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THE PROPERTY B(P,α)-REFINABILITY AND ITS RELATIONSHIP TO GENERALIZED PARACOMPACT TOPOLOGICAL SPACES

by

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(ABSTRACT)

The property B(P,α)-refinability is studied and is used to obtain new covering characterizations of paracompactness, collectionwise normality, subparacompactness, d-paracompactness, d-normality, mesocompactness, and related concepts. These new characterizations both generalize and unify many well-known results.

The property B(P,α)-refinability is strictly weaker than the property θ-refinability. A B(P,α)-refinement is a generalization of a σ-locally finite-closed refinement. Here α is a fixed ordinal which dictates the number of "levels" in a given refinement, and P represents a property such as discreteness or local finiteness which each "level" must satisfy relative to a certain subspace.

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CHAPTER I

INTRODUCTION AND DEFINITIONS

§1. Introduction

There are certain important classes of topological spaces which are fundamental to the content of this thesis. We begin with a brief history of these spaces.

Ever since the class of metric spaces was introduced by M. Frechet in 1906, the notion of metrizability has been extensively studied. One property that all metric spaces possess is normality. The importance of the class of normal spaces was widely recognized in 1925 after both Tietze's Extension Theorem and Urysohn's Lemma were published. As pointed out by M. E. Rudin [68], there was a gap between normality and metrizability which mathematicians of the 1920's tried in vain to fill; in fact, almost a quarter of a century passed before this gap was filled by the class of paracompact spaces. P. J. Dieudonne [25], J. W. Tukey [84], and A. H. Stone [81] played leading roles in initially establishing paracompactness as the candidate for the desired property. In fact, Dieudonne defined paracompactness in 1944 as a natural generalization of compactness and then proved that certain restricted classes of metrizable spaces are paracompact. Several years earlier Tukey had defined full normality, a property stronger than normality, and proved that every metrizable space is fully normal. The stage was then set for A. H. Stone, who in 1948 proved that the properties paracompactness and full normality are equivalent in the class of T_1- spaces, and hence every metrizable space is paracompact. After Stone's discovery, paracompactness was used to generalize many theorems from analysis involving metrizability.

1.1.1. <u>Diagram</u>.

$$
\begin{tikzcd}
& \text{metrizable} \arrow[dl] \arrow[dr] & \\
\text{paracompact} \arrow[r] \arrow[d] \arrow[dr] & \text{subparacompact} & \text{developable} \arrow[l] \\
\text{metacompact} & \text{CWN} \arrow[r] & \text{normal}
\end{tikzcd}
$$

Collectionwise normality (CWN), metacompactness, subparacompactness, and developability are four fundamental properties which were introduced in a single paper of R. H. Bing [6] in 1951 entitled "Metrization of Topological spaces." Their relationships to paracompactness are given in the diagram above. There is a very close tie between Bing's paper and earlier works of R. L. Moore and F. B. Jones. Indeed, in 1916 R. L. Moore [58] defined the class of Moore spaces--that is, the class of developable

T_3- spaces. Although Moore spaces satisfy a rather stringent base property that guarantees $1^{\underline{st}}$ countability, they are in general far from being metrizable in the sense that not every Moore space is normal. Since paracompactness implies normality but not $1^{\underline{st}}$ countability, the class of paracompact spaces and the class of Moore spaces are not related in general. In 1937 F. B. Jones [39] published a paper which contained what is now referred to as the "normal Moore space conjecture," asking whether every normal Moore space is metrizable. In the same paper Jones proved that every separable normal Moore space is metrizable provided a certain axiom of set theory independent of the usual Zermel-Fraenkel with Choice (ZFC) axioms is assumed. Over the last fifty years the study of metrizability of normal Moore spaces has been vast. The nature of the problem continues to change as new axioms of set theory are incorporated in solution attempts. It is still unknown whether there exists a normal nonmetrizable Moore space under the usual ZFC set theory assumptions. R. H. Bing gave a partial solution of the normal Moore space conjecture in his 1951 paper by introducing the class of developable spaces and the property collectionwise normality (CWN). He then proved that every CWN, developable, T_1- space is metrizable. The notion of developability also led to Bing's introduction of metacompactness and subparacompactness in 1951. What we now refer to as metacompactness was originally called "point-wise paracompactness" by Bing, who showed that metacompact, developable, T_1- spaces are not necessarily paracompact. R. Arens and J. Dugundji [2] independently introduced the notion of metacompactness in 1950 and proved that countable compactness and compactness are equivalent in this class of metacompact spaces. In 1955, E. Michael [57] and K. Nagami [61] independently proved that every CWN metacompact space is paracompact. The notion of subparacompactness was implicitly introduced in Bing's 1951 paper where it was shown that every open cover of a developable space has a σ-discrete-closed refinement; hence, every developable space is subparacompact. L. F. McAuley [56] has referred to this property as F_σ-screenability. A related notion called σ-paracompactness was introduced in 1966 by A. V. Arkhangel'skii [4]. In 1969, D. K. Burke [16] proved that F_σ-screenability and σ-paracompactness are equivalent properties which characterize what he referred to as subparacompactness. He also provided examples to show in general that subparacompactness and metacompactness are not related.

In 1965, J. M. Worrell and H. Wicke [87] introduced θ-refinability, a generalization of both metacompactness and subparacompactness, and proved that a space X is developable iff X is θ-refinable and $2^{\underline{nd}}$ countable. Later, θ-refinability was called submetacompactness by H. J. K. Junnila [40], and a number of important results were obtained. In this thesis we will continue to use the name θ-refinability. In 1972, H. R. Bennett and D. J. Lutzer [5] generalized θ-refinability to weak θ-refinability and proved that in a perfect space, weak θ-refinability and subparacompactness are equivalent. It is also known that countable compactness and compactness are equivalent in the class of weak θ-refinable spaces. In 1975, J. C. Smith [70] introduced the notion of weak $\bar{\theta}$-refinability, a property which lies strictly between weak θ-refinability and θ-refinability, and showed that the class of metacompact spaces is exactly the class of almost expandable, weak $\bar{\theta}$-refinable spaces. Also, Smith [74] proved that CWN, weak $\bar{\theta}$-refinable spaces are paracompact.

In 1965, A. V. Arkhangel'skii [3] implicitly introduced the notion of mesocompactness in an attempt to find a relationship between the class of k-spaces and the class of paracompact spaces. He proved that a normal k-space X is paracompact iff X is mesocompact. J. R. Boone [7] later gave the concept mesocompactness its name and showed that mesocompactness lies strictly between paracompactness and metacompactness.

Both normality and paracompactness have external characterizations which depend upon the existence of special continuous maps to metrizable spaces (see chapter IV). The natural "d-versions" of these properties are obtained by changing the range spaces from metrizable spaces to developable spaces. C. M. Pareek [65] introduced the class of d-paracompact spaces in 1972, and H. Brandenburg [12] the class of d-normal spaces in 1981. H. Brandenburg [10, 11, 12, 13, 14, 15], J. Chaber [23], and N. C. Heldermann [35] have recently discovered a number of internal characterizations of d-normal and d-paracompact spaces which are analogous to well-known characterizations of normal and paracompact spaces, respectively.

The diagram at the end of this section includes general relationships between the covering properties mentioned thus far.

For the last forty years, literature dealing with covering properties in general topology has been saturated with generalizations of paracompactness as well as new refinement techniques which have evolved from studying paracompactness. Much of our work not only generalizes, but also offers a unified approach to well-known results in this area. The aim of this thesis is to characterize the covering properties in diagram 1.1.2 below using weaker conditions than those appearing in existing characterizations. The leading role in this endeavor is played by the notion of a $B(P,\alpha)$-refinement defined by J. C. Smith [77] in 1980. This concept is a generalization of a σ-locally finite refinement. Here α represents a fixed ordinal and P is any one of several properties which collections of sets may satisfy, such as discreteness (D) or local finiteness (LF). The results in this thesis should help establish the usefulness of the property $B(P,\alpha)$-refinability when dealing with a variety of covering properties.

In chapter II we study the class of $B(P,\alpha)$-refinable spaces. The strongest $B(P,\alpha)$-property which we deal with is $B(D,\omega)$-refinability and the weakest is $B(C,\alpha)$-refinability. Most of the general relationships given in diagram 1.1.2 at the end of this section, which involve $B(P,\alpha)$-refinability, were established by J. C. Smith [77].

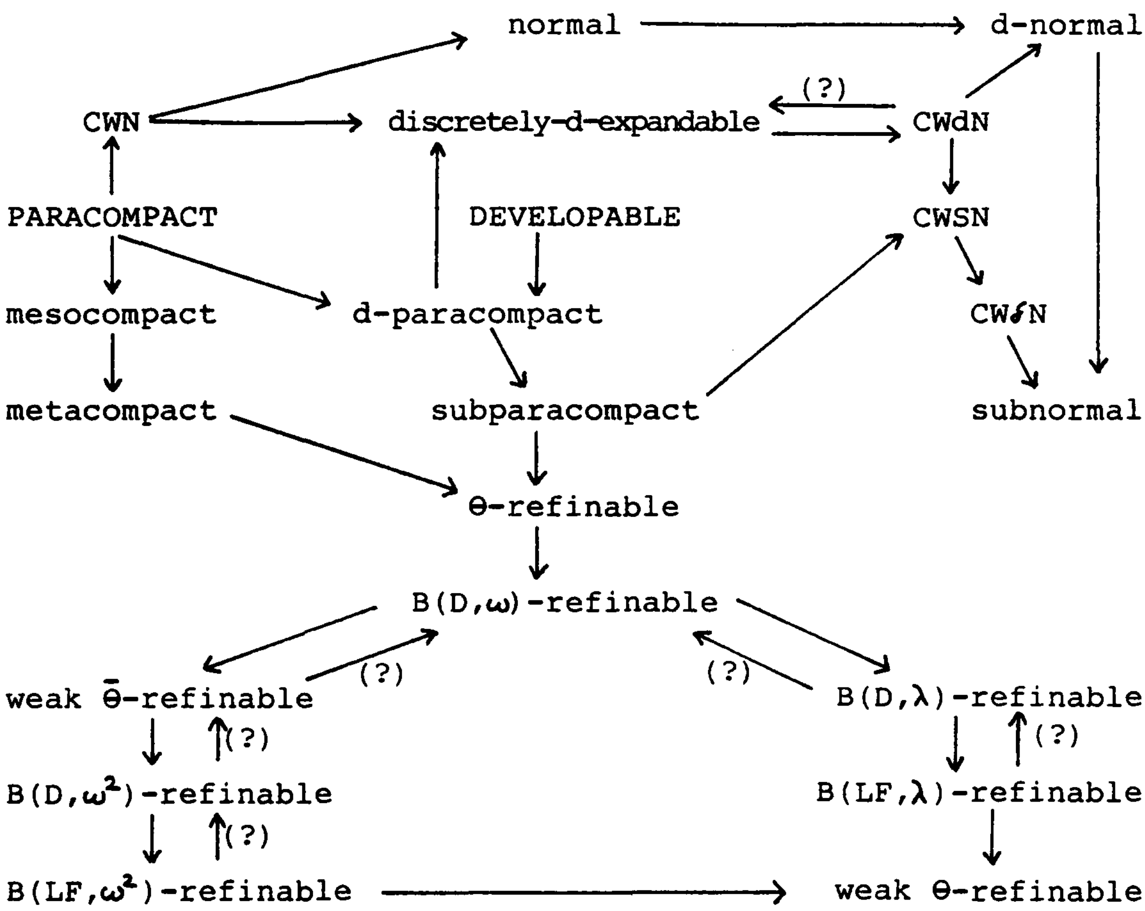
In §1 of chapter II we obtain a weak $\bar{\theta}$-type characterization of $B(D,\omega)$-refinability and use it to show that

$$\theta\text{-refinable} \underset{\not\longleftarrow}{\longrightarrow} B(D,\omega)\text{-refinable}.$$

We also establish conditions under which $B(LF,\alpha)$-refinability and $B(D,\alpha)$-refinability are equivalent. In §2 we prove the equivalence of countable paracompactness to countable $B(C,\lambda)$-refinability in the class of normal spaces and show that a CWN space X is paracompact iff X is $B(LF,\lambda)$-refinable. In §3 of chapter II we establish several fundamental sum and mapping theorems which $B(P,\alpha)$-refinable spaces satisfy.

Our main objective in the remaining three chapters of this thesis is to establish useful $B(P,\alpha)$-characterizations of subparacompactness (chapter III), d-paracompactness and d-normality (chapter IV), and mesocompactness (chapter V).

1.1.2. Diagram.



The diagram above illustrates the general relationships between covering properties studied in this thesis. It is known that other implications, not indicated here, are not generally true. Those with a "?" represent open problems.

§2. Definitions

In this section we explain the terminology and notation used in this thesis and give definitions and related lemmas which are basic to the subject matter. For the meaning of concepts used without definition in this work, we refer the reader to the texts [29] and [85].

Throughout the following, the word "space" always refers to a T_1- topological space--that is, a space in which each singleton is a closed set. If H is a subset of a space X, we denote the closure of H by "cl(H)" and the interior of H by "int(H)."

The abbreviations "TFAE" and "iff" are used to represent the phrases "the following statements are equivalent," and "if and only if," respectively.

We denote the cardinality of a set A by "$|A|$." The symbol "N" represents the set of positive integers. The cardinality of the set of real numbers is c, and $\aleph_0 = |N|$. The first infinite ordinal is ω , and ω_1 is the first uncountable ordinal. We will represent ordinal numbers with lower case Greek letters, and elements of N with lower case English letters. The letter λ will always denote a countably infinite ordinal, and α a general infinite ordinal.

1.2.1. Set operations. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of a space X, and let $A \subset X$. Then

(a) $\cup\mathcal{U} = \cup\{U : U \in \mathcal{U}\}$,

(b) $\cap\mathcal{U} = \cap\{U : U \in \mathcal{U}\}$,

(c) $\mathrm{st}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \text{ hits } A\}$,

(d) $\mathrm{st}(x,\mathcal{U}) = \mathrm{st}(\{x\},\mathcal{U})$ for each $x \in X$, and

(e) $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

1.2.2. <u>Definition</u>. Let $\mathcal{H}$ and $\mathcal{U}$ be collections of sets. The collection $\mathcal{H}$ <u>partially refines</u> $\mathcal{U}$ provided every member of $\mathcal{H}$ is contained in some member of $\mathcal{U}$. If $\cup\mathcal{H} = \cup\mathcal{U}$ is also the case, we call $\mathcal{H}$ a <u>refinement</u> of $\mathcal{U}$.

1.2.3. <u>Properties satisfied by collections of sets</u>. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. For each $x \in X$, define

$$\mathrm{ord}(x,\mathcal{H}) = |\{H \in \mathcal{H} : x \in H\}|.$$

(a) $\mathcal{H}$ is <u>point finite (PF)</u> provided $\mathrm{ord}(x,\mathcal{H})$ is finite for every $x \in X$.

(b) $\mathcal{H}$ is <u>locally finite (LF)</u> provided every $x \in X$ has a neighborhood that hits at most finitely many members of $\mathcal{H}$.

(c) $\mathcal{H}$ is <u>bded-LF</u> provided there exists $n \in N$ such that every $x \in X$ has a neighborhood that hits at most n members of $\mathcal{H}$, in which case $\mathcal{H}$ is <u>n-bded-LF</u>.

(d) $\mathcal{H}$ is <u>discrete (D)</u> provided $\mathcal{H}$ is 1-bded-LF.

(e) $\mathcal{H}$ is <u>locally countable (LC)</u> provided every $x \in X$ has a neighborhood that hits at most countably many members of $\mathcal{H}$.

(f) $\mathcal{H}$ is <u>closure-preserving (CP)</u> provided for every $A' \subset A$, $cl(\cup \{H_\alpha : \alpha \in A'\}) = \cup \{cl(H_\alpha) : \alpha \in A'\}$.

(g) $\mathcal{H}$ is <u>hereditarily closure-preserving (HCP)</u> provided every partial refinement of $\mathcal{H}$ is CP.

(h) $\mathcal{H}$ is <u>closed (C)</u> provided every member of $\mathcal{H}$ is a closed set.

(i) $\mathcal{H}$ is <u>directed</u> provided for every $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $H_\alpha \cup H_\beta \subset H_\gamma$.

(j) $\mathcal{H}$ is <u>well-monotone</u> provided the subset relation "$\subset$" is a well-order on $\mathcal{H}$.

1.2.4. <u>Diagram</u>.

$$\begin{tikzcd} & & & \mathrm{LC} & \\ \mathrm{D} \arrow[r] & \text{bded-LF} \arrow[r] & \mathrm{LF} \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{HCP} \arrow[r] & \mathrm{CP} \\ & & & \mathrm{PF} & \end{tikzcd}$$

$$\text{well-monotone} \longrightarrow \text{directed}$$

The diagram above illustrates general relationships between properties which collections of sets may satisfy.

1.2.5. <u>Remark</u>.

(a) Let $\mathcal{F}$ be a collection of closed subsets of a space X. Then $\mathcal{F}$ is LF iff $\mathcal{F}$ is CP and PF.

(b) Let P represent one of the following properties: D, bded-LF, LF, LC, HCP, or CP. If $\mathcal{H}$ is a P collection of subsets of a space X, then $\{cl(H) : H \in \mathcal{H}\}$ is also a P collection.

1.2.6. <u>Definition</u>. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ and $\mathcal{U}$ be collections of subsets of a space X. The collection $\mathcal{H}$ is <u>cushioned</u> in $\mathcal{U}$ provided we can index $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that for every $A' \subset A$, $\mathrm{cl}(\cup\{H_\alpha : \alpha \in A'\}) \subset (\cup\{U_\alpha : \alpha \in A'\})$.

1.2.7. <u>Remark</u>. If $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is a CP collection of closed subsets of a space X, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an arbitrary collection of subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$, then it should be clear that $\mathcal{F}$ is cushioned in $\mathcal{U}$.

1.2.8. <u>Definition</u>. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ and $\mathcal{U}$ be collections of subsets of a space X, and let P represent any property which collections of sets may satisfy.

(a) $\mathcal{H}$ is a <u>P-(partial) refinement</u> of $\mathcal{U}$ provided $\mathcal{H}$ is a P collection which (partially) refines $\mathcal{U}$.

(b) $\mathcal{H}$ is a <u>P-closed (open) refinement</u> of $\mathcal{U}$ provided $\mathcal{H}$ is a P-refinement of $\mathcal{U}$ and every member of $\mathcal{H}$ is a closed (open) set.

(c) If $\mathcal{H}$ refines $\mathcal{U}$, we refer to $\mathcal{H}$ as a <u>one-to-one refinement</u> of $\mathcal{U}$ provided we can index $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $H_\alpha \subset U_\alpha$ for every $\alpha \in A$.

(d) A space X is <u>P-refinable</u> provided every open cover of X has a P-refinement.

1.2.9. Definition.

(a) Let A be any set. We call $\mathcal{B} = \{B_\gamma : \gamma \in \Gamma\}$ a partition of A provided $\cup\mathcal{B} = A$ and $\mathcal{B}$ is pairwise disjoint.

(b) Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. A collection $\mathcal{K}$ of subsets of X is an amalgamation of $\mathcal{H}$ provided there exists a partition $\{B_\gamma : \gamma \in \Gamma\}$ of A such that $\mathcal{K} = \{\cup\{H_\alpha : \alpha \in B_\gamma\} : \gamma \in \Gamma\}$.

1.2.10. Amalgamation Lemma. Let P represent one of the following properties: D, bded-LF, LF, LC, PF, HCP, or CP. If $\mathcal{H}$ is a P collection of subsets of a space X and $\mathcal{K}$ is an amalgamation of $\mathcal{H}$, then $\mathcal{K}$ is also a P collection.

Proof. (We prove only the PF case. The remaining cases can be proved in a similar fashion.) Assume that $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ is PF, and let $\mathcal{K} = \{K_\gamma : \gamma \in \Gamma\}$ be an amalgamation of $\mathcal{H}$. Since $\mathcal{H}$ is PF, for each $x \in X$ there exists $n(x) \in N$ such that $\mathrm{ord}(x,\mathcal{H}) = n(x)$. Since $\mathcal{K}$ is a partition of $\mathcal{H}$, it should be clear that $\mathrm{ord}(x,\mathcal{K}) \leq n(x)$. It thus follows that $\mathcal{K}$ is PF. □

1.2.11. Corollary. Let P represent one of the following properties: D, bded-LF, LF, LC, PF, HCP, or CP. If a cover $\mathcal{U}$ of a space X has a P-refinement, then $\mathcal{U}$ has a one-to-one P-refinement.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, and suppose that $\mathcal{H}$

is a P-refinement of $\mathcal{U}$. Assume that A is well-ordered, and for each $H \in \mathcal{H}$ define $f(H) = \min(\{\alpha \in A : H \subset U_\alpha\})$, and $\mathcal{K} = \{K_\alpha = \cup\{H : f(H) = \alpha\} : \alpha \in A\}$. It should be clear that $\mathcal{K}$ is both a one-to-one refinement of $\mathcal{U}$ and an amalgamation of $\mathcal{H}$ as well. By the amalgamation lemma above, it follows that $\mathcal{K}$ is a one-to-one P-refinement of $\mathcal{U}$. □

1.2.12. Corollary. Let P represent one of the following properties: D, bded-LF, LF, LC, PF, HCP, or CP. If $\mathcal{U}$ is a cover of a space X, and $\mathcal{U}$ has a P-open refinement, then $\mathcal{U}$ has a one-to-one P-open refinement.

Proof. This result follows from 1.2.11 above and the fact that unions of open sets are open. □

1.2.13. Corollary. Let P represent one of the following properties: D, bded-LF, LF, HCP, or CP. If $\mathcal{U}$ is a cover of a space X, and $\mathcal{U}$ has a P-closed refinement, then $\mathcal{U}$ has a one-to-one P-closed refinement.

Proof. This result follows from 1.2.10 above and the fact that each property P above is CP. □

1.2.14. Corollary. Let $\mathcal{U}$ be a cover of a space X, and suppose that $\mathcal{U}$ has a CP-closed refinement. Then $\mathcal{U}$ has a cushioned refinement.

Proof. By 1.2.13 above, $\mathcal{U}$ must have a one-to-one CP-closed refinement, and any such refinement is cushioned in $\mathcal{U}$. □

1.2.15. Remark. Throughout this thesis we will often use results 1.2.11 - 1.2.13 above, and assume that refinements are one-to-one whenever we can do so without loss of generality.

1.2.16. Special types of refinements. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{U}$, and $\mathcal{V}$ be collections of subsets of a space X, and let P represent any property which collections of sets may satisfy.

(a) If $\mathcal{F}$ is a one-to-one P-(partial) refinement of $\mathcal{U}$, we refer to $\mathcal{F}$ as a P-(partial) shrink of $\mathcal{U}$.

(b) $\mathcal{V}$ is a star refinement of $\mathcal{U}$ provided $\{st(V,\mathcal{V}) : V \in \mathcal{V}\}$ refines $\mathcal{U}$.

(c) $\mathcal{V}$ is a pt-star refinement of $\mathcal{U}$ provided $\{st(x,\mathcal{V}) : x \in X\}$ refines $\mathcal{U}$.

(d) $\mathcal{G}$ is a σ-P-refinement of $\mathcal{U}$ provided we can write $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ such that $\mathcal{G}_n$ is a P-partial refinement of $\mathcal{U}$ for each $n \in N$.

1.2.17. Special types of continuous maps. Let $f : X \longrightarrow Y$ be a map from a space X to a space Y.

(a) If f(H) is an open (closed) subset of Y for every open (closed) subset H of X, we refer to f as an open (closed) map.

(b) The map f is <u>perfect (quasi-perfect)</u> provided f is continuous, closed, and $f^{-1}(y)$ is compact (countably compact) for every $y \in Y$.

(c) We refer to f as a <u>finite-to-one map</u> provided $|f^{-1}(y)|$ is finite for every $y \in Y$.

(d) The map f is <u>bded</u> provided there exists $n \in N$ such that $|f^{-1}(y)| \leq n$ for every $y \in Y$, in which case f is an <u>n-bded map</u>.

1.2.18. <u>Remark</u>. It should be clear that every bded-map is finite-to-one, and every closed, continuous, finite-to-one map is perfect.

1.2.19. <u>Map notation</u>. Let $f : X \longrightarrow Y$ be a map from a space X to a space Y, and let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of X and Y, respectively. Define

(i) $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$, and

(ii) $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$.

1.2.20. <u>Definition</u>. Let $f : X \longrightarrow Y$ be a map from a space X to a space Y, and let $H \subset X$. We refer to H as an <u>f-saturated set</u> provided $f^{-1}(f(H)) = H$.

The following lemma is used often in our work since most of the maps that we deal with are continuous and closed.

1.2.21. <u>Lemma</u>. Let $f : X \longrightarrow Y$ be a closed, continuous map from a space X to a space Y. Let H be an f-saturated subset of X, and U an open subset of X such that $H \subset U$. Then there exists an f-saturated open subset V of X such that $H \subset V \subset U$.

