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# A UNIQUENESS RESULT FOR $p$-MONOTONE VISCOSITY SOLUTIONS OF HAMILTON–JACOBI EQUATIONS IN BOUNDED DOMAINS*

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**Abstract.** We consider a class of Hamilton–Jacobi equations $H(x, Du(x)) = 0$ with no $u$-dependence and with continuity properties consistent with recent applications in queueing theory. Continuous viscosity solutions are considered in a compact polyhedral domain, with oblique derivative (Neumann-type) boundary conditions. Comparison and uniqueness results are presented, which use monotonicity of $H(x, p)$ in the $p$ variable for values of $p$ in the appropriate sub- and superdifferential sets of the solution $u(x)$. Several examples illustrate the results.

**Key words.** viscosity solution, Hamilton–Jacobi equation, uniqueness

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## 1. Introduction.

The theory of viscosity solutions to first order partial differential equations provides a satisfying approach to Hamilton–Jacobi equations for many types of optimal control problems and differential games. Bardi and Capuzzo-Dolcetta [4] give an extensive introduction to the basic theory and its application to a variety of optimization problems. At the heart of the theory are the fundamental comparison and uniqueness results, which identify the optimal value function as the unique viscosity solution of the appropriate Hamilton–Jacobi equation. Those comparison and uniqueness results generally depend on some monotonicity property of the Hamiltonian $H$. For instance, in the case of discounted infinite horizon problems, the Hamilton–Jacobi equation includes a term $\lambda u$ ($\lambda > 0$ being the discount rate). This provides monotonicity in $u$ which is the key to the proof of the typical comparison result, such as [4, Theorem II.3.1].

In this paper we consider problems of the form

$$H(x, Du(x)) = 0,$$

in which the Hamiltonian $H(x, p)$ has no $u$-dependence. It is well known that without some additional property, solutions may be nonunique. (See Example 6 in section 5, for instance.) Ishii [18] provides an approach which assumes convexity of $p \mapsto H(x, p)$ and the existence of a special smooth subsolution $\varphi$. (See also [4, section II.5.3].) The idea is to perturb a given subsolution by a (small) convex combination with $\varphi$ to obtain a "strict" subsolution. A basic comparison result (very like our Lemma 2) then implies the desired inequality. An elementary example is the eikonal equation

$$H(x, p) = |p| - h(x),$$

where $h$ is continuous and strictly positive on the spatial domain $\Omega$. This category of problems can also be treated using the transformation of Kružkov. This can be applied generally when there is a strictly positive lower bound for the running cost $L$

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of (19) below. See Bardi and Soravia [5] and the references in [18]. Our $p$-monotone approach is also applicable to such problems; see Example 2 below.

Another approach is that of Camilli and Siconolfi [7]. They are interested in equations of the form

$$H(x,p) - f(x) = 0$$

and seek to identify maximal subsolutions. (In some control problems this is the standard characterization of the desired viscosity solution; see Soravia [21].) They obtain a definitive characterization of maximal subsolutions in terms of a special singular solution property. Their approach is rather technical, using convexity of the sets $\{p : H(x,p) - f(x) \le 0\}$ and a special topology in $\Omega$ associated with them. Among their few simple hypotheses on the Hamiltonian is the assumption that $t \mapsto H(x, tp)$ is strictly increasing in $t \in [0,1]$ for all $p$. We note that this is essentially the $p$-monotone property that we exploit below. We would comment that our results also provide a simple sufficient condition for a viscosity solution to be the maximal subsolution, namely that it be a $p$-monotone supersolution.

We are motivated by a growing body of work using control problems and differential games for asymptotic analysis of queueing networks. These problems often involve oblique-derivative boundary conditions on some part of $\partial\Omega$. (Although only Dirichlet conditions were considered in [18] and [7], presumably generalizations are possible.) These examples typically do not have the convexity needed for either the approach of [18] or [7]; see Examples 4 and 5 in section 5. However, the literature does contain some uniqueness results for certain problems of this type. The germ of our $p$-monotone argument can be found in the proof of Theorem 5 of Atar, Dupuis, and Schwartz [2] (see their equation (37)). Although it is not a viscosity solution result, the structured verification theorem of Day [12] uses a "positive storage condition" which is related to $p$-monotonicity (as we will see in Example 5). The essential feature underlying these results is monotonicity of $t \mapsto H(x, tp)$, not necessarily for all $p$ but just for those $p = \zeta \in D^{\pm}u(x)$ that are not accounted for by the boundary conditions. Our intent here is to develop comparison and uniqueness results based on this property for problems with oblique-derivative boundary conditions, such as are typical in queueing applications. This class of problems also motivates our regularity hypotheses on $H$.

There are a few other comparison results in the literature which employ properties of the $p$-dependence of $H$. For instance, the development in Crandall, Ishii, and Lions [8] assumes that a special test function $\mu(x)$ exists for which $\lambda \mapsto H(x, p + \lambda D\mu(x)) - H(x,p)$ satisfies a certain lower bound; see their (H2). We note that such a hypothesis is entirely a property of the Hamiltonian and depends on the existence of $\mu(x)$. In general our notion of a $p$-monotone solution depends on the specific solution $u(x)$, not solely on $H$.

In section 2 we pose the specific type of boundary value problem we will address, using oblique-derivative conditions on the boundary of a compact polyhedral domain. Section 3 presents a basic comparison result (Lemma 2) for sub- and supersolutions to a pair of "strictly separated" equations. (That strict separation generally implies a comparison result is well known; see Crandall, Ishii, and Lions [9].) The $p$-monotone results are then developed in section 4. Our main result (Theorem 4) implies that when a $p$-monotone solution exists it is the unique viscosity solution—the "complete solution" in the terminology of [4]. We conclude by looking at several examples in section 5.

**2. Preliminaries and hypotheses.** We consider a domain $\Omega$ which is assumed to be a compact convex polyhedron in $\mathbb{R}^n$, defined by a finite collection of $m$ linear constraints,

$$\Omega = \{x \in \mathbb{R}^n : \; n_i \cdot x \geq c_i \text{ for each } i = 1, \dots, m\}. \tag{1}$$

The $n_i$ are unit vectors (inward normals) and the $c_i$ are constants. For $x \in \partial\Omega$ (the boundary of $\Omega$) we define the set of active constraints as

$$I(x) = \{i : \; n_i \cdot x = c_i\}$$

and take $I(x) = \emptyset$ for $x \in \Omega^\circ$ (the interior of $\Omega$). We consider a closed subset $\mathcal{T} \subseteq \Omega$ on which Dirichlet data will be prescribed. This could be part of the boundary, but that it not necessary. Values for $u$ are prescribed on $\mathcal{T}$ by a continuous function $g : \mathcal{T} \to \mathbb{R}$,

$$u(x) = g(x), \; x \in \mathcal{T}. \tag{2}$$

It will be convenient to use the notation

$$\Omega_{\delta,\mathcal{T}} = \{x \in \Omega : \; \operatorname{dist}(x, \mathcal{T}) > \delta\}$$

to refer to the part of $\Omega$ at least $\delta > 0$ away from $\mathcal{T}$. (We allow $\mathcal{T} = \emptyset$, in which case $\Omega_{\delta,\mathcal{T}} = \Omega$.) On the rest of the boundary, $\partial\Omega \setminus \mathcal{T}$, we want to require oblique-derivative boundary conditions using a collection of vectors $d_i$, $i = 1, \dots, m$,

$$-d_i \cdot Du(x) = 0, \; i \in I(x). \tag{3}$$

In $\Omega \setminus \mathcal{T}$ itself we consider a Hamilton–Jacobi equation,

$$H(x, Du(x)) = 0. \tag{4}$$

If $\mathcal{T} = \partial\Omega$, we have a standard Dirichlet problem. If $\mathcal{T} = \emptyset$ we have a typical Neumann-type problem. In general the problem is a mixture of these two types.

