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A STATE SPACE ERROR ESTIMATE FOR POD-DEIM NONLINEAR MODEL REDUCTION*

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Abstract. This paper derives state space error bounds for the solutions of reduced systems constructed using proper orthogonal decomposition (POD) together with the discrete empirical interpolation method (DEIM) recently developed for nonlinear dynamical systems [*SIAM J. Sci. Comput.*, 32 (2010), pp. 2737–2764]. The resulting error estimates are shown to be proportional to the sums of the singular values corresponding to neglected POD basis vectors both in Galerkin projection of the reduced system and in the DEIM approximation of the nonlinear term. The analysis is particularly relevant to ODE systems arising from spatial discretizations of parabolic PDEs. The derivation clearly identifies where the parabolicity is crucial. It also explains how the DEIM approximation error involving the nonlinear term comes into play.

Key words. nonlinear model reduction, proper orthogonal decomposition, empirical interpolation methods, nonlinear partial differential equations

AMS subject classifications. 65L02, 65M02

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1. Introduction.

Model order reduction (MOR) can effectively reduce computational costs for simulation of dynamical systems. Such reductions are usually achieved through some approximation scheme that produces a reduced dimension dynamical system with nearly the same response characteristics as the original. Although there are many instances where the primary interest is in preserving input-output relations, there is generally an interest in obtaining good approximations to the state variables as well. Here, we analyze the state approximation error for the discrete empirical interpolation method (DEIM) which is a dimension reduction method for large systems of nonlinear ordinary differential equations (ODEs). In particular, we analyze the DEIM state approximation error of systems of ODEs arising from a spatial discretization of time dependent nonlinear parabolic partial differential equations (PDEs) (e.g., convection-diffusion-reaction equations). We provide a bound on the global state space approximation error as well as a uniform bound on the pointwise error over a specified time interval. The analysis shows precisely where the parabolicity is crucial and it shows exactly how the DEIM approximation error involving the nonlinear term comes into play. The ultimate result lends rigor to the heuristic notion that the error in trajectory approximation should be proportional to the first neglected singular value in the truncated singular value decomposition (SVD) used to obtain the reduced basis.

The DEIM can be viewed as a modification of proper orthogonal decomposition (POD). In POD, a reduced basis is constructed from a truncated SVD of a matrix of sampled trajectory vectors (snapshots). The standard POD–Galerkin approach with this reduced basis can provide a significant reduction in the number of variables,

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but for many problems (including the broad class of ODEs coming from spatial discretization of convection-diffusion-reaction equations), the complexity of evaluating the reduced dimension nonlinear terms remains that of the original high dimensional system. As a result, the reduced system is just as expensive to solve as the original, and there is little or no reduction in simulation time. The DEIM resolves this complexity issue. It does so by replacing the orthogonal projection of POD with an interpolatory projection. The result enables the nonlinear term to be evaluated at just a few selected component functions and thus greatly reduces the complexity.

Recently, DEIM has been successfully used for nonlinear model reduction in the application of neural modeling of full Hodgkin–Huxley models for realistic spiking neurons [15] and in the application of two-phase miscible flow in porous media with varying Peclet number, both with and without chemistry at the interface of the different fluids [2]. Galbally et al. [8] applied a closely related method called masked projection and obtained reductions of order 10^5 in both dimension and CPU time for a convection diffusion reaction problem. Hinze and Kunkel [12] apply the DEIM to a POD MOR of drift-diffusion equations in electrical networks. Their results also verify this preservation of accuracy and they achieve significant reduction in simulation times over the original POD reduced system.

DEIM is a discrete variant of the empirical interpolation method (EIM) proposed by Barrault, Maday, Nguyen, and Patera [1], where an a priori error bound and an a posteriori error estimate were proposed for EIM coefficient-function approximation with respect to the basis from a greedy selection process. The proposed a priori error bound is expressed in terms of a constant times the optimal error in the L_∞-norm. Their proposed a posteriori error estimate is theoretically applicable only for particular parametrized functions that lie in the space of snapshots. A more general a posteriori EIM error bound was also recently proposed in [6], where the derivation was based on Taylor's expansion around certain parameter values. This error bound only requires functions to be parametrically smooth. From the numerical examples in [6], obtaining a sharp bound could be expensive, since it would require a high-degree Taylor expansion with exact evaluations of corresponding function derivatives at many samples in the parameter domain. The incorporation of the error estimates from [1] for EIM function approximation into the reduced basis framework [23, 26, 22] to obtain error bounds of the outputs from nonlinear systems or linear systems with nonaffine parameter dependence can be found in [21, 20, 9]. An error analysis for an application of EIM is also given in [11] for a time-dependent evolution scheme with explicit discretization operators.

Error estimates for reduced systems constructed from projection methods, particularly those based on a POD–Galerkin technique, have been studied in a number of existing works. In [19], the authors applied the dual-weighted-residual method which uses the solution of a dual or adjoint system to obtain an error estimate for the solutions from POD reduced models of nonlinear systems. In [24], the error bounds of solutions from a POD reduced system were derived and the effects of small perturbations on the set of snapshots used for constructing the POD basis were studied. Subsequent work [14] proposed an alternative error estimation based on an adjoint method combined with the method of small sample statistical condition estimation. It also analyzed further the effect of perturbations in both the initial conditions and parameters on the resulting POD reduced system. However, the analysis in [14] is based on linearization, and hence, large perturbations may require some knowledge of the solution of the perturbed system. Some related works on error estimations such as in [30, 7, 19, 13] can be found in the extensive review from [14].

In [16, 17], Kunisch and Volkwein derive error estimates for a POD reduced system in a function space setting for a class of nonlinear parabolic PDEs. Their analysis considers cases where the snapshots and the POD basis are in general Hilbert space. Kunisch and Volkwein also considered a snapshot set that included finite difference quotients of the snapshots in response to their theoretical error bounds derived for the state solutions from the POD–Galerkin reduced system. The approximation errors were expressed as the contributions from the POD subspace approximation error and from time discretization error. The theoretical results in [17] provide asymptotic error estimates that do not depend on the snapshot set and demonstrate the effect of two different time discretizations used to produce the set of snapshots and for the numerical integration of the reduced system. Nonlinear problems with Lipschitz continuous nonlinearities are considered in [16] and extended to the Navier–Stokes equations in [17]. Similar approaches for deriving the error estimates in function space setting from [16, 17] were later applied to the finite dimensional Euclidean space setting in [31].

Here we extend the error analysis of Kunisch and Volkwein in [31] for POD reduced systems to the POD-DEIM reduced systems presented in [3] for ODEs with Lipschitz continuous nonlinearities. In our earlier work [3], we derived an error bound for the DEIM approximation of a nonlinear vector-valued function which shows it is nearly as accurate as the optimal orthogonal projection in POD approximation. This error bound is used in the analysis presented here to establish the global accuracy of the POD-DEIM reduced system. To isolate the effect of time discretization error, we first compare the exact solutions of the resulting POD-DEIM reduced system with the exact solutions of the original full-order system. Of course, the exact solutions are generally impossible to obtain in practice. We therefore also compare the *numerical solutions* from the discretized reduced system obtained from the same numerical scheme as the original discretized system in which the snapshots are collected. Since these snapshots are also *numerical solutions*, the resulting error bounds here may not have a clear connection with the truncation error of the time integration scheme as obtained in those previous works [16, 17, 31], which compared the solutions from discretized POD reduced system with the *exact solutions* from the original full-order ODE. Our analysis is illustrated here for the case when the implicit Euler scheme is used for time discretization. Similar error bounds can be obtained for other discretization schemes.

We shall use $\|\cdot\|$ to denote the 2-norm in Euclidean space throughout this paper. The 2-norm error estimates presented here are shown to be proportional to the sums of the singular values corresponding to neglected POD basis vectors both in Galerkin projection of the reduced system and in DEIM approximation of the nonlinear term. The separate POD basis used in DEIM to approximate the nonlinearity is very closely related to the Kunisch–Volkwein inclusion [31] of finite difference snapshots $(\mathbf{y}_{j+1} - \mathbf{y}_j)/h$ into the snapshot set, since $(\mathbf{y}_{j+1} - \mathbf{y}_j)/h \approx \dot{\mathbf{y}}(t_j) = \mathbf{f}(\mathbf{y}_j)$, where $\mathbf{y}_j \approx \mathbf{y}(t_j)$ and $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$.

Two different forms of vector fields in the nonlinear ODEs will be considered here. One is in a form that separates linear and nonlinear terms; the other is in a form that does not (i.e., with one single nonlinear term that possibly contains a linear part). As shown in sections 3 and 4, the only difference of the resulting POD-DEIM reduced systems from these two settings is in the linear term. In particular, in the case of separated linear/nonlinear vector field, only POD approximation will be applied to the linear term, while in the case of a nonseparated vector field, both POD and DEIM approximation will be applied to the linear term. Therefore, the POD-

DEIM reduced system constructed from the setting with a separated vector field is generally more accurate than the one constructed from the nonseparated form. The error analysis for this separated form is given in section 3. This separation allows us to apply the notion of *logarithmic norm* [4] to the linear term, which can further give a computable realistic error bound when the Hermitian part of the linear coefficient matrix is negative definite. In section 4, error estimates are given for the second case when the linear/nonlinear decomposition in the vector field is not available. The derivations for error bounds in this case are based on a generalized concept of *logarithmic Lipschitz constant* [27, 28]. This concept has been used previously in [25] to derive error estimates of the trajectory piecewise-linear approach for nonlinear model reduction in circuit simulations.

2. Problem formulation. We shall first consider systems of nonlinear ODEs of the form

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{F}(t, \mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{y}_0, \qquad \text{for } t \in [0, T], \tag{2.1}$$

where the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is constant and the nonlinear function $\mathbf{F} : [0, T] \to \mathcal{Y}$ is assumed to be uniformly Lipschitz continuous with respect to the second argument with Lipschitz constant $L_f > 0$ and $\mathcal{Y} \subseteq \mathbb{R}^n$. That is, for $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$ and for all $t \in [0, T]$

$$\|\mathbf{F}(t, \mathbf{y}_1) - \mathbf{F}(t, \mathbf{y}_2)\| \leq L_f \|\mathbf{y}_1 - \mathbf{y}_2\|. \tag{2.2}$$

Although the decomposition into a linear term $\mathbf{A}$ and nonlinear term $\mathbf{F}$ above may seem arbitrary, we are thinking of the matrix $\mathbf{A}$ as the discretization of the part of the right-hand side that is naturally linear (such as the Laplacian). Ideally, we would want to separate all the linear terms from the nonlinear terms. This is because in the POD-DEIM approach [3], DEIM will be used further to approximate anything that is treated as a part of the nonlinear term and may destroy structure in the linear term that is normally preserved by the POD approximation. For the ODE systems arising from the discretization of PDEs, the linear coefficient matrices often come from the mass matrix, stiffness matrix, or discrete Laplacian matrix, depending on the spatial discretization schemes. These matrices often have special structures with some desired properties for the error analysis that shall be developed here.