Proof. Let $V = X - f^{-1}(f(X - U))$. It is easy to check that V satisfies the desired conditions. □

1.2.22. Definition. Let $\{X_\alpha : \alpha \in A\}$ be a collection of spaces (not necessarily pairwise disjoint). For each $\alpha \in A$, let $X_\alpha^* = \{(x,\alpha) : x \in X\}$, and define a topology on X_α^* in the obvious way to make X_α^* homeomorphic to X_α. By construction, $\{X_\alpha^* : \alpha \in A\}$ is pairwise disjoint. Let $Y = \cup\{X_\alpha^* : \alpha \in A\}$, and define a subset U of Y to be open iff $U \cap X_\alpha^*$ is open in X_α^* for every $\alpha \in A$. We refer to the space Y as the disjoint sum of $\{X_\alpha : \alpha \in A\}$, denoted by $\oplus\{X_\alpha : \alpha \in A\}$.

1.2.23. Definition. Let $\{F_\alpha : \alpha \in A\}$ be a cover of a space X. The canonical map $f : \oplus\{F_\alpha : \alpha \in A\} \longrightarrow X$ is defined by $f(x,\alpha) = x$ for each $x \in F_\alpha$ and $\alpha \in A$.

1.2.24. Remark. The canonical map defined above is both onto and continuous.

1.2.25. Sum theorems. Let Q^* represent some topological property such as paracompactness, and let P represent one of the following properties: D, bded-LF, HCP, or CP.

(a) Property Q^* satisfies the P Sum Theorem provided for every space X, if $\{F_\alpha : \alpha \in A\}$ is a P-closed cover of X, and F_α satisfies property Q^* for every $\alpha \in A$, then X satisfies property Q^*.

(b) Property Q^* satisfies the Countable Sum Theorem provided for every space X, if $\{F_n : n \in N\}$ is a countable closed cover of X such that F_n satisfies property Q^* for every $n \in N$, then X satisfies property Q^*.

1.2.26. Lemma. Let Q^* represent a topological property which is preserved under both disjoint sums and closed, continuous, (bded-) finite-to-one maps. Then Q^* satisfies the (bded-) LF Sum Theorem.

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a (bded-) LF-closed cover of a space X such that each F_α satisfies property Q^*. Now $Y = \oplus \{F_\alpha : \alpha \in A\}$ satisfies property Q^*, and it is easy to see that the canonical map $f : Y \longrightarrow X$ (see 1.2.23 above) is closed, continuous, (bded-) finite-to-one, and onto. Thus, X satisfies property Q^*. □

1.2.27. Definition. Let $\mathcal{U}^* = \{\mathcal{U}_n : n \in N\}$ be a countable family of open covers of a space X. $\mathcal{U}^*$ is a normal family of covers provided $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ for every $n \in N$. An open cover $\mathcal{U}$ of a space X is a normal cover of X provided $\mathcal{U}$ is a member of some normal family of open covers of X.

1.2.28. Definition. A space X is fully normal provided every open cover of X is a normal cover.

1.2.29. Remark. A. H. Stone [81] proved that a space X is paracompact iff X is fully normal.

1.2.30. Definition. A space X is paracompact, (subparacompact, resp. metacompact) provided every open cover of X has a LF-open (σ-discrete-closed, resp. PF-open) refinement.

1.2.31. Definition. A space X is irreducible provided every open cover of X has an open refinement which is a minimal cover of X.

1.2.32. Definition. A space X is a k-space provided a subset F of X is a closed set iff $F \cap K$ is compact for every compact subset K of X.

1.2.33. Lemma. Every locally compact, T_2- space is a k-space.

Proof. Assume that X is a locally compact, T_2- space. If F is a closed subset of X and K is a compact subset of X, then K is closed since X is T_2, and hence $F \cap K$ is compact since compactness is closed hereditary.

Next, suppose that $A \subset X$, and $A \cap K$ is compact for every compact subset K of X. Let $x \in cl(A)$. There exists a compact neighborhood K_x of x, and since $x \in cl(A)$, $K_x \cap A$ is closed and nonempty. If $x \notin K_x \cap A$, it follows that $x \notin cl(A)$, a contradiction; hence, it must be the case that $x \in A$. Therefore, A is a closed set. □

1.2.34. Definition. Let $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ and $\mathcal{U}$ be collections of subsets of a space X. We call $\mathcal{U}$ an expansion of $\mathcal{H}$ provided we can index $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ such that $H_\alpha \subset U_\alpha$ for every $\alpha \in A$.

1.2.35. Definition. Let H be a subset of a space X. Then H is a cozero set provided there exists a continuous map $f : X \longrightarrow [0,1]$ such that $H = f^{-1}((0,1])$.

1.2.36. Definition. A space X is collectionwise normal (CWN) provided every discrete collection of closed subsets of X has a pairwise disjoint open expansion.

We refer the reader to [29] for the proof of our next lemma.

1.2.37. Lemma. A space X is CWN iff every discrete collection of closed subsets of X has a discrete-open expansion consisting of cozero sets.

1.2.38. Definition. Let $\mathcal{G}$ be a collection of open subsets of a space X. We refer to $\mathcal{G}$ as a θ-collection (almost Θ-collection) provided we can write $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ such that for every $x \in X$, there exists $n(x) \in N$ such that $\mathcal{G}_{n(x)}$ is LF (PF) at x.

1.2.39. Definition. Let $\mathcal{U}$ be an open cover of a space X, and let $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ be an open refinement of $\mathcal{U}$.

(a) $\mathcal{G}$ is a weak Θ-refinement of $\mathcal{U}$ provided $\mathcal{G}$ is an almost Θ-collection.

(b) $\mathcal{G}$ is a weak $\bar{\Theta}$-refinement of $\mathcal{U}$ provided $\mathcal{G}$ is an almost Θ-collection, and $\{\cup\mathcal{G}_n : n \in N\}$ is PF.

(c) $\mathcal{G}$ is a Θ-refinement of $\mathcal{U}$ provided $\mathcal{G}$ is an almost Θ-collection and $\mathcal{G}_n$ covers X for every $n \in N$.

1.2.40. Expandability definitions.

(a) A space X is expandable (almost expandable) provided every LF collection of closed subsets of X has a LF (PF)-open expansion.

(b) A space X is discretely-expandable (almost discretely-expandable) provided every discrete collection of closed subsets of X has a LF (PF)-open expansion.

(c) A space X is bded-expandable (almost bded-expandable) provided every bded-LF collection of closed subsets of X has a LF (PF)-open expansion.

(d) Let $\mathcal{F}$ be a collection of subsets of a space X. We call $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ an (almost) Θ-expansion of $\mathcal{F}$

provided

(i) $\mathcal{G}_n$ is an open expansion of $\mathcal{F}$ for every $n \in N$, and

(ii) $\mathcal{G}$ is an (almost) Θ-collection.

(e) A space X is <u>(almost) Θ-expandable</u> provided every LF collection of closed subsets of X has an (almost) Θ-expansion.

(f) A space X is <u>(almost) discretely-Θ-expandable</u> provided every discrete collection of closed subsets of X has an (almost) Θ-expansion.

1.2.41. <u>Remark</u>. As mentioned earlier, throughout this thesis we use "α" to represent a general infinite ordinal and "λ" a countable infinite ordinal.

1.2.42. <u>Definition</u>. A space X is <u>B(C,α)-refinable</u> provided every open cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ of X has a refinement $\mathcal{E} = \cup \left\{ \mathcal{E}_\beta = \{E(\beta,\gamma) : \gamma \in \Gamma\} : \beta < \alpha \right\}$ which satisfies

(i) $E(\beta,\gamma) \subset U_\gamma$ for every $\gamma \in \Gamma$, $\beta < \alpha$,

(ii) $\{\cup \mathcal{E}_\beta : \beta < \alpha\}$ partitions X,

(iii) for every $\beta < \alpha$, $\mathcal{E}_\beta$ is a collection of relatively closed subsets of the subspace $X - \cup\{\cup \mathcal{E}_\mu : \mu < \beta\}$, and

(iv) for every $\beta < \alpha$, $\cup\{\cup \mathcal{E}_\mu : \mu < \beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a B(C,α)-refinement of $\mathcal{U}$.

1.2.43. Definition. Let P represent one of the following properties: D, bded-LF, LF, LC, PF, HCP, or CP. A space X is B(P,α)-refinable provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E} = \cup\{\mathcal{E}_\beta : \beta < \alpha\}$ which satisfies

(i) $\{\cup\mathcal{E}_\beta : \beta < \alpha\}$ partitions X,

(ii) for every $\beta < \alpha$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X - \cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$, and

(iii) for every $\beta < \alpha$, $\cup\{\cup\mathcal{E}_\mu : \mu < \beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a B(P,α)-refinement of $\mathcal{U}$.

1.2.44. Remark. In definition 1.2.43 above, it should be clear by the amalgamation lemma (1.2.10) that if $\mathcal{U}$ has a B(P,α)-refinement, then $\mathcal{U}$ will have a B(P,α)-refinement as above such that $\mathcal{E}_\beta$ is a one-to-one partial refinement of $\mathcal{U}$--as in 1.2.42 above--for every $\beta < \alpha$. Since the amalgamation of a collection of closed sets is not necessarily closed, we must add the "one-to-one" condition (i) in definition 1.2.42 above to guarantee the existence of a B(C,α)-refinement which satisfies this "one-to-one" property, a condition which is necessary for future applications.

CHAPTER II

THE PROPERTY $B(P,\alpha)$-REFINABILITY

The notion of a "$B(LF,\omega)$-refinement" of an open cover--a generalization of a σ-LF-closed refinement--was introduced by J. Chaber [24] in the mid-1970's under the name "property b_1." J. Chaber proved that

(i) a space X is metacompact iff X is $B(LF,\omega)$-refinable and almost expandable, and

(ii) a space X is θ-refinable iff X is $B(LF,\omega)$-refinable and almost θ-expandable.

In 1980, J. C. Smith [77] generalized this notion further by defining the concept of a $B(P,\alpha)$-refinement (see definition 1.2.43). In this chapter we introduce the property $B(P,\alpha)$-refinability, and use it throughout this thesis to generalize many theorems involving the properties weak $\bar{\theta}$-refinability and θ-refinability.

As diagram 1.1.2 indicates, weak $\bar{\theta}$-refinability is closely related to $B(D,\lambda)$-refinability and $B(LF,\lambda)$-refinability. The property weak $\bar{\theta}$-refinability, introduced by J. C. Smith [70] in 1975, has the distinction of being the first such property which was used to obtain open cover characterizations of CWN analogous to several well-known characterizations of paracompactness; for example,

J. C. Smith [74] showed that a space X is CWN iff every weak $\bar{\theta}$-cover of X is a normal cover. Recall that a space X is paracompact iff every open cover of X is a normal cover. Weak $\bar{\theta}$-refinability has also played an important role in the study of metacompactness and paracompactness.

In §1 of this chapter, we obtain a weak $\bar{\theta}$-type characterization of $B(D,\omega)$-refinability and use this characterization to show that $B(D,\omega)$-refinability is strictly weaker than θ-refinability. We also establish conditions under which $B(D,\alpha)$-refinability and $B(LF,\alpha)$-refinability are equivalent.

In §2 we generalize known characterizations of normality, CWN, and paracompactness by showing that

(i) a space X is normal iff every open cover of X which has a $B(C,\lambda)$-refinement also has a closed shrink,

(ii) a space X is CWN iff every open cover of X which has a $B(D,\lambda)$-refinement is a normal cover, and

(iii) a space X is paracompact iff X is CWN and $B(LF,\lambda)$-refinable.

Sum and mapping theorem results have been fundamental in the study of any given topological property. In §3 we establish a number of such results for the class of $B(P,\alpha)$-refinable spaces.

§1. Characterizations of $B(D,\alpha)$-refinability

We begin with a lemma and then give a characterization of $B(D,\alpha)$-refinability which will be used several times in this chapter.

2.1.1. Lemma. Let $\mathcal{F}$ be an n-bded-LF collection of closed subsets of a space X. Then $\mathcal{F}$ has a refinement $\mathcal{K} = \cup \{\mathcal{K}_i : 1 \le i \le n\}$ satisfying

(i) $\{\cup \mathcal{K}_i : 1 \le i \le n\}$ is a partition of $\cup \mathcal{F}$, and

(ii) $\mathcal{K}_i$ is a relatively discrete collection of closed subsets of the subspace $X - \cup\{\cup \mathcal{K}_j : 1 \le j < i\}$ for $1 \le i \le n$.

Proof. For each i, $1 \le i \le n$, define

$$\mathcal{E}_i = \{\cap \mathcal{F}' : \mathcal{F}' \subset \mathcal{F}, \ |\mathcal{F}'| = i\} \text{ , and}$$

$$\mathcal{K}_i = \{E - \cup\{\cup \mathcal{K}_j : 1 \le j < i\} : E \in \mathcal{E}_{n+1-i}\} .$$

It is easy to see that $\mathcal{K} = \cup \{\mathcal{K}_i : 1 \le i \le n\}$ is a refinement of $\mathcal{F}$ which by construction satisfies condition (i) above. We assert that $\mathcal{K}$ also satisfies condition (ii). Clearly, $\mathcal{K}_1 = \mathcal{E}_n$ is a discrete collection of closed subsets of X since $\mathcal{F}$ is n-bded-LF. Now let $x \in X$ and i be fixed, $1 < i \le n$. If $\operatorname{ord}(x,\mathcal{F}) > n+1-i$, then $x \in \cup\{\cup \mathcal{K}_j : 1 \le j < i\}$. If $\operatorname{ord}(x,\mathcal{F}) \le n+1-i$, then there exists a neighborhood $V(x)$ of x which hits at most $n+1-i$ members of $\mathcal{F}$ and therefore hits at most one member of $\mathcal{K}_i$. It thus follows that condition (ii) above is satisfied. □

2.1.2. Theorem. A space X is B(D,α)-refinable iff X is B(bded-LF,α)-refinable.

Proof. The necessity is clear. Let $\mathcal{E} = \cup\{\mathcal{E}_\gamma : \gamma < \alpha\}$ be a B(bded-LF,α)-cover of X. It suffices to show that $\mathcal{E}$ has a B(D,α)-refinement. By 2.1.1 above, it should be clear for each $\gamma < \alpha$ that there exists $n(\gamma) \in N$ and a refinement $\mathcal{F}_\gamma = \cup\{\mathcal{F}(\gamma,j) : 1 \leq j \leq n(\gamma)\}$ of $\mathcal{E}_\gamma$ satisfying

(i) $\{\cup\mathcal{F}(\gamma,i) : 1 \leq i \leq n(\gamma)\}$ is a partition of $\cup\mathcal{E}_\gamma$, and

(ii) $\mathcal{F}(\gamma,i)$ is a relatively discrete collection of closed subsets of the subspace
$X - \cup\{\cup\mathcal{F}(\gamma,j) : 1 \leq j < i\} - \cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$.

Now define a well-ordering "<" on the set $A = \{(\gamma,i) : \gamma < \alpha, 1 \leq i \leq n(\gamma)\}$ such that for every $(\gamma,i), (\beta,j) \in A$,

$(\gamma,i) < (\beta,j)$ iff

(a) $\gamma < \beta$, or

(b) $\gamma = \beta$ and $i < j$.

Let $g : A \to \{\gamma : \gamma < \alpha\}$ be the unique bijection which preserves "<". For each $\gamma < \alpha$, let

$$\mathcal{K}_\gamma = \mathcal{F}(\beta,i) \text{ such that } g(\beta,i) = \gamma .$$

It is easy to check that $\mathcal{K} = \cup\{\mathcal{K}_\gamma : \gamma < \alpha\}$ is a B(D,α)-refinement of $\mathcal{E}$. □

Originally, our work with "bded-weak $\bar{\theta}$-covers" was motivated by the desire to obtain new characterizations of subparacompactness (see chapter III). Later, we were somewhat surprised to discover that the class of bded-weak $\bar{\theta}$-refinable spaces is precisely the class of $B(D,\omega)$-refinable spaces.

2.1.3. Definition. An open cover $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ of a space X is a bded-weak $\bar{\theta}$-cover provided

(i) $\mathcal{G}$ is a weak $\bar{\theta}$-cover of X (1.2.39 (b)), and

(ii) for every $n \in N$, there exists an integer $k(n)$ such that $X = \{x : 0 < \operatorname{ord}(x, \mathcal{G}_n) \leq k(n), n \in N\}$.

2.1.4. Definition. Let k be a positive integer. An open cover $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ of a space X is a k-bded-weak $\bar{\theta}$-cover provided

(i) $\mathcal{G}$ is a weak $\bar{\theta}$-cover of X, and

(ii) $X = \{x : 0 < \operatorname{ord}(x, \mathcal{G}_n) \leq k, n \in N\}$.

2.1.5. Theorem. A space X is bded-weak $\bar{\theta}$-refinable iff X is 1-bded-weak $\bar{\theta}$-refinable.

Proof. (H. R. Bennett and D. J. Lutzer [5] proved that every weak θ-cover has a "1-bded-weak θ-refinement." Our proof is similar.) The sufficiency is clear. Let $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ be a bded-weak $\bar{\theta}$-cover of X with $k(n)$ defined as in 2.1.3 above for each $n \in N$. Recall that

$\{\cup \mathcal{G}_n : n \in N\}$ is PF. We now construct a 1-bded-weak $\bar{\theta}$-refinement of $\mathcal{G}$. For each $x \in X$ and every $n, j \in N$, define

$$W(n,x) = \cap \{G \in \mathcal{G}_n : x \in G\}, \text{ and}$$

$$\mathcal{W}(n,j) = \{W(n,x) : \mathrm{ord}(x,\mathcal{G}_n) = j\}.$$

Clearly, if $\mathrm{ord}(x,\mathcal{G}_n) = j$, then $\mathrm{ord}(x,\mathcal{W}(n,j) = 1$. Define

$$\mathcal{W} = \cup \{\mathcal{W}(n,j) : 0 < j \leq k(n), n \in N\}.$$

We assert that $\mathcal{W}$ is a 1-bded-weak $\bar{\theta}$-refinement of $\mathcal{G}$. It should be clear that $\mathcal{W}$ is an open refinement of $\mathcal{G}$, and $X = \{x : \mathrm{ord}(x,\mathcal{W}(n,j)) = 1, 0 < j \leq k(n), n \in N\}$. It remains only to show that

$$\{\cup \mathcal{W}(n,j) : 0 < j \leq k(n), n \in N \quad \text{is PF.}$$

Indeed, let $x \in X$. Now $\{\cup \mathcal{G}_n : n \in N\}$ is PF, and so there exists an integer M such that $x \notin \{\cup \mathcal{G}_n : n > M\}$. By construction, $x \notin \{\cup \mathcal{W}(n,j) : n > M\}$. Since $\{\cup \mathcal{W}(n,j) : 0 < j \leq k(n), n \leq M$ is finite, it thus follows that x is covered by at most finitely many "levels" of $\mathcal{W}$. Therefore, $\{\cup \mathcal{W}(n,j) : 0 < j \leq k(n), n \in N\}$ is PF, and our proof is complete. □

2.1.6. Remark. From now on in this thesis we will assume that every bded-weak $\bar{\theta}$-refinement is a 1-bded-weak $\bar{\theta}$-refinement.

The equivalence of the properties $B(D,\omega)$-refinability and bded-weak $\bar{\theta}$-refinability will follow from the next two theorems.

2.1.7. Remark. J. C. Smith [77] proved that

$$B(D,\omega)\text{-refinable} \longrightarrow \text{weak } \bar{\theta}\text{-refinable}.$$

His proof actually gives us the following stronger result.

2.1.8. Theorem. If a space X is $B(D,\omega)$-refinable, then X is bded-weak $\bar{\theta}$- refinable.

Proof. Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a $B(D,\omega)$-refinement $\mathcal{E} = \cup\{\mathcal{E}_n = \{E(\alpha,n) : \alpha \in A\} : n \in N\}$. We construct a 1-bded-weak $\bar{\theta}$-refinement of $\mathcal{U}$. For each $\alpha \in A$ and $n \in N$, choose $U(\alpha,n) \in \mathcal{U}$ such that $E(\alpha,n) \subset U(\alpha,n)$, and define

$$G(\alpha,n) = U(\alpha,n) - \cup\{E(\beta,n) : \beta \neq \alpha\} - \cup\{\cup\mathcal{E}_k : k<n\},$$
$$\mathcal{G}_n = \{G(\alpha,n) : \alpha \in A\}, \text{ and}$$
$$\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}.$$

It is easy to see that $\mathcal{G}$ is a 1-bded-weak $\bar{\theta}$-refinement of $\mathcal{U}$. □

2.1.9. Theorem. If a space X is bded-weak $\bar{\theta}$-refinable, then X is $B(D,\omega)$-refinable.

Proof. Let $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ be a 1-bded-weak $\bar{\theta}$-cover of X. We construct a $B(D,\omega)$-refinement of $\mathcal{G}$.

Notation

(1) Let $\mathcal{G}^* = \{\cup\mathcal{G}_n : n \in N\}$, a PF collection.

(2) For each $n \in N$, define $C_n = \{x : \operatorname{ord}(x,\mathcal{G}^*) = n\}$.

(3) For each $n \in N$, define

$F_n = \{f : \{1,2,\ldots,n\} \longrightarrow N,\ f(1)<f(2)<\cdots<f(n)\}$.

(4) For each $n \in N$ and $x \in C_n$, let f_x represent the unique member of F_n such that $x \in W(x)$, where $W(x) = \cap \{\cup \mathcal{G}_{f_x(i)} : 1 \leq i \leq n\}$.

PART I. By induction, for each $n \in N$ we construct a family $\mathcal{H}_n = \cup \{\mathcal{H}(n,m) : 1 \leq m \leq n\}$ of collections of sets such that

(a_1) $\mathcal{H}(n,m)$ is a partial refinement of $\mathcal{G}$ for $1 \leq m \leq n$,

(a_2) $C_n = \cup\{\cup \mathcal{H}(n,m) : 1 \leq m \leq n\}$ for each $n \in N$,

(a_3) for $1 \leq m \leq n$, $(\cup \mathcal{H}(n,m)) \cap E(n,m) = \emptyset$, where $E(n,m) = \cup\{C_k : k < n\} \cup (\cup\{\cup \mathcal{H}(n,r) : 1 \leq r < m\})$, and

(a_4) $\mathcal{H}(n,m)$ is a relatively discrete collection of closed subsets of the subspace $X - E(n,m)$ for $1 \leq m \leq n$.

Construction 1. For the case $n = 1$, define $\mathcal{H}(1,1) = \{C_1 \cap G : G \in \mathcal{G}\}$. Now $E(1,1) = \emptyset$. It should be clear that $\mathcal{H}(1,1)$ satisfies conditions (a_1) - (a_3) above. We assert that $\mathcal{H}(1,1)$ is a discrete collection of closed subsets of X and hence satisfies (a_4). Indeed, let $x \in X$. If $x \in C_k$ for some $k > 1$, then there exist two members of $\mathcal{G}^*$ which contain x and whose intersection is a neighborhood of x that misses C_1 and hence misses $\cup \mathcal{H}(1,1)$.

If $x \in C_1$, then $x \in C_1 \cap G$ for some $G \in \mathcal{G}$. It is easy to check that G is a neighborhood of x that misses every member of $\mathcal{H}(1,1)$ except $C_1 \cap G$.

Construction 2. Now let n be fixed and assume that $\mathcal{H}_k$ has been constructed such that $\mathcal{H}_k$ satisfies (a_1) - (a_4) above for each k, $1 \leq k < n$. We construct $\mathcal{H}_n$. For each $k \in N$ and $1 \leq m \leq n$, define

$$C(n,m,k) = \{x \in C_n : m = \min(\{r : \operatorname{ord}(x, \mathcal{G}_{f_x(r)}) = 1\}), \text{ and } f_x(m) = k\},$$

$$\mathcal{H}(n,m,k) = \{C(n,m,k) \cap G : G \in \mathcal{G}_k\},$$

$$\mathcal{H}(n,m) = \cup\{\mathcal{H}(n,m,k) : k \in N\}, \text{ and}$$

$$\mathcal{H}_n = \cup\{\mathcal{H}(n,m) : 1 \leq m \leq n\}.$$

The following properties are easy to verify.

Properties

(i) $C(n,m,k) = \cup\, \mathcal{H}(n,m,k)$ for each $k \in N$ and $1 \leq m \leq n$.

(ii) If $(n,m,k) \neq (r,s,t)$, then

$C(n,m,k) \cap C(r,s,t) = \emptyset$. In particular,

$[\cup\, \mathcal{H}(n,m,k)] \cap [\cup\, \mathcal{H}(r,s,t)] = \emptyset$.

(iii) If $j \neq k$ and $x \in C(n,m,k)$, then

$W(x)$ is a neighborhood of x such that

$W(x) \cap C(n,m,j) = \emptyset$. In particular,

$W(x) \cap (\cup\, \mathcal{H}(n,m,j)) = \emptyset$.

