Results of True-Anomaly Regularization in Orbital Mechanics

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(ABSTRACT)

Presented herein are some analytical results available from regularization of the differential equations of satellite motion. True-anomaly regularization is developed as a special case of a more general Sundman-type transformation of the independent variable (time) in the equations of motion. Constants of the unperturbed motion are introduced as extra state variables, and regularization with several types of coordinates is considered. Because analytical results are sought, those regularizing transformations which produce rigorously linear governing equations are of main interest. When solutions of the linear regular equations in the true-anomaly domain are examined, it is found that the initial value and boundary value problems of unperturbed motion, typically requiring iterative solutions of the time equation, can be solved with only a single transcendental function evaluation per iteration cycle. Various means are described which can accelerate the evaluation of this function. The time equation developed in this study is a new universal relation between time of flight and true anomaly, and applies uniformly to all types of orbits, including rectilinear ones. It is a well-behaved function, the zero of which can be found reliably by Newton's method or other typical iteration methods. Once this time equation has been solved, the initial and final state vectors on the transfer arc can be related to each other by rational algebraic formulae; no other transcendental function is needed. When the two problems are generalized by variation of parameters to the case of oblate-gravity perturbed motion, it is found that, to first order, the corrections of the unperturbed solution can be obtained by direct, noniterative formulae valid for all types of orbits. Moreover, it is possible to compute these corrections with only a single extra evaluation of the same transcendental function used in the unperturbed problem. Additional results are also presented, including exact solutions of the first-order averaged differential equations gov-

erning secular variations of the regular orbital elements in the true-anomaly domain. Complete universal expressions are given for the Keplerian state transition matrix in terms of the orbital transfer angle, and a simple midcourse guidance scheme is rederived in terms of universal variables valid for all non-rectilinear transfer orbits.

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General Introduction

Background Basic to the Problem

Many of the most significant modern results in celestial mechanics have been connected with regularization of the differential equations of motion. In the most general usage of the term, "regularization" means the introduction of new independent or dependent variables so that certain singularities in the original equations do not occur in the transformed equations. In celestial mechanics, the unwanted singularities arise from the inverse-square attractive force. For example, a regularization of the two-body problem seeks to remove the singularity which occurs in the gravitational acceleration when the distance between the bodies vanishes. When more than two attracting bodies are involved, the regularization procedures can become quite complicated due to the variety of collision configurations which must be considered. However, this study will be concerned only with topics related to two-body regularization; third-body effects could appear as perturbations of the two-body (Keplerian) motion but the transformations themselves will pertain only to the motion of the satellite with respect to its primary attracting center.

The results of introducing a suitable regularizing transformation are extremely useful both numerically and analytically. For reasons noted later, one finds that the transformed differential equations of motion are usually much better suited for numerical integration than are the original equations. In particular, perturbed orbits making one or more close approaches to the attracting center can be followed reliably, even over many revolutions. When analytical results are pursued, one finds that regularizing transformations sometimes permit new theoretical developments. In fact, regularization in its modern guise arose in a purely theoretical context. In a famous memoir, K. F. Sundman (1912) was able by means of a regularizing transformation to examine the existence and behavior of post-collision trajectories in the restricted three-body problem.

From an engineering point of view, one of the most intriguing aspects of regularization is that several transformations of the unperturbed two-body problem not only remove the collision singularity but also result in rigorously linear governing equations. It has been known for some years that either of two regularizing time transformations, corresponding to the introduction of eccentric anomaly or true anomaly as independent variable, will result in a set of uncoupled, linear, constant-coefficient differential equations for the coordinates and a quadrature for the time. If a small net perturbing force is included in the problem then the differential equations are coupled through small nonlinear terms. It is easy to see that the basic linearity and regularity of the transformed problem might offer great advantage in the development of practical applications of the equations of motion. In fact, the advantage accrues for at least three different reasons. First, it becomes a straightforward job to derive complete sets of two-body solution formulae which are "universal", that is, valid for all initial conditions and, in particular, for all values of the energy. (Certain provisos on this statement are at the core of the present study.) Second, it becomes a simple matter to introduce regular orbital elements into the problem. As classically conceived, orbital elements are parameters which remain constant in unperturbed motion and which may be used in conjunction with the elapsed time to describe the position and velocity of the satellite. The use of elements is also an especially convenient way to describe the geometry of the orbital path in space. For example, one may specify at some epoch the semimajor axis and eccentricity of the orbit, the inclination of the orbital plane relative to some reference plane, the longitude of the line of nodes, the longitude of the pericenter and the time of the pericenter passage. Then, with the path of the satellite established and the elapsed time from pericenter passage given, one can deduce the position of the satellite along its orbit and ultimately it true position in space. The details of such a computation are not important here. The point is to realize that there are serious pitfalls in using this or any closely related set of elements: the semimajor axis is infinite for a parabolic orbit; the line of nodes does not exist for a zero-inclination orbit; a circular orbit does not have a unique pericenter. Sometimes ambiguous cases can be resolved by using special combinations of these elements, as discussed by Geyling and Westerman (1971, chapter 6, section 5) or Brouwer and Clemence (1961, chapter 11, section 7). However, all such difficulties can be avoided from the outset by adopting as elements various parameters which appear in the solution of the linear regular equations of motion. These regular elements typically are valid for all kinds of orbits, avoid the geometrical indeterminancies inherent in many of the classical sets of elements, and are simply, sometimes trivially, related to the dynamical initial conditions and boundary conditions of the problem. Third, the transformed differential equations governing perturbed two-body motion have forms which are well suited for treatment by any of several analytical perturbation techniques. Especially, the derivation of element differential equations by the variation-of-parameters method is vastly simplified compared to classical procedures which begin with the untransformed nonlinear equations of motion. Motivated by these desirable features, this study deals with analytical results obtainable in the true-anomaly domain for some of the linear, regular, transformed systems having small (or no) perturbations.

Problem Statement and Purpose of the Study

The problem to be investigated in this study is posed in the form of the following two questions.

1. How can the results of true-anomaly regularization be adapted to solve typical orbital initial-value and boundary-value problems?

2. For each particular problem treated, what advantages and limitations arise in the proposed solution method based on true-anomaly regularization as opposed to other current solution methods?

The purpose of this study is, therefore, to derive completely by means of true-anomaly regularization all the formulae needed for a universal treatment of the orbital initial-value and boundary-value problems defined more specifically later. Because the true anomaly is only one of many possible regularizing variables, the derivation is to be presented in the context of a general transformation of the two-body problem in which the true-anomaly regularization appears as an important special case.

Significance of the Problem and Need for the Study

Most of the present-day engineering applications of celestial mechanics are in the realm of tracking, guidance, navigation and control of space vehicles. Naturally, the trend in this field has been for requirements on spacecraft guidance system performance to become more stringent as space operations become simultaneously more ambitious and more routine. Furthermore, as newer spacecraft are designed to operate more autonomously, it becomes increasingly necessary to develop guidance solutions that are computationally concise as well as completely general. The need for conciseness becomes especially apparent if the guidance program must include explicit calculation of orbital perturbing effects, such as those due to Earth oblateness. Of course, a prerequisite for developing any particular guidance mechanization is that one have available efficient and general algorithms for solving the basic orbital problems involved. In practice, it is desirable to have several independent algorithms for a given problem, if possible, because advantageous features of several schemes may sometimes be combined effectively for a particular application. Thus, new analytical formulations are always of interest even if existing methods work well.

For many years, completely general formulae have been available for solving the initial-value and boundary-value problems associated with unperturbed two-body motion. These formulae are cast in terms of universally valid variables and transcendental functions (Stumpff functions) which arise from an eccentric-anomaly regularization of the differential equations of motion. This type of regularization leads to the older universal formulae of Stumpff (1947, 1959, 1962), Sperling (1961), Herrick (1965), Battin (1964), Goodyear (1965a), Pitkin (1968) and others, as well as to newer formulations based on the KS theory of Kustaanheimo (1964), Kustaanheimo and Stiefel (1965), Stiefel and Scheifele (1971), Jezewski (1976), Kriz (1976) and others. Much more recently, eccentric-anomaly regularization in its KS form has been used to develop analytical solution methods for some perturbed-orbit initial-value and boundary-value problems (see Andrus, 1977; Engels and Junkins, 1981; and Hand, 1982).

In the meantime, little has been done toward using true-anomaly regularization to develop alternative solution methods for these problems. Of course, true anomaly as an independent variable has found extensive use in celestial mechanics, but primarily in perturbation theories of Earth satellites in which, for example, it is very useful for calculating changes in the orbital elements averaged over one revolution. Direct use of true anomaly for problems involving time is rare, even when perturbations are not considered. The reason is that the unperturbed time of flight is a more complicated function of true anomaly than of eccentric anomaly. On one hand, the quadrature for time in terms of eccentric anomaly is quite elementary and is readily generalized to a universal form.

On the other hand, the quadrature for time in terms of true anomaly can be obtained in terms of elementary functions but is fundamentally more complicated. Heretofore, it has not been presented in a form valid for all types of orbits, with the exceptions of a few highly specialized "near-parabolic" series expansions. Without a universal time equation one cannot develop universal methods for the initial-value and boundary-value problems, even though all the other formulae obtained by true-anomaly regularization are in fact valid for all kinds of orbits. Needless to add, universal formulations based on true-anomaly regularization of the corresponding perturbed problems are not to be found, either. Yet there are at least two important reasons why one might prefer a true-anomaly approach for analytical work.

1. For all non-rectilinear orbits, the unperturbed motion can be expressed in terms of circular trigonometric functions, rather than Stumpff functions, of the true anomaly, greatly simplifying the handling of lengthy formulae.

2. Terms representing zonal geopotential perturbations can be developed without approximation into finite Fourier series in the true anomaly, so that auxiliary series expansions of the perturbing force expression are not usually required in Earth satellite problems.

It is worth emphasizing that perturbation solutions which are at once finite, non-averaged and universally valid are indeed rare in the practice of orbital mechanics. The main reason for this situation, at least in theories of artificial satellite motion, is the difficulty of relating time to true anomaly in a universal way. Evidently, then, any means of generalizing the time equation in terms of true anomaly ought to be investigated.

The above considerations have to do mainly with the use of true anomaly as the independent variable in the solution formulae of two-body motion. There is another important consideration having to do with the use of true anomaly as the independent variable in the governing differential equations. Burdet (1969) showed that the governing equations can be made not only regular but also linear in unperturbed motion in the true anomaly domain, provided that the reciprocal radius and the unit radial vector are used as coordinates. Later, Vitins (1978) found another linear regular true-anomaly formulation using a different type of coordinates. Kamel (1983) discussed yet another, slightly less general, linear true-anomaly formulation. The question naturally arises, what is the class of coordinates for which true-anomaly regularization leads to linear regular governing equations? Conversely, for given coordinates, what independent variables besides true anomaly lead to linear regular equations? More generally, the question might be how to characterize the relation between true-anomaly regularization and eccentric-anomaly regularization, since the latter also leads to linear regular governing equations in several types of coordinates. These are difficult questions for which complete answers are not yet available. Useful partial answers can be offered, however. A recent analysis by Szebehely and Bond (1983) shows that, for a set of time transformations involving powers of the radius, the equations of unperturbed planar motion are transformed into linear harmonic oscillator equations only when the independent variable is the eccentric anomaly, the true anomaly or a simple generalized version of either. Due to the nature of the technique used by Szebehely and Bond, the validity of this result in three dimensions is not immediately obvious. Later, Bond (1985) transformed the spherical-coordinate equations of unperturbed motion into a three-dimensional linear oscillator form. He began with a time transformation involving a general function of the radius and found that the oscillator form of the governing equations was possible only when the independent variable corresponds to the true anomaly. Bond's result depends on the spherical coordinates themselves and it is not clear whether other kinds of coordinates can lead to linear oscillator equations. In this study, the governing equations will be handled in vector form to avoid specializing the results to a particular coordinate system. The time transformations to be considered here will be less general than those considered by Bond (1985), however. In this analysis, only powers of the radius will be included in the time transformation. Also, as in most studies of regularization, first integrals of the unperturbed motion will be introduced to provide the necessary redundancy in the governing system equations. Within this class of regularizing transformations, it is found that for a variety of space coordinates only the eccentric anomaly and the true anomaly lead to linear oscillator-type governing equations. Specific conclusions are reached not only for the three-parameter Cartesian coordinates (as used by Burdet, 1967, 1968; and Szebehely, 1976b) but also for some four-parameter Burdet-type coordinates (like those used by Burdet, 1969), some four-parameter KS-type coordinates (like those used by Kustaanheimo and Stiefel, 1965; and Stiefel and Scheifele, 1971), and some five-parameter coordinates which involve the Euler parameters (like those used by Broucke, *et al.*, 1971; Vitins, 1978; and Junkins and Turner, 1979). Taken together, this collection of transformations constitutes the majority of two-body regularizations in use today. Several new details are added to existing formulations by reason of the general radius-power-law time transformation.

Basic Assumptions and Scope of the Study

The mathematical model underlying this study is that of the perturbed two-body problem: two point masses moving subject to mutual inverse-square attraction while an additional net perturbing force acts on the satellite. In deriving the various forms of the equations of motion, no restriction is made concerning the type or magnitude of the perturbing force. In deriving approximate solutions of these equations, it will be assumed at least that the perturbing force does not depend explicitly on the time and that its magnitude always remains small compared to the primary inverse-square attraction. In practice, the perturbation solutions will be limited to include only the first-order effects of the dominant oblateness term (J_2) in the Earth's gravitational potential function. Restriction to at most a first-order J_2 problem is made primarily for convenience of discussion, given, of course, the known importance of this perturbation in Earth satellite problems. It will be clear how effects due to some higher-order zonal terms could be included without altering the basic features of the formulae. Effects due to sectoral and tesseral terms are not considered in this study, consistent with the assumption that the perturbing force does not depend explicitly on the time. All other perturbing forces, such as rocket thrust, atmospheric drag, third-body gravitation and solar radiation pressure, time-dependent or not, are likewise neglected. Actually, in terms of practical consequences, the assumption that the perturbing force is independent of the time is the most important restriction on the analysis. The reason is that independent variables other than time will be introduced into the equations of motion and time will become a dependent variable with its own governing differential equation. Under the transformations to be considered in this study, if the perturbing force is independent of time then the differential equation of time is not coupled to the rest of the governing equations, but rather can always be treated as a quadrature.

