ANALYTICAL SOLUTIONS FOR THE STATICS AND DYNAMICS OF RECTANGULAR LAMINATED COMPOSITE PLATES USING SHEARING DEFORMATION THEORIES

by

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Dissertation Submitted to the Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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October 1986

Blacksburg, Virginia

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(ABSTRACT)

The Levy-type analytical solutions in conjunction with the state-space concept are developed for symmetric laminated composite rectangular plates. Combinations of simply-supported, free and clamped boundary conditions are considered. The solutions are obtained for the first-order and higher-order theories in predicting the transverse deflections and stresses. Numerical results are presented for various boundary conditions, aspect ratios, lamination schemes and different loadings.

The developments of these theories accomplished in general coordinates allow one to fulfill both the invariance requirements and to derive the relevant equations in any convenient planar systems of coordinates.

The dynamic response problems are analyzed in the framework of higher order theories where the effects of transverse normal stress and rotary inertia forces are evaluated.

The comparison between the theories as well as previously reported results is reported.

ACKNOWLEDGEMENTS

The author would like to extend his appreciation to Dr. J. N. Reddy, the chairman of my committee, for his thoughtfulness, guidance and counselling throughout the duration of this study.

I wish to thank Dr. L. Librescu for many helpful ideas during the study and for his time and effort. Special thanks are also expressed to Dr. D. Frederick, Dr. C. W. Smith and Dr. R. H . Plaut for giving their remarks and time in reading this dissertation.

I would like to express my gratitude to my parents, brothers and sisters whose love and support made all this possible.

The research reported in this dissertation was supported by the Mathematical Sciences Division of the Army Research Office through Grant DAAG-29-85-K-0007 and also was conducted under a Grant NAG-1-459 from NASA-Langley Research Center. The supports are gratefully acknowledged.

Lastly, I am thankful to Mrs. Vanessa McCoy for her patience and skillful typing of this manuscript.

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NOMENCLATURE

$(a \times b)$ or $(\ell_1 \times \ell_2)$	Dimensions of the rectangular plate.
h	Plate thickness
V_i	Displacement vector components
U	Total strain energy of the body
$W(e_{ij})$	Strain energy density
K	Kinetic energy of the body
A	Potential energy of external loadings
H^i	Components of the body force vector per unit mass
σ^i	Stress vector components
σ^{ij}	Stress tensor components
e_{ij}	Strain tensor components
n_i	Components of the outward unit normal vector
τ	Total volume
ρ	Mass density
t_o, t_1	Two arbitrary instants of time t
δ	Variation sign
g_{ij}, g^{ij}	Covariant and contravariant components, respectively, of the spatial metric tensor
$L^{\alpha\beta}_{(n)}, L^{\alpha 3}_{(n)}, L^{33}_{(n)}$	n^{th} order stress couples
$F^{\alpha}_{(n)}, F^{3}_{(n)}$	n^{th} order couples of the body forces
$f^{\alpha}_{(n)}, f^{3}_{(n)}$	n^{th} order couples of the inertia forces
ν_β	Components of the outward unit vector normal to the edge Γ of the boundary lines of the midplane Ω of the plate
$P^{\alpha}_{(n)}, P^{3}_{(n)}$	n^{th} order couples of the external loads

NOMENCLATURE (CONT.)

E^{ijmn}, F_{ijmn}	Tensors of elasticity moduli
δ_{σ}^{ω}	Kronecker delta
$\delta_A, \delta_B, \delta_C$	Tracers, identifying the terms which characterize the effects of σ^{33}, $f^{\pi}_{(1)}$ and that of the inertia forces in σ^{33}, respectively
${}^{(n)}\psi_{\omega}$	Coefficients of $(x^3)^n$ of the expansion of $\sigma^{\omega 3}$ as defined by Eq. (2.55)
$K^2_{(\alpha)}$	Transverse shear correction factors associated to the directions x^{α}
E_1, E_2, E_3	Young's moduli
G_{12}, G_{13}, G_{23}	Shear moduli
$\nu_{12}, \nu_{13}, \nu_{23}$	Poisson's ratios
E, E'	Young's moduli
μ, μ'	Poisson's ratios
G'	Transverse shear modulus
ϕ	Potential function
$\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}$	Permutation symbols
K_0^2, K_1^2, K_2^2	Transverse shear flexibility parameters
$\underset{\sim}{Y}$	State-space column vector
$\underset{\sim}{b}$	Column input load vector
$\underset{\sim}{K}$	Constant column vector
λ_i	Eigenvalues
$[C]$	Matrix of eigenvectors as defined in Eqs. (3.13) and (3.29)
$[C]^{-1}$	Inverse of the eigenvectors matrix
q	Distributed transverse load

NOMENCLATURE (CONT.)

ψ_1, ψ_2	Rotations of the normal to mid-plane about the x_2 and x_1 axis, respectively
$A_{ij}, D_{ij}, F_{ij}, H_{ij}$	Plate stiffness coefficients
$Q_{ij}^{(k)}$	Reduced orthotropic moduli of the K-th lamina
M_i, P_i, Q_i, R_i	Stress resultants
T_{11}, T_{22}	In-plane edge loads
$\overline{V}_{1mn}, \overline{V}_{2mn}, \overline{\phi}_{1mn}, \overline{\phi}_{2mn}, \overline{W}_{mn}$	Eigenmodes
$q_{mn}(t)$	Generalized coordinates
$Q_{mn}(t)$	Generalized force
N_{mn}	Norm
ω_{mn}	Natural frequency
$m_i (i = 0,2,4,6)$	Mass inertias
[C]	Stiffness matrix as defined in Eqs. (3.44) and (3.54)
[S]	Mass matrix
[R]	Stiffness matrix associated to in-plane loading (for buckling case)
DT	Theory based on initial displacement statements
ST	Theory based on initial stress statements
FSDT	First-order theory
HSDT	Higher-order theory
UN	Uniformly distributed load
TR	Triangular distributed load
PL	Point load
SS	Simply supported at $x_2 = -b/2$ and $x_2 = b/2$

NOMENCLATURE (CONT.)

CC	Clamped at $x_2 = -b/2$ and $x_2 = b/2$
FF	Free at $x_2 = -b/2$ and $x_2 = b/2$
SC	Simply supported at $x_2 = -b/2$ and clamped at $x_2 = b/2$
SF	Simply supported at $x_2 = -b/2$ and free at $x_2 = b/2$
CF	Clamped at $x_2 = -b/2$ and free at $x_2 = b/2$
$(1 \rightleftarrows 2)$	Sign indicating that the relations not explicitly written may be obtained by replacing subscript 1 by 2 and vice-versa
$(\ )'$	Ordinary derivative with respect to the coordinate x_2
$(\ )_{,i}$	Partial derivative with respect to the coordinate x^i
$(\ )\|\|_i$	Covariant derivative of space vectors and tensors
$(\ )\|_\alpha$	Covariant derivative of surface vectors and tensors
$(\dot{\ })$	Partial derivative with respect to the time t

Unless mentioned otherwise, Greek indices take the values 1, 2, while Latin indices the values 1, 2, 3.

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CHAPTER I

INTRODUCTION

1.1 Motivation

The advent of new composite materials and their increasing use in various fields of advanced technology has generated a new interest in the development and solution of consistent refined theories of anisotropic plates. This interest is due to the fact that the classical plate theory, in terms of its basic assumptions, comes in conflict with real behavior of these new materials. For example, in contrast to the basic assumption of infinite rigidity in transverse shear in the classical plate theory, the new composite materials exhibit a finite rigidity in transverse shear. This property requires the incorporation of transverse shear deformation effects.

In addition to other shortcomings, the classical plate theory involves a contradiction between the number of boundary conditions physically required to be fulfilled on a free boundary and the number available in theory, which is to be consistent with the order of the associated governing equations (see Stoker [88]). The non-fulfillment of boundary conditions on the bounding planes constitutes another feature of the classical theory. In recent years attempts were made to refine the classical theory by: (i) incorporating transverse shear effects, (ii) removing the contradiction which concerns the number of boundary conditions to be prescribed at each edge, and (iii) fulfilling the boundary conditions on the bounding surfaces and, in the case of laminated composite plates, the continuity conditions at the interfaces between the contiguous layers. In addition, the refined, transverse shear deformation theories can be used to model such anisotropic plates whose material exhibits high degrees of anisotropy, and are not restricted to the thinness requirement implied by the classical theory. Another feature of a refined plate theory concerns the adequate incorporation of the dynamical effects allowing the evaluation of lowest and the higher natural frequencies, as well.

1.2 Background

The shear deformation plate theories known in the literature can be grouped into two classes: (1) stress-based theories, and (2) displacement-based theories. The first stress-based transverse shear deformable plate theory is due to Reissner [70-72]. The theory is based on a linear distribution of the inplane normal and shear stresses through the thickness,

$$\sigma_{11} = \frac{M_{11}}{(h^2/6)} \frac{x_3}{(h/2)} \;, \quad \sigma_{22} = \frac{M_{22}}{(h^2/6)} \frac{x_3}{(h/2)} \;, \qquad \sigma_{12} = \frac{M_{12}}{(h^2/6)} \frac{x_3}{(h/2)} \tag{1.1}$$

where $(\sigma_{11},\sigma_{22})$ and σ_{12} are the normal and shear stresses, (M_{11},M_{22}) and M_{12} are the associated bending moments (which are functions of the inplane coordinates x_1 and x_2), $|x_3| \leq \frac{h}{2}$, $x_3 = 0$ denotes the mid-plane of the plate and h is the uniform thickness of the plate. The distribution across the thickness of the transverse normal and shear stresses $(\sigma_{33},\sigma_{23}$ and $\sigma_{13})$ is determined through integration over the segment $[0,x_3)$ of the equilibrium equations of the 3-D elasticity theory. The associated field equations and boundary conditions expressed in terms of 2-D quantities can be determined by using the variational principles of the 3-D elasticity theory, or by considering the moments of n^{th} order of the basic equations of 3-D elasticity theory. Both methods allow the reduction of the 3-D problems to a 2-D equivalent one.

The pioneering work of the displacement-based theories is due to Basset [4], who used the following representation of the displacement field across the plate thickness:

$$u_1(x_1,x_2,x_3) = u(x_1,x_2) + \sum_{n=1}^{N} x_3^n \psi_1^{(n)}(x_1,x_2) \tag{1.2}$$

where x_1 and x_2 are the cartesian coordinates in the middle plane of the plate, and the functions $\psi_1^{(n)}$ have the meaning

$$\psi_1^{(n)}(x_1,x_2) = \left.\frac{d^n u_1}{dx_3^n}\right|_{x_3 = 0} \quad , \quad n = 0,1,2,... \tag{1.3}$$

Based on Basset's representation of displacement field, Hildebrand, Reissner and Thomas [21] developed a variationally consistent first order theory [i.e., n = 1 in Eq. (1.2)] for shells. They used the following representation for the displacement field:

$$\begin{aligned} u_1(x_1,x_2,x_3) &= u(x_1,x_2) + x_3\psi_1(x_1,x_2) \\ u_2(x_1,x_2,x_3) &= v(x_1,x_2) + x_3\psi_2(x_1,x_2) \\ u_3(x_1,x_2,x_3) &= w(x_1,x_2) \end{aligned} \tag{1.4}$$

The field equations were derived using the principle of minimum total potential energy. This results in five equilibrium equations expressed in terms of five displacement quantities, u, v, w, ψ_1, ψ_2.

By using the displacement representation of Eq. (1.4) Mindlin [48] extended Hencky's theory [20] of isotropic plates to the dynamic case. Historical evidence (from the review of the literature) points out that the basic idea of the displacement-based first-order shear deformation theory came from Hildebrand, Reissner and Thomas [21] and Hencky [20]. Mindlin should be credited with the extension to the dynamic case. The shear deformation theory based on the displacement field given by Eq. (1.4) is referred as the first-order transverse shear deformation theory (see Reddy [63]).

Following these works, many extensions and applications of the two classes of theories were reported in the literature (see e.g. Ambartsumian [1-3], Bert [5-6], Boal and Reissner [7], Bolle [8], Cheng [9], Gol'denveizer [14-16], Green [17-19], Kromm [25,26], Levinson [27,28], Librescu [29-39], Librescu and Reddy [40,41], Lo, Christensen and Wu [44], Medwadowski [46], Murthy [49], Nelson and Lorch [50], Pagano [51,52], Reddy [56-68], Rehfield and Valisetty [69], Reissner [73-80], Schmidt [84], Shirakawa [85], Volterra [96], Whitney [100-102], Wilson and Boresi [104], Yu [107]).

Gol'denveizer [14] has generalized Reissner's theory [70-72] by replacing the linear distribution of stresses through thickness [see Eq. (1.1)] by a distribution represented by an arbitrary function $\phi(x_3)$:

$$\sigma_{11} = M_{11}\phi(x_3) \quad , \quad \sigma_{22} = M_{22}\phi(x_3) \quad , \quad \sigma_{12} = M_{12}\phi(x_3). \qquad (1.5)$$

After Hildebrand, Reissner, and Thomas [21], Librescu [29-31] made the first attempt to formulate a higher-order theory of anisotropic plates and shells, of general shape, while in [32-35] the higher-order theory of anisotropic shells and plates was substantiated in a rigorous manner. It was shown also that, similar to the classical plate theory, there is an exact splitting of the stretching and bending states of stress in the higher-order theory. These two independent states of stress associated with the higher-order theory are appropriately defined in [32]. Some implications of this splitting of the state of stress into "basic state of stress" and "boundary layer effect" were emphasized in [34-35] and will be considered in this work too.

Kromm [25,26] presented a shear deformation theory that is a special case of Gol'denveizer's extension of Reissner's theory. In Kromm's theory the function $\phi(x_3)$, instead of being arbitrary, is determined such that the transverse shear stresses vanish on the bounding planes of the plate. The displacement field in Kromm's theory is of the form:

$$u_1 = u - x_3 \frac{\partial w}{\partial x_1} + \frac{3}{2} x_3\left(1 - \frac{4}{3} \frac{x_3^2}{h^2}\right)\psi_1$$

$$u_2 = v - x_3 \frac{\partial w}{\partial x_2} + \frac{3}{2} x_3\left(1 - \frac{4}{3} \frac{x_3^2}{h^2}\right)\psi_2$$

$$u_3 = w \tag{1.6}$$

where u, v, w, ψ_1, ψ_2, are displacement functions which are functions of x_1 and x_2 only. Schmidt [84] presented an extension of Kromm's theory by accounting for moderately large deflections (i.e., in the Von-Karman sense).

Extension of the displacement-based theory to the moderately large deflections case is due to Medwadowski [46] and the extension to laminated plates is due to Whitney [100] and Whitney and Pagano [102].

The second- and higher-order displacement-based shear deformation theories have been investigated by Nelson and Lorch [50], Librescu [29-39], Lo, Christensen and Wu [44], Levinson [27], Murthy [49] and Reddy [62,63]. Levinson [27] and Murthy [49] presented a third-order theory that assumes transverse inextensibility. The nine displacement functions were reduced to five by requiring that the transverse shear stresses vanish on the bounding planes of the plate. However, both authors (and also Schmidt [84]) used the equilibrium equations of the first-order theory in their analysis and so they are variationally inconsistent. As a consequence, the higher-order terms of the displacement field are accounted for only in the calculation of the strains but not in the governing differential equations or in the boundary conditions.

Recently, Reddy [60,61,64] corrected these theories by deriving the governing differential equations by means of the virtual work principle. The theory presented in [60] accounts for the von Karman strains but is limited to orthotropic plates, while that in [61] deals with the small-deflection theory of laminated plates. The third-order theory presented in [64] accounts for moderately large rotations, in the same manner as described in [38].

Besides the works quoted in the preceding pages, there exists numerous other works pertaining to refined theories of elastic plates. Thus, in addition to the works of Basset, Reissner, Hencky and Mindlin that have inspired the elaboration of a great number of works, we mention the symbolic method advanced by Lur'e [45] which has generated a series of works devoted to the refined theory of the isotropic flat plates (see e.g., Westbrook [98,99]). Starting from the three-dimensional field equations of elasticity, methods based on the expansion of all the functions characterizing the plane state of stress into power series over the plate thickness (see Schipper [83]), into Legendre's polynomials (see Cicala [11]), into biharmonic polynomials (see Teodorescu [92]), or that resulting by considering the moments of the n^{th} order of the equations of three-dimensional elasticity (see Tiffen [93], Tiffen and Lowe [94]) are developed.

Another method which leads to the static and dynamic theory of isotropic and anisotropic plates is the method of asymptotic integration of the three-dimensional equations of elasticity (see Gol'denveizer [15,16], Reissner [75], and Green and Naghdi [18]). Other methods are of mixed character which use first, the method of expanding the functions (defining the stress and strain state) into Legendre's polynomials, and then the asymptotic integration method (see, for example, Poniatovskii [54]).

The analysis of laminated composite plates subjected to static and dynamic loads has received widespread attention in recent years. 3-D elasticity solutions for the bending (see Pagano [52]), vibration and buckling of simply supported thick orthotropic rectangular plates and laminates were obtained by Srinivas and his co-workers [22,86,87].

The Navier solution of simply supported rectangular plates was developed by Whitney and Leissa [101] for classical laminate theory, Pagano [51], Whitney [100], Bert and Chen [5] and Reddy and Chao [56,65], for the first-order transverse shear deformation theory, and by Reddy and his co-workers [62-67] for a refined shear deformation theory.

The available research works indicate that closed-form solutions based on refined plate theories are developed mainly for simply supported boundary conditions, this case being often used as a test case. The absence of a general algorithm allowing to obtain closed form solutions for other types of boundary conditions provides motivation for the present study.

Papers dealing with the response of plates excited by dynamic loads of known time history have been less in evidence. Librescu [35,36] obtained some conclusions concerning the influence of rotary inertia terms of a moderately thick rectangular flat plate, built up of a transversely-isotropic material. Warburton [97] presented the response of isotropic plates to dynamic loading while Yu [106] and Sun and Whitney [89,91] had analyzed the response of anisotropic plates in cylindrical bending using Mindlin's [48] theory to account for normal shear stiffness. Sun and Chattopadhyay [90] used the plate equations developed by Whitney and Pagano [102] to analyze a specially orthotropic plate subject to a center impact. Dobyns [12] presented an analysis of simply-supported orthotropic plates subjected to static and dynamic loading conditions. The vibration frequencies and mode shapes were then determined and solutions for plate deflections, bending strains, and normal shear forces due to several transient loads were obtained. Reddy [58] obtained the exact form of the spatial variation of the solution to forced motions of rectangular composite plates for two different lamination schemes, under appropriate boundary conditions and sinusoidal distribution of the transverse load. He reduced the problem to the solution of a system of ordinary differential equations in time, which are integrated numerically using Newmark's direct integration method. In [59] he investigated the forced motion of laminated composite plates using a finite element that accounts for the transverse shear strains and rotary inertia.

As the literature in the field fully attests (see [12, 58, 59, 89-91, 103, 106]), the only refined theory used in the treatment of the dynamic response of flat orthotropic plates is the first order transverse shear deformation theory (FSDT). Related to this theory, it is well known that: i) it requires the introduction of a transverse shear correction factor, whose determination, in the case of a composite anisotropic structure, becomes a rather challenging problem (see e.g. [41]); ii) it disregards the effect of transverse normal stress σ_{33}; iii) it involves a constant distribution of transverse shear stresses σ_{13} and σ_{23} across the thickness which prevents from fulfilling the tangential conditions on the bounding planes ($x_3 = \pm h/2$) (while in the case of multilayered composite structures the same shortcoming appears in the fulfillment of continuity conditions at the interfaces between the contiguous layers).

It will be one of the goals of the study to analyze the dynamic response problems in the framework of higher order theories and to compare the results to the ones obtained as per FSDT.

1.3 Objectives of the Present Study

The major objective of this study is to investigate the first-order and higher-order theories in predicting the deflections, stresses, natural frequencies, buckling loads and transient response for laminated composite and orthotropic rectangular plates. More specifically, four major objectives are identified:

1. Theoretical formulations in general coordinates using tensor notations, so that the results could be transcribed easily for any convenient coordinate system as well as to extend the transverse shear deformable plate model by incorporating some effects not included in the analysis so far. These are the full anisotropy of the material and the full dynamic and σ^{33} effects.
2. Development of Levy-type analytical solutions [24,42,68] in conjunction with the state-space concept for symmetric rectangular laminated plates with two opposite edges simply supported and the remaining edges subjected to a combination of free, simply supported and clamped boundary conditions which are not reported in the literature yet.
3. Analyze the dynamic response problems [43] in the framework of higher order theories and to compare the results to the ones obtained as per FSDT. Evaluate the effects of rotary inertias on the dynamic response of orthotropic plates. Finally, the statement prompted in [3,35,36], according to which the interior solution for the transversely-isotropic simply-supported plates yields identical results to the ones obtained

on the basis of the full system of equations, will be proven both for the dynamic response and the static stability problems.

4. Comparison between the theories as well as with previously reported results.

CHAPTER II

MATHEMATICAL FORMULATION IN GENERALIZED COORDINATES

2.1 The Hamilton Principle

The Hamilton principle states that of all displacements (dynamic path) $V_i = V_i(x^1,x^2,x^3,t)$ that satisfy the boundary conditions

$$V_i = \underset{\sim}{V}_i \text{ over } \Omega_v \text{ for } \forall t \; (t_o \leq t \leq t_1) \tag{2.1}$$

(which implies that $\delta V_i = 0$ over Ω_v)

and fulfill also the condition

$$\delta V_i = 0 \text{ at } t = t_o \text{ and } t = t_1 \text{ for } \forall x^i \epsilon \tau \tag{2.2}$$

the actual path (i.e. that satisfies the equations of motion, boundary conditions on Ω_σ, and the constitutive equations) is distinguished by that which renders the following integral stationary:

$$J = \int_{t_o}^{t_1} (U - K + A)dt \tag{2.3}$$

where

$$U = \int_\tau W(e_{ij})d\tau \tag{2.4}$$

denotes the total strain energy of the body, $W(e_{ij})$ being the strain energy density,

$$K = \frac{1}{2} \int_\tau \rho \dot{V}^i \dot{V}_i d\tau \tag{2.5}$$

denotes the kinetic energy of the body ($\dot{V}^i \equiv \frac{\partial}{\partial t} V^i$),

$$A = - \int_\tau \rho H^i V_i d\tau - \int_{\Omega_\sigma} \underset{\sim}{\sigma}^i V_i d\Omega \tag{2.6}$$

denotes the potential energy of external loadings, and

H^i - the components of the body force vector per unit mass;

$\underset{\sim}{\sigma}^i = \underset{\sim}{\sigma}^{ij} n_j$ - components of the stress vector acting on the unit area of the body prescribed on the part Ω_σ of the boundary (here tilda under variables indicates a prescribed quantity);

n_i - components of the outward unit normal vector;

Ω_V - the remaining part of the boundary on which the displacement vector $\vec{V}$ is prescribed;

τ - total volume;

ρ - the mass density;

t_o, t_1 - two arbitrary instants of time t.

The stationarity condition $\delta J = 0$, by virtue of Eqs. (2.3-2.6), can be written as:

$$\delta J = \int_{t_o}^{t_1} [\int_\tau \sigma^{ij} \delta e_{ij} d\tau - \delta K - \int_{\Omega_\sigma} \underset{\sim}{\sigma}^i \delta V_i d\Omega - \int_\tau \rho H^i \delta V_i d\tau] dt = 0 \quad (2.7)$$

where δ stands for the variational symbol. Allowing for the symmetry of the stress tensor, as well as the divergence theorem, we successively obtain:

$$\int_\tau \sigma^{ij} e_{ij} d\tau = \int_\tau \frac{1}{2} \sigma^{ij} (V_i \|_j + V_j \|_i) d\tau = \int_\tau \sigma^{ij} V_i \|_j d\tau$$

$$= \int_\tau [(\sigma^{ij} V_i) \|_j - V_i \sigma^{ij} \|_j] d\tau$$

$$= \int_\Omega \sigma^{ij} V_i n_j d\Omega - \int_\tau \sigma^{ij} \|_j V_i d\tau \quad , \quad (\Omega = \Omega_\sigma U \Omega_V) \qquad (2.8)$$

hence the relation

$$\int_\tau \sigma^{ij}\delta e_{ij}d\tau = \int_{\Omega_\sigma} \sigma^{ij}n_j\delta V_i d\Omega - \int_\tau \sigma^{ij}\|_j \delta V_i d\tau \tag{2.9}$$

where the index j, preceded by a double vertical line, denotes covariant differentiation with respect to the coordinate x^i, using the space metric tensor.

On the other hand, from (2.5), we obtain

$$\delta K = \int_\tau \rho\delta\dot{V}_i\dot{V}^i d\tau = \int_\tau \rho \frac{\partial}{\partial t} (\delta V_i\dot{V}^i)d\tau - \int_\tau \rho\delta V_i\ddot{V}^i d\tau \tag{2.10}$$

Integrating (2.10) over the time interval $[t_0,t_1]$, we get

$$\begin{aligned}\int_{t_0}^{t_1} \delta K dt &= \int_{t_0}^{t_1} [\int_\tau \frac{\partial}{\partial t} (\rho\delta V_i\dot{V}^i)d\tau]dt - \int_{t_0}^{t_1} [\int_\tau \rho\delta V_i\ddot{V}^i d\tau]dt \\ &= \int_\tau [\rho\delta V_i\dot{V}^i]_{t_0}^{t_1} d\tau - \int_{t_0}^{t_1} [\int_\tau \rho\delta V_i\ddot{V}^i d\tau]dt \\ &= - \int_{t_0}^{t_1} dt \int_\tau \rho\ddot{V}^i\delta V_i d\tau, \end{aligned} \tag{2.11}$$

where the assumption that the virtual displacements vanish for $t = t_0$ and $t = t_1$ implies that $[\rho\delta V_i\dot{V}^i]_{t_0}^{t_1} = 0$.

Taking (2.9) and (2.11) into consideration, the variational equation (2.7) may be expressed as follows:

$$\delta J \equiv \delta J_1 + \delta J_2 = 0, \tag{2.12}$$

where

$$\begin{aligned}\delta J_1 &= \int_{t_0}^{t_1} dt \int_\tau [\sigma^{ij}\|_j + \rho(H^i - \ddot{V}^i)]\delta V_i d\tau, \\ \delta J_2 &= \int_{t_0}^{t_1} dt \int_{\Omega_\sigma} (\sigma^i - \underset{\sim}{\sigma}^i)\delta V_i d\Omega \end{aligned} \tag{2.13}$$

It can be shown that the equations of motion as well as the boundary conditions of the three-dimensional theory of linear elasticity can be obtained from (2.12).

Allowing for the steady character of the functional J in the time interval $[t_0,t_1]$ and the fact that the variations δV_i are arbitrary throughout the volume τ and on the boundary Ω, their coefficients in the two integrands in equation (2.12) must vanish independently. This leads to the result

$$\begin{aligned} &\sigma^{ij}\|_j + \rho(H^i - \ddot{V}^i) = 0 \text{ in } \tau \\ &\underset{\sim}{\sigma}^i = n_j\sigma^{ij} \qquad \text{on } \Omega_\sigma, \\ &\delta V_i = 0 \qquad \text{on } \Omega_V, \end{aligned} \tag{2.14}$$

2.2 Modelling of the Higher Order Theory of Flat Plates

2.2.1 Equations of Motion

To obtain the equations of motion of the plate theory we expand the displacement across the thickness as (see [36,39])

$$V_\alpha(x^\omega,x^3,t) = \sum_{n=0}^{N} (x^3)^n \overset{(n)}{V}_\alpha \quad , \quad V_3(x^\omega,x^3,t) = \sum_{n=0}^{N} (x^3)^n \overset{(n)}{V}_3 \tag{2.15}$$

where $\overset{(n)}{V}_i \equiv \overset{(n)}{V}_i(x^\omega,t)$. In Eq. (2.15) The Latin indices run over the range 1,2,3 while the Greek indices may take the values 1,2.

Substituting (2.15) into (2.13) and using the relation

$$d\tau = \sqrt{g}\, dx^1dx^2dx^3 = d\sigma dx^3$$

where $g = \det(g_{ij})$, and g_{ij} denote the components of the spatial metric tensor, we have for arbitrary variations $\delta \overset{(n)}{V}_\alpha$, $\delta \overset{(n)}{V}_3$

$$\delta \overset{(n)}{V}_\alpha;\ \int_{-h/2}^{h/2} [\sigma^{\alpha\omega}|_\omega + \sigma^{\alpha 3}_{,3} + \rho(H^\alpha - \ddot{V}^\alpha)](x^3)^n dx^3 = 0$$

$$\overset{(n)}{\delta V_3};\ \int_{-h/2}^{h/2} [\sigma^{3\alpha}|_\alpha + \sigma^{33}_{,3} + \rho(H^3 - \ddot{V}^3)](x^3)^n dx^3 = 0 \qquad (2.16)$$

Using in (2.16) the integrations by parts

$$\int_{-h/2}^{h/2} \sigma^{\alpha 3}_{,3}(x^3)^n dx^3 = [(x^3)^n \sigma^{\alpha 3}]^{h/2}_{-h/2} - n \int_{-h/2}^{h/2} (x^3)^{n-1} \sigma^{\alpha 3} dx^3$$

$$\int_{-h/2}^{h/2} \sigma^{33}_{,3}(x^3)^n dx^3 = [(x^3)^n \sigma^{33}]^{h/2}_{-h/2} - n \int_{-h/2}^{h/2} (x^3)^{n-1} \sigma^{33} dx^3 \qquad (2.17)$$

the dynamic equilibrium equations for the higher-order plate theory become [36]:

$$L^{\alpha\beta}_{(n)}|_\beta - nL^{\alpha 3}_{(n-1)} + p^\alpha_{(n)} + F^\alpha_{(n)} - f^\alpha_{(n)} = 0$$

$$L^{\alpha 3}_{(n)}|_\alpha - nL^{33}_{(n-1)} + p^3_{(n)} + F^3_{(n)} - f^3_{(n)} = 0 \quad n = (\overline{0,N}) \qquad (2.18)$$

where

$$L^{\alpha\beta}_{(n)} = \int_{-h/2}^{h/2} \sigma^{\alpha\beta}(x^3)^n dx^3 \ ; \quad L^{\alpha 3}_{(n)} = \int_{-h/2}^{h/2} \sigma^{\alpha 3}(x^3)^n dx^3$$

$$L^{33}_{(n)} = \int_{-h/2}^{h/2} \sigma^{33}(x^3)^n dx^3 \qquad (2.19)$$

are the n-th order stress couples;

$$F^\alpha_{(n)} = \int_{-h/2}^{h/2} \rho H^\alpha (x^3)^n dx^3 \ ; \quad F^3_{(n)} = \int_{-h/2}^{h/2} \rho H^3 (x^3)^n dx^3 \qquad (2.20)$$

are the n-th order couples of the body forces;

$$f^\alpha_{(n)} = \int_{-h/2}^{h/2} \rho h^\alpha (x^3)^n dx^3 \ ; \quad f^3_{(n)} = \int_{-h/2}^{h/2} \rho h^3 (x^3)^n dx^3 \qquad (2.21)$$

are the n-th order couples of the inertia forces.

Concerning the inertia terms we have $h^i = \ddot{V}^i$, which leads to

$$h^\alpha = \sum_{p=0}^{N} \overset{(\ddot{p})}{V}{}^\alpha (x^3)^p$$

$$h^3 = \sum_{r=0}^{N'} \overset{(\ddot{r})}{V}{}^3 (x^3)^r \tag{2.22}$$

Substituting (2.22) into (2.21) we get [36]

$$f^\alpha_{(n)} = \sum_{p=0}^{N} \rho\eta(n + p + 1) \overset{(\ddot{p})}{V}{}^\alpha$$

$$f^3_{(n)} = \sum_{r=0}^{N'} \rho\eta(n + r + 1) \overset{(\ddot{r})}{V}_3 \tag{2.23}$$

where

$$\eta(n + p + 1) = \begin{cases} \dfrac{h^{n+p+1}}{(n + p + 1)2^{n+p}} & \text{for } n+p+1 \text{ odd} \\ 0 & \text{for } n+p+1 \text{ even} \end{cases} \tag{2.24}$$

Equations (2.23) may be written also under the form:

$$f^\alpha_{(n)} = \sum_{p=0}^{N} m_{p+n+1} \overset{(\ddot{p})}{V}{}^\alpha$$

$$f^3_{(n)} = \sum_{r=0}^{N'} m_{r+n+1} \overset{(\ddot{r})}{V}_3 \tag{2.25}$$

where $m_n = \rho\eta(n)$ denotes the reduced mass of the plate.

At this point it should be underlined that the same equations of motion could be obtained by taking the n^{th} order moments of the equations of motion of the three dimensional elasticity theory.

2.2.2 Boundary Conditions

$$\text{on } \Omega_\sigma: \quad \nu_\beta L^{\beta\alpha}_{\tilde{\beta}(n)} = \nu_\beta L^{\beta\alpha}_{(n)}$$

$$\nu_\beta L^{\beta 3}_{\tilde{\beta}(n)} = \nu_\beta L^{\beta 3}_{(n)} \tag{2.26}$$

$$\text{on } \Omega_V: \quad \overset{(n)}{V_\alpha} = \overset{(n)}{\underset{\sim}{V}_\alpha} \; ; \quad \overset{(n)}{V_3} = \overset{(n)}{\underset{\sim}{V}_3} \quad n = (\overline{0,N}) \tag{2.27}$$

where ν_β denote the components of the outward unit vector normal to the edge Γ of the boundary lines of the midplane Ω of the plate.

The boundary conditions on the bounding planes $S^{\pm}$ of the plate read:

$$[\sigma^{\alpha 3}(x^3)^n]^{h/2}_{-h/2} = [\underset{\sim}{\sigma}^{\alpha 3}(x^3)^n]^{h/2}_{-h/2}$$

$$[\sigma^{33}(x^3)^n]^{h/2}_{-h/2} = [\underset{\sim}{\sigma}^{33}(x^3)^n]^{h/2}_{-h/2} \tag{2.28}$$

where

$$p^\alpha_{(n)} \equiv [\underset{\sim}{\sigma}^{\alpha 3}(x^3)^n]^{h/2}_{-h/2}$$

$$p^3_{(n)} \equiv [\underset{\sim}{\sigma}^{33}(x^3)^n]^{h/2}_{-h/2} \qquad n = (\overline{0,N}) \tag{2.29}$$

denote the n-th order couples of the external loads.

2.2.3 Constitutive Equations

Assuming that there exist no stresses in the body in the undeformed state, the stress-strain relations written for a physically linear elastic anisotropic body are given by the generalized Hooke's law (see e.g. [36 section 3 of Chapter III]):

$$\sigma^{ij} = E^{ijkm} e_{km} \tag{2.30}$$

Under a more detailed form eq. (2.30) writes

$$\sigma^{\alpha\beta} = E^{\alpha\beta\omega\rho}e_{\omega\rho} + E^{\sigma\beta 33}e_{33} + 2E^{\alpha\beta\alpha 3}e_{\alpha 3} \tag{2.31}$$

We shall now deduce the relations by considering the material anisotropy of the elastic symmetry type with respect to the surface x^3 = const.

Accordingly, the constitutive equations are given by

$$\sigma^{\alpha\beta} = \tilde{E}^{\alpha\beta\omega\rho}e_{\omega\rho} + \frac{E^{\alpha\beta 33}}{E^{3333}}\sigma^{33}$$

$$\sigma^{33} = E^{33\alpha\beta}e_{\alpha\beta} + E^{3333}e_{33}$$

$$\sigma^{\alpha 3} = 2E^{\alpha 3\lambda 3}e_{\lambda 3} \quad , \quad (e_{\lambda 3} = 2F_{\lambda 3\alpha 3}\sigma^{\alpha 3}) \tag{2.32}$$

where

$$\tilde{E}^{\alpha\beta\omega\rho} = E^{\alpha\beta\omega\rho} - \frac{E^{\alpha\beta 33}E^{33\omega\rho}}{E^{3333}} \tag{2.33}$$

satisfy the symmetry relations

$$\tilde{E}^{\alpha\beta\omega\rho} = \tilde{E}^{\beta\alpha\omega\rho} = \tilde{E}^{\alpha\beta\rho\omega} = \tilde{E}^{\beta\alpha\rho\omega} \tag{2.34}$$

where E^{ijmn} and F_{ijmn} denote the anisotropic elastic and compliance tensors respectively.

With the transverse-isotropy type body (the surface of isotropy coinciding in all points with the surface x^3 = const.), the expression (2.33) of the tensors of elasticity moduli are defined by [32,36]

$$\tilde{E}^{\alpha\beta\omega\rho} = \frac{E}{1+\mu}\left[\frac{1}{2}\left(g^{\omega\alpha}g^{\rho\beta} + g^{\alpha\rho}g^{\beta\omega}\right) + \frac{\mu}{1-\mu}g^{\alpha\beta}g^{\omega\rho}\right]$$

and

$$E^{\beta 3\omega 3} = G'g^{\beta\omega}$$

$$F_{\beta 3\omega 3} = \frac{1}{4G'}g_{\beta\omega}$$

$$\frac{E^{\alpha\beta 33}}{E^{3333}} = \frac{E\mu'}{E'(1-\mu)}g^{\beta\omega} \tag{2.35}$$

2.2.4 Strain-Displacement Relations

Within the linear theory of three-dimensional elasticity, the geometric strain-displacement relations are given by:

$$e_{ij} = \frac{1}{2} (V_{i|j} + V_{j|i})$$

thereby yielding

$$e_{\alpha\beta} = \frac{1}{2} (V_{\alpha|\beta} + V_{\beta|\alpha})$$

$$e_{\alpha 3} = \frac{1}{2} (V_{\alpha|3} + V_{3|\alpha})$$

$$e_{33} = V_{3,3} \tag{2.36}$$

According to higher-order plate theory

$$e_{ij} = \sum_{n=0}^{N} (x^3)^n \overset{(n)}{e}_{ij}$$

$$e_{\alpha\beta} = \sum_{n=0}^{N} (x^3)^n \overset{(n)}{e}_{\alpha\beta}$$

where

$$\overset{(n)}{e}_{\alpha\beta} = \frac{1}{2} (\overset{(n)}{V}_{\alpha|\beta} + \overset{(n)}{V}_{\beta|\alpha})$$

$$\overset{(n)}{e}_{\alpha 3} = \frac{1}{2} ((n+1) \overset{(n+1)}{V}_{\alpha} + \overset{(n)}{V}_{3,\alpha})$$

$$\overset{(n)}{e}_{33} = (n+1) \overset{(n+1)}{V}_{3} \tag{2.37}$$

2.3 Displacement-Based Theory

2.3.1 The Displacement Field Across the Plate Thickness

A higher-order shear deformation bending theory of elastic anisotropic flat plates is presented. In its substantiation a nonlinear representation of the tangential displacement field across the thickness is postulated. As per this theory the effects of transverse shear deformation $e_{\omega 3}$ and of transverse normal stress σ^{33} have been incorporated while the tangential static conditions on the bounding planes $x^3 = \pm h/2$ are identically fulfilled. An adequate representation of the displacement field, consistent to the above requirements, writes:

$$V_\alpha(x^\omega, x^3, t) = x^3 \overset{(1)}{V}_\alpha + (x^3)^3 \overset{(3)}{V}_\alpha$$

$$V_3(x^\omega, x^3, t) = \overset{(0)}{V}_3$$

where

$$\overset{(i)}{V}_\alpha \equiv \overset{(i)}{V}_\alpha(x^\omega, t) \quad ; \quad \overset{(0)}{V}_3 \equiv \overset{(0)}{V}_3(x^\omega, t) \tag{2.38}$$

Equations (2.38) result in a nonvanishing $e_{\alpha 3}$ (which represents the effect of transverse shear deformations) and in a zero transverse normal strain e_{33}. By using (2.38) in conjunction with (2.36) and $(2.32)_3$, the bending counterpart of transverse shear stresses assumes the form

$$\sigma^{\alpha 3} = E^{\alpha 3 \omega 3}(\overset{(1)}{V}_\omega + \overset{(0)}{V}_{3,\omega} + 3(x^3)^2 \overset{(3)}{V}_\omega) \tag{2.39}$$

Fulfillment, with the help of (2.39), of static boundary conditions

$$p^\alpha_{(1)} = [\sigma^{\alpha 3} x^3]^{h/2}_{-h/2} \tag{2.40}$$

and employment of the relationship

$$4E^{\alpha 3 \omega 3} F_{\alpha 3 \sigma 3} = \delta^\omega_\sigma \tag{2.41}$$

where δ^ω_σ denotes the Kronecker delta, allow to obtain

$$\overset{(3)}{V}_\sigma = -\frac{4}{3h^2}(V_{3,\sigma} + \overset{(1)}{V}_\sigma - \frac{4}{h} F_{\alpha 3 \sigma 3} p^\alpha_{(1)}) \tag{2.42}$$

which, in the case $p^\alpha_{(1)} \to 0$, reduces to the ones obtained in

[6,27,61,62]. As concerns (2.42), two remarks are in order: i) by fulfilling the condition (2.40) on $x^3 = \pm h/2$, the initial representation (2.38) involving five unknown functions (i.e. $\overset{(1)}{V}_\alpha$, $\overset{(3)}{V}_\alpha$, $\overset{(0)}{V}_3$) reduces to only three unknowns, namely $\overset{(1)}{V}_\alpha$ and $\overset{(0)}{V}_3$ like in the case of a first order transverse shear deformation theory (FSDT). However, in contrast to FSDT, in the present case a non-linear variation of V_α across the plate thickness is considered; ii) by assuming an infinite rigidity in the transverse direction, which yields $F_{\alpha 3\sigma 3} \to 0$, and by considering

$$\overset{(1)}{V}_\alpha = - \overset{(0)}{V}_{3,\alpha} \tag{2.43}$$

from (2.42) we obtain $\overset{(3)}{V}_\alpha \to 0$, which reduces (2.38) to a form appropriate to the Kirchhoff type model.

By virtue of (2.42), (2.38) modifies as:

$$V_\alpha(x^\omega, x^3, t) = x^3 \overset{(1)}{V}_\alpha - \frac{4(x^3)^3}{3h^2}\left(V_{3,\alpha} + \overset{(1)}{V}_\alpha - \frac{4}{h} p^\lambda_{(1)} F_{\lambda 3\alpha 3}\right) \tag{2.44}$$

2.3.2 Determination of Strain Quantities

The strain components associated to (2.38) may easily be determined from the strain-displacement relationships (2.36, 2.37). They may be written as

$$\begin{aligned} e^{\alpha\beta} &= x^3 \overset{(1)}{e}_{\alpha\beta} + (x^3)^3 \overset{(3)}{e}_{\alpha\beta} \\ e_{\alpha 3} &= \overset{(0)}{e}_{\alpha 3} + (x^3)^2 \overset{(2)}{e}_{\alpha 3} \\ e_{33} &= 0, \end{aligned} \tag{2.45}$$

where the strain measures $\overset{(r)}{e}_{ij}$ are expressed in terms of the displacement quantities in the form

$$\overset{(1)}{e}_{\alpha\beta} = \frac{1}{2}(\overset{(1)}{V}_{\alpha|\beta} + \overset{(1)}{V}_{\beta|\alpha})$$

$$\overset{(3)}{e}_{\alpha\beta} = -\frac{2}{3h^2}(2V_{3,\alpha\beta} + \overset{(1)}{V}_{\alpha|\beta} + \overset{(1)}{V}_{\beta|\alpha} - \frac{4}{h}p^{\sigma}_{(1)|\alpha}F_{\sigma3\beta3} - \frac{4}{h}p^{\sigma}_{(1)|\beta}F_{\sigma3\alpha3})$$

$$\overset{(0)}{e}_{\alpha3} = \frac{1}{2}(\overset{(1)}{V}_{\alpha} + \overset{(0)}{V}_{3,\alpha}) \tag{2.46}$$

$$\overset{(2)}{e}_{\alpha3} = -\frac{2}{h^2}(\overset{(0)}{V}_{3,\alpha} + \overset{(1)}{V}_{\alpha} - \frac{4}{h}F_{\sigma3\alpha3}p^{\sigma}_{(1)})$$

2.3.3 Determination of σ^{33}

The integration of the 3-D equation of motion

$$\sigma^{3\alpha}|_{\alpha} + \sigma^{33}_{,3} = \rho\ddot{V}_3 \tag{2.47}$$

across the segment $[0,x^3)$, considered in conjunction with (2.39) yields the expression of σ^{33} as

$$\sigma^{33} = x^3\overset{(1)}{\sigma}{}^{33} + (x^3)^3\overset{(3)}{\sigma}{}^{33} \tag{2.48}$$

where

$$\overset{(1)}{\sigma}{}^{33} = \delta_c\rho\overset{(0)}{\ddot{V}}_3 - E^{\alpha3\omega3}(\overset{(1)}{V}_{\omega|\alpha} + \overset{(0)}{V}_{3|\omega\alpha})$$

$$\overset{(3)}{\sigma}{}^{33} = -E^{\alpha3\omega3}\overset{(3)}{V}_{\omega|\alpha} \tag{2.49}$$

In (2.49) the tracer δ_c identifies the dynamic effect in σ^{33} (taking the values one or zero according to whether this effect is included or disregarded, respectively).

Employment of (2.46) and (2.49) in $(2.32)_1$ results in the bending counterpart of σ^{33} given by

$$\sigma^{\alpha\beta} = \tilde{E}^{\alpha\beta\omega\rho}(x^3 \overset{(1)}{e}_{\omega\rho} + (x^3)^3 \overset{(3)}{e}_{\omega\rho}) + \delta_A \frac{E^{\alpha\beta 33}}{E^{3333}} (x^3 \overset{(1)}{\sigma}{}^{33} + (x^3)^3 \overset{(3)}{\sigma}{}^{33}), \tag{2.50}$$

where the tracer δ_A identifies the contribution brought by σ^{33} in $\sigma^{\alpha\beta}$. The equations (2.44)-(2.50) are basic in the formulation of the refined theory of elastic anisotropic flat plates.

2.3.4 The Equations Governing the Bending Theory

In order to determine a full system of equations expressed in terms of the three basic unknown quantities $\overset{(1)}{V}_\alpha$ and $\overset{(0)}{V}_3$, it will suffice to consider the first three macroscopic equations of motion associated to the bending state of stress. The pertinent equations extracted from (2.18) are:

$$\begin{aligned} &L^{\alpha\rho}_{(1)|\rho} - L^{\alpha 3}_{(0)} - p^{\alpha}_{(1)} - \delta_\beta f^{\alpha}_{(1)} = 0 \\ &L^{\alpha 3}_{(0)|\alpha} + p^{3}_{(0)} - f^{3}_{(0)} = 0, \end{aligned} \tag{2.51}$$

where δ_β is a tracer identifying the contribution brought by the rotary inertia terms.

These three equations of motion are to be expressed in terms of the basic unknowns $\overset{(1)}{V}_\alpha$ and $\overset{(0)}{V}_3$. As a first step towards this end we are to express the stress couples involved in (2.51) in terms of these quantities. Appropriate use of (2.50) and(2.39) in (2.19) yields:

$$L^{\alpha\beta}_{(1)} = \tilde{E}^{\alpha\beta\omega\rho} [\frac{h^3}{30} (\overset{(1)}{V}_{\omega|\rho} + \overset{(1)}{V}_{\rho|\omega}) - \frac{h^3}{60} \overset{(0)}{V}_{3|\omega\rho} + \frac{h^2}{30} (p^{\sigma}_{(1)|\rho} F_{\rho 3\omega 3}$$

$$+ p^{\sigma}_{(1)|\omega}F_{\sigma 3\rho 3})] - \delta_A \frac{h^3}{15} \frac{E^{\alpha\beta 33}}{E^{3333}} E^{\pi 3\gamma 3}(\overset{(0)}{V}_{3|\pi\gamma} + \overset{(1)}{V}_{\gamma|\pi})$$

$$+ \delta_C \frac{h^3}{12} \frac{E^{\alpha\beta 33}}{E^{3333}} \rho \overset{(\ddot{0})}{V}_3 - \delta_A \frac{h^2}{60} \frac{E^{\alpha\beta 33}}{E^{3333}} p^{\pi}_{(1)|\pi} \tag{2.52}$$

$$L^{\alpha 3}_{(0)} = \frac{2}{3} h\, E^{\alpha 3\omega 3}(\overset{(1)}{V}_{\omega} + \overset{(0)}{V}_{3,\omega}) + \frac{1}{3} p^{\alpha}_{(1)}$$

In addition, making use of (2.44) and (2.23), one obtains for the inertia terms the expressions

$$f^{\beta}_{(1)} = \frac{\rho h^3}{60} g^{\beta\lambda}(4 \overset{(\ddot{1})}{V}_{\lambda} - \overset{(\ddot{0})}{V}_{3,\lambda} + \frac{4}{h} \ddot{p}^{\sigma}_{(1)} F_{\sigma 3\lambda 3})$$

$$f^{3}_{(0)} = \rho h \overset{(\ddot{0})}{V}_3, \tag{2.53}$$

where it was supposed that $p^{\lambda}_{(1)} \equiv p^{\lambda}_{(1)}(x^{\omega},t)$ and $p^{3}_{(0)} \equiv p^{3}_{(0)}(x^{\omega},t)$. We now turn our attention to the problem of the explicit determination of the governing equations. Substitution of (2.52) and (2.53) into (2.51), and making use of the various symmetry relations implying $\tilde{E}^{\alpha\beta\omega\rho}$, yields the system of governing equations expressed by:

$$\tilde{E}^{\alpha\beta\omega\rho}\overset{(0)}{V}_{3|\omega\rho\alpha} - 4\tilde{E}^{\alpha\beta\omega\rho}\overset{(1)}{V}_{\omega|\rho\alpha} + 4\delta_A E^{\pi 3\gamma 3} \frac{E^{\alpha\beta 33}}{E^{3333}} (\overset{(0)}{V}_{3|\pi\gamma\alpha} + \overset{(1)}{V}_{\gamma|\pi\alpha})$$

$$- 5\delta_C \frac{E^{\alpha\beta 33}}{E^{3333}} \rho \overset{(0)}{V}_{3,\alpha} + \frac{40}{h^2} E^{\beta 3\omega 3}\overset{(1)}{V}_{\omega} + \frac{40}{h^2} E^{\beta 3\omega 3}\overset{(0)}{V}_{3,\omega}$$

$$+ \delta_A \frac{1}{h} \frac{E^{\alpha\beta 33}}{E^{3333}} p^{\sigma}_{(1)|\sigma\alpha} - \frac{40}{h^3} p^{\beta}_{(1)} - \frac{4}{h} \tilde{E}^{\alpha\beta\omega\rho}F_{\sigma 3\omega 3}p^{\sigma}_{(1)|\rho\alpha}$$

$$+ \delta_{\beta}\rho g^{\beta\lambda}(4 \overset{(\ddot{1})}{V}_{\lambda} - \overset{(\ddot{0})}{V}_{3,\lambda} + \frac{4}{h} \ddot{p}^{\sigma}_{(1)} F_{\sigma 3\lambda 3}) = 0$$

$$\frac{2}{3}\, h\, E^{\alpha 3\omega 3}\left(\overset{(1)}{V}_{\omega|\alpha} + \overset{(0)}{V}_{3,\omega\alpha}\right) + \frac{1}{3}\, p^{\alpha}_{(1)|\alpha} + p^{3}_{(0)} - f^{3}_{(0)} = 0 \qquad (2.54)$$

Equations (2.54) constitute a sixth order governing equation system whose solutions are to be determined in conjunction with the prescribed boundary conditions (in number of three at each edge). At this point the similarity with the (FSDT), which also results in a sixth order governing equation system, is worthy to be underlined. For the sake of brevity the above theory will be referred to as DT.