(Proof of (iii): If $y \in C(n,m,j)$, then

$f_y(m) = j \neq k = f_x(m)$; hence,

$$\{\mathcal{G}_{f_y(i)} : 1 \leq i \leq n\} \neq \{\mathcal{G}_{f_x(i)} : 1 \leq i \leq n\}.$$

Since $\operatorname{ord}(y,\mathcal{G}^*) = n$, it thus follows that $y \notin W(x)$.)

<u>Verification of (a_1) - (a_4) above for $\mathcal{H}_n$.</u>

(<u>1</u>). For each $k \in N$ and $1 \leq m \leq n$, since $\mathcal{H}(n,m,k)$ is a partial refinement of $\mathcal{G}_k$, it should be clear that $\mathcal{H}(n,m)$ is a partial refinement of $\mathcal{G}$, and so (a_1) is satisfied.

(<u>2</u>). Since $C_n = \cup \{C(n,m,k) : k \in N,\ 1 \leq m \leq n\}$, by property (i) above it thus follows that $C_n = \cup\{\cup \mathcal{H}(n,m) : m \leq n\}$, and hence (a_2) is satisfied.

(<u>3</u>). By property (ii) above, it should be clear that (a_3) is satisfied.

(<u>4</u>). Let m' be fixed, $1 \leq m' \leq n$, and let $x \in X - E(n,m')$.

<u>Case(i)</u>. Suppose $x \in C_t$ for some $t > n$. It should be clear that $W(x)$ is a neighborhood of x such that $W(x) \cap C_n = \emptyset$. In particular, $W(x) \cap (\cup \mathcal{H}(n,m')) = \emptyset$ since $\cup \mathcal{H}(n,m') \subset C_n$.

<u>Case (ii)</u>. Suppose $x \in \cup \mathcal{H}(n,m')$. Then there exists some $k' \in N$ and $G \in \mathcal{G}_{k'}$ such that $x \in C(n,m',k') \cap G \in \mathcal{H}(n,m',k')$. Since every point in $C(n,m',k')$ has order 1 with respect to $\mathcal{G}_{k'}$, G is a neighborhood of x which misses every member of $\mathcal{H}(n,m',k')$ except $H(n,m',k') \cap G$. By property (iii) above, $W(x)$ is a

neighborhood of x such that $W(x) \cap (\cup \mathcal{H}(n,m',j)) = \emptyset$ for every $j \neq k'$. It thus follows that $G \cap W(x)$ is a neighborhood of x which hits exactly one member of $\mathcal{H}(n,m')$.

Case (iii). Suppose $x \in \cup \mathcal{H}(n,r')$, $m' < r' \leq n$. Then $\mathrm{ord}(x, \mathcal{G}_{f_x(m')}) > 1$, and so there exist two members of $\mathcal{G}_{f_x(m')}$, say G_1 and G_2 such that $U(x) = G_1 \cap G_2$ is a neighborhood of x. Now every point in $C(n,m',f_x(m'))$ has order 1 with respect to $\mathcal{G}_{f_x(m')}$ and so $U(x) \cap C(n,m',f_x(m')) = \emptyset$. In particular, $U(x) \cap (\cup \mathcal{H}(n,m',f_x(m')) = \emptyset$. Now let $s' \in N$ be fixed for any $s' \neq f_x(m')$. We assert that $W(x) \cap C(n,m',s') = \emptyset$, and hence $W(x) \cap (\cup \mathcal{H}(n,m',s')) = \emptyset$. Indeed, if $y \in C(n,m',s')$, then $f_y(m') = s' \neq f_x(m')$. As before, $\{\mathcal{G}_{f_x(i)} : 1 \leq i \leq n\} \neq \{\mathcal{G}_{f_y(i)} : 1 \leq i \leq n\}$. Now $\mathrm{ord}(y, \mathcal{G}^*) = n$, and so $y \notin W(x)$ must be the case. It thus follows that $U(x) \cap W(x)$ is a neighborhood of x which misses every member of $\mathcal{H}(n,m')$.

Cases (i) - (iii) above imply that $\mathcal{H}(n,m')$ is a relatively discrete collection of closed subsets of the subspace $X - E(n,m')$, and so (a_4) is satisfied. Our construction is now complete.

PART II. Define a well-order "<" on the set $S = \{(n,m) : 1 \leq m \leq n,\ n \in N\}$ such that for every (n,m), $(k,r) \in S$,

$(n,m) < (k,r)$ iff

(a) $n < k$, or

(b) $n = k$ and $m < r$.

Let $g : S \longrightarrow N$ be the unique bijection which preserves this order.

For each $n \in N$, define

$\mathcal{F}_n = \mathcal{H}(k,r)$ such that $g(k,r) = n$, and

$\mathcal{F} = \cup \{ \mathcal{F}_n : n \in N \}$.

From the fact that $X = \cup \{ C_n : n \in N \}$ and that $\mathcal{H}(n,m)$ satisfies conditions (a_1) - (a_4) above for every $n \in N$ and $1 \leq m \leq n$, it is easy to see that $\mathcal{F}$ is a $B(D,\omega)$-refinement of $\mathcal{G}$. □

2.1.10. <u>Corollary</u>. A space X is $B(D,\omega)$-refinable iff X is bded-weak $\bar{\theta}$-refinable.

<u>Proof</u>. This result follows immediately from 2.1.8 and 2.1.9. □

The proof of the following lemma is a technique which is frequently used for decomposing an open cover of a space into a countable number of relatively discrete partial refinements. One application of this lemma will be used in the proof of theorem 2.1.12.

2.1.11. <u>Lemma</u>. Let $\mathcal{U}$ be an open cover of a space X. For each $n \in N$, define $X_n = \{ x : \mathrm{ord}(x,\mathcal{U}) = n \}$, and

$\mathcal{E}_n = \{X_n \cap (\cap \mathcal{V}) : \mathcal{V} \subset \mathcal{U}, |\mathcal{V}| = n\}$. Then $\mathcal{E}_n$ is a relatively discrete closed partial refinement of $\mathcal{U}$ with respect to the subspace $X - \cup\{X_i : 1 \leq i < n\}$. Furthermore, $X_n = \cup \mathcal{E}_n$.

Proof. Let $n \in N$ be fixed. It should be clear by construction that $\mathcal{E}_n$ is a partial refinement of $\mathcal{U}$ such that $X_n = \cup \mathcal{E}_n$.

(i) Suppose $x \in X_j$ for some $j > n$. Then there exist n+1 members of $\mathcal{U}$, say $\mathcal{V}'$, which contain x. Clearly, $\cap \mathcal{V}'$ is a neighborhood of x such that $(\cap \mathcal{V}') \cap X_n = \emptyset$. In particular, $\cap \mathcal{V}'$ misses every member of $\mathcal{E}_n$.

(ii) Suppose $x \in X_n$ and let $\mathcal{V} = \{U \in \mathcal{U} : x \in U\}$. Then $x \in X_n \cap (\cap \mathcal{V}) \in \mathcal{E}_n$. It is easy to see that $\cap \mathcal{V}$ is a neighborhood of x which hits only one member of $\mathcal{E}_n$, namely $X_n \cap (\cap \mathcal{V})$.

From cases (i) and (ii) above it follows that $\mathcal{E}_n$ is a relatively discrete collection of closed subsets of the subspace $X - \cup\{X_i : i < n\}$. □

We now apply 2.1.11 and use a "diagonal process" to prove our next theorem. A similar technique was used in [77] to prove that

$$\text{weak } \bar{\theta}\text{-refinable} \longrightarrow B(D,\omega^2)\text{-refinable}.$$

2.1.12. Theorem. If a space X is θ-refinable, then X is $B(D,\omega)$-refinable.

Proof. Let $\mathcal{U}$ be an open cover of X. There exists a θ-refinement $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ of $\mathcal{U}$. We construct a B(D,ω)-refinement of $\mathcal{G}$. By 2.1.11 above, for each $n \in N$ there exists a family $\{\mathcal{E}(n,k) : k \in N\}$ of partial refinements of $\mathcal{G}_n$ such that for every $k \in N$,

(i) $\mathcal{E}(n,k)$ is a relatively discrete collection of closed subsets of the subspace $X - \cup\{\cup\mathcal{E}(n,j) : j < k\}$, and

(ii) $\cup\mathcal{E}(n,k) = \{x \in X : \mathrm{ord}(x,\mathcal{G}_n) = k\}$.

We define a well-order "<" on the set $\mathcal{J} = \{(i,j) : i,j \in N\}$ such that for every $(i,j), (k,m) \in \mathcal{J}$,

$(i,j) < (k,m)$ iff

(a) $i + j < k + m$, or

(b) $i + j = k + m$ and $j < m$.

Let $f : \mathcal{J} \longrightarrow N$ be the unique bijection which preserves this order. The following diagram should indicate the connection between "<" and what we refer to as the "diagonal process."

$$\begin{tikzcd}
f(1,4) = 10 & & & \\
f(1,3) = 6 & f(2,3) = 9 \arrow[ul] & & \\
f(1,2) = 3 & f(2,2) = 5 \arrow[ul] & f(3,2) = 8 \arrow[ul] & \\
f(1,1) = 1 \arrow[r] & f(2,1) = 2 \arrow[ul] & f(3,1) = 4 \arrow[ul] & f(4,1) = 7 \arrow[ul]
\end{tikzcd}$$

For each $n \in N$, define

$\mathcal{H}_n = \mathcal{E}(i,j)$ such that $f(i,j) = n$, and

$$\mathcal{K}_n = \{H - \cup\{\cup \mathcal{H}_k : 1 \leq k < n\} : H \in \mathcal{H}_n\}.$$

By observing the pattern established in the above diagram and condition (i) above, we have that $\mathcal{K} = \cup\{\mathcal{K}_n : n \in N\}$ is a $B(D,\omega)$-partial refinement of $\mathcal{U}$. It remains only to show that $\mathcal{K}$ covers X. Indeed, let $x \in X$. Since $\mathcal{G}$ is a Θ-refinement, there exists $n(x), k(x) \in N$ such that $ord(x, \mathcal{G}_{n(x)}) = k(x)$. By condition (ii) above, $x \in \cup\mathcal{E}(n(x), k(x))$. Let $m(x) = f(n(x), k(x))$. If $x \notin \cup\{\cup\mathcal{H}_j : 1 \leq j < m(x)\}$, then $x \in \cup\mathcal{K}_{m(x)}$. If $x \in \cup\{\cup\mathcal{H}_j : 1 \leq j < m(x)\}$, let $j(x) = \min(\{j : x \in \cup\mathcal{H}_j\})$. By construction, $x \in \mathcal{K}_{j(x)}$. In any case, $x \in \cup\mathcal{K}$. Therefore, $\mathcal{K}$ is a $B(D,\omega)$-refinement of $\mathcal{G}$. In particular, $\mathcal{K}$ refines $\mathcal{U}$. □

The next example is a space X which is bded-weak $\bar{\Theta}$-refinable, and hence $B(D,\omega)$-refinable; however, X is not Θ-refinable. Thus,

$$\Theta\text{-refinable} \underset{\longrightarrow}{\not\longleftarrow} B(D,\omega)\text{-refinable}.$$

2.1.13. <u>Example</u> (H. R. Bennett and D. J. Lutzer [5]). Let ω_2 represent the first ordinal of cardinality 2^c, and define $X = \Pi\{\{0,1\}_\alpha : \alpha < \omega_2\}$. For each $\alpha < \omega_2$, assume that $\{0,1\}_\alpha$ is the two point discrete space, and define

$f_\alpha \in X$ by

(i) $f_\alpha(\alpha) = 1$, and

(ii) $f_\alpha(\beta) = 0$ if $\beta \neq \alpha$.

Let $M = \{f_\alpha : \alpha < \omega_2\}$. The topology $\mathcal{T}$ that we assign to X agrees with the usual product topology on M, and every singleton in $X - M$ is open.

In [5] it is shown that X is a Tychonoff, weak θ-refinable space which is not θ-refinable. We assert that X is bded-weak $\bar{\theta}$-refinable. Indeed, let $\mathcal{U}$ be an open cover of X. Since each $f_\alpha \in M$ is contained in some $U_\alpha \in \mathcal{U}$, there exists a basic open neighborhood $V_\alpha \subset U_\alpha$ of f_α such that $g(\alpha) = 1$ for every $g \in V_\alpha$; hence, if $\beta \neq \alpha$, then $f_\beta \notin V_\alpha$. Define $\mathcal{V}_1 = \{V_\alpha : \alpha < \omega_2\}$, and $\mathcal{V}_2 = \{\{g\} : g \in X - M\}$. By construction, $\operatorname{ord}(f_\alpha, \mathcal{V}_1) = 1$ for every $\alpha < \omega_2$, and $\operatorname{ord}(g, \mathcal{V}_2) = 1$ for every $g \in X - M$. It follows that $\mathcal{V}_1 \cup \mathcal{V}_2$ is a bded-weak $\bar{\theta}$-refinement of $\mathcal{U}$. □

It is known that all metacompact spaces [2], θ-refinable spaces [87], and weak $\bar{\theta}$-refinable spaces [9, 73] are irreducible (1.2.31). Furthermore, if P represents either D, LF, or HCP, all $B(P,\alpha)$-refinable spaces [77] share this property. On the other hand, weak θ-refinable spaces are not irreducible in general. Indeed, the following example is such a space. Therefore, weak θ-refinability is strictly weaker than $B(LF,\lambda)$-refinability, $B(D,\lambda)$-refinability, and weak $\bar{\theta}$-refinability. The next example also shows in general that

$$\text{weak } \theta\text{-refinable} \not\longrightarrow B(HCP,\alpha)\text{-refinable}.$$

2.1.14. Example (E. K. van Douwen and H. Wicke [26]). We refer the reader to [26] for the construction of a regular weak θ-refinable space X which is not irreducible. The space X is the set of real numbers with a finer topology than the "usual" topology. There exists a countable partition $\{L_n : n \in N\}$ of X such that for each $n \in N$ and $x \in L_n$,

$$V(x,n) = \{x\} \cup (\cup \{L_i : i < n\})$$

is an open set. Using this property, it is easy to show that X is weak θ-refinable. Indeed, let $\mathcal{U}$ be an open cover of X. For each $n \in N$ and $x \in L_n$, choose $U(x) \in \mathcal{U}$ such that $x \in U(x)$, and define $W(x,n) = U(x) \cap V(x,n)$, and $\mathcal{W}_n = \{W(x,n) : x \in L_n\}$. By construction $\mathcal{W}_n$ partially refines $\mathcal{U}$, and $\mathrm{ord}(x, \mathcal{W}_n) = 1$ for every $x \in L_n$. It follows that $\mathcal{W} = \cup \{\mathcal{W}_n : n \in N\}$ is a weak θ-refinement of $\mathcal{U}$.

See [26] for a proof that X is regular and not irreducible. □

The next theorem shows that $B(D,\alpha)$-refinability and $B(LF,\alpha)$-refinability are closely related in the class of hereditarily countably metacompact spaces.

2.1.15. Lemma [38]. A space X is countably metacompact iff every countable monotone open cover of X has a closed shrink.

2.1.16. Theorem. Let X be a hereditarily countably metacompact space. If X is B(LF,α)-refinable, then X is B(D,$\omega\times\alpha$)-refinable.

Proof. Let $\mathcal{U}$ be an open cover of X, and $\mathcal{E} = \cup\{\mathcal{E}_\gamma : \gamma < \alpha\}$ a B(LF,α)-refinement of $\mathcal{U}$. For each $n \in N$ and $\gamma < \alpha$, define

$$S(\gamma,n) = \{x : \text{ord}(x,\mathcal{E}_\gamma) \leq n\} - \cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}, \text{ and}$$
$$\mathcal{S}_\gamma = \{S(\gamma,n) : n \in N\}.$$

Now $\mathcal{S}_\gamma$ is a countable monotone open cover of the subspace $X - \cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$. By 2.1.15 above, $\mathcal{S}_\gamma$ has a relatively closed shrink $\mathcal{F}_\gamma = \{F(\gamma,n) : n \in N\}$ which covers the subspace $X - \cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$ with $F(\gamma,n) \subset S(\gamma,n)$ for each $n \in N$.

For every $\gamma < \alpha$ and $n \in N$, define

$$\mathcal{H}(\gamma,n) = \{E \cap F(\gamma,n) : E \in \mathcal{E}_\gamma\}, \text{ and}$$
$$\mathcal{H}_\gamma = \cup\{\mathcal{H}(\gamma,n) : n \in N\}.$$

Since each member of $\mathcal{H}(\gamma,n)$ is contained in $S(\gamma,n)$, it follows that $\mathcal{H}(\gamma,n)$ is a relatively n-bded-LF collection of closed subsets of the subspace $X - \cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$. Furthermore, $\mathcal{H}(\gamma,n)$ partially refines $\mathcal{E}_\gamma$, and $\cup\mathcal{H}_\gamma = \cup\mathcal{E}_\gamma$.

For every $\gamma < \alpha$ and $n \in N$, define

$$\mathcal{K}(\gamma,n) = \{H - \cup\{\cup\mathcal{H}(\gamma,j) : j<n\} : H \in \mathcal{H}(\gamma,n)\}, \text{and}$$
$$\mathcal{K}_\gamma = \cup\{\mathcal{K}(\gamma,n) : n \in N\}.$$

Define a well-order "<" on $\mathcal{J} = \{(\gamma,n) : \gamma < \alpha, n \in N\}$ such that for every $(\beta,m), (\gamma,n) \in \mathcal{J}$,

$(\beta, m) < (\gamma, n)$ iff

(a) $\beta < \gamma$, or

(b) $\beta = \gamma$ and $m < n$.

Let $f : \mathcal{J} \to \{\mu : \mu < \omega \times \alpha\}$ be the unique bijection which preserves this order. For each $\mu < \omega \times \alpha$, define

$$\mathcal{L}_\mu = \mathcal{K}(\gamma, n) \quad \text{such that } f(\gamma, n) = \mu.$$

By construction it is easy to see that $\mathcal{L} = \cup\{\mathcal{L}_\mu : \mu < \omega \times \alpha\}$ is a B(bded-LF,$\omega \times \alpha$)-refinement of $\mathcal{U}$; therefore, by 2.1.2 X is B(D,$\omega \times \alpha$)-refinable. □

2.1.17. Corollary. Let X be a hereditarily countably metacompact space.

(a) If α is an uncountable limit ordinal and X is B(LF,α)-refinable, then X is B(D,α)-refinable.

(b) If X is B(LF,ω^n)-refinable for some $n \in N$, then X is B(D,ω^{n+1})-refinable.

Proof. Since $\omega \times \alpha = \alpha$ for any uncountable limit ordinal α, and $\omega \times \omega^n = \omega^{n+1}$ for each $n \in N$, by 2.1.16 above our result follows immediately. □

§2. Characterizations of normality and CWN

The following theorem is a summary of several open cover characterizations of normality which involve the existence of closed shrinks.

2.2.1. Theorem. For any space X, TFAE.

(a) X is normal.

(b) Every finite open cover of X has a closed shrink.

(c) [53] Every PF-open cover of X has a closed shrink.

(d) [76] Every weak $\bar{\theta}$-cover of X has a closed shrink.

(e) [77] Every open cover of X which has a B(HCP,λ)-refinement also has a closed shrink.

The question of whether in general all countable open covers of normal spaces have closed shrinks--or equivalently, whether all normal spaces are countably paracompact--was posed by C. H. Dowker [28] in 1951 and remained an outstanding problem for 20 years. This question was answered negatively by M. E. Rudin [67] in 1971. The following theorem was proved independently by C. H. Dowker [28] and M. Katetov [46] in 1951.

2.2.2. Theorem ([28], [46]). A normal space X is countably paracompact iff every countable open cover of X has a closed shrink.

In order to generalize 2.2.1 above we need the following two lemmas. Recall that λ represents a countably infinite ordinal in this thesis.

2.2.3. <u>Lemma</u> [77]. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a space X. Suppose $\mathcal{U}$ has a refinement $\cup\{\mathcal{H}_\gamma = \{H(\gamma,\alpha) : \alpha \in A\} : \gamma < \lambda\}$ which satisfies

(i) $cl(H(\gamma,\alpha)) \subset U_\alpha$ for every $\alpha \in A$, $\gamma < \lambda$, and

(ii) $H^*_\gamma = \cup \mathcal{H}_\gamma$ is a cozero set for each $\gamma < \lambda$.

Then $\mathcal{U}$ has a closed shrink.

<u>Proof</u>. By condition (ii) above, $\mathcal{H}^* = \{H^*_\gamma : \gamma < \lambda\}$ is a countable cozero cover of X; hence, $\mathcal{H}^*$ has a LF-open refinement $\mathcal{W}^* = \{W^*_\gamma : \gamma < \lambda\}$ such that $W^*_\gamma \subset H^*_\gamma$ for each $\gamma < \lambda$. (See [62] for a proof of our last assertion.) For every $\gamma < \lambda$ and $\alpha \in A$, define

$K(\gamma,\alpha) = W^*_\gamma \cap H(\gamma,\alpha)$, and

$K_\alpha = \cup \{K(\gamma,\alpha) : \gamma < \lambda\}$.

Now $\{K(\gamma,\alpha) : \gamma < \lambda\}$ partially refines the LF collection $\mathcal{W}^*$, implying that $\{K(\gamma,\alpha) : \gamma < \lambda\}$ is LF. Also, $cl(K(\gamma,\alpha)) \subset cl(H(\gamma,\alpha)) \subset U_\alpha$ by condition (i) above. It thus follows that $cl(K_\alpha) = \cup\{cl(K(\gamma,\alpha) : \gamma < \lambda\} \subset U_\alpha$ for each $\alpha \in A$. Also, by construction $\{K_\alpha : \alpha \in A\}$ covers X. Therefore $\{cl(K_\alpha) : \alpha \in A\}$ is a closed shrink of $\mathcal{U}$. □

2.2.4. <u>Lemma</u>. Let X be a normal space. Suppose $\{U_\alpha : \alpha \in A\}$ is a collection (not necessarily a cover) of open subsets of X and $\{F_\alpha : \alpha \in A\}$ is a collection of closed subsets of X satisfying

(i) $F_\alpha \subset U_\alpha$ for each $\alpha \in A$, and

(ii) $F^* = \cup \{F_\alpha : \alpha \in A\}$ is a closed set.

Then for each $\alpha \in A$ there exists a cozero set H_α such that $F_\alpha \subset H_\alpha \subset cl(H_\alpha) \subset U_\alpha$, and $\cup\{H_\alpha : \alpha \in A\}$ is a cozero set.

Proof. Since X is normal there exists for each $\alpha \in A$ a cozero set K_α such that $F_\alpha \subset K_\alpha \subset cl(K_\alpha) \subset U_\alpha$. Now F^* is a closed set and $F^* \subset \cup\{K_\alpha : \alpha \in A\}$, so by normality there exists a cozero set K^* such that

$$F^* \subset K^* \subset cl(K^*) \subset \cup\{K_\alpha : \alpha \in A\}.$$

By construction, $\{H_\alpha = K_\alpha \cap K^* : \alpha \in A\}$ is a cozero expansion of $\{F_\alpha : \alpha \in A\}$ such that $cl(H_\alpha) \subset U_\alpha$ for each $\alpha \in A$. Also, $\cup\{H_\alpha : \alpha \in A\} = K^*$, a cozero set. □

We now generalize 2.2.1 above by using a modification of the proof of 2.2.1(e) given by J. C. Smith [77].

2.2.5. Theorem. A space X is normal iff every open cover of X which has a $B(C,\lambda)$-refinement also has a closed shrink.

Proof. Sufficiency. To show that X is normal we need only to establish (e) of 2.2.1, but this is immediate.

Necessity. Suppose that X is normal and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open cover of X which has a $B(C,\lambda)$-refinement $\mathcal{E} = \cup\{\mathcal{E}_\gamma = \{E(\gamma,\alpha) : \alpha \in A\} : \gamma < \lambda\}$. By transfinite induction we construct for every $\gamma < \lambda$ a collection $\mathcal{H}_\gamma = \{H(\gamma,\alpha) : \alpha \in A\}$ of cozero subsets of X satisfying

(i) $H^*_\gamma = \cup \mathcal{H}_\gamma$ is a cozero set, and

(ii) $F(\gamma,\alpha) = (E(\gamma,\alpha) - \cup\{H^*_\beta : \beta < \gamma\}) \subset H(\gamma,\alpha)$

$\subset cl(H(\gamma,\alpha)) \subset U_\alpha$ for every $\alpha \in A$.

For fixed $\gamma < \lambda$ assume that the collections $\mathcal{H}_\beta$ with the above properties have been constructed for all $\beta < \gamma$. Now $\cup\{H^*_\beta : \beta < \gamma\}$ is an open set which by condition (ii) above contains $\cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$; hence, $\{F(\gamma,\alpha) : \alpha \in A\}$ is a collection of closed subsets of X such that

$$F^* = \cup\{F(\gamma,\alpha) : \alpha \in A\}$$

is a closed set. Also, $F(\gamma,\alpha) \subset U_\alpha$ for each $\alpha \in A$. By 2.2.4 above, for each $\alpha \in A$ there exists a cozero set $H(\gamma,\alpha)$ such that $F(\gamma,\alpha) \subset H(\gamma,\alpha) \subset cl(H(\gamma,\alpha) \subset U_\alpha$ where H^* is a cozero set, and the construction is complete.

