

## GROWTH OF FUNCTIONS IN *CERCLES DE REMPLISSAGE*

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### Abstract

Suppose that  $f$  is meromorphic in the plane, and that there is a sequence  $z_n \rightarrow \infty$  and a sequence of positive numbers  $\epsilon_n \rightarrow 0$ , such that  $\epsilon_n |z_n| f^\#(z_n) / \log |z_n| \rightarrow \infty$ . It is shown that if  $f$  is analytic and non-zero in the closed discs  $\Delta_n = \{z : |z - z_n| \leq \epsilon_n |z_n|\}$ ,  $n = 1, 2, 3, \dots$ , then, given any positive integer  $K$ , there are arbitrarily large values of  $n$  and there is a point  $z$  in  $\Delta_n$  such that  $|f(z)| > |z|^K$ . Examples are given to show that the hypotheses cannot be relaxed.

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### 1. Introduction

For a meromorphic function  $f$ , discs of the form

$$\Delta(z_n, \epsilon_n |z_n|) = \{z : |z - z_n| < \epsilon_n |z_n|\},$$

where  $z_n \rightarrow \infty$  and  $\epsilon_n \rightarrow 0$ , are called *cercles de remplissage* if  $f$  takes every extended complex value with at most two exceptions infinitely often in any infinite subcollection of them. Lehto [2] pointed out the close connection between the spherical derivative:

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

and *cercles de remplissage*: if, for a sequence  $z_n \rightarrow \infty$ , there exist positive numbers  $\epsilon_n \rightarrow 0$  such that

$$\epsilon_n |z_n| f^\#(z_n) \rightarrow \infty,$$

then  $\Delta(z_n, 2\epsilon_n|z_n|)$  is a sequence of *cercles de remplissage*; and conversely, if  $\Delta(z_n, \epsilon_n|z_n|)$  is a sequence of *cercles de remplissage*, then for each  $n$  there is a point  $z'_n \in \Delta(z_n, 2\epsilon_n|z_n|)$  such that

$$\epsilon_n|z'_n|f^{\#}(z'_n) \rightarrow \infty.$$

This note is concerned with the way in which the growth of  $f$  in *cercles de remplissage* is related to the growth of  $f^{\#}$ . We will prove:

**THEOREM 1.** *Suppose that  $f$  is meromorphic in the plane, and that there is a sequence  $z_n \rightarrow \infty$  and a sequence of positive numbers  $\epsilon_n \rightarrow 0$ , such that*

$$(1) \quad \epsilon_n|z_n|f^{\#}(z_n)/\log|z_n| \rightarrow \infty.$$

*If  $f$  is analytic and non-zero in the closed discs*

$$\Delta_n = \{z : |z - z_n| \leq \epsilon_n|z_n|\}, \quad n = 1, 2, 3, \dots,$$

*then  $f$  grows transcendently there, that is, given any positive integer  $K$ , there are arbitrarily large values of  $n$  and there is a point  $z$  in  $\Delta_n$  such that*

$$(2) \quad |f(z)| > |z|^K.$$

(Notice that, from Lehto's theorem, the discs  $\Delta_n$  form a sequence of *cercles de remplissage*; in fact the radius could be reduced to  $2\epsilon_n|z_n|/\log|z_n|$ .) Theorem 1 cannot be significantly improved. As we will show, there is a function  $f$ , and a sequence  $z_n \rightarrow \infty$ , such that

$$(3) \quad 0 < \lim_{n \rightarrow \infty} |z_n|f^{\#}(z_n)/\log|z_n| < \infty,$$

and such that  $f$  is non-zero in  $\Delta(z_n, |z_n|/2)$  and satisfies  $f(z) = O(z)$  in  $\Delta(z_n, |z_n|/2)$ . Thus (1) cannot be relaxed. Also, there is a function  $f$ , and there are sequences  $z_n \rightarrow \infty$  and  $\epsilon_n \rightarrow 0$ , such that (1) just holds, while  $f(z) = O(z)$  in  $\Delta(z_n, \epsilon_n|z_n|)$ . Every such  $z_n$  is a zero of  $f$ , and thus the hypothesis that  $f$  does not vanish in  $\Delta_n$  cannot be omitted from Theorem 1.

Two remarks are in order. First of all, in [1], the authors use the fact that all transcendental entire functions satisfy (1) to prove the existence of *cercles de remplissage* in which (2) holds. One need not assume that the function is non-zero in these *cercles*. Theorem 1 together with the second example mentioned above show that the situation is quite different for meromorphic functions.