In the POD-DEIM approach [3], two POD bases are derived. The first of these consists of the columns of the $n \times k$ orthogonal matrix $\mathbf{V}$ obtained from a truncated SVD of the trajectory snapshot matrix $\mathbb{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n_s}] \in \mathbb{R}^{n\times n_s}$ while the second consists of the columns of the $n \times m$ orthogonal matrix $\mathbf{U}$ obtained from a truncated SVD of the nonlinear snapshot matrix $\mathbb{F} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{n_s}] \in \mathbb{R}^{n\times n_s}$, where $\mathbf{y}_j \approx \mathbf{y}(t_j)$ and $\mathbf{f}_j = \mathbf{F}(t_j, \mathbf{y}_j)$ with $k, m \leq \min\{n, n_s\}$. The corresponding POD-DEIM reduced system is constructed by applying Galerkin projection on the space spanned by columns of the POD basis matrix $\mathbf{V}$ and then applying the DEIM approximation to the nonlinear function using interpolation projection onto the column space of the POD basis matrix $\mathbf{U}$. The resulting reduced system is then given by

$$\frac{d}{dt}\widehat{\mathbf{y}}(t) = \widehat{\mathbf{A}}\widehat{\mathbf{y}}(t) + \mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}(t)), \qquad \widehat{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0, \qquad \text{for } t \in [0, T], \tag{2.3}$$

where $\widehat{\mathbf{A}} := \mathbf{V}^T\mathbf{A}\mathbf{V} \in \mathbb{R}^{k\times k}$, $\mathbb{P} := \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T \in \mathbb{R}^{n\times m}$, and $\mathbf{P} \in \mathbb{R}^{n\times m}$ is a matrix whose columns come from some selected columns of the identity matrix corresponding to the DEIM indices. Note that in actual computation, the quantity

$\mathbf{V}^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1} \in \mathbb{R}^{k\times m}$ in the nonlinear term would be precomputed and stored, so that the computational cost in solving (2.3) is only proportional to the reduced dimensions k and m (and not the original dimension n) as explained in [3]. However, for error analysis purposes, we will consider the nonlinear term written in the form as given in (2.3). Notice that if $m = n$, then $\mathbb{P}$ is equal to the n-by-n identity matrix and the system in (2.3) is just a reduced system constructed with the standard POD–Galerkin approach. Hence, the error analysis presented here is also applicable to the POD–Galerkin approach. Recall that the Lipschitz continuity assumption on $\mathbf{F}$ in the original system (2.1) will guarantee the existence and uniqueness of the solution from the original system (by, e.g., the Picard–Lindelöf theorem). The Lipschitz continuity of $\mathbf{F}$ is inherited by the reduced order nonlinear term $\widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t)) := \mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}(t))$, since $\|\widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}_1(t)) - \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}_2(t))\| = \|\mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}_1(t)) - \mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}_2(t))\| \le L_f\|\mathbb{P}\|\|\widehat{\mathbf{y}}_1(t) - \widehat{\mathbf{y}}_2(t)\|$ for all $t \in [0, T]$, where $\|\mathbb{P}\|$ is a bounded constant as shown in [3] and we have used the fact that $\mathbf{V}$ has orthonormal columns. Thus, existence and uniqueness of the solution to the POD-DEIM reduced system (2.3) will also be inherited.

The solution $\mathbf{y}(t)$ of the original full-order system (2.1) is then approximated by $\mathbf{V}\widehat{\mathbf{y}}$, where $\widehat{\mathbf{y}}$ is the solution from the POD-DEIM reduced system (2.3). The accuracy of this approximation therefore can be measured by considering the error $\|\mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)\|$ for $t \in [0, T]$. The bounds for this DEIM state space error will be the main focus of our analysis. The derivation framework of the error bounds which will be presented later in this paper can be applied to the case when other matrices with orthonormal columns are used in place of these POD basis matrices. This derivation also can be extended to a more general class of parametrized ODE systems. We shall begin with a brief review, along with the development of some relevant notations, for POD and DEIM approximations.

2.1. POD. POD constructs an orthonormal basis that can represent dominant characteristics of the *space of expected solutions* which is defined as Range($\mathbb{Y}$), the span of the snapshots. Recall that snapshots are numerically sampled values of the trajectory $\mathbf{y}(\cdot)$ at particular time steps and at particular parameter values. Let $r = \operatorname{rank}\{\mathbb{Y}\}$. POD gives an optimal set of basis vectors $\{\mathbf{v}_i\}_{i=1}^k \subset \mathbb{R}^n$ minimizing the mean square error associated with approximating these snapshots for $k < r$. In particular, the POD basis matrix $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_k] \in \mathbb{R}^{n\times k}$ solves the minimization problem

$$\min_{\operatorname{rank}\{\mathbf{V}\}=k} \sum_{j=1}^{n_s} \|\mathbf{y}_j - \mathbf{V}\mathbf{V}^T\mathbf{y}_j\|^2 \qquad \text{s.t.} \qquad \mathbf{V}^T\mathbf{V} = \mathbf{I}_k, \tag{2.4}$$

where $\mathbf{I}_k \in \mathbb{R}^{k\times k}$ is an identity matrix. We recommend [17] for more detail on POD. Here, we shall work in a finite dimensional setting where POD is essentially the same as a truncated SVD. Specifically, a POD basis of dimension k for (2.4) is just a set of left singular vectors corresponding to the first k dominant singular values of $\mathbb{Y}$, the snapshot matrix. The minimum 2-norm error from approximating the snapshots using the POD basis is given by

$$\sum_{j=1}^{n_s} \|\mathbf{y}_j - \mathbf{V}\mathbf{V}^T\mathbf{y}_j\|^2 = \sum_{i=k+1}^{r} \sigma_i^2 = \sum_{i=k+1}^{r} \lambda_i \tag{2.5}$$

for $k < r$, where $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_k] \in \mathbb{R}^{n\times k}$; $\mathbf{v}_1, \mathbf{v}_1, \dots, \mathbf{v}_r \in \mathbb{R}^n$ are the singular vectors corresponding to the nonzero singular values $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$ of

$\mathbb{Y}$; and $\lambda_i = \sigma_i^2$, $i = 1, \ldots, r$. In the large-scale setting, we recommend using the MATLAB routine `svds` (or ARPACK) to just compute the dominant singular values and vectors of $\mathbb{Y}$. If $n \leq n_s$ one only need compute matrix-vector products of the form $\mathbf{w} = \mathbb{Y}(\mathbb{Y}^T\mathbf{v})$, while if $n > n_s$, it is usually more efficient to compute the dominant singular values and vectors of $\mathbb{Y}^T$ which will only require matrix-vector products of the form $\mathbf{w} = \mathbb{Y}^T(\mathbb{Y}\mathbf{v})$.

Ultimately, we shall analyze the discrete setting and include effects of numerical approximation. However, it is illuminating to first consider an ideal setting where we assume the entire continuous trajectory is available on the interval $[0, T]$. This will isolate the contribution to the error resulting solely from application of the POD-DEIM model reduction technique without introducing the effect from discretization error or the choice of snapshots. Our analysis will require the POD basis to satisfy the minimization problem

$$\min_{\text{rank}\{\mathbf{V}\}=k} \int_0^T \|\mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)\|^2 dt \qquad \text{s.t.} \qquad \mathbf{V}^T\mathbf{V} = \mathbf{I}_k, \tag{2.6}$$

as in, e.g., [16, 17, 31]. It is well known [17] that the POD basis which solves (2.6) is the set of first k dominant eigenvectors of the symmetric matrix $\mathbf{R} := \int_0^T \mathbf{y}(t)\mathbf{y}(t)^T dt \in \mathbb{R}^{n\times n}$. Using the notation established in [17], let $r = \text{rank}\{\mathbf{R}\}$ and let $\lambda_1^\infty \geq \lambda_2^\infty \geq \cdots \geq \lambda_r^\infty > 0$ be the nonzero eigenvalues of $\mathbf{R}$ with the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in \mathbb{R}^n$. Then, as in (2.5), the POD basis matrix for (2.6) is $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_k] \in \mathbb{R}^{n\times k}$ for $k < r$, and the corresponding minimum 2-norm error is

$$\int_0^T \|\mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)\|^2 dt = \sum_{i=k+1}^{r} \lambda_i^\infty. \tag{2.7}$$

The connection between (2.5) and (2.7) was demonstrated in [17] when the sampled snapshots used for (2.4) are sufficiently dense in $[0, T]$. In particular, for a fixed time step, $\sum_{i=k+1}^r \lambda_i \leq 2\sum_{i=k+1}^r \lambda_i^\infty$ when $n_s > \bar{n}_s$ for some sufficient large value $\bar{n}_s$.

The application of DEIM also requires use of the POD basis matrix $\mathbf{U} \in \mathbb{R}^{n\times m}$ of nonlinear snapshots $\mathbf{f}_j = \mathbf{F}(t_j, \mathbf{y}_j)$, $t_j \in [0, T]$. For the nonlinear snapshot matrix $\mathbb{F} = [\mathbf{f}_1, \ldots, \mathbf{f}_{n_s}] \in \mathbb{R}^{n\times n_s}$, the columns of $\mathbf{U}$ are the eigenvectors of $\mathbb{F}\mathbb{F}^T$ corresponding to the first m dominant eigenvalues $s_1 \geq \cdots \geq s_m \geq \cdots \geq s_{r_s} > 0$, $r_s := \text{rank}\{\mathbb{F}\}$. The 2-norm error is given by

$$\sum_{j=1}^{n_s} \|\mathbf{f}_j - \mathbf{U}\mathbf{U}^T\mathbf{f}_j\|^2 = \sum_{i=m+1}^{r_s} s_i. \tag{2.8}$$

As with the trajectory, we also consider the POD basis matrix $\mathbf{U}$ satisfying the condition (2.6) for the entire set of continuous nonlinear snapshots $\mathbf{f}(t) = \mathbf{F}(t, \mathbf{y}(t))$, $t \in [0, T]$. Let $s_1^\infty \geq s_2^\infty \geq \cdots \geq s_{r_s}^\infty > 0$ be the r_s nonzero eigenvalues of $\int_0^T \mathbf{f}(t)\mathbf{f}(t)^T dt \in \mathbb{R}^{n\times n}$ so that we have

$$\int_0^T \|\mathbf{f}(t) - \mathbf{U}\mathbf{U}^T\mathbf{f}(t)\|^2 dt = \sum_{i=m+1}^{r_s} s_i^\infty. \tag{2.9}$$

2.2. DEIM. DEIM was introduced in [3] to provide an approximation to the nonlinear terms of ODE systems in a form that enables precomputation of certain matrices so that the computational cost of evaluating the nonlinear terms is greatly decreased and is also independent of the original dimension n. DEIM is a discrete variant of the EIM proposed by Barrault et al. in [1]. Evaluating the approximate nonlinear term from DEIM does not require a prolongation of the reduced state variables back to the original high dimensional state space as is required to evaluate the nonlinearity in the original POD approximation. Only a few entries of the original nonlinear term corresponding to the specially selected interpolation indices from the DEIM must be evaluated at each time step.