It should be clear that $\cup\{\mathcal{H}_\gamma : \gamma < \lambda\}$ is a refinement of $\mathcal{U}$ which satisfies conditions (i) and (ii) of 2.2.3 above; therefore, $\mathcal{U}$ must have closed shrink. $\square$

We now have the following new results.

2.2.6. <u>Corollary</u>. Let X be a normal space.

(a) If X is $B(C,\lambda)$-refinable, then every open cover of X has a closed shrink.

(b) If X is countably $B(C,\lambda)$-refinable, then every countable open cover of X has a closed shrink.

Proof. These results follow immediately from 2.2.5 above. □

2.2.7. Corollary. Let X be a normal space. Then X is countably paracompact iff X is countably $B(C,\lambda)$-refinable.

Proof. Since every θ-cover has a $B(D,\lambda)$-refinement (2.1.12), and in particular, a $B(C,\lambda)$-refinement, the necessity should be clear. The sufficiency follows immediately from 2.2.6(b) and 2.2.2. □

Recall that a space X is paracompact iff every open cover of X is a normal cover. J. C. Smith [74] obtained the following analogous characterization of CWN.

2.2.8. Theorem [74]. A space X is CWN iff every weak $\bar{\theta}$-cover of X is a normal cover.

2.2.9. Corollary [74]. A CWN space X is paracompact iff X is weak $\bar{\theta}$-refinable.

We use the properties $B(D,\lambda)$-refinability and $B(LF,\lambda)$-refinability to obtain more general characterizations of CWN and paracompactness. To prove our next lemma we use a technique similar to that which J. C. Smith [74] used to prove the "sufficiency" in 2.2.8 above.

2.2.10. Lemma. If every bded-weak $\bar{\theta}$-cover of a space X is a normal cover, then X is CWN.

Proof. Let $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X. For each $\alpha \in A$, define $U_\alpha = X - \cup\{D_\beta : \beta \neq \alpha\}$. Then $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open cover of X such that

(i) $D_\alpha \subset U_\alpha$ for each $\alpha \in A$, and

(ii) $U_\alpha \cap D_\beta = \emptyset$ for $\alpha \neq \beta$.

It is easy to see that $\mathcal{U}^* = \mathcal{U} \cup \{X - \cup\mathcal{D}\}$ is a bded-weak $\bar{\theta}$-cover of X, and thus $\mathcal{U}^*$ is a normal cover. In particular, $\mathcal{U}^*$ has a pt-star open refinement $\mathcal{W}$. For each $\alpha \in A$, define $G_\alpha = st(D_\alpha, \mathcal{W})$. By conditions (i) and (ii) above, it should be clear that $\{G_\alpha : \alpha \in A\}$ is a pairwise disjoint open expansion of $\mathcal{D}$. Therefore X is CWN. □

2.2.11. Lemma [47]. A space X is CWN iff every bded-LF collection of closed subsets of X has a LF-open expansion.

2.2.12. Theorem.

(a) A space X is CWN iff every open cover of X which has a $B(D,\lambda)$-refinement is a normal cover.

(b) A countably metacompact space X is CWN iff every open cover of X which has a $B(LF,\lambda)$-refinement is a normal cover.

Proof. (We prove only part (b). Part (a) can be proved in a similar fashion.) Since every bded-weak $\bar{\theta}$-cover has a $B(D,\lambda)$-refinement, and in particular a $B(LF,\lambda)$-refinement, sufficiency follows immediately from 2.2.10 above.

Now assume that X is countably metacompact and CWN. Let $\mathcal{U}$ be an open cover of X which has a B(LF,λ)-refinement $\mathcal{E} = \cup\{\mathcal{E}_\gamma : \gamma < \lambda\}$. We will show that $\mathcal{U}$ has a LF-open refinement, which implies $\mathcal{U}$ is a normal cover of X since X is a normal space. By transfinite induction we construct for every $\gamma < \lambda$ a family $\{\mathcal{H}(\gamma,n) : n \in N\}$ of collections of subsets of X satisfying

(i) $\mathcal{H}(\gamma,n)$ is a LF collection of cozero sets for each $n \in N$,

(ii) $\mathcal{H}(\gamma,n)$ partially refines $\mathcal{U}$ for each $n \in N$, and

(iii) $\cup \mathcal{F}_\gamma \subset H^*_\gamma = \cup\{\cup \mathcal{H}(\gamma,n) : n \in N\}$, where

$$\mathcal{F}_\gamma = \{E - \cup\{H^*_\beta : \beta < \gamma\} : E \in \mathcal{E}_\gamma\}.$$

For fixed $\gamma < \lambda$, assume $\mathcal{H}(\beta,n)$ has been constructed such that conditions (i) - (iii) above are satisfied for all $\beta < \gamma$. Let $T = X - \cup\{H^*_\beta : \beta < \gamma\}$. Now $\mathcal{F}_\gamma$ is a LF-closed partial refinement of $\mathcal{U}$ whose union is contained in the closed, countably metacompact subspace T. For each $n \in N$, define

$$S(\gamma,n) = \{x : \mathrm{ord}(x, \mathcal{F}_\gamma) \leq n\} \cap T \text{, and}$$

$$\mathcal{S}_\gamma = \{S(\gamma,n) : n \in N\}.$$

Now $\mathcal{S}_\gamma$ is a countable monotone open cover of the countably metacompact subspace T. By 2.1.15 $\mathcal{S}_\gamma$ has a closed shrink

$$\mathcal{K}_\gamma = \{K(\gamma,n) : n \in N\}$$

such that $K(\gamma,n) \subset S(\gamma,n)$ for each $n \in N$.

For each $n \in N$, define

$$\mathcal{L}(\gamma,n) = \{F \cap K(\gamma,n) : F \in \mathcal{F}_\gamma\} \text{, and}$$

$$\mathcal{L}_\gamma = \cup \{\mathcal{L}(\gamma,n) : n \in N\} .$$

Since each member of $\mathcal{L}(\gamma,n)$ is contained in $S(\gamma,n)$, it follows that $\mathcal{L}(\gamma,n)$ is an n-bded-LF collection of closed subsets of X; therefore, by 2.2.11 $\mathcal{L}(\gamma,n)$ must have a LF-cozero-expansion $\mathcal{H}(\gamma,n)$ for each $n \in N$, which partially refines $\mathcal{U}$. It is easy to see that $\{\mathcal{H}(\gamma,n) : n \in N\}$ satisfies conditions (i) - (iii) above, and our construction is complete.

Since $\mathcal{H}(\gamma,n)$ is a LF collection of cozero sets, $\cup\, \mathcal{H}(\gamma,n)$ must be a cozero set for every $\gamma < \lambda$ and $n \in N$; hence, $\mathcal{H}^* = \{\cup \mathcal{H}(\gamma,n) : \gamma < \lambda,\ n \in N\}$ is a countable cozero cover of X. Now such a cover $\mathcal{H}^*$ has a LF-open refinement $\mathcal{W} = \{W(\gamma,n) : \gamma < \lambda,\ n \in N\}$ such that $W(\gamma,n) \subset \cup \mathcal{H}(\gamma,n)$ for every $\gamma < \lambda$, $n \in N$.

Define $\mathcal{V}(\gamma,n) = \{W(\gamma,n) \cap H : H \in \mathcal{H}(\gamma,n)\}$ for every $\gamma < \lambda$, $n \in N$, and $\mathcal{V} = \cup \{\mathcal{V}(\gamma,n) : \gamma < \lambda,\ n \in N\}$. It is easy to see that $\mathcal{V}$ is a LF-open refinement of $\mathcal{U}$, and hence $\mathcal{U}$ must be a normal cover of X. □

2.2.13. <u>Corollary</u>. A space X is paracompact iff X is CWN and B(LF,λ)-refinable.

<u>Proof</u>. The necessity should be clear. Now assume that X is CWN and B(LF,λ)-refinable. Since B(LF,λ)-refinability implies B(C,λ)-refinability, by 2.2.7 X must be countably metacompact. From 2.2.12 (b) above it now immediately follows that every open cover of X is normal.

Therefore, X is paracompact. □

After the next lemma we show that paracompactness and B(LF,λ)-refinability are also equivalent in any expandable space.

2.2.14. Lemma [51].

(a) Every paracompact space is expandable.

(b) A space X is countably paracompact iff X is countably expandable.

2.2.15. Theorem. A space X is paracompact iff X is B(LF,λ)-refinable and expandable.

Proof. By 2.2.14 above, the necessity is clear. To prove the sufficiency, assume that X is expandable and B(LF,λ)-refinable. Let $\mathcal{U}$ be an open cover of X, and $\mathcal{E} = \cup \{\mathcal{E}_\gamma : \gamma < \lambda\}$ a B(LF,λ)-refinement of $\mathcal{U}$. We use induction to construct a family $\mathcal{V}^* = \{\mathcal{V}_\gamma : \gamma < \lambda\}$ of collections of subsets of X satisfying

(i) $\mathcal{V}_\gamma$ is a LF-open partial refinement of $\mathcal{U}$ for each $\gamma < \lambda$, and

(ii) $\cup\{\cup \mathcal{E}_\beta : \beta < \gamma\} \subset \cup\{\cup \mathcal{V}_\beta : \beta < \gamma\}$ for each $\gamma < \lambda$.

Let $\gamma < \lambda$ be fixed, and assume that collections $\mathcal{V}_\beta$ have been constructed such that conditions (i) and (ii) above are satisfied for all $\beta < \gamma$. Define

$V^* = \cup\{\cup \mathcal{V}_\beta : \beta < \gamma\}$, and

$$\mathcal{F}_\gamma = \{ E - V^* : E \in \mathcal{E}_\gamma \} .$$

Now $\mathcal{F}_\gamma$ is a LF-closed partial refinement of $\mathcal{U}$, and X is expandable; hence, $\mathcal{F}_\gamma$ has a LF-open expansion $\mathcal{V}_\gamma$ which partially refines $\mathcal{U}$. It should be clear that $\cup \{ \cup \mathcal{E}_\beta : \beta < \gamma \} \subset \cup \{ \cup \mathcal{V}_\beta : \beta < \gamma \}$, and our construction is complete. Now define $\mathcal{V} = \cup \{ \mathcal{V}_\gamma : \gamma < \lambda \}$.

Since $\mathcal{E} = \cup \{ \mathcal{E}_\gamma : \gamma < \lambda \}$ covers X, conditions (i) and (ii) above imply that $\mathcal{V}$ is a σ-LF-open refinement of $\mathcal{U}$. Now $\{ \cup \mathcal{V}_\gamma : \gamma < \lambda \}$ is a countable open cover of X. By 2.2.14 (b) above, X is countably paracompact, and so $\{ \cup \mathcal{V}_\gamma : \gamma < \lambda \}$ has a LF-open refinement $\{ W_\gamma : \gamma < \lambda \}$ such that $W_\gamma \subset \cup \mathcal{V}_\gamma$ for each $\gamma < \lambda$. For each $\gamma < \lambda$, define

$\mathcal{G}_\gamma = \{ W_\gamma \cap V : V \in \mathcal{V}_\gamma \}$, and

$\mathcal{G} = \cup \{ \mathcal{G}_\gamma : \gamma < \lambda \}$.

It is easy to see that $\mathcal{G}$ is a LF-open refinement of $\mathcal{U}$. Therefore, X is paracompact. $\square$

§3. Sum and mapping theorems

Let Q^* represent a topological property, and suppose that $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is a cover of a space X such that F_α satisfies property Q^* for each $\alpha \in A$. Sum theorems address the "local to global" problem of establishing conditions on the collection $\mathcal{F}$ which will guarantee that X satisfies property Q^*. Mapping theorems are also connected with the preservation of the property Q^* under various kinds of maps. Results of either type are fundamental in the study of any given topological property Q^*. In this section we obtain such results for the property $B(P,\alpha)$-refinability. (For definitions of maps and sum theorems used below see 1.2.19 and 1.2.25.)

J. C. Smith [77] has shown that the property $B(D,\omega)$-refinability satisfies the Countable Sum Theorem. We now prove a more general result.

2.3.1. Theorem. Let P represent one of the following properties: D, LF, LC, PF, HCP, CP, or C. Then the property $B(P,\omega)$-refinability satisfies the Countable Sum Theorem.

Proof. Let $\{F_n : n \in N\}$ be a closed cover of a space X where each F_n is $B(P,\omega)$-refinable. Let $\mathcal{U}$ be an open cover of X, and define $\mathcal{U}_n = \{U \cap F_n : U \in \mathcal{U}\}$. By our assumption there exists a $B(P,\omega)$-refinement $\mathcal{E}_n = \cup\{\mathcal{E}(n,j) : j \in N\}$ of $\mathcal{U}_n$ which covers F_n. Now

$\mathcal{E}(n,j)$ is a relatively P collection of closed subsets of the subspace $F_n - \cup\{\cup\mathcal{E}(n,i) : i < j\}$ for every $n, j \in N$. Since F_n is a closed subspace of X, it should be clear that $\mathcal{E}(n,j)$ is a relatively P collection of closed subsets of the subspace $X - \cup\{\cup\mathcal{E}(n,i) : i < j\}$ for every $n, j \in N$. As in the proof of 2.1.12 we now use the same "diagonal process" to construct a $B(P,\omega)$-refinement $\mathcal{K}$ of $\mathcal{U}$. Indeed, define a well-order "<" on the set $\mathcal{J} = \{(i,j) : i, j \in N\}$ such that for every $(i,j), (k,m) \in \mathcal{J}$,

$(i,j) < (k,m)$ iff

(a) $i+j < k+m$, or

(b) $i+j = k+m$ and $j < m$.

Let $f : \mathcal{J} \longrightarrow N$ be the unique bijection which preserves this order. For each $n \in N$, define

$$\mathcal{H}_n = \mathcal{E}(i,j) \text{ such that } f(i,j) = n, \text{ and}$$

$$\mathcal{K}_n = \{H - \cup\{\cup\mathcal{H}_k : k < n\} : H \in \mathcal{H}_n\} .$$

As before, it follows that $\mathcal{K} = \cup\{\mathcal{K}_n : n \in N\}$ is a $B(P,\omega)$-refinement of $\mathcal{U}$. □

We now establish two lemmas which will be used to show that closed, continuous maps preserve the property $B(P,\alpha)$-refinability, where P represents HCP, CP, or C.

2.3.2. <u>Lemma</u>. Let $f : X \longrightarrow Y$ be a closed, continuous map from a space X into a space Y. If $A \subset X$, then $f(cl(A)) = cl(f(A))$.

Proof. It is an easy exercise to show that $cl(f(A)) \subset f(cl(A))$ follows from f being a closed map, and continuity implies $f(cl(A)) \subset cl(f(A))$. □

2.3.3. Lemma. Let $f : X \longrightarrow Y$ be a continuous, closed map from a space X into a space Y. If $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ is a CP (HCP) collection of subsets of X, then $f(\mathcal{H})$ is a CP (HCP) collection of subsets of Y.

Proof. (We prove only the CP case. The HCP case can be proved in a similar fashion.) Assume that $\mathcal{H}$ is CP, and let $A' \subset A$. From 2.3.2 above and the fact that $cl(\cup \{H_\alpha : \alpha \in A'\}) = \cup \{ cl(H_\alpha) : \alpha \in A'\}$ we have

$$
\begin{aligned}
f(cl(\cup \{H_\alpha : \alpha \in A'\}) &= f(\cup \{ cl(H_\alpha) : \alpha \in A'\}) \\
&= \cup \{f(cl(H_\alpha)) : \alpha \in A'\} \\
&= \cup \{cl(f(H_\alpha)) : \alpha \in A'\} \\
&\subset cl(\cup \{f(H_\alpha) : \alpha \in A'\}) \\
&= f(cl(\cup \{H_\alpha : \alpha \in A'\})).
\end{aligned}
$$

It follows that $cl(\cup \{f(H_\alpha) : \alpha \in A'\})$

$= \cup \{cl(f(H_\alpha)) : \alpha \in A'\}$. Therefore, $f(\mathcal{H})$ is CP. □

2.3.4. Theorem. Let $f : X \longrightarrow Y$ be a continuous, closed map from a space X onto a space Y, and let P represent one of the following properties: HCP, CP, or C. If X is $B(P,\alpha)$-refinable, then so is Y.

Proof. (We prove only the CP case. The other cases can be proved in a similar fashion.) Assume that X is B(CP,α)-refinable and let $\mathcal{U}$ be an open cover of Y. There exists a B(CP,α)-refinement $\mathcal{E} = \cup \{\mathcal{E}_\gamma : \gamma < \alpha\}$ of $\mathcal{V} = f^{-1}(\mathcal{U})$. For each $\gamma < \alpha$, define

$$\mathcal{F}_\gamma = f(\mathcal{E}_\gamma),$$
$$\mathcal{H}_\gamma = \{F - \cup\{\cup \mathcal{F}_\beta : \beta < \gamma\} : F \in \mathcal{F}_\gamma\}, \text{ and}$$
$$\mathcal{H} = \cup\{\mathcal{H}_\gamma : \gamma < \alpha\}.$$

For each $\gamma < \alpha$, $\cup\{\mathcal{H}_\beta : \beta < \gamma\}$ is closed in Y since $\cup\{\cup \mathcal{E}_\beta : \beta < \gamma\}$ is closed in X and $f(\cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}) = \cup\{\cup \mathcal{H}_\beta : \beta < \gamma\}$. By construction, $\{\cup \mathcal{H}_\gamma : \gamma < \alpha\}$ partitions Y, and $\mathcal{H}$ refines $\mathcal{U}$. We assert that $\mathcal{H}$ is a B(CP,α)-refinement of $\mathcal{U}$. Indeed, it only remains to show that for fixed $\gamma < \alpha$, $\mathcal{H}_\gamma$ is a relatively CP collection of closed subsets of the subspace $Y - \cup\{\cup \mathcal{H}_\beta : \beta < \gamma\}$. Now $\mathcal{E}_\gamma$ is a relatively CP collection of closed subsets of the subspace $X - \cup\{\cup \mathcal{E}_\beta : \beta < \gamma\}$, and $\cup\{\cup \mathcal{E}_\beta : \beta < \gamma\}$ is closed in X; hence,

$$\mathcal{E}_\gamma^* = \{E \cup (\cup\{\cup \mathcal{E}_\beta : \beta < \gamma\}) : E \in \mathcal{E}_\gamma\}$$

is a CP collection of closed subsets of X. By 2.3.3 above, $\mathcal{F}_\gamma^* = f(\mathcal{E}_\gamma^*)$ is a CP collection of closed subsets of Y, and by construction,

$$\mathcal{H}_\gamma = \{F^* - \cup\{\cup \mathcal{H}_\beta : \beta < \gamma\} : F^* \in \mathcal{F}_\gamma^*\}.$$

It follows that $\mathcal{H}_\gamma$ is a relatively CP collection of closed subsets of the subspace $Y - \cup\{\cup \mathcal{H}_\beta : \beta < \gamma\}$, and therefore Y is B(CP,α)-refinable. $\square$

J. Chaber [24] stated without proof that perfect maps preserve $B(LF,\omega)$-refinability. J. C. Smith [77] observed that in general, perfect maps preserve $B(LF,\alpha)$-refinability; actually, these maps only need to be quasi-perfect, a result included in our next theorem. First, we prove a lemma.

2.3.5. Lemma. Let $f : X \longrightarrow Y$ be a continuous, closed map from a space X onto a space Y. Let S be a subset of X, and $\mathcal{H}$ a collection of subsets of X - S. Suppose

(i) $f^{-1}(y)$ is countably compact (Lindelöf) for each $y \in Y - f(S)$, and

(ii) $\mathcal{H}$ is a relatively LF (LC) collection of subsets of the subspace X - S.

Then $\{f(H) - f(S) : H \in \mathcal{H}\}$ is a relatively LF (LC) collection of subsets of the subspace $Y - f(S)$.

Proof. (We prove only the LC case. The proof of the LF case is similar, and uses the fact that LF collections are finite in countably compact spaces.) Let $\mathcal{H}$ be a relatively LC collection of subsets of the subspace X - S, and let $y \in Y$. Assume that $f^{-1}(y)$ is Lindelöf. Since $f^{-1}(y) \subset X - S$ is Lindelöf and $\mathcal{H}$ relatively LC in X - S, it follows that there exists an open subset W of X such that $f^{-1}(y) \subset W$, and W hits at most countably many members of $\mathcal{H}$. We can assume that W is f-saturated since f is a closed, continuous map. It now follows that f(W) is a neighborhood of y which hits at most countably many members of

$\{f(H) - f(S) : H \in \mathcal{H}\}$, and the proof is complete. □

2.3.6. Theorem. Let $f : X \longrightarrow Y$ be a continuous, closed map from a space X onto a space Y.

(a) ([24], [77]) If f is quasi-perfect and X is $B(LF,\alpha)$-refinable, then Y is $B(LF,\alpha)$-refinable.

(b) If $f^{-1}(y)$ is Lindelöf for each $y \in Y$ and X is $B(LC,\alpha)$-refinable, then Y is $B(LC,\alpha)$-refinable.

Proof. (We prove only (b).) Assume that $f^{-1}(y)$ is Lindelöf for each $y \in Y$, and that X is $B(LC,\alpha)$-refinable. Let $\mathcal{U}$ be an open cover of Y. There exists a $B(LC,\alpha)$-refinement $\mathcal{E} = \cup\{\mathcal{E}_\gamma : \gamma < \alpha\}$ of $\mathcal{V} = f^{-1}(\mathcal{U})$. For each $\gamma < \alpha$, let $\mathcal{F}_\gamma = f(\mathcal{E}_\gamma)$. Define

$$\mathcal{H}_\gamma = \{F - \cup\{\cup \mathcal{F}_\beta : \beta < \gamma\} : F \in \mathcal{F}_\gamma\}, \text{ and}$$
$$\mathcal{H} = \cup\{\mathcal{H}_\gamma : \gamma < \alpha\}.$$

By 2.3.5, for each $\gamma < \alpha$, $\mathcal{F}_\gamma$ is a relatively LC collection of subsets of the subspace $X - \cup\{\cup \mathcal{F}_\beta : \beta < \gamma\}$; therefore, by the same argument as used in 2.3.4, it is easy to show that $\mathcal{H}$ is a $B(LC,\alpha)$-refinement of $\mathcal{U}$. □

2.3.7. Corollary. Let X be a hereditarily countably metacompact space, and let $f : X \longrightarrow Y$ be a quasi-perfect map from X onto a space Y. If X is $B(D,\alpha)$-refinable, then Y is $B(D,\omega\times\alpha)$-refinable.

Proof. From the characterization 2.1.15 of countable metacompactness, it is easy to see that countable metacompactness is preserved under closed, continuous maps; hence, Y is hereditarily countably metacompact. Now X is certainly $B(LF,\alpha)$-refinable, and so Y is $B(LF,\alpha)$-refinable by 2.3.6(a) above. It thus follows by 2.1.16 that Y is $B(D,\omega\times\alpha)$-refinable. $\square$

D. K. Burke [21] recently showed that the property weak θ-refinability is preserved under perfect maps; however, his technique fails to give us an analogous result for weak $\bar{\theta}$-refinability. H. J. K. Junnila [40] proved that θ-refinability is preserved under continuous, closed maps. It is still an open question whether weak $\bar{\theta}$-refinability is preserved under bded-maps which are closed and continuous, or more generally, under perfect maps. We are able to show however that bded-weak $\bar{\theta}$-refinability is preserved under closed, continuous, bded-maps. This result will follow easily from the following lemma.

2.3.8. Lemma. Let $f : X \longrightarrow Y$ be a continuous, closed, and bded-map from a space X onto a space Y. Let $S \subset X$, and suppose that $\mathcal{H}$ is a relatively discrete collection of subsets of the subspace $X - S$. Then $\{f(H) - f(S) : H \in \mathcal{H}\}$ is a relatively bded-LF collection of subsets of the subspace $Y - f(S)$.

Proof. Assume that f is an n-bded-map for some $n \in N$. Let $y \in Y - f(S)$ so that $f^{-1}(y) \subset X - S$, and $|f^{-1}(y)| \leq n$.

Now each point in $f^{-1}(y)$ has a neighborhood which hits at most one member of $\mathcal{H}$. It follows that there exists an f-saturated open subset W of X such that $f^{-1}(y) \subset W$, and W hits at most n members of $\mathcal{H}$; hence, f(W) is a neighborhood of y which hits at most n members of $\{f(H) - f(S) : H \in \mathcal{H}\}$. Therefore, $\{f(H) - f(S) : H \in \mathcal{H}\}$ is a relatively n-bded-LF collection of subsets of the subspace Y - f(S). □

2.3.9. Theorem. Let $f : X \longrightarrow Y$ be a continuous, closed, and bded-map from a space X onto a space Y. If X is $B(D,\alpha)$-refinable, then Y is $B(D,\alpha)$-refinable.

Proof. Assume that X is $B(D,\alpha)$-refinable. From 2.3.8 above we have that Y must be $B(\text{bded-LF},\alpha)$-refinable; thus, by 2.1.2, Y is $B(D,\alpha)$-refinable. □

2.3.10. Question. Is $B(D,\alpha)$-refinability preserved under continuous, closed, finite-to-one maps?

Lemma 1.2.26 is a well-known result which indicates how mapping theorems can be used to establish sum theorems. We use this lemma in the proof of the following corollary.