Of course, some justification beyond mere convenience can be offered for these restrictions of the problem, at least in the case of a non-thrusting satellite. It is well known that for many Earth-satellite orbits (roughly, those with altitude between 160 and 1000 kilometers) the effects due to all other forces are comparable in magnitude to second-order J_2 effects. (See, for example, the discussions in Claus and Lubowe, 1963; and in Hori and Kozai, 1973. There is a famous exception to this rule: the Project Echo balloon satellite had such a low mass density that it was strongly perturbed by solar radiation pressure and the solar wind.) Likewise, all sectoral and tesseral effects can be relegated to second-order in J_2 if the orbital period is not commensurable in small integers with the Earth's rotational period. Thus, perturbation expressions valid only to first order in J_2 do constitute important and useful information by themselves.

The particular problems to be considered in this study are, as indicated previously, the orbital initial-value and boundary-value problems. The initial-value problem, which sometimes bears the name of Kepler, is to compute position and velocity at a specified instant of time given position and velocity at another instant of time. As is well known, the solution procedure in the unperturbed case can be reduced to the inversion of a single transcendental equation involving time of flight and constants of the motion. The boundary-value problem to be considered is that associated with the names of Gauss and Lambert: given two positions and the time of flight between them, compute the velocity vector at the initial position. Although other types of boundary-value problems often need to be solved in guidance applications, solution of the Gauss/Lambert problem is fundamental to many spacecraft targeting schemes and orbit determination methods. Again, it is well known that the solution procedure in the unperturbed case can be reduced to the inversion of a single transcendental equation involving time of flight and constants of the motion. Of course, there are many algorithms for solving the problems of Kepler and Gauss. This study will discuss existing methods only to the extent necessary for comparison purposes and will focus on developing new time of flight equations in the true-anomaly domain. Of special interest will be the generalization of the time equations to include the aforementioned J_2 perturbing effects.

Throughout, the primary goal of this study is to obtain analytical results. This means in particular that all integrations are to be presented ultimately in explicit literal terms. Therefore, in transforming the governing equations of motion, the concern will be not so much with obtaining complete regularization *per se* as with exploiting those regularizing transformations which lead to analytically tractable differential equations. Practically speaking, the only tractable differential systems are linear ones. However, the linear systems considered here are all regular as well, and this circumstance is not merely fortuitous. Linearity and regularity of the transformed equations are later seen to be related properties for the set of transformations considered here. Now within this restricted set of systems for which explicit solutions can be offered, the underlying regularity of the equations usually means that those solutions will be valid for orbits of all types and orientations. For the purposes of this study, certain mild restrictions on orbit geometry, such as restriction to non-rectilinear orbits, are permitted in many formulae. These restrictions are noted and discussed as they arise in the analysis. In addition, certain bounds on the allowable magnitude of the orbital transfer angle are implied in some formulae. These bounds do not affect the universality of the formulae, but do have a controlling influence on the way the formulae must be implemented.

Numerical aspects of the transformed differential equations will be noted but not discussed in detail. In particular, questions of numerical stability and the possibility of stabilizing certain integrations by further transformations have been discussed extensively throughout the literature and will not be pursued here. On the other hand, numerical aspects of the analytical solution formulae, including certain features traceable to the stability of the governing equations, are discussed in all necessary detail.

Hypothesis

With the above considerations and restrictions in mind, this study seeks to verify the following statement:

True-anomaly regularization of the differential equations governing perturbed two-body motion leads to complete sets of analytical formulae permitting universal treatment of the orbital initial-value and boundary-value problems both in unperturbed cases and in perturbed cases arising from zonal (J_2) Earth-gravitational effects.

Chapter 1. The Governing Differential Equations

The significance of true-anomaly regularization of the differential equations of motion will be appreciated most thoroughly in light of more general transformations of the governing equations. This chapter introduces the untransformed differential equations themselves. Later chapters will discuss the various regularizing transformations in a sequence which leads naturally to the case of interest.

Using a coordinate system centered in the primary body, the perturbed relative motion of the satellite body is described by

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = \underline{P} \tag{1.1}$$

together with appropriate initial conditions or boundary conditions. Here $\underline{r}$ is the position vector, the overdot symbol denotes a derivative with respect to time, μ is the gravitational parameter for the two body system, r is the radius from the attracting center, and $\underline{P}$ is the perturbing force per unit mass. This form of the equation of motion implies that the coordinate system is not rotating relative to an inertial reference frame, though the primary body does experience translational accelerations in the inertial frame. In case $\underline{P}$ happens to be a conservative force at least one first integral can be found, namely, the total energy. Other integrals may appear for certain specialized forms of $\underline{P}$. If $\underline{P}$ is not conservative then equation (1.1) does not in general permit any first integrals to be written down.

Now if $\underline{P} = \underline{0}$ identically then equation (1.1) describes unperturbed (i.e., Keplerian) motion. In that case the equation yields three well-known first integrals (comprised of seven scalar components), namely, the energy, the angular momentum vector and the Laplace vector. Although these quantities are not constants when perturbations are present, they nonetheless play crucial roles in all the analysis to follow. Furthermore, it is an easy matter to derive their rates of change due to perturbations. The same manipulations which produce these constants of motion in the unperturbed case now produce their differential equations in the perturbed case.

Form the scalar product of equation (1.1) with $\dot{\underline{r}}$:

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} + \frac{\mu}{r^3} \underline{r} \cdot \dot{\underline{r}} = \underline{P} \cdot \dot{\underline{r}} \tag{1.2}$$

Since $\underline{r} \cdot \dot{\underline{r}} = r\dot{r}$ identically, equation (1.2) can be written as

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} + \frac{\mu}{r^2} \dot{r} = \underline{P} \cdot \dot{\underline{r}} \tag{1.3}$$

Upon introducing the energy E defined by

$$E = \frac{1}{2} \dot{\underline{r}} \cdot \dot{\underline{r}} - \frac{\mu}{r} \tag{1.4}$$

it becomes clear that the left side of equation (1.3) is precisely $\dot{E}$:

$$\dot{E} = \underline{P} \cdot \dot{\underline{r}} \tag{1.5}$$

Here the quantity E is sometimes known as the Keplerian energy in order to distinguish it from the total energy mentioned above.

Now form the vector product of equation (1.1) with $\underline{r}$:

$$\underline{r}\times\ddot{\underline{r}}+\frac{\mu}{r^3}\underline{r}\times\underline{r}=\underline{r}\times\underline{P} \tag{1.6}$$

The second term vanishes identically. Upon introducing the angular momentum defined by

$$\underline{h}=\underline{r}\times\dot{\underline{r}} \tag{1.7}$$

it becomes apparent that the left side of equation (1.6) is just $\dot{\underline{h}}$:

$$\dot{\underline{h}}=\underline{r}\times\underline{P} \tag{1.8}$$

Finally, form the vector product of equation (1.1) with $\underline{h}$:

$$\ddot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})+\frac{\mu}{r^3}\underline{r}\times(\underline{r}\times\dot{\underline{r}})=\underline{P}\times(\underline{r}\times\dot{\underline{r}}) \tag{1.9}$$

Since $\underline{r}\times(\underline{r}\times\dot{\underline{r}})=\underline{r}(\underline{r}\cdot\dot{\underline{r}})-\dot{\underline{r}}(\underline{r}\cdot\underline{r})$ identically, it follows that

$$\ddot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})+\frac{\mu}{r^3}\left[\underline{r}r\dot{r}-\dot{\underline{r}}r^2\right]=\underline{P}\times(\underline{r}\times\dot{\underline{r}}) \tag{1.10}$$

$$\ddot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})-\mu\left[\dot{\underline{r}}\frac{1}{r}-\underline{r}\frac{\dot{r}}{r^2}\right]=\underline{P}\times(\underline{r}\times\dot{\underline{r}}) \tag{1.11}$$

Now observe that

$$\frac{d}{dt}\left[\dot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})\right]=\ddot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})+\dot{\underline{r}}\times(\underline{r}\times\ddot{\underline{r}}) \tag{1.12}$$

Upon adding the latter term of equation (1.12) to both sides of equation (1.11) and introducing the Laplace vector $\underline{B}$ defined by

$$\underline{B}=\dot{\underline{r}}\times(\underline{r}\times\dot{\underline{r}})-\frac{\mu}{r}\underline{r} \tag{1.13a}$$

$$\underline{B} = \underline{r}(\dot{\underline{r}}\cdot\dot{\underline{r}}) - \dot{\underline{r}}(\underline{r}\cdot\dot{\underline{r}}) - \frac{\mu}{r}\underline{r} \tag{1.13b}$$

it is found that

$$\dot{\underline{B}} = \underline{P}\times(\underline{r}\times\dot{\underline{r}}) + \dot{\underline{r}}\times(\underline{r}\times\ddot{\underline{r}}) \tag{1.14}$$

Furthermore, according to equation (1.6), this is

$$\dot{\underline{B}} = \underline{P}\times(\underline{r}\times\dot{\underline{r}}) + \dot{\underline{r}}\times(\underline{r}\times\underline{P}), \tag{1.15}$$

or

$$\dot{\underline{B}} = 2\underline{r}(\underline{P}\cdot\dot{\underline{r}}) - \dot{\underline{r}}(\underline{P}\cdot\underline{r}) - \underline{P}(\underline{r}\cdot\dot{\underline{r}}) \tag{1.16}$$

after the vector products are carried out.

From these results it is obvious that E , $\underline{h}$ and $\underline{B}$ are constants of unperturbed motion since their rates of change are zero when $\underline{P} = \underline{0}$. It happens that, of the seven scalar quantities so represented, only five are independent. Direct calculation from the definitions given above will verify that the two scalar relations

$$\underline{h}\cdot\underline{B} = 0 \tag{1.17}$$

$$\mu^2 + 2E(\underline{h}\cdot\underline{h}) - \underline{B}\cdot\underline{B} = 0 \tag{1.18}$$

hold identically.[1] Thus at least one more first integral (involving time) remains to be found in order to complete the solution for the unperturbed motion. More important to the present purpose, though, is the fact that the extra differential equations (1.5), (1.8) and (1.16) are not in principle

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[1] Equation (1.18) is not usually presented in a universal form, even though it is a fundamental relation in orbital mechanics. Its more familiar form, which is not universal, is used later in this study as equation (5.57). A less familiar, but universal, form appears in equation (5.208). The general relation is discussed by Goldstein (1980, section 3-9).

needed in generating the solution of (1.1). E , $\underline{h}$ and $\underline{B}$ are redundant in the sense that $\underline{r}$ and $\dot{\underline{r}}$ could be obtained from equation (1.1) alone by some numerical process. At first sight, it might seem useless to try to introduce E, $\underline{h}$ and $\underline{B}$ into the governing equations in some more intricate way, thereby increasing the order of the system to be integrated. Yet just such a technique was used with great success by Sperling (1961) and later researchers. This unusual situation in which a system of higher order can be treated more successfully than a system of lower order has been studied by Broucke, *et al.* (1971), and by Stiefel and Scheifele (1971, section 45; see also Stiefel, 1973). The latter authors have shown that a regularization of the two-body equations of motion must necessarily involve redundant variables and an increase beyond six in the order of the system to be integrated. Stiefel's result does not by itself indicate what redundant variables ought to be used, but one finds that the algebraic combinations of $\underline{r}$ and $\dot{\underline{r}}$ represented by E , $\underline{h}$ and $\underline{B}$ have precisely the properties needed to establish regular equations of perturbed two-body motion. Even more remarkably, some of the regular systems so derived actually reduce to uncoupled, linear, constant-coefficient differential equations when the motion is unperturbed. The details of how to introduce E and $\underline{B}$ into equation (1.1), while not perfectly straightforward, are well known and are discussed in the next chapter. Several new details involving $\underline{h}$ arise in the general case treated in this study.

Subsequent chapters deal with other sets of redundant variables which have been proposed. By the introduction of these other variables into equations which are already regularized, it will be seen how it is possible (and sometimes necessary) to eliminate one or more of the quantities E , $\underline{h}$ and $\underline{B}$ in favor of the new variables. This procedure provides a convenient way to interrelate many of the various proposals scattered throughout the literature.

A crucial aspect of all the following developments is the additional introduction of some new independent variable in place of the time. This step in turn creates the important subsidiary problem of relating changes in this abstract independent variable to the physically more significant passage of time. Later chapters of the present study examine this problem in depth for the special case when the new independent variable corresponds to the true anomaly.

Chapter 2. The Generalized Sundman Time Transformation

Introduction

A successful attempt at regularizing the basic equation of motion (1.1) can be made by introducing some new independent variable in place of the time. It is especially noteworthy that a transformation of the coordinates is not required in order to effect the regularization, although any time transformation necessarily implies some magnitude-scaling of the velocity vector. Of the various independent variables which have been proposed, the most famous is that of Sundman (1912), denoted s and related to time by the differential relation

$$dt = r \, ds \tag{2.1}$$

Although this relation was employed already in 1765 by Euler in his study of the rectilinear three-body problem (see Stiefel and Scheifele, 1971, chapter 1; Szebehely, 1967, section 3.12; and Szebehely, 1974), it is usual nowadays to refer to Sundman's name when treating three-dimensional motion. A generalization of (2.1) which has found extensive numerical use is

$$dt = C\ r^n\ ds \tag{2.2}$$

where n is a constant and C is ordinarily taken as a constant but may represent any general function. The advantage of this generalization is that, by choosing C and n carefully for a given problem, a useful "automatic stepsize control" is achieved in the time domain during numerical integration of the transformed differential equations of motion. (See, for example, Stiefel and Scheifele, 1971; Velez, 1974, 1975; Alfriend and Velez, 1975; Baumgarte, 1976). For many numerical purposes it is not necessary to carry the transformation of the differential equations all the way to complete regularization. That is, sufficient gains in the accuracy of integration may be available merely by the introduction of the automatic stepsize control even though negative powers of the radius r remain in the formulae. Also, if close approaches to the attracting center do not actually occur, then regularization of the differential equations (and the associated increase in the order of the system) may not be a practical necessity. Consequently, completely regularized governing equations based on the transformation (2.2) are not usually found except for special values of C and n. Still less common are discussions of the analytical (versus purely numerical) possibilities of equation (2.2). The following analysis will derive transformed governing equations for general values of C and n, and will reveal a family of regularized equations, some of which reduce to rigorously linear equations in the case of unperturbed motion. By retaining general values of C and n in the formulae, it becomes easier to see the rationale behind each of the many particular forms of equation (2.2) which have been proposed.