2.4 An Equivalent Formulation of the DT Theory (Stress-Based Theory)

2.4.1 Variation of $\sigma^{\omega 3}$ Across the Plate Thickness

An alternative refined theory (referred to in the following as ST), allowing one also to incorporate the various effects mentioned above, starts with the following representation of transverse shear stresses $\sigma^{\omega 3}$:

$$\sigma^{\omega 3} = \overset{(0)}{\psi}{}^{\omega} + (x^3)^2\, \overset{(2)}{\psi}{}^{\omega} \qquad (2.55)$$

where $\overset{(n)}{\psi}{}^{\omega} = \overset{(n)}{\psi}{}^{\omega}(x^{\lambda},t)$ are unknown functions characterizing the variation of $\sigma^{\omega 3}$ in the thickness direction.

Fulfillment of the condition (2.40) on the bounding planes modifies (2.55) in the form:

$$\sigma^{\omega 3} = p^{\omega}_{(1)/h} + [(x^3)^2 - h^2/4]\, \overset{(2)}{\psi}{}^{\omega} \qquad (2.56)$$

Such a representation was used in [2] to formulate the refined theory of anisotropic plates and shown in [32] (see also [36]) to be consistent with the theory of moderately thick plates.

2.4.2 Expressions of Displacements and Strain Measures

Making use, in conjunction with (2.56), of the constitutive equation $(2.32)_3$ and strain-displacement relation (2.37), we obtain the expression of the components $\overset{(2S+1)}{V}_{\alpha}$ (associated to the bending theory) in the form

$$\overset{(2r+1)}{V}_{\alpha} = \frac{1}{2r+1}\left(4F_{\alpha 3\rho 3}\overset{(2r)}{\sigma}{}^{\rho 3} - \overset{(2r)}{V}_{3,\alpha}\right) \tag{2.57}$$

By considering further that $e_{33} = 0$ implying that $\overset{(2r)}{V}_{3} \equiv 0$ for $\forall_r \geq 1$, from (2.57) we obtain the expressions of the non-vanishing $\overset{(2r+1)}{V}_{\alpha}$:

$$\begin{aligned} \overset{(1)}{V}_{\alpha} &= \frac{4}{h} p^{\rho}_{(1)} F_{\alpha 3\rho 3} - h^2 F_{\alpha 3\rho 3}\overset{(2)}{\psi}{}^{\rho} - V_{3,\alpha} \\ \overset{(3)}{V}_{\alpha} &= \frac{4}{3} F_{\alpha 3\rho 3}\overset{(2)}{\psi}{}^{\rho} \end{aligned} \tag{2.58}$$

Taking into consideration the group of equations (2.37) and (2.58), we derive the expressions for $\overset{(i)}{e}_{\alpha\beta}$ associated to the bending theory:

$$\begin{aligned} \overset{(1)}{e}_{\alpha\beta} &= -V_{3|\alpha\beta} + \frac{2}{h}\left(F_{\alpha 3\rho 3}p^{\rho}_{(1)|\beta} + F_{\beta 3\rho 3}p^{\rho}_{(1)|\alpha}\right) \\ &\quad - \frac{h^2}{2}\left(F_{\alpha 3\rho 3}\overset{(2)}{\psi}{}^{\rho}|_{\beta} + F_{\beta 3\rho 3}\overset{(2)}{\psi}{}^{\rho}|_{\alpha}\right) \\ \overset{(3)}{e}_{\alpha\beta} &= \frac{2}{3}\left(F_{\alpha 3\rho 3}\overset{(2)}{\psi}{}^{\rho}|_{\beta} + F_{\beta 3\rho 3}\overset{(2)}{\psi}{}^{\rho}|_{\alpha}\right) \end{aligned} \tag{2.59}$$

2.4.3 Determination of σ^{33} and $\sigma^{\alpha\beta}$

By following the same pattern as in Section 2.3.3 we may express σ^{33} and $\sigma^{\alpha\beta}$ in a similar form as in Eqs. (2.48), (2.50). However in the present case $\overset{(i)}{\sigma}{}^{33}$ assume the form:

$$\overset{(1)}{\sigma}{}^{33} = \frac{p^3_{(0)}}{h} + \frac{h^2}{12} \overset{(2)}{\psi}{}^{\alpha}|_{\alpha}$$

$$\overset{(3)}{\sigma}{}^{33} = - \overset{(2)}{\psi}{}^{\rho}|_{\rho}/3 \tag{2.60}$$

2.4.4 The Governing Equations

In order to determine the three unknowns of the problem (i.e., $\overset{(2)}{\psi}{}^{\omega}$, V_3), we shall consider the equations of motion appropriate to the bending theory. As in the previous case, they are expressed by Eqs. (2.51).

With the help of (2.48), (2.50), (2.56), (2.59), (2.60) and (2.19) we may express the stress-couples involved in the 2-D equilibrium equations as:

$$L^{\omega\pi}_{(1)} = - \frac{h^3}{12} \tilde{E}^{\omega\pi\alpha\beta} V_{3|\alpha\beta} - \frac{h^5}{30} \tilde{E}^{\omega\pi\alpha\beta}(F_{\alpha3\rho3} \overset{(2)}{\psi}{}^{\rho}\Big|_{\beta} + F_{\beta3\rho3} \overset{(2)}{\psi}{}^{\rho}|_{\alpha})$$

$$+ \delta_A \frac{h^5}{360} \frac{E^{\omega\pi33}}{E^{3333}} \overset{(2)}{\psi}{}^{\rho}|_{\rho} + \frac{h^2}{6} \tilde{E}^{\omega\pi\alpha\beta}(F_{\alpha3\rho3}p^{\rho}_{(1)}|_{\beta} + F_{\beta3\rho3}p^{\rho}_{(1)}|_{\alpha})$$

$$+ \delta_A \frac{h^2}{12} \frac{E^{\omega\pi33}}{E^{3333}} p^3_{(0)}$$

$$L^{\alpha3}_{(0)} = p^{\alpha}_{(1)} - (h^3/6) \overset{(2)}{\psi}{}^{\alpha} \tag{2.61}$$

Making use of (2.58) and (2.23), one obtains for the inertia terms the expressions

$$f^{\omega}_{(1)} = \rho \frac{h^3}{12} g^{\omega\lambda} [\frac{4}{h} F_{\lambda3\sigma3}\ddot{p}^{\sigma}_{(1)} - \frac{4}{5} h^2 F_{\lambda3\sigma3} \overset{(\ddot{2})}{\psi}{}^{\sigma} - \ddot{V}_{3,\lambda}]$$

$$f^3_{(0)} = \rho h \ddot{V}_3 \tag{2.62}$$

The sixth order governing equations expressed in terms of the unknowns $\overset{(2)}{\psi}{}^{\rho}$ and V_3 are obtained as in the previous case. They read [36]:

$$\tilde{E}^{\omega\pi\alpha\beta}V_{3|\alpha\beta\omega} + \frac{4}{5}h^2\tilde{E}^{\omega\pi\alpha\beta}F_{\alpha3\rho3}\overset{(2)}{\psi}{}^{\rho}|_{\beta\omega} - 2\overset{(2)}{\psi}{}^{\pi} - \delta_A\frac{h^2}{30}\frac{E^{\omega\pi33}}{E^{3333}}\overset{(2)}{\psi}{}^{\rho}|_{\rho\omega}$$

$$- \frac{4}{h}\tilde{E}^{\omega\pi\alpha\beta}F_{\alpha3\rho3}p^{\rho}_{(1)|\beta\omega} - \delta_A\frac{1}{h}\frac{E^{\omega\pi33}}{E^{3333}}p^3_{(0),\omega}$$

$$- \delta_B 4\rho g^{\pi\lambda}(\frac{1}{4}\ddot{V}_{3,\lambda} + \frac{h^2}{5}F_{\lambda3\rho3}\overset{(\ddot{2})}{\psi}{}^{\rho} - \frac{F_{\lambda3\sigma3}}{h}\ddot{p}^{\sigma}_{(1)}) = 0$$

$$\frac{h^3}{6}\overset{(2)}{\psi}{}^{\pi}|_{\pi} - p^3_{(0)} - p^{\alpha}_{(1)|\alpha} + \rho h\ddot{V}_3 = 0 \qquad (2.63)$$

Insertion of $\overset{(2)}{\psi}{}^{\pi}|_{\pi}$ obtained from $(2.63)_2$ into $(2.63)_1$ yields an alternative form of $(2.63)_1$:

$$\tilde{E}^{\omega\pi\alpha\beta}V_{3|\alpha\beta\omega} + \frac{4}{5}h^2\tilde{E}^{\omega\pi\alpha\beta}F_{\alpha3\rho3}\overset{(2)}{\psi}{}^{\rho}|_{\beta\omega} - 2\overset{(2)}{\psi}{}^{\pi} - \delta_B\frac{4}{5}\rho h^2 g^{\pi\lambda}F_{\lambda3\rho3}\overset{(\ddot{2})}{\psi}{}^{\rho}$$

$$- \delta_A\frac{6}{5h}\frac{E^{\omega\pi33}}{E^{3333}}p^3_{(0),\omega} - \frac{4}{h}\tilde{E}^{\omega\pi\alpha\beta}F_{\alpha3\rho3}p^{\rho}_{(1)|\beta\omega} - \delta_A\frac{1}{5h}\frac{E^{\omega\pi33}}{E^{3333}}p^{\rho}_{(1)|\rho\omega}$$

$$+ \delta_B\frac{4\rho}{h}g^{\pi\lambda}F_{\lambda3\sigma3}\ddot{p}^{\sigma}_{(1)} + \delta_C\frac{\rho}{5}\frac{E^{\omega\pi33}}{E^{3333}}\ddot{V}_{3,\omega} - \delta_B\rho g^{\pi\lambda}\ddot{V}_{3,\lambda} = 0 \qquad (2.64)$$

2.5 The First Order Transverse Shear Deformation Bending Theory (FSDT)

For the sake of comparison, several elements concerning (FSDT) will shortly be presented. As per this theory, the following representation for the displacement field is postulated:

$$V_{\alpha}(x^{\omega},x^3,t) = x^3\overset{(1)}{V}_{\alpha}(x^{\omega},t)$$

$$V_3(x^{\omega},x^3,t) = \overset{(0)}{V}_3(x^{\omega},t) \qquad (2.65)$$

Equations (2.65), used in conjunction with the constitutive equations and the strain-displacement relationships, result, through the neglect of σ^{33} in the following expressions for $L^{\alpha\beta}_{(1)}$ and $L^{\alpha 3}_{(0)}$:

$$L^{\alpha\beta}_{(1)} = \frac{h^3}{12}\tilde{E}^{\alpha\beta\omega\rho}\overset{(1)}{V}_{\omega|\rho}$$

$$L^{\alpha 3}_{(0)} = hK^2_{(\alpha)}E^{\alpha 3\lambda 3}(\overset{(1)}{V}_{\lambda} + V_{3,\lambda}) \tag{2.66}$$

where $K^2_{(\alpha)}$ denote the transverse shear correction factors associated to the directions x^{α}.

Making use of the general procedure, followed previously, we may obtain the pertinent governing equations expressed in terms of V_3 and $\overset{(1)}{V}_{\alpha}$. However, they may be obtained also by linearizing the ones in ([36], Chapter IV, Section 5) derived for the non-linear case.

Their specialized counterparts assume the form:

$$\frac{h^3}{12}\tilde{E}^{\alpha\beta\omega\rho}\overset{(1)}{V}_{\omega|\rho\alpha} - K^2_{(\beta)}hE^{\beta 3\lambda 3}(\overset{(1)}{V}_{\lambda} + V_{3,\lambda}) + p^{\beta}_{(1)} - \delta_{\beta}\frac{h^3}{12}\rho\overset{(\ddot{1})}{V}{}^{\beta} = 0$$

$$K^2_{(\alpha)}hE^{\alpha 3\lambda 3}(\overset{(1)}{V}_{\lambda|\alpha} + V_{3|\lambda\alpha}) + p^3_{(0)} - \rho h\ddot{V}_3 = 0 \tag{2.67}$$

Equations (2.67) constitute a sixth order differential equation system. The foregoing theories in terms of their associated equations will be compared in the process of our analysis.

2.6 The Governing Equations of Orthotropic Rectangular Plate Theories

The previously recorded equations will be specialized for a Cartesian orthogonal system of coordinates (x_1,x_2) and for an orthotropic medium. Moreover, it will be assumed that the axes of orthotropy at each point are parallel to the geometrical axes.

Having in view that in a Cartesian and orthogonal system of coordinates i) there is no distinction between covariant, contrvariant and mixed components of a tensor and that ii) the covariant differentiation reduces to the ordinary partial differentiation, and recalling that for an orthotropic medium the components of elastic moduli (E^{ij}_{mn}) and compliance (F^{ij}_{mn}) tensors containing the index 1 once or three times vanish, the orthotropic counterpart of Eqs. (9) (associated to the DT and specialized for $p^{\alpha}_{(1)} = 0$) reads:

$$\tilde{E}^{11}_{11} V_{3,111} + (\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12})V_{3,122} - 4[\tilde{E}^{11}_{11} \overset{(1)}{V}_{1,11} + \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,12}$$

$$+ \tilde{E}^{12}_{12} (\overset{(1)}{V}_{2,12} + \overset{(1)}{V}_{1,22})] + 4\delta_A \frac{E^{11}_{33}}{E^{33}_{33}} [E^{13}_{13} (V_{3,111} + \overset{(1)}{V}_{1,11})$$

$$+ E^{23}_{23} (V_{3,122} + \overset{(1)}{V}_{2,12})] + \frac{40}{h^2} E^{13}_{13} (\overset{(1)}{V}_1 + V_{3,1})$$

$$- 5\delta_C \frac{E^{11}_{33}}{E^{33}_{33}} \rho\ddot{V}_{3,1} + \delta_B\rho(4 \overset{(\ddot{1})}{V}_1 - \ddot{V}_{3,1}) = 0 \qquad (1 \rightleftarrows 2)$$

$$\frac{2}{3} h[E^{13}_{13} (\overset{(1)}{V}_{1,1} + V_{3,11}) + E^{23}_{23} (\overset{(1)}{V}_{2,2} + V_{3,22})] + p_3 - \rho h\ddot{V}_3 = 0. \qquad (2.68)$$

Throughout this section the sign ($1 \rightleftarrows 2$) accompanying certain relations indicates that the remaining equations, not explicitly written, may be obtained by interchanging the index 1 by 2 and vice-versa, while the partial differentiation is denoted by a comma $(\ )_{,i}$ ($\equiv \partial(\ )/\partial x^i$). For computational purposes the converted expressions of 3-D stress components are recorded below:

$$\sigma_{11} = x_3 \, [(\tilde{E}^{11}_{11} \overset{(1)}{V}_{1,1} + \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,2}) \, (1 - \frac{4x_3^2}{3h^2}) - \frac{4x_3^2}{3h^2} \, (\tilde{E}^{11}_{11} \, V_{3,11} + \tilde{E}^{11}_{22} \, V_{3,22})$$

$$- \delta_A \frac{E^{11}_{33}}{E^{33}_{33}} \{E^{13}_{13} \, (\overset{(1)}{V}_{1,1} + V_{3,11}) + E^{23}_{23} \, (\overset{(1)}{V}_{2,2} + V_{3,22})\} \, (1 - \frac{4x_3^2}{3h^2})$$

$$+ \delta_C \frac{E^{11}_{33}}{E^{33}_{33}} \rho \ddot{V}_3] \qquad (1 \rightleftarrows 2)$$

$$\sigma_{12} = x^3 \, \tilde{E}^{12}_{12} \, [(\overset{(1)}{V}_{1,2} + \overset{(1)}{V}_{2,1}) \, (1 - \frac{4x_3^2}{3h^2}) - \frac{8x_3^2}{3h^2} \, V_{3,12}]$$

$$\sigma_{13} = E^{13}_{13}[(\overset{(1)}{V}_1 + V_{3,1})(1 - \frac{4x_3^2}{h^2})] \qquad (1 \rightleftarrows 2)$$

$$(2.69)$$

as well as the stress couples given by:

$$L^{11}_{(1)} = \frac{h^3}{15} \tilde{E}^{11}_{11} \overset{(1)}{V}_{1,1} + \frac{h^3}{15} \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,2} - \frac{h^3}{60} \tilde{E}^{11}_{11} V_{3,11} - \frac{h^3}{60} \tilde{E}^{11}_{22} V_{3,22}$$

$$- \delta_A \frac{E^{11}_{33}}{E^{33}_{33}} \frac{h^3}{15} \, [E^{13}_{13} \, (V_{3,11} + \overset{(1)}{V}_{1,1}) + E^{23}_{23} \, (V_{3,22} + \overset{(1)}{V}_{2,2})]$$

$$+ \delta_C \frac{E^{11}_{33}}{E^{33}_{33}} \rho \, \frac{h^3}{12} \, \ddot{V}_3 \qquad (1 \rightleftarrows 2)$$

$$L^{12}_{(1)} = \tilde{E}^{12}_{12} \frac{h^3}{30} \, [2(\overset{(1)}{V}_{1,2} + \overset{(1)}{V}_{2,1}) - V_{3,12}] \qquad (2.70)$$

$$L^{13}_{(0)} = \frac{2}{3} \, h \, E^{13}_{13} \, (\overset{(1)}{V}_1 + V_{3,1}). \qquad (1 \rightleftarrows 2)$$

The conversion of Eqs. (2.63), (2.64) associated to the ST yields:

$$\tilde{E}^{11}_{11}V_{3,111} + (\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12})V_{3,122} + \frac{4}{5}h^2(\tilde{E}^{11}_{11}F^{13}_{13}\psi_{1,11} + \tilde{E}^{12}_{12}F^{13}_{13}\psi_{1,22}$$

$$+ \tilde{E}^{11}_{22}F^{23}_{23}\psi_{2,12} + \tilde{E}^{12}_{12}F^{23}_{23}\psi_{2,12}) - 2\psi_1 - \delta_A \frac{6}{5h}\frac{E^{11}_{33}}{E^{33}_{33}}p_{3,1}$$

$$- \delta_\beta \rho(\frac{4}{5}h^2F^{13}_{13}\ddot{\psi}_1 + \ddot{V}_{3,1}) + \delta_C \frac{1}{5}\frac{E^{11}_{33}}{E^{33}_{33}}\rho\ddot{V}_{3,1} = 0 \qquad (1 \rightleftarrows 2)$$

$$\frac{h^3}{6}(\psi_{1,1} + \psi_{2,2}) - p_3 + \rho h\ddot{V}_3 = 0 \qquad (2.71)$$

while the specialized counterparts of (2.69), (2.61) and (2.58) read:

$$\sigma_{11} = x_3[-\tilde{E}^{11}_{11}V_{3,11} - \tilde{E}^{11}_{22}V_{3,22} + (\tilde{E}^{11}_{11}F^{13}_{13}\psi_{1,1} + \tilde{E}^{11}_{22}F^{23}_{23}\psi_{2,2})(\frac{4x_3^2}{3} - h^2)$$

$$+ \delta_A \frac{p^3_{(0)}}{h}\frac{E^{11}_{33}}{E^{33}_{33}}(\frac{3}{2} - \frac{2x_3^2}{h^2})] \qquad (1 \rightleftarrows 2)$$

$$\sigma_{12} = 2x_3\tilde{E}^{12}_{12}[-V_{3,12} + (F^{13}_{13}\psi_{1,2} + F^{23}_{23}\psi_{2,1})(\frac{2x_3^2}{3} - \frac{h^2}{2})]$$

$$\sigma_{13} = (x_3^2 - \frac{h^2}{4})\psi_1 \qquad (1 \rightleftarrows 2)$$

$$L^{11}_{(1)} = -\frac{h^3}{12}(E^{11}_{11}V_{3,11} + \tilde{E}^{11}_{22}V_{3,22}) - \frac{h^5}{15}(\tilde{E}^{11}_{11}F^{13}_{13}\psi_{1,1} + \tilde{E}^{11}_{22}F^{23}_{23}\psi_{2,2})$$

$$+ \delta_A \frac{h^2E^{11}_{33}}{10E^{33}_{33}}p_3 - \delta_C \frac{h^3}{60}\frac{E^{11}_{33}}{E^{33}_{33}}\rho\ddot{V}_3 \qquad (1 \rightleftarrows 2) \quad (2.72)$$

$$L^{12}_{(1)} = -\frac{h^3}{6}\tilde{E}^{12}_{12}V_{3,12} - \frac{h^2}{15}(\tilde{E}^{12}_{12}F^{23}_{23}\psi_{2,1} + \tilde{E}^{12}_{12}F^{13}_{13}\psi_{1,2})$$

$$L^{13}_{(0)} = -\frac{h^3}{6}\psi_1 \qquad (1 \rightleftarrows 2)$$

and

$$\overset{(1)}{V}_1 = - h^2 F^{13}_{13} \psi_1 - V_{3,1} \qquad (1 \rightleftarrows 2)$$

$$\overset{(3)}{V}_1 = \frac{4}{3} F^{13}_{13} \psi_1$$

In the case of FSDT the converted counterpart of Eqs. (2.67) becomes:

$$\frac{h^3}{12} [\tilde{E}^{11}_{11} \overset{(1)}{V}_{1,11} + \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,12} + \tilde{E}^{12}_{12} (\overset{(1)}{V}_{2,12} + \overset{(1)}{V}_{1,22})]$$

$$- K^2_{(1)} h E^{13}_{13} (\overset{(1)}{V}_1 + V_{3,1}) - \delta_\beta \frac{h^3}{12} \rho \overset{(\ddot{1})}{V}_1 = 0 \qquad (1 \rightleftarrows 2) \qquad (2.73)$$

$$K^2_{(1)} h E^{13}_{13} (\overset{(1)}{V}_{1,1} + V_{3,11}) + K^2_{(2)} h E^{23}_{23} (\overset{(1)}{V}_{2,2} + V_{3,22}) + p_3 - \rho h \ddot{V}_3 = 0$$

while the relevant stresses and stress couples write

$$\sigma_{11} = x_3 [\tilde{E}^{11}_{11} \overset{(1)}{V}_{1,1} + \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,2}] \qquad (1 \rightleftarrows 2)$$

$$\sigma_{12} = x_3 \tilde{E}^{12}_{12} [\overset{(1)}{V}_{1,2} + \overset{(1)}{V}_{2,1}]$$

$$\sigma_{13} = K^2_{(1)} E^{13}_{13} [\overset{(1)}{V}_1 + V_{3,1}] \qquad (1 \rightleftarrows 2)$$

$$L^{11}_{(1)} = \frac{h^3}{12} (\tilde{E}^{11}_{11} \overset{(1)}{V}_{1,1} + \tilde{E}^{11}_{22} \overset{(1)}{V}_{2,2}) \qquad (1 \rightleftarrows 2)$$

$$L^{12}_{(1)} = \frac{h^3}{12}\tilde{E}^{12}_{12}\,(\overset{(1)}{V}_{1,2} + \overset{(1)}{V}_{2,1}) \qquad (2.74)$$

$$L^{13}_{(0)} = K^2_{(1)} h E^{13}_{13}(\overset{(1)}{V}_1 + V_{3,1}) \qquad (1 \rightleftarrows 2)$$

It is also useful to express the components of the elastic moduli E^{ij}_{mn} and compliance F^{ij}_{mn} tensors in terms of the associated engineering orthotropic characteristics. The relevant correspondences read:

$$\tilde{E}^{11}_{11} \rightarrow \frac{E_1}{1 - \nu_{12}\nu_{21}} \qquad (1 \rightleftarrows 2)$$

$$\tilde{E}^{11}_{22} = \tilde{E}^{22}_{11} \rightarrow \frac{E_2\nu_{12}}{1 - \nu_{12}\nu_{21}} = \frac{E_1\nu_{21}}{1 - \nu_{12}\nu_{21}}$$

$$\tilde{E}^{12}_{12} \equiv E^{12}_{12} \rightarrow G_{12}$$

$$E^{13}_{13} \rightarrow G_{13} \qquad (1 \rightleftarrows 2)$$

$$F^{11}_{11} \rightarrow \frac{1}{E_1} \quad ; \quad F^{13}_{13} \rightarrow \frac{1}{4G_{13}} \qquad (1 \rightleftarrows 2)$$

$$F^{33}_{33} \rightarrow \frac{1}{E_3} \quad ; \quad F^{11}_{22} \rightarrow -\frac{\nu_{12}}{E_2} = -\frac{\nu_{21}}{E_1} \qquad (2.75)$$

$$F^{11}_{33} \rightarrow -\frac{\nu_{13}}{E_3} = -\frac{\nu_{31}}{E_1} \qquad (1 \rightleftarrows 2)$$

$$\frac{E^{11}_{33}}{E^{33}_{33}} = \frac{E_1}{E_3}\,\frac{\nu_{13} + \nu_{21}\nu_{23}}{1 - \nu_{12}\nu_{21}} \qquad (1 \rightleftarrows 2)$$

Here E_1, E_2, E_3; G_{12}, G_{13}, G_{23}, ν_{12}, ν_{21}; ν_{13}, ν_{23}, ν_{32} denote the

Young's and shear moduli and Poisson's ratios, respectively. By virtue of the symmetry relationships

$$\nu_{12}E_2 = \nu_{21}E_1 \; ; \; \nu_{13}E_3 = \nu_{31}E_1 \; ; \; \nu_{23}E_3 = \nu_{32}E_2$$

only nine independent elastic characteristics will be involved in the foregoing recorded equations.

2.7 The Case of Transversely-Isotropic Plates

In the case of a transversely-isotropic plate (the plane of isotropy at each point being assumed parallel to the mid-plane of the plate), the elastic moduli and compliance tensors are defined in eq. (2.35) and could be found in [32,36]. As a result of their employment, the transversal-isotropic counterpart of (2.54) (associated to the DT) becomes

$$- \frac{E}{1-\mu^2} V_3|_{\alpha}^{\alpha\beta} + \frac{2E}{1-\mu^2} [(1+\mu) \overset{(1)}{V}_{\omega}|^{\omega\beta} + (1-\mu) \overset{(1)}{V}{}^{\beta}|_{\rho}^{\rho}] - \frac{40}{h^2} G' \overset{(1)}{V}{}^{\beta}$$
$$- 4\delta_A \frac{\mu' E G'}{E'(1-\mu)} (V_3|_{\pi}^{\pi\beta} + \overset{(1)}{V}{}^{\pi}|_{\pi}^{\beta}) - \frac{40}{h^2} G' g^{\beta\omega} V_{3,\omega} + \delta_C \rho \frac{5E\mu'}{E'(1-\mu)} g^{\alpha\beta} \ddot{V}_{3|\alpha}$$
$$- \delta_A \frac{1}{h} \frac{E\mu'}{E'(1-\mu)} p^{\sigma}_{(1)}|_{\sigma}^{\beta} + \frac{1}{h} \frac{E}{2G'(1-\mu^2)} [(1+\mu) p^{\sigma}_{(1)}|_{\sigma}^{\beta} + (1-\mu) p^{\beta}_{(1)}|_{\rho}^{\rho}]$$
$$+ \frac{40}{h^3} p^{\beta}_{(1)} - \rho\delta_B (4 g^{\beta\lambda} \overset{(\ddot{1})}{V}_{\lambda} - g^{\lambda\beta} \ddot{V}_{3,\lambda} + \frac{1}{hG'} \ddot{p}^{\beta}_{(1)}) = 0 \qquad (2.76)$$

$$\frac{2}{3} hG' g^{\alpha\omega} (\overset{(1)}{V}_{\omega|\alpha} + V_{3,\omega\alpha}) + \frac{1}{3} p^{\alpha}_{(1)|\alpha} + p^{3}_{(0)} - \rho h \ddot{V}_3 = 0$$

The conversion of Eqs. (2.63), (2.64) to the ST yields:

$$\frac{E}{1-\mu^2} V_3|_{\alpha}^{\alpha\pi} + \frac{h^2}{10} \frac{E}{G'(1-\mu^2)} [(1+\mu) \overset{(2)}{\psi}{}^{\sigma}|_{\sigma}^{\pi} + (1-\mu) \overset{(2)}{\psi}{}^{\pi}|_{\sigma}^{\sigma}] - 2 \overset{(2)}{\psi}{}^{\pi}$$

$$- \delta_A \frac{6}{5h} \frac{E\mu'}{E'(1-\mu)} p^3_{(0)}|^\pi - \frac{1}{2h} \frac{E}{G'(1-\mu^2)} [(1+\mu)p^\sigma_{(1)}|^\pi_\sigma + (1-\mu)p^\pi_{(1)}|^\sigma_\sigma]$$

$$- \delta_A \frac{1}{5h} \frac{E\mu'}{E'(1-\mu)} p^\rho_{(1)}|^\pi_\rho + \delta_{c^\rho} \frac{E\mu'}{5E'(1-\mu)} \ddot{V}_3|^\pi$$

$$- \rho\delta_\beta g^{\pi\lambda}[\ddot{V}_{3,\lambda} + \frac{h^2}{5G'} g_{\lambda\rho} \overset{(2)}{\ddot{\psi}}{}^\rho - \frac{1}{hG'} g_{\lambda\sigma}\ddot{p}^\sigma_{(1)}] = 0$$

$$\frac{h^3}{6} \overset{(2)}{\psi}{}^\pi|_\pi - p^3_{(0)} - p^\alpha_{(1)|\alpha} + \rho h\ddot{V}_3 = 0 \qquad (2.77)$$

In the case of FSDT the converted counterpart of Eqs. (2.67) becomes:

$$\frac{2E}{(1-\mu^2)} [(1+\mu) \overset{(1)}{V}{}^\alpha|^\beta_\alpha + (1-\mu) \overset{(1)}{V}{}^\beta|^\alpha_\alpha] - \frac{48}{h^2} G'g^{\beta\lambda}(\overset{(1)}{V}_\lambda + V_{3,\lambda})$$

$$+ \frac{48}{h^3} p^\beta_{(1)} - 4\delta_\beta\rho g^{\beta\lambda} \overset{(\ddot{1})}{V}_\lambda = 0$$

$$hG'g^{\alpha\lambda}(\overset{(1)}{V}_{\lambda|\alpha} + V_{3,\lambda\alpha}) + p^3_{(0)} - \rho h\ddot{V}_3 = 0 \qquad (2.78)$$

In Eqs. (2.76-2.78), E, μ, and E', μ', and G' denote Young's modulus, Poisson's ratio, and the transverse shear modulus, corresponding to the plane of isotropy and normal to the isotropy plane, respectively.

2.8 Alternative Representation of the Governing Equations (2.76-2.78)

At this point we shall make use of the procedure developed in [34,36] allowing to recast the total state of stress governed by Eqs.(2.76-2.78) in terms of two independent ones; i.e., in terms of the interior state of stress and the edge effect solutions. Towards this goal we shall postulate, for convenience, a harmonic time variation for the field quantities, so yielding the representation

$$F(x^\alpha,t) = \overline{F}(x^\alpha)\exp(j\omega t) \quad , \quad j = \sqrt{-1} \tag{2.79}$$

Here F denotes a generic function entering the governing equations. Furthermore, we shall express $\overset{(1)}{V}{}^\beta$ (for DT and FSDT), $\overset{(2)}{\psi}{}^\pi$ for ST in terms of V_3 and a potential function $\phi \equiv \phi(x^\lambda,t)$ as

DT

$$S\overset{(1)}{V}{}^\beta = -\frac{E}{1-\mu^2}V_3|_\alpha^{\alpha\beta} - \varepsilon^{\omega\beta}\phi|_\omega + \frac{6}{5}\frac{E}{(1-\mu^2)G'}\rho\ddot{V}_3|^\beta - \delta_C\frac{\rho}{5}\frac{E\mu'}{E'(1-\mu)}\ddot{V}_3|^\beta$$

$$+\frac{1}{5h}\left[2\delta_A\frac{E\mu'}{E'(1-\mu)} - \frac{2E}{(1-\mu^2)G'} + \frac{E}{2G'(1-\mu)}\right]p^\sigma_{(1)}|^\beta_\sigma + \frac{G}{5G'h}p^\beta_{(1)}|^\rho_\rho$$

$$+\delta_A\frac{6\mu'E}{5hE'(1-\mu)}p^3_{(0)}|^\beta + \frac{8}{h^3}p^\beta_{(1)} - \frac{6}{5}\frac{E}{G'h(1-\mu^2)}p^3_{(0)}|^\beta - \frac{8}{h^2}G'V_3|^\beta$$

$$+\frac{1}{5}\delta_\beta\rho\ddot{V}_3|^\beta - \delta_\beta\frac{\rho}{5hG'}\ddot{p}^\beta_{(1)} \tag{2.80}$$

where

$$S = \frac{8}{h^2}G' + \frac{4}{5}\delta_\beta\rho\omega^2$$

FSDT

$$A\overset{(1)}{V}{}^\beta = \varepsilon^{\beta\lambda}\phi|_\lambda + \frac{h^3}{12}\frac{E}{1-\mu^2}V_3|_\alpha^{\alpha\beta} - \frac{h^3}{12}\frac{E}{hG'(1-\mu^2)}(\rho h\ddot{V}_3|^\beta - p^3_{(0)}|^\beta)$$

$$+ hG'V_3|^\beta - p^\beta_{(1)}$$

where

$$A = -hG' + \delta_\beta\frac{h^3}{12}\rho\omega^2 \tag{2.81}$$

ST

$$B\psi^{(2)}{}^{\pi} = -\frac{E}{(1-\mu^2)} V_3|_{\omega}^{\omega\pi} - \varepsilon^{\omega\pi}\phi|_{\omega} - \frac{6}{5}\frac{E}{hG'(1-\mu^2)} [p^3_{(0)}|^{\pi} + p^{\lambda}_{(1)}|^{\pi}_{\lambda}$$

$$- \rho h\ddot{V}_3|^{\pi}] + \delta_A \frac{6}{5h}\frac{E\mu'}{E'(1-\mu)} p^3_{(0)}|^{\pi} + \frac{1}{2h}\frac{E}{G'(1-\mu^2)} [(1+\mu)p^{\sigma}_{(1)}|^{\pi}_{\sigma}$$

$$+ (1-\mu)p^{\pi}_{(1)}|^{\sigma}_{\sigma}] + \delta_A \frac{1}{5h}\frac{E\mu'}{E'(1-\mu)} p^{\rho}_{(1)}\Big|^{\pi}_{\rho} - \delta_\beta \frac{\rho}{hG'} \ddot{p}^{\pi}_{(1)}$$

$$+ \delta_\beta \rho g^{\pi\lambda}\ddot{V}_{3,\lambda} - \delta_C \frac{E\mu'}{5E'(1-\mu)} \rho\ddot{V}_3|^{\pi} \qquad (2.82)$$

where $B = \delta_\beta \frac{\rho h^2}{5G'}\omega^2 - 2$,

while $\varepsilon^{\alpha\beta}(\varepsilon_{\alpha\beta})$ stand for the 2D permutation symbols ($\varepsilon^{12} = -\varepsilon^{21} = 1$; $\varepsilon^{\alpha\alpha} = \varepsilon_{\alpha\alpha} = 0$; $\varepsilon_{12} = -\varepsilon_{21} = 1$). Substitution of (2.80), (2.81), (2.82) into (2.76), (2.78), (2.77), respectively, yields, after some manipulations (for more details see [34,36]), the following set of governing equations:

DT

$$\phi - \frac{h^2}{10}\frac{G}{G'}\phi|^{\omega}_{\omega} = -\delta_\beta\rho\frac{h^2}{10G'}\ddot{\phi} + \varepsilon_{\mu\lambda}\left(\frac{4}{5h}\frac{G}{G'}p^{\mu}_{(1)}|^{\lambda} + \frac{h}{50}\left(\frac{G}{G'}\right)^2 p^{\mu}_{(1)}|^{\lambda\rho}_{\rho}\right)$$

$$- \delta_\beta\varepsilon_{\mu\lambda}\frac{\rho}{50} h \frac{G}{(G')^2}\ddot{p}^{\mu}_{(1)}|^{\lambda}$$

$$\frac{Eh^3}{12(1-\mu^2)} V_3|^{\alpha\rho}_{\alpha\rho} - [p^3_{(0)} - \frac{h^2}{10(1-\mu)} K_o^2 p^3_{(0)}|^{\alpha}_{\alpha}] - [p^{\beta}_{(1)}|_{\beta}$$

$$- \frac{h^2}{60(1-\mu)} K_1^2 p^{\beta}_{(1)}|^{\rho}_{\rho\beta}] + \rho h[\ddot{V}_3 - \frac{h^2}{10(1-\mu)} K_2^2\ddot{V}_3|^{\alpha}_{\alpha}] - \delta_\beta \frac{h^3}{12}\rho\ddot{V}_3|^{\rho}_{\rho}$$

$$+ \delta_\beta \frac{h^3}{10G'} \rho^2 \ddot{V}_3 - \delta_\beta \frac{h^2}{60} \frac{\rho}{G'} \ddot{p}^\beta_{(1)}|_\beta - \delta_\beta \frac{h^2}{10} \frac{\rho}{G'} \ddot{p}^3_{(0)} = 0 \qquad (2.83)$$

where

$$K_o^2 = 2\frac{G}{G'} - \delta_A \frac{\mu' E}{E'}$$

$$K_1^2 = \frac{2G}{G'} - 2\delta_A \frac{\mu' E}{E'}$$

$$K_2^2 = \frac{2G}{G'} - \delta_C \frac{\mu' E}{6E'}$$

FSDT

$$\phi - \frac{h^2}{12}\frac{G}{G'}\phi|^\sigma_\sigma = -\frac{h^2}{12}\frac{G}{G'}\varepsilon_{\mu\lambda}p^\mu_{(1)}|_\lambda - \delta_\beta \frac{h^2}{12G'}\rho\ddot{\phi}$$

$$\frac{Eh^3}{12(1-\mu^2)} V_3|^{\alpha\beta}_{\alpha\beta} - [p^3_{(0)} - \frac{h^2}{12(1-\mu)}\frac{2G}{G'}p^3_{(0)}|^\alpha_\alpha] + \rho h[\ddot{V}_3 - \frac{h^2}{12(1-\mu)}\frac{2G}{G'}\ddot{V}_3|^\alpha_\alpha]$$

$$- p^\beta_{(1)}|_\beta - \delta_\beta \frac{h^3}{12}\rho\ddot{V}_3|^\alpha_\alpha + \delta_\beta \frac{h^3}{12G'}\rho^2\ddot{V}_3 - \delta_\beta \frac{h^2}{12G'}\rho\ddot{p}^3_{(0)} = 0 \quad (2.84)$$

ST

$$\phi - \frac{h^2}{10}\frac{G}{G'}\phi|^\sigma_\sigma = -\delta_\beta\rho\frac{h^2}{10G'}\ddot{\phi} + \varepsilon_{\mu\lambda}\frac{h}{40}(\frac{G}{G'})^2 p^\mu_{(1)}|^{\sigma\lambda}_\sigma$$

$$- \delta_\beta\varepsilon_{\mu\lambda}\frac{\rho}{40} h \frac{G}{(G')^2}\ddot{p}^\mu_{(1)}|^\lambda$$

$$\frac{Eh^3}{12(1-\mu^2)} V_3|^{\alpha\beta}_{\alpha\beta} - [p^3_{(0)} - \frac{h^2}{10(1-\mu)} K_o^2 p^3_{(0)}|^\alpha_\alpha] - [p^\alpha_{(1)}|_\alpha$$

$$- \frac{h^2}{60(1-\mu)} K_o^2 p^\alpha_{(1)}|^\pi_{\alpha\pi}] + \rho h[\ddot{V}_3 - \frac{h^2}{10(1-\mu)} K_2^2 \ddot{V}_3|^\alpha_\alpha] - \delta_\beta \frac{h^3}{12}\rho\ddot{V}_3|^\lambda_\lambda$$

$$+ \delta_\beta \frac{h^3}{10G'} \rho^2 \ddot{V}_3 - \delta_\beta \frac{h^2}{60} \frac{\rho}{G'} \ddot{P}^{\pi}_{(1)}|_\pi - \delta_\beta \frac{h^2}{10} \frac{\rho}{G'} \ddot{P}^3_{(0)} = 0 \qquad (2.85)$$

Here $(\ )|^\alpha_\alpha$ and $(\ )\ |^{\alpha\beta}_{\alpha\beta}$ denote the harmonic and bi-harmonic operators in an arbitrary in-plane system of coordinates, respectively. These equations (for the case of DT and ST) constitute the generalized counterpart of Reissner's equations governing the bending theory of transversely-isotropic flat plates [70-72,80].

While the Eqs. $(2.83\text{-}2.85)_1$ define the edge effect solution (i.e. such a state of stress decaying rather rapidly as one proceeds from the edge to the inside of the plate region), the Eqs. $(2.83\text{-}2.85)_2$ define the interior solution. These latter equations may be used very accurately to predict the global response characteristics (as e.g. the static or dynamic stability behavior and the eigen-frequency characteristics, as well). A detailed analysis assessing this statement is contained in the following chapters.

When $P^\alpha_{(1)} \to 0$, DT and ST yield the same equations for the case of transversely-isotropic flat plates.

CHAPTER III

THE SOLUTION PROCEDURE

3.1 The Solution Technique for the Static Problem

A generalized Levy type solution considered in conjunction with the state space concept will be used in order to determine the state of stress of plates (a x b). The plate edges $x_1 = 0,a$ will be considered simply supported (SS), while the other two edges ($x_2 = \pm\, b/2$) - see Fig. 3.1 - arbitrary from the edge conditions viewpoint. Following Levy's procedure we shall conveniently represent the unknown quantities so as to satisfy identically the boundary conditions on both x_1-edges. Their substitution into the relevant governing equations will result in an ordinary system of differential equations in the x_2-variable. By using the state space concept this obtained set of differential equations will be solved for arbitrary boundary conditions (B.C.) at $x_2 = \pm\, b/2$. This approach will be applied simultaneously to four theories, denoted for brevity sakes as DT, ST, FSDT, and HSDT the last one being developed in [60-62].

3.1.1 DT, ST, and FSDT Theories

According to the above mentioned scheme, the unknown quantities involved in the three theories will be expressed as:

For DT and FSDT:

$$\overset{(1)}{V_1}(x_1,x_2) = \sum_{m=1}^{\infty} V_{1m}(x_2)\cos\alpha x_1$$

$$\overset{(1)}{V_2}(x_1,x_2) = \sum_{m=1}^{\infty} V_{2m}(x_2)\sin\alpha x_1$$

$$V_3(x_1,x_2) = \sum_{m=1}^{\infty} W_m(x_2)\sin\alpha x_1 \qquad (3.1)$$

while for ST as:

$$\psi_1(x_1,x_2) = \sum_{m=1}^{\infty} \phi_{1m}(x_2)\cos\alpha x_1$$

$$\psi_2(x_1,x_2) = \sum_{m=1}^{\infty} \phi_{2m}(x_2)\sin\alpha x_1 \qquad (3.2)$$

whereas V_3 assumes the same representation as in $(3.1)_3$. For the three considered cases we may represent the load p_3 in a unitary form as:

$$p_3(x_1,x_2) = \sum_{m=1}^{\infty} p_m(x_2)\sin\alpha x_1 \qquad (3.3)$$

where $\alpha \equiv m\pi/a$.

Substitution of (3.1) into (2.68) and respectively in (2.73) and of (3.2) in (2.71) results in a set of three differential equations in the x_2 - variable associated to each theory. Under a generic form they may be expressed as

$$C_1R_m'' + C_2R_m + C_3S_{1m}'' + C_4S_{1m} + C_5S_{2m}' + C_6p_m = 0$$

$$C_7R_m''' + C_8R_m' + C_9S_{1m}' + C_{10}S_{2m}'' + C_{11}S_{2m} + C_{12}p_m' = 0$$

$$C_{13}R_m'' + C_{14}R_m + C_{15}S_{1m}'' + C_{16}S_{1m} + C_{17}S_{2m}' + C_{18}p_m = 0 \tag{3.4}$$

where C_i ($i = \overline{1,18}$) are coefficients rendered explicitly in the Appendix (see Eqs. A1-A3) and the primes on the variables indicate ordinary differentiation with respect to x_2.

In the case of DT and FSDT we are to assimilate the generic state-variables as:

$$R_m \to W_m \;;\; S_{1m} \to V_{1m} \text{ and } S_{2m} \to V_{2m} \tag{3.5}$$

while for ST we are to consider

$$R_m \to W_m \;;\; S_{1m} \to \phi_{1m} \;;\; S_{2m} \to \phi_{2m} \tag{3.6}$$

At this point we shall proceed to a modification of Eqs. (3.4) rendering them more suitable to the state-space approach. This modified form obtained after elementary manipulations reads:

For ST and DT

$$\begin{aligned} R_m''' &= e_1R_m' + e_2S_{1m}' + e_3S_{2m} + e_4p_m' \\ S_{1m}'' &= e_5R_m'' + e_6R_m + e_7S_{1m} + e_8p_m \\ S_{2m}' &= e_9R_m'' + e_{10}R_m + e_{11}S_{1m} + e_{12}p_m \end{aligned} \tag{3.7}$$

while for FSDT

$$R_m'' = e_1 R_m + e_2 S_{1m} + e_3 S_{2m}' + e_4 p_m$$

$$S_{1m}'' = e_5 R_m + e_6 S_{1m} + e_7 S_{2m}'$$

$$S_{2m}'' = e_8 R_m' + e_9 S_{1m}' + e_{10} S_{2m} \tag{3.8}$$

The coefficients e_i appearing in (3.7) and (3.8) are given in the Appendix (Eqs. A5-A7).

A further step consists of the representation of the system of ordinary differential equations (3.7) and (3.8) in the form of a single matrix differential equation [13]

$$\underset{\sim}{Y}' = A\underset{\sim}{Y} + \underset{\sim}{b} \tag{3.9}$$

Here $\underset{\sim}{Y}$ designates the state-space 6 x 1 column vector whose elements are defined as:

For DT and ST:

$$Y_1 \equiv R_m \ ; \ Y_2 \equiv R_m' \ ; \ Y_3 = R_m''$$

$$Y_4 \equiv S_{1m} \ ; \ Y_5 \equiv S_{1m}' \ ; \ Y_6 \equiv S_{2m} \tag{3.10}$$

while for FSDT they are defined as

$$Y_1 \equiv R_m \; ; \; Y_2 \equiv R_m' \; ; \; Y_3 \equiv S_{1m} \; ;$$
$$Y_4 \equiv S_{1m}' \; ; \; Y_5 = S_{2m} \; ; \; Y_6 \equiv S_{2m}' \tag{3.11}$$

$\underset{\sim}{b}$ denotes the 6 x 1 column input vector defined in the Appendix (Eqs. A9).

Needless to say that in Eqs. (3.10) and (3.11) the sense of the generic variables is to be adjusted with their pertinent meaning explained in (3.5) and (3.6).

The solution of Eq. (3.9) is [13]

$$\underset{\sim}{Y}(x_2) = e^{Ax_2}\underset{\sim}{K} + e^{Ax_2}\int_0^{x_2} e^{-A\xi_2}\underset{\sim}{b}\,d\xi_2 \tag{3.12}$$

where $\underset{\sim}{K}$ is a constant column vector to be determined with the help of B.C. while e^{Ax_2} is defined as

$$e^{Ax_2} = [C]\begin{bmatrix} e^{\lambda_1 x_2} & & & 0 \\ & e^{\lambda_2 x_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_6 x_2} \end{bmatrix}[C]^{-1} \tag{3.13}$$

where [C] is the matrix of eigenvectors; λ_i $(i = \overline{1,6})$ denote the eigenvalues associated to the 6 x 6 matrix A defined in the Appendix (A11); $[C]^{-1}$ is the inverse of the matrix [C]. The following B.C. are used at the two edges $x_2 = \pm b/2$:

For DT:

$$\text{SS:}\quad V_3 = 0\ ;\ \overset{(1)}{V}_1 = 0\ ;\ L^{22}_{(1)} = 0$$

$$\text{CL:}\quad V_3 = 0\ ;\ \overset{(1)}{V}_1 = 0\ ;\ 4\,\overset{(1)}{V}_2 - V_{3,2} = 0$$

$$\text{F:}\quad L^{22}_{(1)} = 0\ ;\ L^{12}_{(1)} = 0\ ;\ L^{23}_{(0)} = 0 \tag{3.14}$$

For ST:

$$\text{SS:}\quad V_3 = 0\ ;\ \psi_1 = 0\ ;\ L^{22}_{(1)} = 0$$

$$\text{CL:}\quad V_3 = 0\ ;\ \psi_1 = 0\ ;\ V_{3,2} + \frac{4}{5}\,h^2 F^{23}_{23}\psi_2 = 0 \tag{3.15}$$

For FSDT:

$$\text{SS:}\quad V_3 = 0\ ;\ L^{22}_{(1)} = 0\ ;\ \overset{(1)}{V}_1 = 0$$

$$\text{CL:}\quad V_3 = 0\ ;\ \overset{(1)}{V}_1 = 0\ ;\ \overset{(1)}{V}_2 = 0 \tag{3.16}$$

In the ST and FSDT instances the free edge conditions express similarly as in the DT case.