2.3.11. Corollary.

(a) The property $B(D,\alpha)$-refinability satisfies the bded-LF Sum Theorem.

(b) Let P represent one of the following properties: LF, LC, HCP, CP, or C. Then $B(P,\alpha)$-refinability satisfies the LF Sum Theorem.

Proof. It should be clear that $B(P,\alpha)$-refinability is preserved under disjoint sums for any of the properties P mentioned above. Since $B(D,\alpha)$-refinability is preserved under continuous, closed, bded-maps, and $B(P,\alpha)$-refinability is preserved under maps which are weaker than continuous, closed, finite-to-one maps for the remaining properties P above, results (a) and (b) above follow immediately from 1.2.26. $\square$

CHAPTER III

GENERALIZATIONS OF COLLECTIONWISE NORMALITY

In the class of CWN spaces, paracompactness is equivalent to a number of weaker properties which include metacompactness ([57], [61]), subparacompactness [16], Θ-refinability [87], and weak $\bar{\Theta}$-refinability [74]. In this chapter we study a more general class of spaces by way of the properties collectionwise subnormality (CWSN), collectionwise δ-normality (CWδN), and strong-CWSN, all of which are weaker than CWN but still imply the equivalence of Θ-refinability and subparacompactness. These generalizations of CWN involve the expansion of discrete collections of sets to special pairwise disjoint collections of G_δ-sets.

J. Chaber [22] proved that Θ-refinability and subparacompactness are equivalent in any CWSN space. H. J. K. Junnila [44] recently introduced the notion CWδN and then generalized Chaber's result by proving that a space X is subparacompact iff X is CWδN and Θ-refinable. In §1 of this chapter we obtain a B(D,ω)-characterization of CWSN, and then use it to prove that a space X is subparacompact iff X is CWSN and B(D,ω)-refinable--thus generalizing Chaber's result mentioned above. It remains an open problem as to whether Junnila's result can be generalized by weakening "Θ-refinable" to "$B(D,\omega)$-refinable."

In §2 we introduce the notion of strong-CWSN, a property which is equivalent to CWN in any normal space. We show that if a space X is strong-CWSN and $B(D,\omega)$-refinable, then X must be both subparacompact and metacompact. We also show that paracompactness and $B(LF,\lambda)$-refinability are equivalent in any strong-CWSN space which is also countably paracompact.

§1. Collectionwise subnormality

The notions of subnormality, collectionwise $\mathcal{S}$-normality (CW$\mathcal{S}$N), and collectionwise subnormality (CWSN) are all generalizations of CWN which play important roles in the study of subparacompactness. We give their definitions below and state several known theorems. Our main result is to show that $B(D,\omega)$-refinability is equivalent to subparacompactness in any CWSN space. See diagram 1.1.2 for the general relationships between these properties and others.

The definition of subnormality was motivated by the following theorem.

3.1.1. Theorem. Let X be a normal space. Then TFAE.

(a) X is countably paracompact.

(b) X is countably subparacompact.

(c) X is countably metacompact.

Proof. The implication (a) $\longrightarrow$ (b) should be clear. For a proof of (b) $\longrightarrow$ (c), see [52]. The implication (c) $\longrightarrow$ (a) follows from results 2.2.1(c) and 2.2.2. Note that normality is needed only in the proof of (c) $\longrightarrow$ (a). □

T. R. Kramer [52] discovered the property subnormality in his attempt to find a condition weaker than normality under which countable metacompactness and countable subparacompactness are equivalent. J. Chaber [22] independently introduced an equivalent concept which he also referred to as subnormality.

3.1.2. Definition. A space X is subnormal iff either of the following two equivalent conditions is satisfied.

(a) [52] Every finite open cover of X has an F_σ-shrink.

(b) [22] Any two disjoint closed subsets of X have a disjoint G_δ-expansion.

3.1.3. Theorem [52]. A space X is countably subparacompact iff X is subnormal and countably metacompact.

The next theorem lists several properties which are known to be equivalent to paracompactness in a CWN space.

3.1.4. Theorem. Let X be a CWN space. Then TFAE.

(a) X is paracompact.

(b) ([57], [61]) X is metacompact.

(c) [16] X is subparacompact.

(d) [87] X is θ-refinable.

(e) [74] X is weak $\bar{\theta}$-refinable.

(f) (2.2.13) X is $B(LF,\lambda)$-refinable.

Y. Katuta [50] defined the following concept which he called discrete-subexpandability; however, we prefer to use J. Chaber's terminology collectionwise subnormality (CWSN). J. Chaber [22] has shown that subparacompactness and θ-refinability are equivalent in this class of spaces.

3.1.5. Definition [50]. A space X is CWSN provided every discrete collection $\mathcal{D}$ of closed subsets of X has a pairwise disjoint G_δ-expansion which is also an almost θ-expansion of $\mathcal{D}$.

3.1.6. Theorem [22]. A space X is subparacompact iff X is CWSN and θ-refinable.

In generalizing J. Chaber's result above, H. J. K. Junnila [44] weakened the CWSN condition to the following notion.

3.1.7. Definition [44]. A space X is CWδN provided every discrete collection of closed subsets of X has a pairwise disjoint G_δ-expansion.

3.1.8. Theorem [44]. A space X is subparacompact iff X is CWδN and θ-refinable.

Later in this section we will generalize J. Chaber's result 3.1.6 above by weakening the θ-refinability condition to $B(D,\omega)$-refinability; unfortunately, our proof requires CWSN. It remains an open question whether 3.1.6 can be further generalized by simultaneously weakening the CWSN and θ-refinability conditions.

H. J. K. Junnila [44] proved the following result 3.1.9(a) and then used this result to easily prove 3.1.10(a) from which 3.1.8 above follows. We use analogous proofs to verify 3.1.9(b) and 3.1.10(b) from which the sufficiency in our main result 3.1.15 will follow.

3.1.9. Lemma.

(a) [44] Let X be a CWδN space. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a PF-open cover of X, and $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ a discrete-closed partial refinement of $\mathcal{U}$ such that $D_\alpha \subset U_\alpha$ for each $\alpha \in A$. Then there exists a G_δ-set K and a σ-discrete-closed partial refinement $\mathcal{E}$ of $\mathcal{U}$ such that $\cup\mathcal{D} \subset K \subset \cup\mathcal{E}$.

(b) Let X be a CWSN space. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X, and $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ a discrete-closed partial refinement of $\mathcal{U}$ such that $D_\alpha \subset U_\alpha$ for each $\alpha \in A$. Then there exists a G_δ-set K and a σ-discrete-closed partial refinement $\mathcal{E}$ of $\mathcal{U}$ such that $\cup\mathcal{D} \subset K \subset \cup\mathcal{E}$.

Proof. We prove only part (b). Part (a) is proved in a similar fashion. Note that in part (a), the condition CWδN guarantees that $\mathcal{D}$ has a pairwise disjoint G_δ-expansion which we can assume is PF, since $\mathcal{U}$ is PF. Hence, every $x \in X$ will have finite order with respect to some "level" of the expansion. H. J. K. Junnila's proof depends on this last property.

Assume that X is CWSN. Then there exists a pairwise disjoint G_δ-expansion

$$\mathcal{Q} = \{Q_\alpha = \cap\{Q(\alpha,n) : n \in N\} : \alpha \in A\}$$

of $\mathcal{D}$ such that

$$\cup\ \{\{Q(\alpha,n) : \alpha \in A\} : n \in N\}$$

is an almost Θ-expansion of $\mathcal{D}$, and $D_\alpha \subset Q(\alpha,n) \subset U_\alpha$ for every $\alpha \in A$, $n \in N$.

For each $n \in N$, let

$$\mathcal{Q}_n = \{Q(\alpha,n) : \alpha \in A\} \text{ , and define}$$

$$C_n = X - \cup \mathcal{Q}_n.$$

Then C_n and $\cup\mathcal{D}$ are disjoint closed sets, so by 3.1.2(b) there exist disjoint G_δ-sets H_n and K_n such that

$$C_n \subset H_n \text{ , and}$$

$$\cup\mathcal{D} \subset K_n.$$

For each n, we denote

$$H_n = \cap \{H(n,j) : j \in N\}$$

such that $H(n,j)$ is open for every $j \in N$.

For every $n, j \in N$, define

$$\mathcal{V}(n,j) = \{H(n,j) \cap U_\alpha : \alpha \in A\} \cup \mathcal{Q}_n.$$

By construction, $\mathcal{V}(n,j)$ is an open refinement of $\mathcal{U}$. Now define

$$E(n,j) = \{x : \mathrm{ord}(x, \mathcal{V}(n,j) = 1\} ,$$

$$\mathcal{E}(n,j) = \{E(n,j) \cap V : V \in \mathcal{V}(n,j)\} \text{ , and}$$

$$\mathcal{E} = \cup \{\mathcal{E}(n,j) : n, j \in N\} .$$

It should be clear that $\mathcal{E}$ is a σ-discrete-closed partial refinement of $\mathcal{U}$ since each $\mathcal{V}(n,j)$ is an open cover of X which refines $\mathcal{U}$. Define

$$K = \cap \{K_n : n \in N\} .$$

By construction, K is a G_δ-set such that $\cup\mathcal{D} \subset K$. It remains only to show that $K \subset \cup\mathcal{E}$. Indeed, let $x \in K$.

Since $\mathcal{Q}$ is pairwise disjoint and $\cup\{\mathcal{Q}_n : n \in N\}$ is an almost Θ-collection, there exists $n(x) \in N$ such that $\mathrm{ord}(x, \mathcal{Q}_{n(x)}) \leq 1$. Now $K_{n(x)} \subset \cup\mathcal{Q}_{n(x)}$ from above, so that $K \subset \cup\mathcal{Q}_{n(x)}$. Thus, $x \in \cup\mathcal{Q}_{n(x)}$ and hence $\mathrm{ord}(x, \mathcal{Q}_{n(x)}) = 1$. Since $H_{n(x)} \cap K_{n(x)} = \emptyset$ and $x \in K_{n(x)}$ there exists $j(x) \in N$ such that $x \notin H(n(x), j(x))$. Therefore, $\mathrm{ord}(x, \mathcal{V}(n(x), j(x))) = \mathrm{ord}(x, \mathcal{Q}_{n(x)}) = 1$, implying that $x \in \cup\mathcal{E}(n(x), j(x))$, and hence $x \in \cup\mathcal{E}$.

It follows that $\cup\mathcal{D} \subset K \subset \cup\mathcal{E}$ and the proof is complete. $\square$

3.1.10. <u>Theorem</u>.

(a) [44] If X is a CWδN space, then every Θ-cover of X has a σ-discrete-closed refinement.

(b) If X is a CWSN space, and $\mathcal{U}$ is an open cover of X which has a $B(D,\omega)$-refinement, then $\mathcal{U}$ has a σ-discrete-closed refinement.

<u>Proof</u>. We use 3.1.9(b) to prove part (b). The proof of part (a) follows from 3.1.9(a) in a similar fashion.

Let $\mathcal{E} = \cup\{\mathcal{E}_n : n \in N\}$ be a $B(D,\omega)$-refinement of $\mathcal{U}$. By induction we construct for each $n \in N$,

(a_1) a G_δ-set $Q_n = \cap\{Q(n,j) : j \in N\}$, and

(a_2) a σ-discrete-closed partial refinement $\mathcal{F}_n$ of $\mathcal{U}$, such that

(a_3) $\cup\{\cup\mathcal{E}_k : 1 \leq k \leq n\} \subset Q_n \subset \cup\mathcal{F}_n$.

Then $\mathcal{F} = \cup \{\mathcal{F}_n : n \in N\}$ will be the desired σ-discrete closed refinement of $\mathcal{U}$, and our proof will be complete.

The above conditions are vacuously satisfied for $n = 1$. Now let $n > 1$ be fixed, and assume that Q_i and $\mathcal{F}_i$ have been constructed satisfying the above conditions for $1 \leq i < n$. For each $j \in N$, define

$$\mathcal{D}(n,j) = \{ E - Q(n-1,j) : E \in \mathcal{E}_n \} .$$

It should be clear that $\mathcal{D}(n,j)$ is a discrete-closed partial refinement of $\mathcal{U}$. By 3.1.9(b) above, there exists a G_δ-set $K(n,j)$ and a σ-discrete-closed partial refinement $\mathcal{F}(n,j)$ of $\mathcal{U}$ such that

$$\cup \mathcal{D}(n,j) \subset K(n,j) \subset \cup \mathcal{F}(n,j).$$

Define

$$Q(n,j) = K(n,j) \cup Q(n-1,j),$$

$$Q_n = \cap \{ Q(n,j) : j \in N\} \text{, and}$$

$$\mathcal{F}_n = \mathcal{F}_{n-1} \cup (\cup\{\mathcal{F}(n,j) : j \in N\}) .$$

By construction, Q_n and $\mathcal{F}_n$ satisfy conditions (a_1) and (a_2) above, so it remains only to show that (a_3) is satisfied.

(i) Let $x \in \cup\{\cup \mathcal{E}_k : 1 \leq k \leq n\}$. We show that $x \in Q_n$. Now if $x \in \cup \{ \cup \mathcal{E}_k : 1 \leq k < n \}$, then $x \in Q_{n-1}$, in which case $x \in Q(n,j)$ for every j, implying $x \in Q_n$. Next, suppose that $x \in \cup \mathcal{E}_n$ and let $j' \in N$ be fixed. If $x \in Q(n-1,j')$, then $x \in Q(n,j')$. If $x \notin Q(n-1,j')$, then $x \in E - Q(n-1,j')$ for some $E \in \mathcal{E}_n$ and so $x \in \cup \mathcal{D}(n,j')$. Hence, $x \in K(n,j') \subset Q(n,j')$. In any case, $x \in Q(n,j)$ for every j. Therefore, $x \in Q_n$, and it thus follows that $\cup \{\cup \mathcal{E}_k : 1 \leq k \leq n\} \subset Q_n$.

(ii) To see that $Q_n \subset \cup \mathcal{F}_n$ let $x \in Q_n$. If $x \in Q(n-1,j)$ for some $j \in N$, then $x \in Q_{n-1}$ so that $x \in \cup \mathcal{F}_{n-1}$. Now suppose there exists some $j' \in N$ such that $x \notin Q(n-1,j')$. Then $x \in K(n,j')$ must be the case, and so $x \in \cup \mathcal{F}(n,j')$. Since $\cup \mathcal{F}_{n-1} \subset \cup \mathcal{F}_n$ and $\cup \mathcal{F}(n,j) \subset \cup \mathcal{F}_n$ for each j, in either case it follows that $x \in \cup \mathcal{F}_n$ and hence $Q_n \subset \cup \mathcal{F}_n$.

Our construction is now complete, and it should be clear that $\mathcal{F} = \cup \{\mathcal{F}_n : n \in N\}$ is a σ-discrete-closed refinement of $\mathcal{U}$. $\square$

3.1.11. Questions.

(a) In 3.1.10(a) above, can we replace θ-cover by weak $\bar{\theta}$-cover?

(b) In 3.1.10(b), can we replace ω by an arbitrary countable ordinal λ?

3.1.12. Theorem. Let X be a space with the property that every open cover of X which has a $B(D,\omega)$-refinement also has a σ-cushioned refinement. Then X is CWSN.

Proof. Let $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X. Define

$$\mathcal{U}_1 = \{U_\alpha = X - \cup \{D_\beta : \beta \neq \alpha\} : \alpha \in A\},$$

$$\mathcal{U}_2 = \{X - \cup\mathcal{D}\} \quad , \text{ and}$$
$$\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \; .$$

Now every $x \in \cup\mathcal{D}$ has order 1 with respect to $\mathcal{U}_1$, and every $x \notin \cup\mathcal{D}$ has order 1 with respect to $\mathcal{U}_2$. Hence, $\mathcal{U}$ is a bded-weak $\bar{\theta}$-cover of X and has a $B(D,\omega)$-refinement. Let $\mathcal{F} = \cup\{\mathcal{F}_n = (\{H_n\} \cup \{F(n,\alpha) : \alpha \in A\}) : n \in N\}$ be a σ-cushioned refinement of $\mathcal{U}$ such that

$$H_n \subset X - \cup\mathcal{D} \quad , \text{ and}$$
$$F(n,\alpha) \subset U_\alpha \text{ for every } n \in N, \; \alpha \in A.$$

For every $n \in N$, $\alpha \in A$, define

$$W(n,\alpha) = X - cl(\cup\mathcal{F}_n) \text{ if } F(n,\alpha) \cap D = \emptyset, \text{ and}$$
$$W(n,\alpha) = X - cl(\cup(\mathcal{F}_n - \{F(n,\alpha)\})) \text{ if } F(n,\alpha) \cap D_\alpha \neq \emptyset.$$

For each $n \in N$ recall that $\mathcal{F}_n$ is cushioned in $\mathcal{U}$, and that $U_\beta \cap D_\alpha = \emptyset$ whenever $\beta \neq \alpha$. It follows that $\mathcal{W}_n = \{W(n,\alpha) : \alpha \in A\}$ is an open expansion of $\mathcal{D}$ such that $D(n,\alpha) \subset W(n,\alpha)$ for each $\alpha \in A$. Hence, $\mathcal{W} = \{W_\alpha = \cap\{W(n,\alpha) : n \in N\} : \alpha \in A\}$ is a G_δ-expansion of $\mathcal{D}$.

In order to verify that $\mathcal{W}$ is pairwise disjoint and that $\cup\{\mathcal{W}_n : n \in N\}$ is an almost θ-expansion of $\mathcal{D}$, it suffices to show for each $x \in X$ that there exists some $n(x) \in N$ such that $ord(x,\mathcal{W}_{n(x)}) \leq 1$. Now let $x \in X$ be fixed. Since $\mathcal{F}$ covers X, there exists some $n(x) \in N$ such that either

(i) $x \in F(n(x),\gamma)$ for some $\gamma \in A$, or

(ii) $x \in H_{n(x)}$.

Assume case (i). Now $F(n(x),\gamma) \subset U_\gamma$, and hence $F(n(x),\gamma) \cap D_\alpha = \emptyset$ for every $\alpha \neq \gamma$, so by construction $F(n(x),\gamma)$ misses $W(n(x),\alpha)$ for every $\alpha \neq \gamma$. It follows that $\operatorname{ord}(x,\mathcal{W}_{n(x)}) \leq 1$. Next assume case (ii) above. It should be clear that $H_{n(x)}$ misses every member of $\mathcal{W}_{n(x)}$ and so $\operatorname{ord}(x,\mathcal{W}_{n(x)}) = 0$. Therefore, in any case $\operatorname{ord}(x,\mathcal{W}_{n(x)}) \leq 1$, and our proof is complete. □

3.1.13. <u>Corollary</u>. For any space X, TFAE.

(a) X is CWSN.

(b) Every open cover of X which has a $B(D,\omega)$-refinement also has a σ-discrete-closed refinement.

(c) Every open cover of X which has a $B(D,\omega)$-refinement also has a σ-LF-closed refinement.

(d) Every open cover of X which has a $B(D,\omega)$-refinement also has a σ-HCP-closed refinement.

(e) Every open cover of X which has a $B(D,\omega)$-refinement also has a σ-CP-closed refinement.

(f) Every open cover of X which has a $B(D,\omega)$-refinement also has a σ-cushioned-refinement.

<u>Proof</u>. The implications (b) $\longrightarrow$ (c) $\longrightarrow$ (d) $\longrightarrow$ (e) $\longrightarrow$ (f) should be clear. By 3.1.10, (a) $\longrightarrow$ (b), and 3.1.12 implies (f) $\longrightarrow$ (a). □

From the following characterization of subparacompactness given by D. K. Burke [20], it immediately follows that subparacompactness implies Θ-refinability.

3.1.14. Theorem [20]. A space X is subparacompact iff every open cover $\mathcal{U}$ of X has a Θ-refinement $\mathcal{G} = \cup\{\mathcal{G}_n : n \in N\}$ such that for every $x \in X$, there exists $n(x) \in N$ such that $\mathrm{ord}(x, \mathcal{G}_{n(x)}) = 1$.

3.1.15. Theorem. A space X is subparacompact iff X is CWSN and $B(D,\omega)$-refinable.

Proof. The sufficiency follows immediately from 3.1.10(b). Now assume that X is subparacompact. By 3.1.14 X is Θ-refinable and hence $B(D,\omega)$-refinable. Also, every open cover of X--and in particular, every open cover of X which has a $B(D,\omega)$-refinement--has a σ-discrete closed refinement, so by 3.1.13(b) X is CWSN. $\square$

3.1.16. Questions.

(a) Can CWSN be weakened to CW$\mathcal{I}$N in 3.1.15 above?

(b) Can $B(D,\omega)$-refinable be weakened to weak $\bar{\Theta}$-refinable or $B(LF,\omega)$-refinable in 3.1.15 ?

§2. Strong-collectionwise subnormality

In the definition of CWSN given in the last section, if the notion of "almost Θ-expansion" is strengthened to "Θ-expansion" we get a property which we will refer to as strong-CWSN. We will show that the properties $B(D,\omega)$-refinability, metacompactness, and subparacompactness are eqivalent in any strong-CWSN space.

3.2.1. Definition. A space X is strong-CWSN provided every discrete collection $\mathcal{D}$ of closed subsets of X has a pairwise disjoint G_δ-expansion which is also a Θ-expansion of $\mathcal{D}$.

Recall the if $\mathcal{G} = \cup \{ \mathcal{G}_n : n \in N \}$ is a Θ-expansion in a space X and $x \in X$, then there exists $n(x) \in N$ such that $\mathcal{G}_{n(x)}$ is LF at x. Clearly,

$$\text{CWN} \longrightarrow \text{strong-CWSN} \longrightarrow \text{CWSN}.$$

Later we show that CWSN is strictly weaker than strong-CWSN and that the properties CWN and strong-CWSN are equivalent in any normal space; however, it is not known in general whether the properties CWN and strong-CWSN are equivalent.

We now obtain characterizations of both CWSN and strong-CWSN.

3.2.2. Theorem.

(a) A space X is CWSN iff X is CWδN and almost discretely-Θ-expandable.

(b) A space X is strong-CWSN iff X is CWδN and discretely-Θ-expandable.

Proof. (We prove only part (b). The proof of (a) follows in a similar fashion.) The necessity is clear. To prove the sufficiency, let $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X. Since X is CWδN and discretely-Θ-expandable, $\mathcal{D}$ has a pairwise disjoint G_δ-expansion

$$\mathcal{H} = \{H_\alpha = \cap \{H(\alpha,n) : n \in N\} \quad : \alpha \in A\}$$

and a Θ-expansion

$$\mathcal{G} = \cup \{\mathcal{G}_n = \{G(\alpha,n) : \alpha \in A\} \quad : n \in N\}.$$

We may assume that $\{G(\alpha,n) : n \in N\}$ and $\{H(\alpha,n) : n \in N\}$ are nonincreasing for every $\alpha \in A$. For each $\alpha \in A$ and $n \in N$, define

$$K(\alpha,n) = G(\alpha,n) \cap H(\alpha,n).$$

It is easy to see that

$$\mathcal{K} = \{K_\alpha = \cap \{K(\alpha,n) : n \in N\} \quad : \alpha \in A\}$$

is a pairwise disjoint collection of G_δ-sets and

$$\cup \{\{K(\alpha,n) : \alpha \in A\} \quad : n \in N\}$$

is a Θ-expansion of $\mathcal{D}$. $\square$

Our next example shows that in general

$$\text{CWSN} \not\rightarrow \text{strong-CWSN} \not\leftarrow \text{subparacompact}.$$

3.2.3. Example. Let S be the Sorgenfrey line (the real line with basic neighborhoods of the form [a,b)). Then

$X = S \times S$ (see [80]) is a Tychonoff space which is not metacompact. D. J. Lutzer [54] has shown that X is subparacompact, and hence X is CWSN. By theorem 3.2.9 below, X cannot be strong-CWSN, since subparacompactness and metacompactness are equivalent in any strong-CWSN space. $\square$

Since every strong-CWSN space is discretely-θ-expandable, by the following result of J. C. Smith [72] we immediately observe that CWN and strong-CWSN are equivalent in any normal space.

3.2.4. Theorem [72]. A space X is CWN iff X is normal and discretely-θ-expandable.

3.2.5. Corollary. A space X is CWN iff X is normal and strong-CWSN.

3.2.6. Example. Let X be any countable set with the cofinite topology (see [80]). It is easy to see that X is a strong-CWSN space which is T_1 but not T_2. Hence, in general

$$\text{strong-CWSN} \not\rightarrow \text{normal.}$$

3.2.7. Question. Does there exist a Tychonoff space which is strong-CWSN but not CWN?

Our next lemma will be used to show that the properties $B(D,\omega)$-refinability, metacompactness, and subparacompactness are equivalent in any strong-CWSN space.