Secondly, Theorem 1 is connected with the existence of Hayman directions. A direction  $\theta \in [0, 2\pi]$  is said to be a *Hayman direction* for a meromorphic function

$f$  if, given  $\epsilon > 0$ , either  $f$  takes all complex values infinitely often in the region  $D = \{z : |\arg z - \theta| < \epsilon\}$  or else all its derivatives take all complex values, except possibly zero, infinitely often there. The authors have shown [1, Theorem 2] that every transcendental entire function has a Hayman direction. The proof depends on the fact that every transcendental entire function has a sequence of *cercles de remplissage* in which it grows transcendently [1, Theorem 1], and it is easily seen that, in view of Theorem 1 of the present paper, the same argument can be used to prove the following theorem:

**THEOREM 2.** *Every meromorphic function satisfying (1) has a Hayman direction.*

This complements a result of Yang Lo [5], who showed that a meromorphic function  $f$  has a Hayman direction if

$$(4) \quad \limsup_{r \rightarrow \infty} T(r, f) / (\log r)^3 = \infty.$$

The two statements appear to be independent.

## 2. Proof of Theorem 1

If Theorem 1 is false then there is a meromorphic function  $f$ , analytic and non-zero in the union of the discs  $\Delta_n$ , and a positive number  $K$ , such that

$$(5) \quad |f(z)| \leq |z|^K, \quad z \in \Delta_n,$$

for all large  $n$ . For  $z \in \Delta_n$ , write  $f(z) = e^{p_n(z)}$ , where  $p_n$  is analytic in  $\Delta_n$ , so that

$$(6) \quad f^{\#}(z) = |p'_n(z)| \frac{e^{u_n(z)}}{1 + e^{2u_n(z)}},$$

where  $u_n = \text{Re } p_n$ . Since  $0 \leq t/(1 + t^2) \leq 1/2$  if  $t \geq 0$ ,  $f^{\#}(z) \leq (1/2)|p'_n(z)|$ , and therefore

$$(7) \quad |p'_n(z_n)| \geq 2M_n \log |z_n|/|z_n|,$$

where

$$M_n = |z_n| f^{\#}(z_n) / \log |z_n|.$$

Writing  $p_n(z) = \sum_{j=0}^{\infty} a_j(n)(z - z_n)^j$  and defining  $A_n(t) = \max_{|z-z_n|=t} u_n(z)$ , we have, from (5),  $A_n(t) \leq K \log |z_n|$ , for  $0 \leq t \leq \epsilon_n |z_n|$ . Further [4, page 86],

$$|a_j(n)| t^j \leq 4 \max\{A_n(t), 0\} - 2u_n(z_n),$$

for all  $j$ . With  $j = 1$  and  $t = \epsilon_n |z_n|$ , we obtain

$$(8) \quad \begin{aligned} |p'_n(z_n)| &\leq (4 \max\{A_n(\epsilon_n |z_n|), 0\} - 2u_n(z_n)) / (\epsilon_n |z_n|) \\ &\leq (4K \log |z_n| - 2u_n(z_n)) / (\epsilon_n |z_n|), \end{aligned}$$

and therefore, in view of (7),

$$\epsilon_n M_n \log |z_n| \leq 2K \log |z_n| - u_n(z_n).$$

Since  $\epsilon_n M_n \rightarrow \infty$ , from (1), we deduce that  $u_n(z_n) \leq -(1 + o(1))\epsilon_n M_n \log |z_n|$ , and further, from (8),

$$|p'_n(z_n)| \leq (2 + o(1))|u_n(z_n)| / (\epsilon_n |z_n|).$$

Returning to (6), we obtain, with  $z = z_n$ ,

$$\epsilon_n |z_n| f^*(z_n) \leq (2 + o(1))|u_n(z_n)| e^{u_n(z_n)} \rightarrow 0.$$

But  $\epsilon_n |z_n| f^*(z_n) = \epsilon_n M_n \log |z_n| \rightarrow \infty$ , a contradiction, which proves the theorem. □

### 3. Two examples

The first example is

$$(9) \quad f(z) = \prod_{n=1}^{\infty} \frac{z + e^{\sqrt{n}}}{z - e^{\sqrt{n}}}.$$