3.1.2 HSDT Theory

Consider a laminated plate composed of N orthotropic layers, symmetrically located with respect to the midplane of the laminate. The governing equations of the refined theory considered here can be found in [60-62]. For symmetrical cross-ply laminated plates, the following stiffness coefficients vanish [62]:

$$B_{ij} = E_{ij} = 0 \text{ for } i,j = 1,2,4,5,6$$

$$A_{16} = A_{26} = D_{16} = D_{26} = F_{16} = F_{26} = H_{16} = H_{26} = 0 \qquad (3.17)$$

$$A_{45} = D_{45} = F_{45} = 0,$$

and as a result the stretching and bending states of stress are uncoupled. For such laminates the governing equations are given by:

$$\frac{4}{3h^2} [F_{11}\psi_{1,111} + H_{11}(-\frac{4}{3h^2})(\psi_{1,111} + W_{,1111}) + F_{12}\psi_{2,112}$$
$$+ H_{12}(-\frac{4}{3h^2})(\psi_{2,112} + W_{,1122}) + F_{12}\psi_{1,122} +$$
$$H_{12}(-\frac{4}{3h^2})(\psi_{1,122} + W_{,1122}) + F_{22}\psi_{2,222} + H_{22}(-\frac{4}{3h^2})(\psi_{2,222}$$
$$+ W_{,2222}) + 2F_{66}(\psi_{2,112} + \psi_{1,122}) + 2H_{66}(-\frac{4}{3h^2})(\psi_{1,221} + \psi_{2,112}$$
$$+ 2W_{,1122})] - \frac{4}{h^2} [D_{55}(\psi_{1,1} + W_{,11}) + F_{55}(-\frac{4}{h^2})(\psi_{1,1} + W_{,11})$$
$$+ D_{44}(\psi_{2,2} + W_{,22}) + F_{44}(-\frac{4}{h^2})(\psi_{2,2} + W_{,22})] + [A_{55}(\psi_{1,1} + W_{,11})$$
$$+ D_{55}(-\frac{4}{h^2})(\psi_{1,1} + W_{,11}) + A_{44}(\psi_{2,2} + W_{,22}) + D_{44}(-\frac{4}{h^2})(\psi_{2,2} +$$
$$+ W_{,22})] + q = 0 \qquad (3.18)_1$$

$$D_{11}\psi_{1,11} + D_{12}\psi_{2,12} + F_{11}(-\frac{4}{3h^2})(\psi_{1,11} + W_{,111}) + F_{12}(-\frac{4}{3h^2})(\psi_{2,12}$$
$$+ W_{,122}) + D_{66}(\psi_{1,22} + \psi_{2,12}) + F_{66}(-\frac{4}{3h^2})(\psi_{1,22} + \psi_{2,12} + 2W_{,122})$$

$$- [A_{55}(\psi_1 + W_{,1}) + D_{55}(- \frac{4}{h^2})(\psi_1 + W_{,1})] - \frac{4}{3h^2} [F_{11}\psi_{1,11}$$

$$+ H_{11}(- \frac{4}{3h^2})(\psi_{1,11} + W_{,111}) + F_{12}\psi_{2,12} + H_{12}(- \frac{4}{3h^2})(\psi_{2,12} + W_{,122})$$

$$+ F_{66}(\psi_{1,22} + \psi_{2,12}) + H_{66}(- \frac{4}{3h^2})(\psi_{1,22} + \psi_{2,12} + 2W_{,122})]$$

$$+ \frac{4}{h^2} [D_{55}(\psi_1 + W_{,1}) + F_{55}(- \frac{4}{h^2})(\psi_1 + W_{,1})] = 0 \qquad (3.18)_2$$

$$D_{66}(\psi_{1,12} + \psi_{2,11}) + F_{66}(- \frac{4}{3h^2})(\psi_{1,12} + \psi_{2,11} + 2W_{,112}) + D_{12}\psi_{1,12}$$

$$+ D_{22}\psi_{2,22} + F_{12}(- \frac{4}{3h^2})(\psi_{1,12} + W_{,112}) + F_{22}(- \frac{4}{3h^2})(\psi_{2,22} + W_{,222})$$

$$- [A_{44}(\psi_2 + W_{,2}) + D_{44}(- \frac{4}{h^2})(\psi_2 + W_{,2})] - \frac{4}{3h^2} [F_{66}(\psi_{1,12} + \psi_{2,11})$$

$$+ H_{66}(- \frac{4}{3h^2})(\psi_{1,12} + \psi_{2,11} + 2W_{,112}) + F_{12}\psi_{1,12} + H_{12}(- \frac{4}{3h^2})(\psi_{1,12}$$

$$W_{,112}) + F_{22}\psi_{2,22} + H_{22}(- \frac{4}{3h^2})(\psi_{2,22} + W_{,222})] + \frac{4}{h^2} [D_{44}(\psi_2 + W_{,2})$$

$$+ F_{44}(- \frac{4}{h^2})(\psi_2 + W_{,2})] = 0 \qquad (3.18)_3$$

Here W denotes the transverse displacement, ψ_1 and ψ_2 are the rotations of the normal to midplane about the x_2 and x_1 axis, respectively, q is the distributed transverse load, and A_{ij}, D_{ij}, F_{ij}, H_{ij} are the plate stiffnesses, defined by

$$(D_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{x_3k}^{x_3k+1} Q_{ij}^{(k)} \; (x_3^2, x_3^4, x_3^6) dx_3 \quad (i,j = 1,2,6)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{x_3k}^{x_3k+1} Q_{ij}^{(k)} (1, x_3^2, x_3^4) dx_3 \quad (i,j = 4,5) \qquad (3.19)$$

Here $Q_{ij}^{(k)}$ denote the reduced orthotropic moduli of the k-th lamina (see Fig. 3.2) in the laminate coordinates. The boundary conditions of the refined theory are of the form:

specify

$$\left.\begin{array}{l} W \text{ or } Q_n \\ W_{,n} \text{ or } P_n \\ \psi_n \text{ or } M_n \\ \psi_{ns} \text{ or } M_{ns} \end{array}\right\} \text{on } \Gamma \tag{3.20}$$

where Γ is the boundary of the midplane Ω of the plate, and

$$\begin{aligned}
M_n &= \hat{M}_1 n_1^2 + \hat{M}_2 n_2^2 + 2M_6 n_1 n_2 \\
M_{ns} &= (\hat{M}_2 - \hat{M}_1) n_1 n_2 + \hat{M}_6 (n_1^2 - n_2^2) \\
P_n &= P_1 n_1^2 + P_2 n_2^2 + 2P_6 n_1 n_2 \\
P_{ns} &= (P_2 - P_1) n_1 n_2 + P_6 (n_1^2 - n_2^2) \\
Q_n &= \hat{Q}_1 n_1 + \hat{Q}_2 n_2 - \frac{4}{3h^2} P_{ns,s} \\
\hat{M}_i &= M_i - \frac{4}{3h^2} P_i \quad (i = 1,2,6) \\
\hat{Q}_i &= Q_i - \frac{4}{h^2} R_i \quad (i = 1,2) \\
\frac{\partial}{\partial n} &= n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} \;, \quad \frac{\partial}{\partial s} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}
\end{aligned} \tag{3.21}$$

The stress resultants appearing in Eq. (3.21) can be expressed in terms of the generalized displacements (ψ_1, ψ_2, W) as:

$$\begin{aligned}
M_1 &= D_{11}\psi_{1,1} + D_{12}\psi_{2,2} + F_{11}\left(-\frac{4}{3h^2}\right)(\psi_{1,1} + W_{,11}) \\
&\quad + F_{12}\left(-\frac{4}{3h^2}\right)(\psi_{2,2} + W_{,22})
\end{aligned}$$

$$M_2 = D_{12}\psi_{1,1} + D_{22}\psi_{2,2} + F_{12}(-\frac{4}{3h^2})(\psi_{1,1} + W_{,11})$$
$$+ F_{22}(-\frac{4}{3h^2})(\psi_{2,2} + W_{,22})$$
$$M_6 = D_{66}(\psi_{1,2} + \psi_{2,1}) + F_{66}(-\frac{4}{3h^2})(\psi_{1,2} + \psi_{2,1} + 2W_{,12})$$
$$Q_1 = A_{55}(\psi_1 + W_{,1}) + D_{55}(-\frac{4}{h^2})(\psi_1 + W_{,1})$$
$$Q_2 = A_{44}(\psi_2 + W_{,2}) + D_{44}(-\frac{4}{h^2})(\psi_2 + W_{,2}) \quad (3.22)$$
$$P_1 = F_{11}\psi_{1,1} + F_{12}\psi_{2,2} + H_{11}(-\frac{4}{3h^2})(\psi_{1,1} + W_{,11})$$
$$+ H_{12}(-\frac{4}{3h^2})(\psi_{2,2} + W_{,22})$$
$$P_2 = F_{12}\psi_{1,1} + F_{22}\psi_{2,2} + H_{12}(-\frac{4}{3h^2})(\psi_{1,1} + W_{,11})$$
$$+ H_{22}(-\frac{4}{3h^2})(\psi_{2,2} + W_{,22})$$
$$P_6 = F_{66}(\psi_{1,2} + \psi_{2,1}) + H_{66}(-\frac{4}{3h^2})(\psi_{1,2} + \psi_{2,1} + 2W_{,12})$$
$$R_1 = D_{55}(\psi_1 + W_{,1}) + F_{55}(-\frac{4}{3h^2})(\psi_1 + W_{,1})$$
$$R_2 = D_{44}(\psi_2 + W_{,2}) + F_{44}(-\frac{4}{3h^2})(\psi_2 + W_{,2})$$

Following the Levy type procedure, we assume the following representation of the displacements and loading:

$$W(x_1,x_2) = \sum_{m=1}^{\infty} W_m(x_2)\sin\alpha x_1$$

$$\psi_1(x_1,x_2) = \sum_{m=1}^{\infty} X_m(x_2)\cos\alpha x_1$$

$$\psi_2(x_1,x_2) = \sum_{m=1}^{\infty} Z_m(x_2)\sin\alpha x_1 \tag{3.23}$$

$$q(x_1,x_2) = \sum_{m=1}^{\infty} Q_m(x_2)\sin\alpha x_1$$

Substituting Eqs. (3.23) into Eqs. (3.18), we obtain

$$\begin{aligned}
&C_1W_m'''' + C_2W_m'' + C_3W_m + C_4X_m'' + C_5X_m + C_6Z_m''' + C_7Z_m' + Q_m = 0\\
&C_8W_m'' + C_9W_m + C_{10}X_m'' + C_{11}X_m + C_{12}Z_m' = 0\\
&C_{13}W_m''' + C_{14}W_m' + C_{15}X_m' + C_{16}Z_m'' + C_{17}Z_m = 0,
\end{aligned} \tag{3.24}$$

where $C_i(i = \overline{1,17})$ are coefficients rendered explicitly in the Appendix (see Eqs. A4).

Equations (3.24) can be written as:

$$\begin{aligned}
W_m'''' &= e_1W_m'' + e_2W_m + e_3X_m + e_4Z_m' + e_5Q_m\\
X_m'' &= e_6W_m'' + e_7W_m + e_8X_m + e_9Z_m'\\
Z_m'' &= e_{10}W_m''' + e_{11}W_m' + e_{12}X_m' + e_{13}Z_m
\end{aligned} \tag{3.25}$$

The coefficients e_i appearing in (3.25) are given in the Appendix (Eqs. A8).

The linear system of ordinary differential equations (3.25) with constant coefficients can be reduced to a single matrix differential equation using the state-space concept (see [13]) similar to (3.9):

$$\underset{\sim}{Y}' = A\underset{\sim}{Y} + \underset{\sim}{b} \tag{3.26}$$

Here $\underset{\sim}{Y}$ designates the state-space 8 x 1 column vector whose elements are defined as:

$$Y_1 \equiv W_m \; ; \; Y_2 \equiv W_m' \; ; \; Y_3 \equiv W_m'' \; ; \; Y_4 \equiv W_m''' \; ;$$

$$Y_5 \equiv X_m \; ; \; Y_6 \equiv X_m' \; ; \; Y_7 \equiv Z_m \; ; \; Y_8 \equiv Z_m' \tag{3.27}$$

$\underset{\sim}{b}$ denotes the 8 x 1 column input vector defined in the Appendix (Eqs. A10).

The solution of Eq. (3.26) is given as in (3.12) by

$$\underset{\sim}{Y}(x_2) = e^{Ax_2}\underset{\sim}{K} + e^{Ax_2}\int_0^{x_2} e^{-A\xi_2}\,\underset{\sim}{b}\,d\xi_2 \tag{3.28}$$

where $\underset{\sim}{K}$ is an 8 x 1 constant vector to be determined with the help of B.C., while e^{Ax_2} is defined as

$$e^{Ax_2} = [C]\begin{bmatrix} e^{\lambda_1 x_2} & & & 0 \\ & e^{\lambda_2 x_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_8 x_2} \end{bmatrix}[C]^{-1} \tag{3.29}$$

where [C] is the matrix of eigenvectors; $\lambda_i (i = \overline{1,8})$ denote the eigenvalues associated to the 8 x 8 matrix A defined in the Appendix (A12); $[C]^{-1}$ is the inverse of the matrix [C]. The following B.C. are used at the two edges $x_2 = \pm\, b/2$:

SS: $W = \psi_1 = P_2 = M_2 = 0$

CL: $W = W_{,2} = \psi_1 = \psi_2 = 0$

F: $P_2 = M_2 = 0$ (3.30)

$$M_6 - \frac{4}{3h^2} P_6 = 0$$

$$Q_2 - \frac{4}{h^2} R_2 - \frac{4}{3h^2} P_{6,1} = 0$$

3.2 Dynamic Response of Orthotropic Rectangular Plates

In the foregoing developments we shall analyze the dynamic response of orthotropic rectangular thin panels simply supported along all four edges to DT and ST theories while for FSDT and HSDT see [12,58,59,89-91,103] and [55], respectively.

It is assumed that the plate is restrained against rigid-body motion and that a disturbing transversal force (per unit area) $p_3(x_1, x_2, t)$ is applied at the top surface of the plate. The disturbance force p_3 (x_1, x_2, t) is assumed to be given explicitly with respect to both its space and time variation. In addition, in the case of the plate subjected to the uniform in-plane edge loads T_{11} and T_{22} (applied in the directions x_1 and x_2, respectively, and considered positive in tension) the extra term

$$\hat{P}_3 = T_{11}V_{3,11} + T_{22}V_{3,22} \tag{3.31}$$

is to be added to the load P_3. In dealing with the dynamic response problem, we shall express the displacement quantities in terms of the free vibration modes of the associated plate by the expansions:

For DT

$$\begin{aligned} V_1(x_1, x_2, t) &= \sum_{m,n=1}^{\infty} \overline{V}_{1mn}\, q_{mn} \\ V_2(x_1, x_2, t) &= \sum_{m,n=1}^{\infty} \overline{V}_{2mn}\, q_{mn} \\ V_3(x_1, x_2, t) &= \sum_{m,n=1}^{\infty} \overline{W}_{mn}\, q_{mn} \end{aligned} \tag{3.32}$$

while for ST as:

$$\psi_1(x_1,x_2,t) = \sum_{m,n=1}^{\infty} \bar{\phi}_{1mn} q_{mn}$$

$$\psi_2(x_1,x_2,t) = \sum_{m,n=1}^{\infty} \bar{\phi}_{2mn} q_{mn} \tag{3.33}$$

whereas V_3 assumes the same representation as in $(3.32)_3$. Here $\bar{V}_{\alpha mn} \equiv \bar{V}_{\alpha mn}(x_1, x_2)$, $\bar{\phi}_{\alpha mn} = \bar{\phi}_{\alpha mn}(x_1,x_2)$, $\bar{W}_{mn} \equiv \bar{W}_{mn}(x_1, x_2)$, $(\alpha = 1,2)$ denote the natural mode shapes assumed to be known while $q_{mn} \equiv q_{mn}(t)$ stand for the generalized coordinates whose determination constitutes the central goal of the dynamic response problem.

These may be determined in the usual way (see [47]), by making use of the orthogonality property of modes of free vibration. In the case of the real distinct natural frequencies $\omega_{mn} \equiv \omega$ and $\omega_{rs} \equiv \hat{\omega}$ the orthogonality condition of modes corresponding to ω and $\hat{\omega}$ may be obtained by following the lines used in [36, p. 374]. For this case the orthogonality property of eigenmodes reads:

For DT

$$\int_0^{\ell_1} \int_0^{\ell_2} \{m_0 \hat{\bar{W}}\, \bar{W} + m_2\, (\bar{V}_1 \hat{\bar{V}}_1 + \bar{V}_2 \hat{\bar{V}}_2) - \frac{4}{3h^2}\, m_4\, [\bar{V}_1(\hat{\bar{W}}_{,1} + \hat{V}_1) + \bar{V}_2(\hat{\bar{W}}_{,2} + \hat{\bar{V}}_2) + \hat{\bar{V}}_1(\bar{W}_{,1} + \bar{V}_1) + \hat{\bar{V}}_2(\bar{W}_{,2} + \bar{V}_2)] + (\frac{4}{3h^2})^2 m_6\, [(\bar{W}_{,1} + \bar{V}_1)(\hat{\bar{W}}_{,1} + \hat{\bar{V}}_1) + (\bar{W}_{,2} + \bar{V}_2)(\hat{\bar{W}}_{,2} + \hat{\bar{V}}_2)]\}\, dx_1\, dx_2 = 0 \tag{3.34}$$

while for ST reads:

$$\int_0^{\ell_1} \int_0^{\ell_2} \{m_o \hat{\bar{W}}\, \bar{W} + m_2[(h^2 F_{13}^{13} \bar{\phi}_1 + \bar{W}_{,1})(h^2 F_{13}^{13} \hat{\bar{\phi}}_1 + \hat{\bar{W}}_{,1}) + (h^2 F_{23}^{23} \bar{\phi}_2$$

$$+ \bar{W}_{,2})(h^2 F^{23}_{23}\hat{\bar{\phi}}_2 + \hat{\bar{W}}_{,2})] - \frac{4}{3} m_4[F^{13}_{13}\hat{\bar{\phi}}_1(h^2 F^{13}_{13}\bar{\phi}_1 + \bar{W}_{,1})$$

$$+ F^{23}_{23}\hat{\bar{\phi}}_2(h^2 F^{23}_{23}\bar{\phi}_2 + \bar{W}_{,2}) + F^{13}_{13}\bar{\phi}_1(h^2 F^{13}_{13}\hat{\bar{\phi}}_1 + \hat{\bar{W}}_{,1}) + F^{23}_{23}\bar{\phi}_2(h^2 F^{23}_{23}\hat{\bar{\phi}}_2$$

$$+ \hat{\bar{W}}_{,2})] + \frac{16}{9} m_6[{F^{13}_{13}}^2\bar{\phi}_1\hat{\bar{\phi}}_1 + {F^{23}_{23}}^2\bar{\phi}_2\hat{\bar{\phi}}_2]\}dx_1 dx_2 \tag{3.35}$$

Here m_i $(i = 0, 2, 4, 6)$ denote the mass terms defined as:

$$(m_0, m_2, m_4, m_6) = \int_{-h/2}^{h/2} \rho(1, x_3^2, x_3^4, x_3^6)dx_3. \tag{3.36}$$

By virtue of the representation (3.32), (3.33) of the solution and having in mind that $\bar{W}_{mn}$, $\bar{V}_{1mn}$, $\bar{V}_{2mn}$, $\bar{\phi}_{1mn}$ and $\bar{\phi}_{2mn}$ play the role of eigenmodes, the Eqs. (2.68), (2.71) reduce with the help of (3.34), (3.35), after some manipulations to:

$$\ddot{q}_{mn} + \omega_{mn}^2 q_{mm} = Q_{mn}(t) \tag{3.37}$$

where the generalized force Q_{mn} is expressed as

$$Q_{mn}\ (t) = \frac{\int_0^{\ell_1}\int_0^{\ell_2} p_3\ \bar{W}_{mn}\ dx_1\ dx_2}{N_{mn}} \tag{3.38}$$

N_{mn} stands for the generalized mass expressed as:

For DT

$$N_{mn} \equiv \int_0^{\ell_1}\int_0^{\ell_2}\{m_0\bar{W}_{mn}^2 + m_2\ (\bar{V}_{1mn}^2 + \bar{V}_{2mn}^2) - \frac{8}{3h^2} m_4\ [\bar{V}_{1mn}^2 + \bar{V}_{2mn}^2 + \bar{V}_{1mn}\ \bar{W}_{mn,1}$$

$$+ \bar{V}_{2mn}\bar{W}_{mn,2}] + (\frac{4}{3h^2})^2 m_6\ [\bar{V}_{1mn}^2 + \bar{V}_{2mn}^2 + 2\bar{V}_{1mn}\bar{W}_{mn,1} + 2\bar{V}_{2mn}\bar{W}_{mn,2}$$

$$+ (\bar{W}_{mn,1})^2 + (\bar{W}_{mn,2})^2]\}\ dx_1\ dx_2 \tag{3.39}$$

while for ST

$$N_{mn} \equiv \int_0^{\ell_1}\int_0^{\ell_2} \{m_0\bar{W}_{mn}^2 + m_2[(h^2F_{13}^{13}\bar{\phi}_{1mn} + \bar{W}_{mn,1})^2 + (h^2F_{23}^{23}\bar{\phi}_{2mn} + \bar{W}_{mn,2})^2]$$

$$- \frac{8}{3} m_4[h^2F^2{}_{13}^{13}\bar{\phi}_{1mn}^2 + F_{13}^{13}\bar{\phi}_{1mn}\bar{W}_{mn,1} + h^2F^2{}_{23}^{23}\bar{\phi}_{2mn}^2 + F_{23}^{23}\bar{\phi}_{2mn}\bar{W}_{mn,2}]$$

$$+ \frac{16}{9} m_6[F^2{}_{13}^{13}\bar{\phi}_{1mn}^2 + F^2{}_{23}^{23}\bar{\phi}_{2mn}^2]\}dx_1dx_2 \tag{3.40}$$

The expression (3.39), (3.40) may be obtained from (3.34), (3.35) for the case $\hat{\omega} \equiv \omega$, i.e. when $\bar{V}_{\alpha mn} \equiv \hat{\bar{V}}_{\alpha mn}$, $\bar{\phi}_{\alpha mn} = \hat{\bar{\phi}}_{\alpha mn}$, and $\bar{W}_{mn} \equiv \hat{\bar{W}}_{mn}$. In this case from the mathematical viewpoint N_{mn} plays the role of a norm. For zero initial conditions the solution to (3.37) reads:

$$q_{mn}(t) = \frac{1}{\omega_{mn}} \int_0^t Q_{mn}(\tau)\sin\omega_{mn}(t-\tau)d\tau. \tag{3.41}$$

In order to determine completely the solution of the dynamic response problem, the eigenfrequencies and eigenmodes of the associated free vibration problem are to be obtained.

3.2.1 The Free Vibration Problem of Rectangular Orthotropic Flat Plates

In the case of the free vibration problem, the generalized coordinates $q_{mn}(t)$ intervening in (3.32), (3.33) and the eigenmodes $\bar{V}_{1mn}$, $\bar{V}_{2mn}$, $\bar{\phi}_{1mn}$, $\bar{\phi}_{2mn}$, $\bar{W}_{mn}$ associated to the homogeneous system of governing equations and fulfilling the SSSS edge conditions* may be expressed as:

---

*The SS edge conditions associated to a boundary whose normal and tangential directions are n and s, respectively, write: $V_3 = 0$; $\overset{(1)}{V_s} = 0$; $L^{nn} = 0$.

$$q_{mn}(t) = e^{j\omega_{mn}t} \quad (j = \sqrt{-1})$$

$$\begin{Bmatrix} \overline{V}_{1mn} \\ \overline{\Phi}_{1mn} \end{Bmatrix} = A_{mn} \cos \alpha_1 x_1 \sin \alpha_2 x_2$$

$$\begin{Bmatrix} \overline{V}_{2mn} \\ \overline{\Phi}_{2mn} \end{Bmatrix} = B_{mn} \sin \alpha_1 x_1 \cos \alpha_2 x_2$$

$$\overline{W}_{mn} = C_{mn} \sin \alpha_1 x_1 \sin \alpha_2 x_2 \qquad (3.42)$$

where $\alpha_1 \equiv \frac{m\Pi}{\ell_1}$, $\alpha_2 = \frac{n\Pi}{\ell_2}$; A_{mn}, B_{mn} and C_{mn} being arbitrary constants.

Substitution of (3.32), (3.33) considered in conjunction with (3.42) in the equations (3.68), (2.71) rendered homogeneous by dropping the transversal load term yields a system of algebraic equations expressed in a compact form as:

$$[L] \{V\} = 0 \qquad (3.43)$$

where

$$[L] = [C] - \omega_{mn}^2 [S] \qquad (3.44)$$

and

$$\{V\}^T = \{A_{mn}, B_{mn}, C_{mn}\};$$

The elements of the 3×3 matrices [C] and [S] are expressed as:

For DT

$$C_{11} = 4\tilde{E}_{11}^{11}\alpha_1^2 + 4\tilde{E}_{12}^{12}\alpha_2^2 - 4\delta_A \alpha_1^2 E_{13}^{13} \frac{E_{33}^{11}}{E_{33}^{33}} + \frac{40}{h^2} E_{13}^{13} \qquad (1 \rightleftarrows 2)$$

$$C_{12} = 4\alpha_1\alpha_2(\tilde{E}_{22}^{11} + \tilde{E}_{12}^{12} - \delta_A E_{23}^{23} \frac{E_{33}^{11}}{E_{33}^{33}}) \qquad (1 \rightleftarrows 2)$$

$$C_{13} = -\alpha_1^3\left(\tilde{E}^{11}_{11} + 4\delta_A E^{13}_{13}\,\frac{E^{11}_{33}}{E^{33}_{33}}\right) - \alpha_1\alpha_2^2\left(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12}\right.$$

$$\left. + 4\delta_A E^{23}_{23}\,\frac{E^{11}_{33}}{E^{33}_{33}}\right) + \frac{40}{h^2}\,\alpha_1 E^{13}_{13} \qquad (1 \rightleftarrows 2) \qquad (3.45)$$

$$C_{31} = -\frac{2}{3}\,h\alpha_1 E^{13}_{13} \qquad (1 \rightleftarrows 2)$$

$$C_{33} = -\frac{2}{3}\,h(\alpha_1^2 E^{13}_{13} + \alpha_2^2 E^{23}_{23})$$

$$S_{11} = S_{22} = 4\delta_\beta\rho;\; S_{12} = S_{21} = 0$$

$$S_{13} = -\rho\alpha_1\left(\delta_\beta + 5\delta_C\,\frac{E^{11}_{33}}{E^{33}_{33}}\right) \qquad (1 \rightleftarrows 2)$$

$$S_{31} = S_{32} = 0;\; S_{33} = -\rho h \qquad (3.46)$$

For ST

$$C_{11} = \frac{4}{5}\,h^2 F^{13}_{13}(\alpha_1^2\tilde{E}^{11}_{11} + \alpha_2^2\tilde{E}^{12}_{12}) + 2 \qquad (1 \rightleftarrows 2)$$

$$C_{12} = \frac{4}{5}\,h^2\alpha_1\alpha_2 F^{23}_{23}(\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12}) \qquad (1 \rightleftarrows 2)$$

$$C_{13} = \alpha_1^3\tilde{E}^{11}_{11} + \alpha_1\alpha_2^2(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12}) \qquad (1 \rightleftarrows 2)$$

$$C_{31} = \frac{h^3}{6}\,\alpha_1 \qquad (1 \rightleftarrows 2)$$

$$C_{33} = 0 \qquad (3.47)$$

$$S_{11} = \frac{4}{5}\,h^2\delta_\beta\rho F^{13}_{13} \qquad (1 \rightleftarrows 2)$$

$$S_{12} = 0 \qquad (1 \rightleftarrows 2)$$

$$S_{13} = \rho\alpha_1\left(\delta_\beta - \frac{1}{5}\,\delta_c\,\frac{E_{33}^{11}}{E_{33}^{33}}\right) \qquad (1 \rightleftarrows 2)$$

$$S_{31} = 0 \qquad (1 \rightleftarrows 2)$$

$$S_{33} = -\,\rho h \tag{3.48}$$

The eigenfrequencies are determined for each m,n pair by setting the determinant of the matrix [L] equal to zero. The eigenfunctions associated to the natural frequencies ω_{mn}^2 write as

$$A_{mn} = \frac{L_{12}L_{23} - L_{13}L_{22}}{L_{11}L_{22} - L_{12}L_{21}}\,C_{mn}$$

$$B_{mn} = \frac{L_{21}L_{13} - L_{11}L_{23}}{L_{11}L_{22} - L_{12}L_{21}}\,C_{mn} \tag{3.49}$$

when normalized with respect to C_{mn}.

3.2.2. The Dynamic Response

Equations (3.32), (3.33) and (3.41) may be used to determine the dynamic response of plates exposed to the prescribed time-dependent lateral load $p_3(x_1,x_2,t)$. In order to put into evidence merely the time-dependent character of the external loads we shall assume a spatial uniformly distributed load, i.e.

$$p_3(x_1,x_2,t) \equiv p_0F(t). \tag{3.50}$$

where p_0 is a constant valued quantity while F(t) is an arbitrary function of time.

By virtue of (3.38) and (3.50) we obtain

$$Q_{mn}(t) = \frac{4\ell_1\ell_2 p_0}{mn\pi^2 N_{mn}} F(t). \tag{3.51}$$

so that the solution may be expressed formally as:

$$\begin{Bmatrix} V_3(x_1,x_2,t) \\ \begin{Bmatrix} \overset{(1)}{V_1}(x_1,x_2,t) \\ \psi_1(x_1,x_2,t) \end{Bmatrix} \\ \begin{Bmatrix} \overset{(1)}{V_2}(x_1,x_2,t) \\ \psi_2(x_1,x_2,t) \end{Bmatrix} \end{Bmatrix} = \frac{4\ell_1\ell_2 p_0}{\pi^2} \sum_{m,n=1}^{\infty} \begin{Bmatrix} \dfrac{\sin\alpha_1 x_1 \sin\alpha_2 x_2}{mnN_{mn}\omega_{mn}} \\ \dfrac{\cos\alpha_1 x_1 \sin\alpha_2 x_2}{mnN_{mn}\omega_{mn}} \\ \dfrac{\sin\alpha_1 x_1 \cos\alpha_2 x_2}{mnN_{mn}\omega_{mn}} \end{Bmatrix}$$

$$\times \begin{Bmatrix} 1 \\ \dfrac{L_{12}L_{23} - L_{13}L_{22}}{L_{11}L_{22} - L_{12}L_{21}} \\ \dfrac{L_{21}L_{13} - L_{11}L_{23}}{L_{11}L_{22} - L_{12}L_{21}} \end{Bmatrix} \int_0^t F(\tau)\sin \omega_{mn}(t-\tau)d\tau \tag{3.52}$$

With the help of the equations developed before it is possible to determine the dynamic response of orthotropic plates subject to a lateral external force of known time history and to a system of uniform tensile loadings.

3.3. Some Results on the Static Buckling Problem

The obtained equations allow to derive some results which concern the buckling of simply-supported rectangular flat plates acted by a system of uniform in-plane compressive loads T_{11} and T_{22}. In this case, starting with the equations (2.68), (2.71), where the inertia as well as the transversal load terms are dropped and considering solutions of the form (3.42) we obtain a system of algebraic equations similar to (3.43), i.e.

$$[L]\{V\} = 0 \tag{3.53}$$

where

$$[L] = [C] - \lambda[R] \tag{3.54}$$

while

$$T_{11} = -\lambda \; ; \; T_{22} = -\gamma\lambda$$

$$(\gamma \equiv T_{22}/T_{11}) \tag{3.55}$$

The matrices $\{V\}$ and $[C]$ are defined as in Eqs. (3.44) and (3.45), (3.47), respectively, while the elements R_{ij} of $[R]$ are given by

$$\begin{gathered} R_{11} = R_{12} = R_{13} = R_{21} = R_{22} = R_{23} = R_{31} = R_{32} = 0 \; ; \\ R_{33} = -\alpha_1^2 - \gamma\alpha_2^2 \end{gathered} \tag{3.56}$$

The condition $\det(L_{ij}) = 0$ yields the buckling load λ_{cr}.

CHAPTER IV

NUMERICAL RESULTS AND DISCUSSION

4.1 Static Bending Analysis

Numerical results for a number of plates with different boundary conditions and loadings are presented here. Each lamina is assumed to be orthotropic. In order to put into evidence the influence of the degree of orthotropy, two sets of material properties are considered in the calculations, associated to the Material I and Material II, as:

Material I:

$E_1 = 20.83 \times 10^6$ psi, $E_2 = 10.94 \times 10^6$ psi, $E_3 = 10 \times 10^6$ psi

$G_{12} = 6.10 \times 10^6$ psi, $G_{13} = 3.71 \times 10^6$ psi, $G_{23} = 6.19 \times 10^6$ psi

$\nu_{12} = .44$, $\nu_{13} = .44$, $\nu_{23} = .23$

Material II:

$E_1 = 19.2 \times 10^6$ psi, $E_2 = 1.56 \times 10^6$ psi, $E_3 = 1.56 \times 10^6$ psi

$G_{12} = .82 \times 10^6$ psi, $G_{13} = .82 \times 10^6$ psi, $G_{23} = .523 \times 10^6$ psi

$\nu_{12} = .24$, $\nu_{13} = .24$, $\nu_{23} = .49$

The shear correction factors are taken to be $K_1^2 = K_2^2 = 5/6$. Three different types of symmetric laminates are considered here: single layer (0°), three-layer cross-ply (0°/90°/0°) and four layer angle-ply (-45°/45°/45°/-45°). Three types of transverse loads are used: uniformly distributed load (UN), triangular distributed load (TR), and concentrated load (PL) as represented in Figure 4.1.

The designations SS; CC; FF; SC; SF and CF refer to the edge conditions associated to $x_2 = \pm\, b/2$, only. The edges $x_1 = o,a$ are assumed to be simply supported throughtout this work.

The obtained results concern the values of the non-dimensional center deflection ($\bar{W}$) displayed in Tables (4.1-4.16) and of the maximum axial bending stress ($\bar{\sigma}_{11}$), Tables (4.17-4.32), for the various theories, boundary conditions, loadings and lamination schemes. The numerical results are presented in tabular form to facilitate the use of them as reference for comparison by designers and developers of plate bending finite elements.

To bring out the effect of transverse shear strains on the deflections and stresses, plots of nondimensionalized center deflection $\bar{W}$, and center stress $\bar{\sigma}_{11}$, versus side to thickness ratio using FSDT theory are presented in Figures 4.2 through 4.7. From these figures it is observed that the effect of shear deformation is more pronounced on deflections than on stresses and that the effect is increasingly higher for boundary conditions SC, SS, CF, SF and FF. It is also interesting to note that for certain boundary conditions (SC, SS, and SF) the shear deformation has the effect of decreasing the stresses for thick orthotropic and cross-ply laminates.

As it appears from Figures (4.8-4.10), the transverse shear deformation effect is highly pronounced both in the cross-ply arrangement and in the case of homogeneous orthotropic plates. However, in the former case the rigidity in the transverse shear deformation appears to be higher than in the latter case. At the same time, Figures (4.8-4.10) reveal that the first order shear deformation theory (FSDT) over-predicts deflections relative to the higher order theory (HSDT).

Figure 4.11 contains plots of the transverse stresses $\bar{\sigma}_{13}$ through laminate thickness for various boundary conditions. The stresses were computed using lamina constitutive relations. The transverse shear stresses are constant and parabolic through the thickness of each lamina, for FSDT and HSDT theories, respectively. The discontinuity at the interface of lamina is due to the mismatch of the material properties.

Nondimensionalized center stress ($\bar{\sigma}_{11}$) for SS and FF edge conditions are shown in Figures 4.12 and 4.13 using HSDT theory. The shear deformation effect is quite significant for a/h ratios smaller than 10.

The inclusion of the σ^{33} effect shows an increasing influence on $\bar{w}$ once h/a is increasing. In this sense Figs. 4.14 and 4.15 reveal that this pattern of variation is similar, regardless of the considered edge conditions. In contrast to the previous case, Figs. 4.16 and 4.17, depicting the variation of $\bar{\sigma}_{11}$, show for thicker plates ($\frac{a}{h} < 5$) a strong dependence on σ^{33} coupled to the edge conditions.

The prediction of the axial stress component $\bar{\sigma}_{11}$ as resulting from DT (or ST), in the case $\delta_A = 0$, yields rather close results to the ones obtained as per FSDT (see Figs. 4.18 through 4.23). However when the σ^{33} effect is incorporated ($\delta_A = 1$), the predicted values of $\bar{\sigma}_{11}$ differ appreciably from the ones obtained with the neglect of σ^{33}. This conclusion is more striking in the case of some edge conditions (SS, CC, SC, FC), (Figs. 4.18-4.21), and declines in some other ones (e.g. FS and FF), (Figs. 4.22, 4.23). The same trend appears valid for the variation of $\bar{\sigma}_{13}$ through the thickness (see Figs. 4.24, 4.25). The results displayed in these figures (4.18-4.25) show that this theory gives results close to the other theories involving also the neglect of σ^{33}.

Another point worth underlining is the difference in the response characteristics of plates whose materials exhibit high degrees and low degrees of anisotropy (for the sake of brevity HDO and LDO, respectively). Figures 4.26 and 4.27 reveal the drastic increase of the maximum deflection at relatively low values of a/h for HDO (Material II) as compared to the mild increase of $\bar{w}$ for the plates characterized by LDO (Material I). A similar trend which concerns HDO was put into evidence in [52,69].

4.2 Dynamic Response, Free Vibration and Buckling

In order to put into evidence the influence of rotary inertias, of the transversal normal stress component and of orthotropic and transversal-isotropic material characteristics, we shall consider several numerical applications. These will be done for the case of the orthotropic materials (e.g., for Material I and Material II) as well as for the transversely-isotropic materials defined by:

$$E = 20.83 \times 10^6 \text{ psi},\ E' = 10 \times 10^6 \text{ psi},\ G' = 3.71 \times 10^6 \text{ psi}$$

$$\nu = 0.44,\ \nu' = 0.44$$

and

$$E = 19.20 \times 10^6 \text{ psi},\ E' = 1.56 \times 10^6 \text{ psi},\ G' = 0.82 \times 10^6 \text{ psi}$$

$$\nu = 0.24,\ \nu' = 0.24$$

and referred to as the Material A and Material B, respectively. The dynamic response characteristics (which concern the plate center deflection $\bar{V}_3(= V_3(\ell_1/2,\ \ell_2/2)p_0) \times 10^6$, the axial stress $\bar{\sigma}_{11}(= \sigma_{11}(\ell_1/2,\ \ell_2/2,\ \pm h/2)/p_0)$ and the transverse shear stress $\bar{\sigma}_{13}(= \sigma_{13}(0,\ \ell_2/2,\ 0)/p_0)$ versus time (sec)), were determined for three cases of forcing functions dsepicted in Fig. 4.28. They correspond to a sine pulse, a stepped pulse and to a triangular pulse defined respectively as:

$$F(t) = \begin{cases} F_0 \sin \pi t/t_1 & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases} \text{sine pulse}$$

$$\text{for} \quad \begin{cases} F_0 & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases} \text{stepped pulse}$$

$$\begin{cases} F_0(1 - t/t_1) & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases} \text{triangular pulse}$$

In addition, the following input data were used in the numerical applications:

$$\ell_1 = \ell_2 = 200 \text{ in} \ ; \ \rho = 0.002 \text{ lb sec}^2/\text{in}^4;$$

$$F_0 = 1.0 \ ; \ t_1 = 0.05 \text{ sec.}$$

On each curve involved in Figs. (4.29-4.40), as well as in the captions associated to Figs. (4.41-4.47), the kind of approximation used is properly indicated. In addition, Figs. 4.31, 4.32, 4.35, 4.36 and(4.45-4.47) allow to put into evidence the response characteristcs obtained for a transversely-isotropic medium (Material A) and for the input data exhibited above.

Tables (4.33-4.36) display the eigenfrequencies and critical buckling loads obtained in the framework of various theories. Here Exact and CL identify the results obtained in [87] as per the 3D theory of elasticity (for $\delta_A = \delta_C = 0$) and of the Kirchhoff theory of plates, respectively.

Table 4.36 emphasizes the great influence of σ^{33} and rotary inertia forces on the eigenfrequencies, which are similar to the ones discussed in [36, pp. 439-443], in the same respect, by the degree of orthotropy of the material. A similar strong influence appears also in the case of the buckling problem, (see Tables 4.33 and 4.34). Figures (4.29-4.40) emphasize the influence played by the rotary inertias and the transversal normal stress on the dynamic response characteristics (i.e. on $\overline{V}_3$, $\overline{\sigma}_{11}$, $\overline{\sigma}_{13}$), both for orthotropic (Figs. 4.29, 4.30, 4.33, 4.34, 4.37-4.40) and transversely-isotropic plate materials, (Figs. 4.31, 4.32, 4.35, 4.36). These results allow us to infer that the rotary inertia terms have an insignificant influence on the dynamic response characteristics, when in their evaluation, the lowest branch of the frequency spectrum is involved (as it is the case in this numerical example). This conclusion holds valid both for thick plates and for the plates whose materials are weak in transverse shear. In the same context it appears that σ^{33} plays a not negligible role in diminishing the amplitude of the dynamic response. For a fixed thickness ratio (ℓ_1/h), its influence increases in importance in the case of the materials exhibiting high degrees of orthotropicity (or for high E/G' transversely-isotropic plate materials).

The pattern of variation of the various response characteristics associated to different pulse shapes appears in Figs. (4.29-4.47). It clearly emerges from the same figures that the amplitude of the dynamic response is higher for the materials weaker in transverse shear than for the more rigid ones (in which the instances $G_{\alpha 3} \to \infty$, or $G' \to \infty$ corresponding to the Kirchhoff plate theory, appear as a lower limit). Figures 4.43 and 4.44 show the beneficial effect of the tensile loads on the dynamic response characteristics.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary of the Present Study

The Levy-type analytical solutions in conjunction with the state-space concept are developed for symmetric, laminated composite and orthotropic rectangular plates. Combinations of simply-supported, free and clamped boundary conditions are considered. The solutions are obtained for the first order transverse shear deformation theory (FSDT) and three higher order theories (HSDT, DT, ST) in predicting the transverse deflection and stresses. The analysis concerns the solution of these theories for rectangular plates with two opposite edges simply supported and the remaining edges subjected to a combination of free, simply supported and clamped boundary conditions. Numerical results are presented for various boundary conditions, aspect ratios, lamination schemes and different loadings. The results should serve as a reference for designers and numerical analysts.

The developments of these theories (DT, ST, FSDT) accomplished in general coordinates allow one to fulfill both the invariance requirements and to derive the relevant equations in any convenient planar system of coordinates.

The dynamic response problems are analyzed in the framework of higher order theories (DT, ST) and the results are compared to the ones obtained as per FSDT. The effects of σ^{33} and rotary inertia forces on the dynamic response of orthotropic plates are evaluated.

The comparison between the theories as well as previously reported results according to deflections, stresses, natural frequencies, buckling loads and transient response is reported.

5.2 Conclusions

The tables and diagrams previously displayed enable us to draw a series of conclusions.

One of them concerns the fact that the theories denoted here as DT and ST yield identical results. This expected result (see in this sense also [40]), allows one to conclude that the theory promoted in [2] and generalized in [32] and respectively the one developed in [40] on the basis of the statements formulated in [27] represent nothing but two different formulations of the same theory, i.e. of the moderately thick plate theory [32,36].

Another conclusion concerns the fact that DT and ST, specialized for $\delta_A = 0$, yield identical results with FSDT (in which framework the transverse shear correction factors $K^2_{(1)} = K^2_{(2)} = 5/6$ are to be incorporated) regarding the transverse deflection $\overline{W}$, the dynamic response, eigenfrequencies and buckling loads.

In addition to the conclusions which concern the influence of σ^{33} and rotary inertia forces on the eigenfrequencies (which are similar to the ones discussed in [36, pp. 439-443]) and on the dynamic response characteristics (i.e. on $\overline{V}_3$, $\overline{\sigma}_{11}$, $\overline{\sigma}_{13}$), these results allow us to infer that the rotary inertia terms have an insignificant influence on the dynamic response characteristics. This conclusion holds valid for thick plates and for the plates whose materials are weak in transverse shear as well as for the case of thin plates. In the same context it appears that σ^{33} plays a not negligible role in diminishing the amplitude of the dynamic response.

We underline the fact that the dynamic response problem (treated for $\delta_A = \delta_\beta = \delta_C = 0$) as well as the eigenfrequency and buckling problems approached (for $\delta_\beta = 0$) on the basis of the full system of equations (2.68, 2.71) and correspondingly, on the basis of the single equation $(2.83)_2$ (characterizing the interior solution), yield identical results. This allows us to infer (at least for the case of simply supported edge conditions) that equation $(2.83)_1$ identifying the boundary layer effect is automatically fulfilled and so, in this case, it does not enter into play. A similar result was obtained previously [3,36] in the framework of ST.

Finally, concerning the state-space concept used in conjunction with the Levy type method, one may assert that it constitutes a tool of great computational convenience and a compact vehicle for analytic manipulation. In addition, it provides a general framework allowing us to solve the envisioned problem for various edge conditions, which, as is well known from the specialized literature, constitutes a rather difficult task, not attained so far.

REFERENCES

1. Ambartsumian, S. A., "On the Theory of Anisotripic Shells and Plates," IUTAM Symp. on non-homogenity in elasticity and plasticity, Warsaw, Pergamon Press, 1958.

2. Ambartsumian, S. A., "Theory of Anisotropic Shells," NASA Report TT F-118, 1964.

3. Ambartsumian, S. A., "Theory of Anisotropic Plates," Izdat. Nauka, Moskva (1967), English transl. by Technomic Publishing Co. (1969).

4. Basset, A. B., "On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells," Phil. Trans. Royal Soc., (London), Ser. A, Vol. 181, No. 6, pp. 433-480, 1890.

5. Bert, C. W. and Chen, T. L. C., "Effect of Shear Deformation of Antisymmetric Angle-Ply Laminated Rectangular Plates," Int. J. Solids Struct., Vol. 14, pp. 465-473, 1978.

6. Bert, C. W., "A Critical Evaluation of New Plate Theories Applied to Laminated Composites," Composite Structures, Vol. 2, pp. 329-347, 1984.

7. Boal, J. L. and Reissner, E., "Three-dimensional Theory of Elastic Plates with Transverse Inextensibility," J. Math. Phys., Vol. 39, pp. 161-181, 1960.

8. Bolle, L.. "Contribution an Probleme Lineaire de Flexion Dune Plaque Elastique," Bull. Technique de la Suisse Romande, Parts 1 and 2, Vol. 73, pp. 281-285 and 293-298, 1947.

9. Cheng, S., "Elasticity Theory of Plates and a Refined Theory," J. Appl. Mech., Vol. 60, pp. 644-650, 1979.

10. Chou, P. C. and Carleone, J., "Transverse Shear in Laminated Plate Theories," AIAA J., Vol. 11, No. 9, pp. 1333-1336, 1973.

11. Cicala, P., "Consistent Approximations in Shell Theory," J. Engng. Mech. Div. Proc. Amer. Soc. Civil Eng., 1962, 88, 4.

12. Dobyns, A. L., "Analysis of Simply-Supported Orthotropic Plates Subject to Static and Dynamic Loads," AIAA J., Vol. 19, No. 5, pp. 642-650, 1981.

13. Franklin, J. N., Matrix Theory, Prentice-Hall, Englewood Cliffs, N.J., 1968.

14. Gol'denveizer, A. L., "O Teorii Izgiba Plastinok Reissnera (On Reissner's Theory of the Bending of Plates)," Izvestiya AN SSSR, OTN, No. 4, pp. 102-109, 1958 (translation of the paper available as NASA Technical Translation F-27, May 1960).

15. Gol'denveizer, A. L., "Derivation of an Approximate Theory of Bending of a Plate by the Method of Asymptotic Integration of the Equations of the Theory of Elasticity," Priklad matem. i mechan, 26, 4, 1962.

16. Gol'denveizer, A. L., "Derivation of an Approximate Theory of Bending of a Plate by the Method of Asymptotic Integration of the Equations of the Theory of Elasticity," Priklad, matem. i mechan, 27, 4, 1963.

17. Green, A. E., "On Reissner's Theory of Bending of Elastic Plates," Q. Appl. Math., Vol. 7, No. 2, pp. 223-228, 1949.

18. Green, A. E. and Naghdi, P. M., "Some Remarks on the Linear Theory of Shells," Quart. Journ. Mech. and Appl. Math., 18, 3, 257-276, 1965.