**2.1. Continuity hypothesis on the Hamiltonian.** Appropriate continuity hypotheses for the Hamiltonian $H$ are important. The examples we have in mind use a Hamiltonian of the form (19) below, with $f = f(a, b)$ independent of state and running cost $L = h(x) + \ell(a, b)$ with separate state and player components. This leads to a Hamiltonian of separated form, $H(x, p) = H_0(p) - h(x)$. But all we really need are continuity hypotheses consistent with that. We assume there exist $m : [0, \infty) \to [0, \infty)$ with $m(0) = 0$ and continuous at 0, and $M : [0, \infty)^2 \to [0, \infty)$ with $M(0, R) = 0$ and $M(\cdot, R)$ continuous at 0 for each $R < \infty$, such that for all $x, y \in \Omega$ and $p, q \in \mathbb{R}^d$ with $|p|, |q| \leq R$, we have

$$|H(x, p) - H(y, q)| \leq m(|x - y|) + M(|p - q|, R). \tag{5}$$

**2.2. Technical hypotheses on $\Omega$ and $d_i$.** The oblique-derivative boundary conditions (3) are closely associated with the Skorokhod problem for $\Omega$; see Dupuis and Ishii [14]. Control problems for systems including a Skorokhod problem in their dynamics are common in queueing theory and lead to Hamilton–Jacobi equations with boundary conditions (3); see Lions [19], Dupuis and Ishii [15], and Day [11]. Although the Skorokhod problem does not appear in our results below, hypotheses from [14] regarding $\Omega$ and the $d_i$ of the boundary conditions are important ingredients for the proof of Lemma 2. For that purpose we assume the following.

• *$B$-hypothesis* [14, Assumption 2.1]. There exists a compact, convex $B \subseteq \mathbb{R}^n$ with $0 \in B^\circ$ and the following property: If $z \in \partial B$ and $|z \cdot n_i| < 1$, then $\nu \cdot d_i = 0$ for all unit outward normals to $B$ at $z$. ($\nu$ is an outward normal to $B$ at $z$ if $\nu \cdot (z - x) \geq 0$ for all $x \in B$.) For further discussion of this hypothesis and an illustrative figure, see Dupuis and Ramanan [17].

• *Coercivity hypothesis.* For each $x \in \partial\Omega$, and any $a_i \in \mathbb{R}$,

$$\left(\sum_{I(x)} a_i d_i\right) \cdot \left(\sum_{I(x)} a_i n_i\right) \geq 0, \tag{6}$$

with equality only if $a_i = 0$ for all $i \in I(x)$. It is shown in Day [10] that this, together with the $B$-hypothesis, implies [14, Assumption 3.1] concerning the existence of a discrete projection map. Moreover, it implies that, for each $x \in \partial\Omega$, the $d_i$, $i \in I(x)$, are linearly independent, which is needed for Lemma 1 below. We might have assumed [14, Assumption 3.1] along with this linear independence property, but (6) is a convenient sufficient condition for both and is easy to verify in examples, since it reduces to checking positive definiteness of a small number of matrices.

These hypotheses provide the following technical result, which will be needed for the proof of Lemma 2.

LEMMA 1. *Assume the $B$-hypothesis and the coercivity hypothesis.*

(a) *There exists a $C^1$ function $\mu : \Omega \to [0,1]$ with the property that $d_i \cdot D\mu(x) < 0$ whenever $x \in \partial\Omega$ and $i \in I(x)$.*

(b) *There exists a $C^1$ function $\xi : \mathbb{R}^n \to [0, \infty)$ with the properties that*

(i) *$\xi^{1/2}$ is a norm on $\mathbb{R}^d$, and*

(ii) *for any $x \in \mathbb{R}^n$ and $i = 1, \ldots, m$, $x \cdot n_i \geq 0$ $[\leq 0]$ implies $d_i \cdot D\xi(x) \geq 0$ $[\leq 0]$.*

*Proof.* Part (a) is Lemma 3.2 of Dupuis and Ishii [15]. Their hypothesis (B.6) follows from the independence of $d_i$, $i \in I(x)$, pointed out above. The other hypotheses are simple to check in our setting.

Part (b) follows from arguments given in Atar and Dupuis [1], which we outline. (See their remark on page 1109.) First, it is shown that the property of $B$ is equivalent to an extended property, namely that if $z \in \partial B$ and $\nu$ is an outward normal to $B$ at $z$, then

$$z \cdot n_i \geq -1 [\leq 1] \text{ implies } d_i \cdot \nu \geq 0 [\leq 0].$$

(Although [1] only considers $\Omega = \mathbb{R}^n_+$, the extension argument based on Dupuis and Ramanan [17] applies in general.) Next, given that the set $B$ exists, it is argued that $B$ can be assumed symmetric with a smooth boundary, in the sense that the unit outward normal $\nu(x)$ is uniquely determined and continuous as a function of $x \in \partial B$. Such a $B$ determines a (smooth) norm on $\mathbb{R}^n$, defined by

$$\|x\|_B = \inf\{r > 0 : \ x \in rB\}.$$

$B$ is the closed unit ball with respect to $\|\cdot\|_B$. We define $\xi(x) = \|x\|_B^2$. It follows that $\xi$ is $C^1$, and for a given $x$,

$$D\xi(x) = b\|x\|_B \, \nu,$$

where $b = b(x) > 0$ is a scalar function and $\nu$ the unit outward normal to $B$ at $z = x/\|x\|_B \in \partial B$. Therefore if $x \cdot n_i \geq 0$, then $-1 < 0 \leq z \cdot n_i$, so that the extended property of $B$ above implies $d_i \cdot \nu \geq 0$, which in turn implies $d_i \cdot D\xi(x) \geq 0$. The other case is proven analogously, or by appeal to symmetry. $\square$

As a consequence of (a), observe that there exists a constant $\mu_0 > 0$ such that

$$\mu_0 < -d_i \cdot D\mu(x) \text{ for all } x \in \partial\Omega,\ i \in I(x). \tag{7}$$

**2.3. Viscosity solutions.** In the proof of Lemma 2 we will use the generalization of (3) to

$$C - d_i \cdot Du(x) = 0,\ i \in I(x), \tag{8}$$

where $C$ is a constant. We want to state carefully what it means to be a viscosity sub- or supersolution of (4) with boundary conditions (8) on $\Omega \setminus \mathcal{T}$. Note that the definitions will not refer to (2) on $\mathcal{T}$; we prefer to express that separately by referring to "subsolutions with $u(x) \leq g(x)$ on $\mathcal{T}$" as needed.

We will consider only continuous functions $u : \Omega \to \mathbb{R}$ as possible solutions. For $x \in \Omega$ the superdifferential set $D^+u(x)$ consists of those $\zeta \in \mathbb{R}^n$ which occur as the value $\zeta = D\phi(x)$ for some $C^1$ function $\phi : \mathbb{R}^n \to \mathbb{R}$ with the property that $u(x) - \phi(x) \geq u(y) - \phi(y)$ for all $y \in \Omega$ sufficiently close to $x$. For the correct viscosity-sense understanding of (8) it is important to note that $x$ is a local maximum of $u - \phi$ *only relative* to $\Omega$. For $x \in \partial\Omega$ this means that even if $u$ is smooth, $D^+u(x)$ can contain many $\zeta$ other that $Du(x)$ itself. (See Lemma 7 in section 5.) Similarly, $D^-v(x)$ consists of $\zeta$ arising as $\zeta = D\phi(x)$ for some $C^1$ function $\phi(x)$ such that $v - \phi$ has a local minimum at $x$ relative to $\Omega$. The function $u(x) \in C(\Omega)$ is called a *subsolution* of

$$H(x, Du(x)) = 0 \text{ on } \Omega \setminus \mathcal{T} \text{ with } C - d_i \cdot Du(x) = 0 \text{ on } \partial\Omega \setminus \mathcal{T} \tag{9}$$

provided the following hold for all $\zeta \in D^+u(x)$:

(i) if $x \in \Omega^\circ \setminus \mathcal{T}$, then $H(x, \zeta) \leq 0$;

(ii) if $x \in \partial\Omega \setminus \mathcal{T}$, then either $H(x, \zeta) \leq 0$ or $C - d_i \cdot \zeta \leq 0$ for some $i \in I(x)$.

