(\cdot) \in H^1(-r, 0; \mathbb{R}^n) \quad \text{and} \quad \phi(0) = \eta\}$$

by

$$A \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix} = \begin{bmatrix} A_0\eta + A_1\varphi(-r) \\ \phi'(\cdot) \end{bmatrix}.$$

Also, let $B : \mathbb{R}^m \rightarrow H$ be defined by

$$Bu = \begin{bmatrix} B_0u \\ 0 \end{bmatrix},$$

and let $C : H \rightarrow \mathbb{R}^m$ be given by

$$Cz = C \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix} = C_0\eta.$$

It is well known (see [3]) that A generates a C_0 -semigroup $S(t) : H \rightarrow H$, $t \geq 0$, such that

$$(8.4) \quad S(t)(\eta, \phi(\cdot)) = (x(t), x_t(\cdot)) \in H$$

for all $(\eta, \phi(\cdot)) \in H$, where $x(t)$ is the solution to (8.1)–(8.2) and $x_t(s) = x(t+s)$ for all $-r < s < 0$. Moreover, the delay system is equivalent to the infinite dimensional system in H defined by

$$(8.5) \quad \dot{z}(t) = Az(t) + Bu(t), \quad t > 0,$$

with initial data

$$(8.6) \quad z(0) = \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix},$$

and the LQR cost function has the form

$$J(u) = \int_0^{+\infty} \{\|Cz(s)\|^2 + \|u(s)\|^2\} ds.$$

Finally, the Hilbert adjoint A^* is defined on the domain

$$D(A^*) = \{(\xi, \psi(\cdot)) \in H : \psi(\cdot) \in H^1(-r, 0; \mathbb{R}^n), \psi(-r) = A_1^T \xi\}$$

by

$$A^* \begin{bmatrix} \xi \\ \psi(\cdot) \end{bmatrix} = \begin{bmatrix} A_0^T \xi + \psi(0) \\ -\psi'(\cdot) \end{bmatrix}.$$

Observe that the linear operator A is not normal; this can cause problems when approximating the LQR control problem (see [8], [13], [16], [15], and [35]).

8.1. Approximations of the delay system. We consider two different numerical schemes for approximating the LQR control problem for the delay system. Since the B and C operators act only on the finite dimensional part of the state, the main issue is how to approximate A . We focus on a finite volume method known as the “AVE” scheme in [4] and a conforming finite element scheme first described by Banks and Kappel in [5], and hence we do not give the details here. The key difference between these two schemes is how they approximate the initial condition ϕ in (8.1). The “AVE” scheme uses an averaging technique and characteristic functions, while the Banks–Kappel scheme uses a continuous finite element technique. Although both schemes are convergent, only the “AVE” scheme is dual convergent, and hence produces convergent approximations of the operator

$$\mathcal{A}(\Pi) = A^* \Pi + A \Pi, \quad \Pi \in D(\mathcal{A}).$$

The papers by Rosen (see [43], [44], and [45]) provide considerable insight in this problem.

The AVE/finite volume scheme. For each $N > 1$, create a partition on $[-r, 0]$ by defining $\tau_j^N = -jr/N$, where $j = 0, \dots, N$. On $[-r, 0]$, define $\chi_j^N(\cdot)$ to be the characteristic function on $[\tau_j^N, \tau_{j-1}^N)$ for $j = 2, \dots, N$, and define $\chi_1^N(\cdot)$ to be the

characteristic function on $[\tau_1^N, \tau_0^N]$. Define the finite dimensional subspace H_{AVE}^N of H by

$$(8.7) \quad H_{AVE}^N \equiv \left\{ (\eta, \phi^N(\cdot)) \in H : \phi^N(s) = \sum_{j=1}^N v_j^N \chi_j^N(s), v_j^N \in \mathbb{R}^n \right\}.$$

The projection P^N of H into H_{AVE}^N is defined by

$$P^N(\eta, \phi(\cdot)) = \left(\phi_0^N, \sum_{j=1}^N \phi_j^N \chi_j^N(\cdot) \right),$$

where

$$\phi_0^N \equiv \eta \quad \text{and for } j = 1, \dots, N, \quad \phi_j^N \equiv \frac{N}{r} \int_{\tau_j^N}^{\tau_{j-1}^N} \phi(s) ds.$$