19. Green, A. E. and Naghdi, P. M., "A Theory of Laminated Composite Plates," IMA J. Appl. Mathem., Vol. 29, pp. 1-23, 1982.

20. Hencky, H., "Über Die Berücksichtigung Der Schubverzerrung in Ebenen Platten," Ing. Arch., Vol. 16, pp. 72-76, 1947.

21. Hildebrand, F. B., Reissner, E. and Thomas, G. B., "Notes on the Foundations of the Theory of Small Displacements of Orthotropic Shells," NACA Technical Note No. 1833, March 1949.

22. Hussainy, S. A. and Srinivas, S., "Flexure of Rectangular Composite Plates," Fibre Science and Technology, Vol. 8, pp. 59-76, 1975.

23. Irretier, H., "The Influence of Mid-Plane Normal Stress on Cross Section Deformation in Free Vibrating Plates," Mech. Res. Commun., Vol. 10, pp. 53-61, 1983.

24. Khdeir, A. A., Reddy, J. N. and Librescu, L., "Analytical Solution of a Refined Shear Deformation Theory for Rectangular Composite Plates," Intl. J. Solids and Struct., in review.

25. Kromm, A., "Verallgeneinerte Theorie der Plattenstatik," Ing. Arch., Vol. 21, pp. 266-286, 1953.

26. Kromm, A., "Über die RandquerKrafte bei gestutzten Platten," ZAMM, Vol. 35, pp. 231-242, 1955.

27. Levinson, M., "An Accurate Simple Theory of the Statics and Dynamics of Elastic Plates," Mech. Res. Commun., Vol. 7, pp. 343-350, 1980.

28. Levinson, M., "A New Rectangular Beam Theory," Journ. Sound and Vibration, Vol. 74, pp. 81-87, 1981.

29. Librescu, L., "On the Theory of Elastic Anisotropic Shells and Plates," Inzh. Zhurnal 4, 3, 1964.

30. Librescu, L., "On the Thermoelastic Problem of Shells Treated by Eliminating the Love-Kirchhoff Hypothesis," in Non-Classical Shell-Problems, North-Holland Publ. Co., Amsterdam, P.W.N., Warsaw, 1964.

31. Librescu, L., "The Thermoelastic Problem of Shells and Plates Approached by Eliminating the Love-Kirchhoff Hypothesis," St. Cer. Mec. Apl. 21, 2, 351-365, 1966.

32. Librescu, L., "On the Theory of Anisotropic Elastic Shells and Plates," Intern. Journal of Solids and Structures, Vol. 3, pp. 53-68, 1967.

33. Librescu, L., "Substantiation of A Theory of Elastic and Anisotropic Shells and Plates," St. Cerc. Mec. Apl., Vol. 1, pp. 105-128, 1968.

34. Librescu, L., "Sur les equations de la theorie linearie des plaques elastiques anisotropes," C. R. Acad. Sc. Paris, Vol. 267, Sept. 1968, pp. 443-446.

35. Librescu, L., "The Elasto-Kinetic Problems in the Theory of Anisotropic Shells and Plates," Part II, Plates Theory, Rev. Roum. Sci. Tech. Mec. Appl. 7, 3, 1969.

36. Librescu, L., Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures, Noordhoff, Leyden, The Nethrlands, 1975.

37. Librescu, L., "On the Refined Theory of Multilayered Anisotropic Shells," Mechanika Polimerov, Part I, 6, 1974, Part II, 1, 1975, (U.S. translated by Plenum Press).

38. Librescu, L., Schmidt, R., "Higher-Order Moderate Rotation Theories for Elastic Anisotropic Plates," in Finite Rotations in Structural Mechanics, Proc. Euromech-Colloquim, 197, Jablona, 1985, Ed. W. PietraszKiewicz, Berlin, Springer-Verlag, 1986, (in print).

39. Librescu, L., "The Foundations of the Theory of Plates and Shells," Seminar Notes at Virginia Polytechnic Institute and State University, Oct. 1985-Feb. 1986.

40. Librescu, L., and Reddy, J. N., A Critical Review and Generalization of Transverse Shear Deformable Anisotropic Plate Theories, Euromech Colloquium 219, Refined Dynamical Theories of Beams, Plates and Shells and their Applications, Kassel, Sept. 23-26, 1986 (invited paper).

41. Librescu, L. and Reddy, J. N., "A Generalization of the Theory of Anisotropic Laminated Composite Plates," Paper presented at the First Conference on Composite Materials, October 7-9, 1986, Dayton, Ohio.

42. Librescu, L., Khdeir, A. A. and Reddy, J. N., "A Comprehensive Analysis of the State of Stress of Elastic Anisotropic Flat Plates Using Refined Theories," Acta Mechanica, accepted for publication.

43. Librescu, L., Khdeir, A. A. and Reddy, J. N., "Further Results Concerning the Dynamic Response of Transverse Shear Deformable Elastic Orthotropic Plates," AIAA J., in review.

44. Lo, K. H., Christensen, R. M. and Wu, E. M., "A High-Order Theory of Plate Deformation," Parts 1 and 2, J. Appl Mech., Vol. 44, pp. 663-676, 1977.

45. Lur'e, A. I., "Three-Dimensional Problems of Elasticity Theory," Gostehizdat, Moskva, 1955.

46. Medwadowski, S. J., "A Refined Theory of Elastic, Orthotropic Plates," J. Appl. Mech., Vol. 25, pp. 437-443, 1958.

47. Meirovitch, L., Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

48. Mindlin, R. D., "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates," J. Appl. Mech., Vol. 18, pp. 31-38, 1951.

49. Murthy, M. V. V., "An Improved Transverse Shear Deformation Theory for Laminated Anisotropic Plates," NASA TP-1903, Nov. 1981.

50. Nelson, R. B. and Lorch, D. R., "A Refined Theory for Laminated Orthotropic Plates," J. Appl. Mech., Vol. 41, pp. 177-183, 1974.

51. Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," J. Composite Materials, Vol. 3, No. 3, 398-411, 1969.

52. Pagano, N. J., "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," J. Composite Materials, Vol. 4, pp. 20-34, 1970.

53. Phan, N. D., "Linear Analysis of Laminated Composite Plates Using a Higher-Order Shear Deformation Theory," M.S. Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Va., U.S.A, 1984.

54. Poniatovskii, V. V., "On the Bending Theory of Anisotropic Plates," Prikladmatem, i mechan, 28, 6, 1964.

55. Putcha, N. S., "A Mixed Shear Flexible Finite Element for Geometrically Nonlinear Analysis of Laminated Plates," Ph.D. Thesis, Virginia Polytechnic Institute & State University, VA, USA, 1984.

56. Reddy, J. N., "Free Vibration of Antisymmetric, Angle-Ply Laminated Plates Including Transverse Shear Deformation by the Finite Element Method," J. Sound & Vibration, Vol. 66, No. 4, pp. 565-576, 1979.

57. Reddy, J. N., "A Penalty-Bending Element for the Analysis of Laminated Anisotropic Plates," Int. J. Numer. Meth. Engng., Vol. 15, pp. 1187-1206, 1980.

58. Reddy, J. N., "On the Solutions to Forced Motions of Rectangular Composite Plates," J. Appl. Mech., Vol. 49, pp. 403-408, 1982.

59. Reddy, J. N., "A Note on Symmetry Considerations in the Transient Response of Symmetrically Laminated Anisotropic Plates," Int. J. Num. Engng., Vol. 20, pp. 175-181, 1984.

60. Reddy, J. N., "A Refined Nonlinear Theory of Plates with Transverse Shear Deformation," Int. J. Solids Struct., Vol. 20, No. 9/10, pp. 881-896, 1984.

61. Reddy, J. N., "A Simple Higher-Order Theory for Laminated Composite Plates," J. Appl. Mech., Vol. 51, pp. 745-752, 1984.

62. Reddy, J. N., Energy and Variational Methods in Applied Mechanics, John Wiley, New York, 1984.

63. Reddy, J. N., "A Review of the Literature on Finite-Element Modeling of Laminated Composite Plates," The Shock and Vibration Digest, Vol. 17, No. 4, April 1985, pp. 3-8.

64. Reddy, J. N., "On Kinematic and Constitutive Models of Laminated Composite Plates," Proc. ARO Workshop on Constitutive Models, Virginia Polytechnic Institute and State University, Blacksburg, VA, March 24-26, 1986, SIAM Publications, Philadelphia, to appear.

65. Reddy, J. N. and Chao, C. W., "A Comparison of Closed Form and Finite Element Solutions of Thick Laminated Anisotropic Rectangular Plates," Nuclear Engng. Design, Vol. 64, pp. 153-167, 1981.

66. Reddy, J. N. and Rasmussen, M. L., Advanced Engineering Analysis, John Wiley, New York, 1982.

67. Reddy, J. N. and Phan, N. D., "Stability and Vibration of Isotropic, Orthotropic and Laminated Plates According to a Higher-Order Shear Deformation Theory," J. Sound & Vibration, Vol. 98, No. 2, pp. 157-170, 1985.

68. Reddy, J. N., Khdeir, A. A. and Librescu, L., "The Levy Type Solutions for Symmetric Rectangular Composite Plates Using the First-Order Shear Deformation Theory," J. Appl. Mech., in review.

69. Rehfield, W. L., and Valisetty, R. R., "A Simple Refined Theory for Bending and Stretching of Homogeneous Plates," AIAA Journal, Vol. 22, Jan. 1984, pp. 90-95.

70. Reissner, E., "On the Theory of Bending of Elastic Plates," J. Math. Phy., Vol. 23, pp. 184-191, 1944.

71. Reissner, E., "The Effect of Transferse Shear Deformation on the Bending of Elastic Plates," J. Appl. Mech., Vol. 12, No. 1, pp. A-69 to A-77, 1945.

72. Reissner, E., "On Bending of Elastic Plates," Q. Appl. Math., Vol. 5, pp. 55-68, 1947.

73. Reissner, E., "Finite Deflections of Sandwich Plates," J. Aero. Sci., Vol. 15, pp. 435-440, 1948.

74. Reissner, E., "On Variational Theorem for Finite Elastic Deformation," J. Math. Phys., Vol. 32, pp. 129-135, 1953.

75. Reissner, E., "On the Foundations of the Theory of Elastic Shells," Proc. 11th Intl. Congr. Appl. Mech., Münich, 1964, 20-30, Springer 1966, Editor H. Görtler.

76. Reissner, E., "A Consistent Treatment of Transverse Shear Deformations in Laminated Anisotropic Plates," AIAA J., Vol. 10, No. 5, pp. 716-718, 1972.

77. Reissner, E., "On Transverse Bending Including the Effect of Transverse Shear Deformation," Int. Journ. Solids and Structures, Vol. 11, pp. 569-573, 1975.

78. Reissner, E., "On the Theory of Transverse Bending of Elastic Plates," Int. Journ. Solids and Structures, Vol. 12, pp. 545-554, 1976.

79. Reissner, E., "Note on the Effect of Transverse Shear Deformation in Laminated Anisotropic Plates," Comput. Meth. Appl. Mech. Engng., Vol. 20, No. 2, pp. 203-209, 1979.

80. Reissner, E., "Reflections on the Theory of Elastic Plates," Appl. Mech. Rev., Vol. 38, No. 11, 1985.

81. Salemo, V. and Goldberg, M. A., "Effect of Shear Deformations on the Bending of Rectangular Plates," J. Appl. Mech., Vol. 27, No. 1, pp. 54-58, 1960.

82. Schäfer, V. M., "Über eine verfeinerung der Klassischen theorie dunner schwach gebogener platten," ZAMM, Vol. 32, No. 6, pp. 161-171, 1952.

83. Schipper, J. F., "An Exact Formulation of the Linear Equation for Thick Orthotropic Shells with Arbitrary, Imposed Temperature and Force Fields and Temperature-Dependent Parameters," Developments in Theoretical Mechanics, Vol. 3, Edited by W. A. Shaw, pp. 255-278, 1967.

84. Schmidt, R., "A Refined Nonlinear Theory of Plates with Transverse Shear Deformations," J. Industrial Mathematics Society, Vol. 27, Part 1, pp. 23-28, 1977.

85. Shirakawa, K., "Bending of Plates Based on Improved Theory," Mech. Res. Commun., Vol. 10, pp. 205-211, 1983.

86. Srinivas, C. V., Rao, J. and Rao, A. K., "An Exact Analysis for Vibration of Simply Supported Homogeneous and Laminated Thick Rectangular Plates," J. Sound & Vibration, Vol. 12, pp. 187-199, 1970.

87. Srinivas, S. and Rao, A. K., "Bending, Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates," Intl. J. Solids & Struct., Vol. 6, pp. 1463-1481, 1970.

88. Stoker, J. J., "Mathematical Problems Connected with the Bending and Buckling of Elastic Plates," Bull. Amer. Math. Soc., Vol. 48, pp. 247-261, 1942.

89. Sun, C. T. and Whitney, J. M., "Forced Vibrations of Laminated Composite Plates in Cylindrical Bending," J. Acoust. Soc. Am., Vol. 55, pp. 1003-1008, 1974.

90. Sun, C. T. and Chattopadhyay, S., "Dynamic Response of Anisotropic Plates Under Initial Stress Due to Impact of a Mass," J. Appl. Mech., Vol. 42, pp. 693-698, 1975.

91. Sun, C. T. and Whitney, J. M., "Dynamic Response of Laminated Composite Plates Under Initial Stress," AIAA Journal, Vol. 14, pp. 268-270, 1976.

92. Teodorescu, P. P., "Spatial Problems in the Theory of Elasticity," Edit. Acad. RSR, Bucuresti, 349-360, 1970.

93. Tiffen, R., "An Investigation of the Transverse Displacement Equations of Elastic Plate Theory," Quart. J. Mech. & Appl. Math., 14, 59, 1961.

94. Tiffen, R. and Lowe, P. G., "An Exact Theory of Plane Stress," Journ. of London Math. Soc., 40, 72, 1962.

95. Vinson, J. R. and Chou, T. W., Composite Materials and Their Use in Structures, John Wiley & Sons, New York, Toronto, 1974.

96. Volterra, E., "Effect of Shear Deformations on Bending of Rectangular Plates," J. Appl. Mech., Vol. 27, No. 3, pp. 594-596, 1960.

97. Warburton, G. B., The Dynamical Behavior of Structures, Pergamon Press, Oxford, 1964.

98. Westbrook, D. R., "A Symbolic Approach to Dynamical Problems in Plates," Research Paper No. 22, June 1967, Univ. of Calgary, Canada.

99. Westbrook, D. R., "Symbolic Approach to Dynamical Problems in Plates," Journal of Acoustical Soc. of America, 44, 4, 1083-1092, 1968.

100. Whitney, J. M., "The Effect of Transverse Shear Deformation on the Bending of Laminated Plates," J. Comp. Materials, Vol. 3, No. 3, pp. 534-547, 1969.

101. Whitney, J. M. and Leissa, A. W., "Analysis of Heterogeneous Anisotropic Plates," J. Appl. Mech., Vol. 36, pp. 261-266, 1969.

102. Whitney, J. M. and Pagano, N. J., "Shear Deformation in Hetergeneous Anisotropic Plates," J. Appl. Mech., Vol. 37, pp. 1031-1036, 1970.

103. Whitney, J. M. and Sun, C. T., "Transient Response of Laminated Composite Plates Subjected to Transverse Dynamic Loading," J. Acoust. Soc. Am., Vol. 61, pp. 101-104, 1977.

104. Wilson, P. E. and Boresi, A. P., "Annotated Bibliography of Selected References on the Theory of Elastic Plates," T.A.M., Report 158, University of Illinois, 1960.

105. Yang, P. C., Norris, C. H. and Stavsky, Y., "Elastic Wave Propagation in Heterogeneous Plates," Intl. J. Solids & Struct., Vol. 2, No. 4, pp. 665-684, 1966.

106. Yu, Y. Y., "Forced Flexural Vibrations of Sandwich Plates in Plane Strain," J. Appl. Mech., Vol. 27, pp. 535-540, 1960.

107. Yu, Yi-Yuan, "On Linear Equations of Anisotropic Elastic Plates," Quart. Appl. Mathemat., 22, 4, 357-360, 1965.

Appendix

The coefficients C_i associated to DT:

$$C_1 = \alpha(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12} + 4\delta_A \frac{E^{11}_{33}}{E^{33}_{33}} E^{23}_{23}) \ , \ C_2 = -\alpha^3(\tilde{E}^{11}_{11} + 4\delta_A \frac{E^{11}_{33}}{E^{33}_{33}} E^{13}_{13}) + \frac{40}{h^2}\alpha E^{13}_{13},$$

$$C_3 = -4\tilde{E}^{12}_{12} \ , \ C_4 = 4\alpha^2(\tilde{E}^{11}_{11} - \delta_A \frac{E^{11}_{33}}{E^{33}_{33}} E^{13}_{13}) + \frac{40}{h^2} E^{13}_{13} \ ,$$

$$C_5 = -4\alpha(\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12} - \delta_A \frac{E^{11}_{33}}{E^{33}_{33}} E^{23}_{23}) \ , \ C_6 = 0 \ , \ C_7 = \tilde{E}^{22}_{22} + 4\delta_A \frac{E^{22}_{33}}{E^{33}_{33}} E^{23}_{23} \ ,$$

$$C_8 = -\alpha^2(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12} + 4\delta_A \frac{E^{22}_{33}}{E^{33}_{33}} E^{13}_{13}) + \frac{40}{h^2} E^{23}_{23} \ ,$$

$$C_9 = 4\alpha(\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12} - \delta_A \frac{E^{22}_{33}}{E^{33}_{33}} E^{13}_{13}) \ , \ C_{10} = 4(\delta_A \frac{E^{22}_{33}}{E^{33}_{33}} E^{23}_{23} - \tilde{E}^{22}_{22}),$$

$$C_{11} = 4\alpha^2\tilde{E}^{12}_{12} + \frac{40}{h^2} E^{23}_{23} \ , \ C_{12} = 0 \ , \ C_{13} = \frac{2}{3} hE^{23}_{23} \ , \ C_{14} = -\frac{2}{3} h\alpha^2 E^{13}_{13} \ ,$$

$$C_{15} = 0 \ , \ C_{16} = \alpha C_{14} \ , \ C_{17} = C_{13} \ , \ C_{18} = 1 \qquad \text{(A1)}$$

associated to ST:

$$C_1 = \alpha(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12}) \ , \ C_2 = -\alpha^3\tilde{E}^{11}_{11} \ , \ C_3 = \frac{4}{5} h^2\tilde{E}^{12}_{12}F^{13}_{13}$$

$$C_4 = -2(1 + \frac{2}{5}\alpha^2 h^2\tilde{E}^{11}_{11}F^{13}_{13}) \ , \ C_5 = \frac{4}{5} h^2\alpha F^{23}_{23}(\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12}),$$

$$C_6 = -\frac{6}{5h}\alpha\delta_A \frac{E^{11}_{33}}{E^{33}_{33}} \ , \ C_7 = \tilde{E}^{22}_{22} \ , \ C_8 = -\alpha^2(\tilde{E}^{11}_{22} + 2\tilde{E}^{12}_{12}),$$

$$C_9 = -\frac{4}{5} h^2\alpha F^{13}_{13}(\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12}) \ , \ C_{10} = \frac{4}{5} h^2 F^{23}_{23}\tilde{E}^{22}_{22},$$

$$C_{11} = -2(1 + \frac{2}{5} h^2\alpha^2\tilde{E}^{12}_{12}F^{23}_{23}) \ , \ C_{12} = - \frac{6}{5h} \delta_A \frac{E^{22}_{33}}{E^{33}_{33}} \ , \ C_{13} = 0 \ , \ C_{14} = 0 \ ,$$

$$C_{15} = 0 \ , \ C_{16} = - \frac{h^3\alpha}{6} \ , \ C_{17} = \frac{h^3}{6} \ , \ C_{18} = - 1 \qquad \text{(A2)}$$

associated to FSDT:

$$C_1 = hK^2_{(2)}E^{23}_{23} \ , \ C_2 = -\alpha^2hK^2_{(1)}E^{13}_{13} \ , \ C_3 = 0 \ , \ C_4 = \frac{C_2}{\alpha} \ , \ C_5 = C_1, \ C_6 = 1 \ ,$$

$$C_7 = 0 \ , \ C_8 = - C_1 \ , \ C_9 = - \alpha \frac{h^3}{12} (\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12}) \ , \ C_{10} = \frac{h^3}{12} \tilde{E}^{22}_{22} \ ,$$

$$C_{11} = - (C_1 + \alpha^2 \frac{h^3}{12} \tilde{E}^{12}_{12}) \ , \ C_{12} = 0 \ , \ C_{13} = 0 \ , \ C_{14} = C_4 \ , \ C_{15} = \frac{h^3}{12} \tilde{E}^{12}_{12} \ ,$$

$$C_{16} = - (\frac{h^3}{12} \alpha^2\tilde{E}^{11}_{11} + hK^2_{(1)}E^{13}_{13}) \ , \ C_{17} = -C_9 \ , \ C_{18} = 0.$$

For symmetric laminated composite plates the coefficients C_i associated to FSDT are:

$$C_1 = A_{44}, \ C_2 = -\alpha^2A_{55}, \ C_3 = 0, \ C_4 = \frac{C_2}{\alpha}, \ C_5 = C_1, \ C_6 = 1$$

$$C_7 = 0, \ C_8 = -C_1, \ C_9 = -\alpha(D_{12} + D_{66}), \ C_{10} = D_{22}, \ C_{11} = -(A_{44} + \alpha^2D_{66})$$

$$C_{12} = 0, \ C_{13} = 0, \ C_{14} = C_4, \ C_{15} = D_{66}, \ C_{16} = -(A_{55} + \alpha^2D_{11})$$

$$C_{17} = -C_9, \ C_{18} = 0 \qquad \text{(A3)}$$

where

$$A_{ij} = \sum_{k=1}^{N} \int_{x_3k}^{x_3k+1} K_iK_jQ^{(k)}_{ij} \ dx_3 \quad (i,j = 4,5)$$

$$D_{ij} = \sum_{k=1}^{N} \int_{x_3k}^{x_3k+1} Q_{ij}^{(k)} x_3^2 dx_3 \quad (i,j = 1,2,6)$$

Here $Q_{ij}^{(k)}$ denote the reduced orthotropic moduli of the k-th lamina in the laminate coordinates, and K_i, K_j are the shear correction factors.

Associated to HSDT:

$$C_1 = -(\frac{4}{3h^2})^2 H_{22},\ C_2 = 2(\frac{4}{3h^2})^2\alpha^2(H_{12} + 2H_{66}) + A_{44} - \frac{8}{h^2} D_{44} + (\frac{4}{h^2})^2 F_{44},$$

$$C_3 = -\alpha^2[(\frac{4}{3h^2})^2\alpha^2 H_{11} + (\frac{4}{h^2})^2 F_{55} - \frac{8}{h^2} D_{55} + A_{55}],$$

$$C_4 = \alpha \frac{4}{3h^2} [\frac{4}{3h^2} H_{12} + \frac{8}{3h^2} H_{66} - F_{12} - 2F_{66}],$$

$$C_5 = \alpha^3 \frac{4}{3h^2} (F_{11} - \frac{4}{3h^2} H_{11}) + \alpha[\frac{8}{h^2} D_{55} - (\frac{4}{h^2})^2 F_{55} - A_{55}],$$

$$C_6 = \frac{4}{3h^2} (F_{22} - \frac{4}{3h^2} H_{22}),$$

$$C_7 = \alpha^2 \frac{4}{3h^2} [\frac{4}{3h^2} (H_{12} + 2H_{66}) - F_{12} - 2F_{66}] - \frac{8}{h^2} D_{44} + (\frac{4}{h^2})^2 F_{44} + A_{44}$$

$$C_8 = C_4,\ C_9 = C_5,\ C_{10} = D_{66} - \frac{8}{3h^2} F_{66} + (\frac{4}{3h^2})^2 H_{66},$$

$$C_{11} = \alpha^2[\frac{8}{3h^2} F_{11} - D_{11} - (\frac{4}{3h^2})^2 H_{11}] + \frac{8}{h^2} D_{55} - (\frac{4}{h^2})^2 F_{55} - A_{55},$$

$$C_{12} = \alpha[D_{12} + D_{66} - \frac{8}{3h^2} (F_{12} + F_{66}) + (\frac{4}{3h^2})^2 (H_{12} + H_{66})],$$

$$C_{13} = -C_6,\ C_{14} = -C_7,\ C_{15} = -C_{12},\ C_{16} = D_{22} - \frac{8}{3h^2} F_{22} + (\frac{4}{3h^2}) H_{22},$$

$$C_{17} = \alpha^2[\frac{8}{3h^2} F_{66} - D_{66} - (\frac{4}{3h^2})^2 H_{66}] + \frac{8}{h^2} D_{44} - (\frac{4}{h^2})^2 F_{44} - A_{44} \qquad (A4)$$

The coefficients e_i associated to DT:

$$e_1 = \frac{C_{10}C_{14} - C_8C_{17}}{C_7C_{17} - C_{10}C_{13}} \;,\quad e_2 = \frac{C_{10}C_{16} - C_9C_{17}}{C_7C_{17} - C_{10}C_{13}} \;,\quad e_3 = \frac{C_{11}C_{17}}{C_{10}C_{13} - C_7C_{17}} \;,$$

$$e_4 = \frac{C_{10}}{C_7C_{17} - C_{10}C_{13}} \;,\quad e_5 = \frac{C_5C_{13} - C_1C_{17}}{C_3C_{17}} \;,\quad e_6 = \frac{C_5C_{14} - C_2C_{17}}{C_3C_{17}} \;,$$

$$e_7 = \frac{C_5C_{16} - C_4C_{17}}{C_3C_{17}} \;,\quad e_8 = \frac{C_5}{C_3C_{17}} \;,\quad e_9 = -\frac{C_{13}}{C_{17}} \;,\quad e_{10} = -\frac{C_{14}}{C_{17}} \;,$$

$$e_{11} = -\frac{C_{16}}{C_{17}} \;,\quad e_{12} = -\frac{1}{C_{17}} \qquad \text{(A5)}$$

associated to ST:

$$e_1 = -\frac{C_8}{C_7} \;,\quad e_2 = \frac{C_{10}C_{16} - C_9C_{17}}{C_7C_{17}} \;,\quad e_3 = -\frac{C_{11}}{C_7} \;,\quad e_4 = -\frac{(C_{10} + C_{12}C_{17})}{C_7C_{17}} \;,$$

$$e_5 = -\frac{C_1}{C_3} \;,\quad e_6 = -\frac{C_2}{C_3} \;,\quad e_7 = \frac{C_5C_{16} - C_4C_{17}}{C_3C_{17}} \;,\quad e_8 = -\frac{(C_5 + C_6C_{17})}{C_3C_{17}} \;,$$

$$e_9 = 0 \;,\quad e_{10} = 0 \;,\quad e_{11} = -C_{16}/C_{17} \;,\quad e_{12} = \frac{1}{C_{17}} \qquad \text{(A6)}$$

associated to FSDT:

$$e_1 = \alpha^2 \frac{K^2_{(1)}}{K^2_{(2)}} \frac{E^{13}_{13}}{E^{23}_{23}} \;,\quad e_2 = \frac{e_1}{\alpha} \;,\quad e_3 = -1 \;,\quad e_4 = \frac{-1}{hK^2_{(2)}E^{23}_{23}} \;,$$

$$e_5 = \frac{12\alpha K^2_{(1)}E^{13}_{13}}{h^2\tilde{E}^{12}_{12}} \;,\quad e_6 = \alpha^2 \frac{\tilde{E}^{11}_{11}}{\tilde{E}^{12}_{12}} + \frac{12K^2_{(1)}E^{13}_{13}}{h^2\tilde{E}^{12}_{12}} \;,\quad e_7 = -\alpha\left(\frac{\tilde{E}^{11}_{22} + \tilde{E}^{12}_{12}}{\tilde{E}^{12}_{12}}\right) \;,$$

$$e_8 = \frac{12K_{(2)}^2 E_{23}^{23}}{h^2 \tilde{E}_{22}^{22}} \quad , \quad e_9 = -\frac{\tilde{E}_{12}^{12}}{\tilde{E}_{22}^{22}} e_7 \quad , \quad e_{10} = \alpha^2 \frac{\tilde{E}_{12}^{12}}{\tilde{E}_{22}^{22}} + \frac{12K_{(2)}^2 E_{23}^{23}}{h^2 \tilde{E}_{22}^{22}}$$

For symmetric laminated composite plates the coefficients e_i associated to FSDT are:

$$e_1 = \alpha^2 \frac{A_{55}}{A_{44}},\ e_2 = \frac{e_1}{\alpha},\ e_3 = -1,\ e_4 = -\frac{1}{A_{44}},\ e_5 = \alpha \frac{A_{55}}{D_{66}},$$

$$e_6 = \frac{A_{55} + \alpha^2 D_{11}}{D_{66}},\ e_7 = -\alpha \frac{(D_{12} + D_{66})}{D_{66}},\ e_8 = \frac{A_{44}}{D_{22}},\ e_9 = -\frac{D_{66}}{D_{22}} e_7,$$

$$e_{10} = \frac{A_{44} + \alpha^2 D_{66}}{D_{22}} \tag{A7}$$

associated to HSDT:

$$e_1 = \left(\frac{C_4^2}{C_{10}} + \frac{C_4 C_6 C_{12}}{C_{10} C_{16}} - \frac{C_6 C_7}{C_{16}} - C_2\right)/\left(C_1 + \frac{C_6^2}{C_{16}}\right),$$

$$e_2 = \left(\frac{C_4 C_5}{C_{10}} + \frac{C_5 C_6 C_{12}}{C_{10} C_{16}} - C_3\right)/\left(C_1 + \frac{C_6^2}{C_{16}}\right),$$

$$e_3 = \left(\frac{C_4 C_{11}}{C_{10}} + \frac{C_6 C_{11} C_{12}}{C_{10} C_{16}} - C_5\right)/\left(C_1 + \frac{C_6^2}{C_{16}}\right),$$

$$e_4 = \left(\frac{C_6 C_7}{C_{16}} + \frac{C_4 C_{12}}{C_{10}} + \frac{C_6 C_{12}^2}{C_{10} C_{16}} - C_7\right)/\left(C_1 + \frac{C_6^2}{C_{16}}\right),\ e_5 = -\frac{C_{16}}{C_1 C_{16} + C_6^2},$$

$$e_6 = -\frac{C_4}{C_{10}},\ e_7 = -\frac{C_5}{C_{10}},\ e_8 = -\frac{C_{11}}{C_{10}},\ e_9 = -\frac{C_{12}}{C_{10}},\ e_{10} = \frac{C_6}{C_{16}},$$

$$e_{11} = \frac{C_7}{C_{16}},\ e_{12} = \frac{C_{12}}{C_{16}},\ e_{13} = -\frac{C_{17}}{C_{16}}. \tag{A8}$$

The vector $\underline{b}$:

$$\underline{b} \equiv \begin{array}{l} \{0,0,e_4p_m',0,e_8p_m,e_{12}p_m\}^T \text{ for DT and ST} \\ \{0,e_4p_m,0,0,0\}^T \text{ for FSDT} \end{array} \quad \text{(A9)}$$

$$\underline{b} \equiv \{0,0,0,e_5Q_m,0,0,0,0\}^T \text{ for HSDT} \quad \text{(A10)}$$

The matrix A:

For DT

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 & e_2 & e_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ e_6 & 0 & e_5 & e_7 & 0 & 0 \\ e_{10} & 0 & e_9 & e_{11} & 0 & 0 \end{bmatrix},$$

For ST

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 & e_2 & e_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ e_6 & 0 & e_5 & e_7 & 0 & 0 \\ 0 & 0 & 0 & e_{11} & 0 & 0 \end{bmatrix}$$

For FSDT (A11)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ e_1 & 0 & e_2 & 0 & 0 & e_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ e_5 & 0 & e_6 & 0 & 0 & e_7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & e_8 & 0 & e_9 & e_{10} & 0 \end{bmatrix}$$

The matrix A for HSDT:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ e_2 & 0 & e_1 & 0 & e_3 & 0 & 0 & e_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ e_7 & 0 & e_6 & 0 & e_8 & 0 & 0 & e_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & e_{11} & 0 & e_{10} & 0 & e_{12} & e_{13} & 0 \end{bmatrix} \qquad \text{(A12)}$$

Table 4.1
Center Deflections* Of Orthotropic Plates
(Material I)
(FSDT, DT, ST, δ_A =0)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{W}$					
1	0.2	UN	89.99	58.21	208.37	72.31	143.52	117.71
		TR	58.98	38.73	134.43	47.72	93.10	76.66
		PL	0.8009	0.5504	1.7339	0.6617	1.2230	1.0198
	0.14	UN	231.31	141.22	548.51	180.82	374.58	301.00
		TR	150.62	93.24	352.59	118.47	241.85	194.99
		PL	1.9912	1.2824	4.484	1.5941	3.1171	2.5386
2	0.2	UN	20.076	10.038	219.07	13.823	80.20	44.60
		TR	13.956	7.467	141.23	9.925	52.51	29.79
		PL	0.2302	0.1476	1.8171	0.1792	0.7140	0.4320
	0.14	UN	48.19	19.88	577.58	29.87	204.73	106.42
		TR	33.41	15.014	371.06	21.56	133.38	70.60
		PL	0.5261	0.2892	4.711	0.3748	1.7688	0.9890
3	0.2	UN	6.398	3.5325	223.89	4.6774	50.02	18.133
		TR	4.8032	2.8895	144.31	3.6623	33.02	12.629
		PL	0.1020	0.0760	1.8554	0.0867	0.4644	0.2103
	0.14	UN	13.840	5.9782	591.78	8.687	123.91	40.46
		TR	10.672	5.1460	380.13	7.107	81.477	27.978
		PL	0.2168	0.1346	4.8226	0.1652	1.1134	0.4436
4	0.2	UN	2.7902	1.7743	225.77	2.2371	34.579	8.563
		TR	2.2414	1.5365	145.53	1.8614	22.968	6.289
		PL	0.0594	0.0492	1.8710	0.0540	0.3323	0.1235
	0.14	UN	5.4003	2.7728	597.64	3.7540	82.89	17.791
		TR	4.6356	2.5457	383.91	3.3480	54.98	13.059
		PL	0.1187	0.0815	4.8706	0.0963	0.7729	0.2430

* $\bar{w} \equiv [w(\frac{a}{2}, 0)/q_0] \times 10^6$ for distributed load cases (UN and TR)

$\bar{w} \equiv [w(\frac{a}{2}, 0)/p] \times 10^6$ for the point load case (PL)

a = 200 in

Table 4.2
Center Deflections Of Orthotropic Plates
(Material I)
(DT, ST, δ_A=1)

a/b	h/a	Load	$\bar{W}$ SS	CC	FF	SC	SF	CF
1	0.2	UN	85.42	55.93	200.44	69.02	137.45	113.27
		TR	56.33	37.526	129.64	45.87	89.50	74.08
		PL	0.7792	0.5465	1.6857	0.6499	1.1894	0.9989
	0.14	UN	226.88	140.22	539.40	178.31	368.05	296.91
		TR	146.34	91.15	345.31	115.41	236.22	190.92
		PL	1.9118	1.2307	4.3670	1.5301	3.0209	2.4620
2	0.2	UN	18.305	9.6345	211.63	12.899	76.91	43.421
		TR	12.975	7.323	136.72	9.465	50.60	29.206
		PL	0.2264	0.1531	1.7712	0.1813	0.6994	0.4334
	0.14	UN	47.232	20.651	569.04	30.065	201.78	106.52
		TR	31.647	14.573	364.17	20.642	130.20	69.47
		PL	0.4851	0.2698	4.5985	0.3469	1.7064	0.9548
3	0.2	UN	5.5599	3.4280	216.62	4.2784	48.182	17.931
		TR	4.3221	2.8552	139.94	3.4479	32.008	12.623
		PL	0.1008	0.0797	1.8113	0.0884	0.4593	0.2168
	0.14	UN	13.608	6.431	583.56	8.933	122.71	41.463
		TR	9.770	4.991	373.42	6.682	79.61	27.739
		PL	0.1947	0.1286	4.7117	0.1528	1.0712	0.4265
4	0.2	UN	2.3275	1.7415	218.54	2.0095	33.460	8.582
		TR	1.9535	1.5239	141.22	1.7216	22.407	6.401
		PL	0.0583	0.0514	1.8285	0.0546	0.3321	0.1302
	0.14	UN	5.2911	2.9940	589.59	3.8589	82.50	18.792
		TR	4.1062	2.4802	377.27	3.1038	53.79	13.040
		PL	0.1062	0.0802	4.7594	0.0906	0.7419	0.2329

Table 4.3
Center Deflections Of Orthotropic Plates
(Material I)
(HSDT)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\overline{W}$					
3	0.2	UN	6.266	3.178	216.60	4.362	48.86	16.767
		TR	4.810	2.649	139.59	3.493	32.337	11.740
		PL	0.1039	0.0707	1.792	0.0842	0.4553	0.1963
	0.14	UN	14.546	6.024	581.07	8.952	123.46	39.127
		TR	10.518	4.826	373.22	6.814	80.70	26.761
		PL	0.2019	0.1221	4.733	0.1512	1.0916	0.4182
4	0.2	UN	2.710	1.525	214.98	2.021	33.243	7.528
		TR	2.242	1.343	138.55	1.724	22.175	5.596
		PL	0.0609	0.0438	1.778	0.0513	0.3224	0.1112
	0.14	UN	5.905	2.723	580.54	3.882	82.08	16.87
		TR	4.528	2.314	372.90	3.138	53.91	12.091
		PL	0.1095	0.0732	4.729	0.0876	0.7463	0.2203

Table 4.4
Center Deflections Of Orthotropic Plates
(Material I)
(CL*)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{W}$					
1	0.2	UN	69.08	38.533	168.70	51.76	113.93	88.66
		TR	44.53	25.068	107.96	33.492	73.09	56.99
		PL	0.5663	0.3259	1.3494	0.4300	0.9189	0.7201
	0.14	UN	201.39	112.34	491.82	150.89	332.15	258.47
		TR	129.81	73.08	314.76	97.64	213.08	166.16
		PL	1.6510	0.9501	3.9340	1.2536	2.6789	2.0995
2	0.2	UN	13.402	3.9180	176.95	6.977	59.481	27.607
		TR	8.921	2.7939	113.21	4.7845	38.342	18.000
		PL	0.1229	0.0449	1.4140	0.0706	0.4880	0.2358
	0.14	UN	39.072	11.423	515.88	20.341	173.41	80.49
		TR	26.01	8.145	330.06	13.949	111.78	52.48
		PL	0.3582	0.1310	4.1225	0.2059	1.4229	0.6873
3	0.2	UN	3.4958	0.7969	181.62	1.5530	33.617	9.235
		TR	2.4539	0.6322	116.20	1.1582	21.783	6.175
		PL	0.0386	0.0132	1.4509	0.0210	0.2812	0.0863
	0.14	UN	10.192	2.3232	529.51	4.5277	98.01	26.924
		TR	7.1543	1.8433	338.76	3.3768	63.51	18.002
		PL	0.1125	0.0384	4.2302	0.0611	0.8198	0.2516
4	0.2	UN	1.2036	0.2509	183.92	0.5003	20.905	3.6031
		TR	0.8947	0.2127	117.67	0.4002	13.614	2.4942
		PL	0.0163	0.0055	1.4694	0.0088	0.1782	0.0381
	0.14	UN	3.5090	0.7314	536.22	1.4587	60.95	10.505
		TR	2.6083	0.6201	343.06	1.1668	39.692	7.272
		PL	0.0476	0.0160	4.2839	0.0256	0.5196	0.1110

*Classical plate theory

Table 4.5
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material I)
(FSDT)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{W}$					
1	0.2	UN	87.85	58.54	204.41	71.61	140.52	116.49
		TR	57.47	38.79	131.71	47.12	91.02	75.71
		PL	0.7734	0.5426	1.6901	0.6456	1.1877	0.9987
	0.14	UN	228.30	141.18	547.40	179.48	372.20	300.12
		TR	148.37	92.86	351.59	117.27	240.02	194.11
		PL	1.9479	1.2622	4.4571	1.5638	3.0796	2.5127
2	0.2	UN	19.900	10.734	214.87	14.306	78.24	45.04
		TR	13.863	7.931	138.35	10.253	51.18	30.00
		PL	0.2256	0.1500	1.7717	0.1798	0.6910	0.4283
	0.14	UN	48.08	21.176	576.21	30.861	202.66	106.98
		TR	32.88	15.503	369.90	21.790	131.68	70.63
		PL	0.5025	0.2817	4.6821	0.3624	1.7321	0.9753
3	0.2	UN	6.4876	3.8888	219.59	4.9834	48.28	18.744
		TR	4.9349	3.1678	141.37	3.9189	31.892	12.999
		PL	0.1030	0.0783	1.8091	0.0890	0.4462	0.2109
	0.14	UN	14.381	6.7392	590.16	9.498	121.82	41.30
		TR	10.624	5.4723	378.81	7.3627	79.76	28.272
		PL	0.2035	0.1320	4.7926	0.1591	1.0780	0.4364
4	0.2	UN	2.8890	1.9824	221.45	2.4235	32.940	9.056
		TR	2.3733	1.7116	142.57	2.0341	21.945	6.620
		PL	0.0615	0.0512	1.8244	0.0562	0.3172	0.1252
	0.14	UN	5.8952	3.2042	595.97	4.2683	80.88	18.611
		TR	4.6901	2.7646	382.56	3.5399	53.33	13.389
		PL	0.1117	0.0816	4.8401	0.0941	0.7396	0.2388

Table 4.6
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material I)
(HSDT)

a/b	h/a	Load	$\overline{W}$ SS	CC	FF	SC	SF	CF
3	0.2	UN	6.853	3.856	193.77	5.096	46.465	17.930
		TR	5.116	3.092	124.77	3.937	30.636	12.384
		PL	0.1044	0.0729	1.596	0.0851	0.4247	0.1970
	0.14	UN	14.876	6.896	510.79	9.806	117.84	40.268
		TR	10.901	5.539	328.06	7.524	77.110	27.513
		PL	0.2028	0.1285	4.155	0.1568	1.038	0.4202
4	0.2	UN	3.122	1.871	187.04	2.427	31.269	8.355
		TR	2.497	1.594	120.48	1.999	20.790	6.082
		PL	0.0608	0.0457	1.543	0.0526	0.2977	0.1126
	0.14	UN	6.216	3.197	490.49	4.376	77.87	17.827
		TR	4.889	2.733	315.15	3.591	51.32	12.803
		PL	0.1120	0.0777	3.996	0.0919	0.7087	0.2254

Table 4.7
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material I)
(CL)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	69.09	38.029	171.73	51.41	115.20	89.10
		TR	44.55	24.755	109.91	33.281	73.91	57.28
		PL	0.5668	0.3223	1.3737	0.4276	0.9293	0.7240
	0.14	UN	201.44	110.87	500.68	149.88	335.87	259.76
		TR	129.87	72.17	320.43	97.03	215.48	167.00
		PL	1.6525	0.9396	4.0048	1.2468	2.7093	2.1108
2	0.2	UN	13.167	3.8043	180.11	6.806	59.84	27.356
		TR	8.775	2.7215	115.24	4.6772	38.577	17.844
		PL	0.1212	0.0441	1.4393	0.0694	0.4912	0.2340
	0.14	UN	38.388	11.091	525.11	19.842	174.46	79.75
		TR	25.584	7.934	335.97	13.636	112.47	52.02
		PL	0.3533	0.1285	4.1963	0.2023	1.4319	0.6822
3	0.2	UN	3.4061	0.7711	184.77	1.5060	33.730	9.064
		TR	2.3968	0.6141	118.21	1.1271	21.860	6.067
		PL	0.0379	0.0129	1.4761	0.0206	0.2823	0.0850
	0.14	UN	9.930	2.2482	538.68	4.3908	98.34	26.425
		TR	6.988	1.7904	344.63	3.2860	63.73	17.689
		PL	0.1106	0.0377	4.3034	0.0600	0.8230	0.2479
4	0.2	UN	1.1682	0.2427	187.03	0.4843	20.949	3.5160
		TR	0.8711	0.2063	119.66	0.3887	13.645	2.4387
		PL	0.0160	0.0054	1.4943	0.0086	0.1787	0.0374
	0.14	UN	3.4057	0.7076	545.29	1.4121	61.07	10.251
		TR	2.5397	0.6016	348.86	1.1334	39.780	7.110
		PL	0.0467	0.0157	4.3565	0.0251	0.5210	0.1091

Table 4.8
Center Deflections For Angle-Ply (-45/45/45/-45) Laminates
(Material I)
(FSDT)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	86.80	55.70	254.01	69.16	159.13	127.27
		TR	56.86	37.04	163.36	45.62	102.93	82.64
		PL	0.7664	0.5216	2.0818	0.6276	1.3355	1.0851
	0.14	UN	226.53	129.28	695.17	170.19	428.37	327.45
		TR	147.54	85.57	446.03	111.65	276.11	211.83
		PL	1.9454	1.1795	5.6316	1.5019	3.5334	2.7397
2	0.2	UN	17.651	10.409	266.70	13.370	83.15	44.32
		TR	12.408	7.729	171.44	9.648	54.34	29.577
		PL	0.2065	0.1470	2.1813	0.1716	0.7301	0.4235
	0.14	UN	41.21	19.160	731.07	27.171	217.79	102.19
		TR	28.560	14.275	468.87	19.495	141.53	67.78
		PL	0.4507	0.2681	5.9129	0.3356	1.8589	0.9454
3	0.2	UN	5.8649	3.9544	271.35	4.8553	49.30	17.693
		TR	4.4983	3.2134	174.42	3.8216	32.553	12.363
		PL	0.0965	0.0787	2.2185	0.0872	0.4542	0.2033
	0.14	UN	12.199	6.4817	745.14	8.7095	125.75	37.076
		TR	9.195	5.3201	477.89	6.8504	82.40	25.713
		PL	0.1851	0.1306	6.0254	0.1527	1.1140	0.4084
4	0.2	UN	2.7298	2.0692	273.10	2.4361	32.850	8.607
		TR	2.2427	1.7745	175.55	2.0339	21.883	6.3555
		PL	0.0594	0.0521	2.2331	0.0561	0.3161	0.1220
	0.14	UN	5.1671	3.2139	750.67	4.0830	81.57	16.604
		TR	4.1827	2.7827	481.46	3.4125	53.85	12.189
		PL	0.1046	0.0825	6.0711	0.0926	0.7484	0.2263

Table 4.9
Center Deflections For Angle-Ply (-45/45/45/-45) Laminates
(Material I)
(CL)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	68.95	32.587	223.31	47.315	134.95	96.48
		TR	44.60	21.415	142.91	30.807	86.64	62.13
		PL	0.5720	0.2852	1.7860	0.4014	1.0912	0.7884
	0.14	UN	201.03	95.01	651.06	137.94	393.45	281.30
		TR	130.03	62.44	416.66	89.82	252.59	181.14
		PL	1.6676	0.8314	5.2070	1.1704	3.1813	2.2987
2	0.2	UN	10.758	2.7772	234.32	5.187	64.86	24.710
		TR	7.286	2.0659	149.92	3.6648	41.874	16.229
		PL	0.1047	0.0366	1.8723	0.0580	0.5353	0.2166
	0.14	UN	31.366	8.097	683.16	15.122	189.09	72.04
		TR	21.243	6.023	437.08	10.685	122.08	47.31
		PL	0.3053	0.1066	5.4587	0.1692	1.5605	0.6314
3	0.2	UN	2.5692	0.5494	239.30	1.0901	35.128	7.357
		TR	1.8623	0.4545	153.10	0.8478	22.816	5.004
		PL	0.0317	0.0107	1.9118	0.0171	0.2964	0.0731
	0.14	UN	7.490	1.6017	697.66	3.1782	102.41	21.449
		TR	5.429	1.3252	446.35	2.4717	66.52	14.590
		PL	0.0925	0.0312	5.5738	0.0498	0.8642	0.2130
4	0.2	UN	0.8521	0.1730	241.53	0.3463	21.390	2.6922
		TR	0.6583	0.1511	154.54	0.2878	13.967	1.9159
		PL	0.0133	0.0044	1.9300	0.0071	0.1843	0.0314
	0.14	UN	2.4842	0.5042	704.18	1.0096	62.36	7.849
		TR	1.9193	0.4406	450.54	0.8390	40.720	5.586
		PL	0.0388	0.0129	5.6267	0.0207	0.5372	0.0915

Table 4.10
Center Deflections Of Orthotropic Plates
(Material II)
(FSDT,DT,ST,δ_A=0)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{W}$					
2	0.2	UN	146.04	91.61	382.63	114.44	274.73	203.01
		TR	98.98	64.02	249.78	78.70	180.99	135.17
		PL	1.5009	1.0611	3.3652	1.2465	2.5147	1.9451
	0.14	UN	331.01	176.27	844.08	236.87	610.15	433.22
		TR	220.57	121.46	547.19	160.34	398.27	285.33
		PL	3.1589	1.9201	7.1868	2.4077	5.3504	3.9490
3	0.2	UN	56.80	37.96	384.35	46.56	216.73	120.84
		TR	41.10	28.64	250.86	34.35	143.56	82.22
		PL	0.7570	0.5896	3.3784	0.6669	2.0384	1.2733
	0.14	UN	120.43	64.66	848.69	86.30	478.35	248.16
		TR	84.95	48.35	550.11	62.66	313.61	166.44
		PL	1.4440	0.9605	7.2222	1.1526	4.2856	2.4519
4	0.2	UN	27.105	20.084	385.15	23.795	175.50	71.83
		TR	21.021	16.147	251.38	18.724	116.75	50.36
		PL	0.4720	0.3989	3.3849	0.4375	1.6899	0.8594
	0.14	UN	52.91	31.994	850.98	40.96	385.67	140.37
		TR	40.017	25.637	551.58	31.861	253.73	96.71
		PL	0.8305	0.6195	7.2404	0.7128	3.5196	1.5572