In other words, at boundary points only one of the inequalities $H(x, \zeta) \leq 0$, $C - d_i \cdot \zeta \leq 0$ $(i \in I(x))$ needs to hold. This is the, now standard, viscosity formulation of first order equations with "Neumann-type" boundary conditions (see Barles and Lions [6]), generalized to consider different boundary conditions $C - d_i \cdot Du(x) = 0$ on the different planar faces of $\partial\Omega$. We can express this subsolution definition succinctly by writing

$$H(x, \zeta) \wedge \min_{i \in I(x)} (C - d_i \cdot \zeta) \leq 0 \text{ for all } x \in \Omega \setminus \mathcal{T} \text{ and } \zeta \in D^+u(x) \tag{10}$$

and using the convention that $\min_{i \in I(x)} = +\infty$ if $I(x) = \emptyset$. The definition of a *supersolution* is obtained by reversing all the inequalities in (i) and (ii) and considering $\zeta \in D^-u(x)$ instead. We would replace (10) by $H(x, \zeta) \vee \max_{i \in I(x)}(C - d_i \cdot \zeta) \geq 0$.

**3. A basic comparison result for strictly separated equations.** The task of this section is to establish a basic comparison result for oblique-derivative boundary conditions (3) analogous to that of Ishii [18, Lemma 1] . The comparison argument of Atar, Dupuis, and Schwartz [2] is close to ours and is the source of our approach to handling the boundary conditions. The use of a norm such as $\xi$ in the function $\Phi_\epsilon$ of the proof below originated in Dupuis, Ishii, and Soner [16].

LEMMA 2. *Assume that $g : \mathcal{T} \to \mathbb{R}$ is continuous and that $u, v \in C(\Omega)$ with $u \leq g \leq v$ on $\mathcal{T}$ are such that*

(a) *$u$ is a subsolution of $H(x, Du(x)) + \eta_+(x) = 0$ on $\Omega \setminus \mathcal{T}$, with $-d_i \cdot Du(x) = 0$ on $\partial\Omega \setminus \mathcal{T}$; and*

(b) *$v$ is a supersolution of $H(x, Dv(x)) - \eta_-(x) = 0$ on $\Omega \setminus \mathcal{T}$, with $-d_i \cdot Du(x) = 0$ on $\partial\Omega \setminus \mathcal{T}$,*

*where $\eta_\pm : \Omega \to \mathbb{R}$ have the property that for each $\delta > 0$,*

$$\inf_{x \in \Omega_{\delta,\mathcal{T}}} \eta_+(x) + \inf_{x \in \Omega_{\delta,\mathcal{T}}} \eta_-(x) > 0. \tag{11}$$

*Then $u(x) \le v(x)$ for all $x \in \Omega$.*

We will say that the $u$ and $v$ of this lemma are viscosity sub- and supersolutions to a *strictly separated* pair of equations. Note that because of (11) this notion of strict separation depends on the choice of $\mathcal{T}$. Also observe that we have made no regularity assumption on the $\eta_\pm$. The inequality (11) is all the proof needs. An alternate hypothesis would be to assume that $\inf_{\Omega_{\delta,\mathcal{T}}} [\eta_+(x) + \eta_-(x)] > 0$ along with continuity of (one of) the $\eta_\pm$.

*Proof.* Let $0 < c_\epsilon < 1$ be a family of constants with $c_\epsilon \to 0$ as $\epsilon \downarrow 0$. Near the end of the proof we will be more specific about how $c_\epsilon$ should be chosen, but that detail is not needed yet. Given $\epsilon > 0$, define

$$u_\epsilon(x) = u(x) - c_\epsilon \mu(x), \quad v_\epsilon(x) = v(x) + c_\epsilon \mu(x),$$

where $\mu(x)$ is as in Lemma 1 above. It follows that $\zeta_\epsilon \in D^+ u_\epsilon(x)$ iff $\zeta = \zeta_\epsilon + c_\epsilon D\mu(x) \in D^+ u(x)$. Notice that

$$-d_i \cdot \zeta = -d_i \cdot (\zeta_\epsilon + c_\epsilon D\mu(x)) \ge -d_i \cdot \zeta_\epsilon + c_\epsilon \mu_0,$$

where $\mu_0$ is as in (7). Therefore, the subsolution hypothesis of (a) implies that for all $\zeta \in D^+ u(x)$,

$$[H(x, \zeta_\epsilon + c_\epsilon D\mu(x)) + \eta_+(x)] \wedge \min_{i \in I(x)} (c_\epsilon \mu_0 - d_i \cdot \zeta_\epsilon) \le 0.$$

In other words, $u_\epsilon$ is a subsolution of

$$H(x, Du_\epsilon(x) + c_\epsilon D\mu(x)) + \eta_+(x) = 0 \text{ on } \Omega \setminus \mathcal{T} \text{ with } c_\epsilon \mu_0 - d_i \cdot Du_\epsilon(x) = 0 \text{ on } \partial\Omega \setminus \mathcal{T}. \tag{12}$$

Similarly, $v_\epsilon$ is a supersolution of

$$H(x, Dv_\epsilon(x) - c_\epsilon D\mu(x)) - \eta_-(x) = 0 \text{ on } \Omega \setminus \mathcal{T} \text{ with } -c_\epsilon \mu_0 - d_i \cdot Du_\epsilon(x) = 0 \text{ on } \partial\Omega \setminus \mathcal{T}. \tag{13}$$

Now suppose that $\sup_\Omega (u(x) - v(x)) > 0$. Then because $\mu(x)$ is bounded and $c_\epsilon \to 0$, there is a positive constant $\rho$ so that for all sufficiently small $\epsilon > 0$,

$$0 < \rho < \sup_\Omega [u_\epsilon(x) - v_\epsilon(x)]. \tag{14}$$

We now give a version of the usual argument leading to a contradiction. Define

$$\Phi_\epsilon(x, y) = u_\epsilon(x) - v_\epsilon(y) - \epsilon^{-1} \xi(x - y),$$

where $\xi(\cdot)$ is as in Lemma 1, and let $(x_\epsilon, y_\epsilon) \in \Omega \times \Omega$ be a maximizing pair for $\Phi_\epsilon$. By comparison to $x = y$, we have

$$\Phi_\epsilon(x_\epsilon, y_\epsilon) \ge \rho. \tag{15}$$

From $\Phi_\epsilon(x_\epsilon, x_\epsilon) \le \Phi_\epsilon(x_\epsilon, y_\epsilon)$ it follows that

$$\epsilon^{-1}\xi(x_\epsilon - y_\epsilon) \le v_\epsilon(x_\epsilon) - v_\epsilon(y_\epsilon). \tag{16}$$

Since $v$ and $\mu$ are bounded, and $0 < c_\epsilon < 1$, it follows that $v_\epsilon$ is bounded (independent of $\epsilon$). We deduce that $\xi(x_\epsilon - y_\epsilon) = \mathrm{O}(\epsilon)$. Since all norms on $\mathbb{R}^n$ are equivalent, we have

$$\|x_\epsilon - y_\epsilon\| = \mathrm{O}(\epsilon^{1/2}). \tag{17}$$

Next, we claim that none of the limit points of $x_\epsilon$ (as $\epsilon \downarrow 0$) can be in $\mathcal{T}$. Indeed, if (along a sequence of $\epsilon \downarrow 0$) we had $x_\epsilon \to z \in \mathcal{T}$, then by (17) $y_\epsilon \to z$ as well. It follows that

$$\lim_\epsilon [u_\epsilon(x_\epsilon) - v_\epsilon(y_\epsilon)] \le g(z) - 0\mu(z) - [g(z) + 0\mu(z)] = 0.$$

Since $v$ and $\mu$ are continuous, $v_\epsilon$ is equicontinuous with respect to $\epsilon$. This, together with (16), implies that $\epsilon^{-1}\xi(x_\epsilon - y_\epsilon) \to 0$. Therefore $\Phi_\epsilon(x_\epsilon, y_\epsilon) \to 0$, contrary to (15), and this proves our claim. The claim means that there exists $\delta > 0$ so that $x_\epsilon, y_\epsilon \in \Omega_{\delta,\mathcal{T}}$ for all sufficiently small $\epsilon$. By hypothesis (11), there exists $\eta_0 > 0$ so that

$$\eta_0 \le \eta_+(x_\epsilon) + \eta_-(y_\epsilon),$$

for all $\epsilon > 0$ sufficiently small.