To approximate the operator A , we first define $L^N : H_{AVE}^N \rightarrow \mathbb{R}^n$ and $D^N : H_{AVE}^N \rightarrow L^2(-r, 0; \mathbb{R}^n)$ by

$$L^N \left(\eta, \sum_{j=1}^N v_j^N \chi_j^N(\cdot) \right) = A_0 \eta + A_1 v_N^N$$

and

$$D^N \left(\eta, \sum_{j=1}^N v_j^N \chi_j^N(\cdot) \right) = \frac{N}{r} \sum_{j=1}^N \{v_{j-1}^N - v_j^N\} \chi_j^N(\cdot),$$

respectively, where $v_0^N = \eta$. The AVE approximation $A_{AVE}^N : H_{AVE}^N \rightarrow H_{AVE}^N \subseteq H$ is given by

$$(8.8) \quad A_{AVE}^N(\eta, \psi) \equiv (L^N(\eta, \psi), D^N(\eta, \psi)).$$

In order to complete the approximation scheme, we define

$$(8.9) \quad B_{AVE}^N = P^N B \quad \text{and} \quad C_{AVE}^N = CP^N,$$

and this yields the AVE approximation scheme $(H_{AVE}^N, A_{AVE}^N, B_{AVE}^N, C_{AVE}^N)$.

Observe that

$$B_{AVE}^N u = P^N \begin{bmatrix} B_0 u \\ 0 \end{bmatrix} = \begin{bmatrix} B_0 u \\ 0 \end{bmatrix} = Bu$$

and

$$C_{AVE}^N \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix} = CP^N \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix} = C \begin{bmatrix} \eta \\ \phi^N(\cdot) \end{bmatrix} = C_0 \eta = C \begin{bmatrix} \eta \\ \phi(\cdot) \end{bmatrix}.$$

Hence, the operators B and C are compact and satisfy the conditions in Theorem 7.6. Norm convergence of the input and output operators is trivial for this approximation.

Since $\|B^N[B^N]^*P^N\| = \|BB^*\|$ for all $N \geq 1$, it follows that $\beta(N)$ defined in (7.16) satisfies

$$\beta(N) = \|B^N[B^N]^*P^N\| - \|BB^*\| = 0,$$

and hence $\delta(N) = \frac{M_3^2}{2\omega_3}\beta(N) = 0$ for all $N \geq 1$. Moreover, the AVE scheme satisfies all the assumptions in Theorem 7.6 above, and the following convergence and mesh independence result holds (see pages 164–166 in [33]).

THEOREM 8.1. *The AVE approximation scheme $(H_{AVE}^N, A_{AVE}^N, B_{AVE}^N, C_{AVE}^N)$ satisfies all the assumptions in Theorem 7.6. There exist constants \hat{M} and α independent of N such that*

$$\|X_\infty^N - P^N X_\infty P^N\| \leq \frac{\hat{M}}{\sqrt{N}},$$

and the finite dimensional Kleinman–Newton iterates satisfy

$$\|X_{k+1}^N - P^N X_\infty P^N\| \leq \alpha \|X_k^N - X_\infty^N\|^2 + \frac{\hat{M}}{\sqrt{N}}.$$

Note that the overall convergence rate for the AVE scheme is $O(1/\sqrt{N})$. In order to improve this rate, several “high order” spline-based schemes were proposed. The first of these schemes was developed by Banks and Kappel in [5]. Because this spline scheme failed to produce strongly convergent Riccati operators, several modifications were developed to overcome this issue. A nice summary of these schemes and their properties can be found in Kappel’s survey paper [35]. We briefly describe the scheme below.

The Banks–Kappel (BK) spline scheme. We now describe the “BK” finite element spline-based scheme first proposed by Banks and Kappel in [5]. For each $N > 1$, create a partition on $[-r, 0]$ by defining $\tau_j^N = -jr/N$, where $j = 0, \dots, N$. For ease of notation we set $\tau_{N+1}^N = -r$ and $\tau_{-1}^N = 0$. On $[-r, 0]$, define the standard linear B-splines by

$$B_j^N(s) = \begin{cases} \frac{N}{r}(s - \tau_{j+1}^N), & s \in [\tau_{j+1}^N, \tau_j^N], \\ \frac{N}{r}(\tau_{j-1}^N - s), & s \in [\tau_j^N, \tau_{j-1}^N], \\ 0 & \text{otherwise.} \end{cases}$$