Table 4.11
Center Deflections Of Orthotropic Plates
(Material II)
(DT,ST,δ_A =1)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\overline{W}$					
2	0.2	UN	131.21	86.72	356.10	105.39	254.04	191.29
		TR	88.72	60.26	231.99	72.21	166.96	126.94
		PL	1.3364	0.9822	3.1057	1.1312	2.3027	1.8073
	0.14	UN	309.29	170.10	806.33	224.63	580.13	416.76
		TR	205.59	116.58	521.94	151.49	377.98	273.77
		PL	2.9209	1.8127	6.8201	2.2484	5.0458	3.7552
3	0.2	UN	49.64	36.28	359.53	42.39	202.23	116.09
		TR	35.76	27.08	234.14	31.06	133.48	78.50
		PL	0.6530	0.5425	3.1293	0.5933	1.8736	1.1923
	0.14	UN	110.00	62.83	813.24	81.15	456.74	241.78
		TR	77.19	46.46	526.26	58.47	298.65	161.39
		PL	1.2944	0.8970	6.8684	1.0540	4.0443	2.3401
4	0.2	UN	23.233	19.376	361.09	21.415	165.23	70.20
		TR	17.897	15.341	235.18	16.692	109.46	48.75
		PL	0.3980	0.3658	3.1419	0.3829	1.5611	0.8075
	0.14	UN	47.335	31.282	816.60	38.172	369.81	138.55
		TR	35.515	24.679	528.44	29.366	242.54	94.73
		PL	0.7244	0.5745	6.8944	0.6402	3.3262	1.4873

Table 4.12
Center Deflections Of Orthotropic Plates
(Material II)
(HSDT)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	56.62	33.641	379.24	43.47	218.78	113.42
		TR	40.65	25.421	247.09	31.973	144.44	77.00
		PL	0.7057	0.5008	3.280	0.5900	2.000	1.159
	0.14	UN	120.33	60.15	843.02	82.95	485.39	240.91
		TR	84.62	45.06	546.10	60.17	317.76	161.42
		PL	1.400	0.8772	7.129	1.081	4.295	2.347
4	0.2	UN	26.872	16.670	378.29	21.318	176.91	63.19
		TR	20.629	13.478	246.46	16.760	117.28	44.34
		PL	0.4268	0.3182	3.271	0.3687	1.650	0.7349
	0.14	UN	52.77	28.488	842.68	38.287	392.87	130.56
		TR	39.726	22.941	545.86	29.808	258.02	90.03
		PL	0.7914	0.5433	7.125	0.6476	3.533	1.431

Table 4.13
Center Deflections Of Orthotropic Plates
(Material II)
(CL)

			$\widetilde{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
2	0.2	UN	81.02	29.488	200.36	48.218	145.44	95.08
		TR	52.28	19.453	128.23	31.389	93.28	61.21
		PL	0.6665	0.2609	1.6029	0.4084	1.1720	0.7760
	0.14	UN	236.21	85.97	584.14	140.58	424.02	277.20
		TR	152.43	56.71	373.86	91.51	271.96	178.45
		PL	1.9432	0.7606	4.6733	1.1908	3.4170	2.2625
3	0.2	UN	26.472	6.545	201.59	12.435	110.79	48.57
		TR	17.498	4.6765	129.01	8.493	71.19	31.52
		PL	0.2360	0.0744	1.6124	0.1232	0.8988	0.4076
	0.14	UN	77.18	19.080	587.72	36.252	323.01	141.60
		TR	51.01	13.634	376.13	24.761	207.56	91.88
		PL	0.6879	0.2169	4.7008	0.3591	2.6203	1.1883
4	0.2	UN	9.661	2.0257	202.28	4.0751	86.12	23.765
		TR	6.669	1.5956	129.45	2.9970	55.43	15.615
		PL	0.0992	0.0311	1.6178	0.0509	0.7027	0.2086
	0.14	UN	28.167	5.9058	589.75	11.881	251.09	69.29
		TR	19.443	4.6519	377.42	8.738	161.60	45.52
		PL	0.2892	0.0906	4.7166	0.1484	2.0488	0.6081

Table 4.14
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material II)
(FSDT)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	46.212	31.411	416.85	38.410	227.58	108.61
		TR	34.455	24.446	272.34	29.200	150.94	74.77
		PL	0.6913	0.5499	3.6803	0.6177	2.1570	1.2047
	0.14	UN	96.11	52.24	906.09	69.47	496.64	220.32
		TR	69.69	40.34	587.81	51.99	326.00	149.18
		PL	1.2831	0.8781	7.7411	1.0426	4.4803	2.2726
4	0.2	UN	21.793	16.602	417.47	19.491	184.33	61.41
		TR	17.489	13.756	272.74	15.831	122.74	43.907
		PL	0.4342	0.3732	3.6853	0.4069	1.7863	0.7967
	0.14	UN	41.44	26.003	907.87	32.890	401.28	117.82
		TR	32.477	21.497	588.95	26.448	264.24	82.58
		PL	0.7468	0.5722	7.7554	0.6527	3.6840	1.4070

Table 4.15
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material II)
(HSDT)

			$\overline{W}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	46.31	26.784	409.32	35.148	233.73	98.79
		TR	34.410	21.087	267.42	26.846	154.76	68.24
		PL	0.6528	0.4623	3.579	0.5461	2.1654	1.0807
	0.14	UN	96.48	47.54	864.91	66.05	496.84	207.28
		TR	70.06	37.088	561.87	49.73	326.36	140.89
		PL	1.2661	0.8059	7.403	0.9873	4.466	2.145
4	0.2	UN	21.595	12.988	400.14	16.879	188.84	50.94
		TR	17.239	10.921	261.53	13.805	125.51	36.821
		PL	0.3963	0.2901	3.505	0.3395	1.7832	0.6627
	0.14	UN	41.428	22.387	841.87	30.169	402.18	104.79
		TR	32.480	18.735	547.13	24.447	276.01	74.10
		PL	0.7239	0.4997	7.219	0.5960	3.674	1.2712

Table 4.16
Center Deflections For Cross-Ply (0/90/0) Laminates
(Material II)
(CL)

a/b	h/a	Load	$\overline{W}$ SS	CC	FF	SC	SF	CF
3	0.2	UN	20.398	4.6313	208.93	9.077	111.55	41.838
		TR	13.636	3.4201	133.71	6.337	71.69	27.222
		PL	0.1886	0.0581	1.6710	0.0963	0.9054	0.3546
	0.14	UN	59.47	13.502	609.11	26.464	325.22	121.98
		TR	39.756	9.971	389.82	18.476	209.02	79.37
		PL	0.5500	0.1694	4.8718	0.2807	2.6397	1.0337
4	0.2	UN	7.061	1.4104	2Q9.49	2.8753	86.85	19.053
		TR	4.9915	1.1539	134.06	2.1908	55.90	12.596
		PL	0.0781	0.0244	1.6754	0.0400	0.7088	0.1709
	0.14	UN	20.585	4.1121	610.75	8.383	253.22	55.55
		TR	14.553	3.3641	390.85	6.387	162.97	36.723
		PL	0.2277	0.0711	4.8845	0.1165	2.0665	0.4983

Table 4.17
Max Center Stress*For Orthotropic Plates
(Material I)
(FSDT)

a/b	h/a	Load	$\bar{\sigma}_{11}$ SS	CC	FF	SC	SF	CF
1	0.2	UN	8.573	5.596	18.496	6.917	13.048	10.659
		TR	5.985	4.081	12.338	4.926	8.851	7.326
		PL	0.0998	0.0761	0.1791	0.0866	0.1356	0.1167
	0.14	UN	17.900	11.217	37.58	14.154	26.765	21.420
		TR	12.530	8.261	25.074	10.138	18.182	14.772
		PL	0.2091	0.1560	0.3643	0.1794	0.2791	0.2368
2	0.2	UN	2.2953	1.0423	18.859	1.5138	7.049	3.4732
		TR	1.7920	0.9735	12.578	1.2838	4.9343	2.6443
		PL	0.0421	0.0315	0.1823	0.0356	0.0843	0.0558
	0.14	UN	4.7754	1.8401	38.165	2.8639	14.345	6.3418
		TR	3.9088	1.9221	25.450	2.6317	10.142	4.9848
		PL	0.0938	0.0662	0.3689	0.0765	0.1754	0.1101
3	0.2	UN	0.9408	0.3774	18.895	0.6031	3.9313	0.8646
		TR	0.7823	0.4128	12.614	0.5619	2.8570	0.8874
		PL	0.0241	0.0193	0.1834	0.0213	0.0556	0.0309
	0.14	UN	1.8429	0.6174	38.391	1.0334	8.110	1.3410
		TR	1.7302	0.8167	25.613	1.1421	6.0148	1.6066
		PL	0.0550	0.0402	0.3717	0.0458	0.1185	0.0615
4	0.2	UN	0.5201	0.1915	18.860	0.3420	2.2695	0.0897
		TR	0.4382	0.2242	12.594	0.3226	1.7190	0.3089
		PL	0.0161	0.0134	0.1835	0.0146	0.0386	0.0206
	0.14	UN	0.9626	0.3075	38.410	0.5523	4.8735	0.0681
		TR	0.9643	0.4486	25.636	0.6465	3.8184	0.5433
		PL	0.0364	0.0275	0.3728	0.0310	0.0859	0.0422

* $\bar{\sigma}_{11} \equiv [\sigma_{11}(\frac{a}{2}, 0, \frac{h}{2})/q_0]$ for distributed loads

$\bar{\sigma}_{11} \equiv [\sigma_{11}(\frac{a}{2}, 0, \frac{h}{2})/p]$ for point loads

Table 4.18
Max Center Stress For Orthotropic Plates
(Material I)
(HSDT)

a/b	h/a	Load	$\bar{\sigma}_{11}$ SS	CC	FF	SC	SF	CF
3	0.2	UN	0.9994	0.4106	18.833	0.6337	4.0351	0.8221
		TR	0.9863	0.5521	12.754	0.7215	3.1103	1.0110
		PL	0.0497	0.0418	0.2095	0.0451	0.0828	0.0550
	0.14	UN	2.1705	0.8036	38.229	1.2655	8.423	1.4475
		TR	1.8940	0.9307	25.690	1.2680	6.221	1.6871
		PL	0.0782	0.0619	0.3988	0.0682	0.1433	0.0840
4	0.2	UN	0.5711	0.2409	18.523	0.3801	2.3731	0.1074
		TR	0.5995	0.3448	12.549	0.4536	1.9545	0.4406
		PL	0.0380	0.0324	0.2065	0.0349	0.0639	0.0426
	0.14	UN	1.2084	0.4534	37.859	0.7252	5.1961	0.1179
		TR	1.0994	0.5525	25.452	0.7560	3.9815	0.6355
		PL	0.0576	0.0466	0.3955	0.0511	0.1077	0.0621

Table 4.19
Max Center Stress For Orthotropic Plates
(Material I)
(CL)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	9.0156	5.384	18.405	6.955	13.242	10.322
		TR	6.289	3.9661	12.281	4.9719	8.987	7.123
		PL	0.1042	0.0752	0.1785	0.0878	0.1376	0.1145
	0.14	UN	18.399	10.988	37.562	14.195	27.025	21.065
		TR	12.835	8.094	25.064	10.147	18.341	14.536
		PL	0.2126	0.1535	0.3642	0.1791	0.2809	0.2337
2	0.2	UN	2.5465	0.8570	18.510	1.3899	7.014	2.6798
		TR	2.0360	0.8774	12.342	1.2566	4.9502	2.1442
		PL	0.0479	0.0312	0.1790	0.0371	0.0858	0.0499
	0.14	UN	5.1969	1.7489	37.775	2.8364	14.315	5.469
		TR	4.1552	1.7905	25.187	2.5645	10.102	4.3760
		PL	0.0978	0.0638	0.3653	0.0756	0.1751	0.1018
3	0.2	UN	1.0164	0.3073	18.612	0.4939	3.9134	0.4369
		TR	0.9193	0.3607	12.404	0.5255	2.8906	0.5895
		PL	0.0293	0.0187	0.1797	0.0224	0.0576	0.0265
	0.14	UN	2.0743	0.6271	37.984	1.0079	7.986	0.8917
		TR	1.8761	0.7362	25.315	1.0724	5.889	1.2030
		PL	0.0598	0.0382	0.3667	0.0457	0.1175	0.0540
4	0.2	UN	0.5332	0.1663	18.664	0.2568	2.3852	0.0980
		TR	0.5166	0.1945	12.438	0.2855	1.8431	0.1473
		PL	0.0204	0.0125	0.1800	0.0153	0.0419	0.0171
	0.14	UN	1.0883	0.3394	38.091	0.5241	4.8677	0.2000
		TR	1.0543	0.3969	25.384	0.5826	3.7615	0.3005
		PL	0.0416	0.0255	0.3674	0.0311	0.0856	0.0350

Table 4.20
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material I)
(FSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	8.7653	5.8509	18.811	7.1502	13.292	10.944
		TR	6.1348	4.2731	12.550	5.1035	9.026	7.5285
		PL	0.1026	0.0795	0.1823	0.0898	0.1385	0.1200
	0.14	UN	18.142	11.472	38.247	14.404	27.186	21.796
		TR	12.684	8.422	25.520	10.296	18.460	15.020
		PL	0.2115	0.1585	0.3708	0.1818	0.2832	0.2406
2	0.2	UN	2.3329	1.0890	19.174	1.5725	7.1634	3.6213
		TR	1.8750	1.0591	12.788	1.3787	5.0444	2.7758
		PL	0.0452	0.0346	0.1854	0.0388	0.0871	0.0588
	0.14	UN	4.9175	1.9724	38.834	3.0236	14.560	6.5110
		TR	3.9473	1.9840	25.892	2.6965	10.259	5.0849
		PL	0.0943	0.0677	0.3753	0.0777	0.1772	0.1121
3	0.2	UN	0.9273	0.3743	19.241	0.6077	4.0100	0.9620
		TR	0.8253	0.4560	12.842	0.6128	2.9522	0.9936
		PL	0.0269	0.0219	0.1866	0.0241	0.0585	0.0339
	0.14	UN	1.9434	0.6819	39.093	1.1322	8.274	1.4916
		TR	1.7433	0.8487	26.076	1.1781	6.0823	1.6863
		PL	0.0559	0.0424	0.3783	0.0476	0.1197	0.0634
4	0.2	UN	0.4975	0.1806	19.221	0.3348	2.3429	0.1437
		TR	0.4609	0.2490	12.833	0.3522	1.8157	0.3902
		PL	0.0184	0.0155	0.1869	0.0169	0.0418	0.0235
	0.14	UN	1.0271	0.3449	39.139	0.6137	5.0484	0.1321
		TR	0.9686	0.4681	26.117	0.6698	3.8797	0.6102
		PL	0.0377	0.0298	0.3796	0.0331	0.0870	0.0440

Table 4.21
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material I)
(HSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	1.0729	0.4735	17.671	0.7198	4.170	1.0624
		TR	1.0538	0.6273	11.973	0.8058	3.1874	1.1742
		PL	0.0477	0.0409	0.1940	0.0439	0.0795	0.0535
	0.14	UN	2.0911	0.7801	34.568	1.2476	8.308	1.5862
		TR	1.9866	1.0339	23.319	1.3871	6.244	1.8725
		PL	0.0789	0.0634	0.3631	0.0695	0.1412	0.0846
4	0.2	UN	0.6117	0.2805	16.894	0.4311	2.5214	0.2670
		TR	0.6486	0.3973	11.473	0.5124	2.0491	0.5644
		PL	0.0372	0.0324	0.1876	0.0346	0.0618	0.0418
	0.14	UN	1.1415	0.4413	32.862	0.7112	5.1397	0.2012
		TR	1.1699	0.6254	22.229	0.8426	4.0654	0.7965
		PL	0.0591	0.0486	0.3496	0.0530	0.1080	0.0638

Table 4.22
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material I)
(CL)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{\sigma}_{11}$					
1	0.2	UN	9.0145	5.3057	18.734	6.902	13.380	10.353
		TR	6.297	3.9245	12.501	4.9469	9.085	7.152
		PL	0.1047	0.0752	0.1817	0.0879	0.1393	0.1153
	0.14	UN	18.397	10.828	38.233	14.086	27.306	21.129
		TR	12.852	8.009	25.512	10.096	18.540	14.595
		PL	0.2137	0.1534	0.3707	0.1794	0.2843	0.2353
2	0.2	UN	2.4942	0.8262	18.863	1.3483	7.039	2.6309
		TR	2.0064	0.8579	12.576	1.2319	4.9724	2.1175
		PL	0.0478	0.0312	0.1824	0.0370	0.0865	0.0499
	0.14	UN	5.0902	1.6860	38.495	2.7516	14.365	5.369
		TR	4.0948	1.7509	25.665	2.5141	10.148	4.3213
		PL	0.0976	0.0637	0.3722	0.0755	0.1765	0.1018
3	0.2	UN	0.9862	0.2963	18.975	0.4765	3.9127	0.4147
		TR	0.9001	0.3517	12.646	0.5130	2.8944	0.5764
		PL	0.0292	0.0187	0.1831	0.0223	0.0579	0.0265
	0.14	UN	2.0126	0.6047	38.725	0.9725	7.985	0.8463
		TR	1.8370	0.7177	25.807	1.0470	5.9069	1.1763
		PL	0.0596	0.0381	0.3737	0.0456	0.1182	0.0541
4	0.2	UN	0.5158	0.1606	19.032	0.2479	2.3804	0.1031
		TR	0.5043	0.1893	12.682	0.2781	1.8426	0.1433
		PL	0.0203	0.0124	0.1835	0.0152	0.0421	0.0172
	0.14	UN	1.0526	0.3278	38.840	0.5060	4.8579	0.2104
		TR	1.0292	0.3863	25.882	0.5676	3.7605	0.2925
		PL	0.0414	0.0254	0.3746	0.0310	0.0860	0.0350

Table 4.23
Max Center Stress For Angle-Ply (-45/45/45/-45) Laminates
(Material I)
(FSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	7.1925	4.5368	18.464	5.6858	12.045	9.442
		TR	5.1156	3.4195	12.319	4.1537	8.218	6.5587
		PL	0.0894	0.0683	0.1789	0.0774	0.1280	0.1074
	0.14	UN	14.679	8.468	37.543	11.079	24.487	18.352
		TR	10.440	6.4669	25.049	8.139	16.711	12.796
		PL	0.1824	0.1329	0.3640	0.1537	0.2604	0.2119
2	0.2	UN	1.7547	0.7996	18.773	1.1894	6.0144	2.5867
		TR	1.4823	0.8572	12.522	1.1138	4.2943	2.1016
		PL	0.0395	0.0314	0.1817	0.0348	0.0771	0.0498
	0.14	UN	3.5821	1.3618	38.09	2.1605	12.108	4.3201
		TR	3.0258	1.5317	25.399	2.0794	8.648	3.6419
		PL	0.0807	0.0601	0.3683	0.0679	0.1557	0.0926
3	0.2	UN	0.6856	0.2684	18.825	0.4651	3.2192	0.5698
		TR	0.6551	0.3752	12.564	0.5077	2.4265	0.7228
		PL	0.0240	0.0203	0.1827	0.0221	0.0512	0.0298
	0.14	UN	1.3989	0.4731	38.295	0.8302	6.570	0.6803
		TR	1.3374	0.6717	25.543	0.9354	4.9413	1.1139
		PL	0.0490	0.0387	0.3707	0.0429	0.1038	0.0546
4	0.2	UN	0.3638	0.1289	18.814	0.2593	1.8443	0.0405
		TR	0.3658	0.2054	12.559	0.2943	1.4723	0.2991
		PL	0.0165	0.0145	0.1828	0.0156	0.0365	0.0216
	0.14	UN	0.7416	0.2422	38.334	0.4636	3.9178	0.1496
		TR	0.7467	0.3745	25.576	0.5421	3.0996	0.4086
		PL	0.0336	0.0276	0.3717	0.0303	0.0754	0.0396

Table 4.24
Max Center Stress For Angle-Ply (-45/45/45/-45) Laminates
(Material I)
(CL)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
1	0.2	UN	7.1925	3.6763	18.378	5.0983	11.973	8.446
		TR	5.1155	2.8559	12.263	3.7717	8.172	5.9159
		PL	0.0894	0.0609	0.1782	0.0725	0.1274	0.0993
	0.14	UN	14.679	7.503	37.506	10.405	24.435	17.236
		TR	10.440	5.828	25.026	7.697	16.678	12.073
		PL	0.1824	0.1244	0.3637	0.1480	0.2600	0.2026
2	0.2	UN	1.7560	0.5376	18.541	0.8909	5.772	1.5550
		TR	1.4830	0.6053	12.360	0.8777	4.1314	1.3835
		PL	0.0395	0.0256	0.1792	0.0304	0.0748	0.0391
	0.14	UN	3.5837	1.0972	37.840	1.8182	11.779	3.1734
		TR	3.0265	1.2352	25.224	1.7912	8.431	2.8234
		PL	0.0807	0.0523	0.3656	0.0621	0.1527	0.0798
3	0.2	UN	0.6859	0.2082	18.639	0.3273	3.0606	0.0695
		TR	0.6556	0.2496	12.421	0.3661	2.3100	0.3002
		PL	0.0240	0.0150	0.1798	0.0182	0.0491	0.0209
	0.14	UN	1.3997	0.4250	38.038	0.6679	6.246	0.1419
		TR	1.3379	0.5094	25.348	0.7470	4.7144	0.6126
		PL	0.0490	0.0306	0.3669	0.0370	0.1002	0.0426
4	0.2	UN	0.3635	0.1149	18.683	0.1757	1.8190	0.1782
		TR	0.3661	0.1347	12.450	0.1988	1.4420	0.0434
		PL	0.0164	0.0098	0.1802	0.0121	0.0354	0.0136
	0.14	UN	0.7418	0.2344	38.128	0.3585	3.7122	0.3637
		TR	0.7471	0.2749	25.408	0.4057	2.9428	0.0886
		PL	0.0336	0.0199	0.3677	0.0247	0.0722	0.0278

Table 4.25
Max Center Stress For Orthotropic Plates
(Material II)
(FSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
2	0.2	UN	6.161	3.8512	18.677	4.8195	12.979	9.710
		TR	4.4355	2.9461	12.455	3.5718	8.802	6.711
		PL	0.0802	0.0614	0.1806	0.0693	0.1348	0.1088
	0.14	UN	13.797	7.037	37.967	9.680	26.958	19.070
		TR	9.889	5.519	25.321	7.235	18.290	13.235
		PL	0.1758	0.1201	0.3674	0.1421	0.2800	0.2168
3	0.2	UN	2.0619	1.3971	18.756	1.7007	10.081	5.712
		TR	1.7018	1.2628	12.507	1.4639	6.900	4.1063
		PL	0.0426	0.0369	0.1813	0.0395	0.1095	0.0748
	0.14	UN	4.5000	2.2393	38.177	3.1108	20.975	10.749
		TR	3.7388	2.2016	25.456	2.8039	14.373	7.805
		PL	0.0929	0.0712	0.3690	0.0800	0.2284	0.1458
4	0.2	UN	0.8386	0.6276	18.775	0.7392	7.788	3.1943
		TR	0.8086	0.6713	12.520	0.7439	5.3871	2.4492
		PL	0.0275	0.0260	0.1815	0.0267	0.0890	0.0525
	0.14	UN	1.7342	0.9453	38.257	1.2811	16.576	5.775
		TR	1.7296	1.1536	25.510	1.4034	11.463	4.5176
		PL	0.0596	0.0505	0.3698	0.0545	0.1887	0.1011

Table 4.26
Max Center Stress For Orthotropic Plates
(Material II)
(DT,ST,δ_A=0)

			$\overline{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
2	0.2	UN	6.359	3.5699	20.347	4.7357	13.973	10.108
		TR	5.8122	3.9778	14.782	4.7493	10.692	8.201
		PL	0.2307	0.2063	0.3429	0.2167	0.2917	0.2601
	0.14	UN	14.067	6.690	36.626	9.569	27.977	19.416
		TR	11.334	6.525	27.640	8.414	20.207	14.700
		PL	0.3279	0.2653	0.5295	0.2902	0.4376	0.3680
3	0.2	UN	1.6497	0.8174	20.437	1.1947	10.700	5.547
		TR	2.5705	1.9636	14.840	2.2425	8.494	5.163
		PL	0.1825	0.1723	0.3436	0.1771	0.2606	0.2179
	0.14	UN	4.0852	1.6010	39.847	2.5507	21.583	10.487
		TR	4.6436	2.8745	27.780	3.5699	15.976	8.804
		PL	0.2352	0.2071	0.5312	0.2187	0.3805	0.2885
4	0.2	UN	0.2798	0.0135	20.464	0.1540	8.198	2.7590
		TR	1.4497	1.2227	14.859	1.3427	6.785	3.2559
		PL	0.1578	0.1524	0.3439	0.1552	0.2343	0.1885
	0.14	UN	1.1535	0.2972	39.936	0.6573	16.949	5.241
		TR	2.3913	1.6829	27.840	1.9916	12.856	5.260
		PL	0.1925	0.1772	0.5320	0.1842	0.3351	0.2365

Table 4.27
Max Center Stress For Orthotropic Plates
(Material II)
(DT,ST,δ_A=1)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
2	0.2	UN	8.897	6.601	22.201	7.562	16.169	12.777
		TR	5.533	4.045	14.055	4.6702	10.189	8.016
		PL	0.1234	0.1045	0.2296	0.1125	0.1814	0.1543
	0.14	UN	16.657	9.999	41.426	12.601	30.158	22.241
		TR	11.084	6.771	26.875	8.465	19.691	14.613
		PL	0.2209	0.1658	0.4158	0.1876	0.3271	0.2635
3	0.2	UN	4.646	4.040	22.374	4.316	13.257	8.614
		TR	2.6013	2.1856	14.170	2.3762	8.232	5.252
		PL	0.0790	0.0737	0.2303	0.0762	0.1529	0.1158
	0.14	UN	7.224	5.095	41.754	5.913	24.173	13.790
		TR	4.759	3.2842	27.080	3.8629	15.728	9.046
		PL	0.1326	0.1112	0.4171	0.1199	0.2727	0.1883
4	0.2	UN	3.4938	3.3385	22.414	3.4205	11.007	6.005
		TR	1.6526	1.5399	14.211	1.5995	6.709	3.4893
		PL	0.0571	0.0564	0.2306	0.0568	0.1293	0.0891
	0.14	UN	4.5180	3.8422	41.872	4.128	19.811	8.746
		TR	2.6903	2.1656	27.172	2.3935	12.807	5.666
		PL	0.0929	0.0840	0.4179	0.0880	0.2300	0.1394

Table 4.28
Max Center Stress For Orthotropic Plates
(Material II)
(HSDT)

a/b	h/a	Load	SS	CC	FF	SC	SF	CF
			$\bar{\sigma}_{11}$					
3	0.2	UN	2.2518	1.2161	20.835	1.6528	11.469	5.7671
		TR	2.4326	1.6601	14.535	1.9937	8.444	4.7463
		PL	0.1228	0.1096	0.2799	0.1155	0.2011	0.1533
	0.14	UN	4.6629	1.9891	40.199	2.9835	22.535	10.735
		TR	4.5300	2.6064	27.501	3.3461	16.091	8.454
		PL	0.1892	0.1584	0.4822	0.1710	0.3369	0.2387
4	0.2	UN	0.8694	0.4838	20.767	0.6558	8.917	2.9243
		TR	1.3165	0.9544	14.488	1.1208	6.710	2.7956
		PL	0.1004	0.0911	0.2791	0.0955	0.1755	0.1237
	0.14	UN	1.7221	0.7542	40.164	1.1334	17.895	5.3756
		TR	2.2804	1.4431	27.475	1.7891	12.971	4.8274
		PL	0.1481	0.1291	0.4816	0.1375	0.2922	0.1858

Table 4.29
Max Center Stress For Orthotropic Plates
(Material II)
(CL)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
2	0.2	UN	7.732	2.6272	18.498	4.4799	13.543	8.619
		TR	5.5065	2.2378	12.340	3.4267	9.195	6.051
		PL	0.0954	0.0546	0.1792	0.0695	0.1406	0.1016
	0.14	UN	15.779	5.3616	37.751	9.143	27.638	17.590
		TR	11.238	4.5669	25.183	6.9933	18.766	12.350
		PL	0.1947	0.1115	0.3656	0.1419	0.2870	0.2074
3	0.2	UN	2.4545	0.4260	18.598	1.0030	10.227	4.0956
		TR	2.1011	0.6982	12.399	1.1206	7.060	3.1011
		PL	0.0525	0.0323	0.1798	0.0389	0.1138	0.0635
	0.14	UN	5.0091	0.8695	37.954	2.0469	20.872	8.358
		TR	4.2880	1.4249	25.303	2.2870	14.407	6.329
		PL	0.1071	0.0660	0.3669	0.0795	0.2322	0.1297
4	0.2	UN	0.8165	0.0886	18.656	0.2517	7.882	1.7708
		TR	0.9565	0.3155	12.434	0.5000	5.5177	1.5306
		PL	0.0358	0.0229	0.1801	0.0275	0.0935	0.0416
	0.14	UN	1.6663	0.1808	38.073	0.5137	16.085	3.6139
		TR	1.9521	0.6438	25.376	1.0205	11.261	3.1236
		PL	0.0731	0.0467	0.3676	0.0561	0.1909	0.0849

Table 4.30
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material II)
(FSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	1.5315	1.0670	19.421	1.2866	10.090	4.8925
		TR	1.3315	1.0179	12.950	1.1668	6.903	3.5763
		PL	0.0371	0.0329	0.1877	0.0349	0.1097	0.0683
	0.14	UN	3.3182	1.6723	39.560	2.3131	21.117	9.231
		TR	2.9296	1.7684	26.377	2.2307	14.463	6.821
		PL	0.0816	0.0641	0.3824	0.0713	0.2300	0.1337
4	0.2	UN	0.6070	0.4728	19.432	0.5475	7.718	2.5672
		TR	0.6238	0.5364	12.958	0.5850	5.3372	2.0349
		PL	0.0238	0.0230	0.1879	0.0235	0.0882	0.0471
	0.14	UN	1.2295	0.7030	39.616	0.9364	16.652	4.6292
		TR	1.3299	0.9242	26.416	1.1074	11.506	3.7581
		PL	0.0521	0.0451	0.3830	0.0483	0.1893	0.0911

Table 4.31
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material II)
(HSDT)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	1.5105	0.7850	20.812	1.0880	11.271	4.5281
		TR	1.9433	1.3262	14.667	1.5945	8.395	3.9924
		PL	0.1244	0.1111	0.2947	0.1172	0.2112	0.1532
	0.14	UN	3.2223	1.3285	39.072	2.0240	21.764	8.609
		TR	3.5829	2.0861	26.959	2.6662	15.690	7.130
		PL	0.1880	0.1591	0.4928	0.1713	0.3463	0.2344
4	0.2	UN	0.5490	0.3029	20.324	0.4105	8.652	2.0444
		TR	1.0505	0.7548	14.348	0.8899	6.596	2.2284
		PL	0.1016	0.0909	0.2904	0.0960	0.1833	0.1234
	0.14	UN	1.1163	0.4945	37.994	0.7395	17.181	3.9085
		TR	1.7908	1.1510	26.261	1.4204	12.581	3.8760
		PL	0.1476	0.1281	0.4837	0.1370	0.2997	0.1829

Table 4.32
Max Center Stress For Cross-Ply (0/90/0) Laminates
(Material II)
(CL)

			$\bar{\sigma}_{11}$					
a/b	h/a	Load	SS	CC	FF	SC	SF	CF
3	0.2	UN	1.7784	0.2289	19.292	0.6349	10.245	3.4388
		TR	1.6748	0.5394	12.860	0.8703	7.083	2.6761
		PL	0.0479	0.0300	0.1864	0.0360	0.1149	0.0586
	0.14	UN	3.6294	0.4672	39.371	1.2958	20.909	7.018
		TR	3.4179	1.1007	26.245	1.7761	14.455	5.4614
		PL	0.0977	0.0612	0.3804	0.0734	0.2344	0.1196
4	0.2	UN	0.5158	0.0425	19.340	0.1320	7.912	1.3478
		TR	0.7459	0.2430	12.890	0.3870	5.5407	1.2445
		PL	0.0332	0.0210	0.1867	0.0255	0.0944	0.0380
	0.14	UN	1.0527	0.0867	39.470	0.2693	16.147	2.7507
		TR	1.5223	0.4959	26.306	0.7897	11.308	2.5398
		PL	0.0679	0.0429	0.3810	0.0520	0.1926	0.0775

TABLE 4.33

Nondimensional Critical Buckling Loads, $P_{cr} = \frac{\lambda_{cr} \ell_1^2}{h^3 E_1}$, for an Orthotropic Square Plate

E_1 belongs to Material I

		MATERIAL I				MATERIAL II			
γ	$\frac{\ell_1}{h}$	DT, ST, FSDT $\delta_A=0$	DT, ST $\delta_A=1$	HSDT	CL	DT, ST, FSDT $\delta_A=0$	DT, ST $\delta_A=1$	HSDT	CL
	2	0.9435	1.0773	0.9581	2.7820	0.1625	0.1913	0.1707	0.9828
	5	2.0978	2.2039	2.0999		0.5221	0.5655	0.5248	
0	10	2.5704	2.6096	2.5706		0.8011	0.8253	0.8016	
	20	2.7258	2.7367	2.7258		0.9296	0.9376	0.9297	
	50	2.7729	2.7747	2.7729		0.9738	0.9752	0.9738	
	2	0.6290	0.7182	0.6388	1.8547	0.1084	0.1275	0.1138	0.6552
	5	1.3986	1.4693	1.3999		0.3481	0.3770	0.3499	
0.5	10	1.7136	1.7397	1.7137		0.5341	0.5502	0.5344	
	20	1.8172	1.8245	1.8172		0.6198	0.6251	0.6198	
	50	1.8486	1.8498	1.8486		0.6492	0.6502	0.6492	
	2	0.4718	0.5386	0.4791	1.3910	0.0813	0.0957	0.0853	0.4914
	5	1.0489	1.1020	1.0500		0.2611	0.2827	0.2624	
1.0	10	1.2852	1.3048	1.2853		0.4006	0.4127	0.4008	
	20	1.3629	1.3684	1.3629		0.4648	0.4688	0.4648	
	50	1.3864	1.3873	1.3864		0.4869	0.4876	0.4869	

TABLE 4.34

Nondimensionl Critical Buckling Loads, $P_{cr} = \frac{\lambda_{cr} \ell_1^2}{h^3 E}$ for a Transversely-Isotropic Square Plate,

E belongs to Material A

γ	$\frac{\ell_1}{h}$	MATERIAL A DT, ST, FSDT $\delta_A=0$	MATERIAL A DT, ST $\delta_A=1$	MATERIAL A HSDT	MATERIAL A CL	MATERIAL B DT, ST, FSDT $\delta_A=0$	MATERIAL B DT, ST $\delta_A=1$	MATERIAL B HSDT	MATERIAL B CL
	2	0.9197	1.1244	0.9486	4.0797	0.2427	0.2837	0.2750	3.2178
	5	2.6325	2.8720	2.6386		1.0864	1.2120	1.1031	
0	10	3.5868	3.6916	3.5875		2.1589	2.2761	2.1631	
	20	3.9442	3.9752	3.9442		2.8663	2.9162	2.8668	
	50	4.0574	4.0626	4.0574		3.1559	3.1654	3.1559	
	2	0.6131	0.7496	0.6324	2.7198	0.1618	0.1891	0.1834	2.1452
	5	1.7550	1.9147	1.7591		0.7243	0.8080	0.7354	
0.5	10	2.3912	2.4611	2.3916		1.4393	1.5174	1.4421	
	20	2.6295	2.6501	2.6295		1.9109	1.9441	1.9112	
	50	2.7049	2.7084	2.7049		2.1039	2.1103	2.1039	
	2	0.4599	0.5622	0.4743	2.0398	0.1213	0.1418	0.1375	1.6089
	5	1.3163	1.4360	1.3193		0.5432	0.6060	0.5516	
1.0	10	1.7934	1.8458	1.7937		1.0795	1.1381	1.0815	
	20	1.9721	1.9876	1.9721		1.4332	1.4581	1.4334	
	50	2.0287	2.0313	2.0287		1.5779	1.5827	1.5779	

TABLE 4.35

Eigenfrequencies, $\bar{\omega} = \omega h^{*} \sqrt{\rho/d_{11}}$, of an orthotropic square plate ($\ell_1/h = 10$) for Material I.

		Exact			HSDT			$\delta_A = 0$ FSDT, DT, ST			$\delta_A = 1$ DT, ST			CL.
m	n	I	II	III	I	II	III	I	II	III	I	II	III	I
1	1	0.0474	1.3077	1.6530	0.0474	1.3086	1.6550	0.0474	1.3159	1.6646	0.0475	1.3149	1.6638	0.0493 (0.0497)‡
1	2	0.1033	1.3331	1.7160	0.1033	1.3339	1.7209	0.1032	1.3410	1.7305	0.1034	1.3404	1.7275	0.1095 (0.1120)
2	1	0.1188	1.4205	1.6805	0.1189	1.4216	1.6827	0.1187	1.4284	1.6921	0.1191	1.4249	1.6908	0.1327 (0.1354)
2	2	0.1694	1.4316	1.7509	0.1695	1.4323	1.7562	0.1692	1.4392	1.7655	0.1698	1.4371	1.7611	0.1924 (0.1987)
1	3	0.1888	1.3765	1.8115	0.1888	1.3772	1.8210	0.1884	1.3841	1.8305	0.1890	1.3838	1.8243	0.2070 (0.2154)
3	1	0.2180	1.5777	1.7334	0.2184	1.5789	1.7361	0.2177	1.5855	1.7450	0.2189	1.5799	1.7415	0.2671 (0.2779)
2	3	0.2475	1.4596	1.8523	0.2477	1.4603	1.8622	0.2469	1.4670	1.8714	0.2481	1.4661	1.8633	0.2879 (0.3029)
3	2	0.2624	1.5651	1.8195	0.2629	1.5658	1.8255	0.2619	1.5726	1.8340	0.2634	1.5699	1.8263	0.3248 (0.3418)
1	4	0.2969	1.4372	1.9306	0.2969	1.4379	1.9466	0.2958	1.4445	1.9559	0.2974	1.4444	1.9457	0.3371 (0.3599)
4	1	0.3319	1.7179	1.8548	0.3330	1.7186	1.8588	0.3310	1.7265	1.8654	0.3336	1.7245	1.8533	0.4471 (0.4773)
3	3	0.3320	1.5737	1.9289	0.3326	1.5744	1.9395	0.3310	1.5811	1.9478	0.3332	1.5804	1.9359	0.4172 (0.4470)
2	4	0.3476	1.5068	1.9749	0.3479	1.5076	1.9912	0.3462	1.5141	2.0001	0.3484	1.5141	1.9877	0.4152 (0.4480)
4	2	0.3070	1.6940	1.9447	0.3720	1.6947	1.9514	0.3695	1.7021	1.9584	0.3724	1.7012	1.9440	0.5018 (0.5415)

*$d_{11} = 23.2 \times 10^6$ psi, ‡Numbers in parenthesis indicate frequencies obtained by omitting the rotary inertia.

TABLE 4.36

Non-Dimensional Fundamental Frequencies of an Orthotropic Plate,

$\bar{\omega} = (\omega \ell_1^2/h)\sqrt{\rho/E_2}$, E_2 belongs to Material I

ℓ_1/ℓ_2	ℓ_1/h	MATERIAL I: DT, ST, FSDT $\delta_A = 0$	MATERIAL I: DT, ST $\delta_A = 1$	MATERIAL II: DT, ST, FSDT $\delta_A = 0$	MATERIAL II: DT, ST $\delta_A = 1$
	2	7.509	7.613	2.611	2.656
	5	12.605	12.684	4.733	4.774
2.0	10	15.021	15.055	5.953	5.974
	20	15.948	15.957	6.483	6.489
	50	16.257	16.269	6.663	6.672
	2	4.067	4.110	1.705	1.727
	5	6.158	6.182	3.087	3.106
1.0	10	6.901	6.910	3.856	3.865
	20	7.144	7.146	4.172	4.175
	50	7.226	7.259	4.284	4.287
	2	2.966	2.995	1.494	1.510
	5	4.321	4.335	2.785	2.800
0.5	10	4.748	4.753	3.492	3.500
	20	4.880	4.882	3.778	3.780
	50	4.963	4.953	3.900	3.894

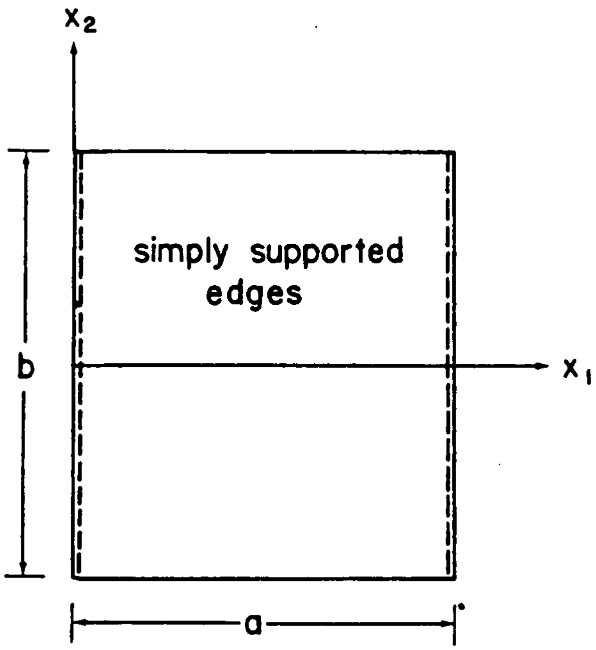
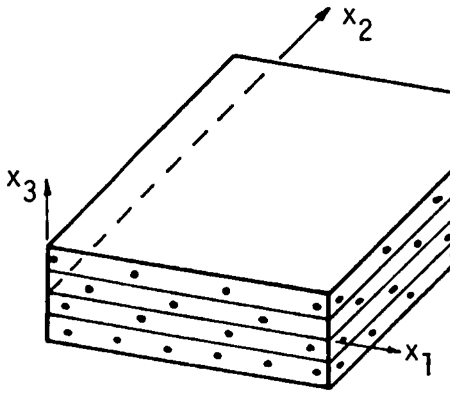
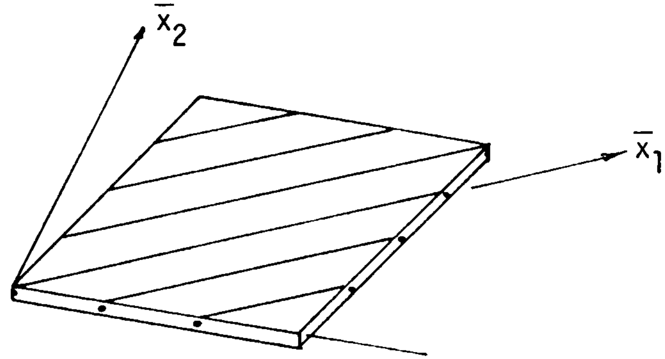


Figure 3.1 Geometry and coordinate system of rectangular plate (a x b).

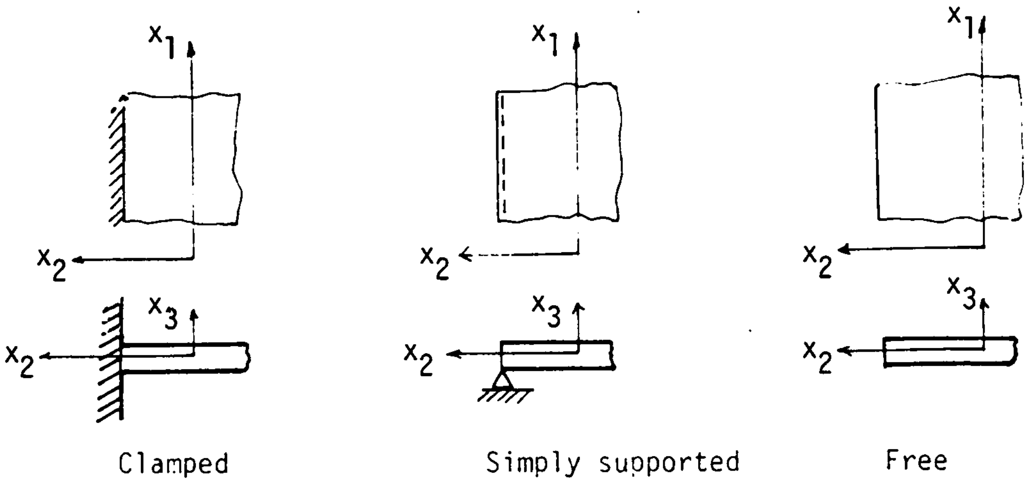


(a) Laminate coordinates

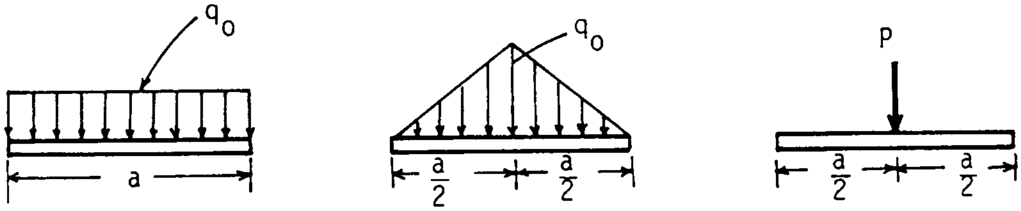


(b) Lamina coordinates

Figure 3.2 Laminate and lamina (or material) coordinate systems



(a) Various types of boundary conditions



(b) Various types of transverse loads

Figure 4.1 Various types of boundary conditions and loads

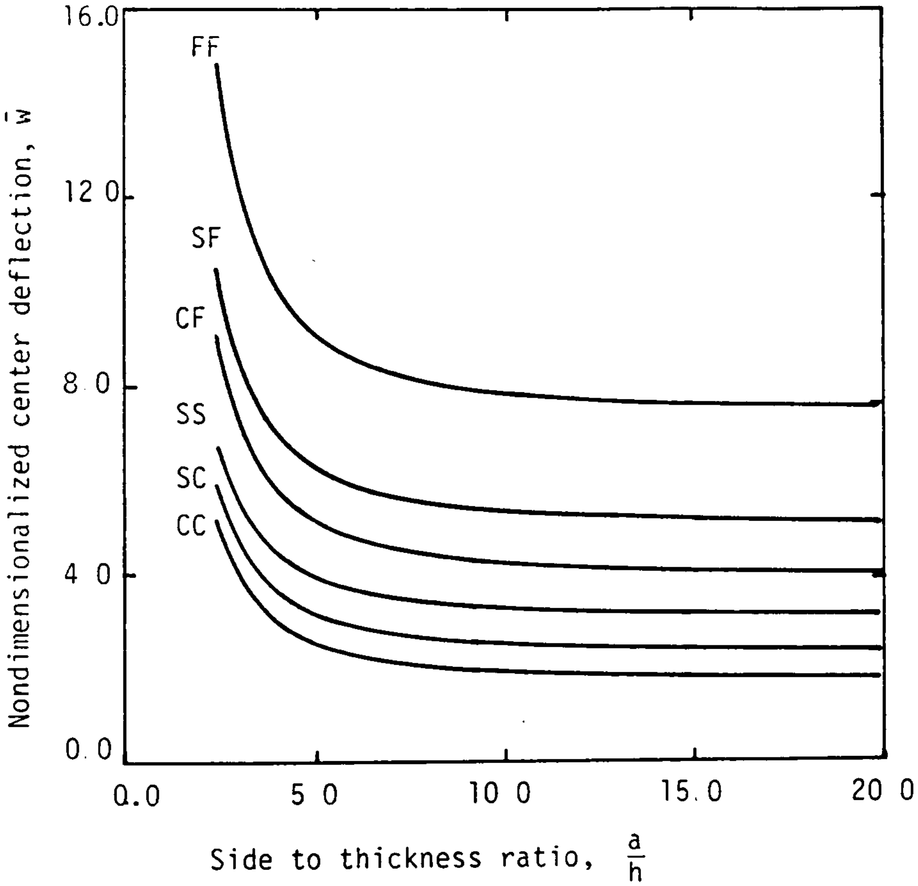


Figure 4.2 Nondimensionalized center deflection versus side to thickness ratio of orthotropic plates under uniformly distributed load (a/b=1) using FSDT theory, (Material I, $\overline{W} = \dfrac{Wh^3E_2 10^2}{a^4 q_o}$).