Let $\mathbf{f}(t) \in \mathbb{R}^n$ be a nonlinear vector-valued function for $t \in [0, T]$ for some positive final time T. Let $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_m] \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}\{\mathbf{U}\} = m$. The DEIM approximation of order $m \le n$ in $\operatorname{span}\{\mathbf{U}\}$ for $\mathbf{f}$ is given by

$$\widehat{\mathbf{f}}(t) := \mathbb{P}\mathbf{f}(t), \tag{2.10}$$

where $\mathbb{P} = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T$ and $\mathbf{P} = [\mathbf{e}_{\wp_1}, \dots, \mathbf{e}_{\wp_m}] \in \mathbb{R}^{n\times m}$ with $\{\wp_1, \dots, \wp_m\}$ being the output index set from Algorithm 1 with the input basis $\{\mathbf{u}_i\}_{i=1}^m$ and the vector $\mathbf{e}_{\wp_j} = [0, \dots, 0, 1, 0, \dots, 0]^T \in \mathbb{R}^n$ denoting the $\wp_j$th column of the n-by-n identity matrix, i.e., having all zero entries except one at the entry $\wp_j$ for $j = 1, \dots, m$. Note that for general definition of the DEIM approximation (2.10), the matrix $\mathbf{U}$ is not required to have orthonormal columns as in the previous sections. The notation `max` in Algorithm 1 is the same as the function `max` in MATLAB. Thus, $[|\rho|,\ \wp_\ell] = \mathtt{max}\{|\mathbf{r}|\}$ implies $|\rho| = |r_{\wp_\ell}| = \max_{i=1,\dots,n}\{|r_i|\}$ with the smallest index taken in case of a tie. The interpolation indices selection process of the DEIM may be interpreted as locally limiting the growth of the matrix $(\mathbf{P}^T\mathbf{U})^{-1}$. When $\mathbf{U}$ has orthonormal columns, there is a bound on the approximation error of the interpolant as given below (see Lemma 3.2 in [3]).

ALGORITHM 1. DEIM.

INPUT: $\{\mathbf{u}_\ell\}_{\ell=1}^m \subset \mathbb{R}^n$ linearly independent
OUTPUT: $\vec{\wp} = [\wp_1, \dots, \wp_m]^T \in \mathbb{R}^m$

$[|\rho|,\ \wp_1] = \mathtt{max}\{|\mathbf{u}_1|\}$
$\mathbf{U} = [\mathbf{u}_1]$, $\mathbf{P} = [\mathbf{e}_{\wp_1}]$, $\vec{\wp} = [\wp_1]$
for $\ell = 2$ to m **do**
    Solve $(\mathbf{P}^T\mathbf{U})\mathbf{c} = \mathbf{P}^T\mathbf{u}_\ell$ for $\mathbf{c}$
    $\mathbf{r} = \mathbf{u}_\ell - \mathbf{U}\mathbf{c}$
    $[|\rho|\ \ \wp_\ell] = \mathtt{max}\{|\mathbf{r}|\}$
    $\mathbf{U} \leftarrow [\mathbf{U}\ \ \mathbf{u}_\ell]$, $\mathbf{P} \leftarrow [\mathbf{P}\ \ \mathbf{e}_{\wp_\ell}]$, $\vec{\wp} \leftarrow \begin{bmatrix} \vec{\wp} \\ \wp_\ell \end{bmatrix}$
end for

LEMMA 2.1. *Using the notation defined earlier, if* $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, *then*

$$\mathbf{f}(t) - \widehat{\mathbf{f}}(t) = (\mathbf{I} - \mathbb{P})\mathbf{w}(t) \quad \textit{and} \quad \|\mathbf{f}(t) - \widehat{\mathbf{f}}(t)\| \le \mathbf{C}_m\, \mathscr{E}_*(\mathbf{f}(t)), \tag{2.11}$$

where $\mathbf{w}(t) := (\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{f}(t)$,

$$\mathbf{C}_m := \|(\mathbf{P}^T\mathbf{U})^{-1}\| \quad \text{and} \quad \mathscr{E}_*(\mathbf{f}(t)) = \|(\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{f}(t)\|. \tag{2.12}$$

$\mathscr{E}_*(\mathbf{f}(t))$ *is the error of the best* 2*-norm approximation for* $\mathbf{f}(t)$ *from the space* Range$(\mathbf{U})$ *and the constant* $\mathbf{C}_m$ *is bounded by*

$$\mathbf{C}_m \leq (1+\sqrt{2n})^{m-1}\|\mathbf{u}_1\|_\infty^{-1}. \tag{2.13}$$

The invertibility of $\mathbf{P}^T\mathbf{U}$ at each iteration of the DEIM procedure follows from the boundedness of its inverse as shown (2.13), which was originally derived in [3]. The a priori bound (2.13) is, of course, useless in practice as it is a gross overestimate. In our computations, we simply evaluate $\|(\mathbf{P}^T\mathbf{U})^{-1}\|_2$ and use this as an a posteriori estimate as given in (2.11). This quantity has been on the order of 100 or less in all the experiments we have conducted. The error bound (2.11) will be used in the next section to analyze the accuracy of the state variables in the POD-DEIM reduced system.

3. Error analysis of POD-DEIM reduced system. This section develops a bound on the state approximation error for numerical solutions obtained from the POD-DEIM reduced system. The derivation will involve an application of the logarithmic norm [4] and Gronwall's lemma [10, 18].

We shall consider the logarithmic norm of $\mathbf{A} \in \mathbb{C}^{n\times n}$ with respect to the 2-norm defined as

$$\mu(\mathbf{A}) := \lim_{h\to 0^+} \frac{\|\mathbf{I}+h\mathbf{A}\|_2 - 1}{h}, \tag{3.1}$$

which has an explicit expression suitable for the calculation given by

$$\mu(\mathbf{A}) = \max\{\mu : \mu \in \sigma\left([\mathbf{A}+\mathbf{A}^*]/2\right)\}, \tag{3.2}$$

where $\sigma\left([\mathbf{A}+\mathbf{A}^*]/2\right)$ is the set of eigenvalues of the Hermitian part $[\mathbf{A}+\mathbf{A}^*]/2$ of $\mathbf{A}$. In our derivation, we will use the following equivalent formulation [28]:

$$\mu(\mathbf{A}) = \sup_{\mathbf{x}\in\mathbb{C}^n\setminus\{0\}} \frac{\mathrm{Re}\langle\mathbf{x},\mathbf{A}\mathbf{x}\rangle}{\langle\mathbf{x},\mathbf{x}\rangle}, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in Euclidean space and Re denotes the real part of a complex number. It is straightforward to show that

$$\mu(\widehat{\mathbf{A}}) \leq \mu(\mathbf{A}), \tag{3.4}$$

where $\widehat{\mathbf{A}} = \mathbf{V}^T\mathbf{A}\mathbf{V} \in \mathbb{R}^{k\times k}$ and $\mathbf{V} \in \mathbb{R}^{n\times k}$, $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

The logarithmic norm was introduced by Dahlquist [4] to provide a mechanism for bounding the growth of the solution to a linear dynamical system of the form

$$\dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{r}(t)$$

whenever $\mathbf{r}$ is a bounded function of t. For $t \geq 0$ the norm of $\mathbf{y}$ satisfies the differential inequality

$$\frac{d}{dt}\|\mathbf{y}(t)\| \leq \mu(\mathbf{A})\|\mathbf{y}(t)\| + \|\mathbf{r}(t)\|. \tag{3.5}$$

As explained by Söderlind [28], the bound (3.5) is able to distinguish between forward and reverse time and it may also be able to distinguish between stable and unstable systems. In fact, $\mu(\mathbf{A})$ may be negative, and when it is, the system is certain to be stable. The opposite assertion ($\mathbf{A}$ stable implies $\mu(\mathbf{A}) < 0$) is *not* true. The nonnormal matrix $\left[\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right]$ provides a counterexample when $-.5 < \mathrm{Real}(\lambda) < 0$. More details on logarithmic norms can be found in, e.g., [4, 29, 5, 28]. Next, we shall derive bounds on the state approximation error provided by POD-DEIM solutions involving the full trajectory of the ODE system.

3.1. Error bounds in the ODE setting. This section compares the solution $\mathbf{y}(t)$ from the original full-order system to the approximation $\mathbf{V}\widehat{\mathbf{y}}(t)$, where $\widehat{\mathbf{y}}$ is the solution of the POD-DEIM reduced system (2.3). We shall show that the resulting error bounds in 2-norm can be approximated by the sums of the singular values corresponding to neglected POD basis vectors both in Galerkin projection of the reduced system and in DEIM approximation of the nonlinear term.