The Transformed Differential Equations

Now according to equation (2.2), time derivatives will be transformed as

$$\frac{d(\ldots)}{dt} = C^{-1} r^{-n} \frac{d(\ldots)}{ds} \tag{2.3}$$

For example, the velocity is transformed as

$$\dot{\underline{r}} = C^{-1}\, r^{-n}\, \underline{r}' \tag{2.4}$$

where $(\ldots)'$ denotes $d(\ldots)/ds$. Then the acceleration is

$$\ddot{\underline{r}} = C^{-1} r^{-n} (C^{-1} r^{-n} \underline{r}')' \tag{2.5}$$

$$\ddot{\underline{r}} = -C^{-3} C' r^{-2n} \underline{r}' - nC^{-2} r^{-2n-1} r' \underline{r}' + C^{-2} r^{-2n} \underline{r}'' \tag{2.6}$$

Now $\ddot{\underline{r}}$ can be replaced in the equation of motion (1.1) to give

$$\underline{r}'' - nr^{-1} r' \underline{r}' + C^2 \mu r^{2n-3} \underline{r} = C^2 r^{2n} \underline{P} + C^{-1} C' \underline{r}' \tag{2.7}$$

Here the quantity C may be any general function and the question of how to find its derivative C' is deferred until specific choices for C itself arise later. In anticipation of later results, the term containing C' has been included on the right-hand side with the perturbation term. In principle, C' certainly need not depend on $\underline{P}$. In practice, the most useful choices for C will turn out to be constants and elements, meaning that either $C' = 0$ identically or else $C' = 0$ when $\underline{P} = \underline{0}$.

It should be noted that, in equation (2.7), a negative power of r occurs which cannot be removed by any choices of C and n. Hence, this differential equation is not regular as it stands. On the other hand, some numerical success has been achieved with this equation (see Velez, 1974, 1975; and Alfriend and Velez, 1975) because, regardless of the singularity, position and velocity vary much more smoothly in the s domain than in the time domain, provided that C and n are chosen judiciously. Actually, as pointed out by Stiefel and Schiefele (1971, chapter 1), one must distinguish between regularization of the solutions of differential equations and regularization of the differential equations themselves. These authors prove by simple example that a nonregular differential equation may still have perfectly regular solutions (though they do not discuss whether the solutions of equation (2.7) are regular). Of course, the desideratum for both analytical and numerical work is to deal with regular differential equations as well as regular solutions. To achieve this end, it will at least be necessary to increase the order of equation (2.7) through the introduction of appropriate redundant variables. A most elegant regularization, of a type first presented by Sperling (1961), can be obtained by introducing the integrals of unperturbed motion, namely, the Keplerian energy, the angular momentum and the Laplace vector.

The said quantities are first written down in s-domain form by transforming their definitions according to the rule given in equation (2.3).

$$E = \frac{1}{2} C^{-2} r^{-2n} (\underline{r}' \cdot \underline{r}') - \mu r^{-1} \tag{2.8}$$

$$\underline{h} = C^{-1} r^{-n} \underline{r} \times \underline{r}' \tag{2.9}$$

$$\underline{B} = C^{-2} r^{-2n} \underline{r}' \times (\underline{r} \times \underline{r}') - \mu r^{-1} \underline{r} \tag{2.10}$$

Likewise, the corresponding rates of change are readily obtained by using equation (2.2).

$$E' = \underline{P} \cdot \underline{r}' \tag{2.11}$$

$$\underline{h}' = C r^{n} \underline{r} \times \underline{P} \tag{2.12}$$

$$\underline{B}' = 2\underline{r}(\underline{P} \cdot \underline{r}') - \underline{r}'(\underline{P} \cdot \underline{r}) - \underline{P}(\underline{r} \cdot \underline{r}') \tag{2.13}$$

Now in order to introduce E and $\underline{B}$ into the transformed equation of motion (2.7), rewrite $\underline{B}$ as

$$\underline{B} = C^{-2} r^{-2n} [\underline{r}(\underline{r}' \cdot \underline{r}') - \underline{r}'(\underline{r} \cdot \underline{r}')] - \mu r^{-1} \underline{r} \tag{2.14}$$

$$\underline{B} = C^{-2} r^{-2n} \underline{r}(\underline{r}' \cdot \underline{r}') - C^{-2} r^{-2n+1} r' \underline{r}' - \mu r^{-1} \underline{r} \tag{2.15}$$

Observe that the first and third terms of (2.15) resemble the terms of the transformed energy equation (2.8), while the middle term resembles the nonregular second term of the equation of motion (2.7). It is this nonregular term that must be replaced by regular expressions involving E and $\underline{B}$. A little experimentation leads to the following important combination of quantities:

$$\underline{B} - 2E\underline{r} = -C^{-2}r^{-2n+1}r'\underline{r}' + \mu r^{-1}\underline{r} \tag{2.16}$$

The first term on the right side of this equation can be made to match the second term of the equation (2.7) by further concocting the combination:

$$nC^2r^{2n-2}(\underline{B} - 2E\underline{r}) = -nr^{-1}r'\underline{r}' + C^2\mu r^{2n-3}\underline{r} + (n-1)C^2\mu r^{2n-3}\underline{r} \tag{2.17}$$

Equation (2.17) has been arranged so that the first two terms on the right side match two terms in the equation of motion (2.7). Direct substitution then produces

$$\underline{r}'' - 2EC^2nr^{2n-2}\underline{r} - (n-1)C^2\mu r^{2n-3}\underline{r} = -nC^2r^{2n-2}\underline{B} + C^2r^{2n}\underline{P} + C^{-1}C'\underline{r}' \tag{2.18}$$

Now the desired regularization is within reach. Suppose for a moment that E and $\underline{B}$ are given as parameters in this equation. (Such will actually be the case for unperturbed motion.) It is clear that then the explicit appearance of negative powers of r can be avoided by a suitable choice of n , say, n sufficiently large, or, perhaps more obviously, n equal to unity. Of course, in general E and $\underline{B}$ are not given as parameters in the problem; in the presence of perturbations they are variables. One might imagine trying to obtain these quantities in terms of $\underline{r}$ and $\underline{r}'$ at any instant from their definitions, equations (2.8) and (2.10). However, a moment's inspection shows that this procedure unavoidably re-introduces the unwanted negative power of r into the calculations. The remedy is quite simple. Rather than calculate E and $\underline{B}$ from their definitions, calculate them by integrating their differential equations (2.11) and (2.13) simultaneously with (2.18). The equations for E' and $\underline{B}'$ are obviously regular for any choices of C and n. The regular s-domain solution so obtained will finally have to be related to the time domain by integrating the original time transformation in the form

$$t' = C\,r^n \tag{2.19}$$

This equation also is obviously regular if C and n are chosen so that equation (2.18) is regular. Hence, the motion can be described in a perfectly nonsingular way and the price to be paid for this convenience is an increase in the order of the system equations from six to eleven. Before this regular system is summarized, an important additional differential equation will be derived.

Observe that computation of the radius r will be required for use in equations (2.18) and (2.19). In a numerical treatment of the equations, extracting the square root of

$$r^2 = \underline{r} \cdot \underline{r} \tag{2.20}$$

is not burdensome. However, in an analytical approach, in which handling of literal algebraic forms is contemplated, this operation can be clumsy and may lead to apparently intractable expressions. In the present case it happens to be easier to obtain r from its own differential equation. This situation will become clearer in the remainder of this chapter. Furthermore, in the next chapter it will be essential to have a separate differential equation for the radius.

Differentiate the identity (2.20) twice to obtain

$$r'^2 + rr'' = \underline{r}' \cdot \underline{r}' + \underline{r} \cdot \underline{r}'' \tag{2.21}$$

Substitute for $\underline{r}''$ from the equation of motion (2.18) to obtain

$$\begin{aligned} r'^2 + rr'' = {} & \underline{r}' \cdot \underline{r}' - nC^2 r^{2n-2} \underline{r} \cdot \underline{B} + C^2 r^{2n} \underline{P} \cdot \underline{r} \\ & + C^{-1} C' r r' + 2EC^2 n r^{2n} + (n-1) C^2 \mu r^{2n-1} \end{aligned} \tag{2.22}$$

Now observe that when r'' is isolated in this equation a negative power of r multiplying both r'^2 and $\underline{r}' \cdot \underline{r}'$ will occur which cannot be removed by any choice of n or C. The only recourse is to eliminate $\underline{B}$ or E (or both) algebraically in an attempt to cancel the offending terms. From equation (2.8) it can be seen that E contains $\underline{r}' \cdot \underline{r}'$ but not r'^2, so the better choice is to eliminate $\underline{B}$. Substitute for $\underline{B}$ from equation (2.15) where it can be seen that $\underline{r} \cdot \underline{B}$ will indeed contain terms in both r'^2 and $\underline{r}' \cdot \underline{r}'$. Then division by r reveals that this procedure is at least partly successful:

$$r'' - 2EC^2 n r^{2n-1} = (2n-1) C^2 \mu r^{2n-2} + (n-1)(r'^2 - \underline{r}' \cdot \underline{r}') r^{-1}$$

$$+ C^2 r^{2n-1}(\underset{\sim}{P}\cdot \underset{\sim}{r}) + C^{-1}C'r' \tag{2.23}$$

Equation (2.23) is regular in the special case $n = 1$, but not otherwise. A little thought shows that this defect can be overcome by introducing the remaining integral of unperturbed motion, namely the angular momentum, as another dependent variable. From equation (2.9), form the scalar product

$$\underset{\sim}{h}\cdot \underset{\sim}{h} = C^{-2} r^{-2n}(\underset{\sim}{r}\times \underset{\sim}{r}')\cdot(\underset{\sim}{r}\times \underset{\sim}{r}') \tag{2.24}$$

Using first a scalar triple-product identity and then a vector triple-product identity, it is easy to derive

$$h^2 = C^{-2} r^{2-2n}(\underset{\sim}{r}'\cdot \underset{\sim}{r}' - r'^2) \tag{2.25}$$

or

$$\underset{\sim}{r}'\cdot \underset{\sim}{r}' - r'^2 = C^2 h^2 r^{2n-2} \tag{2.26}$$

Now the troublesome term in equation (2.23) can be rewritten in a more suitable form, so that that equation becomes

$$r'' - 2EC^2 nr^{2n-1} = (2n-1)C^2 \mu r^{2n-2} + (1-n)C^2 h^2 r^{2n-3} + C^2 r^{2n-1}(\underset{\sim}{P}\cdot \underset{\sim}{r}) + C^{-1}C'r' \tag{2.27}$$

As was the case with E and $\underset{\sim}{B}$ earlier, any attempt to calculate h^2 algebraically in terms of $\underset{\sim}{r}$ and $\underset{\sim}{r}'$, say by (2.24) or (2.25), merely reintroduces the unwanted negative power of r into the system. However, a regular differential equation for h^2 is available.

$$(h^2)' = 2\underset{\sim}{h}\cdot \underset{\sim}{h}' \tag{2.28}$$

From equations (2.9) and (2.12), there results

$$(h^2)' = 2C^{-1} r^{-n}(\underset{\sim}{r}\times \underset{\sim}{r}')\cdot Cr^n(\underset{\sim}{r}\times \underset{\sim}{P}) \tag{2.29}$$

which becomes

$$(h^2)' = 2(\underline{r}\cdot\underline{r})(\underline{P}\cdot\underline{r}') - 2(\underline{r}\cdot\underline{r}')(\underline{P}\cdot\underline{r}) \tag{2.30}$$

after the vector products are carried out. Some algebraic manipulations in the next chapter are shortened if it is observed here that, incidentally,

$$(h^2)' = \underline{r}\cdot\underline{B}' \tag{2.31}$$

Equations (2.27) and (2.30) may be new; the equation of the radius in the s domain is not usually considered except for the special case $n = 1$, in which case the equation happens to be regular without the introduction of angular momentum. Furthermore, in that case, the equation becomes linear if $\underline{P} = \underline{0}$, and this in turn greatly simplifies the integration of equation (2.19) for the time. Here it is noted that other values of n in equation (2.27) also produce regular equations, although these other equations are all nonlinear in the unperturbed case.

The time-transformed differential equations may now be summarized.

$$\underline{r}'' - 2EC^2nr^{2n-2}\underline{r} - (n-1)C^2\mu r^{2n-3}\underline{r} = -nC^2r^{2n-2}\underline{B} + C^2r^{2n}\underline{P} + C^{-1}C'\underline{r}' \tag{2.32}$$

$$\underline{B}' = 2\underline{r}\,(\underline{P}\cdot\underline{r}') - \underline{r}'\,(\underline{P}\cdot\underline{r}) - \underline{P}\,(\underline{r}\cdot\underline{r}') \tag{2.33}$$

$$E' = \underline{P}\cdot\underline{r}' \tag{2.34}$$

$$t' = C\,r^n \tag{2.35}$$

$$r'' - 2EC^2nr^{2n-1} = (2n-1)C^2\mu r^{2n-2} + (1-n)C^2h^2r^{2n-3} + C^2r^{2n-1}(\underline{P}\cdot\underline{r}) + C^{-1}C'r' \tag{2.36}$$

$$(h^2)' = 2(\underline{r}\cdot\underline{r})(\underline{P}\cdot\underline{r}') - 2(\underline{r}\cdot\underline{r}')(\underline{P}\cdot\underline{r}) = \underline{r}\cdot\underline{B}' \tag{2.37}$$

The latter two of these equations are optional, at least numerically, and the rest of the system is of order eleven. Once the factor C is given definite form, it is a simple matter to calculate s-domain initial conditions from the corresponding time-domain data. Since only the velocity needs to be transformed, not the coordinates, solve equation (2.4) for t' and evaluate at the initial instant.

Various special forms of this set of differential equations have been presented by several writers. Practically all of these analyses have dealt the case in which $n = 1$ and $C = 1$ (or $C =$ constant). In this group, for example, Sperling (1961) presented the regularized equations of unperturbed motion, which happen to be solvable in closed form, and cast their solution in a universal form. Burdet (1967, 1968, 1969), who credits Sperling with originating this regularization, gave a more complete derivation of the differential equations and their unperturbed (universal) solutions. He also discussed at length the use of these differential equations in describing perturbed motion numerically. Pitkin (1965) discussed the use of Sundman's time transformation (2.1) in deriving universal solutions of unperturbed motion, even giving a regular differential equation for the radius, but failing to give the transformed differential equations of motion. Mangad (1967) used Pitkin's (1965) results to establish regular equations of unperturbed motion in terms of plane polar coordinates. He also gave the solutions, but not in a universal form. Szebehely (1967, ch. 3) discussed a Sperling-type regularization for the planar unperturbed problem in which position is represented by a complex number instead of an ordinary vector and obeys a linear oscillator equation. He also cited (1967, sec. 3.12) an equivalent, but apparently unpublished, planar formulation by Arenstorf. Blair (1971) reported an independent discovery of the regular differential equations of perturbed motion and used them to analyze a class of trajectories in Earth-Moon space. Heggie (1973) gave a succinct derivation of the same equations and discussed their use in numerical treatments of the N-body problem. Silver (1975) presented yet another derivation of the differential equations, by now called the Sperling-Burdet equations, but treated no applications.