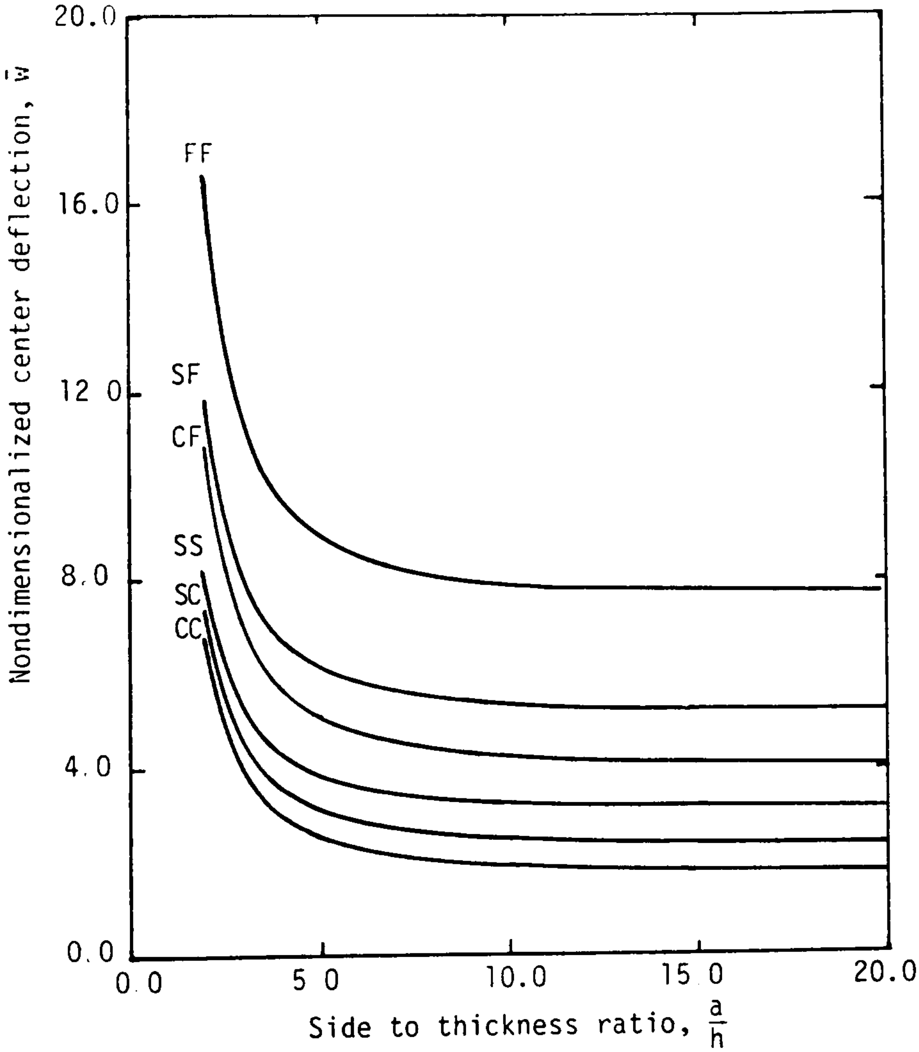


Figure 4.3 Nondimensionalized center deflection versus side to thickness ratio of cross-ply (0/90/0) laminates (a/b = 1) under uniformly distributed transverse load using FSDT theory,

(Material I, $\overline{W} = \dfrac{Wh^3E_2 10^2}{a^4 q_o}$).

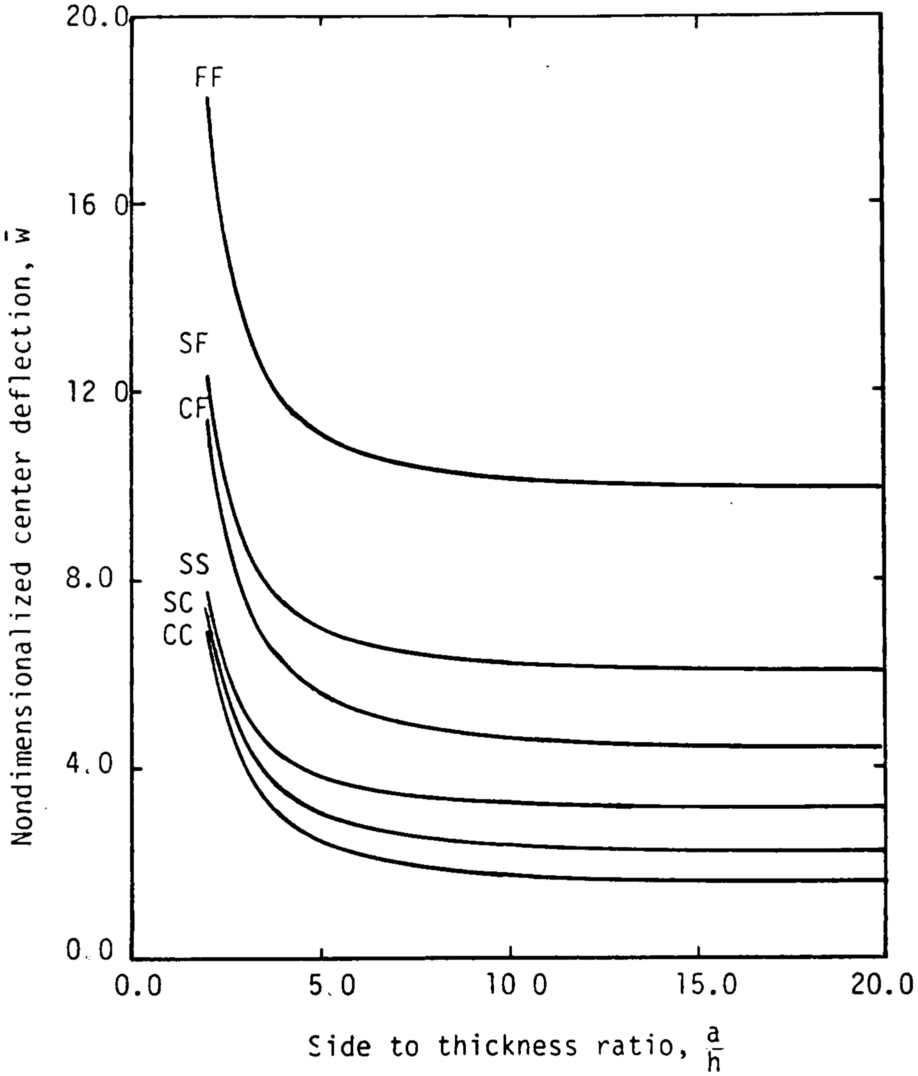


Figure 4.4 Nondimensionalized center deflection versus side to thickness ratio of angle-ply (-45/45/45/-45) square(a/b=1) laminates under uniformly distributed transverse load using FSDT theory,

(Material I, $\overline{W} = \dfrac{Wh^3E_2 10^2}{a^4 q_o}$).

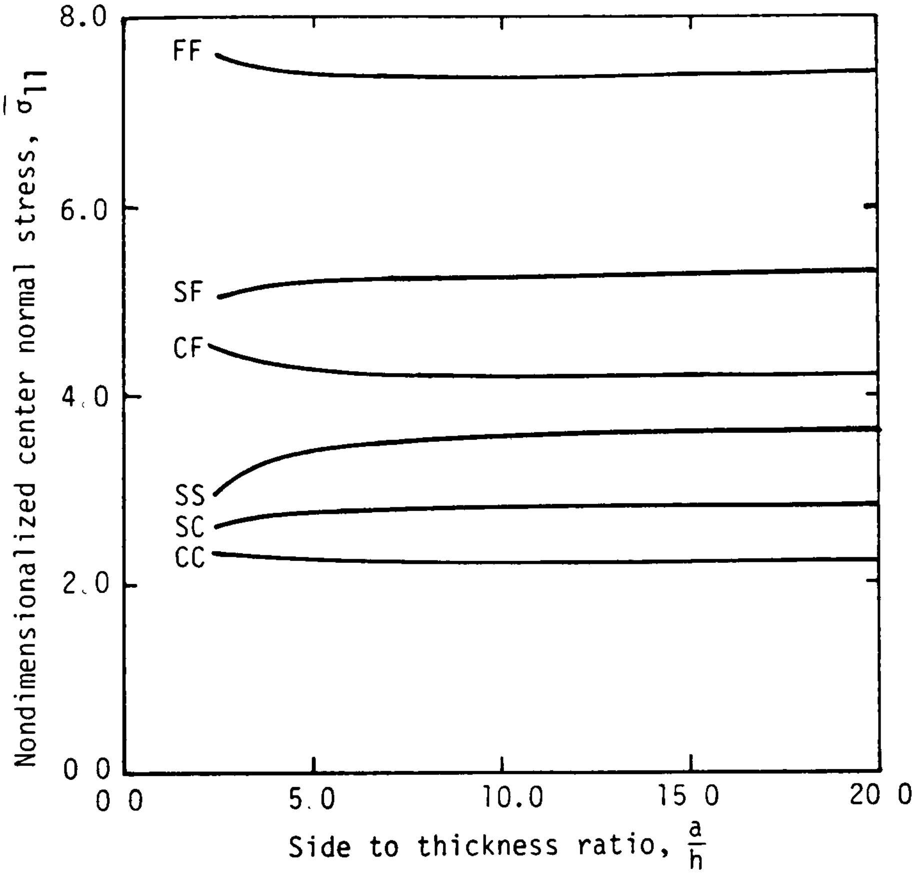


Figure 4.5 Nondimensionalized center normal stress versus side to thickness ratio for orthotropic plates (a/b=1) under uniformly distributed transverse load using FSDT theory,

$$(\text{Material I, } \bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10}{a^2 q_o}).$$

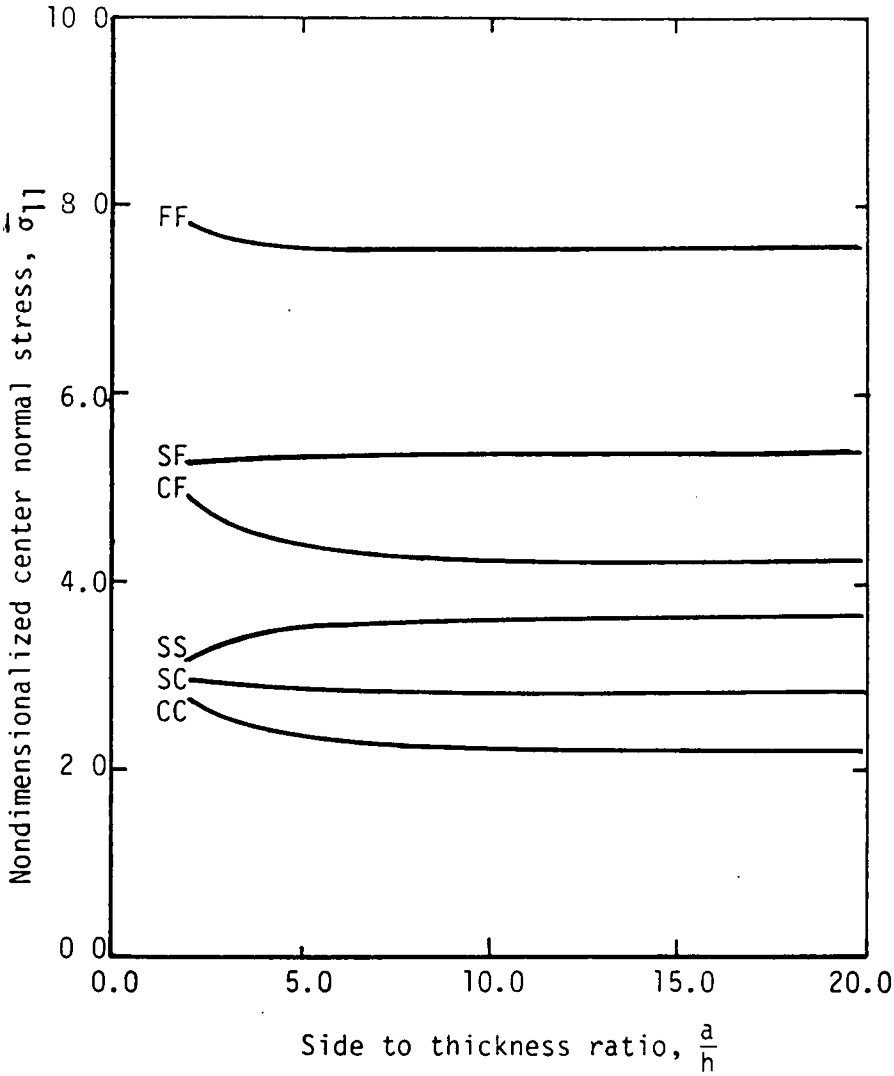


Figure 4.6 Nondimensionalized center normal stress versus side to thickness ratio of cross-ply (0/90/0) square (a/b=1) laminates under uniformly distributed transverse load using FSDT theory,

(Material I, $\bar{\sigma}_{11} = \dfrac{\sigma_{11} h^2 10}{a^2 q_o}$).

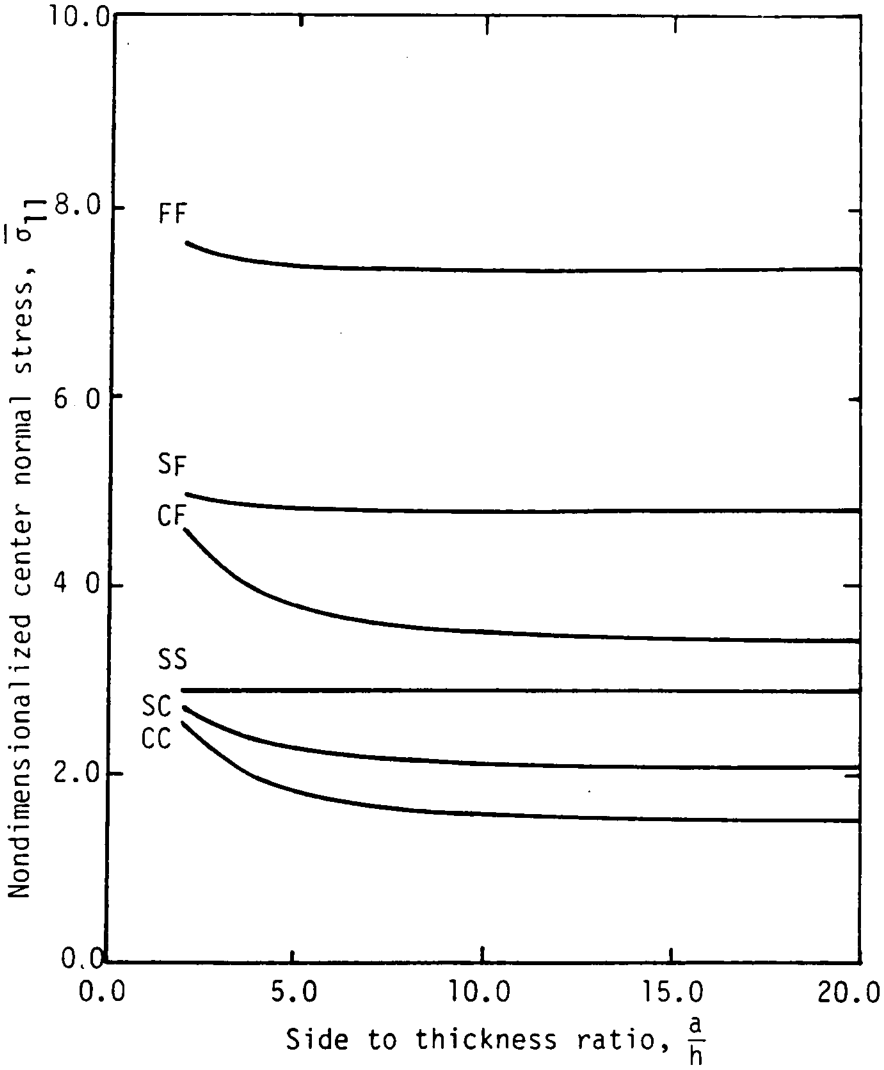


Figure 4.7 Nondimensionalized center normal stress versus side to thickness ratio of angle-ply (-45/45/45/-45) square (a/b=1) laminates under uniformly distributed transverse load using FSDT theory,

(Material I, $\bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10}{a^2 q_o}$).

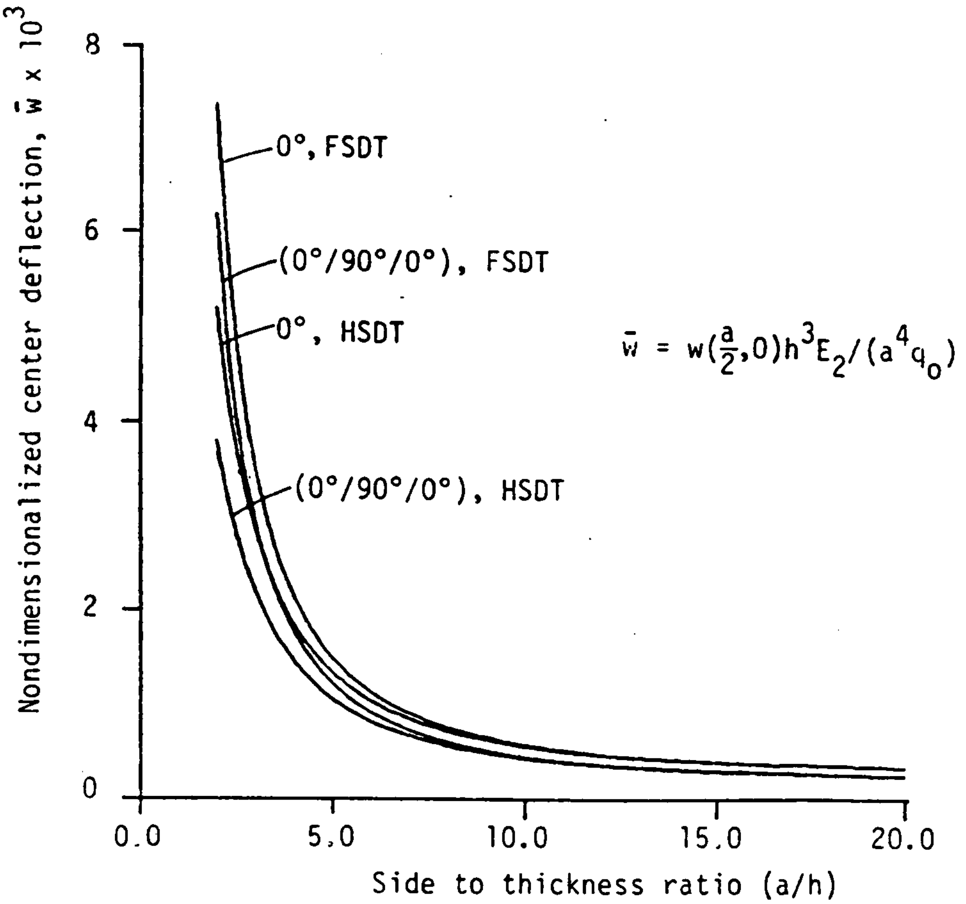


Figure 4.8 Nondimensional center deflection versus side to thickness ratio of SSSC laminates using FSDT and HSDT theories, (Material II, a/b = 4, uniform load).

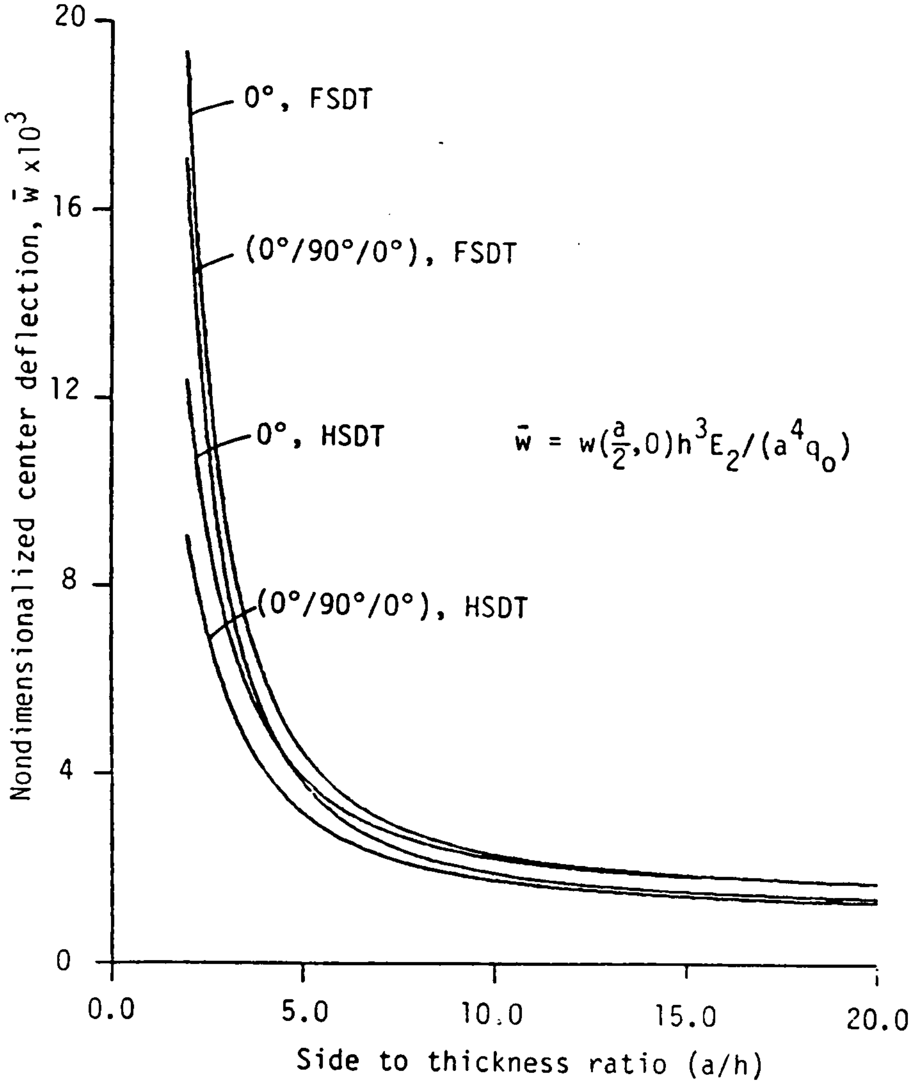


Figure 4.9 Nondimensionalized center deflection versus side to thickness ratio of SSFC laminates using FSDT and HSDT (Material II, a/b = 4, uniform load).

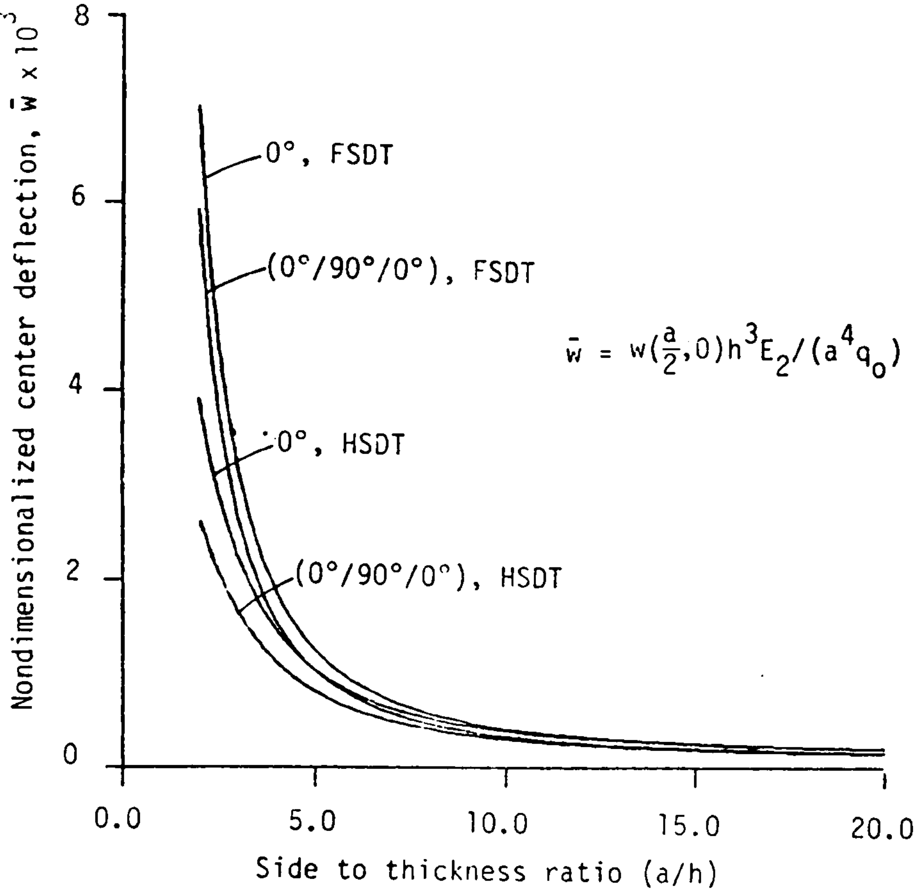


Figure 4.10 Nondimensionalized center deflection versus side to thickness ratio of SSCC laminates using FSDT and HSDT theories (Material II, a/b = 4, uniform load).

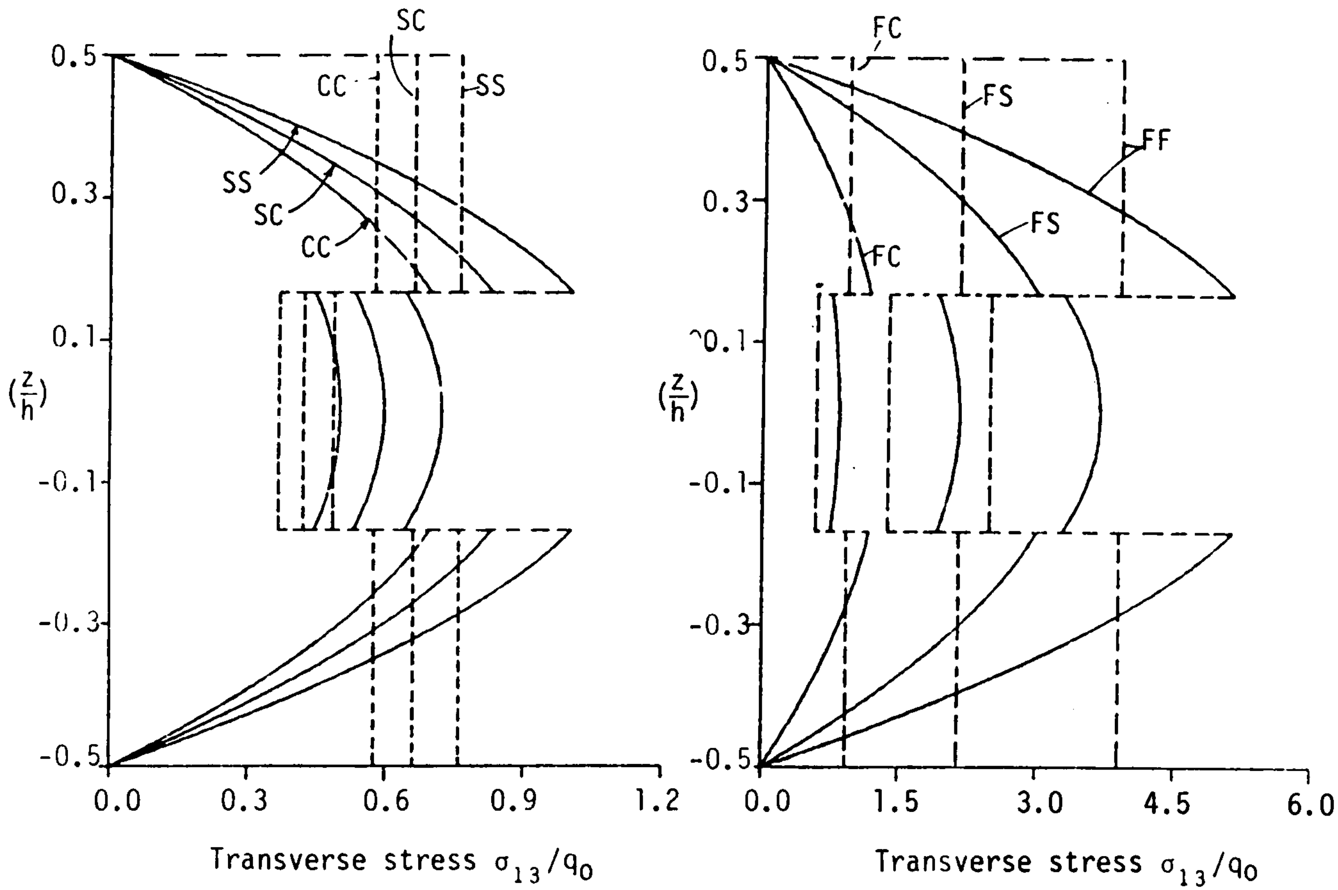


Figure 4.11 Variation of the transverse shear stress through the thickness of cross-ply (0/90/0) laminates under uniform load and subjected to various boundary conditions (a/b=4, Material II, h/a = 0.14). ---- FSDT, —— HSDT.

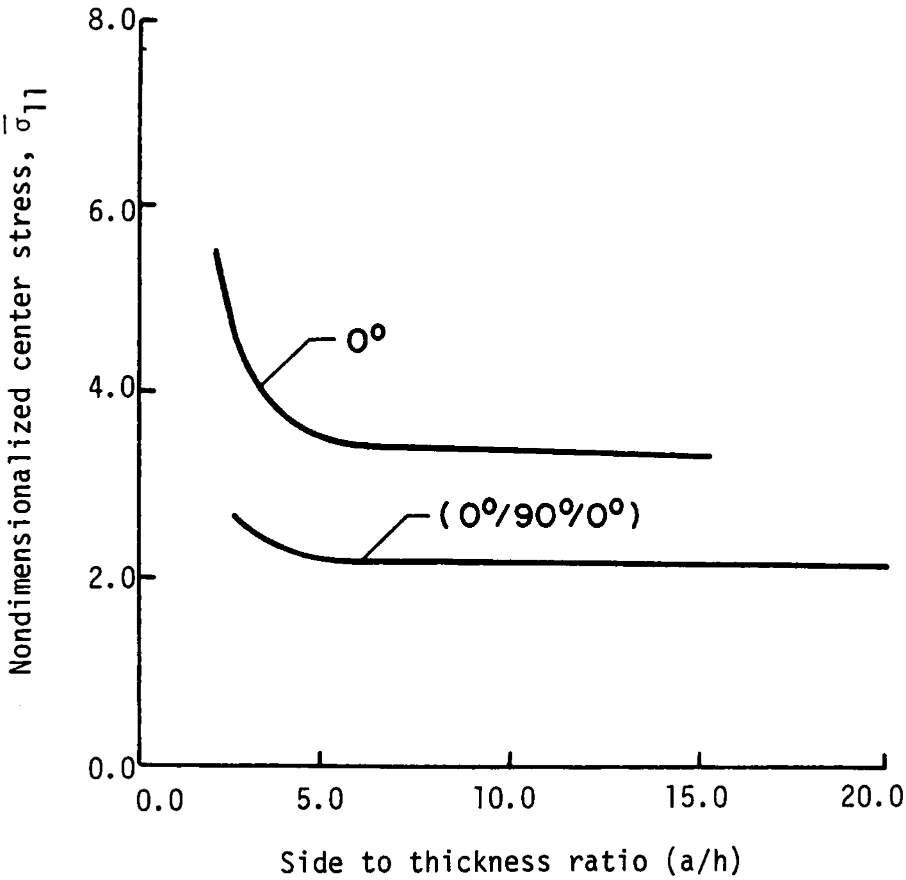


Figure 4.12 Nondimensionalized center axial stress

$$\left(\bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10^2}{a^2 q_o}\right)$$

versus side to thickness rato for simply supported laminates using HSDT theory under uniform load, (a/b = 4, Material II).

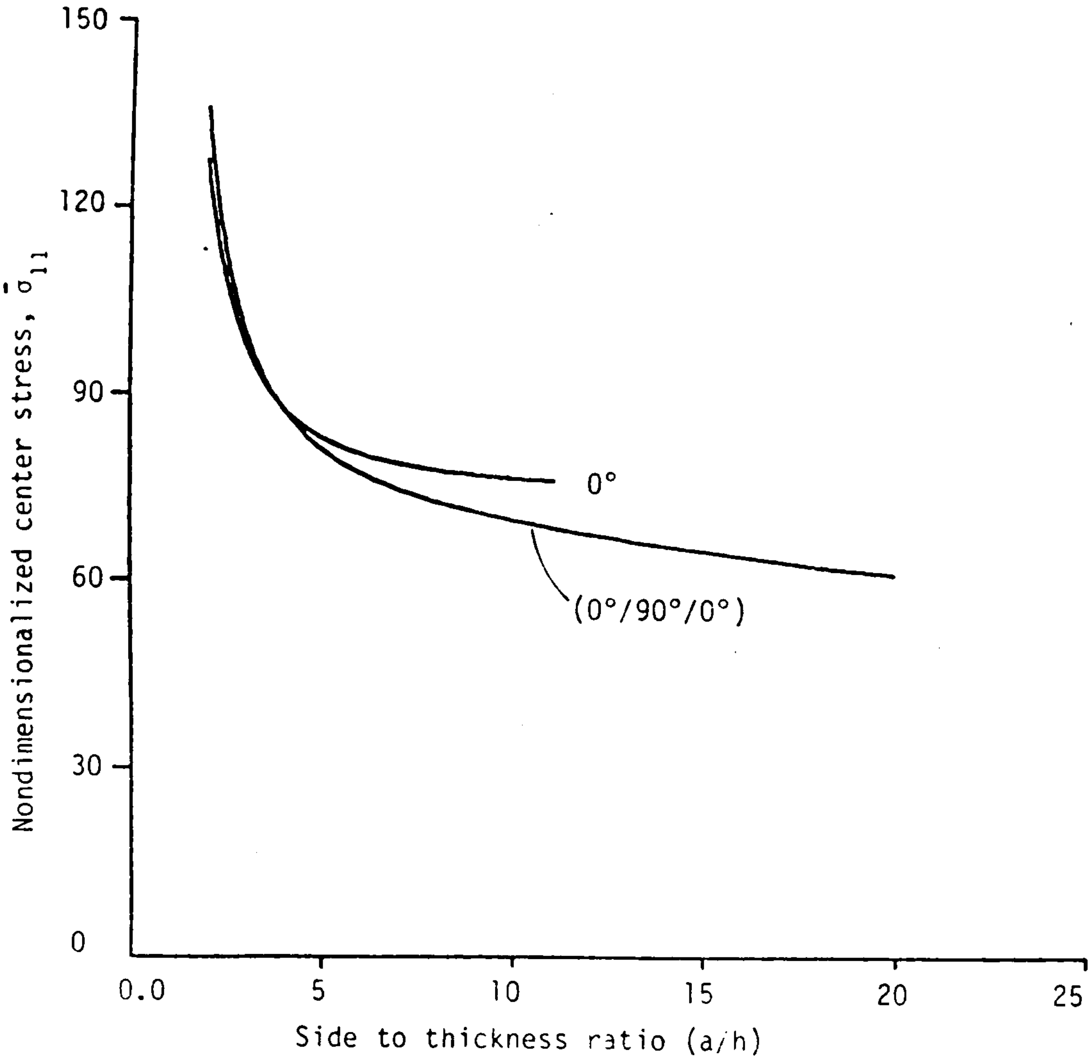


Figure 4.13 Nondimensionalized center axial stress

$$(\bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10^2}{a^2 q_o})$$

versus side to thickness ratio for SSFF laminates using HSDT theory under uniform load, (a/b = 4, Material II).

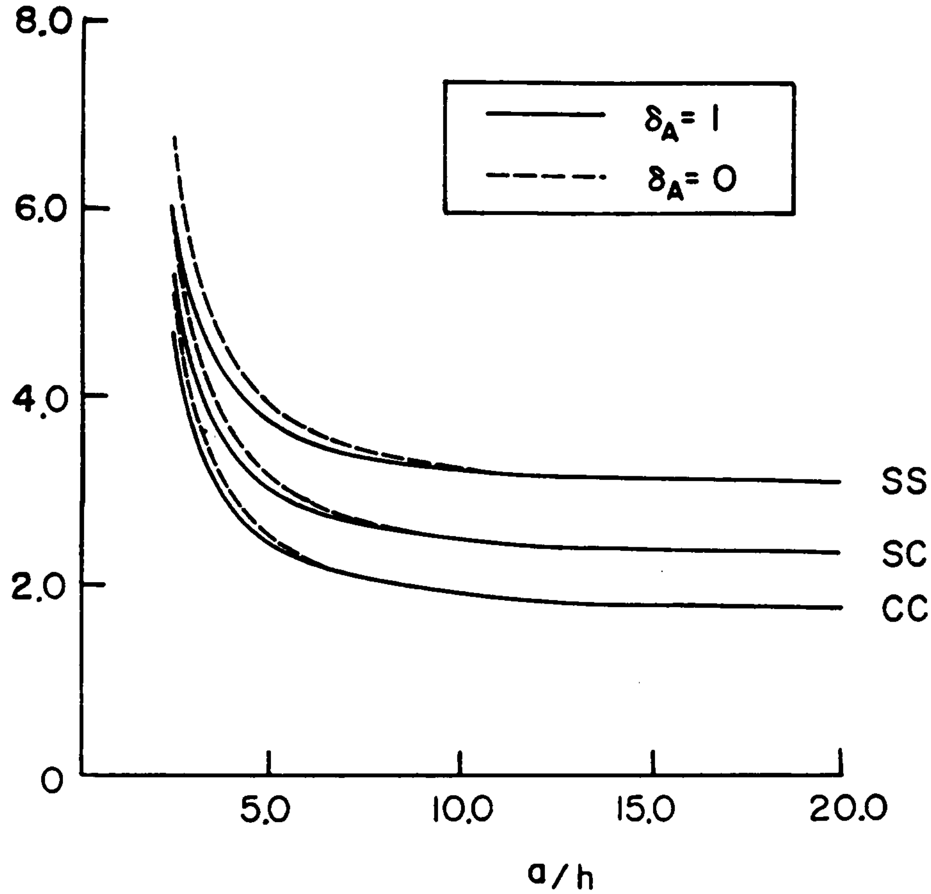


Figure 4.14 Nondimensional center deflection

$$(\overline{W} = \frac{WE_2h^3 10^2}{a^4 q_o})$$

versus side to thickness ratio for DT, ST when $\delta_A = 1$ and for DT, ST, and FSDT when $\delta_A = 0$ for CC, SC, SS edge conditions under UN load, (a/b = 1, Material I).

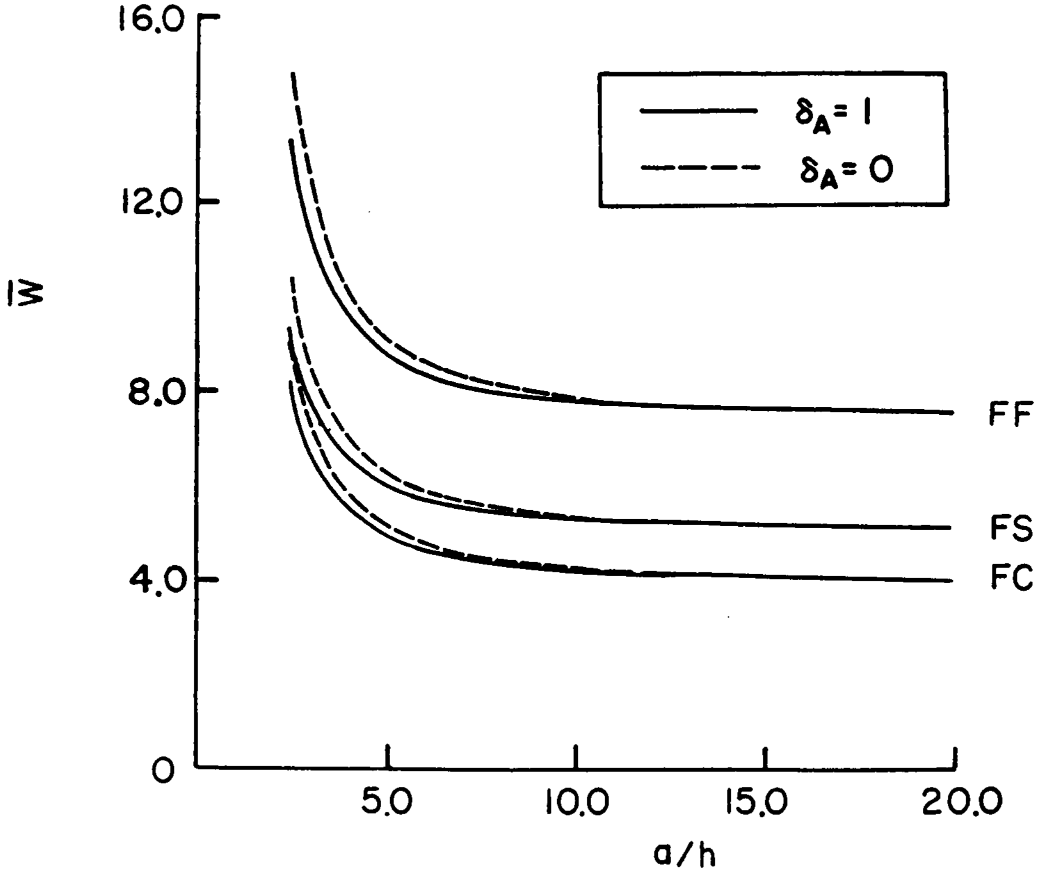


Figure 4.15 Nondimensional center deflection

$$(\overline{W} = \frac{WE_2h^3 10^2}{a^4 q_0})$$

versus side to thickness ratio for DT, ST when $\delta_A = 1$ and for DT, ST and FSDT when $\delta_A = 0$ for FC, FS, FF edge conditions under UN load, (a/b = 1, Material I).

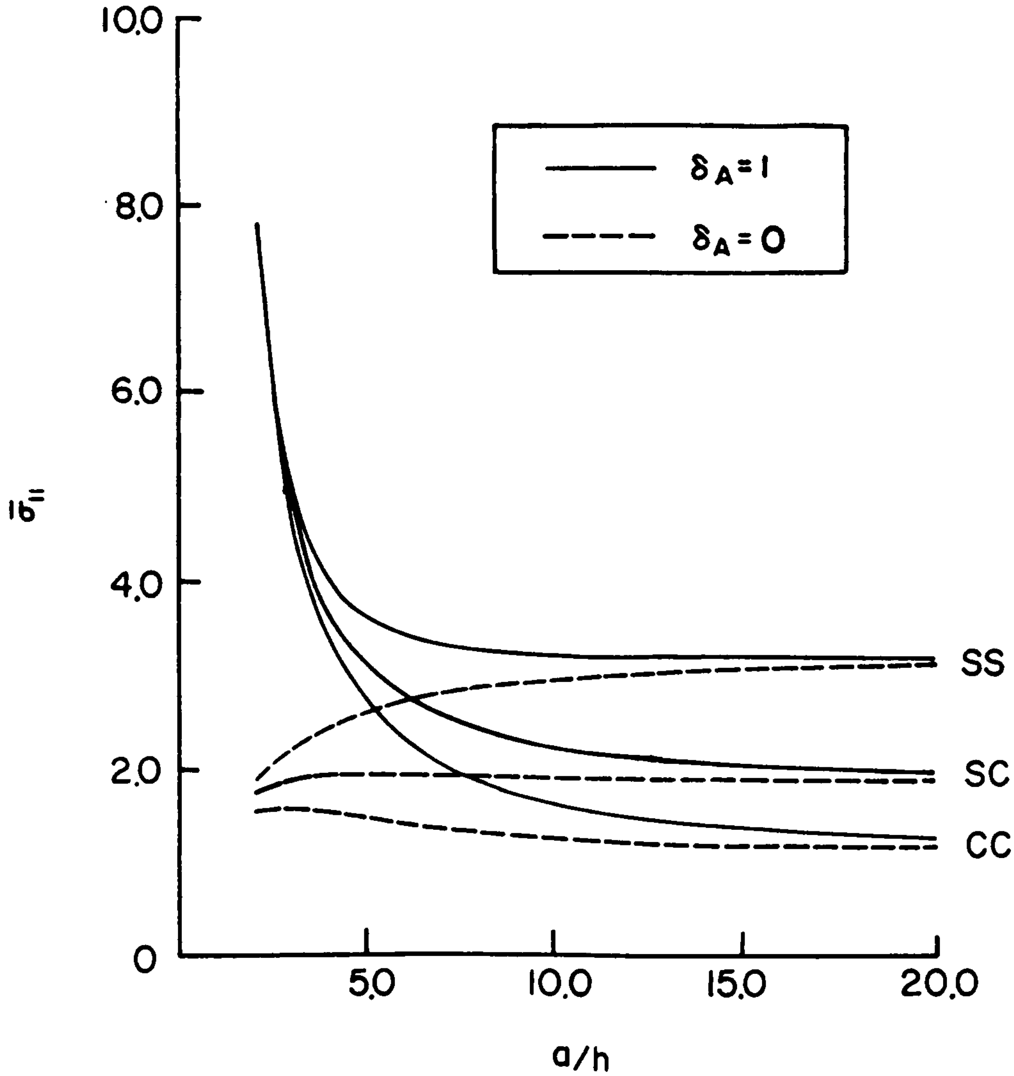


Figure 4.16 Nondimensional center axial stress

$$\left(\bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10}{a^2 q_o}\right)$$

versus side to thickness ratio for DT, ST for CC, SC, SS edge conditions under UN load, (a/b = 2, Material II).

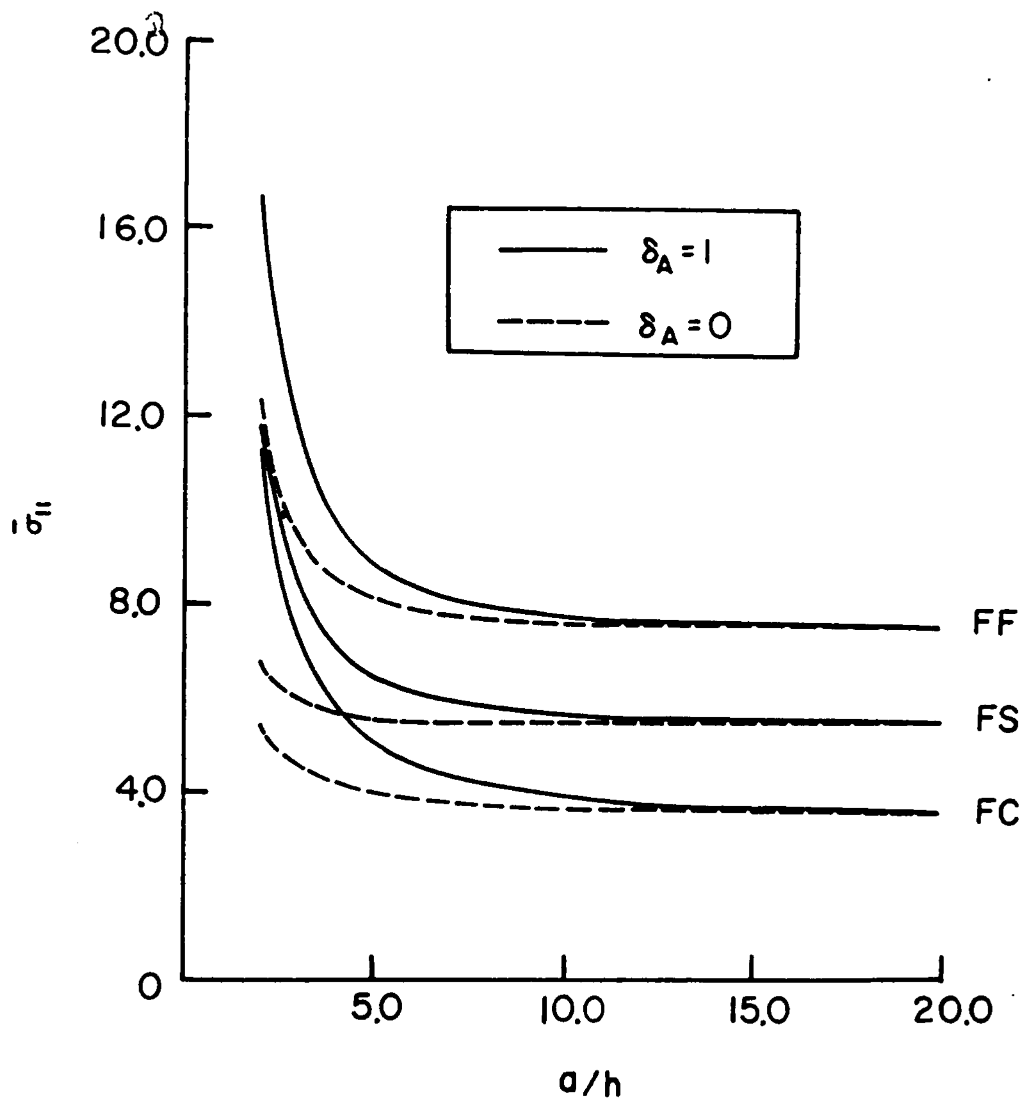


Figure 4.17 Nondimensional center axial stress

$$(\bar{\sigma}_{11} = \frac{\sigma_{11} h^2 10}{a^2 q_o})$$

versus side to thickness ratio for DT, ST for FC, FS, FF edge conditions under UN load, (a/b = 2, Material II).