THEOREM 3.1. *Let $\mathbf{y}(t)$ be the solution of the original full-order system* (2.1) *and $\widehat{\mathbf{y}}$ be the solution of the POD-DEIM reduced system* (2.3) *for $t \in [0,T]$. Let $\mu(\mathbf{A})$ be the logarithmic norm defined in* (3.1) *and assume that $\mathbf{F}(t,\mathbf{y})$ in* (2.1) *is Lipschitz continuous in the second argument with Lipschitz constant L_f as in* (2.2). *Then*

$$\int_0^T \|\mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)\|^2 dt \le \mathbf{C}(T)\ \left(\mathcal{E}_{\mathbf{y}} + \mathcal{E}_{\mathbf{f}}\right), \tag{3.6}$$

where $\mathbf{C}(T) := \max\{1 + c_\mu\ \alpha^2 T, c_\mu\ \beta^2 T\}$,

$$\alpha := \|\mathbf{V}^T\mathbf{A}\| + \|\mathbf{V}^T\mathbb{P}\|L_f, \quad \beta := \|\mathbf{V}^T(\mathbf{I} - \mathbb{P})\|, \quad \gamma := \|\mathbf{V}^T\mathbb{P}\|L_f, \tag{3.7}$$

$$c_\mu := \begin{cases} \frac{1}{a}\left[e^{2aT} - 1\right], & a \ne 0, \\ 2T, & a = 0, \end{cases} \qquad a := \mu(\mathbf{A}) + \gamma, \tag{3.8}$$

$$\mathcal{E}_{\mathbf{y}} := \int_0^T \|\mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)\|^2 dt, \quad \mathcal{E}_{\mathbf{f}} := \int_0^T \|\mathbf{f}(t) - \mathbf{U}\mathbf{U}^T\mathbf{f}(t)\|^2 dt, \tag{3.9}$$

with $\mathbf{f}(t) = \mathbf{F}(t,\mathbf{y}(t))$. In addition,

(i) *If $\mu(\mathbf{A}) + \gamma < 0$, then c_μ in* (3.8) *can be bounded by a constant independent of T, i.e., for $T \ge 0$,*

$$c_\mu < \frac{1}{|\mu(\mathbf{A}) + \gamma|}; \tag{3.10}$$

(ii) *If the POD-DEIM reduced system* (2.3) *is constructed from the POD basis matrices $\mathbf{V} \in \mathbb{R}^{n\times k}$ and $\mathbf{U} \in \mathbb{R}^{n\times m}$ of solution snapshots and nonlinear snapshots, respectively, which satisfy* (2.6)*; then, from* (2.7) *and* (2.9),

$$\mathcal{E}_{\mathbf{y}} = \sum_{\ell=k+1}^{r} \lambda_\ell^\infty, \quad \mathcal{E}_{\mathbf{f}} = \sum_{\ell=m+1}^{r_s} s_\ell^\infty. \tag{3.11}$$

Proof. Define the pointwise error $\mathbf{e}(t) := \mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)$ and write

$$\mathbf{e}(t) = \rho(t) + \theta(t),$$

where $\rho(t) := \mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)$, $\quad \theta(t) := \mathbf{V}\mathbf{V}^T\mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)$. The proof consists of developing a bound for $\|\theta(t)\|$ using Gronwall's lemma. Consider $\dot{\theta}(t) = \mathbf{V}\mathbf{V}^T\dot{\mathbf{y}}(t) - \mathbf{V}\dot{\widehat{\mathbf{y}}}(t)$ with $\dot{\mathbf{y}}(t)$ and $\dot{\widehat{\mathbf{y}}}(t)$ satisfying (2.1) and (2.3). Then

$$\dot{\theta}(t) = \mathbf{V}\mathbf{V}^T\left[\mathbf{A}\left[\rho(t) + \theta(t)\right] + \mathbf{F}(t,\mathbf{y}(t)) - \mathbb{P}\mathbf{F}(t,\mathbf{V}\widehat{\mathbf{y}}(t))\right]. \tag{3.12}$$

Define $\widehat{\theta}(t) := \mathbf{V}^T\theta(t)$. Premultiplying by $\mathbf{V}^T$ throughout (3.12), using orthonormality of the columns of $\mathbf{V}$, and noting that $\theta(t) = \mathbf{V}\widehat{\theta}(t)$ provides

$$\frac{d}{dt}\widehat{\theta}(t) = \widehat{\mathbf{A}}\widehat{\theta}(t) + \mathbf{G}(t), \tag{3.13}$$

where $\mathbf{G}(t) := \mathbf{V}^T\mathbf{A}\rho(t) + \mathbf{V}^T\left[\mathbf{F}(t,\mathbf{y}(t)) - \mathbb{P}\mathbf{F}(t,\mathbf{V}\widehat{\mathbf{y}}(t))\right]$. Recall that $(\mathbf{I}-\mathbb{P})\mathbf{F}(t,\mathbf{y}(t)) = (\mathbf{I}-\mathbb{P})\mathbf{w}(t)$ from Lemma 2.1, where $\mathbf{w}(t) := \mathbf{F}(t,\mathbf{y}(t)) - \mathbf{U}\mathbf{U}^T\mathbf{F}(t,\mathbf{y}(t))$. This together with the Lipschitz continuity of $\mathbf{F}$ implies

$$\begin{aligned}\|\mathbf{G}(t)\| &\le \|\mathbf{V}^T\mathbf{A}\rho(t)\| + \|\mathbf{V}^T(\mathbf{I}-\mathbb{P})\mathbf{F}(t,\mathbf{y}(t))\| + \|\mathbf{V}^T\mathbb{P}\|\|\mathbf{F}(t,\mathbf{y}(t)) - \mathbf{F}(t,\mathbf{V}\widehat{\mathbf{y}}(t))\| \\ &\le \alpha\|\rho(t)\| + \beta\|\mathbf{w}(t)\| + \gamma\|\theta(t)\|,\end{aligned} \tag{3.14}$$

where $\alpha := \|\mathbf{V}^T\mathbf{A}\| + \|\mathbf{V}^T\mathbb{P}\|L_f$, $\beta := \|\mathbf{V}^T(\mathbf{I}-\mathbb{P})\|$, $\gamma := \|\mathbf{V}^T\mathbb{P}\|L_f$. Note that $\|\widehat{\theta}(t)\| = \|\theta(t)\|$ and $\theta(0) = 0$ since $\widehat{\mathbf{y}}_0 = \mathbf{V}^T\mathbf{y}(0)$. Using the fact that $\|\widehat{\theta}(t)\|\frac{d}{dt}\|\widehat{\theta}(t)\| = \frac{1}{2}\frac{d}{dt}(\|\widehat{\theta}(t)\|^2) = \widehat{\theta}(t)^T\dot{\widehat{\theta}}(t)$, we have

$$\begin{aligned}\frac{d}{dt}\|\widehat{\theta}(t)\| &= \frac{1}{\|\widehat{\theta}(t)\|}\left\langle\widehat{\theta}(t),\dot{\widehat{\theta}}(t)\right\rangle = \frac{1}{\|\widehat{\theta}(t)\|}\left\langle\widehat{\theta}(t),\widehat{\mathbf{A}}\widehat{\theta}(t)+\mathbf{G}(t)\right\rangle \\ &= \frac{1}{\|\widehat{\theta}(t)\|^2}\left\langle\widehat{\theta}(t),\widehat{\mathbf{A}}\widehat{\theta}(t)\right\rangle\|\widehat{\theta}(t)\| + \frac{1}{\|\widehat{\theta}(t)\|}\left\langle\widehat{\theta}(t),\mathbf{G}(t)\right\rangle \\ &\le \mu(\widehat{\mathbf{A}})\|\widehat{\theta}(t)\| + \frac{1}{\|\widehat{\theta}(t)\|}\|\widehat{\theta}(t)\|\|\mathbf{G}(t)\| \\ &\le \mu(\mathbf{A})\|\widehat{\theta}(t)\| + \Big(\alpha\|\rho(t)\| + \beta\|\mathbf{w}(t)\| + \gamma\|\theta(t)\|\Big) \\ &= a\|\widehat{\theta}(t)\| + b(t),\end{aligned} \tag{3.15}$$

where $a := \mu(\mathbf{A}) + \gamma$ and $b(t) := \alpha\|\rho(t)\| + \beta\|\mathbf{w}(t)\|$. Above, we have used the fact that $\mu(\widehat{\mathbf{A}}) \ge \frac{\langle\widehat{\mathbf{x}},\widehat{\mathbf{A}}\widehat{\mathbf{x}}\rangle}{\|\widehat{\mathbf{x}}\|^2}$ for all $\widehat{\mathbf{x}} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ from (3.3) and $\mu(\widehat{\mathbf{A}}) \le \mu(\mathbf{A})$ from (3.4). Applying the differential form of Gronwall's lemma (see Lemma 1.8 in [18]) on (3.15) and using the Cauchy–Schwarz inequality give

$$\begin{aligned}\|\widehat{\theta}(t)\| &\le \underbrace{\|\widehat{\theta}(0)\|}_{=0}e^{at} + \int_0^t e^{a(t-s)}b(s)ds \\ &\le \left(\int_0^t e^{2a(t-s)}ds\right)^{1/2}\left(2\int_0^t \alpha^2\|\rho(s)\|^2 + \beta^2\|\mathbf{w}(s)\|^2 ds\right)^{1/2}\end{aligned} \tag{3.16}$$

$$\|\widehat{\theta}(t)\|^2 \le c_\mu(T)\left(\alpha^2\int_0^T\|\rho(t)\|^2dt + \beta^2\int_0^T\|\mathbf{w}(t)\|^2dt\right) \tag{3.17}$$

for all $t \in [0,T]$, where $c_\mu(t) := 2\int_0^t e^{2a(t-s)}ds = \begin{cases}\frac{1}{a}\left[e^{2at}-1\right], & a \ne 0 \\ 2t, & a = 0\end{cases}$. That is,

$$\int_0^T\|\theta(t)\|^2dt \le Tc_\mu(T)\left(\alpha^2\int_0^T\|\rho(t)\|^2dt + \beta^2\int_0^T\|\mathbf{w}(t)\|^2dt\right), \tag{3.18}$$

since $\|\widehat{\theta}(t)\| = \|\theta(t)\|$. Finally, the proof is completed by applying the above bound to $\int_0^T\|\theta(t)\|^2$ on the right in $\int_0^T\|e(t)\|^2dt = \int_0^T\|\rho(t)\|^2dt + \int_0^T\|\theta(t)\|^2dt$. ◻

As noted earlier, the error bound in Theorem 3.1 can be applied to POD–Galerkin reduced system by setting $\mathbb{P} = \mathbf{I}$. In fact, this bound is applicable to any reduced system constructed from Galerkin projection with orthonormal basis matrices $\mathbf{U}$ and $\mathbf{V}$ (not necessarily from POD). From Theorem 3.1, $\mathcal{E}_{\mathbf{y}}$ and $\mathcal{E}_{\mathbf{f}}$ in (3.9) are the least-squares errors from approximating the solution $\mathbf{y}(t)$ and the nonlinear function $\mathbf{F}(t, \mathbf{y}(t))$ by using the given orthonormal projection matrices $\mathbf{V}$ and $\mathbf{U}$, respectively. These errors are minimized when $\mathbf{V}$ and $\mathbf{U}$ are POD basis matrices satisfying (ii) in Theorem 3.1 and therefore the neglected singular values can be used to give lower bounds for $\mathcal{E}_{\mathbf{y}}$ and $\mathcal{E}_{\mathbf{f}}$ as shown in (3.11).