Discussions of the equations for a general value of n are harder to find. Both Pitkin (1965) and Mangad (1967) began with a general value of n in the time transformation. Pitkin did not consider the transformed governing equations, but Mangad presented the equations of unperturbed motion in plane polar coordinates and gave their solutions for $n = 1$ and $n = 2$. These solutions are not in simplest form nor are they universal. In fact, the use of a particular coordinate system in Mangad's analysis prevents one from immediately reaching an important conclusion about planar Keplerian motion finally enunciated some years later by Szbehely and Bond (1983): only $n = 1$ and $n = 2$ will lead to linear harmonic oscillator equations. Szebehely (1967, ch. 3) included a general constant value of n in the regularization of the planar unperturbed problem in the complex plane, and later (1976b) formulated the regular equations of unperturbed and perturbed motion using $C = 1$ and a general constant value of n. He did not formulate the corresponding equation of the radius and consequently did not bring the angular momentum and its differential equation into the system. His main (1976b) conclusion is that if the integrals of motion E and $\underline{B}$ are introduced along with the given time transformation to effect regularization, then the equations of unperturbed motion are regular for $n = 1$ and $n = \frac{3}{2}$ but are linear with constant coefficients in case, and only in case, $n = 1$. A trivial extension of Szebehely's discussion shows that $n \geq \frac{3}{2}$ results in regular but nonlinear equations. The importance of this result for the present study is that if other linear representations of the unperturbed motion are to be developed, then other types of transformations will have to be considered. The possibilities for linearization using only the time transformation $dt = r^n ds$ are already exhausted by the choice $n = 1$. As is well known, the choice $n = 1$ (Sundman's original transformation) in the unperturbed problem corresponds to the introduction of eccentric anomaly as the independent variable. At least since Burdet's (1969) analysis, it has been known that if a time transformation corresponding to the introduction of true anomaly ($n = 2$) is used then linear equations of unperturbed motion again appear. However, in this case a transformation of coordinates is required besides the time transformation, a situation which is examined in the next chapter. Also requiring a coordinate transformation, Bond (1985) used a time transformation of the form $dt = g(r)\, ds$ and transformed the equations of unperturbed motion expressed in spherical coordinates into a linear oscillator form. His main result is that the oscillator form is possible only if $g(r) = r^2$, that is, if the independent variable is the true anomaly.

Now by inspection of equations (2.32) through (2.37) the possibilities for regularization and linearization of this set of transformed equations can be summarized in detail. These are just

Szebehely's (1976b) results, now extended to include the equation of the radius. The presence of a possibly variable factor C does not materially alter the following conclusions about n.

1. For $n < 1$, negative powers of r occur unavoidably so the system cannot be regular. Needless to add, the unperturbed system is not linear, either.

2. For $n = 1$, negative powers of r do not appear explicitly so the system is regular. Furthermore, since in the unperturbed case the integrations for E , h^2 and $\underline{B}$ are trivial, resulting in constants, the remaining equations become not only regular but linear with constant coefficients.

3. For $1 < n < \frac{3}{2}$, negative powers of r once again occur unavoidably so the system is not regular. The unperturbed system also is neither regular nor linear.

4. For $n \geq \frac{3}{2}$, negative powers of r never occur explicitly so the system is always regular. The unperturbed system also is regular but nonlinear.

The reasoning behind these statements assumes that the expression for $\underline{P}$ itself contains no negative powers of r. These negative powers would, of course, be unavoidable in the differential equations (2.32) through (2.37). For some types of perturbation, such as attraction by a remote third body, this assumption is true. On the other hand, most expressions of geophysical perturbations do contain negative powers of r, even indefinitely large negative powers. The general regularization of the differential equations containing perturbing terms of the latter type is a difficult problem which has been investigated by several authors (Belen'kii, 1981; Szebehely and Bond, 1983; Cid, *et al.*, 1983), but with only limited success to date. It is worth noting that in the position equation (2.32) and the equation of the radius (2.36) all finite negative powers of r occurring in $\underline{P}$ can be cancelled merely by choosing n large enough. However, this cancellation is not available in the differential equations for E, $\underline{B}$ and h^2. Additionally, it is worth noting that, as will be proved shortly, the origin cannot be reached in a finite interval of the independent variable s if $n \geq \frac{3}{2}$. Hence, in this case negative powers of r would cause no trouble in the transformed domain, but only because the governing equations inherently do not describe motion near the origin. Treatment of nonregular terms arising solely from $\underline{P}$ is, in any case, beyond the scope of this study.

Another assumption behind the above enumerated statements is that no singularity is introduced by the presence of C. In several important cases discussed in this and the next chapter, C will be a constant of the unperturbed motion. Even though no difficulty at $r = 0$ occurs, still certain geometrical singularities may arise from the combination of elements comprising C. This situation will become clearer below.

Before the discussion turns elsewhere, a note should be inserted about the equation of the radius (2.36). In the unperturbed case E and h^2 are constants so that the differential equation can, in principle, be solved by quadratures for a general value of n. Here it is assumed, in light of later discussion, that C is either strictly constant or else a constant of unperturbed motion. Then if $\underline{P} = \underline{0}$ equation (2.36) reduces to

$$r'' - 2EC^2 n r^{2n-1} = (2n-1)C^2 \mu r^{2n-2} + (1-n)C^2 h^2 r^{2n-3} \tag{2.38}$$

Multiplying both sides by $r'ds$ and integrating produces

$$\frac{1}{2}r'^2 = EC^2 r^{2n} + C^2 \mu r^{2n-1} - \frac{1}{2}C^2 h^2 r^{2n-2} + \frac{1}{2}R \tag{2.39}$$

where the constant of integration R is evaluated as

$$R = r'(0)^2 - 2EC^2 r(0)^{2n} - 2C^2 \mu r(0)^{2n-1} + C^2 h^2 r(0)^{2n-2} \tag{2.40}$$

Then, remembering that $dt = Cr^n ds$, it is a straightforward job to obtain the following separation of variables:

$$dt = \frac{Cr^n dr}{\sqrt{2EC^2 r^{2n} + 2C^2 \mu r^{2n-1} - C^2 h^2 r^{2n-2} + R}} \tag{2.41}$$

It is feasible to integrate this expression in terms of elementary functions for $n = 1$ and in terms of elliptic integrals for $n = \frac{3}{2}$ and $n = 2$ (see, for example, section 8.1 of Gradshteyn and Ryzhik, 1980). The resulting constant of integration is a constant of unperturbed motion representing the origin of time, i.e., a time element. It is interesting in this connection that Nacozy (1975, 1976, 1981) and Kwok and Nacozy (1981) have introduced time elements for just these three values of n, though not by the use of this equation. Nacozy's work was directed toward establishing variation-of-parameters formulae for the differential equation of time (2.35). Zare (1983) was able to define a time element for elliptic orbits and a general value of n, though again not by the use of equation (2.41). His variation-of-parameters formulae reduce to especially simple forms for the three values $n = 1$, $n = \frac{3}{2}$ and $n = 2$.

The quadrature (2.41) can also be used to illustrate an important property of the time transformation (2.2). Consider the case of an unperturbed ejection orbit, that is, a rectilinear orbit originating at the center of attraction, and suppose that the value of the energy is specified. In the time domain, such a problem is ill-posed because, according to equation (1.4), the initial velocity must have infinite magnitude. In the s domain the situation is different. After rewriting the transformed energy equation (2.8) as

$$\underline{r}' \cdot \underline{r}' = 2C^2 r^{2n} E + 2C^2 \mu r^{2n-1} \tag{2.42}$$

it is easy to see that the s-domain "velocity" magnitude is not infinite if r and E are specified, at least provided n has one of its regularizing values. In particular, for any value of E,

$$\underline{r}'(0) \cdot \underline{r}'(0) = 0 \tag{2.43}$$

if $r(0) = 0$. Now since the orbit is rectilinear, $h^2 = 0$ as shown by equation (2.24). Furthermore, equation (2.26) confirms the geometrical intuition that

$$\underline{r}' \cdot \underline{r}' = r'^2 \tag{2.44}$$

so that initially

$$r'(0)^2 = 0 \tag{2.45}$$

if $r(0) = 0$. These relations imply that the constant of integration R evaluated according to equation (2.40) must always be zero if $r(0) = 0$. Then the quadrature (2.41) reduces to

$$dt = \frac{Cr^n dr}{\sqrt{2EC^2 r^{2n} + 2C^2 \mu r^{2n-1}}} \tag{2.46}$$

$$ds = \frac{r^{\frac{1}{2}-n} dr}{C\sqrt{2Er + \mu}} \tag{2.47}$$

The integration of this expression for a general real value of n is still excessively complicated for the present illustration, and it suffices to consider the special case $E = 0$:

$$ds = \frac{r^{\frac{1}{2}-n} dr}{C\sqrt{2\mu}} \tag{2.48}$$

In the limit as $r(0) \rightarrow 0$, the integrated expression

$$s = \frac{r^{\frac{3}{2}-n} - r(0)^{\frac{3}{2}-n}}{C\sqrt{2\mu}\left[\frac{3}{2} - n\right]} \tag{2.49}$$

is finite only if $n < \frac{3}{2}$. In converse terms, a satellite approaching the primary body along a zero-energy rectilinear path cannot reach the attracting center in a finite value of s if $n \geq \frac{3}{2}$. This fact has already been noted by Stiefel and Scheifele (1971, section 17), and it has important consequences for the present study. It means that the regularization effected by $n = 1$ and the regularization effected by $n \geq \frac{3}{2}$ are fundamentally different in nature. In fact, the term "regularization" in regard to values $n \geq \frac{3}{2}$ ought to be used only in a restricted sense. Although the governing differential equations show no singularity at $r = 0$, one cannot follow the motion through an actual collision. Even if merely a close approach occurs, very large values of the independent variable must be pursued near the pericenter.

Since only the zero-energy case has been exhibited here, it may well be asked whether these same conclusions hold for every value of the energy. In Appendix A the integration of equation (2.47) is undertaken for non-zero values of E and for positive integral and half-integral values of n. There it is verified that for these discrete values of n the above conclusions are true regardless of the value of E.

Solutions of the Linear Regular Equations

From a numerical point of view, the regular forms of the transformed governing system (2.32) through (2.37) offer important and well known advantages over the original time-domain equation of motion (1.1). Firstly, near-collision orbits of any energy E can be computed without danger of exceeding the arithmetic range of the computing device. In case $n = 1$ in the time transformation, even that theoretical event of an actual collision of point masses can be handled without modifying the computation in any way. Secondly, the form of the time transformation (2.2) produces an "analytical stepsize control" during numerical integration which tends to improve the accuracy of the result. The basic reason is that constant ds-steps correspond to smaller dt -steps near pericenter, where the coordinates are more rapidly varying, and larger dt-steps near apocenter, where the coordinates are less rapidly varying. The special choice $n = 1$ in the time transformation is interesting numerically for several reasons of its own. The unperturbed equations are linear oscillator equations and the s -domain solution is simple-harmonic. If $|\underline{P}|$ is relatively small, as is often the case in practice, the oscillator system is only weakly nonlinear and the motion is easily followed by any reasonable numerical method. In fact, Stiefel and Scheifele (1971, chapter 7) present several numerical methods which take advantage of the properties of linear oscillators to integrate the unperturbed equations without truncation error. Their methods were presented in terms of KS coordinates but the whole development could as readily be made in terms of the Cartesian coordinates. Furthermore, an s-domain variation-of-parameters method can be developed in a few straightforward steps. The elements so introduced are regular in the same sense that the coordinates are regular, but with the additional advantage that these elements will all be slowly varying when $|\underline{P}|$ is small. Some choices of $n \geq \frac{3}{2}$ have also found application, particularly when $|\underline{P}|$ cannot be considered small so that the introduction of elements is not so useful. In these cases, the system nonlinearities are more pronounced but the analytical stepsize modulation is also more pronounced. The net result for a variety of Earth-gravity-perturbed satellite problems is a gain in numerical accuracy (see, for example, Velez, 1974; and Alfriend and Velez, 1975). Recall that, whereas a complete regular variation-of-parameters formulation is available from the time-transformed equations only for $n = 1$, elements for the time equation are available for $n = 1$, $n = \frac{3}{2}$ and $n = 2$ (Nacozy, 1975, 1976, 1981; Kwok and Nacozy, 1981), or even for a general value of n on elliptic orbits (Zare, 1983). The use of these time elements in following the perturbed motion offers significant gains in numerical accuracy, though particularly when $|\underline{P}|$ is small.

Now if analytical results are sought, only the choice $n = 1$ need be considered. The time-transformed equations (2.32) through (2.36) reduce to

$$\underline{r}'' - 2EC^2\underline{r} = -C^2\underline{B} + C^2r^2\underline{P} + C^{-1}C'\underline{r}' \tag{2.50}$$

$$\underline{B}' = 2\underline{r}(\underline{P}\cdot\underline{r}') - \underline{r}'(\underline{P}\cdot\underline{r}) - \underline{P}(\underline{r}\cdot\underline{r}') \tag{2.51}$$

$$E' = \underline{P}\cdot\underline{r}' \tag{2.52}$$

$$t' = C\,r \tag{2.53}$$

$$r'' - 2EC^2r = C^2\mu + C^2r(\underline{P}\cdot\underline{r}) + C^{-1}C'r' \tag{2.54}$$

Notice that the differential equation for h^2 is not needed since that quantity appears nowhere else in the system when $n = 1$. If the further choice $C = 1$ is made, the formulation known to Sperling, Burdet and many others results:

$$\underline{r}'' - 2E\underline{r} = -\underline{B} + r^2\underline{P} \tag{2.55}$$

$$\underline{B}' = 2\underline{r}\,(\underline{P}\cdot\underline{r}') - \underline{r}'\,(\underline{P}\cdot\underline{r}) - \underline{P}\,(\underline{r}\cdot\underline{r}') \tag{2.56}$$

$$E' = \underline{P}\cdot\underline{r}' \tag{2.57}$$

$$t' = r \tag{2.58}$$

$$r'' - 2Er = \mu + r(\underline{P}\cdot\underline{r}) \tag{2.59}$$

It can be seen that choosing C equal to some constant other than unity will not alter the basic characteristics of the system but only the time scale of the motion. In the case of unperturbed motion, the integrations for $\underline{B}$ and E are trivial, and there results the simple oscillator system in which $\underline{B}$ and E are constants:

$$\underline{r}'' - 2E\underline{r} = -\underline{B} \tag{2.60}$$

$$t' = r \tag{2.61}$$

$$r'' - 2Er = \mu \tag{2.62}$$

Since no assumptions have been made so far about the type of orbit to be considered, this system describes all types of orbits. It is easy to see that the character of the motion is determined solely by the value of the energy E. Bounded oscillations, corresponding to elliptic orbits, occur for $E < 0$, while unbounded exponential-type motion, corresponding to hyperbolic orbits, occurs for $E > 0$. The motion in the limiting case $E = 0$ (parabolic orbits) is also unbounded but not exponential-type.