Now $u_\epsilon(x) - \left[v_\epsilon(y_\epsilon) + \epsilon^{-1}\xi(x - y_\epsilon)\right]$ is maximized at $x = x_\epsilon$. Therefore $\zeta_\epsilon \doteq \epsilon^{-1}D\xi(x_\epsilon - y_\epsilon) \in D^+u_\epsilon(x_\epsilon)$. Since $D\xi$ is continuous and $\Omega$ is compact, it follows that

$$\zeta_\epsilon = \mathrm{O}(\epsilon^{-1}).$$

If it were the case that $x_\epsilon \in \partial\Omega$, then by definition of $\Omega$ we would have $n_i \cdot (x_\epsilon - y_\epsilon) \le 0$ for all $i \in I(x_\epsilon)$. By property (ii) of $\xi$ in Lemma 1, it follows that $d_i \cdot \zeta_\epsilon \le 0$ for all $i \in I(x_\epsilon)$. Therefore,

$$c_\epsilon\mu_0 - d_i \cdot \zeta_\epsilon \ge c_\epsilon\mu_0 > 0.$$

Since we know $x_\epsilon \notin \mathcal{T}$, (12) implies that

$$H(x_\epsilon, \zeta_\epsilon + c_\epsilon D\mu(x_\epsilon)) + \eta_+(x_\epsilon) \le 0.$$

Arguing in the same way, from the fact that $y = y_\epsilon$ maximizes

$$v_\epsilon(y) - \left[u(x_\epsilon) - \epsilon^{-1}\xi(x_\epsilon - y)\right],$$

we are led to the conclusion that

$$H(y_\epsilon, \zeta_\epsilon - c_\epsilon D\mu(y_\epsilon)) - \eta_-(y_\epsilon) \ge 0.$$

Therefore,

$$0 < \eta_0 \le \eta_+(x_\epsilon) + \eta_-(y_\epsilon) \le H(y_\epsilon, \zeta_\epsilon - c_\epsilon D\mu(y_\epsilon)) - H(x_\epsilon, \zeta_\epsilon + c_\epsilon D\mu(x_\epsilon)).$$

Now we know that for some constant $K$ (independent of $\epsilon > 0$), $|\zeta_\epsilon \pm c_\epsilon D\mu| \le \epsilon^{-1}K$. Our continuity hypotheses on $H(x,p)$ imply that the right-hand side of the above expression is bounded above by

$$m(|x_\epsilon - y_\epsilon|) + M(2c_\epsilon|\mu|, \epsilon^{-1}K).$$

The first term converges to 0 because $|x_\epsilon - y_\epsilon| \to 0$. We can choose $c_\epsilon \downarrow 0$ so that the second term $\to 0$ as well. For such choices we have a contradiction to the positive lower bound $\eta_0$. This contradiction implies that $\sup_\Omega[u(x) - v(x)] \le 0$, concluding the proof. $\square$

**4. $p$-monotone uniqueness.** We want to use monotonicity properties of $H$ in the $p$ variable to produce the additional $\eta_\pm(x)$ terms needed for application of Lemma 2. Intuitively, we want to use a property such as

$$H(x, s\zeta) < H(x,\zeta) \text{ for } 0 < s < 1 \quad \text{and} \quad H(x,\zeta) < H(x, s\zeta) \text{ for } 1 < s.$$

However, this is considerably stronger than needed for the proof. For the subsolution case, $0 < s < 1$, we don't really need $H(x, s\zeta) < H(x,\zeta)$, only $H(x, s\zeta) < 0$, but holding uniformly on compacts disjoint from $\mathcal{T}$. We express this as

$$H(x, s\zeta) + \eta_s(x) \leq 0$$

for some function $\eta_s(x)$ which is uniformly positive on each $\Omega_{\delta,\mathcal{T}}$. Moreover, we only need these properties for those $\zeta \in D^+u(x)$ such that the inequality (10) is not satisfied by virtue of the $-d_i \cdot \zeta$ terms. This can be stated succinctly by saying that $u(x)$ is a subsolution of

$$H(x, sDu(x)) + \eta_s(x) = 0 \text{ on } \Omega \setminus \mathcal{T} \text{ with } -d_i \cdot Du(x) = 0 \text{ on } \partial\Omega \setminus \mathcal{T},$$

which is what we need to invoke Lemma 2. The following definition is based on this weakened monotonicity requirement.

Definition 3. *A viscosity subsolution $u(x)$ of*

$$H(x, Du(x)) = 0 \textit{ on } \Omega \setminus \mathcal{T} \textit{ with } -d_i \cdot Du(x) = 0 \textit{ on } \partial\Omega \setminus \mathcal{T} \tag{18}$$

*is called* $p$-monotone *if, for some $\delta_0 > 0$ and each $1 - \delta_0 < s < 1$, there exists a function $\eta_s : \Omega \to [0,\infty)$ with $\inf_{\Omega_{\delta,\mathcal{T}}} \eta_s > 0$ for each $\delta > 0$, so that $u(x)$ is a subsolution of*

$$H(x, sDu(x)) + \eta_s(x) = 0 \textit{ on } \Omega \setminus \mathcal{T} \textit{ with } -d_i \cdot Du(x) = 0 \textit{ on } \partial\Omega \setminus \mathcal{T}.$$

*A viscosity supersolution $v(x)$ of* (18) *is called* $p$-monotone *if, for some $\delta_0 > 0$ and each $1 < s < 1 + \delta_0$, there exists a function $\eta_s : \Omega \to [0,\infty)$ with $\inf_{\Omega_{\delta,\mathcal{T}}} \eta_s > 0$ for each $\delta > 0$, so that $v(x)$ is a supersolution of*

$$H(x, sDv(x)) - \eta_s(x) = 0 \textit{ on } \Omega \setminus \mathcal{T} \textit{ with } -d_i \cdot Dv(x) = 0 \textit{ on } \partial\Omega \setminus \mathcal{T}.$$

*A viscosity solution which is both a $p$-monotone subsolution and a $p$-monotone supersolution is called a* $p$-monotone solution*.*

We observe that $p$-monotonicity concerns $s\zeta$ for $s < 1$ in the case of a subsolution, but $1 < s$ for a supersolution. It is possible for a viscosity solution to have the $p$-monotone property in the supersolution sense but not the subsolution sense. This would be a viscosity solution and a $p$-monotone supersolution, but not a $p$-monotone solution.

We are now ready for our main theorem. The basic idea is that if $u(x)$ is a subsolution, then $p$-monotonicity will imply that $su(x) + (s-1)c$ is a "strict" subsolution. (The constant term $(s-1)c$ is to insure that $su(x) + (s-1)c \leq g$ in case $g(x) < 0$. The fact that $H$ has no $u$-dependence allows us to add such constants with impunity.) We then appeal to Lemma 2 and let $s \uparrow 1$.

Theorem 4. *Suppose $u$ is a $p$-monotone subsolution of* (9) *with $u \leq g$ on $\mathcal{T}$, and $v$ is (any) supersolution with $g \leq v$ on $\mathcal{T}$. Then $u(x) \leq v(x)$ for all $x \in \Omega$. Likewise if $u$ is (any) subsolution and $v$ is a $p$-monotone supersolution with $u \leq g \leq v$ on $\mathcal{T}$,* *then $u(x) \leq v(x)$ for all $x \in \Omega$. If* (9) *has a $p$-monotone solution $v$, then $v$ is the complete solution (i.e., it is the unique solution, the maximal subsolution, and the minimal supersolution).*

COROLLARY 5. *A viscosity solution which is a $p$-monotone supersolution is the maximal subsolution.*

*Proof.* We focus on the $p$-monotone subsolution case. Let

$$-c = \min_{\mathcal{T}} g(x).$$

On $\mathcal{T}$ we have $u(x)+c \leq g(x)+c$. Since $0 \leq g(x)+c$, it follows that (for any $0 < s < 1$) $s(u(x)+c) \leq g(x)+c$ on $\mathcal{T}$. This is equivalent to

$$u_s(x) \doteq su(x) + (s-1)c \leq g(x), \ x \in \mathcal{T}.$$

Now $\zeta_s \in D^+u_s(x)$ iff $\zeta_s = s\zeta$ for some $\zeta \in D^+u(x)$. If $x \in \partial\Omega \setminus \mathcal{T}$ and $-d_i \cdot \zeta_s > 0$ for all $i \in I(x)$, then $-d_i \cdot \zeta > 0$ for all $i \in I(x)$, so by the $p$-monotone subsolution property for $u(x)$,

$$H(x, \zeta_s) + \eta_s(x) = H(x, s\zeta) + \eta_s(x) \leq 0.$$

The same inequality holds for $x \in \Omega^\circ$. We conclude that $u_s$ is a viscosity subsolution of

$$H(x, Du_s(x)) + \eta_s(x) = 0 \text{ on } \Omega \setminus \mathcal{T} \text{ with } -d_i \cdot Du_s(x) = 0 \text{ on } \partial\Omega \setminus \mathcal{T}.$$