The magnification factor $\mathbf{C}(T)$ depends on the original problem through the quantities $\mathbf{A}, \mu(\mathbf{A}), L_f$ as well as the POD and DEIM approximations through $\mathbf{V}$ and $\mathbb{P}$. It can be computed inexpensively without actually solving the reduced system. This factor $\mathbf{C}(T)$ is linear in final integration time T when $a = \mu(A) + \|\mathbf{V}^T\mathbb{P}\|L_f < 0$ as c_μ in (3.10) is uniformly bounded on $[0, T]$ for any $T \geq 0$.

Remark 3.2. Given an arbitrary system $\frac{d}{dt}\mathbf{y}(t) = \mathbf{f}(t, \mathbf{y}(t))$ with Lipschitz constant L_f, it may appear that one could define $\mathbf{F}(t, \mathbf{y}) = \mathbf{f}(t, \mathbf{y}) - \mathbf{A}\mathbf{y}$ for any $\mathbf{A}$ with $\mu(\mathbf{A}) < 0$ and obtain an arbirarily small error estimate for the system $\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{F}(t, \mathbf{y}) = \mathbf{f}(t, \mathbf{y}(t))$. Here, $\mathbf{F}$ would have a Lipschitz constant $L_f + \|\mathbf{A}\|$. However, since $\mu(\mathbf{A}) + \|\mathbf{A}\| \geq 0$ for any $\mathbf{A}$, it turns out that

$$\mu(\mathbf{A}) + \gamma = \mu(\mathbf{A}) + \|\mathbf{V}^T\mathbb{P}\|(L_f + \|\mathbf{A}\|) \geq \mu(\mathbf{A}) + (L_f + \|\mathbf{A}\|) \geq L_f > 0$$

(assuming $\|\mathbf{V}^T\mathbb{P}\| \geq 1$). Thus, in Theorem 3.1, the constant $\mathbf{C}(T)$ would be exponential in T.

The error analysis in this section has illustrated the basic idea concerning how the parabolicity assumption together with the combination of the POD-DEIM approach will lead to a bound on the state approximation error. However, it depends upon the ability to separate out a constant matrix $\mathbf{A}$ on the right-hand side of the ODE system. The key tool in this analysis has been the logarithmic norm. In the next section, we shall utilize a generalization to obtain an error estimate that does not require the constant matrix $\mathbf{A}$.

4. Analysis based on generalized logarithmic norm. A logarithmic norm was used in the previous section to analyze the state approximation error of the POD-DEIM system. That approach required the presence of a constant matrix $\mathbf{A}$. More generally, as is done in [28], one can apply a logarithmic norm argument to a local linearization about the trajectory. In the remainder of our analysis, we shall employ a generalization of the logarithmic norm that avoids the need for a linearization or for the presence of a constant matrix $\mathbf{A}$. The generalization of logarithmic norm to unbounded nonlinear operators was introduced through *logarithmic Lipschitz constants* in [27] to avoid working with linearizations and logarithmic norms that are only applicable to bounded operators. Here, we shall use this analysis tool to develop a *conceptual* framework suitable to analyzing POD-DEIM reduced systems of nonlinear ODEs. We will now consider nonlinear ODEs of the form

$$\dot{\mathbf{y}}(t) = \mathbf{F}(t, \mathbf{y}(t)), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{4.1}$$

where $\mathbf{F} : [0, T] \times \mathcal{Y} \to \mathbb{R}^n$, $\mathcal{Y} \subseteq \mathbb{R}^n$ with POD-DEIM reduced system of the form

$$\dot{\widehat{\mathbf{y}}}(t) = \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t)), \quad \widehat{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0, \tag{4.2}$$

where $\widehat{\mathbf{F}} : [0, T] \times \widehat{\mathcal{Y}} \to \mathbb{R}^k$, $\widehat{\mathcal{Y}} \subseteq \mathbb{R}^k$, $\widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}) = \mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}(t))$ for $\widehat{\mathbf{y}}(t) \in \widehat{\mathcal{Y}}$, $t \in [0, T]$. Note that the POD reduced system can be obtained by replacing $\mathbb{P}$ with the n-by-n

identity matrix. Hence, the error bounds derived in this section also apply to the POD reduced system. We shall work with the Euclidian inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, for some positive integer d, i.e., $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, and its induced norm $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$, $\mathbf{u} \in \mathbb{R}^d$. As in [28], for a map $\mathbf{F} : [0,T] \times \mathcal{Y} \to \mathbb{R}^d$, $\mathcal{Y} \subseteq \mathbb{R}^d$, the (least upper bound) *logarithmic Lipschitz constants* with respect to the inner product $\langle \cdot, \cdot \rangle$ can be defined, uniformly for all $t \in [0,T]$, as

$$M[\mathbf{F}] := \sup_{\mathbf{u} \neq \mathbf{v}} \frac{\langle \mathbf{u} - \mathbf{v}, \mathbf{F}(t, \mathbf{u}) - \mathbf{F}(t, \mathbf{v}) \rangle}{\|\mathbf{u} - \mathbf{v}\|^2}. \tag{4.3}$$

The convergence of the solution as well as the stability of the corresponding POD-DEIM reduced system can be analyzed by using these logarithmic Lipschitz constants. The map $\mathbf{F}$ is called *uniformly negative monotone* if $M[\mathbf{F}] < 0$, in which case it will be shown that the error bound of the reduced-order solution is uniformly bounded on $t \in [0,T]$.

Remark 4.1. If $\mathbf{F}(t, \mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{f}(t, \mathbf{y})$ with L_f the Lipschitz constant for $\mathbf{f}$, then $M[\mathbf{F}] \le \mu(\mathbf{A}) + L_f \le \mu(\mathbf{A}) + \gamma$, where $\gamma = \|\mathbf{V}^T \mathbb{P}\| L_f$ (assuming $\|\mathbf{V}^T \mathbb{P}\| \ge 1$). Thus, the condition $\mu(\mathbf{A}) + \gamma < 0$ in Theorem 3.1 ensures $M[\mathbf{F}] < 0$.

The asymptotic error analysis will be considered first in section 4.1 for the continuous setting where the overall accuracy of the reduced system is only contributed from applying the POD-DEIM technique without other effects, such as the choice of time integration method. Then, a framework for error analysis in the discrete setting for the implicit Euler time integration scheme will be presented in section 4.2. The proposed error bounds in both continuous and discrete settings are derived through an application of generalized logarithmic norms discussed earlier and they are summarized in Theorem 4.2. Note that Lipschitz continuity of $\mathbf{F}$ is the only main assumption used in Theorem 4.2. The resulting error bounds in 2-norm reflect the approximation property of the POD-based scheme through the decay of singular values.

THEOREM 4.2. *Let $M[\widehat{\mathbf{F}}]$ be the logarithmic Lipschitz constant of $\widehat{\mathbf{F}}$ defined as in* (4.3) *and assume that $\mathbf{F}(t, \mathbf{y})$ in* (4.1) *is Lipschitz continuous with Lipschitz constant L_f as in* (2.2)*. Let $\mathbf{y}(t)$ be the solution of the original full-order system* (4.1) *and $\widehat{\mathbf{y}}(t)$ be the solution of the POD-DEIM reduced system* (4.2) *for $t \in [0,T]$. Let Y_j and $\widehat{Y}_j$ be the solutions of the discretized systems of* (4.1) *and* (4.2)*, respectively, obtained from implicit Euler time integration at $t_j = j\triangle t \in [0,T]$ for $j = 0, \dots n_t$, $n_t = T/\triangle t$ with $\triangle t M[\widehat{\mathbf{F}}] < 1$. Then*

$$\int_0^T \|\mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)\|^2 dt \le \mathcal{C}(T) \left(\mathcal{E}_{\mathbf{y}} + \mathcal{E}_{\mathbf{f}} \right), \tag{4.4}$$

$$\sum_{j=0}^{n_t} \|Y_j - \mathbf{V}\widehat{Y}_j\|^2 \le \bar{\mathcal{C}}(T) \left(\bar{\mathcal{E}}_{\mathbf{y}} + \bar{\mathcal{E}}_{\mathbf{f}} \right), \tag{4.5}$$

where $\mathcal{C}(T) := \max\{1 + c_M \alpha^2 T, c_M \beta^2 T\}$ *and* $\bar{\mathcal{C}}(T) := \max\{1 + \bar{c}_M \alpha^2 T, \bar{c}_M \beta^2 T\}$,

$$\alpha := \|\mathbf{V}^T \mathbb{P}\| L_f, \quad \beta := \|\mathbf{V}^T (\mathbf{I} - \mathbb{P})\|, \quad \zeta := \frac{1}{1 - \triangle t M[\widehat{\mathbf{F}}]}, \tag{4.6}$$

$$c_M := \begin{cases} \frac{1}{M[\widehat{\mathbf{F}}]}\left(e^{2M[\widehat{\mathbf{F}}]T} - 1\right), & M[\widehat{\mathbf{F}}] \neq 0, \\ 2T, & M[\widehat{\mathbf{F}}] = 0, \end{cases} \quad \bar{c}_M := \triangle t \zeta^2 \left(\frac{1 - \zeta^{2n_t}}{1 - \zeta^2} \right), \tag{4.7}$$

$$\mathcal{E}_{\mathbf{y}} := \int_0^T \|\mathbf{y}(t) - \mathbf{V}\mathbf{V}^T \mathbf{y}(t)\|^2 dt, \quad \mathcal{E}_{\mathbf{f}} := \int_0^T \|\mathbf{f}(t) - \mathbf{U}\mathbf{U}^T \mathbf{f}(t)\|^2 dt, \tag{4.8}$$

$$\bar{\mathcal{E}}_{\mathbf{y}} := \sum_{j=0}^{n_t} \|Y_j - \mathbf{V}\mathbf{V}^T Y_j\|^2, \quad \bar{\mathcal{E}}_{\mathbf{f}} := \sum_{j=0}^{n_t} \|F_j - \mathbf{U}\mathbf{U}^T F_j\|^2, \tag{4.9}$$

with $\mathbf{f}(t) = \mathbf{F}(t, \mathbf{y}(t))$*,* $F_j = \mathbf{F}(t_j, Y_j)$*. In addition,*