The explicit solutions of these linear equations of Keplerian motion are readily obtained. The complete solution of (2.60) for, say, the elliptic case is

$$\underline{r}(s) = \underline{r}(0)\cos(\sqrt{-2E}\,s) + \underline{r}'(0)\frac{\sin(\sqrt{-2E}\,s)}{\sqrt{-2E}} - \underline{B}\frac{[1-\cos(\sqrt{-2E}\,s)]}{-2E} \tag{2.63}$$

$$\underline{r}'(s) = -\underline{r}(0)\sqrt{-2E}\,\sin(\sqrt{-2E}\,s) + \underline{r}'(0)\cos(\sqrt{-2E}\,s) - \underline{B}\frac{\sin(\sqrt{-2E}\,s)}{\sqrt{-2E}} \tag{2.64}$$

The equation of the radius has the solution

$$r(s) = \frac{\mu}{-2E} + \left[r(0) - \frac{\mu}{-2E}\right]\cos(\sqrt{-2E}\,s) + r'(0)\frac{\sin(\sqrt{-2E}\,s)}{\sqrt{-2E}} \tag{2.65}$$

$$r'(s) = -\left[r(0) - \frac{\mu}{-2E}\right]\sqrt{-2E}\,\sin(\sqrt{-2E}\,s) + r'(0)\cos(\sqrt{-2E}\,s) \tag{2.66}$$

The s-domain solution of Keplerian motion is related to the time domain by means of equation (2.61). Inserting $r(s)$ from (2.65) and integrating, there results

$$t(s) - t(0) = \frac{\mu}{-2E}s + \left[r(0) - \frac{\mu}{-2E}\right]\frac{\sin(\sqrt{-2E}\,s)}{\sqrt{-2E}} + r'(0)\frac{1-\cos(\sqrt{-2E}\,s)}{-2E} \tag{2.67}$$

Notice that this last integration would have been rather cumbersome if one had been forced to integrate the square root of $\underline{r}(s)\cdot\underline{r}(s)$; hence, the advantage of solving a separate equation for the radius becomes apparent. Comparison of these formulae with more traditional two-body formulae shows that the quantity $(\sqrt{-2E}\,s)$ is just the change in eccentric anomaly elapsed since the initial epoch, where this epoch may be located at any point along the orbit. Equation (2.67) is essentially Kepler's equation, where time is reckoned from the arbitrary epoch.

All of these formulae are real-valued only for $E < 0$; that is, for elliptic orbits. The solution procedure could be repeated supposing $E > 0$ in order to obtain real-valued expressions for hyperbolic orbits. Equivalently, the following complex identities allow the elliptic formulae to be converted directly to their hyperbolic analogs.

$$\sin(\sqrt{-2E}\,s) = \sqrt{-1}\,\sinh(\sqrt{2E}\,s) \tag{2.68}$$

$$\cos(\sqrt{-2E}\,s) = \cosh(\sqrt{2E}\,s) \tag{2.69}$$

Then the solutions for hyperbolic orbits can be written down immediately as

$$\underline{z}(s) = \underline{z}(0)\cosh(\sqrt{2E}\,s) + \underline{z}'(0)\frac{\sinh(\sqrt{2E}\,s)}{\sqrt{2E}} + \underline{B}\frac{[1-\cosh(\sqrt{2E}\,s)]}{2E} \tag{2.70}$$

$$\underline{z}'(s) = -\underline{z}(0)\sqrt{2E}\sinh(\sqrt{2E}\,s) + \underline{z}'(0)\cosh(\sqrt{2E}\,s) - \underline{B}\frac{\sinh(\sqrt{2E}\,s)}{\sqrt{2E}} \tag{2.71}$$

$$r(s) = -\frac{\mu}{2E} + \left[r(0) + \frac{\mu}{2E}\right]\cosh(\sqrt{2E}\,s) + r'(0)\frac{\sinh(\sqrt{2E}\,s)}{\sqrt{2E}} \tag{2.72}$$

$$r'(s) = +\left[r(0) + \frac{\mu}{2E}\right]\sqrt{2E}\sinh(\sqrt{2E}\,s) + r'(0)\cosh(\sqrt{2E}\,s) \tag{2.73}$$

$$t(s) - t(0) = -\frac{\mu}{2E}s + \left[r(0) + \frac{\mu}{2E}\right]\frac{\sinh(\sqrt{2E}\,s)}{\sqrt{2E}} + r'(0)\frac{1-\cosh(\sqrt{2E}\,s)}{2E} \tag{2.74}$$

Here the quantity $(\sqrt{2E}\,s)$ is just the change in hyperbolic anomaly elapsed since the (arbitrary) initial epoch.

The presence of E in various denominators in both the elliptic and hyperbolic forms of the Keplerian solutions prevents all these formulae from being valid in the limiting case $E = 0$ (parabolic orbits). However, if $E = 0$ then the differential equations (2.60) through (2.62) have the exceedingly simple form

$$\underline{z}'' = -\underline{B} \tag{2.75}$$

$$t' = r \tag{2.76}$$

$$r'' = \mu \tag{2.77}$$

The complete solution for the parabolic case is

$$\underline{z}(s) = \underline{z}(0) + \underline{z}'(0)s - \frac{1}{2}\underline{B}s^2 \tag{2.78}$$

$$\underline{r}'(s) = \underline{r}'(0) - \underline{B}s \tag{2.79}$$

$$r(s) = r(0) + r'(0)s + \frac{1}{2}\mu s^2 \tag{2.80}$$

$$r'(s) = r'(0) + \mu s \tag{2.81}$$

$$t(s) - t(0) = r(0)s + \frac{1}{2}r'(0)s^2 + \frac{1}{6}\mu s^3 \tag{2.82}$$

Whichever form of solution is required, the s-domain initial conditions are found from the velocity relation (2.4):

$$\underline{r}'(0) = r(0)\dot{\underline{r}}(0) \tag{2.83}$$

$$r'(0) = r(0)\dot{r}(0) = \underline{r}(0) \cdot \dot{\underline{r}}(0) \tag{2.84}$$

The parameters E and $\underline{B}$ follow from their definitions. Then the time-domain velocity vector corresponding to some value of s is obtained from (2.4):

$$\dot{\underline{r}}(s) = \frac{1}{r(s)}\underline{r}'(s) \tag{2.85}$$

Thus, the position vector, the velocity vector and the corresponding time are represented parametrically in terms of s and any initial conditions given at an arbitrary epoch. In the usual initial-value problem, the time and initial conditions are given in order to compute the corresponding position and velocity. For this purpose the time equation for the appropriate type of orbit must be inverted to find the value of s which corresponds to the given time. Then position and velocity follow immediately. In other words, the unperturbed Kepler problem is reduced to the inversion of at most a single transcendental equation. If the motion is parabolic then a cubic equation must be solved.

The Gauss/Lambert boundary value problem is fundamentally more complicated. Actually, the above collection of formulae is not well suited to either the formulation or the solution of this problem. In the first place, the unknown boundary condition $\dot{\underline{r}}(0)$ occurs very inconveniently in many places throughout the formulae, such as in $\underline{B}$ and E. In the second place, the type of orbit is not always known in advance in the boundary-value problem but the use of the above formulae requires *a priori* knowledge of the type of orbit. In particular, the iterative process of determining the unknown boundary condition must be able to proceed smoothly regardless of the sign of E that results from the current guess for $\dot{\underline{r}}(0)$, yet continuous transition among the three groups of formulae in the above collection is not possible. The difficulty occurs for orbits that are nearly, but not exactly, parabolic. When E is small in magnitude, small denominators and ratios of small numbers inevitably occur which make the computations unreliable. The same situation occurs in the initial-value problem for near-parabolic orbits. Fortunately, this difficulty can be overcome by simple manipulations.

The following power series expansions will permit a single set of real-valued expressions to be used for computing Keplerian motion along any type of orbit.

$$\sin(\sqrt{-2E}\,s) = (\sqrt{-2E}\,s) - \frac{(\sqrt{-2E}\,s)^3}{3!} + \frac{(\sqrt{-2E}\,s)^5}{5!} - \cdots \tag{2.86}$$

$$\sin(\sqrt{-2E}\,s) = \sqrt{-2E}\left[s + \frac{(2E)s^3}{3!} + \frac{(2E)^2 s^5}{5!} + \frac{(2E)^3 s^7}{7!} + \cdots\right] \tag{2.87}$$

$$\sinh(\sqrt{2E}\,s) = (\sqrt{2E}\,s) + \frac{(\sqrt{2E}\,s)^3}{3!} + \frac{(\sqrt{2E}\,s)^5}{5!} + \cdots \tag{2.88}$$

$$\sinh(\sqrt{2E}\,s) = \sqrt{2E}\left[s + \frac{(2E)s^3}{3!} + \frac{(2E)^2 s^5}{5!} + \frac{(2E)^3 s^7}{7!} + \cdots\right] \tag{2.89}$$

$$\cos(\sqrt{-2E}\,s) = 1 - \frac{(\sqrt{-2E}\,s)^2}{2!} + \frac{(\sqrt{-2E}\,s)^4}{4!} - \cdots \tag{2.90}$$

$$\cos(\sqrt{-2E}\,s) = 1 + \frac{(2E)s^2}{2!} + \frac{(2E)^2 s^4}{4!} + \frac{(2E)^3 s^6}{6!} + \cdots \tag{2.91}$$

$$\cosh(\sqrt{2E}\,s) = 1 + \frac{(\sqrt{2E}\,s)^2}{2!} + \frac{(\sqrt{2E}\,s)^4}{4!} + \cdots \tag{2.92}$$

$$\cosh(\sqrt{2E}\,s) = 1 + \frac{(2E)s^2}{2!} + \frac{(2E)^2 s^4}{4!} + \frac{(2E)^3 s^6}{6!} + \cdots \tag{2.93}$$

Observe that the series for $\cos(\sqrt{-2E}\,s)$ and $\cosh(\sqrt{2E}\,s)$ are identical, while the series for $\sin(\sqrt{-2E}\,s)$ and $\sinh(\sqrt{2E}\,s)$ differ only by an overall factor of $\sqrt{-1}$. Of course, these properties are evident from the identities (2.68) and (2.69) already employed for these functions. Now, however, the above series themselves can be substituted into the two collections of formulae for Keplerian motion (2.63) through (2.67) and (2.70) through (2.74). Obvious algebraic steps then bring the elliptic and hyperbolic formulae into the following common form.

$$\underline{r}(s) = \underline{r}(0)[1 - (-2E)C(s)] + \underline{r}'(0)[s - (-2E)S(s)] - \underline{B}C(s) \tag{2.94}$$

$$\underline{r}'(s) = -\underline{r}(0)(-2E)[s - (-2E)S(s)] + \underline{r}'(0)[1 - (-2E)C(s)] - \underline{B}[s - (-2E)S(s)] \tag{2.95}$$

$$t(s) - t(0) = \mu S(s) + r(0)[s - (-2E)S(s)] + r'(0)C(s) \tag{2.96}$$

where the functions $S(s)$ and $C(s)$ are to be evaluated by means of

$$S(s) = \frac{s^3}{3!} - \frac{(-2E)s^5}{5!} + \frac{(-2E)^2 s^7}{7!} - \frac{(-2E)^3 s^9}{9!} + \cdots \tag{2.97}$$

$$C(s) = \frac{s^2}{2!} - \frac{(-2E)s^4}{4!} + \frac{(-2E)^2 s^6}{6!} - \frac{(-2E)^3 s^8}{8!} + \cdots \tag{2.98}$$

Note that $S(s)$ and $C(s)$ are actually functions of the two parameters E and s. For some computational purposes, it is convenient to express $S(s)$ and $C(s)$ further in terms of functions of a single combined parameter z :

$$S(s) = s^3 S^*(z) \tag{2.99}$$

$$C(s) = s^2 C^*(z) \tag{2.100}$$

where

$$S^*(z) = \frac{1}{3!} - \frac{z}{5!} + \frac{z^2}{7!} - \frac{z^3}{9!} + \cdots \tag{2.101}$$

$$C^*(z) = \frac{1}{2!} - \frac{z}{4!} + \frac{z^2}{6!} - \frac{z^3}{8!} + \cdots \tag{2.102}$$

$$z = -2Es^2 \tag{2.103}$$

Whatever the final arrangement of terms in the $S(s)$ and $C(s)$ series, it is evident that the series will be uniformly convergent for all real values of E and s. In the combined Keplerian formulae (2.94) through (2.96) the parameter E occurs neither in denominators nor under radicals so these expressions can be used regardless of the value of E. A further check on the validity of these expressions is that they reduce to the correct parbolic forms when $E = 0$. In using these formulae to compute Keplerian motion, initial conditions are still found from (2.83) and (2.84) and the velocity is still computed from (2.85) for all orbits. The initial-value problem is still solved by inverting a single transcendental equation which reduces to a cubic when $E = 0$. In addition, since foreknowledge of the type of orbit is not required in order to use these universal formulae, the Gauss/Lambert boundary-value problem can be formulated and solved successfully using the above expressions or equivalent ones. It is still true, however, that the unknown boundary condition $\dot{z}(0)$ occurs inconveniently throughout the formulae. Considerable insight is required to formulate a successful iterative solution procedure. Since a discussion of the boundary-value problem in the eccentric-anomaly domain is beyond the scope of this study, reference is made to Battin (1964), Pitkin (1968) and Bate, *et al.*, (1971), where the details are discussed at length.