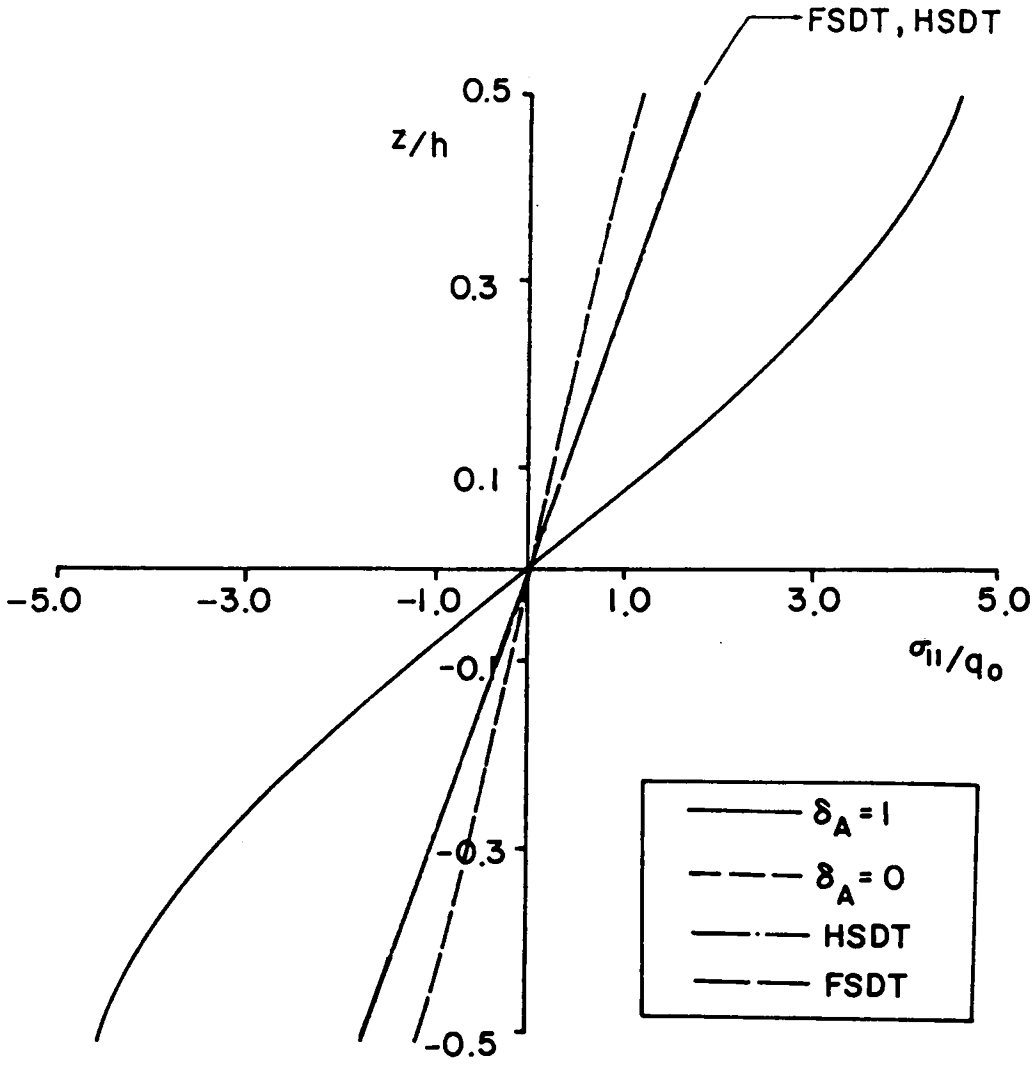


Figure 4.18 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for SS edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

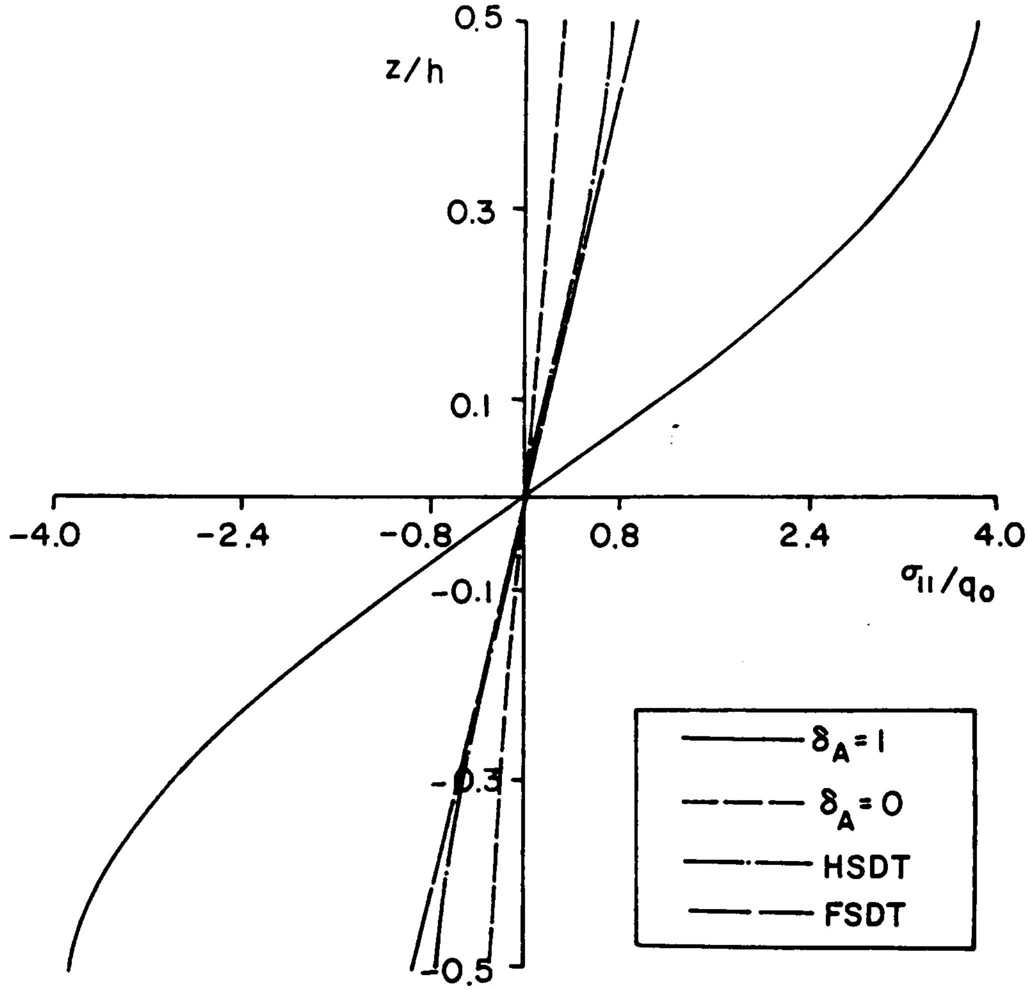


Figure 4.19 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for CC edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

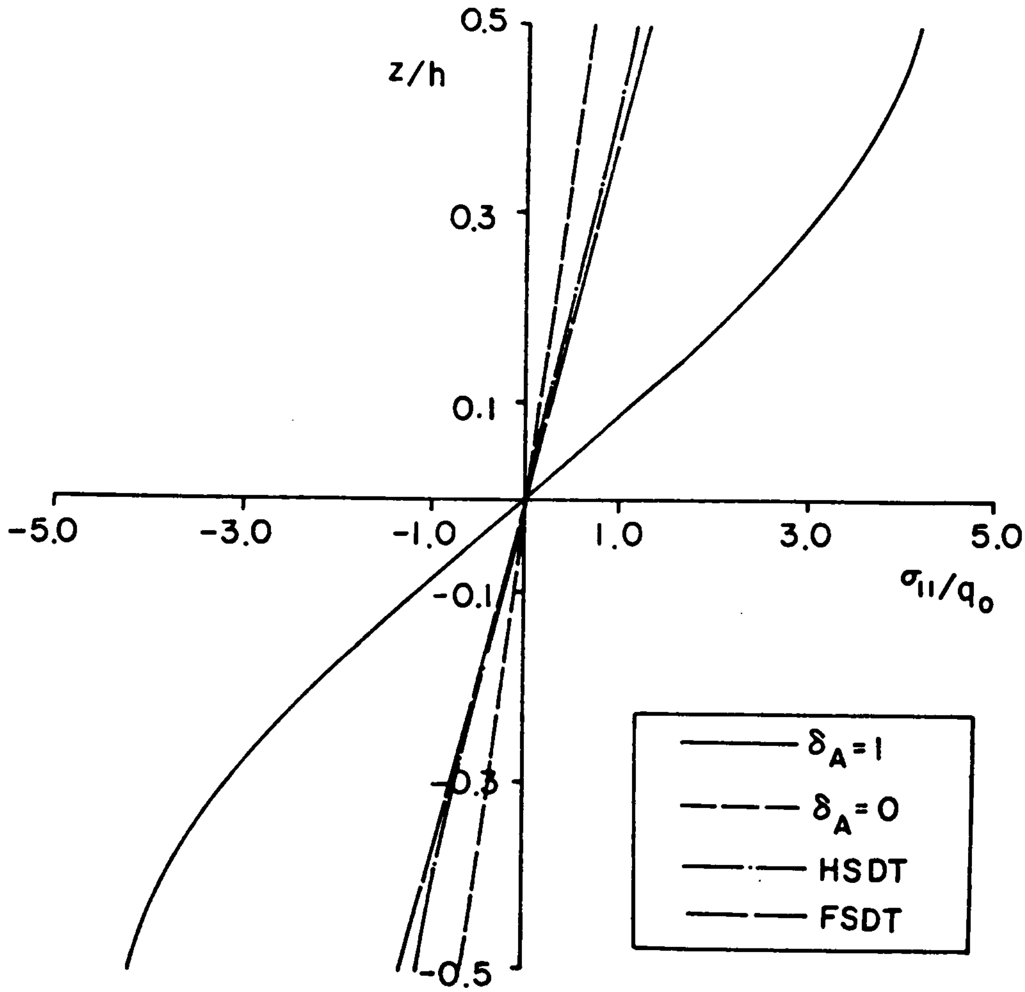


Figure 4.20 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for SC edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

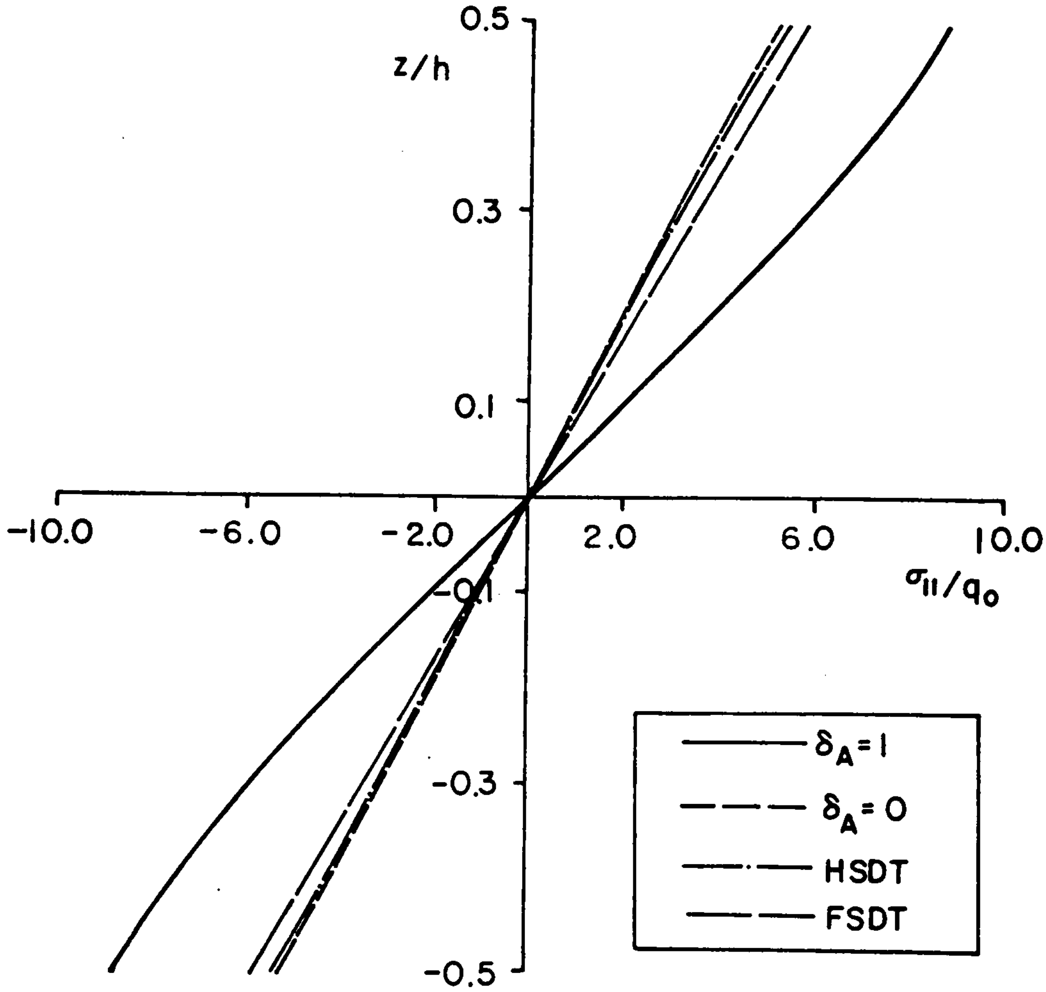


Figure 4.21 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for FC edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

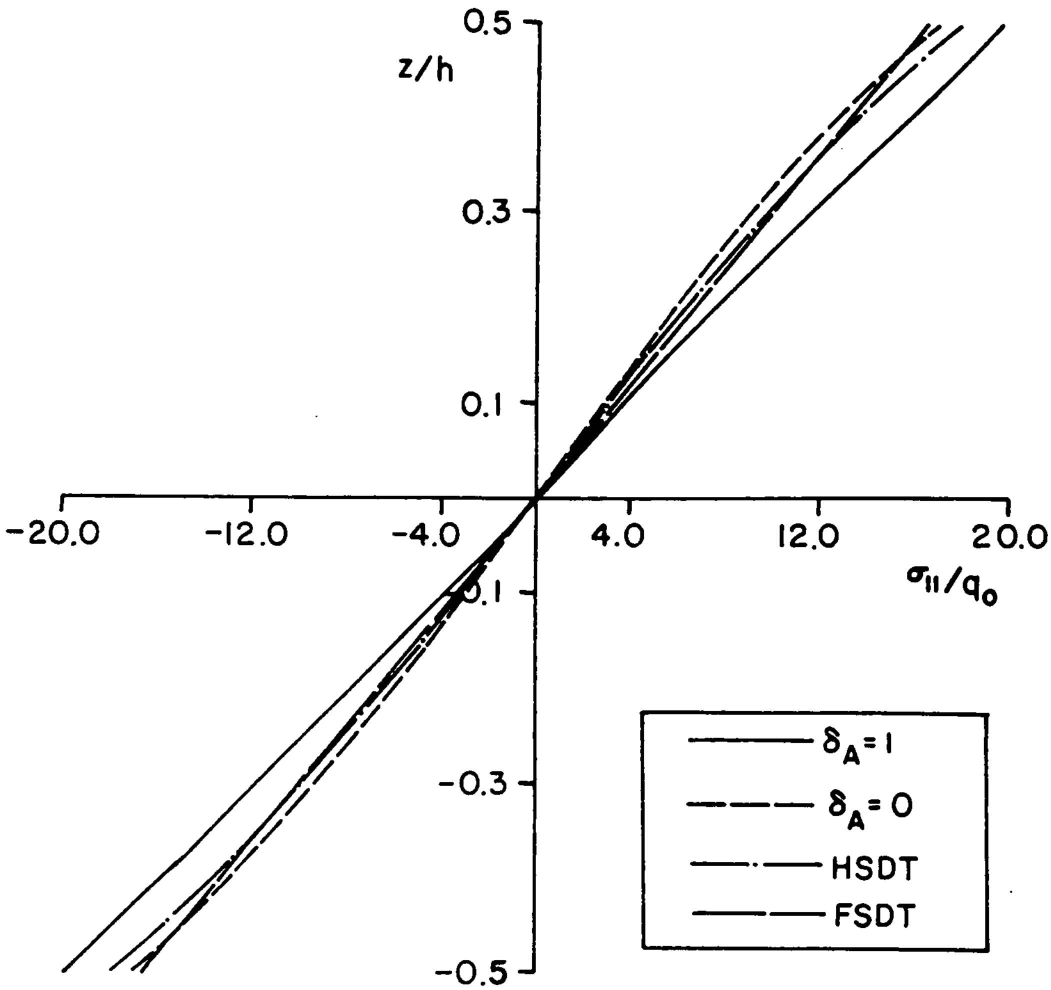


Figure 4.22 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when δ_A = 0 and for DT, ST when δ_A = 1 for FS edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

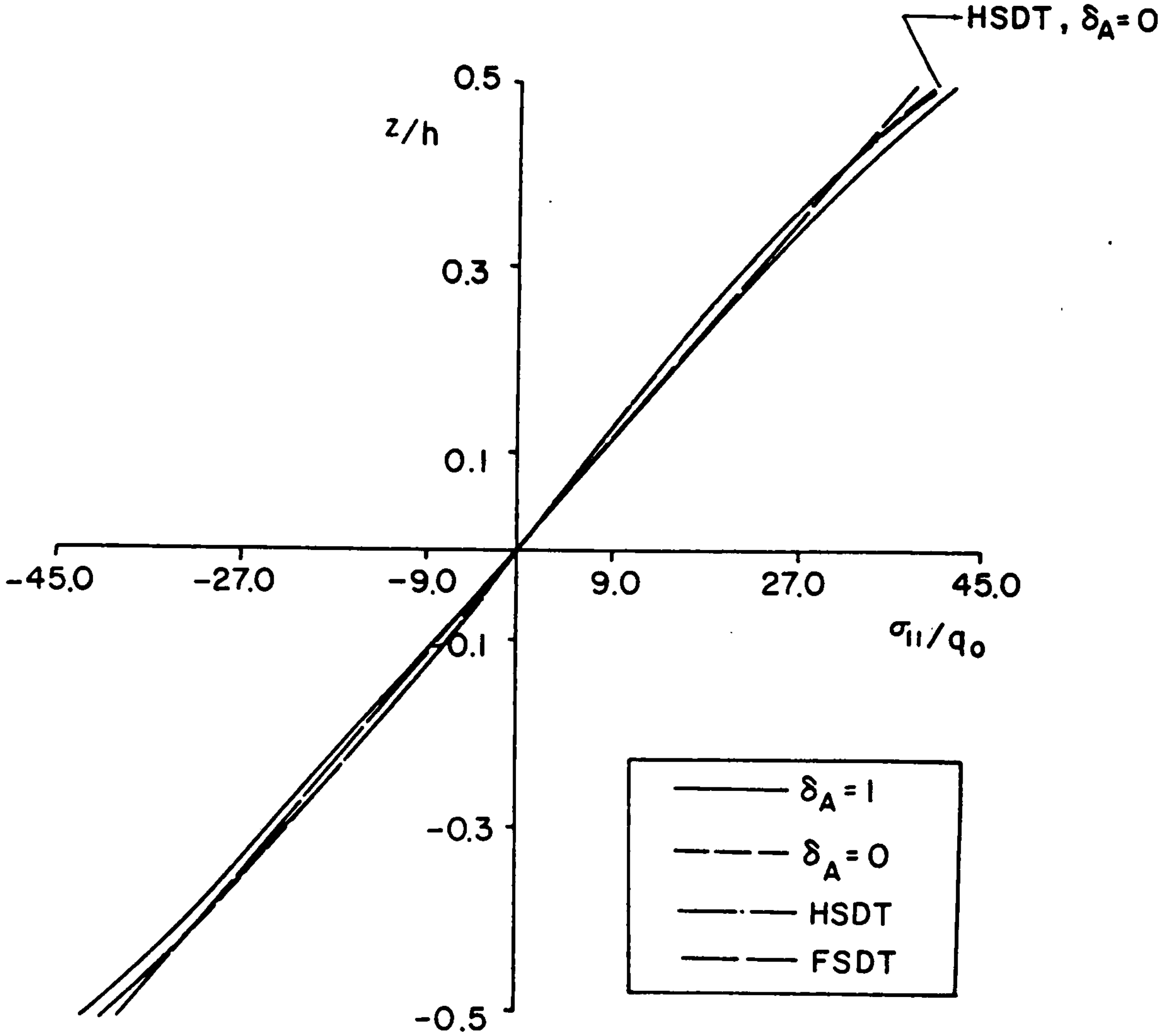


Figure 4.23 Variation of the center normal axial stress through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for FF edge condition under UN load, (a/b = 4, h/a = 0.14, Material II).

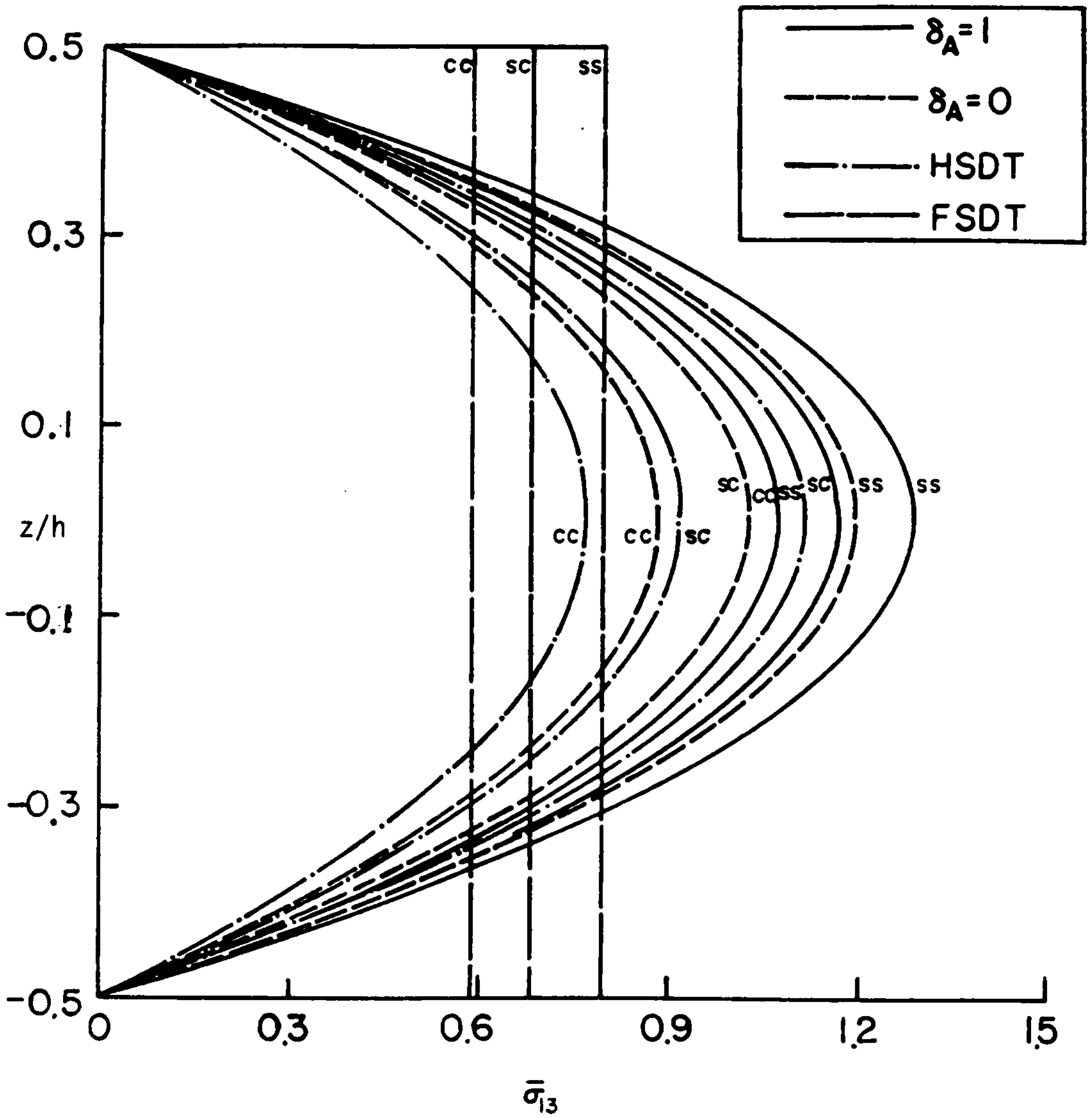


Figure 4.24 Variation of the transverse shear stress ($\bar{\sigma}_{13} = \sigma_{13}/q_o$) through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for CC, SC, SS edge conditions under UN load, (a/b = 4, h/a = 0.14, Material II).

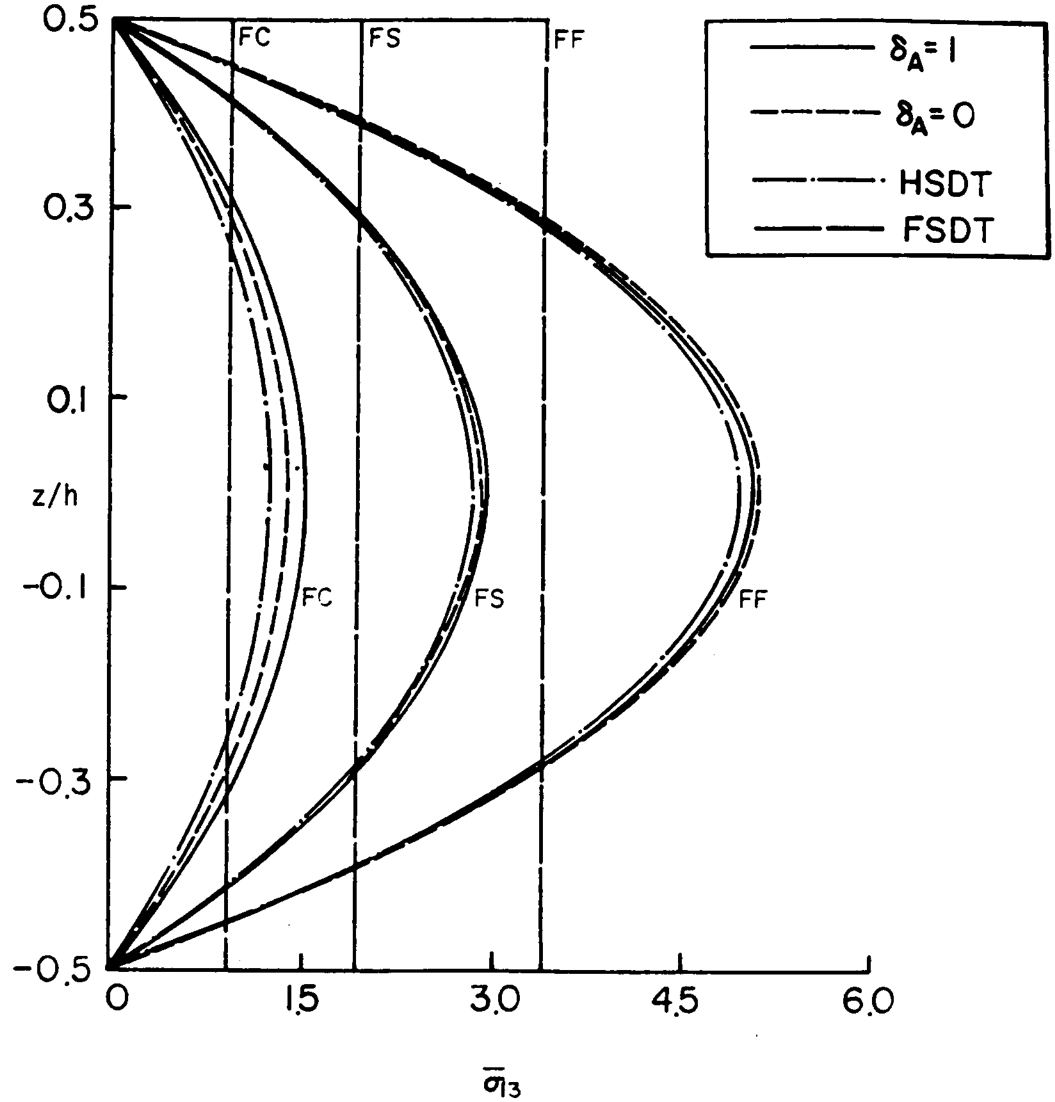


Figure 4.25 Variation of the transverse shear stress ($\bar{\sigma}_{13} = \sigma_{13}/q_o$) through the thickness for DT, ST, FSDT and HSDT when $\delta_A = 0$ and for DT, ST when $\delta_A = 1$ for FC, FS, FF edge conditions under UN load, (a/b = 4, h/a = 0.14, Material II).

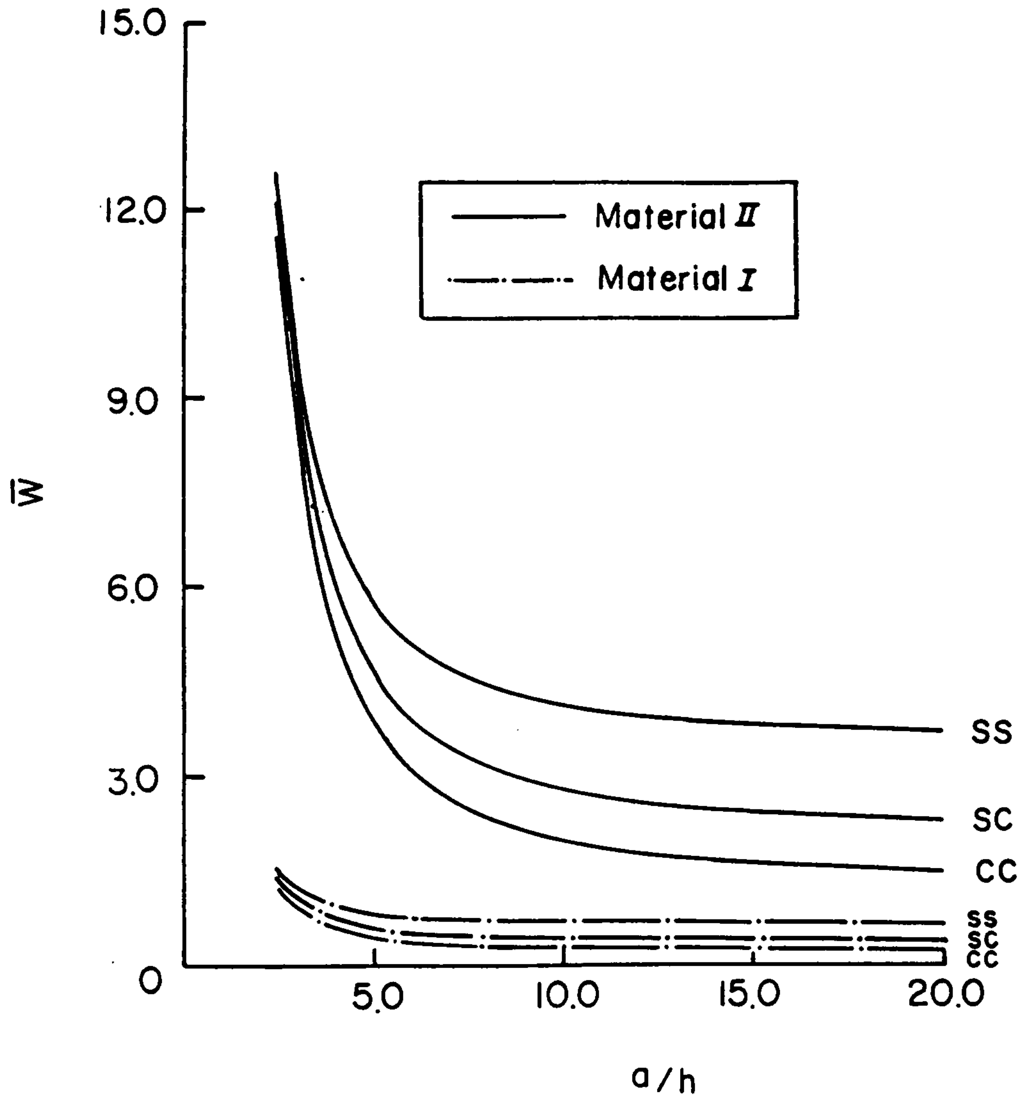


Figure 4.26 Nondimensional center deflection

$$(\overline{W} = \frac{WE_2h^3 10^2}{a^4 q_o},$$

E_2 belonging to Material I) versus side to thickness ratio for DT, ST when δ_A = 1 using two different Materials (I, II) for CC, SC, SS edge conditions under UN load, (a/b = 2).

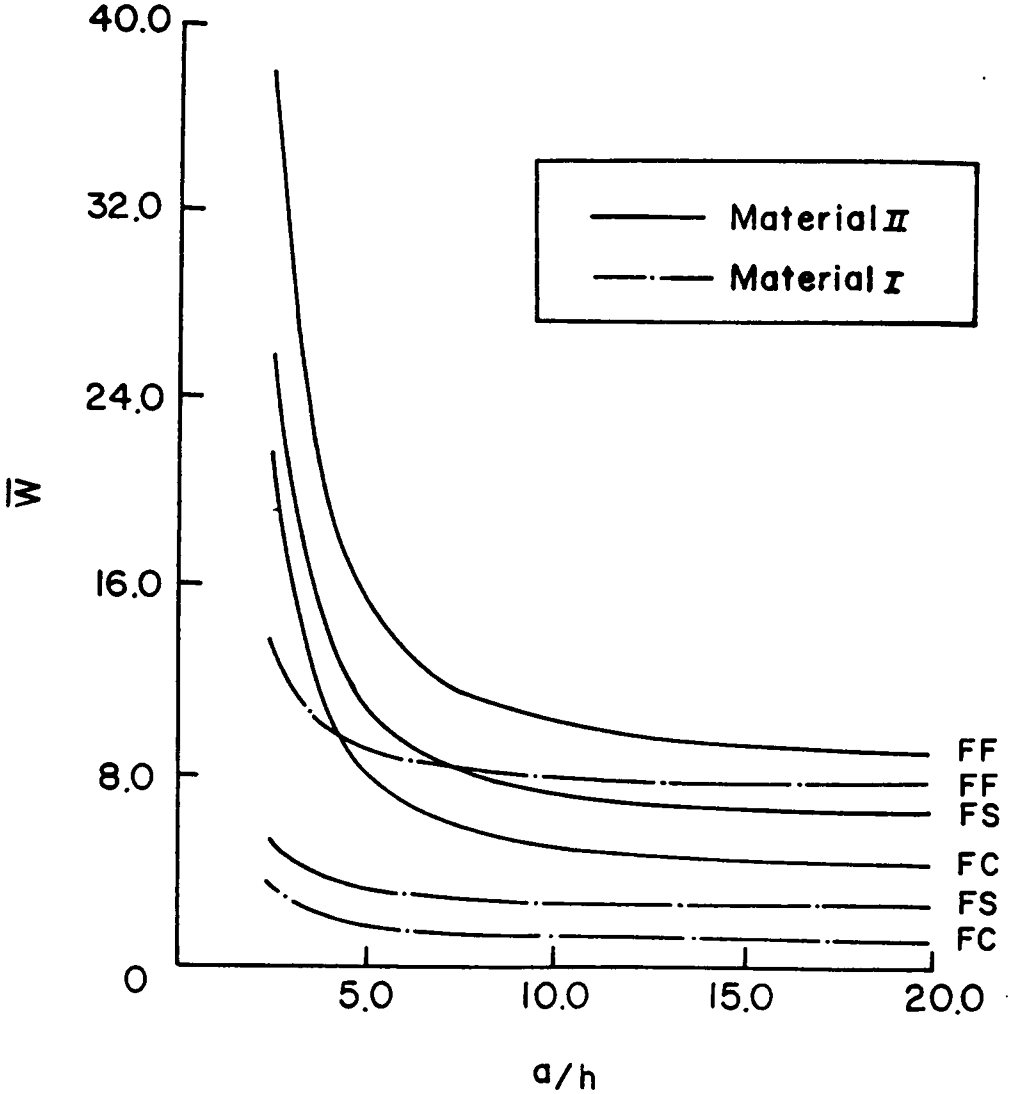


Figure 4.27 Nondimensional center deflection

$$(\overline{W} = \frac{WE_2h^3 10^2}{a^4 q_o},$$

E_2 belonging to Material I) versus side to thickness ratio for DT, ST when $\delta_A = 1$ using two different Materials (I,II) for FC, FS, FF edge conditions under UN load, (a/b = 2).

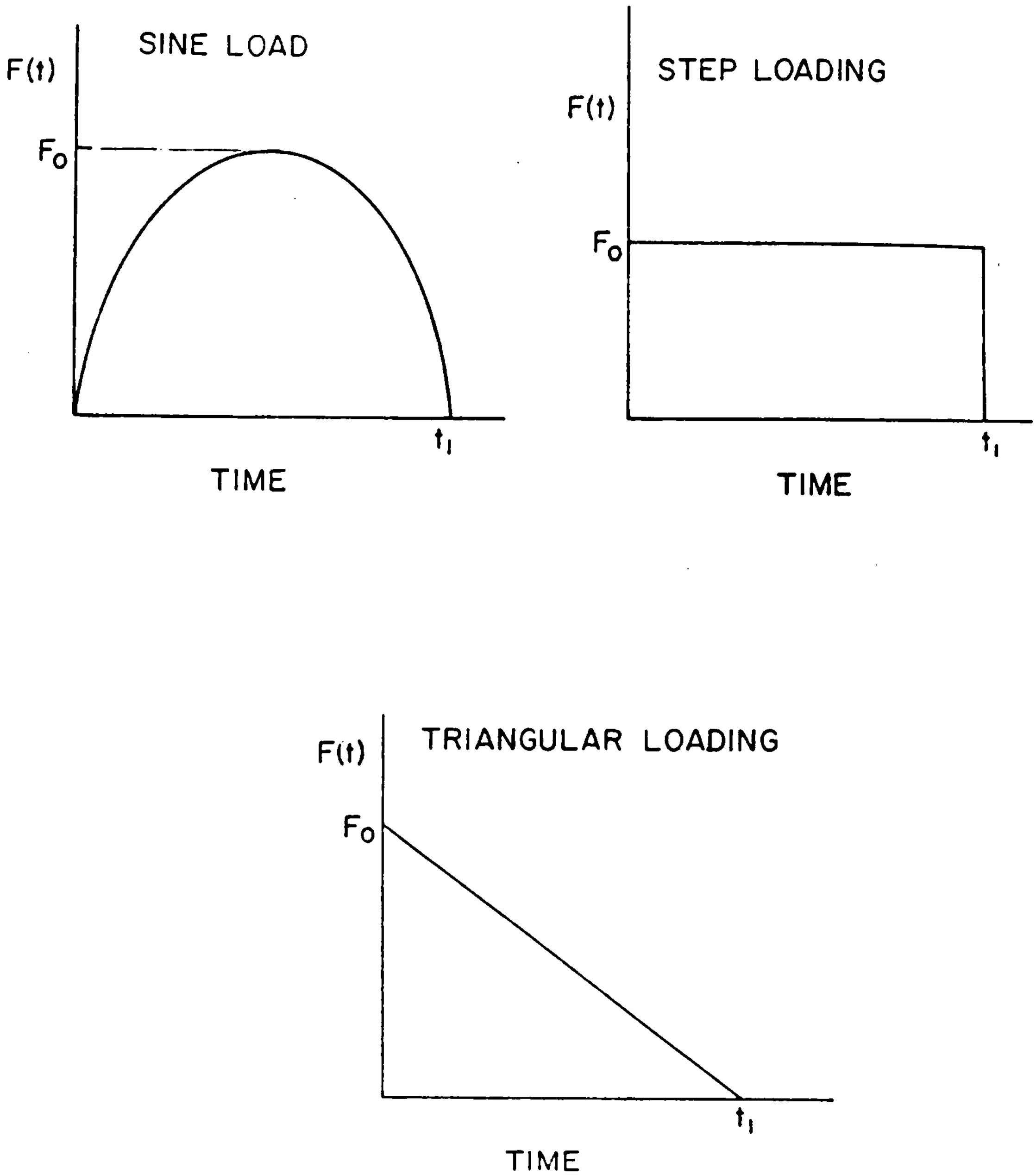


Figure 4.28 Load pulse shapes.

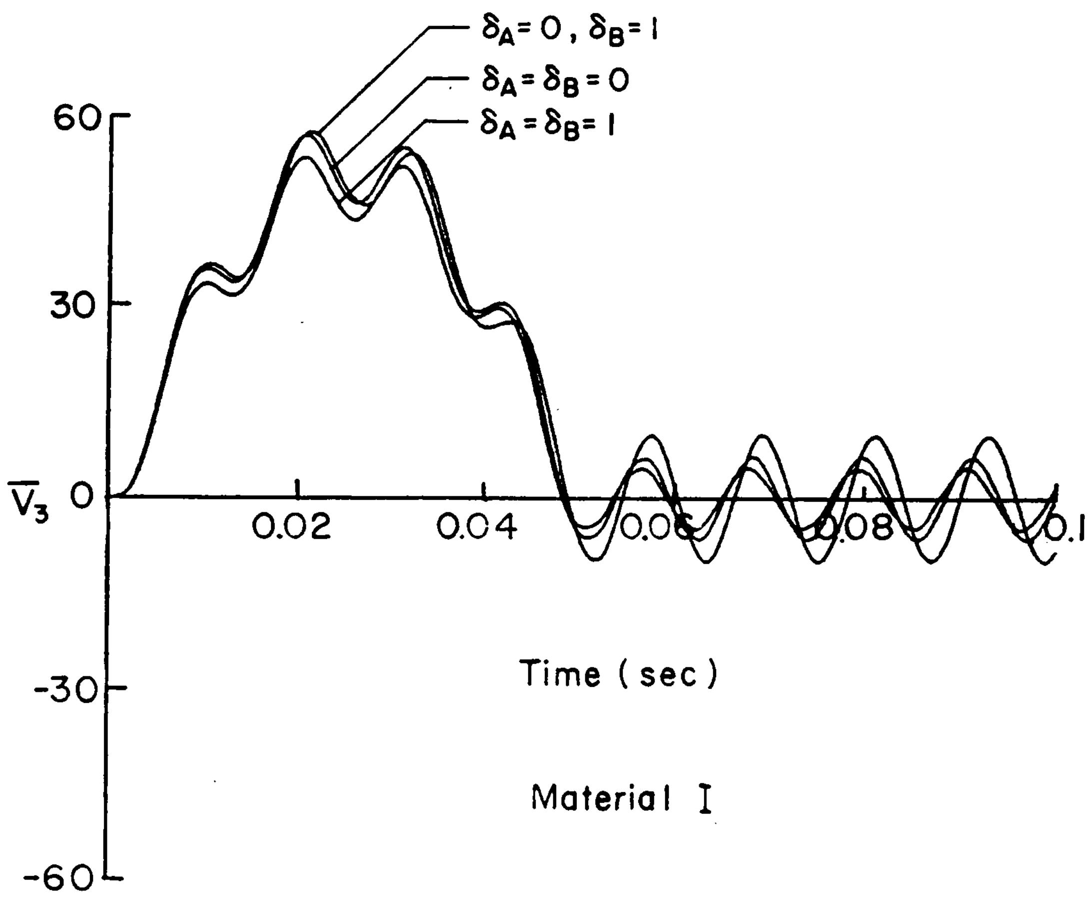


Figure 4.29 Center deflection $\overline{V}_3$ for an orthotropic plate, (sine pulse, $\ell_1/h = 4$).

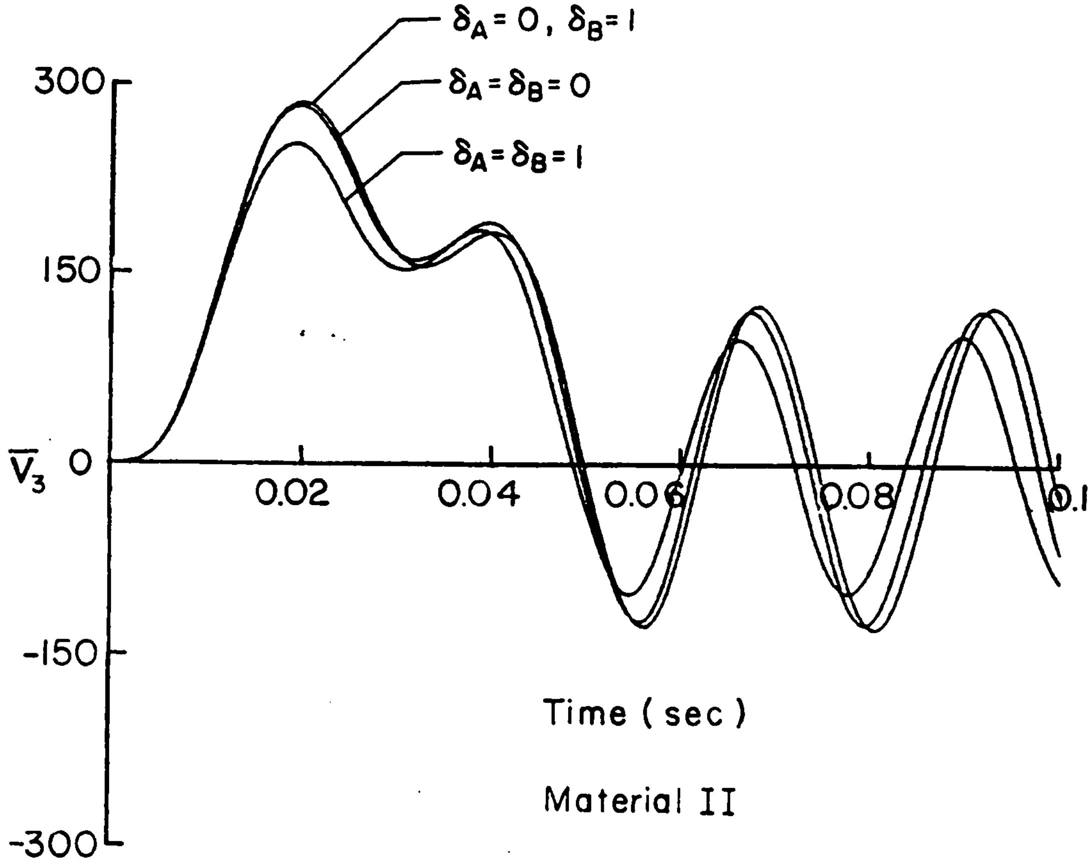


Figure 4.30 Center deflection $\overline{V}_3$ for an orthotropic plate, (sine pulse, $\ell_1/h \geq 4$).

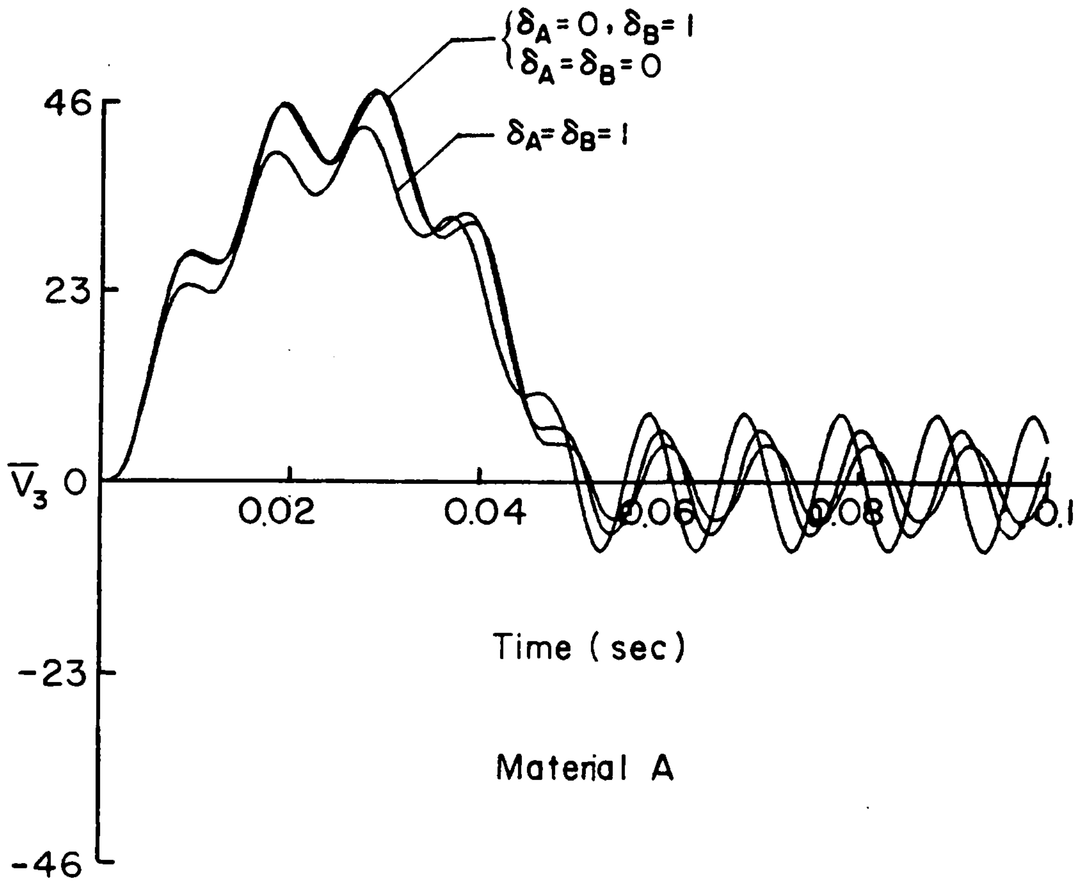


Figure 4.31 Center deflection $\overline{V}_3$ for a transverse-isotropic plate, (sine pulse, $\ell_1/h = 4$).

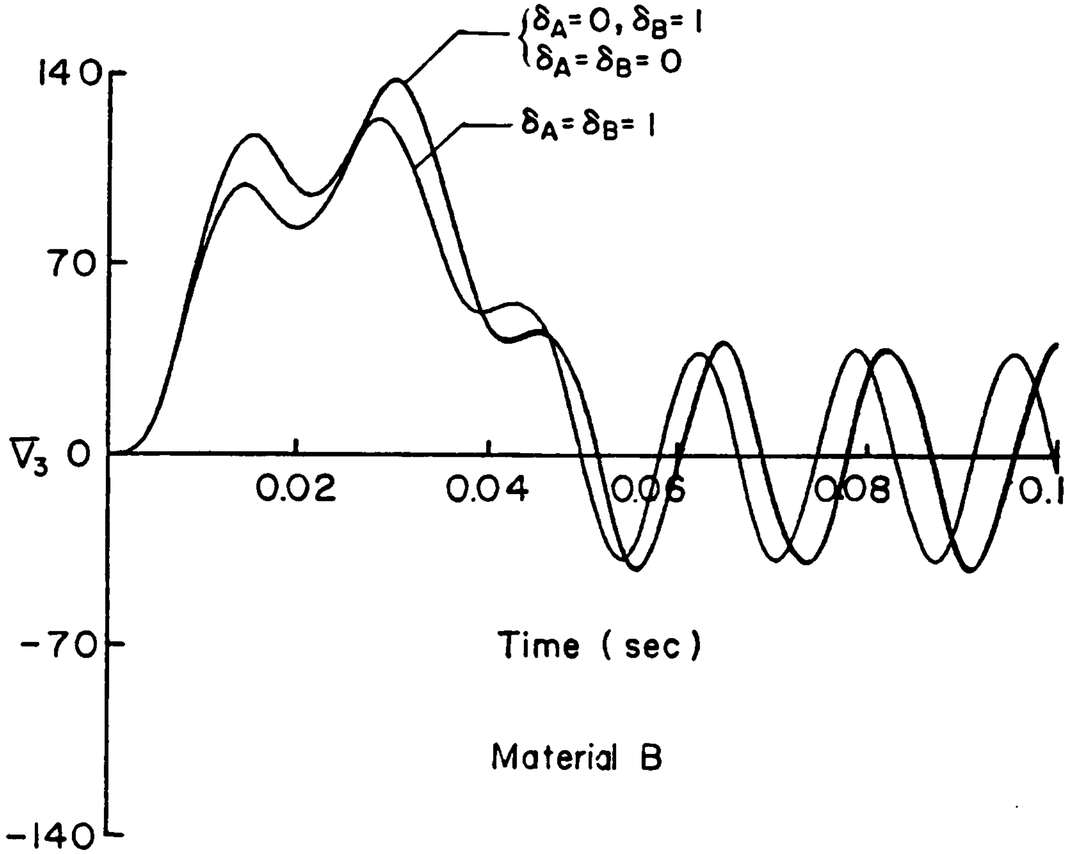


Figure 4.32 Center deflection $\overline{V}_3$ for a transverse-isotropic plate, (sine pulse, $\ell_1/h = 4$).