(i) *If* $M[\widehat{\mathbf{F}}] < 0$*, then* c_M *and* $\bar{c}_M$ *in* (4.7) *are bounded by*

$$c_M < \frac{1}{|M[\widehat{\mathbf{F}}]|} \quad \text{and} \quad \bar{c}_M < \frac{1}{|M[\widehat{\mathbf{F}}]| + \triangle t M[\widehat{\mathbf{F}}]^2/2}; \tag{4.10}$$

(ii) *If the POD basis matrices* $\mathbf{V} \in \mathbb{R}^{n\times k}$ *and* $\mathbf{U} \in \mathbb{R}^{n\times m}$ *used in* (4.2)*, respectively, satisfy* (2.7) *and* (2.9)*, then*

$$\mathcal{E}_{\mathbf{y}} = \sum_{\ell=k+1}^{r} \lambda_\ell^\infty, \quad \mathcal{E}_{\mathbf{f}} = \sum_{\ell=m+1}^{r_s} s_\ell^\infty; \tag{4.11}$$

(iii) *If* $\mathbf{V} \in \mathbb{R}^{n\times k}$ *and* $\mathbf{U} \in \mathbb{R}^{n\times m}$ *used in* (4.2) *are the POD basis matrices of snapshot matrices* $\mathbb{Y} = [Y_1, \ldots, Y_{n_t}]$ *and* $\mathbb{F} = [\mathbf{F}(t_1, Y_1), \ldots, \mathbf{F}(t_{n_t}, Y_{n_t})] \in \mathbb{R}^{n\times n_t}$*, then using* (2.5) *and* (2.8) *gives*

$$\bar{\mathcal{E}}_{\mathbf{y}} = \sum_{\ell=k+1}^{\bar{r}} \lambda_\ell, \quad \bar{\mathcal{E}}_{\mathbf{f}} = \sum_{\ell=m+1}^{\bar{r}_s} s_\ell. \tag{4.12}$$

Remark 4.3. When (i) and (iii) of Theorem 4.2 hold true, the norm of the pointwise error in discrete setting is uniformly bounded at each time step:

$$\|Y_\ell - \mathbf{V}\widehat{Y}_\ell\|^2 \le \bar{c} \left(\sum_{\ell=k+1}^{\bar{r}} \lambda_\ell + \sum_{\ell=m+1}^{\bar{r}_s} s_\ell \right) \quad \text{for all } \ell = 1, \ldots n_t, \tag{4.13}$$

where $\bar{c} = \max\{1 + \bar{q}\alpha^2, \bar{q}\beta^2\}$, $\bar{q} = \frac{1}{|M[\widehat{\mathbf{F}}]| + \triangle t M[\widehat{\mathbf{F}}]^2/2}$.

Analogous to Theorem 3.1, the magnification factors $\mathcal{C}(T)$ and $\bar{\mathcal{C}}(T)$ are linear in final integration time T when $M[\widehat{\mathbf{F}}] < 0$, as c_M and $\bar{c}_M$ in (4.10) are uniformly bounded on $[0, T]$ for any $T \ge 0$ with fixed $\triangle t$. Moreover, for $M[\widehat{\mathbf{F}}] < 0$, it is clear from (4.10) that $\bar{c}_M \to c_M$ as $\triangle t \to 0$ and therefore the factor $\bar{\mathcal{C}}(T)$ in discrete setting converges to $\mathcal{C}(T)$ in continuous setting. The convergence of the discrete error bound (4.5) to the continuous error bound in (4.4) can be obtained by applying some quadrature weights, such as $\triangle t$, throughout (4.5) and including these weights in the least-squares errors $\bar{\mathcal{E}}_{\mathbf{y}}, \bar{\mathcal{E}}_{\mathbf{f}}$ (e.g., in POD computation) as done in [16, 17, 31]. However, unlike these previous analyses, we compare the *numerical solutions* from the reduced-order system to those of the full-order systems at the same time instances and with the same stepsize $\triangle t$ instead of comparing the reduced system approximate state values to the full-order continuous solution. For this reason, the effect of the global truncation error for implicit Euler (proportional to $\triangle t^2$) does not appear in the bound (4.5) as it does in [16, 17, 31].

There are two main differences for the error bounds in the continuous setting from (3.6) of Theorem 3.1 and from (4.4) of Theorem 4.2: one in the quantities $\mu(\cdot)$ and $M[\cdot]$ and one in the terms c_μ and c_M. Note that $\mu(\cdot)$ and $M[\cdot]$ are the same when they are applied to linear operators, and hence there is no need to introduce the notion of logarithmic Lipschitz constant for linear systems. With nonlinearities, however, applying the logarithmic Lipschitz constant $M[\cdot]$ can give a smaller error bound, since $M[\cdot] \le L_f$, which implies $c_M \le c_\mu$ from (4.10) and (3.10). The applications of $M[\cdot]$ for deriving the error bounds in Theorem 4.2 are presented next.

4.1. Error bounds in continuous ODE setting. Consider the error of the solution from the POD-DEIM reduced system of the form

$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{y}_r(t), \quad \mathbf{y}_r(t) := \mathbf{V}\widehat{\mathbf{y}}(t),$$

where $\mathbf{V} \in \mathbb{R}^{n\times k}$ is the POD basis matrix with $\mathbf{y}$ and $\widehat{\mathbf{y}}$ satisfying $\dot{\mathbf{y}}(t) = \mathbf{F}(t, \mathbf{y}(t))$, $\mathbf{y}(0) = \mathbf{y}_0$, and $\dot{\widehat{\mathbf{y}}}(t) = \mathbf{V}^T\mathbb{P}\mathbf{F}(t, \mathbf{V}\widehat{\mathbf{y}}(t))$, $\widehat{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0$ for $t \in [0, T]$. Again, put

$$\mathbf{e}(t) = \rho(t) + \theta(t),$$

where $\rho(t) := \mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)$, $\theta(t) := \mathbf{V}\mathbf{V}^T\mathbf{y}(t) - \mathbf{V}\widehat{\mathbf{y}}(t)$, and note that $\widehat{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0$ implies $\theta(0) = 0$. Note also that $\rho(t)^T\theta(t) = 0$ implies that $\|e(t)\|^2 = \|\rho(t)\|^2 + \|\theta(t)\|^2$. Define $\widehat{\theta}(t) := \mathbf{V}^T\theta(t)$. As before, $\theta(t) = \mathbf{V}\widehat{\theta}(t)$ and hence $\|\theta(t)\| = \|\widehat{\theta}(t)\|$. Now consider

$$\mathbf{V}^T\dot{\mathbf{y}}(t) = \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)) + \widehat{\mathbf{r}}(t), \tag{4.14}$$

$$\dot{\widehat{\mathbf{y}}}(t) = \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t)), \tag{4.15}$$

where

$$\widehat{\mathbf{r}}(t) := \mathbf{V}^T\mathbf{F}(t, \mathbf{y}(t)) - \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)). \tag{4.16}$$

This gives $\dot{\widehat{\theta}}(t) = \mathbf{V}^T\dot{\mathbf{y}}(t) - \dot{\widehat{\mathbf{y}}}(t) = \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)) - \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t)) + \widehat{\mathbf{r}}(t)$. Next, since $\|\widehat{\theta}(t)\|^2 = \widehat{\theta}(t)^T\widehat{\theta}(t)$,

$$\begin{aligned}
\frac{d}{dt}\|\widehat{\theta}(t)\| &= \frac{\langle\widehat{\theta}(t), \dot{\widehat{\theta}}(t)\rangle}{\|\widehat{\theta}(t)\|} \\
&= \frac{\langle\widehat{\theta}(t), \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)) - \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t)) + \widehat{\mathbf{r}}(t)\rangle}{\|\widehat{\theta}(t)\|} \\
&= \frac{\langle\widehat{\theta}(t), \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)) - \widehat{\mathbf{F}}(t, \widehat{\mathbf{y}}(t))\rangle}{\|\widehat{\theta}(t)\|} + \frac{\langle\widehat{\theta}(t), \widehat{\mathbf{r}}(t)\rangle}{\|\widehat{\theta}(t)\|} \\
&\le M[\widehat{\mathbf{F}}]\|\widehat{\theta}(t)\| + \|\widehat{\mathbf{r}}(t)\|.
\end{aligned}$$

Again, we apply the Gronwall lemma. Since $\|\theta(t)\| = \|\widehat{\theta}(t)\|$ and $\|\widehat{\theta}(0)\| = 0$, we have

$$\|\theta(t)\| \le e^{M[\widehat{\mathbf{F}}]t}\|\theta(0)\| + \int_0^t e^{M[\widehat{\mathbf{F}}](t-\tau)}\|\widehat{\mathbf{r}}(\tau)\|d\tau = \int_0^t e^{M[\widehat{\mathbf{F}}](t-\tau)}\|\widehat{\mathbf{r}}(\tau)\|d\tau.$$

We now recast the expression for $\widehat{\mathbf{r}}(t)$ as the sum of differences that can be estimated in terms of the neglected singular values. From (2.11) in Lemma 2.1, for $\mathbf{w}(t) = \mathbf{F}(t, \mathbf{y}(t)) - \mathbf{U}\mathbf{U}^T\mathbf{F}(t, \mathbf{y}(t))$,

$$\begin{aligned}
\widehat{\mathbf{r}}(t) &= \mathbf{V}^T\mathbf{F}(t, \mathbf{y}(t)) - \widehat{\mathbf{F}}(t, \mathbf{V}^T\mathbf{y}(t)) = \mathbf{V}^T[\mathbf{F}(t, \mathbf{y}(t)) - \mathbb{P}\mathbf{F}(t, \mathbf{V}\mathbf{V}^T\mathbf{y}(t))] \\
&= \mathbf{V}^T[\mathbf{F}(t, \mathbf{y}(t)) - \mathbb{P}\mathbf{F}(t, \mathbf{y}(t)) + \mathbb{P}\mathbf{F}(t, \mathbf{y}(t)) - \mathbb{P}\mathbf{F}(t, \mathbf{V}\mathbf{V}^T\mathbf{y}(t))] \\
&= \mathbf{V}^T(\mathbf{I} - \mathbb{P})\mathbf{w}(t) + \mathbf{V}^T\mathbb{P}(\mathbf{F}(t, \mathbf{y}(t)) - \mathbf{F}(t, \mathbf{V}\mathbf{V}^T\mathbf{y}(t))).
\end{aligned}$$