We can now apply Lemma 2 to $u_s$ and $v$, using $\eta_+(x) = \eta_s(x)$ for $u_s$ and $\eta_-(x) \equiv 0$ for $v$. The lemma implies that $u_s(x) \leq v(x)$ all $x \in \Omega$ as follows: for all $1-\delta_0 < s < 1$,

$$su(x) + (s-1)c \leq v(x).$$

Letting $s \uparrow 1$ implies $u(x) \leq v(x)$, as claimed. The supersolution case (using $s \downarrow 1$) is analogous. The rest of the assertions of the theorem and corollary are now elementary. ∎

In general the $p$-monotone property may depend on the specific solution, since the definition only concerns $\zeta \in D^{\pm}u(x)$. However, for some Hamiltonians all (sub- or super-) solutions will be $p$-monotone. We consider in particular Hamiltonians associated with a running cost $L(x, a, b)$,

$$H(x, p) = \inf_{b \in \mathcal{B}} \sup_{a \in \mathcal{A}} \left\{ -p \cdot f(x, a, b) - L(x, a, b) \right\}, \tag{19}$$

still assuming the continuity hypotheses of section 2.1 above. The next lemma shows that uniform positivity of the running cost provides a simple sufficient condition for all solutions to have the $p$-monotone property. (The argument is embedded in the proof of [2, Theorem 5].) When the lemma applies, Theorem 4 becomes a simple comparison and uniqueness theorem for *all* viscosity solutions.

LEMMA 6. *Suppose that $H(x, p)$ is given by* (19)*, and that there exists a function $\sigma : \Omega \to [0, \infty)$ with the property that $0 < \inf_{\Omega_{\delta,\mathcal{T}}} \sigma(x)$ for each $\delta > 0$ and for which*

$$\sigma(x) \leq L(x, a, b)$$

*for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x \in \Omega$. Then every subsolution and every supersolution of* (9) *is $p$-monotone.*

Note that since $\sigma(x)$ is allowed to vanish on $\mathcal{T}$, the choice of $\mathcal{T}$ may affect the applicability of the lemma.

*Proof.* Suppose that $0 < s < 1$ and consider any $\zeta \in \mathbb{R}^n$. We have

$$\begin{aligned} -s\zeta \cdot f(a,b) - L(x,a,b) &= s\left[-\zeta \cdot f(a,b) - L(x,a,b)\right] - (1-s)L(x,a,b) \\ &\le s\left[-\zeta \cdot f(a,b) - L(x,a,b)\right] - (1-s)\sigma(x). \end{aligned}$$

Taking $\inf_{b\in\mathcal{B}} \sup_{a\in\mathcal{A}}$ yields $H(x, s\zeta) \le sH(x,\zeta) - (1-s)\sigma(x)$. Let $\eta_s(x) = (1-s)\sigma(x)$. We have

$$H(x, s\zeta) + \eta_s(x) \le sH(x,\zeta),$$

holding for all $\zeta$. It follows from this that any subsolution is a $p$-monotone subsolution.

The supersolution argument is analogous by using $1 < s$, $\eta_s(x) = (s-1)\sigma(x)$, with the appropriate inequalities reversed. ∎

**5. Examples.** We now discuss several examples, most of which are taken from existing literature, which illustrate the applicability and limitations of the above results. In all the examples, the Hamiltonian has the form $H(x,p) = H_0(p) - h(x)$, for which the hypotheses (5) are easy to verify. We omit those details, as well as the confirmations of the $B$-hypothesis and the coercivity hypothesis.

Numerous optimal control or differential game problems have been posed for "fluid limits" of queueing networks. The most common domain for these examples is the nonnegative orthant $\Omega = \mathbb{R}^d_+$. Being unbounded, this is outside the scope of our results above. Our first example makes the point that our main result, Theorem 4, can fail in unbounded domains.

*Example* 1. In Day [11] an example in two dimensions was considered for the Hamiltonian

$$H(x,p) = \frac{1}{2}\|p\|^2 - \frac{1}{2}\|x\|^2. \tag{20}$$

This arises as in (19) using

$$L(x,a,b) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|a\|^2, \quad f(x,a,b) = a, \tag{21}$$

with $\mathcal{A} = \mathbb{R}^2$. ($\mathcal{B}$ is irrelevant.) With $\mathcal{T} = \{(0,0)\}$ and $\sigma(x) = \frac{1}{2}\|x\|^2$ we see that Lemma 6 applies, and therefore *all* viscosity solutions are $p$-monotone. The equation, however, was considered in the *unbounded* half-space $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : \ x_1 \le 1\}$, using $d = (-1,0)$ ($= -\gamma(x)$ in the notation of [11]) for the boundary condition on $\partial\Omega$, and taking $g(0,0) = 0$. If Theorem 4 were valid for unbounded domains, solutions would be unique. However, in [11] it was shown that both $v(x) = \frac{1}{2}x_1^2 \pm \frac{1}{2}x_2^2$ are viscosity solutions.

The rest of our examples will use compact $\Omega$ as hypothesized. Examples 2–4 illustrate the applicability of Lemma 6.

*Example* 2. The "eikonal" equation

$$|Du(x)| - h(x) = 0, \quad u(x) = g(x) \text{ on } \partial\Omega,$$

with $h(x) > 0$ on $\Omega$ was cited above in reference to the approach of Ishii [18]. We simply observe that $H(x,p) = |p| - h(x)$ is obtained from (19) using $f(x,a,b) = a$, $a \in \mathcal{A} = \{a : |a| \le 1\}$, and $L(x,a,b) = h(x)$. ($\mathcal{B}$ is irrelevant.) Lemma 6 applies, so that all solutions are $p$-monotone, and Theorem 4 provides the usual comparison and uniqueness results for this Hamiltonian on bounded domains, for any choice of $\mathcal{T}$.

*Example* 3. The doctoral dissertation of Menendez [20] considers an example using dynamics of the form

$$f(x, a, b) = \lambda - Ga \tag{22}$$

in a bounded rectangle $\Omega$ in two dimensions. The running cost $L(x, a, b) = \frac{1}{2}|x|^2 + 1.1$ is strictly positive. The control set $\mathcal{A}$ is compact and there is no dependence on $b$. This problem again falls within the scope of Lemma 6 (regardless of $\mathcal{T}$), so that Theorem 4 applies to all viscosity solutions. Although [20] does not employ viscosity solution techniques, our results above show that they would be a viable alternative approach.

*Example* 4. A rather different problem is considered by Atar, Dupuis, and Schwartz [2]. Here a differential game is studied which provides an asymptotic description of a risk-sensitive stochastic control problem. In the stochastic control problem, reaching the target set $\mathcal{T}$ ($\partial_o G$ in their notation) is viewed as an event to be avoided, so the control attempts to maximize the time until this occurs. This becomes the maximizing player in the limiting game. The minimizing player emerges from the asymptotic analysis as the limiting representation of the random fluctuations.