What is probably the first universal Keplerian solution actually used the Sundman variable s (albeit implicitly) and was given by Stumpff (1947). His purpose was to facilitate the computation of ephemerides for high-eccentricity orbits such as cometary orbits. No mention of Sundman or of regularization is found in this (1947) paper, but, as in the above equations, power series are used to evaluate all the transcendental functions which appear and the working formulae turn out to be universal. Stumpff's developments are also found in his textbook (1959) as well as in an English-language exposition (1962). In the latter work Stumpff does make use of some linear regular differential equations in deriving his universal formulae, though he does not pursue regularization *per se*. The special transcendental functions which appear in series form throughout Stumpff's work (and as particular arrangements of terms in the series given above) are nowadays usually referred to as "Stumpff functions". Being derived from trigonometric functions, the Stumpff functions satisfy many analogous relations and can be manipulated in similar fashion (see Stumpff, 1947, 1959, 1962; and Stiefel and Scheifele, 1971, sec. 11).

Following the appearance of Stumpff's results, various authors presented universal Keplerian formulae. These all differ from one another in algebraic details, but all are some rearrangement of the s-domain solution of the linear regular governing equations, whether or not these governing differential equations are acknowledged explicitly. Sperling (1961) actually derived his universal formulae from the s -domain equations of unperturbed motion. With changes of notation, the formulae (2.94) through (2.96) match Sperling's formulae. Most presentations of universal formulae for Keplerian motion have been based on *ad hoc* manipulations of classical formulae, following Stumpff's original approach, rather than on a general solution of the underlying differential equations. Herrick (1965), drawing on previous independent work of his own, undertook to compare many algebraic forms of the universal formulae from the point of view of computational efficiency. He also derived a universal variation-of-parameters scheme for computing perturbed motion in terms of the same formulae. Battin (1964), beginning from Herrick's early work, presented a modified set of universal formulae which was well adapted to solving both the initial-value and Gauss/Lambert boundary-value problems. He likewise developed, based on a method of Pines

(1961), a universal variation-of-parameters scheme using the same formulae. About the same time, Goodyear (1965a, 1966) presented all the formulae for the initial-value problem in a form easy to program on a digital computer, as well as (1965b) a universal variation-of-parameters scheme. A special feature of Goodyear's (1965a, 1966) articles is the inclusion of universal formulae for all the elements of the transition matrix which arises in linearized error analysis and sensitivity analysis of unperturbed trajectories. Herrick (1965) also gives formulae for the elements of the transition matrix, but in a less convenient form. Burdet (1967, 1968) followed Sperling's approach and derived universal Keplerian and variation-of-parameters formulae based on the linear transformed equations. A few years later, Bate, *et al.* (1971) in their textbook rederived Battin's (1964) universal formulations of the initial-value and boundary-value problems of Keplerian motion, even introducing Sundman's time transformation explicitly (though not in the context of transforming the governing equations), and gave particular attention to the details of the solution procedures. For the initial-value problem, they offered starting values of the parameter $x = s\sqrt{\mu}$ to be used in a Newton-type iterative solution. Since different starting estimates for s are given for elliptic, parabolic and hyperbolic orbits, the perfect universality of their solution procedure is compromised. One might think that the harm from this limitation would be more aesthetic than practical since for the initial-value problem the type of orbit can always be ascertained in advance. However, some recent work has addressed this problem of efficiently and reliably solving the s -domain universal time equation of the initial-value problem. Prussing (1977, 1979) derived rigorous upper and lower bounds on the solution values of s. In principle, the use of rigorous bounds to start a *regula falsi* iteration will guarantee convergence to the correct solution. Alternatively, the mean of the bounds can serve as a reasonable starting value for a Newton-type iteration. In practice, actual computation of the bounds given by Prussing requires that the three kinds of orbits be distinguished, so his procedure, while elegant, is not perfectly universal. Burkardt and Danby (1983), who do not cite Prussing's work, reviewed many refinements of Newton's method for solving the problem. They observe that even high-order Newton methods could be made to proceed reliably except that to date no generally efficient and perfectly universal starting value for s has been proposed.

A natural question would be whether any rigorous bounds on the solution have yet been proposed for the boundary-value problem. The answer appears to be no. Perhaps this lack is due to the greater inherent complexity of the boundary-value problem and the many possible choices for the variable of iteration.

Now in actually using the universal solution of Keplerian motion, the advantages of its complete generality become obvious. A subtle disadvantage remains, however, which becomes apparent when the unperturbed motion is followed for long times. For the sake of discussion, suppose that $E < 0$, and consider the s-domain elliptic formulae (2.63) through (2.67). These equations describe displaced simple harmonic oscillations of frequency $\sqrt{-2E}$. If the initial conditions are not known exactly, then errors enter the s-domain solution in two ways: directly, through errors in $\underline{r}(0)$, $\underline{r}'(0)$ and $\underline{B}$ multiplying sine and cosine terms; and indirectly, through errors in the value of E modifying the frequency of the motion. The direct errors in position are not serious in the long run because they are purely periodic and do not tend to grow in magnitude as large values of s develop. The indirect errors in position are more serious because they can grow without limit as s increases. For example, if the value of E is slightly in error then the size and shape of the computed orbit may not be seriously in error but, due to the erroneous frequency being used, the computed angular position along this orbit gradually departs in a secular manner from the true position. A similar in-track error growth occurs in the calculation of time elapsed along the orbit. As long as the frequency of the motion depends on imperfectly known quantities, the in-track error in the solution can be mitigated by highly precise measurement and computation but never entirely eliminated. Burdet (1967, 1968) mentions these error trends, although his discussion is about the numerical (Lyapunov) stability of the transformed governing equations. This error growth characteristic of the universal s-domain solution of Keplerian motion has its counterpart in numerical solutions of the perturbed motion. If the Sperling-Burdet equations of motion (2.55) through (2.59) are examined, it is found that in the perturbed oscillator equations for $\underline{r}$ and r the frequency of oscillations, $\sqrt{-2E}$, is variable. Errors in E now accumulate at every integration step instead of merely at its initial evaluation, leading to a more complicated, but still secular, error trend. In the differential equation for time, the secular error growth due to errors in the value of r is analogous and equally troublesome. One might think of using slowly varying elements to describe the perturbed motion in hope of avoiding inaccuracies of this type, but then the situation is just as difficult. When the variation-of-parameters method is pursued, the fact that the energy E is variable in perturbed motion inevitably leads to element differential equations in which s appears linearly in the coefficients of trigonometric (or Stumpff) functions of s. For example,

$$\frac{d}{ds}\left[\sin(\sqrt{-2E}\,s)\right] = \left[\sqrt{-2E} + (\sqrt{-2E}\,)'s\right]\cos(\sqrt{-2E}\,s) \tag{2.104}$$

Terms of this type sometimes occur in first or second order analytical perturbation solutions for the orbital elements, but here these so-called mixed secular terms occur in the element differential equations and lead to numerical problems akin to those that arise with the coordinate differential equations. Even though the net element rates are expected to remain small if $|\underline{P}|$ is small, individual terms in the element equations eventually achieve large amplitude as s increases. Furthermore, the quadratures of the mixed secular terms, required in analytical treatments, can be difficult or impossible to obtain in terms of elementary functions.

These undesirable features of the s-domain formulation of the motion can be partly overcome by an appropriate choice of the factor C in the generalized Sundman time transformation. Recall that the Sperling-Burdet equations (2.55) through (2.59) were derived by putting $C = 1$ in equations (2.50) through (2.54), recovering Sundman's original transformation. If in these latter equations the alternate choice

$$C^2 = \frac{1}{-2E} \tag{2.105}$$

is made, the frequency of the oscillator-type equations for $\underline{r}$ and r is rendered strictly constant even in perturbed motion. (Clearly, no special advantage attaches to having some constant frequency other than unity; merely the time scale of the solution would be altered.) Since C must be a real number, this choice is valid only for $E < 0$, that is, for elliptic orbits. If $E > 0$, so that the orbit is hyperbolic, then C must be chosen as

$$C^2 = \frac{1}{+2E} \tag{2.106}$$

Of course, neither of these choices can be made if $E = 0$, nor can continuous transition between these two choices be made as E changes sign. The differential equations resulting from either choice of C will be valid only for the given type of orbit and will not apply universally. On the other hand, since the frequency of the motion is independent of any measured quantity, being strictly constant, the secular in-track error growth in the transformed domain has been suppressed; only the similar error in the time computation remains. Mainly for the sake of discussion, the elliptic case $E < 0$ is considered in the remainder of this chapter. The hyperbolic case $E > 0$ could be pursued in a completely analogous way.

Now that a particular choice for the factor C has been made, its derivative can be found. Differentiate equation (2.105), remembering that in perturbed motion the rate of change of energy is given by, say, equation (2.34). Simple steps lead to

$$C^{-1}C = \frac{1}{-2E}(\underline{P} \cdot \underline{r}') \tag{2.107}$$

Then the equations of motion (2.50) through (2.54) take the forms

$$\underline{r}'' + \underline{r} = -\frac{1}{-2E}\underline{B} + \frac{1}{-2E}(\underline{r} \cdot \underline{r})\underline{P} + \frac{1}{-2E}(\underline{P} \cdot \underline{r}')\underline{r}' \tag{2.108}$$

$$\underline{B}' = 2\underline{r}(\underline{P} \cdot \underline{r}') - \underline{r}'(\underline{P} \cdot \underline{r}) - \underline{P}(\underline{r} \cdot \underline{r}') \tag{2.109}$$

$$E' = \underline{P} \cdot \underline{r}' \tag{2.110}$$

$$t' = \frac{r}{\sqrt{-2E}} \tag{2.111}$$

$$r'' + r = \frac{\mu}{-2E} + \frac{1}{-2E}(\underline{P}\cdot\underline{r})r + \frac{1}{-2E}(\underline{P}\cdot\underline{r}')r' \tag{2.112}$$

In the unperturbed case, these equations reduce to

$$\underline{r}'' + \underline{r} = -\frac{1}{-2E}\underline{B} \tag{2.113}$$

$$t' = \frac{r}{\sqrt{-2E}} \tag{2.114}$$

$$r'' + r = \frac{\mu}{-2E} \tag{2.115}$$

Denoting the independent variable by θ, the solution for elliptic Keplerian motion can be written as follows:

$$\underline{r}(\theta) = \underline{r}(0)\cos\theta + \underline{r}'(0)\sin\theta - \frac{1}{-2E}\underline{B}(1-\cos\theta) \tag{2.116}$$

$$\underline{r}'(\theta) = -\underline{r}(0)\sin\theta + \underline{r}'(0)\cos\theta - \frac{1}{-2E}\underline{B}\sin\theta \tag{2.117}$$

$$r(\theta) = \frac{\mu}{-2E} + \left[r(0) - \frac{\mu}{-2E}\right]\cos\theta + r'(0)\sin\theta \tag{2.118}$$

$$r'(\theta) = -\left[r(0) - \frac{\mu}{-2E}\right]\sin\theta + r'(0)\cos\theta \tag{2.119}$$

$$\sqrt{-2E}\,[t(\theta) - t(0)] = \frac{\mu}{-2E}\theta + \left[r(0) - \frac{\mu}{-2E}\right]\sin\theta + r'(0)(1-\cos\theta) \tag{2.120}$$

It is readily established that θ is the classical eccentric anomaly to within an additive constant of unperturbed motion, being just the change in eccentric anomaly elapsed since the epoch. As before, the initial conditions are calculated in terms of time-domain data from the velocity relation (2.4).

$$\underline{r}'(0) = \frac{r(0)}{\sqrt{-2E}}\dot{\underline{r}}(0) \tag{2.121}$$

$$r'(0) = \frac{r(0)}{\sqrt{-2E}}\dot{r}(0) = \frac{1}{\sqrt{-2E}}\underline{r}(0) \cdot \dot{\underline{r}}(0) \tag{2.122}$$

Values for the constants E and $\underline{B}$ follow from their definitions evaluated at the initial time. Then the time-domain velocity vector is avaliable for any value of θ from the velocity relation (2.4).

$$\dot{\underline{r}}(\theta) = \frac{\sqrt{-2E}}{r(\theta)}\underline{r}'(\theta) \tag{2.123}$$

The main advantage of the θ -domain formulation is plain from these formulae, namely, that the frequency of the motion is independent of the initial conditions. Therefore θ-domain errors in calculated position and velocity due to errors in the initial conditions are purely periodic and not troublesome. Examination of Kepler's equation (2.120) reveals that errors in the initial conditions still propagate secularly in the calculation of time, or in the calculation of θ if time is given. This time-related in-track error growth cannot be entirely suppressed. It is simply characteristic of all two-body calculations in both the perturbed and unperturbed cases.

Variation of Parameters

It is worthwhile to exhibit the θ-domain variation-of-parameters formulation as part of the present study for several reasons. First, the technique has intrinsic interest because it can be carried through so easily. Several authors, namely, Pines (1961) and Godal, *et al.* (1971), have presented formulae based on the same variable θ and nearly the same elements used here, but they did not begin with linear governing equations and their manipulations become quite involved. Here the steps are all straightforward. Stiefel and Scheifele (1971, sec.19) used θ as the independent variable in variation of parameters formulae, and they did begin with linear governing equations. However, their concurrent use of the KS coordinate transformation obscures some of the basic simplicity exhibited here where no coordinate transformation has been used. The following developments conform most closely to those of Burdet (1967, 1968) who established universal formulae in terms of the Sundman variable s. Second, the θ-domain formulation will be useful for comparison purposes later when analogous operations are carried out in the true-anomaly domain.