The Lipschitz continuity of $\mathbf{F}$ implies $\|\mathbf{F}(t, \mathbf{y}(t)) - \mathbf{F}(t, \mathbf{V}\mathbf{V}^T\mathbf{y}(t))\| \le L_f\|\mathbf{y}(t) - \mathbf{V}\mathbf{V}^T\mathbf{y}(t)\| = L_f\|\rho(t)\|$ so that

$$\|\widehat{\mathbf{r}}(t)\| \le \alpha\|\rho(t)\| + \beta\|\mathbf{w}(t)\|, \tag{4.17}$$

where $\alpha := \|\mathbf{V}^T\mathbb{P}\|L_f$, $\beta := \|\mathbf{V}^T(\mathbf{I}-\mathbb{P})\|$. Thus for all $t \in [0,T]$ and for $\tilde{q}_M(t) := \int_0^t e^{2M[\widehat{\mathbf{F}}](t-\tau)}d\tau = \begin{cases} \frac{1}{2M[\widehat{\mathbf{F}}]}(e^{2M[\widehat{\mathbf{F}}]t}-1), & M[\widehat{\mathbf{F}}] \neq 0 \\ t, & M[\widehat{\mathbf{F}}] = 0 \end{cases}$, we have

$$\|\theta(t)\|^2 \leq \tilde{q}_M(t)\int_0^t \|\widehat{\mathbf{r}}(\tau)\|^2 d\tau \leq 2\tilde{q}_M(T)\left(\alpha^2\int_0^T \|\rho(t)\|^2dt + \beta^2\int_0^T \|\mathbf{w}(t)\|^2dt\right).$$

Using the above bound for $\|\theta(t)\|$ with $\int_0^T \|e(t)\|^2dt = \int_0^T \|\rho(t)\|^2dt + \int_0^T \|\theta(t)\|^2dt$ will give (4.4) in Theorem 4.2 with $c_M := 2\tilde{q}_M(T)$. In the case when $\int_0^T \|\rho(t)\|^2dt = \sum_{\ell=k+1}^r \lambda_\ell^\infty$ and $\int_0^T \|\mathbf{w}(t)\|^2dt = \sum_{\ell=m+1}^{r_s} s_\ell^\infty$,

$$\int_0^T \|e(t)\|^2dt = \int_0^T \|\rho(t)\|^2dt + \int_0^T \|\theta(t)\|^2dt \leq \mathcal{C}(T)\left(\sum_{\ell=k+1}^{r}\lambda_\ell^\infty + \sum_{\ell=m+1}^{r_s} s_\ell^\infty\right),$$

where $\mathcal{C}(T) = \max\{1 + 2T\tilde{q}_M(T)\alpha^2, 2T\tilde{q}_M(T)\beta^2\}$.

4.2. Error bounds in discretized ODE setting. Using our analysis of the full trajectory as a guide, by analogy to (4.1) and (4.2), we shall analyze the discrete systems obtained from backward Euler time integration corresponding to the full-order system and the POD-DEIM reduced system in the form

$$\frac{Y_j - Y_{j-1}}{\triangle t} = \mathbf{F}(t_j, Y_j), \qquad \frac{\widehat{Y}_j - \widehat{Y}_{j-1}}{\triangle t} = \widehat{\mathbf{F}}(t_j, \widehat{Y}_j), \tag{4.18}$$

for $j = 0, \ldots, n_t$, where $n_t = T/\triangle t$ is the number of time steps with time step $\triangle t$ chosen so that $\triangle t M[\widehat{\mathbf{F}}] < 1$. (This condition is required as shown later in this section.) In (4.18), Y_j and $\widehat{Y}_j$ are approximations of $\mathbf{y}(t_j)$ and $\widehat{\mathbf{y}}(t_j)$, respectively, at $t_j = j\triangle t$. Consider the error

$$E_j = Y_j - \mathbf{V}\widehat{Y}_j,$$

where Y_j is the solution of full-order system and $\widehat{Y}_j$ is the solution of the POD-DEIM reduced system in (4.18) for $j = 1, \ldots, n_t$. Write

$$E_j = \rho_j + \theta_j,$$

where $\rho_j := Y_j - \mathbf{V}\mathbf{V}^T Y_j$, $\theta_j := \mathbf{V}\mathbf{V}^T Y_j - \mathbf{V}\widehat{Y}_j$. Define $\widehat{\theta}_j := \mathbf{V}^T\theta_j$. As before, $\theta_j = \mathbf{V}\widehat{\theta}_j$,$\|\theta_j\| = \|\widehat{\theta}_j\|$ and $\rho_j^T\theta_j = 0$.

From (4.18), we have $\mathbf{V}^T\left(\frac{Y_j - Y_{j-1}}{\triangle t}\right) = \mathbf{V}^T\mathbf{F}(t_j, Y_j)$, and we shall consider

$$\mathbf{V}^T\left(\frac{Y_j - Y_{j-1}}{\triangle t}\right) = \widehat{\mathbf{F}}(t_j, \mathbf{V}^T Y_j) + \widehat{R}_j, \tag{4.19}$$

$$\frac{\widehat{Y}_j - \widehat{Y}_{j-1}}{\triangle t} = \widehat{\mathbf{F}}(t_j, \widehat{Y}_j), \tag{4.20}$$

where

$$\widehat{R}_j = \mathbf{V}^T\mathbf{F}(t_j, Y_j) - \widehat{\mathbf{F}}(t_j, \mathbf{V}^T Y_j) \tag{4.21}$$

so that for $\widehat{\theta}_j = \mathbf{V}^T Y_j - \widehat{Y}_j$, $\frac{\widehat{\theta}_j - \widehat{\theta}_{j-1}}{\triangle t} = \widehat{\mathbf{F}}(t_j, \mathbf{V}^T Y_j) - \widehat{\mathbf{F}}(t_j, \widehat{Y}_j) + \widehat{R}_j$. Then,

$$\begin{aligned}
\frac{\|\widehat{\theta}_j\| - \|\widehat{\theta}_{j-1}\|}{\triangle t} &\le \frac{1}{\triangle t}\left(\frac{\langle \widehat{\theta}_j, \widehat{\theta}_j\rangle}{\|\widehat{\theta}_j\|} - \frac{\langle \widehat{\theta}_j, \widehat{\theta}_{j-1}\rangle}{\|\widehat{\theta}_j\|}\right) \\
&= \frac{1}{\|\widehat{\theta}_j\|}\left\langle \widehat{\theta}_j, \frac{\widehat{\theta}_j - \widehat{\theta}_{j-1}}{\triangle t}\right\rangle \\
&= \frac{1}{\|\widehat{\theta}_j\|}\left\langle \widehat{\theta}_j, \widehat{\mathbf{F}}(t_j, \mathbf{V}^T Y_j) - \widehat{\mathbf{F}}(t_j, \widehat{Y}_j) + \widehat{R}_j\right\rangle \\
&= \frac{1}{\|\widehat{\theta}_j\|}\left\langle \widehat{\theta}_j, \widehat{\mathbf{F}}(t_j, \mathbf{V}^T Y_j) - \widehat{\mathbf{F}}(t_j, \widehat{Y}_j)\right\rangle + \frac{1}{\|\widehat{\theta}_j\|}\langle \widehat{\theta}_j, \widehat{R}_j\rangle \\
&\le M[\widehat{\mathbf{F}}]\|\widehat{\theta}_j\| + \|\widehat{R}_j\|,
\end{aligned}$$

where we have used $\langle \widehat{\theta}_j, \widehat{\theta}_{j-1}\rangle \le \|\widehat{\theta}_j\|\|\widehat{\theta}_{j-1}\|$ in the first inequality and used $\langle \widehat{\theta}_j, \widehat{\mathbf{F}}(\mathbf{V}^T Y_j) - \widehat{\mathbf{F}}(\widehat{Y}_j)\rangle \le M[\widehat{\mathbf{F}}]\|\widehat{\theta}_j\|^2$ from (4.3) and $\langle \widehat{\theta}_j, \widehat{R}_j\rangle \le \|\widehat{\theta}_j\|\|\widehat{R}_j\|$ in the last inequality. That is, using $\|\widehat{\theta}_j\| = \|\theta_j\|$ for $\zeta := \frac{1}{1-\triangle t M[\widehat{\mathbf{F}}]}$,

$$\begin{aligned}
\|\theta_j\| &\le \zeta\left(\|\theta_{j-1}\| + \triangle t\|\widehat{R}_j\|\right) \le \zeta^j\|\theta_0\| + \triangle t\sum_{\ell=1}^{j}\zeta^\ell\|\widehat{R}_{j-\ell+1}\| \\
&\le \triangle t\left(q_j\sum_{\ell=1}^{j}\|\widehat{R}_\ell\|^2\right)^{1/2},
\end{aligned}$$

where $q_j := \sum_{\ell=1}^{j}\zeta^{2\ell}$ and we have used $\theta_0 = 0$. Note that since $\triangle t M[\widehat{\mathbf{F}}] < 1$, we have $\zeta > 0$. As in the continuous case, we recast $\|\widehat{R}_\ell\|$ as a sum of differences that can be estimated using the neglected singular values.

$$\begin{aligned}
\widehat{R}_\ell &= \mathbf{V}^T\mathbf{F}(t_\ell, Y_\ell) - \widehat{\mathbf{F}}(t_\ell, \mathbf{V}\mathbf{V}^T Y_\ell) = \mathbf{V}^T[\mathbf{F}(t_\ell, Y_\ell) - \mathbb{P}\mathbf{F}(t_\ell, \mathbf{V}\mathbf{V}^T Y_\ell)] \\
&= \mathbf{V}^T[\mathbf{F}(t_\ell, Y_\ell) - \mathbb{P}\mathbf{F}(t_\ell, Y_\ell) + \mathbb{P}\mathbf{F}(t_\ell, Y_\ell) - \mathbb{P}\mathbf{F}(t_\ell, \mathbf{V}\mathbf{V}^T Y_\ell)] \\
&= \mathbf{V}^T(\mathbf{I} - \mathbb{P})\mathbf{w}_\ell + \mathbf{V}^T\mathbb{P}(\mathbf{F}(t_\ell, Y_\ell) - \mathbf{F}(t_\ell, \mathbf{V}\mathbf{V}^T Y_\ell)),
\end{aligned}$$

where $\mathbf{w}_\ell = (\mathbf{I} - \mathbf{U}\mathbf{U}^T)\mathbf{F}(t_\ell, Y_\ell)$ from (2.11) in Lemma 2.1. The Lipschitz continuity of $\mathbf{F}$ implies $\|\mathbf{F}(t_\ell, Y_\ell) - \mathbf{F}(t_\ell, \mathbf{V}\mathbf{V}^T Y_\ell)\| \le L_f\|Y_\ell - \mathbf{V}\mathbf{V}^T Y_\ell\| = L_f\|\rho_\ell\|$, and thus

$$\|\widehat{R}_\ell\| \le \alpha\|\rho_\ell\| + \beta\|\mathbf{w}_\ell\|, \tag{4.22}$$

where $\alpha := \|\mathbf{V}^T\mathbb{P}\|L_f$, $\beta := \|\mathbf{V}^T(\mathbf{I} - \mathbb{P})\|$. Let $q := q_{n_t} = \sum_{\ell=1}^{n_t}\zeta^{2\ell}$, so for $j = 0, \ldots, n_t$,

$$\|\theta_j\|^2 \le \triangle t^2 q\left(\sum_{\ell=1}^{j}\|\widehat{R}_\ell\|^2\right) \le \triangle t^2 2q\left(\alpha^2\sum_{\ell=1}^{j}\|\rho_\ell\|^2 + \beta^2\sum_{\ell=1}^{j}\|\mathbf{w}_\ell\|^2\right).$$