The problem fits our format in the case that all the arrival parameters $\lambda_i$ are positive. (If some $\lambda_i = 0$, then different boundary conditions are to be used on some parts of $\partial\Omega \setminus \mathcal{T}$.) We recast their problem in our notation. $\Omega$ (their $G$) is the rectangle $\times_1^d [0, z_i]$ in $\mathbb{R}^d$. $\mathcal{T}$ consists of the portion of the boundary where $x_i = z_i$ for one or more coordinates. The $d_i$ are the $-\tilde{v}_i$ (below) for the respective faces $\partial_i\Omega = \{x : x_i = 0\}$. The maximizing player chooses the control $b = (u_1, \ldots, u_d)$ in a compact polygon $\mathcal{B}$. The minimizing player chooses a vector of rate perturbation factors $a = (\alpha_i^\lambda, \alpha_i^\mu : i = 1, \ldots, d)$, with $a \in \mathcal{A} = [0, \infty)^{2d}$. The state dynamics are

$$f(x, a, b) = \sum_i \lambda_i \alpha_i^\lambda e_i + \sum_i u_i \mu_i \alpha_i^\mu \tilde{v}_i,$$

where $e_i$ are the standard unit vectors in $\mathbb{R}^d$, and $\tilde{v}_i$ are the *service event vectors*, $\tilde{v}_i = e_{i'} - e_i$, where $i \to i'$ indicates the routing sequence in the network. The running cost is

$$L(x, a, b) = c + \sum_i \lambda_i \ell(\alpha_i^\lambda) + \sum_i u_i \mu_i \ell(\alpha_i^\mu),$$
$$\text{where } \ell(\alpha) = \alpha \log(\alpha) - \alpha + 1.$$

Here $c > 0$ is a positive constant, $\lambda_i > 0$, and $\mu_i \geq 0$, so $L(x, a, b) \geq c$. Thus the hypotheses of Lemma 6 are satisfied once again, so that Theorem 4 applies to all viscosity solutions.

Our last two examples are beyond the scope of Lemma 6, and the details are more involved. The following lemma will assist us in checking the boundary conditions for (locally) smooth solutions.

LEMMA 7. *Assume the coercivity condition* (6). *Suppose $x \in \partial\Omega$ and $u$ is continuously differentiable in a neighborhood of $x$.*

(a) *$\zeta \in D^+u(x)$ iff $\zeta = Du(x) + \sum_{i \in I(x)} \beta_i n_i$ for some choice of $\beta_i \geq 0$. Analogously, $\zeta \in D^-u(x)$ iff $\zeta = Du(x) - \sum_{i \in I(x)} \beta_i n_i$ for some $\beta_i \geq 0$.*

(b) *If $-d_i \cdot Du(x) \le 0$ for all $i \in I(x)$, then the viscosity subsolution property with boundary conditions holds as follows: for all $\zeta \in D^+u(x)$,*

$$H(x,\zeta) \wedge \min_{i\in I(x)} (-d_i \cdot \zeta) \le 0.$$

*Analogously, if $-d_i \cdot Du(x) \ge 0$ for all $i \in I(x)$, then for all $\zeta \in D^-u(x)$,*

$$H(x,\zeta) \vee \max_{i\in I(x)} (-d_i \cdot \zeta) \ge 0.$$

*Proof.* The proof of (a) is the first paragraph of the proof of [12, Theorem 2.1]. For (b), suppose that $-d_i \cdot Du(x) \le 0$ for all $i \in I(x)$ and consider any $\zeta \in D^+u(x)$. By (a) we know that $\zeta = Du(x) + \sum_{I(x)} \beta_i n_i$ with $\beta_i \ge 0$. We can assume some $\beta_i > 0$ for some $i \in I(x)$; else $-d_i \cdot \zeta = -d_i \cdot Du(x) \le 0$ follows directly. Observe that

$$\sum_{I(x)} \beta_i\, d_i \cdot \zeta = \left(\sum_{I(x)} \beta_i\, d_i \cdot Du(x)\right) + \left(\sum_{I(x)} \beta_i d_i\right) \cdot \left(\sum_{I(x)} \beta_i n_i\right).$$

By hypothesis, the first term on the right side is nonnegative. The last term is positive by the coercivity hypothesis and our assumption that $\beta_i > 0$ for some $i$. Therefore the left side is positive. This implies that $d_i \cdot \zeta > 0$ for some $i \in I(x)$. Consequently,

$$H(x,\zeta) \wedge \min_{i\in I(x)} (-d_i \cdot \zeta) \le 0,$$

regardless of the value of $H(x,\zeta)$. The supersolution case in (b) is argued analogously. ∎

*Example* 5. The recent papers [3], [13], and [12] of Day and others explore a robust control approach to fluid queueing models, using state dynamics of the form

$$f(x,a,b) = b - Ga,$$

a compact control space $\mathcal{A}$, and opposing quadratic costs for the state and "disturbance" $b \in \mathcal{B} = \mathbb{R}^n$ as follows:

$$L(x,a,b) = \frac{1}{2}\|x\|^2 - \frac{1}{2}\|b\|^2. \tag{23}$$

The resulting Hamiltonian is

$$\begin{aligned} H(x,p) &= \sup_{a\in\mathcal{A}} p \cdot Ga - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|p\|^2 \\ &= \frac{1}{2} \sup_{a\in\mathcal{A}} \left(\|Ga\|^2 - \|p - Ga\|^2 - \|x\|^2\right). \end{aligned} \tag{24}$$

Since $\|Ga\|$, $a \in \mathcal{A}$ is bounded, we see from the second form that $H(x,p) \ge 0$ implies a bound on $\|x\|$. Thus these problems are reasonable to consider *only* in bounded domains $\Omega$. The examples in the literature consider a bounded polygon $\Omega$ consisting of $x \in \mathbb{R}^d$ with $x_i \ge 0$ and $\eta \cdot x \le c$ for a particular vector $\eta$. In [3] and [13] the boundary $\eta \cdot x = c$ is omitted from $\Omega$ and in its place an admissibility condition is imposed on controls, which prohibits the state from approaching this missing boundary. (See the "minimum performance criterion" and its discussion in section 2.4 of [13].) In [12] all of $\partial\Omega$ is included, consistent with our formulation. Section 6 of [12] considers a specific example of the type considered here. We will need to take advantage of certain explicit calculations, which would be cumbersome for that example. Instead, we will consider a simple instance of the example(s) of [3, sections 1–3], modified to include all of $\partial\Omega$ in accordance with our hypotheses here.

We let $G$ be the $2\times 2$ identity matrix. (In [3] this corresponds to $s_i = \gamma = 1$.) The control set is $\mathcal{A} = \{(a_1, a_2) :\ 0 \le a_i,\ a_1 + a_2 = 1\}$. The Hamiltonian (24) simplifies to

$$H(x,p) = \max(p_1,p_2) - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|p\|^2. \tag{25}$$

We consider the planar domain

$$\Omega = \{x \in \mathbb{R}^2_+ :\ x_1 + x_2 \le r\}$$

for $r < 1$. If $r = 1$, then our $\Omega$ would be (the closure of) the domain considered in [3]. With $r < 1$ the domain here is slightly smaller. This reduction of the domain is important for the $p$-monotone property. We identify the faces and respective normal vectors as follows:

$$\begin{aligned}
&\partial_1\Omega = \{x \in \Omega :\ x_1 = 0\}, \quad n_1 = (1,0),\\
&\partial_2\Omega = \{x \in \Omega :\ x_1 = 0\}, \quad n_2 = (0,1),\\
&\partial_3\Omega = \{x \in \Omega :\ x_1 + x_2 = r\}, \quad n_3 = (-1/\sqrt{2}, -1/\sqrt{2}).
\end{aligned}$$

We take $d_i = n_i$ for all the faces. The target set will be the origin, $\mathcal{T} = \{(0,0)\}$, with $g(0,0) = 0$.

The constructions of [3] produce a $C^1$ solution to $H(x, Du(x)) = 0$. We will see that this is a $p$-monotone solution, even though Lemma 6 does not apply. We will first indicate briefly how the viscosity solution properties are verified, and then turn our attention to $p$-monotonicity. The solution is symmetric about the diagonal $x_1 = x_2$. We confine our discussion to the lower-right half of $\Omega$: $0 \le x_2 \le x_1 \le r$. The analysis on the other half follows by symmetry.