Define the finite dimensional subspace H_{BK}^N of H by

$$(8.10) \quad H_{BK}^N \equiv \left\{ (\phi^N(0), \phi^N(\cdot)) \in H : \phi^N(s) = \sum_{j=0}^N v_j^N B_j^N(s), v_j^N \in \mathbb{R}^n \right\}.$$

Let P^N denote the orthogonal projection of H into H_{BK}^N and note that since $H_{BK}^N \subseteq D(A) \subseteq H$, the range of P^N is contained in the domain of A . Therefore, we define the spline approximation $A_{BK}^N : H_{BK}^N \rightarrow H_{BK}^N \subseteq H$ by

$$(8.11) \quad A_{BK}^N = P^N A = P^N A P^N.$$

In order to complete the approximation scheme, we define

$$(8.12) \quad B_{BK}^N = P^N B \quad \text{and} \quad C_{BK} = C P^N,$$

and this yields the BK spline approximation scheme $(H_{BK}^N, A_{BK}^N, B_{BK}^N, C_{BK}^N)$.

Note. The BK spline scheme satisfies Assumption 7.1 and hence yields a convergent numerical scheme in the sense that, for a given initial condition and input function, the approximations of the forward problem converge on finite time intervals. However, unlike the AVE scheme above, the BK spline scheme fails to satisfy the dual convergence assumption, Assumption 7.2 (see [16]), and the uniformly stabilizable and detectable assumption, Assumption 7.3 (see [17] and [18]). The BK spline scheme does satisfy the basic assumption, Assumption 7.7, so that the finite dimensional Kleinman–Newton algorithm converges quadratically with constant c^N possibly depending on N . However, the approximating Riccati operators do not converge strongly so, in particular, norm convergence fails.

In the next section we present numerical results based on these two schemes. The numerical results will confirm (as Theorem 8.1 implies) that the AVE scheme is mesh independent and the approximating Riccati operators converge. However, the numerical results below also show that the BK spline scheme is not mesh independent, although there is a bound on c^N .

9. Numerical results. In this section we illustrate the importance of Assumptions 7.1, 7.2, and 7.3 in obtaining strong convergence (norm convergence) of feedback gain operators as well as strong mesh independence of the Kleinman–Newton iterations.

For this discussion we use the two schemes discussed in section 8.1. The AVE scheme satisfies all the assumptions of Theorems 7.6 and 7.10. Therefore, both forms of strong mesh independence, (2.4) and (2.7), are satisfied and the approximate Riccati operators converge in norm to X_∞ . This is not the case for the BK scheme since it fails to satisfy Assumptions 7.2 and 7.3. In the numerical approximations below, X_∞ is taken as the (converged) fine grid solution of the Riccati equation using the AVE scheme.

In this section we use the following notation: Let \hat{c}_{AVE}^N and \hat{c}_{BK}^N , respectively, denote the values \hat{c}^N if the AVE and BK schemes are used for the approximations. Also, let $\hat{M}_{AVE}^N(\varepsilon, x_0^N)$ and $\hat{M}_{BK}^N(\varepsilon, x_0^N)$ denote the values $\hat{M}^N(\varepsilon, x_0^N)$ for the AVE and BK schemes.

Mesh independence implies that a finite dimensional process behaves asymptotically the same as the underlying infinite dimensional process. Thus, in order to compare the behavior of the approximation schemes, it is necessary that the starting operators in the approximation spaces are the projections of the starting operator in the infinite dimensional space onto the respective approximation spaces. To accomplish this, $X_{0,AVE}^N$ and $X_{0,BK}^N$ are expressed in terms of multiples of the identity operator in the respective approximation spaces. For the other obvious choice, the zero operator, the convergence was too fast to make observations about quadratic convergence or mesh independence. Since the mass matrices, MASS_{AVE} and MASS_{BK} , are the projections of $I_{\mathbb{R} \times L_2(0,1)}$ onto the approximation spaces H_{AVE}^N and H_{BK}^N , respectively, the starting matrices will be multiples of the appropriate mass matrices.

All computations in this section have been performed on a PowerPC G5, 2.7GHz, using MATLAB version 7.0.0. All Lyapunov equations are solved by implementing the MATLAB Lyapunov solver which uses the SLICOT routines SB03MD and SG03AD.