Applying the above bound for $\|\theta_j\|$ to $\sum_{\ell=0}^{n_t}\|E_\ell\|^2 = \sum_{\ell=0}^{n_t}\|\rho_\ell\|^2 + \sum_{\ell=0}^{n_t}\|\theta_\ell\|^2$ gives the error bound (4.5) in Theorem 4.2. In the case when $\sum_{\ell=0}^{n_t}\|\rho_\ell\|^2 = \sum_{\ell=k+1}^{r}\lambda_\ell$ and $\sum_{\ell=0}^{n_t}\|\mathbf{w}_\ell\|^2 = \sum_{\ell=m+1}^{r_s}s_\ell$,

$$\sum_{\ell=0}^{n_t}\|E_\ell\|^2 = \sum_{\ell=0}^{n_t}\|\rho_\ell\|^2 + \sum_{\ell=0}^{n_t}\|\theta_\ell\|^2 \le \bar{\mathcal{C}}(T)\left(\sum_{\ell=k+1}^{r}\lambda_\ell + \sum_{\ell=m+1}^{r_s}s_\ell\right), \tag{4.23}$$

where $\bar{\mathcal{C}}(T) = \max\{1 + \bar{c}_M\alpha^2 T, \bar{c}_M\beta^2 T\}$ and for $\bar{c}_M = 2q\triangle t$, $T = n_t\triangle t$. When $M[\widehat{\mathbf{F}}] < 0$ for all $j = 1, 2, \ldots, n_t$, $q_j = \sum_{\ell=1}^{j}\zeta^{2\ell} \leq \sum_{\ell=1}^{\infty}\zeta^{2\ell} = \sum_{\ell=0}^{\infty}\zeta^{2\ell} - 1 = \frac{1}{1-\zeta^2} - 1 = \frac{1}{(1-\triangle t M[\widehat{\mathbf{F}}])^2-1}$. Therefore, in this case, with fixed $\triangle t$, the norm of the total error $\|E_j\|$ is uniformly bounded for all $j = 1, \ldots, n_t$, as shown below:

$$\|E_\ell\|^2 = \|\rho_\ell\|^2 + \|\theta_\ell\|^2 \quad \leq \bar{c}\left(\sum_{\ell=k+1}^{r}\lambda_\ell + \sum_{\ell=m+1}^{r_s} s_\ell\right), \tag{4.24}$$

where $\bar{c} = \max\{1 + \bar{q}\alpha^2, \bar{q}\beta^2\}$ and $\bar{q} = \frac{2q\triangle t}{(1-\triangle t M[\widehat{\mathbf{F}}])^2-1} = \frac{1}{|M[\widehat{\mathbf{F}}]| + \triangle t M[\widehat{\mathbf{F}}]^2/2}$ as given in (4.13) of Remark 4.3.

5. Conclusion. This paper provides error bounds for the state approximations from POD-DEIM reduced systems of ODEs with Lipschitz continuous nonlinearities. The asymptotic error analysis was considered in the continuous setting where we assumed the availability of the solutions on the entire time interval and the overall accuracy of the reduced system was only contributed from applying the POD-DEIM technique. This paper also presented a framework for error analysis in the discrete setting for the implicit Euler time integration scheme, which can be extended to other numerical methods. The proposed error bounds in both continuous and discrete settings were derived through a standard approach using logarithmic norms as well as through an application of generalized logarithmic norms [27]. The conditions under which the reduction error is uniformly bounded were also discussed. The resulting error bounds in 2-norm reflect the approximation property of POD-based schemes through the decay of the corresponding singular values.

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REFERENCES

[1] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera, *An "Empirical Interpolation" method: Application to efficient reduced-basis discretization of partial differential equations*, Comptes Rendus Math., 339 (2004), pp. 667–672.

[2] S. Chaturantabut and D. C. Sorensen, *Application of POD and DEIM to Dimension Reduction of Nonlinear Miscible Viscous Fingering in Porous Media*, Technical report TR09-25, CAAM, Rice University, Houston, TX, July 2009.

[3] S. Chaturantabut and D. C. Sorensen, *Nonlinear model reduction via discrete empirical interpolation*, SIAM J. Sci. Comput., 32 (2010), pp. 2737–2764.

[4] G. Dahlquist, *Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations*, Trans. Royal Institute of Technology 130, Stockholm, Sweden, 1959.

[5] G. Dahlquist, 33 *years of numerical instability, part* I, BIT, 25 (1985), pp. 188–204.

[6] J. L. Eftang, M. A. Grepl, and A. T. Patera, *A posteriori error bounds for the empirical interpolation method*, Comptes Rendus Math., 348 (2010), pp. 575–579.

[7] F. Ebert, *A Note on POD Model Reduction Methods for DAEs*, Preprint 06-364 (Matheon), Institut für Mathematik, TU Berlin, 2006.

[8] D. Galbally, K. Fidkowski, K. Willcox, and O. Ghattas, *Non-linear model reduction for uncertainty quantification in large-scale inverse problems*, Int. J. Numer. Methods Engrg., 81 (2010), pp. 1581–1608.

[9] M. A. Grepl, *Reduced-Basis Approximations and A Posteriori Error Estimation for Parabolic Partial Differential Equations*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 2005.

[10] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math., 20 (1919), pp. 292–296.

[11] B. Haasdonk, M. Ohlberger, and G. Rozza, *A reduced basis method for evolution schemes with parameter-dependent explicit operators*, Electron. Trans. Numer. Anal., 32 (2008), pp. 145–168.

[12] M. Hinze and M. Kunkel, *Discrete Empirical Interpolation in POD Model Order Reduction of Drift-Diffusion Equations in Electrical Networks*, Hamburger Beiträge zur Angewandten Mathematik, 2010.

[13] C. Homescu, L. R. Petzold, and R. Serban, *Error estimation for reduced-order models of dynamical systems*, SIAM J. Numer. Anal., 43 (2005), pp. 1693–1714.

[14] C. Homescu, L. R. Petzold, and R. Serban, *Error estimation for reduced-order models of dynamical systems*, SIAM Rev., 49 (2007), pp. 277–299.

[15] A. R. Kellems, S. Chaturantabut, D. C. Sorensen, and S. J. Cox, *Morphologically accurate reduced order modeling of spiking neurons*, J. Comput. Neurosci., 2010, DOI:10.1007/s10827-010-0229-4.

[16] K. Kunisch and S. Volkwein, *Galerkin proper orthogonal decomposition methods for parabolic problem*, Numer. Math., 90 (2001), pp. 117–148.

[17] K. Kunisch and S. Volkwein, *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics*, SIAM J. Numer. Anal., 40 (2002), pp. 492–515.

[18] R. Mattheij and J. Molenaar, *Ordinary Differential Equations in Theory and Practice*, SIAM, Philadelphia, 2002.

[19] M. Meyer and H. G. Matthies, *Efficient model reduction in non-linear dynamics using the Karhunen-Loève expansion and dual-weighted-residual methods*, Comput. Mechan., 31 (2003), pp. 179–191.

[20] N. C. Nguyen, *Reduced-Basis Approximation and A Posteriori Error Bounds for Nonaffine and Nonlinear Partial Differential Equations: Application to Inverse Snalysis*, Ph.D. thesis, Singapore-MIT Alliance, National University of Singapore, 2005.

[21] N. C. Nguyen, *A posteriori error estimation and basis adaptivity for reduced-basis approximation of nonaffine-parametrized linear elliptic partial differential equations*, J. Comput. Phys., 227 (2007), pp. 983–1006.

[22] A. T. Patera and G. J. Rozza, *Reduced basis approximation and a posteriori error estimation for parametrized partial differential equations, version* 1.0, Massachusetts Institute of Technology, Cambridge, MA, 2006.

[23] C. Prud'homme, D. V. Rovas, K. Veroy, L. Machiels, Y. Maday, A. T. Patera, and G. Turinici, *Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods*, J. Fluids Engrg., 124 (2002), pp. 70–80.

[24] M. Rathinam and L. R. Petzold, *A new look at proper orthogonal decomposition*, SIAM J. Numer. Anal., 41 (2003), pp. 1893–1925.

[25] M. Rewienski and J. White, *A trajectory piecewise-linear approach to model order reduction and fast simulation of nonlinear circuits and micromachined devices*, IEEE Trans. Computer-Aided Design Integrated Circuits Systems, 22 (2003), pp. 155–170.

[26] G. Rozza, *Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity*, Appl. Numer. Math., 55 (2005), pp. 403–424.

[27] G. Söderlind, *Bounds on nonlinear operators in finite-dimensional banach spaces*, Numer. Math., 50 (1986), pp. 27–44.

[28] G. Söderlind, *The logarithmic norm. History and modern theory*, BIT Numer. Math., 46 (2006), pp. 631–652.

[29] T. Ström, *On logarithmic norms*, SIAM J. Numer. Anal., 12 (1975), pp. 741–753.

[30] S. Utku, J. L. M. Clemente, and M. Salama, *Errors in reduction methods*, Comput. Struc., 21 (1985), pp. 1153–1157.

[31] S. Volkwein, *Model Reduction Using Proper Orthogonal Decomposition*, http://www.uni-graz.at/imawww/volkwein/POD.pdf (April 2008).