In the subregion $0 \le x_2 \le x_1 \le r$ the solution is most conveniently described in terms of the orthogonal basis,

$$\mu = (1/2, -1/2), \quad \eta = (1/2, 1/2).$$

(In the notation of [3, page 335], $\mu = \mu_1 = \eta_{\{1\}} - \eta_{\{1,2\}}$ and $\eta = \mu_2 = \eta_{\{1,2\}}$.) The gradient $Du(x)$ is related to $x$ in terms of parameters $0 \le t_1 \le t_2 \le \pi/2$ by the expressions

$$x = \sin(t_1)\mu + \sin(t_2)\eta, \quad Du(x) = [1 - \cos(t_1)]\mu + [1 - \cos(t_2)]\eta. \tag{26}$$

The parameters can be eliminated to obtain the explicit expressions for $0 \le x_2 \le x_1 \le r$,

$$\begin{aligned}
u(x) &= x_1 - \frac{1}{4}\Big(\sqrt{1-(x_1-x_2)^2}(x_1-x_2) + \sin^{-1}(x_1-x_2)\\
&\qquad\qquad + (x_1+x_2)\sqrt{1-(x_1+x_2)^2} + \sin^{-1}(x_1+x_2)\Big),\\
\frac{\partial u}{\partial x_1} &= 1 - \frac{1}{2}\left(\sqrt{1-(x_1-x_2)^2} + \sqrt{1-(x_1+x_2)^2}\right),\\
\frac{\partial u}{\partial x_2} &= \frac{1}{2}\left(\sqrt{1-(x_1-x_2)^2} - \sqrt{1-(x_1+x_2)^2}\right).
\end{aligned}$$

The parametric representation is more convenient for most purposes. For instance, observe that for $p = Du(x)$, $\max(p_1, p_2) = p_1$ is equivalent to $p\cdot\mu = \frac{1}{2}[1-\cos(t_1)] \geq 0$, which does hold. Therefore,

$$H(x,p) = p_1 - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|p\|^2 = p\cdot(\mu+\eta) - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|p\|^2. \tag{27}$$

It is now straightforward to evaluate this, using the orthogonality of $\mu$ and $\eta$ to confirm that $H(x, Du(x)) = 0$. The explicit formulae provide the easiest way to check that

$$\partial u/\partial x_i \geq 0 \text{ for both } i, \tag{28}$$

because $x_2 \leq x_1$, and

$$\partial u/\partial x_2 = 0 \text{ when } x_2 = 0. \tag{29}$$

By Lemma 7(b), (29) implies that the viscosity boundary conditions are satisfied on $\partial_2\Omega$. On $\partial_3\Omega$ we have from (28) that $-d_3 \cdot Du(x) \geq 0$, so that the supersolution boundary condition is satisfied there, as well as at the corner $(r, 0)$.

The subsolution property on $\partial_3\Omega$ and at the corner takes more careful examination. For these $x$ we need to identify the $\zeta \in D^+u(x)$, for which $-d_i \cdot Du(x) > 0$ for all $i \in I(x)$, and for these we need to check that $H(x,\zeta) \leq 0$ holds. Consider the corner $x = (r, 0)$ specifically. From the explicit formulas, $Du(x) = (1 - \sqrt{1-r^2}, 0)$. By Lemma 7, the $\zeta \in D^+u(x)$ are

$$\zeta = Du(x) + \beta_2 n_2 + \beta_3 n_3, \quad \beta_i \geq 0.$$

One finds that the $\zeta$ with $\beta_i \geq 0$ and $-d_i \cdot \zeta > 0$ comprise the triangle in the $\zeta$-plane with vertices $(0,0)$, $Du(x) = [1-\sqrt{1-r^2}](1,0)$, and $[1-\sqrt{1-r^2}](\frac{1}{2}, -\frac{1}{2})$. For future reference, notice that all such $\zeta$ satisfy

$$\|\zeta\| \leq \|Du(x)\|. \tag{30}$$

What we need at the moment is that $\zeta_1 \leq 0 \leq \zeta_2$, so that just as in (27),

$$H(x,\zeta) = \zeta\cdot(\mu+\eta) - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|\zeta\|^2.$$

For the particular $\zeta$ identified above, this works out to be

$$H(x,\zeta) = \frac{1}{2}\left(-\beta_2{}^2 + \sqrt{2}\beta_3\beta_2 - \beta_3\left(\beta_3 + \sqrt{2}\sqrt{1-r^2}\right)\right),$$

from which one may verify that $H(x,\zeta) \leq 0$ for all $\beta_i \geq 0$. This confirms the viscosity subsolution property at the corner.

For $x \in \partial_3\Omega$ with $x_2 \leq x_1 < r$ the calculations are similar but simpler. The $\zeta \in D^+u(x)$ with $-d_3 \cdot \zeta > 0$ are $\zeta = Du(x) + \beta_3 n_3$ with

$$0 \leq \sqrt{2}\beta_3 < 1 - \cos(t_2). \tag{31}$$

Since $n_3 = -\sqrt{2}\eta$, we have

$$\zeta = [1-\cos(t_1)]\mu + \left[1 - \cos(t_2) - \sqrt{2}\beta_3\right]\eta. \tag{32}$$

Notice that (32) implies that (30) again holds. Since $\mu \cdot \zeta = \mu \cdot Du(x) \geq 0$, we can again work out that

$$\begin{aligned} H(x,\zeta) &= \zeta \cdot (\mu + \eta) - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|\zeta\|^2 \\ &= \frac{-1}{2}\beta_3 \left[\sqrt{2}\cos(t_1) + \beta_3\right] \leq 0 \text{ since } \beta_3 \geq 0. \end{aligned}$$

This completes the verification that $u(x)$ is a viscosity solution to our problem.

We now consider $p$-monotonicity. Note that due to the $-\|b\|^2$ term, there is no finite lower bound for the $L$ of (23). Thus Lemma 6 does *not* apply. Even so, we will see that $u(x)$ *is* a $p$-monotone solution. Observe that for any $s > 0$, we have $\max(s\zeta_1, s\zeta_2) = s\max(\zeta_1, \zeta_2)$. As a consequence we have the following identity:

$$H(x, s\zeta) = sH(x,\zeta) + (s-1)\left[\frac{1}{2}\|x\|^2 - \frac{s}{2}\|\zeta\|^2\right]. \tag{33}$$

Consider the supersolution $p$-monotonicity property first. As observed above, $-d_i \cdot Du(x) \geq 0$ on all boundary faces, so that only the interior points are involved in the $p$-monotonicity supersolution property. Since $H(x, Du(x)) = 0$, we see from (33) that $p$-monotonicity requires that

$$0 < \frac{1}{2}\|x\|^2 - \frac{s}{2}\|Du(x)\|^2 \tag{34}$$

for $x \neq 0$ and $s \approx 1$. For $s = 1$ this is the *positive storage condition* (see [3, (2.25)] and [12, (33)]), which was important for the verification results obtained in those papers. Here we are interested in $1 < s$. The parametric representation of $Du(x)$ allows us to check (34) directly as follows:

$$\frac{1}{2}\|x\|^2 - \frac{s}{2}\|Du(x)\|^2 = \frac{1}{4}[\sin(t_1)^2 - s(1-\cos(t_1))^2] + \frac{1}{4}[\sin(t_2)^2 - s(1-\cos(t_2))^2].$$

Now $0 \leq t_t \leq t_2$ and $\frac{1}{2}\sin(t_2) = \eta \cdot x \leq \frac{r}{2}$. Thus $t_i \leq \sin^{-1}(r) < \frac{\pi}{2}$, since $r < 1$. It is elementary to check that there exists $\delta_0 > 0$, so that

$$\sin(t)^2 - s(1-\cos(t))^2 > 0$$

for all $0 \leq t \leq \sin^{-1}(r)$ and all $0 < s < 1 + \delta_0$. This implies that (34) holds, and so $u(x)$ is indeed a $p$-monotone supersolution, using

$$\eta_s(x) = (s-1)\left[\frac{1}{2}\|x\|^2 - \frac{s}{2}\|Du(x)\|^2\right].$$

Finally, consider the subsolution $p$-monotone property. Based on (33), for $s-1 < 0$, we need to know that, for $\zeta \in D^+u(x)$ with $-d_i \cdot \zeta > 0$ all $i \in I(x)$,

$$0 < \frac{1}{2}\|x\|^2 - \frac{s}{2}\|\zeta\|^2.$$

But we observed in (30) above that for all such $\zeta$, $\|\zeta\| \leq \|Du(x)\|$, and so

$$\frac{1}{2}\|x\|^2 - \frac{s}{2}\|\zeta\|^2 \geq \frac{1}{2}\|x\|^2 - \frac{s}{2}\|Du(x)\|^2.$$