Numerical Example 1. The results presented here are typical for all the runs on a one-dimensional delay equation,

$$\dot{x}(t) = x(t) + x(t-1) + u(t),$$

with cost function

$$J(u(\cdot)) = \int_0^{+\infty} \{10^4[x(t)]^2 + [u(t)]^2\} dt.$$

The starting operator, X_0 , equals 100 times the identity in $H = \mathbb{R} \times L_2(0, 1)$; thus $X_0 = 100I_{\mathbb{R} \times L_2(0,1)}$. The finite dimensional approximations for X_0 using the AVE and BK schemes result in $X_{0,AVE}^N = 100\text{MASS}_{AVE}$ and $X_{0,BK}^N = 100\text{MASS}_{BK}$, respectively. The tolerance is set to be $\|X_k^N - X_\infty^N\| < 10^{-8} = \varepsilon$.

Since the AVE scheme satisfies the criteria for both forms of strong mesh independence, (2.4) and (2.7), we expect mesh-independent behavior of the quantities in Table 9.1. Indeed, we notice that $\hat{c}_{AVE}^N \rightarrow 10^{-2}$ and $\hat{M}_{AVE}^N \rightarrow 3$, confirming the theoretical results.

For the BK scheme this observation cannot be made from the numerical results. This is in line with the fact that the BK scheme fails to satisfy Assumptions 7.2 and 7.3. Note that mesh independence would imply that $\hat{c}_{BK}^N \rightarrow \hat{c} \approx 10^{-2}$ and $\hat{M}_{BK}^N \rightarrow M(10^{-8}, 100I) \approx 3$ based on the results from the AVE scheme. The results for the BK scheme show that $\hat{c}_{BK}^{1024} \approx 1.6 \times 10^2 \gg \hat{c}$ and $\hat{M}_{BK}^N \geq 5 > M(10^{-8}, 100I)$.

A further comparison of the two approximation schemes includes the actual CPU-time per Newton iteration that was used by the MATLAB process. This was computed using the MATLAB function `cputime` as well as `tic` and `toc`. The resulting times were identical. Both schemes use roughly the same amount of CPU-time per iteration. For example, for $N = 256$, the size of the problem is 257. The AVE scheme uses on average 10.9s of CPU-time per iteration, and the BK scheme uses 10.6s. Consequently, the number of iterations that the two schemes use is a direct measure of the total computational time. In general, the BK scheme needs more iterations than the AVE scheme to obtain a specific accuracy, and in some cases significantly more.

Numerical Example 2. We present typical results for the two-dimensional delay equation,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1.6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

with cost function

$$J(u(\cdot)) = \int_0^{+\infty} \left\{ \left(\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} x(t) \right)^2 + [u(t)]^2 \right\} dt.$$

The starting operator, X_0 , equals twice the identity in $H = \mathbb{R}^2 \times L_2(0, 1; \mathbb{R}^2)$; thus $X_0 = 2I_{\mathbb{R}^2 \times L_2(0,1;\mathbb{R}^2)}$. For the AVE and BK schemes, the starting matrices are $X_{0,AVE}^N = 2\text{MASS}_{AVE}$ and $X_{0,BK}^N = 2\text{MASS}_{BK}$, respectively. As before, ε is taken to be 10^{-8} , $\|X_k^N - X_\infty^N\| < 10^{-8}$.

The results presented in Table 9.2 are similar to the results observed in Example 1. The AVE scheme yields strong mesh independence, and the Riccati operators converge strongly, while this is not true for the BK scheme. In particular, the optimal feedback law defined by (5.3) has the form

$$Kz(t) = k_0z(t) + \int_{-1}^0 k_1(s)z(t+s)ds + \int_{-1}^0 k_2(s)z(t+s)ds,$$

where $k_i(s)$, $i = 1, 2$, are called the functional gains. Figures 9.1 and 9.2 illustrate that the AVE scheme (the solid line) yields strong convergence of the gains, while the BK scheme (the dashed line) does not.