Rewrite the unperturbed solution (2.116) through (2.120) in terms of the elements to be used:

$$\underline{r}(\theta) = \underline{r}_0 \cos\theta + \underline{U}_0 \sin\theta + \frac{1}{2E}\underline{B}(1-\cos\theta) \tag{2.124}$$

$$\underline{r}'(\theta) = -\underline{r}_0 \sin\theta + \underline{U}_0 \cos\theta + \frac{1}{2E}\underline{B}\sin\theta \tag{2.125}$$

$$r(\theta) = -\frac{\mu}{2E} + \left[r_0 + \frac{\mu}{2E}\right]\cos\theta + W_0 \sin\theta \tag{2.126}$$

$$r'(\theta) = -\left[r_0 + \frac{\mu}{2E}\right]\sin\theta + W_0 \cos\theta \tag{2.127}$$

$$\begin{aligned} t(\theta) - t_0 = {} & \frac{\mu}{(-2E)\sqrt{-2E}}\theta + \left[\frac{r_0}{\sqrt{-2E}} - \frac{\mu}{(-2E)\sqrt{-2E}}\right]\sin\theta \\ & + \frac{W_0}{\sqrt{-2E}}(1-\cos\theta) \end{aligned} \tag{2.128}$$

By correspondence with the previous formulae (2.116) through (2.120), one sees that in Keplerian motion the elements are $\underline{r}_0 = \underline{r}(0)$, $\underline{U}_0 = \underline{r}'(0)$, $r_0 = r(0)$, $W_0 = r'(0)$, $t_0 = t(0)$, $\underline{B}$ and E. In order to develop a solution of this form for the perturbed system (2.108) through (2.112), assume that the elements $\underline{r}_0$, $\underline{U}_0$, r_0, W_0, $\underline{B}$ and E are functions of θ. Upon differentiating $\underline{r}(\theta)$ in (2.124), it is found that preserving the form of $\underline{r}'(\theta)$ in (2.125) will require that

$$\underline{0} = \underline{r}_0' \cos\theta + \underline{U}_0' \sin\theta + (\underline{B}/2E)'(1-\cos\theta) \tag{2.129}$$

Then differentiating $\underline{r}'(\theta)$ and substituting for $\underline{r}$ and $\underline{r}''$ in the equation of motion (2.108) produces

$$-\underline{r}_0'\sin\theta + \underline{U}_0'\cos\theta + (\underline{B}/2E)'\sin\theta = -\frac{1}{2E}(\underline{r}\cdot\underline{r})\underline{P} - \frac{1}{2E}(\underline{P}\cdot\underline{r}')\underline{r}' \qquad (2.130)$$

These latter two equations can be solved simultaneously for $\underline{r}_0'$ and $\underline{U}_0'$ without any complications.

$$\underline{r}_0' = (\underline{B}/2E)'(1-\cos\theta) + \frac{1}{2E}[(\underline{r}\cdot\underline{r})\underline{P} + (\underline{P}\cdot\underline{r}')\underline{r}']\sin\theta \qquad (2.131)$$

$$\underline{U}_0' = (\underline{B}/2E)'\sin\theta - \frac{1}{2E}[(\underline{r}\cdot\underline{r})\underline{P} + (\underline{P}\cdot\underline{r}')\underline{r}']\cos\theta \qquad (2.132)$$

Wherever $\underline{B}'$ and E' occur, they are to be replaced by (2.109) and (2.110). In particular,

$$(\underline{B}/2E)' = \frac{1}{2E}\underline{B}' - \frac{2}{(2E)^2}\underline{B}E' \qquad (2.133)$$

$$(\underline{B}/2E)' = \frac{1}{2E}[2\underline{r}(\underline{P}\cdot\underline{r}') - \underline{r}'(\underline{P}\cdot\underline{r}) - \underline{P}(\underline{r}\cdot\underline{r}')] - \frac{2}{(2E)^2}\underline{B}(\underline{P}\cdot\underline{r}') \qquad (2.134)$$

Then

$$\underline{r}_0' = \left[\frac{1}{2E}[2\underline{r}(\underline{P}\cdot\underline{r}') - \underline{r}'(\underline{P}\cdot\underline{r}) - \underline{P}(\underline{r}\cdot\underline{r}')] - \frac{2}{(2E)^2}\underline{B}(\underline{P}\cdot\underline{r}')\right](1-\cos\theta)$$

$$+\frac{1}{2E}[(\underline{r}\cdot\underline{r})\underline{P} + (\underline{P}\cdot\underline{r}')\underline{r}']\sin\theta \qquad (2.135)$$

$$\underline{U}_0' = \left[\frac{1}{2E}[2\underline{r}(\underline{P}\cdot\underline{r}') - \underline{r}'(\underline{P}\cdot\underline{r}) - \underline{P}(\underline{r}\cdot\underline{r}')] - \frac{2}{(2E)^2}\underline{B}(\underline{P}\cdot\underline{r}')\right]\sin\theta$$

$$-\frac{1}{2E}[(\underline{r}\cdot\underline{r})\underline{P} + (\underline{P}\cdot\underline{r}')\underline{r}']\cos\theta \qquad (2.136)$$

Of course, it remains to replace $\underline{r}$ and $\underline{r}'$ in terms of elements and θ from (2.124) and (2.125), so that the final expressions will be lengthy. If these equations are to be integrated numerically, the apparent length of the equations is no obstacle. Since several groupings of terms recur, judicious programming would allow the derivatives $\underline{r}_0'$ and $\underline{U}_0'$ to be evaluated efficiently.

The auxiliary elements $\underline{B}$ and E could be evaluated by integrating their differential equations (2.109) and (2.110), where the right-hand sides will have been put in terms of elements and θ. However, one might consider the alternative of evaluating $\underline{B}$ and E algebraically from (2.8) and (2.14), not in terms of the instantaneous values $\underline{r}$ and $\underline{r}'$, but in terms of the epochal values $\underline{r}_0$ and $\underline{U}_0$. This is feasible since $\underline{B}$ and E, being constants of unperturbed motion, can be evaluated anywhere on the osculating orbit; one may as well choose the epoch. If $\underline{r}$ and $\underline{r}'$ are used for this purpose, the undesirable negative powers of r appear, but if $\underline{r}_0$ and $\underline{U}_0$ are used then only negative powers of the epochal radius r_0 appear in the formulae. Changes in epochal position $\underline{r}_0$ occur only in response to perturbations, so that even on a collision orbit it is unlikely that $\underline{r}_0$ would approach the origin if the epoch were chosen at a point on the orbit well away from the origin initially. Thus an integration of order six, plus an integration for the time, would suffice to describe the perturbed elliptic ($E < 0$) motion in a regular manner in all cases except that of ejection at the epoch. If this approach were taken, then $\underline{U}_0$ would be initialized according to the velocity relation (2.4) as

$$\underline{U}_0 = \frac{1}{\sqrt{-2E}} r_0 \underline{V}_0 \tag{2.137}$$

where $\underline{V}_0$ is the time-domain epochal velocity vector and E follows from its definition. Thereafter, at every integration step, E and $\underline{B}$ would be calculated as

$$E = \frac{-\mu}{r_0\left[1 + \frac{1}{r_0^2}\underline{U}_0 \cdot \underline{U}_0\right]} \tag{2.138}$$

$$\underline{B} = \frac{-2E}{r_0^2}[\underline{r}_0(\underline{U}_0 \cdot \underline{U}_0) - \underline{U}_0(\underline{r}_0 \cdot \underline{U}_0)] - \frac{\mu}{r_0}\underline{r}_0 \tag{2.139}$$

These expressions follow from equations (2.8) and (2.14).

Element rates for the equation of the radius are obtained just as readily as for the position equation. Upon differentiating equation (2.126) and comparing the result with (2.127), the following requirement emerges:

$$0 = -(\mu/2E)' + [r_0 + (\mu/2E)]' \cos\theta + W_0' \sin\theta \tag{2.140}$$

Then after differentiating (2.127) and substituting for r and r'' in the equation of the radius (2.112), there results

$$-[r_0 + (\mu/2E)]' \sin\theta + W_0' \cos\theta = -\frac{1}{2E}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')] \tag{2.141}$$

It is a simple matter to solve these two equations for r_0' and W_0'.

$$r_0' = -(\mu/2E)'(1-\cos\theta) + \frac{1}{2E}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')] \sin\theta \tag{2.142}$$

$$W_0' = +(\mu/2E)' \sin\theta - \frac{1}{2E}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')] \cos\theta \tag{2.143}$$

Of course, E', wherever it occurs, is to be replaced by equation (2.110). There results

$$r_0' = +\left[\frac{\mu}{2E^2}\right](\underline{P}\cdot\underline{r}')(1-\cos\theta) + \frac{1}{2E}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')] \sin\theta \tag{2.144}$$

$$W_0' = -\left[\frac{\mu}{2E^2}\right](\underline{P}\cdot\underline{r}') \sin\theta - \frac{1}{2E}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')] \cos\theta \tag{2.145}$$

After $\underline{r}$, $\underline{r}'$, r and r' have been replaced in terms of elements and θ, these equations will be coupled to those for $\underline{r}_0'$ and $\underline{U}_0'$.

The variation-of-parameters approach to the time equation (2.128) requires some extra care. In the unperturbed case, that equation can be represented as

$$t(\theta) - t_0 = F(\theta;\ r_0,\ W_0,\ E) \tag{2.146}$$

so that

$$t' = \frac{dt}{d\theta} = \frac{dF}{d\theta} = \frac{1}{\sqrt{-2E}}\, r(\theta;\ r_0,\ W_0,\ E) \tag{2.147}$$

In the perturbed case, the situation is

$$t(\theta) - t_0(\theta) = F(\ \theta,\ r_0(\theta),\ W_0(\theta),\ E(\theta)\ ) \tag{2.148}$$

so that

$$t' - t_0' = \frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial r_0} r_0' + \frac{\partial F}{\partial W_0} W_0' + \frac{\partial F}{\partial E} E' \tag{2.149}$$

But

$$\frac{\partial F}{\partial \theta} = \frac{1}{\sqrt{-2E}}\, r(\ \theta,\ r_0(\theta),\ W_0(\theta),\ E(\theta)\ ) \tag{2.150}$$

so that, according to the differential equation of time (2.114), it must be that

$$t' = \frac{\partial F}{\partial \theta} \tag{2.151}$$

This leaves the desired element rate to be calculated as

$$-t_0' = +\frac{\partial F}{\partial r_0} r_0' + \frac{\partial F}{\partial W_0} W_0' + \frac{\partial F}{\partial E} E' \tag{2.152}$$

Expressions for r_0' , W_0' and E' have already been derived and the partial derivatives can be calculated straightforwardly from the right-hand side of equation (2.128). One obtains

$$-t_0' = \frac{1}{(-2E)\sqrt{-2E}}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')]\,(1-\cos\theta)$$

$$+\frac{3\mu}{(-2E)^2\sqrt{-2E}}(\theta - \sin\theta)(\underline{P}\cdot\underline{r}')$$

$$+\frac{1}{(-2E)\sqrt{-2E}}[r_0 \sin\theta + W_0(1-\cos\theta)](\underline{P}\cdot\underline{r}') \tag{2.153}$$

A difficulty with this equation is that terms proportional to θ (actually mixed secular terms) occur once $\underline{r}'$ has been replaced in terms of elements and θ. Part of the original motivation for introducing θ as the independent variable was to avoid the appearance of such terms in the element equations. This defect can be avoided in the present case by returning to equation (2.128) and introducing a different time element.

Let a quantity τ be defined by

$$\tau = t_0 + \frac{\mu}{(-2E)\sqrt{-2E}}\theta \tag{2.154}$$

From equation (2.128) it can be seen that

$$t(\theta) = \tau + \left[\frac{r_0}{\sqrt{-2E}} - \frac{\mu}{(-2E)\sqrt{-2E}}\right]\sin\theta + \frac{W_0}{\sqrt{-2E}}(1-\cos\theta) \tag{2.155}$$

Although τ is not a constant of unperturbed motion, it is nevertheless a linear function of θ in unperturbed motion, and common modern usage has been simply to expand the meaning of "element" to include all linear functions of the independent variable. In perturbed motion t_0 and E are functions of θ so that τ will not be a linear function in the general case, but clearly if τ is known for some value of θ then the corresponding value of time follows at once from (2.155), the elements r_0, W_0 and E also being known. The use of τ in place of t_0 in this manner is analogous to the use of mean anomaly in place of time of pericenter passage in classical celestial mechanics. (In fact, the coefficient of θ in equation (2.154) is the reciprocal of the mean motion.) Like mean anomaly in the time domain, the quantity τ in the θ domain is a linear function of the independent variable in unperturbed motion. Furthermore, the rate of change of τ in perturbed motion turns out not to involve terms proportional to θ, just as the time rate of change of mean anomaly contains no terms proportional to time. The situation for mean anomaly is discussed in most textbooks on celestial mechanics; see, for example, Brouwer and Clemence (1961, ch. 11, sec. 6), or Moulton (1914, art. 218). The latter cites Leverrier's classical work in which mean anomaly was introduced for just this reason. In recent times, Stiefel and Scheifele (1971, sec. 19) discussed the use of the time element τ in the θ-domain, but in the context of the KS coordinate transformation.

Now then the differential equation for τ is established by differentiating equation (2.154) and substituting for t_0' and E'.

$$\tau' = \frac{\mu}{(-2E)\sqrt{-2E}} + \frac{1}{(-2E)\sqrt{-2E}}[r(\underline{P}\cdot\underline{r}) + r'(\underline{P}\cdot\underline{r}')](1-\cos\theta)$$

$$+\frac{3\mu}{(-2E)^2\sqrt{-2E}}(\underline{P}\cdot\underline{r}')\sin\theta$$

$$-\frac{1}{(-2E)\sqrt{-2E}}(\underline{P}\cdot\underline{r}')[r_0\sin\theta + W_0(1-\cos\theta)] \tag{2.156}$$

It remains, of course, to replace $\underline{r}$, $\underline{r}'$, r and r' in terms of elements and θ. With that done, it becomes evident that no secular or mixed-secular terms appear. However, the computational problem of errors in the value of E leading to secular error growth in τ is still present in the first term. This feature can not be entirely eliminated from the two-body time calculation.

Several comments should be made about the above variation-of-parameters formulae. As mentioned before, the apparent length of the equations is of no particular concern in a numerical treatment, provided the arithmetic is arranged judiciously. In an analytical treatment, one must deal with all the numerous terms, realizing further that $\underline{P}$ itself is almost always a very extensive expression when put in terms of elements. However, against the tedious algebraic work must be balanced the rather complete generality of the formulae. The equations are free of singularities and geometrical ambiguities for all elliptic orbits, including rectilinear ones, and, being in vector form, can be applied directly in any coordinate system in which the original equation of motion (1.1) is valid.