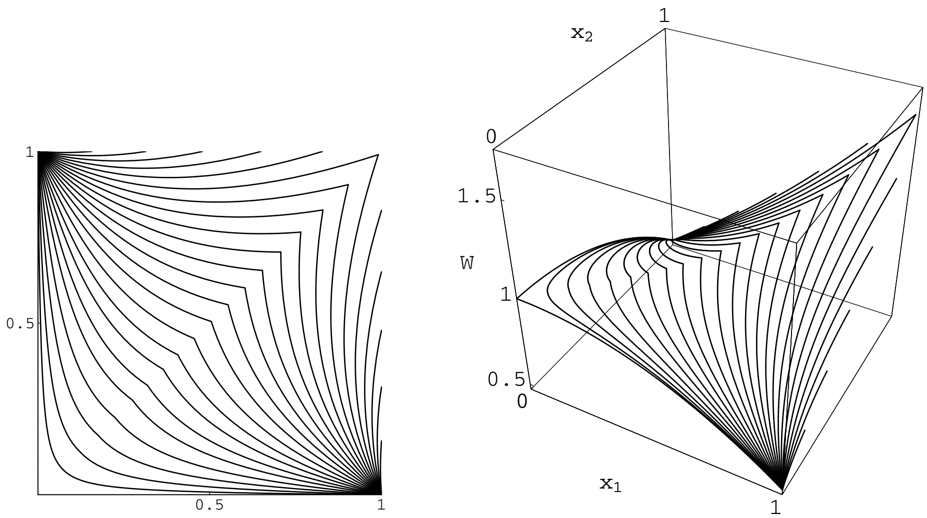


FIG. 1. $w(x)$.

Thus we can again use

$$\eta_s(x) = (s-1)\left[\frac{1}{2}\|x\|^2 - \frac{s}{2}\|Du(x)\|^2\right],$$

which is strictly positive on $\Omega \setminus \mathcal{T}$, as shown above.

In summary, $u(x)$ is a $p$-monotone viscosity solution and hence the complete solution of our problem.

Finally, we offer a new example which exhibits nonuniqueness of solutions when no $p$-monotone solution exists, but for which comparisons based on $p$-monotonicity properties are still possible.

*Example* 6. We return to the Hamiltonian (20) but consider the cube

$$\Omega = \{(x_1, , x_2) : \ 0 \le x_i \le 1\}.$$

We number the boundary faces as

$$\begin{aligned} \partial_1\Omega &= \{x \in \Omega : \ x_1 = 0\}, \quad \partial_2\Omega = \{x \in \Omega : \ x_2 = 0\}, \\ \partial_3\Omega &= \{x \in \Omega : \ x_1 = 1\}, \quad \partial_4\Omega = \{x \in \Omega : \ x_2 = 1\}. \end{aligned}$$

The normals are $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (-1,0)$, and $n_4 = (0,-1)$, and we take $d_i = n_i$ for all faces. Consider the target set consisting of the two off-diagonal corners, $\mathcal{T} = \{(1,0),(0,1)\}$, taking $g = \frac{1}{2}$ at both corners.

It is elementary to check that $v(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is a classical solution of $0 = H(x, Dv(x))$ in the interior of $\Omega$. A second solution $w$ is illustrated in Figure 1. It is symmetric about the diagonal line $\Gamma = \{x \in \Omega : \ x_1 = x_2\}$, but is nondifferentiable on $\Gamma$ (and at the corners in $\mathcal{T}$). In the upper left triangle, $0 \le x_1 \le x_2 \le 1$, it is constructed from the family of characteristics (illustrated in the left pane of Figure 1),

$$\begin{aligned} &\dot{x} = H_p(x,p) = p; && x(0) = (0,1), \\ &\dot{p} = -H_x(x,p) = x; && p(0) = (\cos(\theta), -\sin(\theta)), \ 0 \le \theta \le \pi/2, \\ &\dot{w} = p \cdot \dot{x}; && w(0) = 1/2 = g(x(0)), \end{aligned}$$

and extended by symmetry across $\Gamma$. It turns out that *both* $v$ and $w$ are viscosity solutions of $H(x, Du(x)) = 0$ in $\Omega \setminus \mathcal{T}$ with $-d_i \cdot Du(x) = 0$ on $\partial\Omega \setminus \mathcal{T}$ and $u = g$ on $\mathcal{T}$. Moreover, $v$ satisfies the oblique derivative boundary conditions at the points of $\mathcal{T}$ as well. The verification of these assertions is similar to that of the previous example; we omit it for brevity.

In light of Theorem 4, neither $v$ nor $w$ can be a $p$-monotone solution. Lemma 6 does not apply here since $x = (0, 0)$ does not belong to $\mathcal{T}$ and $L$ of (21) has no positive lower bound at this point. In fact, neither $v$ nor $w$ is a $p$-monotone solution in either the sub- or supersolution sense. In order for $v$ to be a $p$-monotone subsolution we would need, for $0 < s < 1$, a function $\eta_s(x) > 0$ (off $\mathcal{T}$) with

$$H(x, sDv(x)) \leq -\eta_s(x), \quad x \in \Omega^\circ.$$

Now

$$H(x, sp) = s^2 \frac{1}{2}\|p\|^2 - \frac{1}{2}\|x\|^2 = s^2 H(x, p) + \frac{s^2 - 1}{2}\|x\|^2.$$

Since $H(x, Dv(x)) = 0$,

$$H(x, sDv(x)) = \frac{s^2 - 1}{2}\|x\|^2.$$

Thus we would need $\frac{s^2-1}{2}\|x\|^2 \leq -\eta_s(x)$ to be *uniformly* negative in a neighborhood of $(0, 0)$. This is clearly not possible. The same argument applies to $w$ if we keep $x$ off the diagonal. For the supersolution case, we would need $\frac{s^2-1}{2}\|x\|^2 \geq \eta_s(x)$ to be uniformly positive in a neighborhood of $(0, 0)$, which is likewise impossible.

In Figure 2 we have plotted both solutions $v$ and $w$. It is apparent that $v \leq w$. This can be deduced from Theorem 4 by considering the enlarged target set $\mathcal{T}' = \{(0, 0),\ (0, 1),\ (1, 0)\}$. Now Lemma 6 *does* apply; both $v$ and $w$ are $p$-monotone for this $\mathcal{T}'$. If we take $g(0, 0) = 0 = v(0, 0)$, then $v$ is the complete solution of the

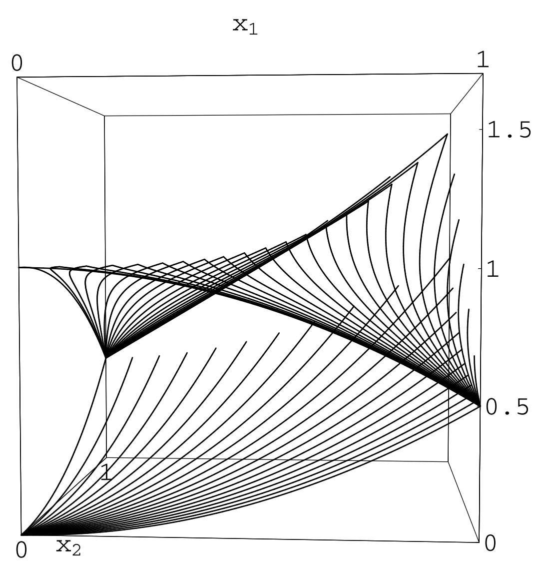


FIG. 2. $v(x) \leq w(x)$.

problem, but since $w(0,0) = 1 > g(0,0)$, $w$ is only a supersolution. Thus $v \leq w$ follows from the comparison theorem. If instead we take $g(0,0) = 1$, then $w$ is the complete solution. We obtain a supersolution by adding a constant to $v$ as follows: $\tilde{v}(x) = 1 + v(x)$. In that case, $\tilde{v} \geq g$ on $\mathcal{T}$, so that the comparison theorem implies $v + 1 \geq w$.

Also consider the target set $\mathcal{T}'' = \{(0,0)\}$ consisting of the origin alone, with $g(0,0) = 0$. As above, Lemma 6 applies, so that $v$ is the complete solution. According to Theorem 4, there can be no other viscosity solutions. Adding a constant, $w - 1$ conforms to $g$ at the origin. But investigation of the corners $(0,1)$ and $(1,0)$ shows that the supersolution condition fails there (details omitted). It is, however, a subsolution, which implies $w - 1 \leq v$, as we already deduced above.

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