TABLE 9.1

N	\hat{c}_{AVE}^N	\hat{M}_{AVE}^N	\hat{c}_{BK}^N	\hat{M}_{BK}^N
8	1.01497×10^{-2}	3	3.98797×10^6	14
16	6.12638×10^{-3}	3	3.31429×10^6	12
32	6.18756×10^{-3}	3	1.81757×10^6	10
64	6.57817×10^{-3}	3	5.04459×10^5	8
128	7.11713×10^{-3}	3	3.04030×10^4	7
256	7.78503×10^{-3}	3	7.82120×10^4	7
512	8.55008×10^{-3}	3	6.02653×10^3	6
1024	9.32057×10^{-3}	3	1.59829×10^2	5

TABLE 9.2

N	\hat{c}_{AVE}^N	\hat{M}_{AVE}^N	\hat{c}_{BK}^N	\hat{M}_{BK}^N
8	4.77751×10^{-2}	8	1.28997×10^6	16
16	4.94581×10^{-2}	8	8.69896×10^5	14
32	4.98913×10^{-2}	8	4.56992×10^4	12
64	5.04197×10^{-2}	9	3.59473×10^3	11
128	5.07046×10^{-2}	9	9.31569×10^3	11
256	5.08547×10^{-2}	9	4.18416×10^1	10
512	5.09322×10^{-2}	9	2.38082×10^1	10

As in Numerical Example 1, the two schemes use roughly the same CPU-time per Newton iteration. For $N = 256$, the problem size is 514, and the average CPU-time per iteration is 102s for the AVE scheme and 105s for the BK scheme.

We note that Figures 9.1 and 9.2 verify the theoretical results in this paper as well as those established in the earlier paper by Burns, Ito, and Propst [16]. In particular, in [16] it was proved that the BK scheme does not produce approximating Riccati operators that converge in norm. Hence the oscillations seen in these figures, which are indicative of weak convergence, are the best one can expect. However, the AVE scheme is norm convergent and this is also illustrated in Figures 9.1 and 9.2.

We close this section with an example that illustrates the need for infinite dimensional feedback. As noted earlier, the convergence theory in Damm and Hinrichsen [23] is easily applied to a wide variety of finite dimensional (matrix) Riccati equations. However, applying this method to infinite dimensional Riccati equations is not straightforward. A special feedback problem for a delay system was used to illustrate their results. They considered the problem of stabilizing a delay system with finite dimensional feedback only, which leads to a finite dimensional Riccati type (matrix) equation. In particular, the control system considered in [23] is given by the delay differential equation

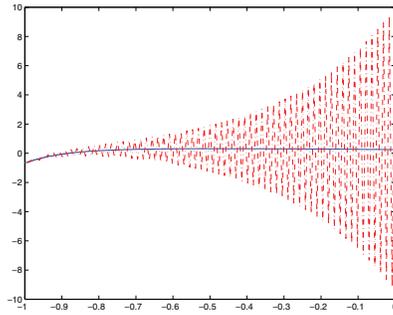
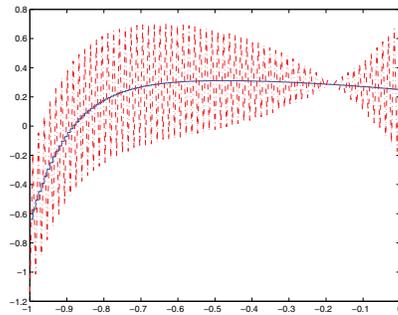
$$\dot{x}(t) = A_0x(t) + A_1x(t-r) + Bu(t).$$

The problem (see problem 5 on page 58 in [23]) is to find a finite dimensional feedback law of the form

$$u(t) = -Kx(t)$$

so that the closed-loop system

$$\dot{x}(t) = [A_0 - BK]x(t) + A_1x(t-r)$$

FIG. 9.1. *Numerical Example 2: Functional gain k_1 .*FIG. 9.2. *Numerical Example 2: Functional gain k_2 .*

is stable for all delays $r > 0$. This leads to a matrix Riccati equation for the matrix K .