Rossi [3] has shown that  $f(z) = O(z)$  in any small sector about the imaginary axis, and  $f$  is evidently non-zero there. We will show that

$$(10) \quad \lim_{t \rightarrow +\infty} t f^*(\pm it) / \log t = \pi/2.$$

Differentiating  $\log f$  we have, with  $z = \pm it$ ,

$$(11) \quad \frac{f'(z)}{f(z)} = \sum_1^{\infty} \frac{2e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}}.$$

Fix  $N$  such that  $e^{\sqrt{N}} \leq t < e^{\sqrt{N+1}}$ . Since  $X/(T^2 + X^2)$  increases for  $0 \leq X \leq T$  and decreases after that,

$$\int_0^{N-1} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx \leq \sum_1^{N-1} \frac{e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}} \leq \int_1^N \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx,$$

and

$$\int_{N+3}^{\infty} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx \leq \sum_{N+2}^{\infty} \frac{e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}} \leq \int_{N+1}^{\infty} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx.$$

Using simple estimates on terms of the form  $e^{\sqrt{w}}/(t^2 + e^{\sqrt{w}})$ , it follows that

$$\sum_1^{\infty} \frac{e^{\sqrt{n}}}{t^2 + e^{2\sqrt{n}}} = \int_0^{\infty} \frac{e^{\sqrt{x}}}{t^2 + e^{2\sqrt{x}}} dx + O(t^{-1}).$$

After a change of variable the integral on the right hand side is

$$\frac{2}{t} \int_{1/t}^{\infty} \frac{\log u + \log t}{1 + u^2} du,$$

and since

$$\int_0^{\infty} \frac{\log u}{1 + u^2} du = 0 \quad \text{and} \quad \int_0^{\infty} \frac{1}{1 + u^2} du = \frac{1}{2}\pi,$$

we obtain, from (11),

$$\frac{f'(z)}{f(z)} = (\pi + o(1)) \frac{\log t}{t}.$$

And (10) follows from this since, for  $z = \pm it$ ,  $|f(z)| = 1$ .

The second example is

$$(12) \quad f(z) = \prod_{n=1}^{\infty} \frac{1 - z/a_n}{1 - z/b_n},$$

where  $a_n = e^{3^n/\psi_n}$ ,  $b_n = 3a_n$ , and  $\psi_n$  is a positive increasing sequence that tends slowly to infinity, and is such that, for all large  $N$ ,

$$(13) \quad 3^n/\psi_n \leq 3^N/(2\psi_N), \quad n < N.$$

Let  $\epsilon_n = 1/\sqrt{\psi_n}$ . Using the fact that  $a_n < \sqrt{a_N}$  if  $n < N$  and  $a_n > \sqrt{a_N}$  if  $n > N$ , it follows from (13) that, for  $|z - a_N| < \epsilon_N a_N$ ,

$$(1 - z/a_n)/(1 - z/b_n) = (1 + O(e^{-3^n/2\psi_n}))b_n/a_n,$$

for  $n < N$ , and

$$(1 - z/a_n)/(1 - z/b_n) = 1 + O(e^{-3^n/2\psi_n}),$$

for  $n > N$ . Thus, if  $|z - a_N| < \epsilon_N a_N$ ,

$$(14) \quad f(z) = (1 + o(1))3^{N-1}(1 - z/a_N)/(1 - z/b_N) = (1 + o(1))3^N(1 - z/a_N).$$

Writing  $F(z) = f(z)/(1 - z/a_N)$ , we obtain  $f'(a_N) = -F(a_N)/a_N$ , and so, by (14),

$$f'(a_N) = -(1 + o(1))3^N/a_N.$$

We conclude that

$$\epsilon_n a_N f'(a_N) / \log a_N = (1 + o(1))3^N \epsilon_n / \log a_N = \sqrt{\psi_N},$$

so that (1) holds with  $z_n = a_n$ . Moreover, from (14), if  $|z - a_N| < \epsilon_N a_N$ , then

$$|f(z)| \leq (1 + o(1))3^N \epsilon_N = (1 + o(1))3^N / \sqrt{\psi_N} \leq (1 + o(1))\sqrt{\psi_N} \log a_N.$$

Thus, with  $\psi_n = n$  say, which satisfies (13) for all  $N \geq 3$ ,  $f$  does not grow transcendently in the discs  $\Delta(a_n, \epsilon_n a_n)$ .

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