3.2.8. Lemma. Let X be a discretely-Θ-expandable space. If $\mathcal{U}$ is a collection (not necessarily a cover) of open subsets of X, and $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ is a discrete-closed partial refinement of $\mathcal{U}$, then $\mathcal{U}$ has a $\prec$-PF-open partial refinement $\mathcal{V}$ such that $\cup\mathcal{D} \subset \cup\mathcal{V}$.

Proof. Since X is discretely-Θ-expandable, there exists a Θ-expansion $\mathcal{G} = \cup\{\mathcal{G}_n = \{G(\alpha,n) : \alpha \in A\} : n \in N\}$ of $\mathcal{D}$ such that $D_\alpha \subset G(\alpha,n)$ for every $\alpha \in A$, $n \in N$, and $\mathcal{G}$ is a partial refinement of $\mathcal{U}$. For each $n \in N$, define

$$V_n = \{x : \mathcal{G}_n \text{ is LF at } x\},$$

$$\mathcal{V}_n = \{V_n \cap G(\alpha,n) : \alpha \in A\}, \text{ and}$$

$$\mathcal{V} = \cup\{\mathcal{V}_n : n \in N\}.$$

By construction, $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ such that $\cup\mathcal{D} \subset \cup\mathcal{V}$. To show that $\mathcal{V}$ is an open collection, it suffices to verify that V_n is open for each $n \in N$. Indeed, if $x \in V_n$ then x has a neighborhood W(x) which hits at most finitely many members of $\mathcal{G}_n$ and so $W(x) \subset V_n$ must be the case. Therefore, V_n is open. Now $\mathcal{G}_n$ is LF on V_n, and in particular, PF on V_n. By construction, it thus follows that $\mathcal{V}_n$ is PF for each $n \in N$, and our proof is complete. □

3.2.9. Theorem. For any strong-CWSN space X, TFAE.

(a) X is subparacompact.

(b) X is metacompact.

(c) X is Θ-refinable.

(d) X is $B(D,\omega)$-refinable.

Proof. Clearly, (b) $\longrightarrow$ (c), and since X is CWSN, by 3.1.15 we have (a) $\longrightarrow$ (c) $\longrightarrow$ (d). It remains only to show that (a) $\longrightarrow$ (b). Assume that X is strong-CWSN and subparacompact, and let $\mathcal{U}$ be an open cover of X. Now $\mathcal{U}$ has a σ-discrete-closed refinement $\mathcal{D} = \cup \{\mathcal{D}_n : n \in N\}$. Since X is strong-CWSN, and hence discretely-θ-expandable, by 3.2.8 above there exists for each $n \in N$ a σ-PF-open partial refinement $\mathcal{V}_n$ of $\mathcal{U}$ such that $\cup \mathcal{D}_n \subset \cup \mathcal{V}_n$. Since $\mathcal{D} = \cup \{\cup \mathcal{D}_n : n \in N\}$ covers X, it thus follows that $\mathcal{V} = \cup \{\mathcal{V}_n : n \in N\}$ is a σ-PF-open refinement of $\mathcal{U}$.

Since X is countably subparacompact and hence countably metacompact, $\mathcal{U}$ must have a PF-open refinement. $\square$

As shown in 3.2.12 below, the list of equivalent properties above can be lengthened if we also assume that X is countably paracompact. To prove 3.2.12, we use the following result of J. C. Smith.

3.2.10. Theorem [72]. A space X is expandable iff X is discretely-θ-expandable and countably paracompact.

3.2.11. Corollary. A space X is paracompact iff X is countably paracompact, discretely-θ-expandable, and B(LF,λ)-refinable.

Proof. The necessity should be clear. To prove the sufficiency, assume that X is countably paracompact, discretely-θ-expandable, and B(LF,λ)-refinable. By 3.2.10

above, X is expandable, and by 2.2.15 every expandable, $B(LF,\lambda)$-refinable space is paracompact. $\square$

3.2.12. Corollary. Let X be any countably paracompact, strong-CWSN space. Then TFAE.

(a) X is paracompact.

(b) X is subparacompact.

(c) X is metacompact.

(d) X is θ-refinable.

(e) X is $B(D,\omega)$-refinable.

(f) X is weak $\bar{\theta}$-refinable.

(g) X is $B(D,\lambda)$-refinable.

(h) X is $B(LF,\lambda)$-refinable.

Proof. Clearly, (a)$\longrightarrow$(b) and (e)$\longrightarrow$(f)$\longrightarrow$(g)$\longrightarrow$(h). By 3.2.9, we have (b)$\longleftrightarrow$(c)$\longleftrightarrow$(d)$\longleftrightarrow$(e). Furthermore, (h)$\longrightarrow$(a) follows from 3.2.11 above since every strong-CWSN space is discretely-θ-expandable. $\square$

CHAPTER IV

CHARACTERIZATIONS OF d-PARACOMPACTNESS AND RELATED PROPERTIES

Both normality and paracompactness have close external relationships to metrizability. From the well-known "Urysohn's Lemma" it immediately follows that a space X is normal iff every disjoint pair of closed subsets of X can be separated by a continuous map from X into some metrizable space Y. Furthermore, a space X is paracompact iff for every open cover $\mathcal{U}$ of X, there exists a special "$\mathcal{U}$-map" from X into some metrizable space Y. The natural "d-versions" of these properties are obtained by changing the range space Y from a metrizable space to a developable space. C. M. Pareek [65] introduced d-paracompactness in 1972. H. Brandenburg ([14], [15]) and J. Chaber [23] later improved Pareek's internal characterizations of d-paracompactness by way of the new concepts "dissectability," "kernel-normal cover," and "CWdN," which are analogues of the important paracompactness related concepts σ-LF-refinement, normal cover, and CWN, respectively.
H. Brandenburg [12] also used the notions "dissectability" and "kernel-normal cover" to obtain characterizations of d-normality as well. See diagram 1.1.2 for the general

relationships between these "d-properties" and other covering properties discussed thus far.

The properties d-normality, d-paracompactness, and countable d-paracompactness are discussed in §1, §2, and §3 of this chapter, respectively. In each section, we give known characterizations of the respective property and then obtain new $B(P,\lambda)$-type generalizations of these results. Applications of the results then follow.

§1. Characterizations of d-normality

By Urysohn's Lemma we immediately get the following characterization of normality.

4.1.1. Theorem. A space X is normal iff for every pair F_1 , F_2 of disjoint closed subsets of X, there exists a metrizable space Y and a continuous map f from X into Y such that $cl(f(F_1)) \cap cl(f(F_2)) = \emptyset$.

In 1981, H. Brandenburg [12] introduced the property d-normality in an attempt to find a class of spaces which shares a relationship to the class of developable spaces analogous to the result above. Here we present results of H. Brandenburg, J. Chaber, and N. C. Heldermann which establish many analogous properties that d-normality and normality have in common. Furthermore, we include new characterizations of d-normality in terms of open covers which have $B(C,\lambda)$-refinements.

4.1.2. Definition. A family $\mathcal{U}^* = \{\mathcal{U}_n : n \in N\}$ of open covers of a space X is a development for X provided $\{st(x,\mathcal{U}_n) : n \in N\}$ is a neighborhood base at x for each $x \in X$. A space X is developable provided X has a development.

4.1.3. Definition [12]. A space X is d-normal provided for every pair F_1 , F_2 of disjoint closed subsets of X, there exists a developable space Y and a continuous map f from X into Y such that $cl(f(F_1)) \cap cl(f(F_2)) = \emptyset$.

It is clear that

$$\text{normal} \longrightarrow \text{d-normal},$$

however the Niemytzki Plane (see [80]) is an example of a d-normal, Tychonoff space which is not normal.

H. Brandenburg [12] introduced the notion of a "kernel-normal cover" as an analogue to "normal cover," and proved that a space X is d-normal iff every LF-open cover of X is kernel-normal. It is well-known that LF-open covers of normal spaces are normal covers. The "kernel-normal" notion involves a special type of "interior" defined below.

4.1.4. Definition [12]. Let $\mathcal{H}^* = \{\mathcal{H}_\alpha : \alpha \in A\}$ be a family of covers of a space X, and let $\mathcal{C}$ be a collection of subsets of X. For each $C \in \mathcal{C}$, define

$$\text{int}_{\mathcal{H}^*}(C) = \{x \in C : st(x, \mathcal{H}_\alpha) \subset C \text{ for some } \alpha \in A\}.$$

We call $\text{int}_{\mathcal{H}^*}(C)$ the interior of C with respect to the family of covers $\mathcal{H}^*$. Define

$$\text{int}_{\mathcal{H}^*}(\mathcal{C}) = \{\text{int}_{\mathcal{H}^*}(C) : C \in \mathcal{C}\},$$

which is called the interior of $\mathcal{C}$ with respect to $\mathcal{H}^*$.

4.1.5. Definition [12]. Let $\mathcal{K}^* = \{\mathcal{K}_\alpha : \alpha \in A\}$ be a family of open covers of a space X. The family $\mathcal{K}^*$ is kernel-normal provided for each $\alpha \in A$, there exists $\beta \in A$ such that $\mathcal{K}_\beta$ refines $\text{int}_{\mathcal{K}^*}(\mathcal{K}_\alpha)$. An open cover $\mathcal{U}$ of a space X is kernel-normal provided $\mathcal{U}$ belongs to a countable kernel-normal family of open covers of X.

It should be clear that every normal cover is kernel-normal.

The definition of "dissectability" given below was motivated by the desire to find a "d-analogue" of the following theorem.

4.1.6. <u>Nagata-Smirnov Metrization Theorem</u>. A space X is metrizable iff X is regular and has a σ-LF base.

4.1.7. <u>Definition</u> [11]. An open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of a space X is <u>dissectable</u> provided for every $\alpha \in A$ and $n \in N$, there exists a subset $D(n,\alpha)$ of X such that

(i) $U_\alpha = \cup \{D(n,\alpha) : n \in N\}$ for each $\alpha \in A$,

(ii) $\mathcal{D}_n = \{D(n,\alpha) : \alpha \in A\}$ is a CP collection of closed sets for each $n \in N$, and

(iii) for every $x \in X$ and $n \in N$, if $x \in \cup \mathcal{D}_n$, then $\cap \{U_\alpha : x \in D(n,\alpha)\}$ is a neighborhood of x.

The collection $\mathcal{D} = \cup \{\mathcal{D}_n : n \in N\}$ is called a <u>dissection</u> of $\mathcal{U}$.

4.1.8. <u>Remark</u>. H. Brandenburg [11] proved that a space X is developable iff X has a σ-dissectable open base. Note that every LF-cover of open F_σ-sets of a space X is dissectable.

The notions "cozero set" and "d-open set" are also closely related, both in their definitions as well as their relationships to normality and d-normality, respectively.

4.1.9. Definition [10]. A subset H of a space X is d-closed (d-open) provided there exists a continuous map f from X into a developable space Y, and a closed (open) subset K of Y such that $H = f^{-1}(K)$.

4.1.10. Remark. Note that a subset H of a space X is d-open iff H is the complement of a d-closed subset of X. Furthermore,

$$\text{cozero set} \longrightarrow \text{d-open set} \longrightarrow \text{open } F_\sigma\text{-set}$$

since every open subset of a developable space is an F_σ-set.

H. Brandenburg [10] gave an internal characterization of d-closed sets in terms of a special "G_δ-collection" defined below. In a similar fashion we can characterize d-open sets, and use this characterization to prove that any countable union of d-open sets is d-open. This fact will be used later.

4.1.11. Definition.

(a) [10] A collection $\mathcal{F}$ of closed subsets of a space X is called a G_δ-collection provided for every $F \in \mathcal{F}$, there exists a countable subcollection $\{F_n : n \in N\}$ of $\mathcal{F}$ such that $F = \cap\{X - F_n : n \in N\}$.

(b) A collection $\mathcal{G}$ of open subsets of a space X is called an F_σ-collection provided for every $G \in \mathcal{G}$, there exists a countable subcollection $\{G_n : n \in N\}$ of $\mathcal{G}$ such that $G = \cup \{X - G_n : n \in N\}$.

4.1.12. Lemma. Let $\mathcal{F}$ be a G_δ-collection in a space X. Then $\mathcal{G} = \{X - F : F \in \mathcal{F}\}$ is an F_σ-collection in X.

Proof. Let $F \in \mathcal{F}$, and define $G = X - F$. There exists a subcollection $\{F_n : n \in N\}$ of $\mathcal{F}$ such that $F = \cap \{X - F_n : n \in N\}$. For each $n \in N$, let $G_n = X - F_n$. Then $\{G_n : n \in N\}$ is a subcollection of $\mathcal{G}$, and we have $F = \cap \{G_n : n \in N\}$. Using DeMorgan's Law, we get $G = X - F = \cup \{X - G_n : n \in N\}$. Therefore, $\mathcal{G}$ is an F_σ-collection in X. □

4.1.13. Theorem.

(a) [10] A subset F of a space X is d-closed iff F belongs to a G_δ-collection in X.

(b) A subset G of a space X is d-open iff G belongs to an F_σ-collection in X.

Proof. We use result (a) above to prove part (b).

Sufficiency. Let $\mathcal{G}$ be an F_σ-collection in X, and assume that $G' \in \mathcal{G}$. By 4.1.12 above, $\mathcal{F} = \{X - G : G \in \mathcal{G}\}$ is a G_δ-collection in X which contains $F' = X - G'$, implying that F' is d-closed. Therefore, $G' = X - F'$ is d-open.

Necessity. Let G' be a d-open subset of X. Then $F' = X - G'$ is d-closed, so there exists a G_δ-collection $\mathcal{F}$

in X which contains F'. By 4.1.12, $\mathcal{G} = \{X - F : F \in \mathcal{F}\}$ is an F_σ-collection which by construction contains G'. □

4.1.14. Lemma.

(a) Any countable intersection of d-closed sets is d-closed.

(b) Any countable union of d-open sets is d-open.

Proof. We prove only part (b). Part (a) is proved in a similar fashion. Let $\mathcal{H} = \{H_n : n \in N\}$ be a countable collection of d-open subsets of X. Then there exists an F_σ-collection $\mathcal{G}_n$ in X which contains H_n and a subcollection $\{G(n,i) : i \in N\}$ of $\mathcal{G}_n$ such that

$$H_n = \cup \{X - G(n,i) : i \in N\}$$

for each $n \in N$. Define

$$\mathcal{G} = \{\cup \mathcal{H}\} \cup (\cup \{\mathcal{G}_n : n \in N\}).$$

By construction, $\{G(n,i) : n, i \in N\}$ is a countable subcollection of $\mathcal{G}$ such that

$$\cup \mathcal{H} = \cup \{X - G(n,i) : n, i \in N\}.$$

It should now be clear that $\mathcal{G}$ is an F_σ-collection in X which contains $\cup \mathcal{H}$, and hence $\cup \mathcal{H}$ is d-open. □

The following is a summary of known results concerning characterizations for d-normal spaces.

4.1.15. Theorem. For any space X, TFAE.

(a) X is d-normal.

(b) [10] For every pair F_1 , F_2 of disjoint closed subsets of X, there exists a pair H_1 , H_2 of disjoint closed G_δ-sets such that $F_1 \subset H_1$ and $F_2 \subset H_2$.

(c) [35] For every closed subset F and open subset U of X such that $F \subset U$, there exists an open F_σ-set K such that $H \subset K \subset U$.

(d) [12] For every closed subset F and open subset U of X such that $F \subset U$, there exists a d-open set G such that $F \subset G \subset U$.

(e) [12] Every LF-open cover of X has a d-open (open F_σ)-shrink.

(f) [12] Every PF-open cover of X has a d-open (open F_σ)-shrink.

(g) [12] Every Θ-cover of X has a d-open (open F_σ)-shrink.

(h) [15] Every weak $\bar{\Theta}$-cover of X has a d-open (open F_σ)-shrink.

(i) [12] Every LF-open cover of X is kernel-normal.

(j) [12] Every countable PF-open cover of X is kernel-normal.

(k) [12] Every countable PF-open cover of X has a dissectable open refinement.

4.1.16. Remark. It follows from (b) above that every d-normal space is subnormal. Examples of subnormal spaces which are not d-normal are found in [23] and [60].

Results 4.1.17 - 4.1.32 which follow will be used to establish several $B(C,\lambda)$-characterizations of d-normality (see 4.1.33).

4.1.17. Lemma. Let H be an F_σ-subset and U an open subset of a d-normal (normal) space X such that $H \subset U$. Then there exists a d-open (cozero) set G such that $H \subset G \subset U$.

Proof. We can write $H = \cup \{ H_n : n \in N\}$ such that H_n is a closed set for each $n \in N$. By 4.1.15(d) above, for every $n \in N$ there exists a d-open set G_n such that $H_n \subset G_n \subset U$. Now $G = \cup \{G_n : n \in N\}$ satisfies $H \subset G \subset U$. Furthermore, G is d-open by 4.1.14(b). (The proof for the case when X is normal is similar.) □

4.1.18. Corollary. Let X be a d-normal space. Then every open F_σ-subset of X is d-open.

Proof. Let H be an open F_σ-subset of a d-normal space X. By 4.1.17 above, there exists a d-open set G such that $H \subset G \subset H$, and hence H is d-open. □

4.1.19. Corollary. Let $\mathcal{U}$ be an open cover of a d-normal space X. If $\mathcal{U}$ has an F_σ-shrink, then $\mathcal{U}$ has a d-open-shrink.

Proof. It should be clear that any F_σ-shrink of $\mathcal{U}$ can be expanded to a d-open-shrink of $\mathcal{U}$ by 4.1.17 above. □

Kernel-normal covers

The following characterization of a kernel-normal open cover can be used to show that countable open covers which have d-open-shrinks are kernel-normal (see 4.1.21).

4.1.20. Theorem [12]. An open cover $\mathcal{U}$ of a space $(X, \mathcal{T})$ is kernel-normal iff there exists a developable topology $\mathcal{T}' \subset \mathcal{T}$ such that $\mathcal{U}$ has a $\mathcal{T}'$-open shrink.

Proof. Sufficiency. Assume there exists a developable topology $\mathcal{T}' \subset \mathcal{T}$ and a $\mathcal{T}'$-open shrink $\mathcal{V}$ of $\mathcal{U}$. Let $\{\mathcal{W}_n : n \in N\}$ be a development for $(X, \mathcal{T}')$. Then

$$\mathcal{W}^* = \{\mathcal{U}\} \cup (\{\mathcal{W}_n : n \in N\})$$

is a countable kernel-normal family of open covers of $(X, \mathcal{T})$ which contains $\mathcal{U}$. Therefore, $\mathcal{U}$ is kernel-normal.

Necessity. Assume that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a kernel-normal open cover of X. There exists a countable kernel-normal family $\mathcal{W}^*$ of open covers of X which contains $\mathcal{U}$. Define

$$\mathcal{G}^* = \{\wedge\{\mathcal{V} : \mathcal{V} \in \mathcal{V}^*\} : \mathcal{V}^* \subset \mathcal{W}^* : 1 \leq |\mathcal{V}^*| < \infty\},$$

and

$$\mathcal{T}' = \{H : H \subset X, \operatorname{int}_{\mathcal{G}^*}(H) = H\} .$$

In [12] it is shown that $\mathcal{T}' \subset \mathcal{T}$, $(X, \mathcal{T}')$ is developable, and $\operatorname{int}_{\mathcal{G}^*}(\mathcal{G})$ is a $\mathcal{T}'$-open cover of X for each $\mathcal{G} \in \mathcal{G}^*$. Furthermore, since $\mathcal{U} \in \mathcal{G}^*$, $\{\operatorname{int}_{\mathcal{G}^*}(U_\alpha) : \alpha \in A\}$ is a $\mathcal{T}'$-open shrink of $\mathcal{U}$. $\square$

4.1.21. <u>Lemma</u> [12]. Let $\mathcal{U} = \{U_n : n \in N\}$ be a countable open cover of a space $(X, \mathcal{T})$, and suppose $\mathcal{U}$ has a d-open-shrink $\mathcal{V} = \{V_n : n \in N\}$ such that $V_n \subset U_n$ for each $n \in N$. Then $\mathcal{U}$ is kernel-normal.

<u>Proof</u>. For each $n \in N$ there exists a continuous map $f_n : X \longrightarrow Y_n$ into a developable space Y_n and an open subset G_n of Y_n such that $f_n^{-1}(G_n) = V_n$. Let $\mathcal{T}'$ be the smallest topology on X such that f_n is continuous for every $n \in N$. Clearly $\mathcal{T}' \subset \mathcal{T}$, and it is not difficult to use the fact that $\{f_n : n \in N\}$ is countable to show that $(X, \mathcal{T}')$ is developable. Thus, $\{V_n : n \in N\}$ is a $\mathcal{T}'$-open shrink of $\mathcal{U}$. By 4.1.20 above, $\mathcal{U}$ must be kernel-normal. □

To prove that every open cover of a developable space is dissectable, we need the following result of J. W. Green [32].

4.1.22. <u>Theorem</u> [32]. Let X be a developable space. Then X has a development $\mathcal{U}^* = \{\mathcal{U}_n : n \in N\}$ such that for every $x' \in X$ and neighborhood V' of x', there exists an integer m' depending on x' and V' such that

(i) $\operatorname{ord}(x', \mathcal{U}_{m'}) = 1$, and

(ii) $\operatorname{st}(x', \mathcal{U}_{m'}) \subset V'$.

4.1.23. <u>Theorem</u> [11]. Every open cover of a developable space is dissectable.

<u>Proof</u>. Assume that X is a developable space, and let

$\mathcal{U}$ be an open cover of X. There exists a development $\mathcal{G}^* = \{\mathcal{G}_n : n \in N\}$ of X satisfying conditions (i) and (ii) of 4.1.22 above. Assume that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$.

For each $n \in N$, define

$$H_n = \{x : \text{ord}(x, \mathcal{G}_n) = 1\}.$$

For each $x \in H_n$, let $V(x,n)$ be the unique member of $\mathcal{G}_n$ which contains x. Define

$$W(x,n) = X - \cup\{G \in \mathcal{G}_n : x \notin G\}, \text{ and}$$

$$R(n,\alpha) = \cup\{W(x,n) : x \in H_n, V(x,n) \subset U_\alpha\}$$

for every $n \in N$ and $\alpha \in A$. In [11] it is shown that

$$\mathcal{R} = \cup\{\mathcal{R}_n = \{R(n,\alpha) : \alpha \in A\} : n \in N\}$$

is a dissection of $\mathcal{U}$. □

4.1.24. Lemma. If an open cover $\mathcal{U}$ of a space $(X,\mathcal{T})$ is kernel-normal, then $\mathcal{U}$ has a dissectable open refinement.

Proof. Assume that $\mathcal{U}$ is a kernel-normal open cover of $(X,\mathcal{T})$. By 4.1.20, there exists a developable topology $\mathcal{T}' \subset \mathcal{T}$ on X and a $\mathcal{T}'$-open shrink $\mathcal{V}$ of $\mathcal{U}$. Now by 4.1.23 above, $\mathcal{V}$ is dissectable, and the proof is complete. □

J. Chaber [23] has used the following concept--which we refer to as "weakly-kernel-normal"--to generalize several theorems of H. Brandenburg.

4.1.25. Definition. An open cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of a space X is weakly-kernel-normal provided there exists

(i) an open refinement $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of $\mathcal{U}$, and

(ii) a countable family $\mathcal{W}^*$ of open covers of X

satisfying $V_\alpha \subset \mathrm{int}_{\mathcal{W}^*}(U_\alpha) \subset U_\alpha$ for each $\alpha \in A$.

It should be clear that every kernel-normal cover is weakly-kernel-normal.

4.1.26. Question. Is every weakly-kernel-normal cover a kernel-normal cover? If not, what conditions will guarantee that every weakly-kernel-normal cover is kernel-normal? (For a partial answer to this question, see 4.1.32 below.)

The next lemma will be used to show that dissectability implies weak-kernel-normality.

4.1.27. Lemma. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a collection of open subsets (not necessarily a cover) of a space X, and assume that $\mathcal{U}$ is dissectable. Then there exists a countable family $\mathcal{G}^*$ of open covers of X such that

$$U_\alpha = \mathrm{int}_{\mathcal{G}^*}(U_\alpha)$$

for every $\alpha \in A$.

Proof. (The following technique was used by H. Brandenburg in his proof of ([14], Theorem 1).) Let $\mathcal{D} = \cup\{\mathcal{D}_n = \{D(n,\alpha) : \alpha \in A\} : n \in N\}$ be a dissection of $\mathcal{U}$ such that $D(n,\alpha) \subset U_\alpha$ for every $n \in N$ and $\alpha \in A$. Recall that

(i) $\mathscr{D}_n$ is a CP collection of closed sets, and

(ii) $\cap \{U_\alpha : x \in D(n,\alpha)\}$ is a neighborhood of x for each $x \in \cup \mathscr{D}_n$, $n \in N$.

It follows that for every $n \in N$ and $x \in \cup \mathscr{D}_n$ that there exists an open set $G(n,x)$ such that $x \in G(n,x)$, and

(a) $G(n,x)$ misses $\cup \{D(n,\beta) : x \notin D(n,\beta)\}$, and

(b) $G(n,x) \subset \cap \{U_\alpha : x \in D(n,\alpha)\}$.