If one considers the retarded delay equation,

$$(9.1) \quad \dot{x}(t) = \alpha x(t) + \beta x(t-r) + \gamma \int_{-r}^0 x(t+s) ds,$$

then one will know (see Corollary 2.8 in [32]) that (9.1) is stable independent of delay if and only if

$$\alpha < 0, \quad \gamma < 0, \quad 0 < -\gamma \leq \frac{\alpha^2 - \beta^2}{2}.$$

If $\gamma > 0$, then (9.1) is not stable for all $r > 0$. If one starts with the control system

$$(9.2) \quad \dot{x}(t) = x(t) + 2x(t-r) + \int_{-r}^0 x(t+s) ds + u(t)$$

and uses only current state feedback

$$u(t) = -kx(t),$$

then the closed-loop system has the form

$$(9.3) \quad \dot{x}(t) = [1 - k]x(t) + 2x(t - r) + \int_{-r}^0 x(t + s)ds,$$

and this system is never stable independent of delay since $\gamma = +1$. On the other hand, the complete state (infinite dimensional) feedback law,

$$u(t) = -k_0x(t) + (k_1 - 1) \int_{-r}^0 x(t + s)ds,$$

leads to the closed loop system

$$(9.4) \quad \dot{x}(t) = [1 - k_0]x(t) + 2x(t - r) + k_1 \int_{-r}^0 x(t + s)ds,$$

which is stable independent of delay if and only if

$$1 < k_0, \quad k_1 < 0, \quad 0 < -k_1 \leq \frac{[1 - k_0]^2 - 4}{2}.$$

If we set $k_0 = 4$ and $k_1 = 5/4 < 5/2$, then (9.4) is stable independent of delay. In particular, the control system (9.2) is stable independent of delay, and the MIP holds if we apply the AVE scheme to this problem. Numerical results on mesh independence and convergence for this problem are almost identical to the previous two numerical examples and will not be presented here.

10. Conclusions. The theoretical results above provide precise conditions on approximation schemes needed to guarantee an MIP. The numerical results are interesting for two reasons. First, they provide numerical support for the mesh independence of the AVE scheme. Also, since the BK scheme does not generate norm convergent Riccati solutions, it is certainly not a mesh-independent scheme. However, the numerical results alone might be used to *incorrectly* justify some type of mesh independence.

There are many PDE control problems in which the linearization is not normal. For example, in channel flow control, when one linearizes about a nonzero equilibrium, the resulting A operator is highly nonnormal. Thus dual convergence is extremely important. Moreover, we have tested the theoretical results above on self-adjoint parabolic PDE control systems such as the ones considered by Banks and Kunisch [7]. Since dual convergence is not an issue and in [7] POES is established for this class of problems, our results imply mesh independence for standard finite element schemes. We have also applied the theory to some non-self-adjoint PDE problems. These PDE results, along with numerical examples, will appear in a forthcoming paper.

We have established a mesh independence result for the infinite dimensional version of the Kleinman-Newton algorithm for solving the algebraic Riccati operator equation associated with the LQR problem in a Hilbert space. We applied the results to systems governed by delay equations and presented numerical examples to illustrate the ideas. The results provide insight into the type of approximation schemes that lead to mesh independence. In particular, we showed that it is sufficient that the approximation be convergent, dual convergent, and uniformly stabilizable and detectable. As noted by Kappel in [35], it is possible to obtain (at least strong) convergence of the approximating Riccati operators without POES. This leaves open the

question of whether or not it is possible to achieve mesh independence without preserving stabilizability and detectability *uniformly* under approximation. However, it is important to note again that mesh independence alone does not imply convergence. We are currently looking into this issue and other issues concerning the numerical conditioning of the finite dimensional approximating Riccati equations.