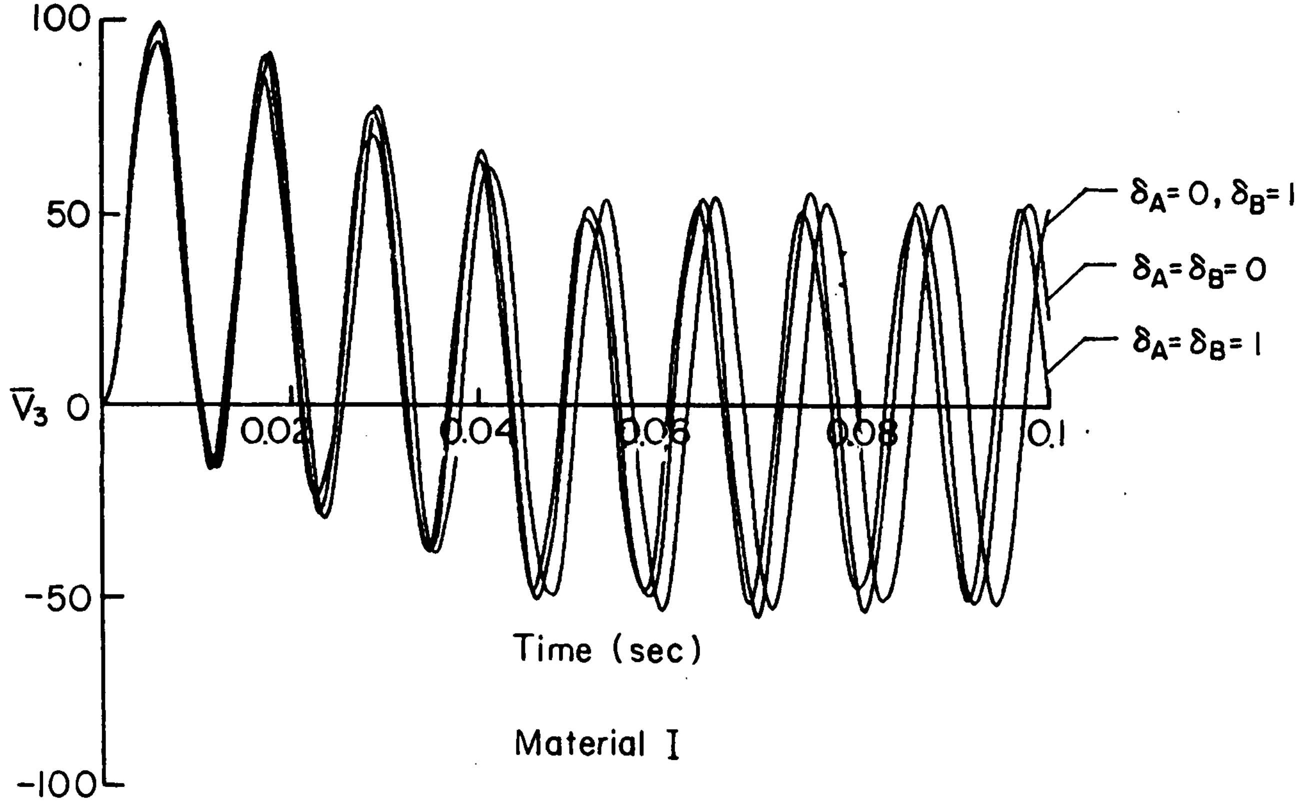


Figure 4.33 Center deflection $\overline{V}_3$ for an orthotropic plate, (triangular pulse, $\ell_1/h = 4$).

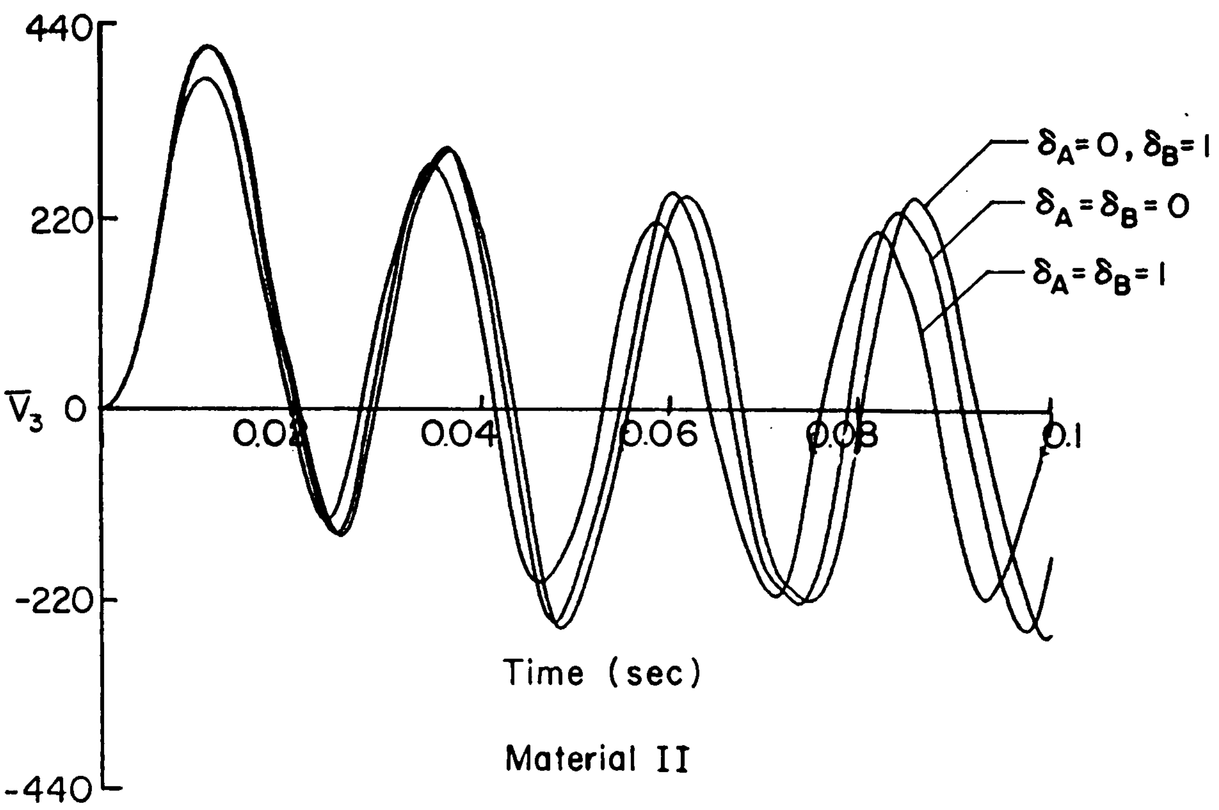


Figure 4.34 Center deflection $\overline{V}_3$ for an orthotropic plate, (triangular pulse, $\ell_1/h = 4$).

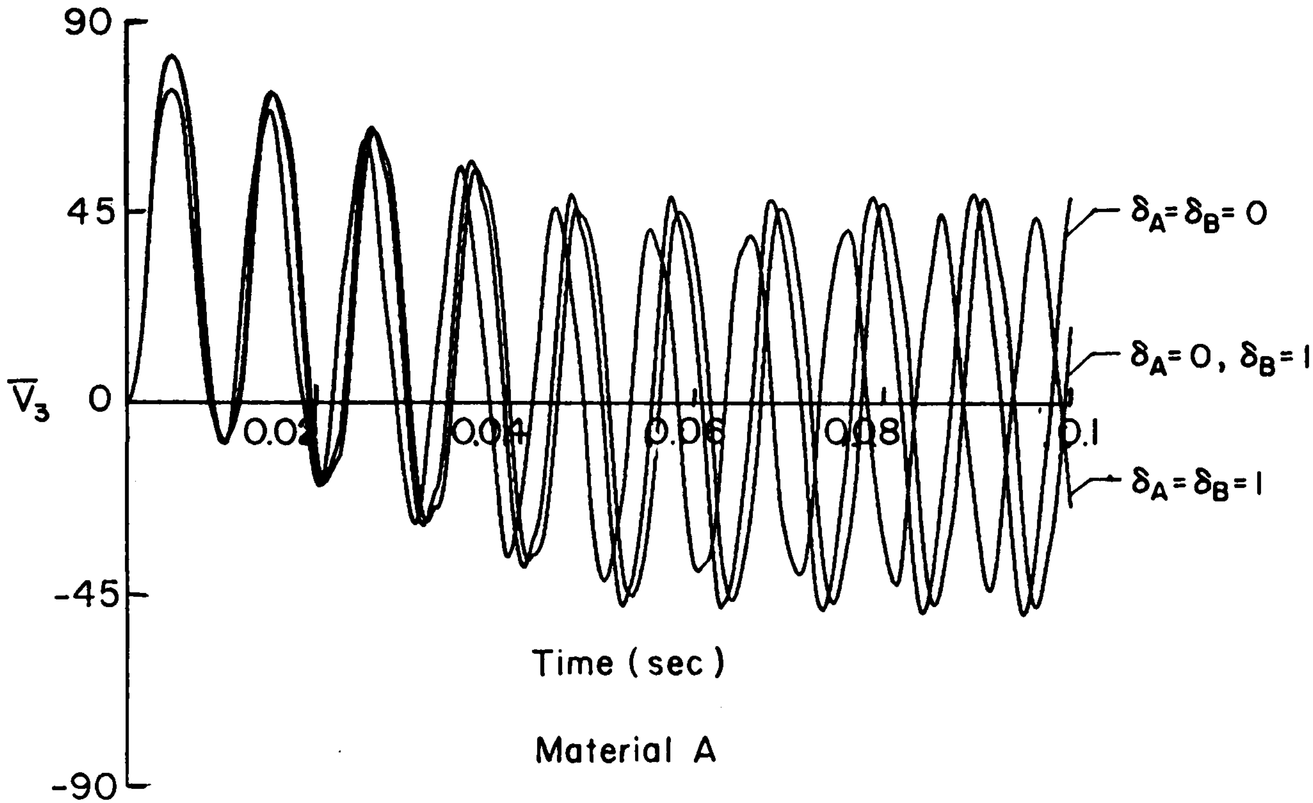


Figure 4.35 Center deflection $\overline{V}_3$ for a transverse-isotropic plate, (triangular pulse, $\ell_1/h = 4$).

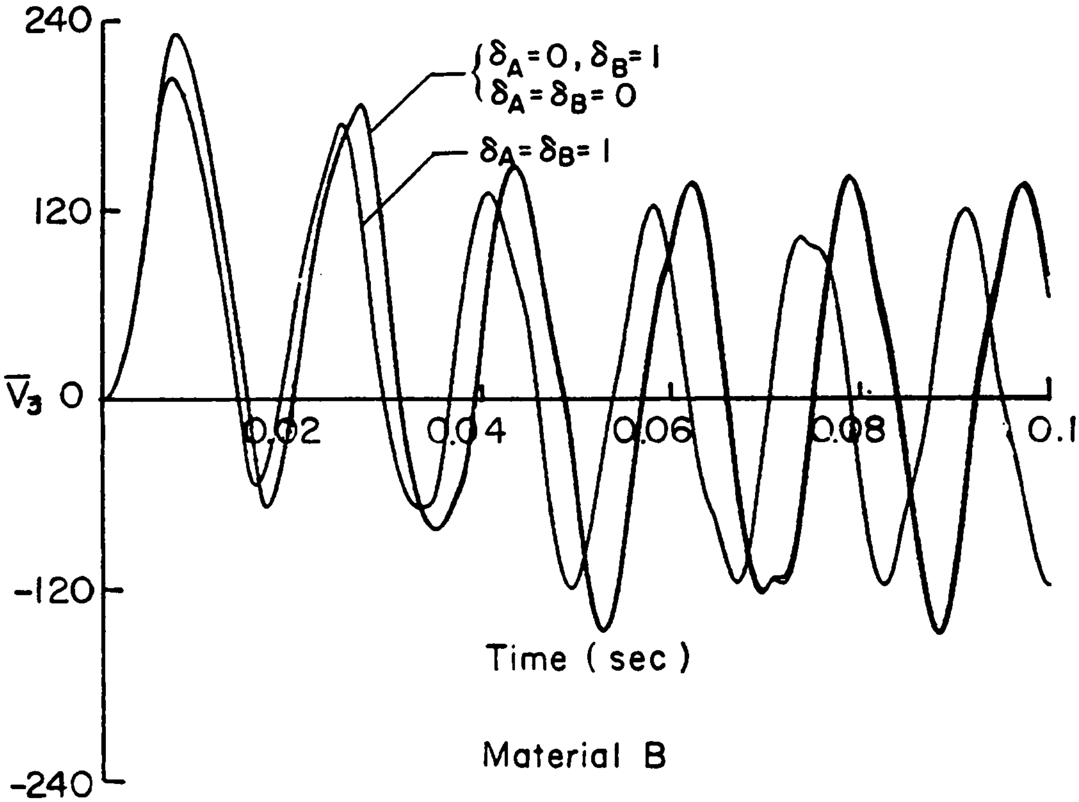


Figure 4.36 Center deflection $\overline{V}_3$ for a transverse-isotropic plate, (triangular pulse, $\ell_1/h = 4$).

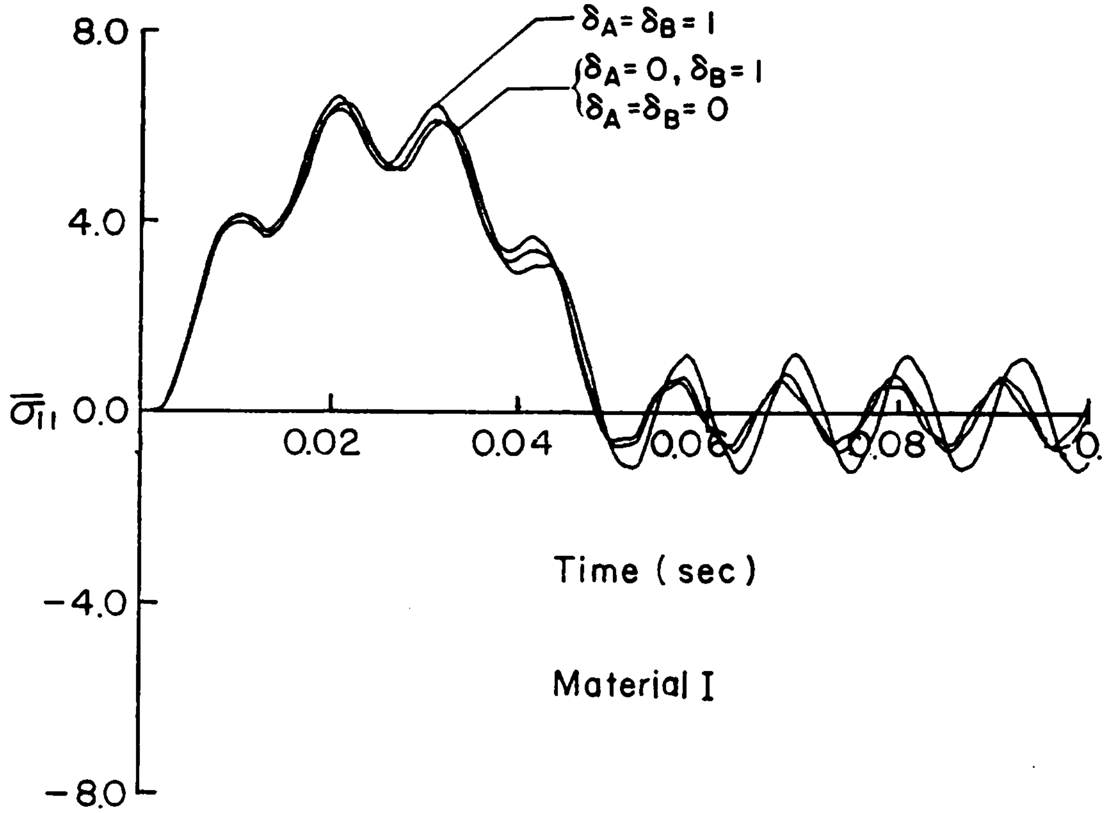


Figure 4.37 Normal axial stress $\bar{\sigma}_{11}$ for an orthotropic plate, (sine pulse, $\ell_1/h = 4$).

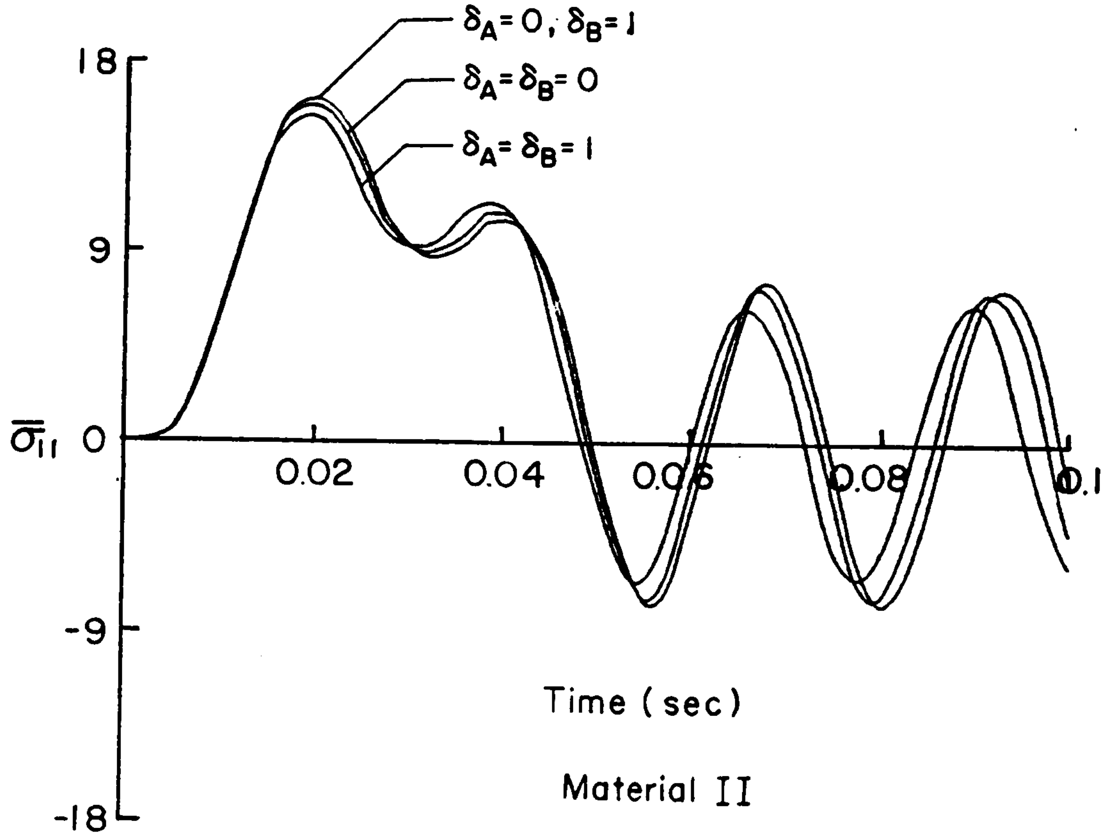


Figure 4.38 Normal axial stress $\bar{\sigma}_{11}$ for an orthotropic plate, (sine pulse, $\ell_1/h = 4$).

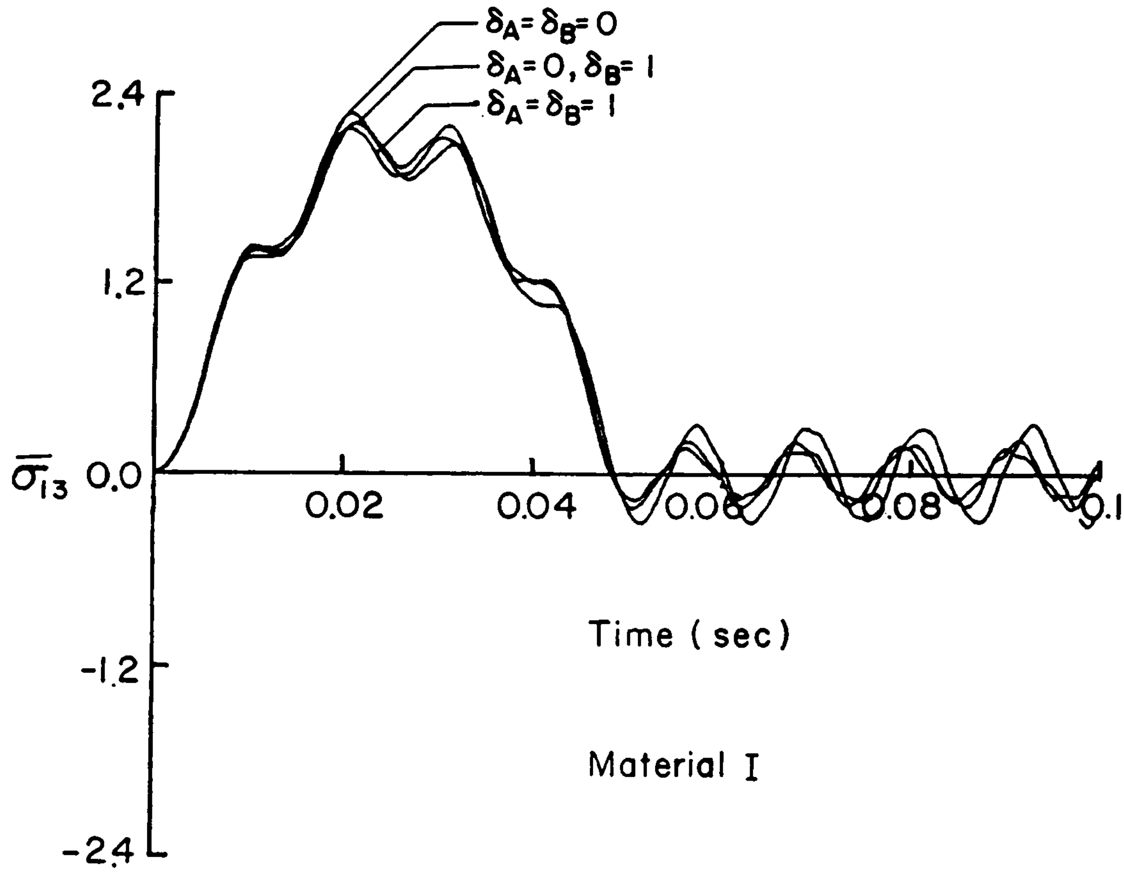


Figure 4.39 Transverse shear stress $\overline{\sigma}_{13}$ for an orthotropic plate, (sine pulse, $\ell_1/h = 4$).

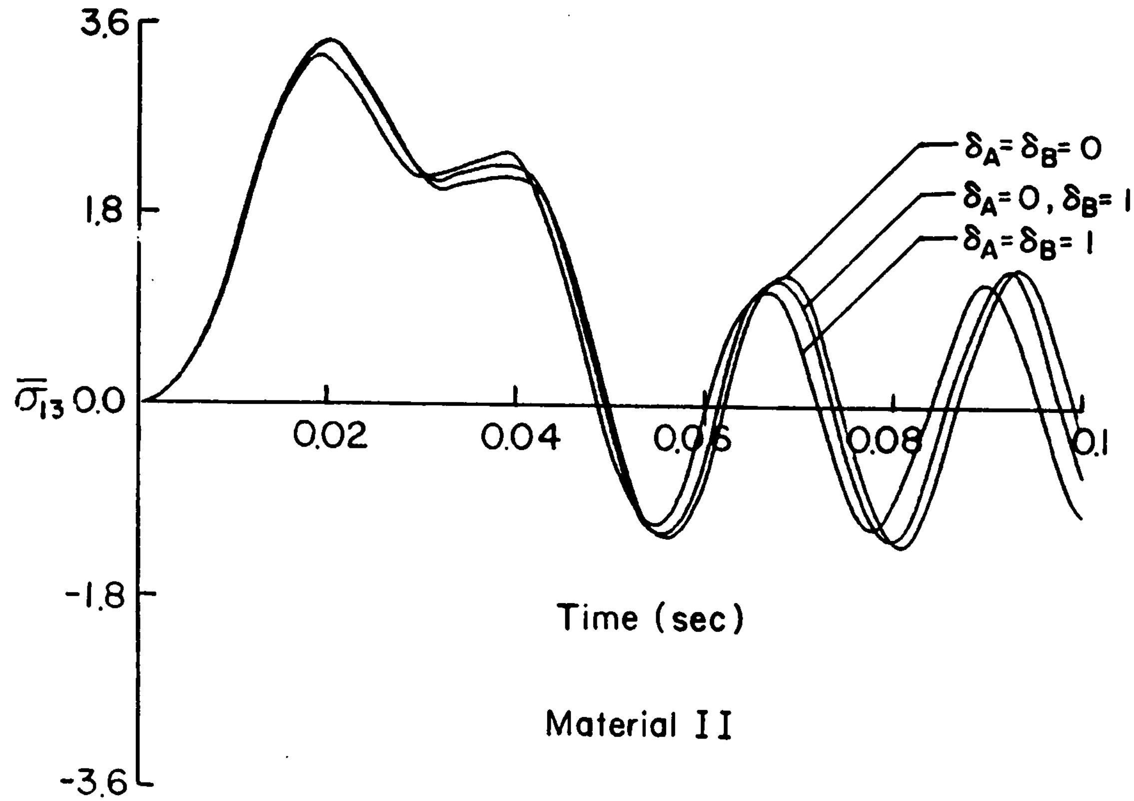


Figure 4.40 Transverse shear stress $\overline{\sigma}_{13}$ for an orthotropic plate, (sine pulse, $\ell_1/h = 4$).

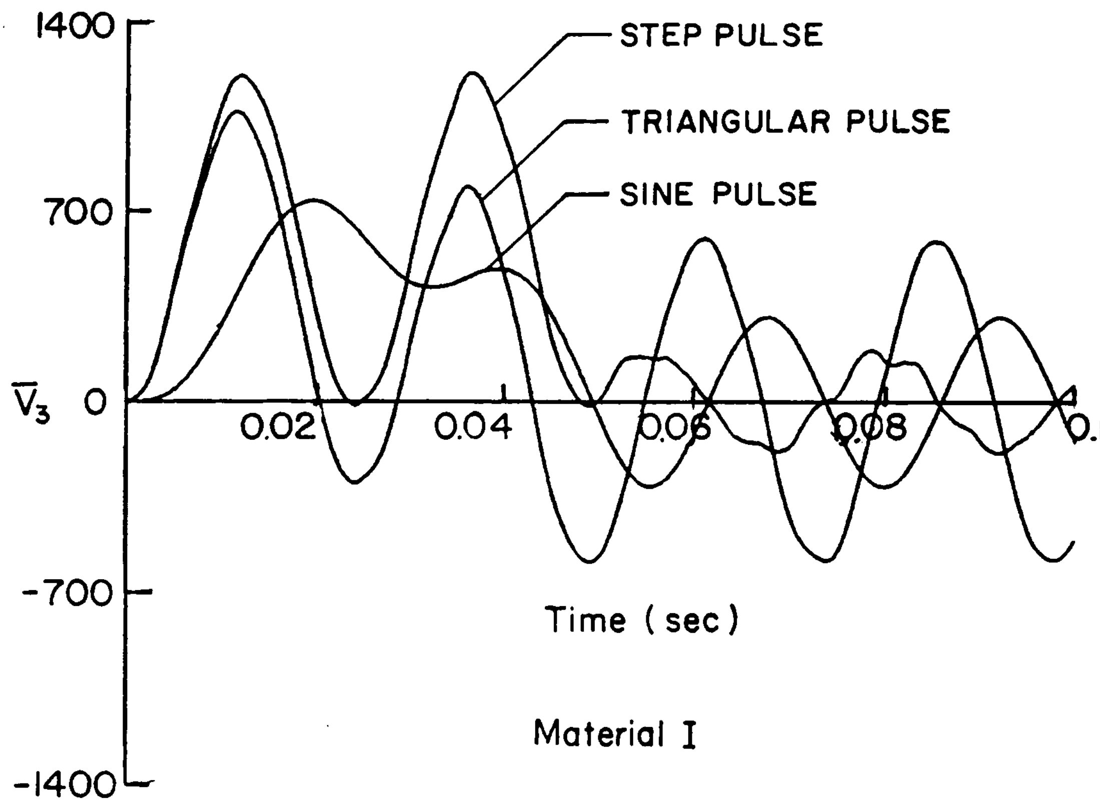


Figure 4.41 Effect of pulse shape on the center deflection $\overline{V}_3$ for an orthotropic plate, ($\delta_A = \delta_\beta = 1$, $\ell_1/h = 10$).

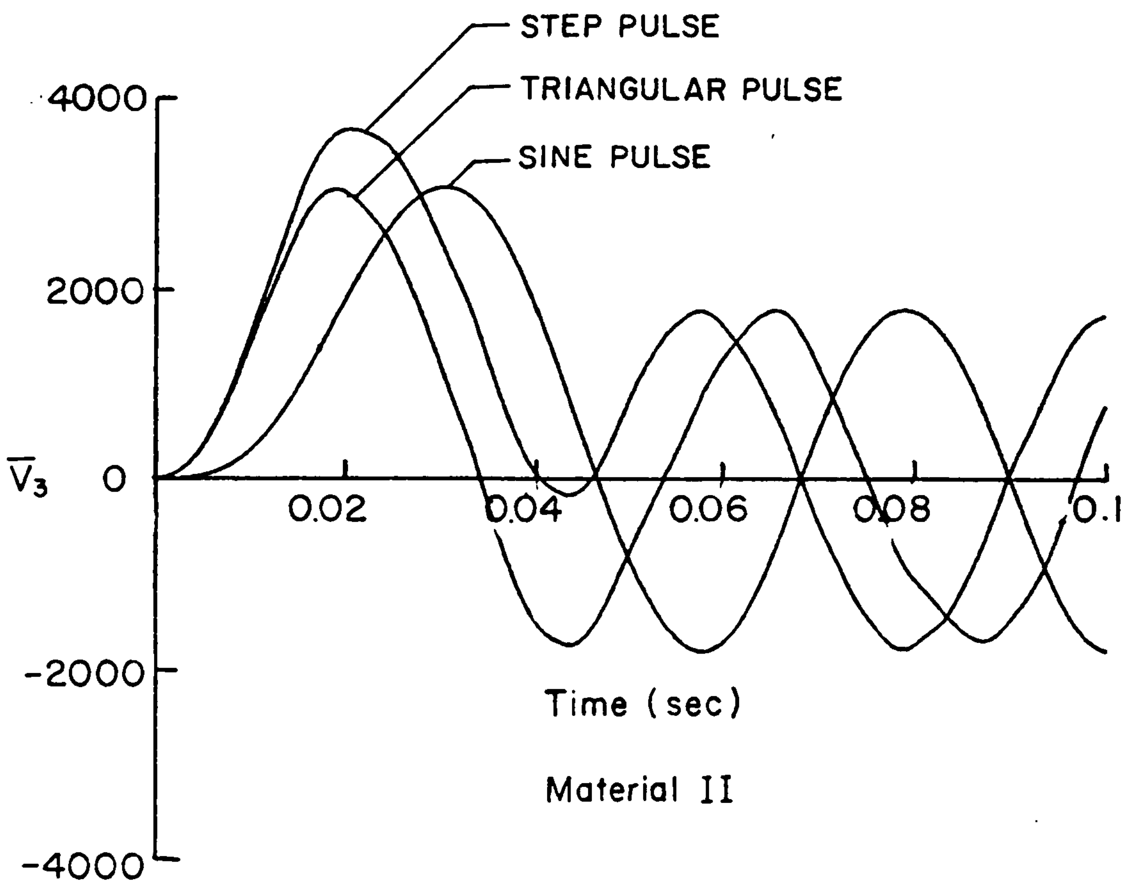


Figure 4.42 Effect of pulse shape on the center deflection $\overline{V}_3$ for an orthotropic plate, ($\delta_A = \delta_\beta = 1$, $\ell_1/h = 10$).

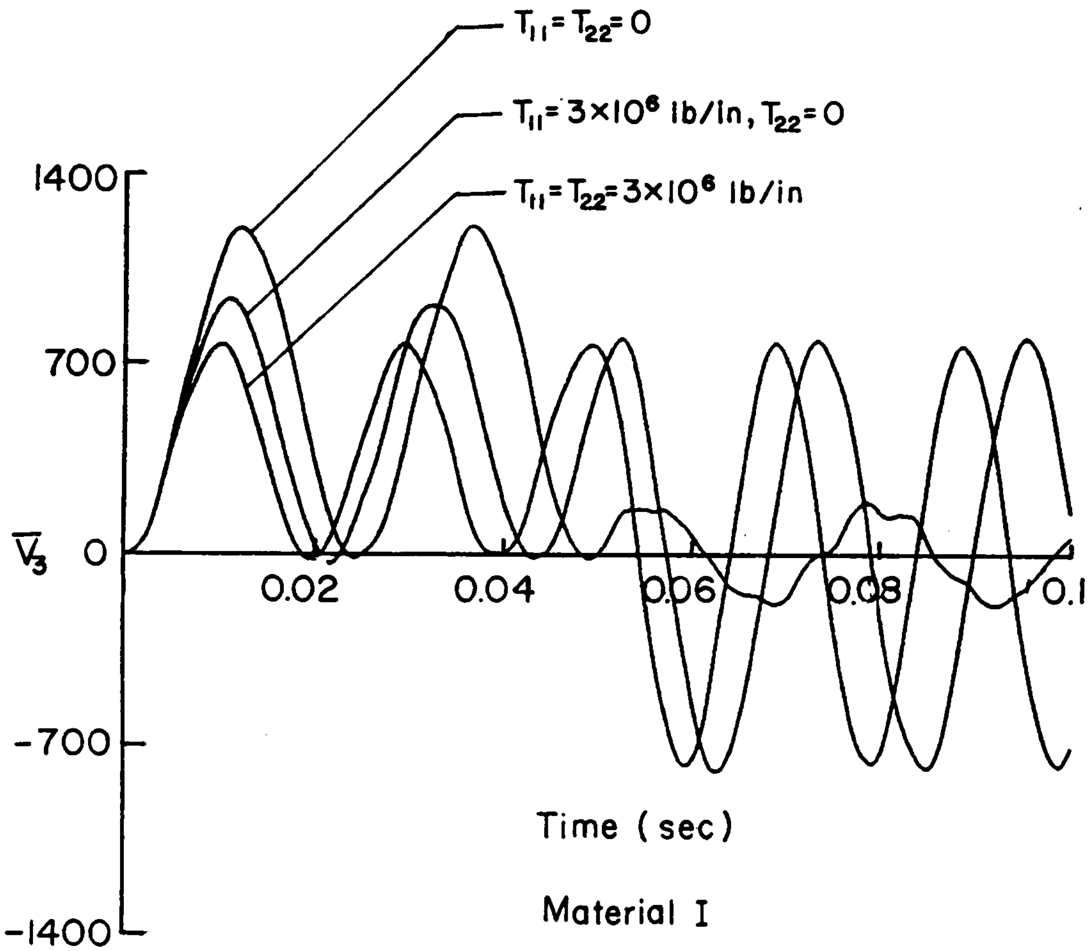


Figure 4.43 Effect of tensile loads on the center deflection $\overline{V}_3$ for an orthotropic plate, (step pulse, $\delta_A = \delta_\beta = 1$, $\ell_1/h = 10$).

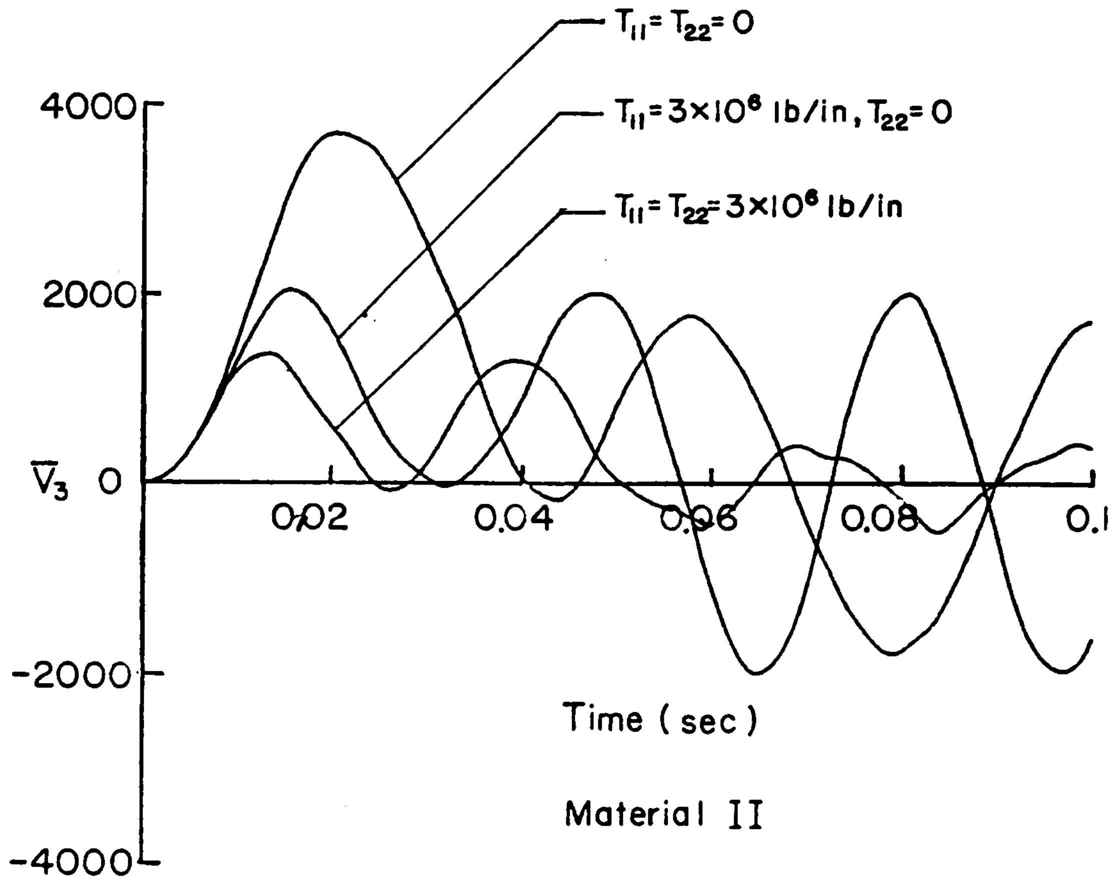


Figure 4.44 Effect of tensile loads on the center deflection $\overline{V}_3$ for an orthotropic plate, (step pulse, $\delta_A = \delta_\beta = 1$, $\ell_1/h = 10$).

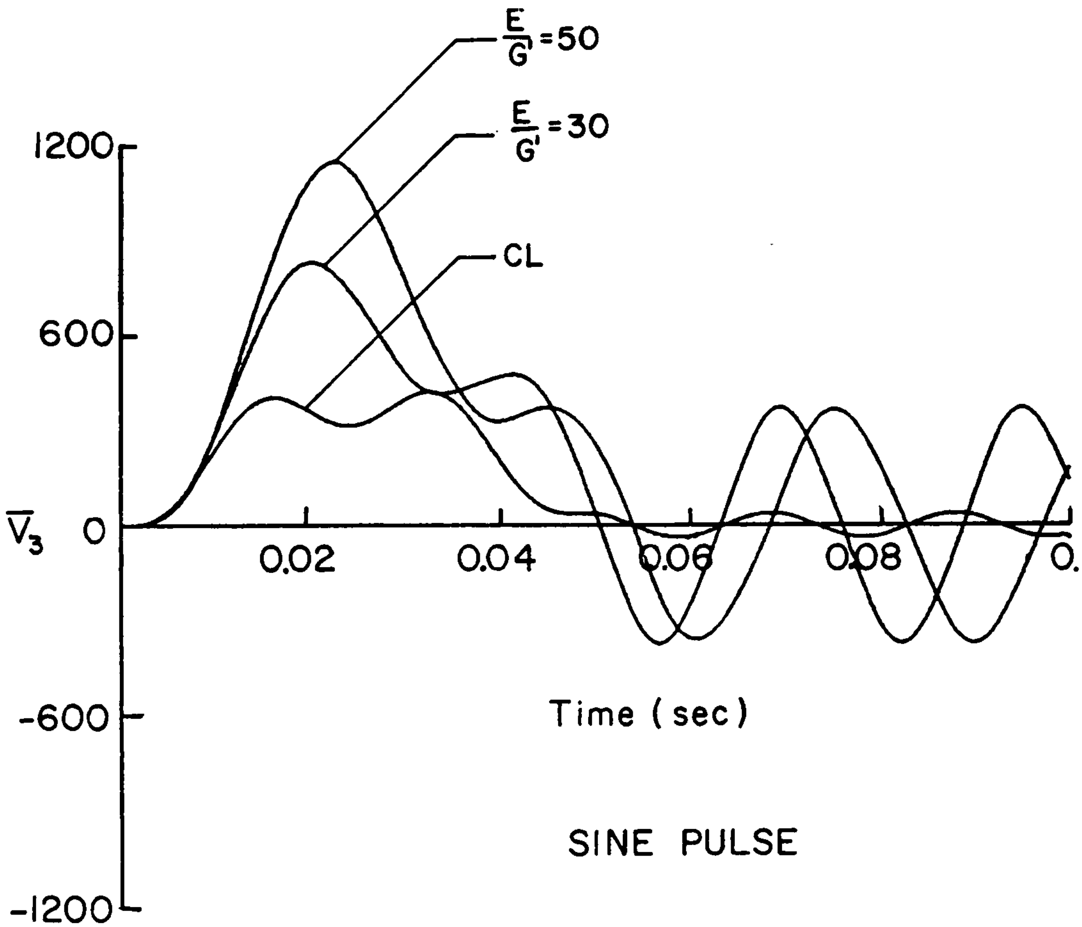


Figure 4.45 Effect of E/G' ratio on the center deflection $\overline{V}_3$ for a transverse-isotropic plate, (Material A, $\delta_A = \delta_\beta = 0$, $\ell_1/h = 10$).

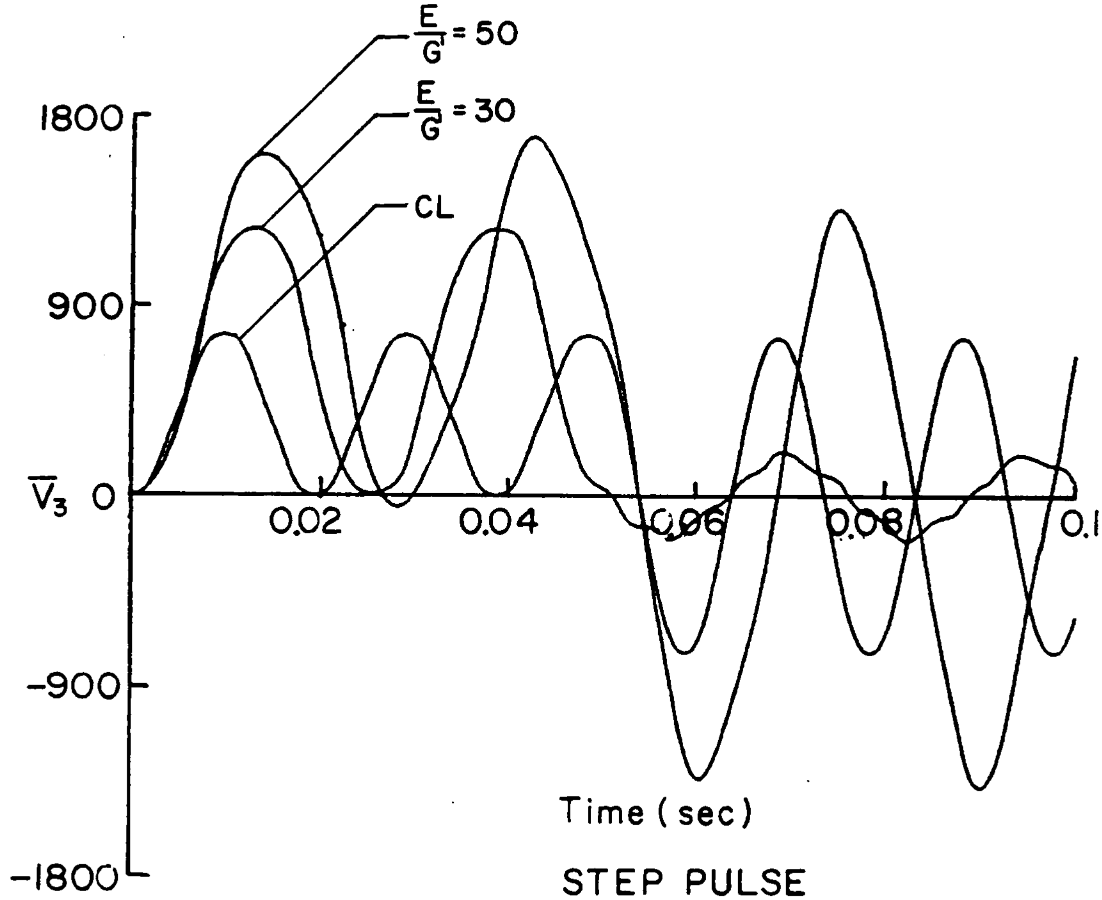


Figure 4.46 Effect of E/G' ratio on the center deflection $\overline{V}_3$ for a transverse-isotropic plate, (Material A, $\delta_A = \delta_\beta = 0$, $\ell_1/h = 10$).

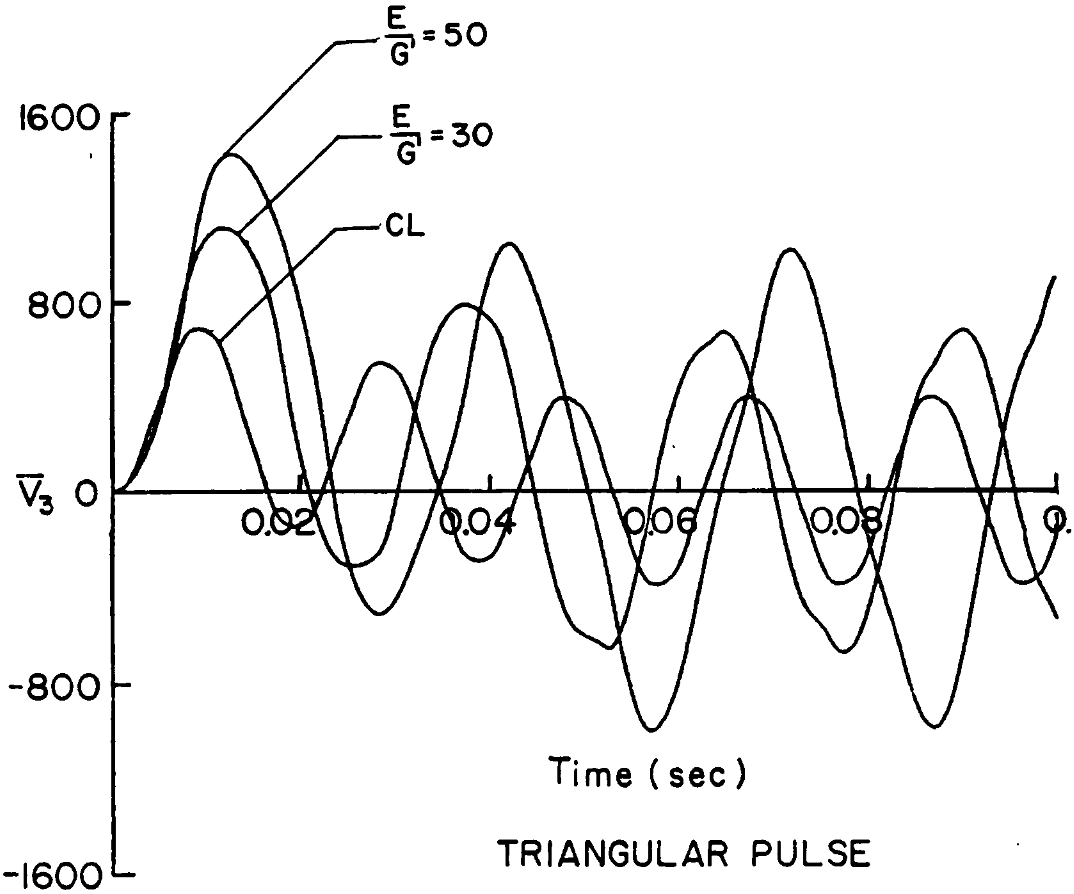


Figure 4.47 Effect of E/G' ratio on the center deflection $\overline{V}_3$ for a transverse-isotropic plate, (Material A, $\delta_A = \delta_\beta = 0$, $\ell_1/h = 10$).

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