For each $n \in N$, define

$$\mathscr{G}_n = \{G(n,x) : x \in \cup \mathscr{D}_n\} \cup \{X - \cup \mathscr{D}_n\},$$

so that $\mathscr{G}^* = \{\mathscr{G}_n : n \in N\}$ is a countable family of open covers of X.

We assert that $U_\alpha = \text{int}_{\mathscr{G}^*}(U_\alpha)$ for each $\alpha \in A$. Clearly, $\text{int}_{\mathscr{G}^*}(U_\alpha) \subset U_\alpha$, so it suffices to show that $U_\alpha \subset \text{int}_{\mathscr{G}^*}(U_\alpha)$. Now assume that $x \in U_\alpha$. Then there exists $n(x) \in N$ such that $x \in D(n(x),\alpha)$. If $x \in G(n(x),y)$ for some $y \in \cup \mathscr{D}_{n(x)}$, by condition (a) above, $y \in D(n(x),\alpha)$ must be the case. It thus follows that $G(n(x),y) \subset U_\alpha$ by condition (b) above, and hence $st(x,\mathscr{G}_{n(x)}) \subset U_\alpha$. Therefore, $\text{int}_{\mathscr{G}^*}(U_\alpha) = U_\alpha$. □

4.1.28. Corollary. Let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a space X. If $\mathscr{U}$ has a dissectable open refinement, then $\mathscr{U}$ is weakly-kernel-normal.

Proof. Let $\mathscr{V} = \{V_\alpha : \alpha \in A\}$ be a dissectable open refinement of $\mathscr{U}$ such that $V_\alpha \subset U_\alpha$ for each $\alpha \in A$. By 4.1.27 above, there exists a countable family $\mathscr{G}^*$ of open covers of X such that $V_\alpha = \mathrm{int}_{\mathcal{G}^*}(V_\alpha)$ for every $\alpha \in A$. Since $V_\alpha = \mathrm{int}_{\mathcal{G}^*}(V_\alpha) \subset \mathrm{int}_{\mathcal{G}^*}(U_\alpha) \subset U_\alpha$ for each $\alpha \in A$, $\mathcal{U}$ is weakly-kernel-normal. □

From results 4.1.24 and 4.1.28, we have

$$\begin{gathered}\mathcal{U} \text{ is a kernel-normal open cover} \\ \downarrow \\ \mathcal{U} \text{ has a dissectable open refinement} \\ \downarrow \\ \mathcal{U} \text{ is a weakly-kernel-normal open cover.}\end{gathered}$$

H. Brandenburg [12] obtained the following lemma by using a straightforward induction argument in his proof.

4.1.29. Lemma [12]. Let $\mathcal{K}^*$ be a family of open covers of a space X satisfying

(i) $\mathcal{K}_1 \wedge \mathcal{K}_2 \in \mathcal{K}^*$ whenever $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{K}^*$, and

(ii) for each $\mathcal{K} \in \mathcal{K}^*$, there exists a countable subfamily $\mathcal{U}^*$ of $\mathcal{K}^*$ and some $\mathcal{U} \in \mathcal{U}^*$ such that $\mathcal{U}$ refines $\mathrm{int}_{\mathcal{U}^*}(\mathcal{K})$.

Then every member of $\mathcal{K}^*$ is kernel-normal.

The following two lemmas will be used to establish conditions under which weakly-kernel-normal open covers are kernel-normal (see 4.1.32).

4.1.30. Lemma. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a (PF-) open cover of a space X, and let $\mathcal{W}^* = \{\mathcal{W}_n : n \in N\}$ be a countable family of open covers of X. For each $n \in N$ and $\alpha \in A$, define

$$E(n,\alpha) = \{x : st(x, \mathcal{W}_n) \subset U_\alpha\} \quad , \text{ and}$$
$$\mathcal{E} = \cup \{\mathcal{E}_n = \{E(n,\alpha) : \alpha \in A\} : n \in N\} \ .$$

Then $\mathcal{E}$ is a (σ-LF-closed) σ-cushioned closed partial refinement of $\mathcal{U}$.

Proof. We prove only the PF case. Indeed, assume that $\mathcal{U}$ is PF, and let $n \in N$. Clearly, $\mathcal{E}_n$ is a PF-partial refinement of $\mathcal{U}$, so it suffices to show that $\mathcal{E}_n$ is a CP collection of closed sets. Now let $A' \subset A$, and let

$$x \in X - (\cup \{E(n,\alpha) : \alpha \in A'\}).$$

Define $B_x = \{\beta \in A : x \in U_\beta\}$, a finite set. For each $\beta \in B_x$, choose $W_\beta \in \mathcal{W}_n$ such that

(i) $x \in W_\beta$ if $\beta \in B_x - A'$, and

(ii) $x \in W_\beta$ such that $W_\beta \not\subset U_\beta$ if $\beta \in A'$,

and define $V_x = \cap \{W_\beta : \beta \in B_x\}$. By construction, V_x is a neighborhood of x which misses $\cup \{E(n,\alpha) : \alpha \in A'\}$. It follows that $\mathcal{E}_n$ is a CP collection of closed sets, and our proof is complete. $\square$

4.1.31. Lemma [23]. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ and $\mathcal{V}$ be (countable PF-) PF-open covers of a space X, and $\mathcal{W}^* = \{\mathcal{W}_n : n \in N\}$ a countable family of open covers of X such that $\mathcal{V}$ refines $\text{int}_{\mathcal{W}^*}(\mathcal{U})$. Then there exists a countable family $\mathcal{G}^* = \{\mathcal{G}_n : n \in N\}$ of (countable PF-) PF-open covers of X such that $\mathcal{V}$ refines $\text{int}_{\mathcal{G}^*}(\mathcal{U})$.

Proof. We prove only the PF case.

Define $\mathcal{E} = \cup \{ \mathcal{E}_n = \{E(n,\alpha) : \alpha \in A\} : n \in N\}$ as in 4.1.30 above, and for each $n \in N$ and $x \in X$, define

$$G(x,n) = \cap \{U_\alpha : x \in E(n,\alpha)\} - \cup \{E(n,\alpha) : x \notin E(n,\alpha)\},$$

$$\mathcal{G}_n = \{G(x,n) : x \in X\} \text{ , and}$$

$$\mathcal{G}^* = \{ \mathcal{G}_n : n \in N\}.$$

By 4.1.30 above, $\mathcal{E}_n$ is a LF-closed partial refinement of $\mathcal{U}$. Therefore, $\mathcal{G}_n$ is a PF-open cover of X for each $n \in N$ such that if $st(x,\mathcal{W}_n) \subset U_\alpha$, then $st(x,\mathcal{G}_n) \subset U_\alpha$ for every $x \in X$ and $\alpha \in A$. It thus follows that $\mathcal{G}^*$ is the desired family of PF-open covers of X. □

4.1.32. <u>Corollary</u>. If every (countable PF-) PF-open cover of X is weakly-kernel-normal, then every such cover is kernel-normal.

<u>Proof</u>. We prove only the PF case. Assume that every PF-open cover of X is weakly-kernel-normal, and let $\mathcal{U}$ be a PF-open cover of X. Then there exists a one-to-one (and hence PF)-open refinement $\mathcal{V}$ of $\mathcal{U}$, and a countable family $\mathcal{W}^*$ of open covers of X such that $\mathcal{V}$ refines $int_{\mathcal{W}^*}(\mathcal{U})$. By 4.1.31 above, we may assume that every member of $\mathcal{W}^*$ is also PF. Now let $\mathcal{K}^*$ be the family of all PF-open covers of X so that condition (i) of 4.1.29 is easily satisfied. However, by 4.1.31, $\mathcal{K}^*$ also satisfies condition (ii) of 4.1.29 as well, and hence every member of $\mathcal{K}^*$ must be kernel-normal. □

The following are characterizations for d-normality in terms of open covers which have a $B(C,\lambda)$-refinement. Note that these results generalize 4.1.15 (e) - (k).

4.1.33. Theorem. For any space X, TFAE.

(a) X is d-normal.

(b) Every open cover of X which has a $B(C,\lambda)$-refinement also has a d-open-shrink.

(c) Every open cover of X which has a $B(C,\lambda)$-refinement also has an open F_σ-shrink.

(d) Every countable open cover of X which has a $B(C,\lambda)$-refinement is kernel-normal.

(e) Every countable open cover of X which has a $B(C,\lambda)$-refinement has a dissectable open refinement.

(f) Every countable open cover of X which has a $B(C,\lambda)$-refinement is weakly-kernel-normal.

Proof. It is clear that (b) $\longrightarrow$ (c). Also, (d) $\longrightarrow$ (e) $\longrightarrow$ (f) follows from 4.1.24 and 4.1.28. The implications (c) $\longrightarrow$ (a) and (b) $\longrightarrow$ (d) follow from 4.1.15(e) and 4.1.21, respectively. Furthermore, (f) $\longrightarrow$ (a) follows from the fact that every PF-open cover has a $B(C,\lambda)$-refinement, and hence every countable PF-open cover $\mathcal{U}$ is weakly-kernel-normal. By 4.1.32, $\mathcal{U}$ is kernel-normal. Therefore, X is d-normal by 4.1.15(j). It remains only to show that

(a) $\longrightarrow$ (b).

Assume that X is d-normal, and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X which has a B(C,λ)-refinement

$$\mathcal{E} = \cup \{\mathcal{E}_\gamma = \{E(\gamma,\alpha) : \alpha \in A\} : \gamma < \lambda\}$$

such that $E(\gamma,\alpha) \subset U_\alpha$ for every $\alpha \in A$ and $\gamma < \lambda$. By transfinite induction we show that for every $\gamma < \lambda$, there exists a collection $\mathcal{G}_\gamma = \{G(\gamma,\alpha) : \alpha \in A\}$ of d-open sets satisfying

(i) $G(\gamma,\alpha) \subset U_\alpha$ for each $\alpha \in A$, and

(ii) $\cup \{\cup \mathcal{E}_\beta : \beta \leq \gamma\} \subset V_\gamma = \cup \{\cup \mathcal{G}_\beta : \beta \leq \gamma\}$.

For fixed $\gamma < \lambda$, assume that $\mathcal{G}_\beta$ with the above properties has been constructed for all $\beta < \gamma$. Define

$V^* = \cup \{V_\beta : \beta < \gamma\}$, and

$\mathcal{F}_\gamma = \{F(\gamma,\alpha) = E(\gamma,\alpha) - V^* : \alpha \in A\}$.

Now V^* is an open set which contains $\cup \{\cup \mathcal{E}_\beta : \beta < \gamma\}$, and so $F(\gamma,\alpha)$ is a closed subset of U_α for each $\alpha \in A$. Since X is d-normal, there exists a d-open set $G(\gamma,\alpha)$ such that $F(\gamma,\alpha) \subset G(\gamma,\alpha) \subset U_\alpha$ for every $\alpha \in A$. It should be clear that $\mathcal{G}_\gamma = \{G(\gamma,\alpha) : \alpha \in A\}$ satisfies conditions (i) and (ii) above, and the construction is complete.

For each $\alpha \in A$, let $G_\alpha = \cup \{G(\gamma,\alpha) : \gamma < \lambda\}$. By construction, $G_\alpha \subset U_\alpha$, and since G_α is a countable union of d-open sets, G_α is d-open by 4.1.14. Also, condition (ii) above implies that $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ covers X, and hence $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ is a d-open-shrink of $\mathcal{U}$. $\square$

The following characterization of countable subparacompactness by T. R. Kramer [52] will be used in our proof of 4.1.35 below.

4.1.34. Theorem [52]. A space X is subparacompact iff every countable open cover of X has an F_{σ}-shrink.

4.1.35. Corollary. Let X be any d-normal space. Then TFAE.

(a) X is countably subparacompact.

(b) X is countably metacompact.

(c) X is countably θ-refinable.

(d) X is countably weak $\bar{\theta}$-refinable.

(e) X is countably $B(C,\lambda)$-refinable.

Proof. Clearly, (a) $\longrightarrow$ (b) $\longrightarrow$ (c) $\longrightarrow$ (d) $\longrightarrow$ (e), and hence it remains only to show that (e) $\longrightarrow$ (a). However, 4.1.33(c) implies that every countable open cover of X which has a $B(C,\lambda)$-refinement also has an open F_{σ}-shrink. Therefore, by Kramer's result above it follows that X is countably subparacompact. □

4.1.36. Remark. In §3 we will consider the property countable-d-paracompactness. We again strengthen 4.1.35 above by showing that countable d-paracompactness and countable $B(C,\lambda)$-refinability are equivalent in any d-normal space.

§2. Characterizations of d-paracompactness

In 1948, C. H. Dowker [27] characterized paracompactness in terms of "$\mathcal{U}$-maps" to metrizable spaces.

4.2.1. Definition. Let $\mathcal{U}$ be an open cover of a space X, and let f be a continuous map from X into a space Y. The map f is a $\mathcal{U}$-map provided there exists an open cover $\mathcal{V}$ of Y such that $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$.

4.2.2. Theorem [27]. A space X is paracompact iff for every open cover $\mathcal{U}$ of X, there exists a metrizable space Y and a $\mathcal{U}$-map f from X into Y.

In 1972, C. M. Pareek [65] introduced the following "d-version" of paracompactness based on 4.2.2 above.

4.2.3. Definition [65]. A space X is d-paracompact provided for every open cover $\mathcal{U}$ of X, there exists a developable space Y and a $\mathcal{U}$-map f from X into Y.

Clearly,

$$\text{paracompact} \longrightarrow \text{d-paracompact} \longleftarrow \text{developable}.$$

However, the Niemytzki Plane (see [80]) is an example of a developable (hence d-paracompact) Tychonoff space which is not normal and hence not paracompact.

C. M. Pareek [65] was the first to internally characterize d-paracompactness. His results were improved by H. Brandenburg ([10], [14], [15]) and J. Chaber [23].

In this section we discuss these results and give new characterizations of d-paracompactness in terms of $B(LF,\lambda)$-refinability.

In the last section we studied the concepts kernel-normal cover (4.1.5), dissectability (4.1.7), and weakly-kernel-normal cover (4.1.25). In particular, we showed how their connection to d-normality closely parallels the way normal covers are related to normality; that is, a space X is normal iff every LF-open cover of X is a normal cover. In 1948, A. H. Stone [81] proved that a space X is paracompact iff every open cover of X is a normal cover. Here we continue this type of cover characterization.

4.2.4. Theorem. For any space X, TFAE.

(a) X is d-paracompact.

(b) [14] Every open cover of X is kernel-normal.

(c) [14] Every open cover of X has a dissectable open refinement.

(d) [23] Every open cover of X is weakly-kernel-normal.

4.2.5. Remark. D. K. Burke [16] proved that a space X is subparacompact iff every open cover of X has a $\sphericalangle$-CP-closed refinement. Since every dissectable refinement has by definition a $\sphericalangle$-CP-closed refinement, from 4.2.4(c) it follows that

$$\text{d-paracompact} \longrightarrow \text{subparacompact}.$$

Example 4.2.12 below shows that the above implication is not reversible.

The concepts discrete-d-expandability and collection-wise d-normality (CWdN), which are defined below, were introduced by H. Brandenburg [14] and J. Chaber [23], respectively. These properties will play the role of CWN in the analogous "d-versions" of the following theorem.

4.2.6. Theorem. For any space X, TFAE.

(a) X is paracompact.

(b) [16] X is CWN and subparacompact.

(c) [87] X is CWN and θ-refinable.

(d) [74] X is CWN and weak $\bar{\theta}$-refinable.

4.2.7. Definition [14]. A space X is discretely-d-expandable provided

(i) for every discrete collection $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ of closed subsets of X, and

(ii) for each open expansion $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of $\mathcal{D}$, there exists a dissectable open expansion $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of $\mathcal{D}$ such that $D_\alpha \subset V_\alpha \subset U_\alpha$ for all $\alpha \in A$.

4.2.8. Remark. H. Brandenburg [14] introduced the concept defined above by the name "d-expandable." In this text we prefer to use the term "discretely-d-expandable" to be consistent with existing terminology.

4.2.9. Definition [23]. A space X is CWdN provided

(i) for every discrete collection $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ of closed subsets of X, and

(ii) for each open expansion $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of $\mathcal{D}$, there exists an open expansion $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of $\mathcal{D}$ and a countable family $\mathcal{W}^*$ of open covers of X satisfying $D_\alpha \subset V_\alpha \subset \operatorname{int}_{\mathcal{W}^*}(U_\alpha) \subset U_\alpha$ for all $\alpha \in A$.

See diagram 1.1.2 for general relationships between the concepts defined above and other covering properties. Later we will verify that discrete-d-expandability implies CWdN (see 4.2.17(b)).

4.2.10. Question. Are the properties discrete-d-expandability and CWdN equivalent?

The following is a summary of results analogous to 4.2.6 which were obtained by H. Brandenburg [14] and J. Chaber [23].

4.2.11. Theorem. For any space X, TFAE.

(a) X is d-paracompact.

(b) [14] X is discretely-d-expandable and subparacompact.

(c) [23] X is CWdN and subparacompact.

(d) [14] X is discretely-d-expandable and θ-refinable.

(e) [23] X is CWdN and Θ-refinable.

(f) [14] X is discretely-d-expandable and weak $\bar{\Theta}$-refinable.

4.2.12. Example. Let S be the Sorgenfrey line (the real line with basic open sets of the form [a,b)), and define $X = S \times S$. R. W. Heath and E. Michael [34] proved that X is perfect. Since every closed subset of X is a G_δ-set, every pair of disjoint closed subsets of X vacuously has a disjoint closed G_δ-expansion, so X is d-normal. D. J. Lutzer [54] proved that X is subparacompact; hence, X is CWSN. However, H. Brandenburg [15] has shown that X is not d-paracompact. Therefore, X is not CWdN by 4.2.11(c) above. Thus,

$$\text{d-normal} \not\rightarrow \text{CWdN} \not\leftarrow \text{CWSN}.$$

4.2.13. Example. The Niemytzki Plane (see [80]) is an example of a developable, and hence d-paracompact space, which is not normal. Thus,

$$\text{discretely-d-expandable} \not\rightarrow \text{normal}.$$

J. C. Smith [74] discovered the following weak $\bar{\Theta}$-cover characterization of CWN.

4.2.14. Theorem [74]. A space X is CWN iff every weak $\bar{\Theta}$-cover of X is a normal cover.

We now obtain analogous characterizations for discrete-d-expandability and CWdN. Later, we use these to establish new characterizations of d-paracompactness.

4.2.15. Lemma. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a space X, and assume that

$$\mathcal{V} = \cup \{\mathcal{V}_\gamma = \{V(\gamma,\alpha) : \alpha \in A\} : \gamma < \lambda\}$$

is an open refinement of $\mathcal{U}$ such that $V(n,\alpha) \subset U_\alpha$ for each $\alpha \in A$.

(a) If $\mathcal{V}_\gamma$ is dissectable for each $\gamma < \lambda$, then $\mathcal{U}$ has a dissectable open refinement.

(b) If for each $\gamma < \lambda$, there exists a countable family $\mathcal{W}^*_\gamma$ of open covers of X such that $V(\gamma,\alpha) \subset \operatorname{int}_{\mathcal{W}^*_\gamma}(U_\alpha)$ for each $\alpha \in A$, then $\mathcal{U}$ is weakly-kernel-normal.

Proof. (a). Since $\mathcal{V}_\gamma$ is dissectable for each $\gamma < \lambda$, there exists a dissection

$$\mathcal{F}_\gamma = \cup \{\mathcal{F}(\gamma,i) = \{F(\gamma,i,\alpha) : \alpha \in A\} : i \in N\}$$

of $\mathcal{V}_\gamma$ such that

$$V(\gamma,\alpha) = \cup \{F(\gamma,i,\alpha) : i \in N\}$$

for each $\gamma < \lambda$ and $\alpha \in A$. For each $\alpha \in A$, define

$$V_\alpha = \cup \{V(\gamma,i,\alpha) : \gamma < \lambda,\ i \in N\} \text{ , and}$$

$$\mathcal{V} = \{V_\alpha : \alpha \in A\} .$$

Let $\mathcal{J} = \{(\gamma,i) : \gamma < \lambda\}$. Now $\mathcal{J}$ is countable, so there exists a bijection $f : \mathcal{J} \longrightarrow N$ from $\mathcal{J}$ onto N. For each $n \in N$, define

$$\mathcal{E}_n = \mathcal{F}(\gamma,i), \text{ where } f(\gamma,i) = m, \text{ and}$$

$$\mathcal{E} = \cup \{\mathcal{E}_n : n \in N\} .$$

It is easy to verify that $\mathcal{E}$ is a dissection of $\mathcal{V}$, and that $\mathcal{V}$ is an open refinement of $\mathcal{U}$.

<u>(b)</u>. Let $\mathcal{V} = \{V_\alpha = \cup \{V(\gamma,\alpha) : \gamma < \lambda\} : \alpha \in A\}$, and $\mathcal{W}^* = \cup\{\mathcal{W}_\gamma^* : \gamma < \lambda\}$. Now $\mathcal{W}^*$ is a countable family of open covers of X, and clearly,

$$\operatorname{int}_{\mathcal{W}_\gamma^*}(U_\alpha) \subset \operatorname{int}_{\mathcal{W}^*}(U_\alpha)$$

for each $\alpha \in A$. By construction, it thus follows that $\mathcal{V}$ is an open refinement of $\mathcal{U}$ such that

$$V_\alpha \subset \operatorname{int}_{\mathcal{W}^*}(U_\alpha) \subset U_\alpha$$

for each $\alpha \in A$. Therefore, $\mathcal{U}$ is weakly-kernel-normal. □

4.2.16. <u>Theorem</u>.

(a) A space X is discretely-d-expandable iff every open cover of X which has a B(D,λ)-refinement also has a dissectable open refinement.

(b) A space X is CWdN iff every open cover of X which has a B(D,λ)-refinement is weakly-kernel-normal.

<u>Proof</u>. (We prove only part (a). Part (b) follows in a similar fashion.)

<u>Sufficiency</u>. Assume that every open cover of X which has a B(D,λ)-refinement also has a dissectable open refinement. Let $\mathcal{D} = \{D_\alpha : \alpha \in A\}$ be a discrete collection of closed subsets of X, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ an open expansion of $\mathcal{D}$ such that $D_\alpha \subset U_\alpha$ for each $\alpha \in A$. We

can also assume that $U_\alpha \cap D_\beta = \emptyset$ for every $\alpha \in A$ and $\beta \neq \alpha$.

Define $\mathcal{U}^* = \mathcal{U} \cup \{X - \cup \mathcal{D}\}$. It is easy to see that $\mathcal{U}^*$ is a bded-weak $\bar{\theta}$-cover of X, and hence $\mathcal{U}^*$ has a $B(D,\lambda)$-refinement by 2.1.9. Thus, $\mathcal{U}^*$ has a dissectable open refinement $\{V_\alpha : \alpha \in A\} \cup \{W\}$ such that $D_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in A$, and $W \subset X - \cup \mathcal{D}$. It follows that $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ is a dissectable open expansion of $\mathcal{D}$ such that $D_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in A$, and hence X is discretely-d-expandable.

Necessity. Assume that X is discretely-d-expandable, and let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X which has a $B(D,\lambda)$-refinement

$$\mathcal{E} = \cup \{\mathcal{E}_\gamma = \{E(\gamma,\alpha) : \alpha \in A\} : \gamma < \lambda\}$$

such that $E(\gamma,\alpha) \subset U_\alpha$ for every $\gamma < \lambda$ and $\alpha \in A$. By transfinite induction we construct for each $\gamma < \lambda$ a dissectable collection $\mathcal{V}_\gamma = \{V(\gamma,\alpha) : \alpha \in A\}$ of open sets such that

(i) $V(\gamma,\alpha) \subset U_\alpha$ for each $\alpha \in A$, and

(ii) $\cup\{\cup \mathcal{E}_\beta : \beta \leq \gamma\} \subset S_\gamma = \cup\{\cup \mathcal{V}_\beta : \beta \leq \gamma\}$.

For fixed $\gamma < \lambda$, assume that collections $\mathcal{V}_\beta$ have been constructed satisfying the above conditions for all $\beta < \gamma$. Define

$$S^* = \cup \{S_\beta : \beta < \gamma\}, \text{ and}$$

$$\mathcal{D}_\gamma = \{D(\gamma,\alpha) = E(\gamma,\alpha) - S^* : \alpha \in A\}.$$

Since S^* is open and contains $\cup\{\cup\mathcal{E}_\beta : \beta < \gamma\}$, then $\mathcal{D}_\gamma$ is a discrete collection of closed subsets of X. Also, $D(\gamma,\alpha) \subset U_\alpha$ for each $\alpha \in A$. Since X is discretely-d-expandable, for each $\alpha \in A$ there exists an open set $V(\gamma,\alpha)$ such that $\mathcal{V}_\gamma = \{V(\gamma,\alpha) : \alpha \in A\}$ satisfies condition (i) above. It should be clear that condition (ii) is also satisfied, and our construction is complete.