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REFERENCES

- [1] E. L. ALLGOWER, K. BÖHMER, F. A. POTRA, AND W. C. RHEINBOLDT, *A mesh-independence principle for operator equations and their discretizations*, SIAM J. Numer. Anal., 23 (1986), pp. 160–169.
- [2] E. L. ALLGOWER AND K. BÖHMER, *Application of the mesh independence principle to mesh refinement strategies*, SIAM J. Numer. Anal., 24 (1987), pp. 1335–1351.
- [3] H. T. BANKS AND J. A. BURNS, *Hereditary control problems: Numerical methods based on averaging approximations*, SIAM J. Control Optim., 16 (1978), pp. 169–208.
- [4] H. T. BANKS, J. A. BURNS, AND E. M. CLIFF, *Parameter estimation and identification for systems with delays*, SIAM J. Control Optim., 19 (1981), pp. 791–828.
- [5] H. T. BANKS AND F. KAPPEL, *Spline approximations for functional differential equations*, J. Differential Equations, 34 (1979), pp. 496–522.
- [6] H. T. BANKS AND K. KUNISCH, *An approximation theory for nonlinear partial differential equations with applications to identification and control*, SIAM J. Control Optim., 20 (1982), pp. 815–849.
- [7] H. T. BANKS AND K. KUNISCH, *The linear regulator problem for parabolic systems*, SIAM J. Control Optim., 22 (1984), pp. 684–698.
- [8] H. T. BANKS, I. G. ROSEN, AND K. ITO, *A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations*, SIAM J. Sci. Stat. Comput., 5 (1984), pp. 830–855.
- [9] P. BENNER, *Solving large-scale control problems*, IEEE Control Systems Magazine, 24 (2004), pp. 44–59.
- [10] P. BENNER AND J. SAAK, *Linear-Quadratic Regulator Design for Optimal Cooling of Steel Profiles*, Preprint SFB393/05, TUChemnitz, Chemnitz, Germany, 2005.
- [11] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, AND S. K. MITTER, *Representation and Control of Infinite Dimensional Systems*, Vol. I, Birkhäuser, Boston, Basel, Berlin, 1992.
- [12] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, AND S. K. MITTER, *Representation and Control of Infinite Dimensional Systems*, Vol. II, Birkhäuser, Boston, Basel, Berlin, 1993.
- [13] J. BORGGAARD, J. A. BURNS, E. VUGRIN, AND L. ZIETSMAN, *On strong convergence of feedback operators for non-normal distributed parameter systems*, in Proceedings of the 43rd IEEE Conference on Decision and Control, Paradise Island, Bahamas, 2004, pp. 1526–1531.
- [14] P. BUBÁK, C. V. M. VAN DER MEE, AND A. C. M. RAN, *Approximation of solutions of Riccati equations*, SIAM J. Control Optim., 44 (2005), pp. 1419–1435.
- [15] J. A. BURNS, *Nonlinear distributed parameter control systems with non-normal linearizations: Applications and approximations*, in Research Directions in Distributed Parameter Systems, R. C. Smith and M. A. Demetriou, eds., SIAM, Philadelphia, 2003, pp. 17–53.
- [16] J. A. BURNS, K. ITO, AND G. PROPST, *On nonconvergence of adjoint semigroups for control systems with delays*, SIAM J. Control Optim., 26 (1988), pp. 1442–1454.
- [17] J. A. BURNS AND G. H. PEICHL, *Preservation of controllability under approximation and controllability radii for hereditary systems*, Differential Integral Equations, 2 (1989), pp. 439–452.
- [18] J. A. BURNS AND G. H. PEICHL, *A note on the asymptotic behavior of controllability radii for a scalar hereditary system*, in Proceedings of the 4th International Conference on Control of Distributed Parameter Systems, F. Kappel, K. Kunisch, and W. Schappacher, eds., Birkhäuser, Basel, 1989, pp. 27–39.
- [19] J. A. BURNS AND G. H. PEICHL, *Control system radii and robustness under approximation*, in Robust Optimization-Directed Design, A. Kurdila, P. Pardalos, and M. Zabaranin, eds., Springer, New York, 2006, pp. 25–61.