Now $\mathcal{E}$ covers X, so by conditions (i) and (ii) above $\mathcal{V} = \cup\{\mathcal{V}_\gamma : \gamma < \lambda\}$ is an open refinement of $\mathcal{U}$ which satisfies all conditions of 4.2.15(a) above, and therefore $\mathcal{U}$ has a dissectable open refinement. □

4.2.17. Remarks.

(a) Since every open cover of a d-paracompact space has a dissectable open refinement, 4.2.16(a) above implies that

$$\text{d-paracompact} \longrightarrow \text{discretely-d-expandable.}$$

(b) Since every open cover which has a dissectable open refinement is weakly-kernel-normal (4.1.28), from 4.2.16 above it immediately follows that

$$\text{discretely-d-expandable} \longrightarrow \text{CWdN.}$$

(c) Let $\mathcal{U}$ be an open cover of a CWN space X. If $\mathcal{U}$ has a $B(D,\lambda)$-refinement, then $\mathcal{U}$ is a normal cover (2.2.12(a)), and hence is kernel-normal. Therefore, $\mathcal{U}$ has a dissectable open refinement (4.1.24). By 4.2.16(a), it follows that X is discretely-d-expandable. Thus,

$$\text{CWN} \longrightarrow \text{discretely-d-expandable.}$$

(d) By 4.2.16(b) above, every open cover of a CWdN space X which has a B(D,ω)-refinement is weakly-kernel-normal, and hence has a $\prec$-cushioned refinement (4.1.30). Therefore, by 3.1.12 X is CWSN. Thus,

$$\text{CWdN} \longrightarrow \text{CWSN.}$$

(e) From 4.1.15(j) and 4.1.33(f), it follows that a space X is d-normal iff every countable open cover of X which has a B(D,λ)-refinement is weakly-kernel-normal. By 4.2.16(b) above, it thus follows that

$$\text{CWdN} \longrightarrow \text{d-normal.}$$

□

It now follows from 4.2.4 and 4.2.16 that a space X is d-paracompact iff X is CWdN and B(D,λ)-refinable. In order to obtain a more general B(LF,λ)-characterization of d-paracompactness, we first prove a lemma.

4.2.18. Lemma. Assume that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a collection (not necessarily a cover) of open subsets of a space X, and $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is a bded-LF collection of closed subsets of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$.

(a) If X is discretely-d-expandable, then $\mathcal{F}$ has a dissectable open expansion $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ such that $F_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in A$.

(b) If X is CWdN, then there exists an open expansion $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of $\mathcal{F}$ and a countable family $\mathcal{W}^*$ of open covers of X such that

$$F_\alpha \subset V_\alpha \subset \mathrm{int}_{\mathcal{W}^*}(U_\alpha) \subset U_\alpha$$

for each $\alpha \in A$.

<u>Proof</u>. (We prove only part (a). Part (b) is proved in a similar fashion.) Assume that X is discretely-d-expandable, with $\mathcal{U}$ and $\mathcal{F}$ as given above. We may assume that $\mathcal{F}$ is n-bded-LF. For $1 \leq i \leq n$, define

$$\mathcal{E}_i = \{ E(i,\alpha) = \{x \in F_\alpha : \mathrm{ord}(x,\mathcal{F}) = i\} : \alpha \in A\} .$$

Now $\mathcal{E} = \cup \{\mathcal{E}_i : 1 \leq i \leq n\}$ is a B(D,n)-partial refinement of $\mathcal{U}$. As in the proof of 4.2.16(a), we can inductively construct an open dissectable expansion $\mathcal{V}_i = \{V(i,\alpha) : \alpha \in A\}$ for $1 \leq i \leq n$ such that $E(i,\alpha) \subset V(i,\alpha) \subset U_\alpha$ for each $\alpha \in A$. Now by the same method used in the proof of 4.2.15(a), we can show that

$$\mathcal{V} = \{ V_\alpha = \cup \{V(i,\alpha) : 1 \leq i \leq n\} : \alpha \in A\}$$

is a dissectable open expansion of $\mathcal{F}$ such that $F_\alpha \subset V_\alpha \subset U_\alpha$ for each $\alpha \in A$. □

Our next result shows that if we assume countable metacompactness, then $B(D,\lambda)$-characterizations for discrete-d-expandability and CWdN (see 4.2.16) can be generalized to $B(LF,\lambda)$-characterizations. This will then be used to obtain a $B(LF,\lambda)$-characterization of d-paracompactness.

4.2.19. <u>Theorem</u>. Let X be a countably metacompact space. Then

(a) X is discretely-d-expandable iff every open cover of X which has a $B(LF,\lambda)$-refinement also has a dissectable open refinement.

(b) X is CWdN iff every open cover of X which has a $B(LF,\lambda)$-refinement is weakly-kernel-normal.

<u>Proof</u>. As before, we prove only part (a). Since every $B(D,\lambda)$-refinement is a $B(LF,\lambda)$-refinement, the sufficiency is clear from 4.2.16(a). To prove the necessity, assume that X is countably metacompact and discretely-d-expandable. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X which has a $B(LF,\lambda)$-refinement

$$\mathcal{E} = \cup \{ \mathcal{E}_\gamma = \{E(\gamma,\alpha) : \alpha \in A\} \quad : \gamma < \lambda\}$$

such that $E(\gamma,\alpha) \subset U_\alpha$ for each $\alpha \in A$. By transfinite induction we construct for every $\gamma < \lambda$ a family $\{\mathcal{V}(\gamma,n) : n \in N\}$ of collections of subsets of X satisfying

(i) $\mathcal{V}(\gamma,n) = \{V(\gamma,n,\alpha) : \alpha \in A\}$ is a dissectable open partial refinement of $\mathcal{U}$ for each $n \in N$ such that $V(\gamma,n,\alpha) \subset U_\alpha$ for each $\alpha \in A$, and

(ii) $\cup \{ \mathcal{V}(\gamma,n) : n \in N\}$ covers $\cup \mathcal{F}_\gamma$, where $\mathcal{F}_\gamma = \{F(\gamma,\alpha) = E(\gamma,\alpha) - R_\gamma : \alpha \in A\}$, and $R_\gamma = \cup \{\cup \mathcal{V}(\beta,n) : \beta < \gamma , n \in N\}$.

For fixed $\gamma < \lambda$, assume that $\mathcal{V}(\beta,n)$ with the above properties has been constructed for all $\beta < \gamma$, $n \in N$. Now $\mathcal{F}_\gamma$ is a LF-closed partial refinement of $\mathcal{U}$ whose union is contained in the closed (countably metacompact) subspace $X - R_\gamma$. For each $n \in N$, define

$$S(\gamma,n) = \{x : \operatorname{ord}(x, \mathcal{F}_\gamma) \leq n\} \cap (X - R_\gamma), \text{ and}$$

$$\mathcal{S}_\gamma = \{S(\gamma,n) : n \in N\} .$$

Now $\mathcal{S}_\gamma$ is a countable monotone open cover of the countably metacompact subspace $X - R_\gamma$. By 2.1.15, $\mathcal{S}_\gamma$ has a closed shrink $\mathcal{K}_\gamma = \{K(\gamma,n) : n \in N\}$ such that $K(\gamma,n) \subset S(\gamma,n)$ for each $n \in N$. For every $n \in N$, define

$$\mathcal{L}(\gamma,n) = \{F(\gamma,\alpha) \cap K(\gamma,n) : \alpha \in A\} , \text{ and}$$

$$\mathcal{L}_\gamma = \cup \{\mathcal{L}(\gamma,n) : n \in N\} .$$

Since each member of $\mathcal{L}(\gamma,n)$ is contained in $S(\gamma,n)$, it follows that $\mathcal{L}(\gamma,n)$ is an n-bded-LF collection of closed subsets of X. Therefore by 4.2.18(a) above, $\mathcal{L}(\gamma,n)$ must have a dissectable open expansion

$$\mathcal{V}(\gamma,n) = \{V(\gamma,n,\alpha) : \alpha \in A\}$$

for each $n \in N$, which partially refines $\mathcal{U}$. It is easy to see that $\{\mathcal{V}(\gamma,n) : n \in N\}$ satisfies conditions (i) and (ii) above, and our construction is complete.

Let $\mathcal{V} = \cup \{\mathcal{V}(\gamma,n) : \gamma < \lambda, n \in N\}$. Since $\mathcal{E}$ covers X, by condition (ii) above $\mathcal{V}$ must also cover X; hence, $\mathcal{V}$ is an open refinement of $\mathcal{U}$. Since $\{(\gamma,n) : \gamma < \lambda, n \in N\}$ is countable, it should be clear that $\mathcal{V}$ can be reindexed so that it satisfies all conditions of 4.2.15(a). Therefore, $\mathcal{U}$ has a dissectable open refinement. $\square$

The following characterizations of d-paracompactness are generalizations of those given in 4.2.11.

4.2.20. <u>Theorem</u>. For any space X, TFAE.

(a) X is d-paracompact.

(b) X is discretely-d-expandable and $B(LF,\lambda)$-refinable.

(c) X is CWdN and $B(LF,\lambda)$-refinable.

<u>Proof</u>. The fact that (a) $\longrightarrow$ (b) $\longrightarrow$ (c) is clear. Assume (c), and let $\mathcal{U}$ be an open cover of X. Since X is CWdN, X is d-normal, and since X is $B(LF,\lambda)$-refinable, X is $B(C,\lambda)$-refinable. Thus, X is countably metacompact by 4.1.35. Now $\mathcal{U}$ has a $B(LF,\lambda)$-refinement, and therefore is weakly-kernel-normal by 4.2.19(b). Hence, every open cover of X is weakly-kernel-normal, and so X is d-paracompact by 4.2.4(d). Therefore, we also have (c) $\longrightarrow$ (a). $\square$

§3. Characterizations of countable d-paracompactness

In this section we show that the relationship between d-normality and countable d-paracompactness is similar to the relationship between CWdN and d-paracompactness. H. Brandenburg [15] gave the following definition of countable d-paracompactness along with several equivalences which we list below.

4.3.1. Definition [15]. A space X is countably d-paracompact provided every countable open cover of X has a dissectable open refinement.

4.3.2. Theorem [15]. For any space X, TFAE.

(a) X is countably d-paracompact.

(b) Every countable open cover of X has a dissectable open refinement.

(c) Every countable open cover of X is kernel-normal.

(d) Every countable open cover of X has an open F_σ (d-open)-shrink.

(e) X is d-normal and countably subparacompact.

(f) X is d-normal and countably metacompact.

4.3.3. Remark. Since a space X is d-normal iff every countable PF-open cover of X has a dissectable open refinement (4.1.15(k)), from 4.3.2(b), it thus follows that

$$\text{countably d-paracompact} \longrightarrow \text{d-normal}, \qquad \text{just as}$$

$$\text{d-paracompact} \longrightarrow \text{CWdN}.$$

By using our $B(C,\lambda)$-characterization of d-normality, the following new $B(C,\lambda)$-characterization of countable d-paracompactness is easy to verify.

4.3.4. <u>Theorem</u>. For any space X, TFAE.

(a) X is countably d-paracompact.

(b) X is d-normal and countably subparacompact.

(c) X is d-normal and countably metacompact.

(d) X is d-normal and countably θ-refinable.

(e) X is d-normal and countably weak $\bar{\theta}$-refinable.

(f) X is d-normal and countably $B(LF,\lambda)$-refinable.

(g) X is d-normal and countably $B(C,\lambda)$-refinable.

<u>Proof</u>. By 4.3.2(e) above, (a) $\longrightarrow$ (b), and clearly, (b) $\longrightarrow$ (c) $\longrightarrow$ (d) $\longrightarrow$ (e) $\longrightarrow$ (f) $\longrightarrow$ (g). To see that (g) $\longrightarrow$ (a), assume (g) and let $\mathcal{U}$ be a countable open cover of X. Then $\mathcal{U}$ has a $B(C,\lambda)$-refinement. Since X is d-normal, $\mathcal{U}$ must have a dissectable open refinement by 4.1.33(e). Therefore, every countable open cover of X has a dissectable open refinement, and hence X is countably d-paracompact. $\square$

In §1 of this chapter we proved that a d-normal space is countably subparacompact iff X is countably $B(C,\lambda)$-refinable. The following corollary, which follows immediately from 4.3.4 above, is a stronger result.

4.3.5. Corollary. Let X be a d-normal space. Then X is countably d-paracompact iff X is countably $B(C,\lambda)$-refinable.

CHAPTER V

MESOCOMPACTNESS

The notion of mesocompactness was implicitly introduced by A. V. Arkhangel'skii [3] in 1965 in order to establish a relationship between the class of k-spaces and the class of paracompact spaces. Recall that every locally compact, T_2-space is a k-space (see 1.2.32 and 1.2.33). A. V. Arkhangel'skii [3] proved that a normal k-space X is paracompact iff X is mesocompact. J. R. Boone, in his dissertation written under the direction of H. Tamano at Texas Christian University, later introduced the name mesocompactness. J. R. Boone [7] proved that a locally compact T_2-space X is paracompact iff X is mesocompact. See diagram 1.1.2 for the general relationships between mesocompactness and other covering properties.

In §1 of this chapter we obtain a $B(LF,\lambda)$-characterization of mesocompactness, and use it in §2 to provide an easy proof that mesocompactness is preserved under perfect maps. K. Kao and L. Wu [45] established this result in 1983 using a different and more involved argument.

§1. Characterizations of mesocompactness

In his dissertation, J. R. Boone gave the following definitions for the concepts introduced by A. V. Arkhangel'skii [3].

5.1.1. Definition. Let $\mathcal{H}$ be a collection of subsets of a space X. $\mathcal{H}$ is compact-finite (CF) provided every compact subset of X hits at most finitely many members of $\mathcal{H}$.

5.1.2. Definition. A space X is mesocompact provided every open cover of X has a CF-open refinement.

Since $LF \longrightarrow CF \longrightarrow PF$, we have that

$$\text{paracompact} \longrightarrow \text{mesocompact} \longrightarrow \text{metacompact}.$$

J. R. Boone [7] also gave examples to show that mesocompactness lies strictly between paracompactness and metacompactness.

L. L. Krajewski ([51], [78]) and J. C. Smith [78] have used various expandability properties to obtain useful characterizations of paracompactness and metacompactness. Below we introduce an analogous concept of CF-expandability and use techniques similar to those of Krajewski and Smith to prove that a space X is mesocompact iff X is CF-expandable and B(LF,λ)-refinable. We now define two notions of "CF-expandability."

5.1.3. Definition.

(a) [8] A space X is discretely-CF-expandable provided every discrete collection of closed subsets of X has a CF-open expansion.

(b) A space X is (countably) CF-expandable provided every (countable LF collection) LF collection of closed subsets of X has a CF-open expansion.

In 1973 J. R. Boone [8] introduced the concept of "property k" which we will call in this paper "discrete-CF-expandability." He obtained the following characterization of normal mesocompact spaces.

5.1.4. Theorem [8]. A normal space X is mesocompact iff X is metacompact and discretely-CF-expandable.

We now establish several lemmas which will be used to prove a new characterization of mesocompactness which does not require normality.

5.1.5. Lemma. Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a LF collection of closed subsets of a space X. If there exists a CF collection $\mathcal{U}$ of open subsets of X such that

(i) $\cup \mathcal{F} \subset \cup \mathcal{U}$, and

(ii) each member of $\mathcal{U}$ hits at most finitely many members of $\mathcal{F}$,

then $\mathcal{F}$ has a CF-open expansion.

Proof. Let $\mathcal{U}$ be a CF collection of open subsets of X, and assume that $\mathcal{U}$ satisfies conditions (i) and (ii) above. For each $\alpha \in A$, define $G_\alpha = st(F_\alpha, \mathcal{U})$, and $\mathcal{G} = \{G_\alpha : \alpha \in A\}$.

By condition (i) above, $\mathcal{G}$ is clearly an open expansion of $\mathcal{F}$. To see that $\mathcal{G}$ is CF, let K be a compact subset of X. Now $\mathcal{U}$ is CF, so K hits at most finitely many members of $\mathcal{U}$. By condition (ii) above, it thus follows by construction that K will hit at most finitely many members of $\mathcal{G}$. Therefore, $\mathcal{G}$ is a CF-open expansion of $\mathcal{F}$. □

5.1.6. Lemma. Let X be a (countably) mesocompact space. Then X is (countably) CF-expandable.

Proof. Assume that X is (countably) mesocompact, and let $\mathcal{F}$ be a (countable) LF collection of closed subsets of X. For each $x \in X$, define

$W(x) = X - \cup \{ F \in \mathcal{F} : x \notin F\}$, and

$\mathcal{W} = \{W(x) : x \in X\}$.

By construction, $\mathcal{W}$ is a (countable) open cover of X such that each member of $\mathcal{W}$ hits at most finitely many members of $\mathcal{F}$. Now $\mathcal{W}$ has a CF-open refinement $\mathcal{U}$. It should be clear that $\mathcal{U}$ satisfies conditions (i) and (ii) of 5.1.5 above, and hence $\mathcal{F}$ has a CF-open expansion. Therefore, X is (countably) CF-expandable. □

Our characterization of countable mesocompactness in the next theorem is analogous to L. L. Krajewski's characterizations of countable metacompactness and countable paracompactness given in part (a).

5.1.7. Theorem.

(a) [51] A space X is countably metacompact (countably paracompact) iff X is almost countably expandable (countably expandable).

(b) A space X is countably mesocompact iff X is countably CF-expandable.

Proof. We prove only part (b). The necessity follows immediately from 5.1.6 above. Now assume that X is countably CF-expandable and let $\mathcal{U} = \{U_n : n \in N\}$ be a countable open cover of X. For each $n \in N$, define

$$H_n = U_n - \cup \{U_i : i < n\}, \text{ and}$$
$$\mathcal{H} = \{H_n : n \in N\}.$$

It is easy to see that $\mathcal{H}$ is a countable LF-cover of X such that $H_n \subset U_n$ for every $n \in N$. Now $\mathcal{H}$ has a CF-open expansion $\mathcal{V} = \{V_n : n \in N\}$ such that $H_n \subset V_n \subset U_n$ for each $n \in N$. Therefore $\mathcal{V}$ is a CF-open refinement of $\mathcal{U}$, and hence X is countably mesocompact. □

5.1.8. Lemma. Let $\mathcal{U}$ be an open cover of a countably mesocompact space X. If $\mathcal{U}$ has an open refinement $\mathcal{V} = \cup \{\mathcal{V}_\gamma : \gamma < \lambda\}$ such that $\mathcal{V}_\gamma$ is CF for each $\gamma < \lambda$, then $\mathcal{U}$ has a CF-open refinement.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be given as above. Since X is countably mesocompact, $\{\cup \mathcal{V}_\gamma : \gamma < \lambda\}$ has a CF-open refinement $\mathcal{W} = \{W_\gamma : \gamma < \lambda\}$ such that $W_\gamma \subset \cup \mathcal{V}_\gamma$ for each $\gamma < \lambda$. For every $\gamma < \lambda$, define

$$\mathcal{G}_\gamma = \{V \cap W_\gamma : V \in \mathcal{V}_\gamma\}, \text{ and}$$

$$\mathcal{G} = \cup \{\mathcal{G}_\gamma : \gamma < \lambda\}$$

so that $\mathcal{G}$ is an open refinement of $\mathcal{U}$. To see that $\mathcal{G}$ is CF, let K be a compact subset of X. Now K hits at most finitely many members of $\mathcal{W}$, and for each $\gamma < \lambda$, K hits at most finitely many members of $\mathcal{V}_\gamma$. By construction, it follows that K hits at most finitely many members of $\mathcal{G}$. Therefore, $\mathcal{G}$ is a CF-open refinement of $\mathcal{U}$. □

5.1.9. Theorem. For any space X, TFAE.

(a) X is mesocompact.

(b) X is metacompact and CF-expandable.

(c) X is θ-refinable and CF-expandable.

(d) X is weak $\bar{\theta}$-refinable and CF-expandable.

(e) X is B(LF,λ)-refinable and CF-expandable.

Proof. By 5.1.6 above, every mesocompact space is CF-expandable, and so the implications (a) $\longrightarrow$ (b) $\longrightarrow$ (c) and (c) $\longrightarrow$ (d) $\longrightarrow$ (e) should be clear. It remains only to establish (e) $\longrightarrow$ (a), so assume that X is B(LF,λ)-refinable and CF-expandable. Let $\mathcal{U}$ be an open cover of X, and $\mathcal{E} = \cup \{\mathcal{E}_\gamma : \gamma < \lambda\}$ a B(LF,λ)-refinement of $\mathcal{U}$. We use transfinite induction to construct for each $\gamma < \lambda$, a CF collection $\mathcal{V}_\gamma$ of open subsets of X such that

(i) $\mathcal{V}_\gamma$ partially refines $\mathcal{U}$, and

(ii) $\cup\{\cup\mathcal{E}_\beta : \beta < \gamma\} \subset V_\gamma$, where

$$V_\gamma = \cup\{\cup\mathcal{V}_\beta \ : \beta \leq \gamma\} .$$

Assume for fixed $\gamma < \lambda$ that collections $\mathcal{V}_\beta$ have been constructed for all $\beta < \gamma$ satisfying the above conditions. Define

$$V^* = \cup\{V_\beta \ : \ \beta < \gamma\} \text{, and}$$

$$\mathcal{F}_\gamma = \{E - V^* \ : E \in \mathcal{E}_\gamma\}.$$

Condition (ii) above guarantees that $\cup\{\cup\mathcal{E}_\beta \ : \beta < \gamma\} \subset V^*$. Since V^* is open, it follows that $\mathcal{F}_\gamma$ is a LF-closed partial refinement of $\mathcal{U}$, and so $\mathcal{F}_\gamma$ has a CF-open expansion $\mathcal{V}_\gamma$ which also partially refines $\mathcal{U}$. Hence $\mathcal{V}_\gamma$ satisfies condition (i) above. It is easy to see that $\mathcal{V}_\gamma$ also satisfies condition (ii) above, so our construction is complete.

Since $\mathcal{E}$ covers X, conditions (i) and (ii) above imply that $\mathcal{V} = \cup\{\mathcal{V}_\gamma \ : \gamma < \lambda\}$ is an open refinement of $\mathcal{U}$ such that $\mathcal{V}_\gamma$ is CF for each $\gamma < \lambda$. Since X is CF-expandable, X is countably mesocompact by 5.1.7. Therefore $\mathcal{U}$ has a CF-open refinement by 5.1.8 above, and hence X is mesocompact. $\square$

In contrast to our last result, K. Kao and L. Wu [45] obtained the following characterization of mesocompactness in 1983 which we state here without proof.

5.1.10. Theorem [45]. A space X is mesocompact iff every directed open cover $\mathcal{U}$ of X has a CP-closed refinement $\mathcal{F}$, such that every compact subset of X is contained in some member of $\mathcal{F}$.

§2. Mapping theorems

In 1970 V. J. Mancuso [55] published a result which stated that mesocompactness is preserved under perfect maps; however, in 1973 J. R. Boone [8] observed that Mancuso's proof was invalid. Boone then proved that discrete-CF-expandability (property k) is preserved under perfect maps, and used this fact along with his result 5.1.4 to conclude that perfect images of normal mesocompact spaces are mesocompact. In 1983, K. Kao and L. Wu [45] improved Boone's result by proving that the closed continuous image of any normal mesocompact space is mesocompact. They also used their characterization 5.1.10 above of mesocompactness to conclude that mesocompactness is preserved under perfect maps. Below we provide an alternate proof that mesocompactness is preserved under perfect maps.

5.2.1. Lemma. The perfect image of any (countably) CF-expandable space is (countably) CF-expandable.

Proof. Let X be a (countably) CF-expandable space, and assume that $f : X \longrightarrow Y$ is a perfect map from X onto a space Y. Let $\mathcal{E}$ be a (countable) LF collection of closed subsets of Y.

Since f is continuous, $\mathcal{F} = f^{-1}(\mathcal{E})$ is a (countable) LF collection of closed subsets of X, and hence there exists a CF-open expansion $\mathcal{U}$ of $\mathcal{F}$. Since f is closed and continuous, we may assume that each member of $\mathcal{U}$ is f-saturated; thus, $\mathcal{V} = f(\mathcal{U})$ is an open expansion of $\mathcal{E}$. To complete the proof, we show that $\mathcal{V}$ is also a CF collection. Indeed, let K be a compact subset of Y. Since f is perfect, $L = f^{-1}(K)$ is a compact subset of X, and hence L hits at most finitely many members of $\mathcal{U}$. It follows that $K = f(L)$ hits at most finitely many members of $\mathcal{V}$, and therefore $\mathcal{V}$ is CF. □

5.2.2. Corollary. The perfect image of any countably mesocompact space is countably mesocompact.

Proof. This result follows immediately from 5.2.1 above and the equivalence of countable CF-expandability and countably mesocompactness (5.1.7). □

5.2.3. Corollary [45]. The perfect image of any mesocompact space is mesocompact.

Proof. This result follows immediately from 5.1.9(e) and the fact that CF-expandability (5.2.1) and $B(LF,\lambda)$-refinability (2.3.6) are preserved under perfect maps. □

5.2.4. Question. Is the closed, continuous image of every mesocompact space mesocompact?

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