- [20] J. A. BURNS AND J. R. SINGLER, *Modeling transition: New scenarios, system sensitivity, and feedback control*, in Transition and Turbulence Control, Lecture Notes Ser. 8, Institute for Mathematical Sciences, National University of Singapore, M. Gad-el-Hak and H. M. Tsai, eds., World Scientific Publishers, Singapore, 2006, pp. 1–37.
- [21] F. CHATELIN, *Spectral Approximation of Linear Operators*, Academic Press, New York, 1983.
- [22] R. F. CURTAIN AND H. J. ZWART, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, New York, Berlin, Heidelberg, 1995.
- [23] T. DAMM AND D. HINRICHSSEN, *Newton's method for concave operators with resolvent positive derivatives in ordered Banach spaces*, Linear Algebra Appl., 363 (2003), pp. 43–64.
- [24] A. DE SANTIS, A. GERMANI, AND L. JETTO, *Approximation of the algebraic Riccati equation in the Hilbert space of Hilbert-Schmidt operators*, SIAM J. Control Optim., 31 (1993), pp. 847–874.
- [25] A. GERMANI, L. JETTO, AND M. PICCIONI, *Galerkin approximation for optimal linear filtering of infinite-dimensional linear systems*, SIAM J. Control Optim., 26 (1988), pp. 1287–1305.
- [26] A. GERMANI, C. MANES, AND P. PEPE, *A twofold spline approximation for finite horizon LQG control of hereditary systems*, SIAM J. Control Optim., 39 (2000), pp. 1233–1295.
- [27] J. S. GIBSON, *The Riccati integral equations for optimal control problems on Hilbert spaces*, SIAM J. Control Optim., 17 (1979), pp. 537–565.
- [28] J. S. GIBSON, *Linear-quadratic optimal control of hereditary differential systems: Infinite dimensional Riccati equations and numerical approximations*, SIAM J. Control Optim., 21 (1983), pp. 95–139.
- [29] J. S. GIBSON AND A. ADAMIAN, *Approximation theory for linear-quadratic-Gaussian optimal control of flexible structures*, SIAM J. Control Optim., 29 (1991), pp. 1–37.
- [30] L. GRASEDYCK, W. HACKBUSCH, AND B. N. KHOROMSKII, *Solution of large scale algebraic matrix Riccati equations by use of hierarchical matrices*, Computing, 70 (2003), pp. 121–165.
- [31] N. J. HIGHAM, M. KONSTANTINOV, V. MEHRMANN, AND P. PETKOV, *The sensitivity of computational control problems*, IEEE Control Systems Magazine, 24 (2004), pp. 28–43.
- [32] C. HUANG AND S. VANDEWALLE, *An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays*, SIAM J. Sci. Comput., 25 (2004), pp. 1608–1632.
- [33] K. ITO, *Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces*, in Distributed Parameter Systems, F. Kappel, K. Kunisch, and W. Schappacher, eds., Springer, New York, 1987, pp. 151–166.
- [34] K. ITO, *Finite-dimensional compensators for infinite-dimensional systems via Galerkin-type approximations*, SIAM J. Control Optim., 28 (1990), pp. 1251–1269.
- [35] F. KAPPEL, *Approximations of LQR-problems for delay systems: A survey*, in Computation and Control II, K. Bowers and J. Lund, eds., Birkhäuser, Cambridge, 1991, pp. 187–224.
- [36] F. KAPPEL AND D. SALAMON, *Spline approximations for retarded systems and the Riccati equation*, SIAM J. Control Optim., 25 (1987), pp. 1082–1117.
- [37] F. KAPPEL AND D. SALAMON, *On the stability properties of spline approximations for retarded systems*, SIAM J. Control Optim., 27 (1989), pp. 407–431.
- [38] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, New York, 1976.
- [39] L. A. LUSTERNIK AND V. J. SOBOLEV, *Elements of Functional Analysis*, Gordon and Breach, New York, 1961.
- [40] V. L. MEHRMANN, *The Autonomous Linear Quadratic Control Problem*, Springer, Berlin, Heidelberg, 1991.
- [41] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, New York, 1983.
- [42] T. PENZL, *A Multi-Grid Method for Generalized Lyapunov Equations*, Preprint SFB393/97, TUChemnitz, Chemnitz, Germany, 1997.
- [43] I. G. ROSEN, *On Hilbert-Schmidt norm convergence of Galerkin approximations for operator Riccati equations*, in Control and Estimation of Distributed Parameter Systems, Internat. Ser. Numer. Math. 91, Birkhäuser, Basel, 1989, pp. 335–349.
- [44] I. G. ROSEN, *Convergence of Galerkin approximations for operator Riccati equations—A nonlinear evolution equation approach*, J. Math. Anal. Appl., 155 (1991), pp. 226–248.
- [45] I. G. ROSEN AND C. WANG, *A multilevel technique for the approximate solution of operator Lyapunov and Riccati equations*, SIAM J. Numer. Anal., 32 (1995), pp. 514–541.
- [46] H. SASAI, *Approximation of optimal control problems governed by nonlinear evolution equations*, Internat. J. Control, 28 (1978), pp. 313–324.

- [47] H. SASAI AND E. SHIMEMURA, *On convergence of approximating solutions of a class of linear optimal control problems of distributed parameter systems*, Bul. Inst. Politehn. Iași (N.S.), 15 (1969), pp. 39–47.
- [48] H. SASAI AND E. SHIMEMURA, *On the convergence of approximating solutions for linear distributed parameter optimal control problems*, SIAM J. Control Optim., 9 (1971), pp. 263–273.