Earlier in this chapter it was mentioned that several authors, notably Battin (1964), Herrick (1965) and Goodyear (1965b), years ago derived universal variation-of-parameters formulae which make use of the Sundman variable s (although they give the time rates, instead of the s -rates, of the elements). Here note is made that the work of none of these authors relies on, or even mentions, the s-domain equations of motion as a basis for the variation of parameters. Instead, a technique of direct "perturbative differentiation" of the Keplerian solution formulae was used. Briefly, the method is illustrated in the time domain as follows. The Keplerian solution of the equation of motion (1.1) is first written in terms of initial conditions in the form

$$\underline{r} = f_1(\underline{r}_0,\ \underline{V}_0,\ t - t_0) \tag{2.157}$$

$$\underline{V} = f_2(\underline{r}_0,\ \underline{V}_0,\ t - t_0) \tag{2.158}$$

For example, the well known "f and g" solution has this form. By mentally reversing the sense of the motion, one deduces that

$$\underline{r}_0 = f_1(\underline{r},\ \underline{V},\ t_0 - t) \tag{2.159}$$

$$\underline{V}_0 = f_2(\underline{r},\ \underline{V},\ t_0 - t) \tag{2.160}$$

Then the Keplerian initial conditions $\underline{r}_0$ and $\underline{V}_0$ serve as elements in the perturbed case. Their rates of change due to perturbation forces are found by differentiating the right-hand sides of (2.159) and (2.160) using the formal rules:

1. the time rate of change of $\underline{r}$ due to $\underline{P}$ is identically zero;
2. the time rate of change of $\underline{V}$ due to $\underline{P}$ is just $\underline{P}$.

It can be shown that these rules coincide with the condition of osculation used in classical celestial mechanics. This technique had been introduced by Pines (1961) for use in a time-domain variation of parameters for near-circular orbits, and the basic idea had been exploited even earlier by Herrick (1948) in the field of special perturbations. The dynamical basis of the technique was discussed more rigorously by Battin (1964) in his textbook where the method of Pines (1961) was generalized to handle all types of orbits. Goodyear (1965b), who gives the technique itself little discussion, presents the method in its clearest formal application. Later, Godal, *et al.* (1971), applied perturbative differentiation in refining the work of Pines for elliptic orbits, and also used θ as the independent variable. In all these cases, the method is conceptually simple but the necessary manipulations become quite complicated because the functions f_1 and f_2 are quite complicated. In fact, the final formulae of Pines, Battin, Herrick, Goodyear and Godal, *et al.* are very difficult to cross-check. By contrast, the manipulations for variation of parameters required in this chapter have all been straightforward. Furthermore, despite the easier algebra, the final expressions given here are no more lengthy than those derived by perturbative differentiation. The same convenience will be found, of course, in any approach based on linear governing equations; the most famous other examples of this fact are found in Stiefel and Scheifele (1971) and in Burdet (1969). These examples require a coordinate transformation in addition to the time transformation. In earlier papers, Burdet (1967, 1968) did begin with linear governing equations and no coordinate transformation, but he used the Sundman variable s instead of θ.

Summary

This chapter has been largely introductory in nature, containing a unified discussion of results available for many years but scattered throughout the literature. The simplest type of regularization is presented, that based on a time transformation only. The original Cartesian coordinates are retained and integrals of the unperturbed motion are introduced as done by Sperling (1961) and Burdet (1967, 1968) to serve as new dependent variables. These extra variables provide the redundancy required in the governing system to permit regularization. Differential time is replaced according to $dt = Cr^n ds$, which is an often-used generalization of Sundman's original (1912) proposal $dt = r ds$. The central result of this chapter is a slight generalization of Szebehely's (1976b) finding that in these circumstances only the value $n = 1$ leads to linear regular equations. This turns out to be the well-known eccentric-anomaly regularization. Regular but nonlinear equations appear for $n \geq \frac{3}{2}$, and the implication is that if linear regular formulations are to be developed for values of n other than $n = 1$ then some kind of coordinate transformation is required. Thus, a linear true-anomaly regularization, which corresponds to $n = 2$, requires a suitable choice of non-Cartesian coordinates besides the time transformation.

A corollary result, thought to be new, is the regularization of the equation of the radius for both $n = 1$ and $n \geq \frac{3}{2}$ by means of the introduction of angular momentum as a new dependent variable. In the case of unperturbed rectilinear motion the equation of the radius can be integrated in closed form for all values of the energy and for positive integral and half-integral values of n. It is found that a collision of the satellite with the primary body is not described in a finite interval of s if $n \geq \frac{3}{2}$, regardless of the value of the energy. A similar theorem by Stiefel and Scheifele (1971) holds only for zero energy but for all real values of $n \geq \frac{3}{2}$ and not just for integral and half-integral values.

Finally, a regular eccentric-anomaly variation-of-parameters formulation is developed. A new and much-simplified derivation is presented for a set of element equations originally developed by Pines (1961). These eccentric-anomaly formulae are valid only for elliptic (negative energy) orbits, but the analogous true-anomaly formulae to be developed in the next chapter will turn out to be valid for all non-rectilinear orbits.

Chapter 3. Regularization Using Burdet-Type Coordinates

Introduction

In the last chapter, regularization and linearization of the equations of motion were examined using a transformation of the independent variable (time) involving a power of the radius. No transformation of the coordinates was introduced. In this chapter, the idea of introducing a power of the radius to achieve regularization and possibly linearization is extended to a transformation of the coordinates. Specifically, in addition to the generalized Sundman time transformation already considered, there is now to be considered a power-law transformation of the radius such that

$$\underline{r} = u^m \underline{\xi} \qquad (3.1)$$

where $\underline{\xi}$ is the unit radial vector and m is a non-zero constant. This choice of coordinates is motivated by Burdet's (1969) success with the particular case of $m = -1$ and by the results of Szebehely and Bond (1983) showing that Burdet's case is an important linearizing transformation in the planar unperturbed problem. Recall from the last chapter that, while linear governing equations for unperturbed motion resulted only when $n = 1$, a whole family of regular governing equations resulted when $n \geq \frac{3}{2}$. Some members of this family have, in turn, already been found useful for numerical work with perturbed motion. Now the question is asked, what values of C, n and m will produce regularization and, more importantly, linearization of the equations of motion? Burdet (1969) used $n = 2$ to arrive at his linear equations, but considered no other time transformation for these coordinates. Bond (1985) showed that with spherical coordinates and a time transformation of the form $dt = g(r)ds$ linear regular governing equations can be derived only when $g(r) = r^2$. Notice that the coordinates now under consideration are really just the vector form of the usual spherical coordinates with, additionally, a power-law transformation of the radius.

Intuitively, one might expect that the coordinates $(u, \underset{\sim}{\xi})$ are good candidates for being regularizing variables. It was noted in Chapter 1 that regularization must necessarily involve redundant variables. That $(u, \underset{\sim}{\xi})$ are redundant is clear since four scalar components are used to represent three independent Cartesian coordinates. The redundancy is expressed by the unit vector identity

$$\underset{\sim}{\xi} \cdot \underset{\sim}{\xi} = 1 \tag{3.2}$$

which also implies

$$\underset{\sim}{\xi} \cdot \underset{\sim}{\xi}' = 0 \tag{3.3}$$

Of course, it remains to be seen yet that this redundancy is actually of any use in regularizing the equations of motion.

The Transformed Differential Equations

Now the Burdet-type coordinates are to be introduced into the already time-transformed system (2.32) through (2.37). The necessary derivatives of $\underline{r}$ are obtained from equation (3.1) as

$$\underline{r}' = mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}' \tag{3.4}$$

$$\underline{r}'' = m(m-1)u^{m-2}u'^2\underline{\xi} + mu^{m-1}u''\underline{\xi} + 2mu^{m-1}u'\underline{\xi}' + u^m\underline{\xi}'' \tag{3.5}$$

For the radius, there results

$$r = u^m \tag{3.6}$$

$$r' = mu^{m-1}u' \tag{3.7}$$

$$r'' = m(m-1)u^{m-2}u'^2 + mu^{m-1}u'' \tag{3.8}$$

Using the above relations, the integrals of unperturbed motion are also transformed from equations (2.8), (2.9) and (2.10).

$$E = \frac{1}{2}C^{-2}(u^m)^{-2n}(mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}')\cdot(mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}') - \mu(u^m)^{-1} \tag{3.9}$$

$$E = \frac{1}{2}C^{-2}u^{2m-2mn}(m^2u^{-2}u'^2 + \underline{\xi}'\cdot\underline{\xi}') - \mu u^{-m} \tag{3.10}$$

Then

$$\underline{h} = C^{-1}(u^m)^{-n}(u^m\underline{\xi}) \times (mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}') \tag{3.11}$$

$$\underline{h} = C^{-1}u^{2m-mn}\underline{\xi} \times \underline{\xi}' \tag{3.12}$$

Also

$$h^2 = C^{-2} u^{4m-2mn} (\underline{\xi} \times \underline{\xi}') \cdot (\underline{\xi} \times \underline{\xi}') \tag{3.13}$$

$$h^2 = C^{-2} u^{4m-2mn} (\underline{\xi}' \cdot \underline{\xi}') \tag{3.14}$$

Then

$$\underline{B} = C^{-2} (u^m)^{-2n} (m u^{m-1} u' \underline{\xi} + u^m \underline{\xi}') \times$$

$$\left[u^m \underline{\xi} \times (m u^{m-1} u' \underline{\xi} + u^m \underline{\xi}') \right] - \mu (u^m)^{-1} (u^m \underline{\xi}) \tag{3.15}$$

$$\underline{B} = \left[C^{-2} u^{3m-2mn} (\underline{\xi}' \cdot \underline{\xi}') - \mu \right] \underline{\xi} - m C^{-2} u^{3m-2mn-1} u' \underline{\xi}' \tag{3.16}$$

Here it should be noted that with the introduction of a coordinate transformation and its associated parameter m a new possibility exists. It was seen in the last chapter that regularization could be achieved by introducing E, h^2 and $\underline{B}$ as extra dependent variables and that any attempt to eliminate them algebraically inevitably reintroduced the unwanted negative power of r. Regularization in that case necessarily meant raising the order of the system to be integrated to at least eleven $(\underline{r}'', \underline{B}', E', t')$. Now, however, with the value of the radial exponent m freely adjustable, the algebraic elimination of E, h^2 and $\underline{B}$ must be re-examined. A suitable choice of m may allow one or more of these variables to be eliminated, lowering the order of the system, while still preserving regularization. In fact, Burdet's (1969) regularization for the special case $n = 2$, $m = -1$, $C = 1$ leads to a tenth-order system in which h^2 , but not E or $\underline{B}$, appears. Observe that when the time-transformed system (2.32) through (2.37) is straightforwardly subjected to the coordinate transformation (3.1) then the resulting system will actually be of order fourteen $(\underline{\xi}'', u'', \underline{B}', E', (h^2)', t')$. The differential equation of the radius will no longer be optional since separate differential equations are required for $\underline{\xi}$ and u. One of the purposes of the following analysis must be, therefore, to examine the circumstances under which a reduction in order is possible while still retaining a regular system. Of special interest is the manner in which Burdet's tenth-order system appears because his equations happen to be rigorously linear for unperturbed motion.

Now the position equation (2.32) is transformed via the above formulae:

$$m(m-1)u^{m-2}u'^2\underline{\xi} + mu^{m-1}u''\underline{\xi} + 2mu^{m-1}u'\underline{\xi}' + u^m\underline{\xi}''$$

$$-2EC^2n(u^m)^{2n-2}(u^m\underline{\xi}) - (n-1)C^2\mu(u^m)^{2n-3}(u^m\underline{\xi})$$

$$= -nC^2(u^m)^{2n-2}\underline{B} + C^2(u^m)^{2n}\underline{P} + C^{-1}C'(mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}') \tag{3.17}$$

Straightforward manipulations bring this equation into the form

$$\underline{\xi}'' + 2mu^{-1}u'\underline{\xi}' + \Big[m(m-1)u^{-2}u'^2 + mu^{-1}u''$$

$$-2EC^2nu^{2mn-2m} + (1-n)C^2\mu u^{2mn-3m}\Big]\underline{\xi}$$

$$= -nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}\underline{P} + C^{-1}C'(mu^{-1}u'\underline{\xi} + \underline{\xi}') \tag{3.18}$$

The Laplace vector equation (2.33) is transformed into

$$\underline{B}' = 2(u^m\underline{\xi})\Big[\underline{P}\cdot(mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}')\Big] - (mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}')\Big[\underline{P}\cdot(u^m\underline{\xi})\Big]$$

$$-\underline{P}\Big[u^m\underline{\xi}\cdot(mu^{m-1}u'\underline{\xi} + u^m\underline{\xi}')\Big] \tag{3.19}$$

which reduces to

$$\underline{B}' = u^{2m}\Big[2(\underline{P}\cdot\underline{\xi}')\underline{\xi} - (\underline{P}\cdot\underline{\xi})\underline{\xi}' - \big[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\big]mu^{-1}u'\Big] \tag{3.20}$$

The energy equation (2.34) is transformed into

$$E' = u^m\Big[mu^{-1}u'(\underline{P}\cdot\underline{\xi}) + (\underline{P}\cdot\underline{\xi}')\Big] \tag{3.21}$$

and the time equation becomes

$$t' = C\ u^{mn} \tag{3.22}$$

As mentioned before, the equation of the radius (2.36) is no longer optional under the present coordinate transformation, and is now rewritten as

$$m(m-1)u^{m-2}u'^2 + mu^{m-1}u'' - 2EC^2 n(u^m)^{2n-1}$$

$$= (2n-1)C^2\mu(u^m)^{2n-2} + (1-n)C^2h^2(u^m)^{2n-3}$$

$$+ C^2(u^m)^{2n-1}(\underline{P}\cdot u^m\underline{\xi}) + C^{-1}C'mu^{m-1}u' \tag{3.23}$$

This equation reduces straightforwardly to the form

$$m(m-1)u^{-2}u'^2 + mu^{-1}u'' - 2EC^2n\,u^{2mn-2m}$$

$$= (2n-1)C^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}$$

$$+ C^2u^{2mn-m}(\underline{P}\cdot\underline{\xi}) + C^{-1}C'mu^{-1}u' \tag{3.24}$$

Here it is opportune to eliminate the occurence of u'' in the $\underline{\xi}$ equation, (3.18). Actually, the entire left-hand side of (3.24) appears in the coefficient of $\underline{\xi}$ in (3.18). The latter equation becomes, after this substitution,

$$\underline{\xi}'' + 2mu^{-1}u'\underline{\xi}' + \left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= -nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.25}$$

Notice that this step automatically eliminates the explicit appearance of E from the $\underline{\xi}$ equation. The elimination of $\underline{B}$ and h^2 will be examined later. Returning now to equation (3.24), u'' is isolated, with the result

$$u'' - 2EC^2\frac{n}{m}u^{2mn-2m+1} + \frac{(n-1)}{m}C^2h^2u^{2mn-4m+1} + (m-1)u^{-1}u'^2$$

$$= \frac{(2n-1)}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}\,(\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{3.26}$$

Finally, the differential equation for h^2 can be cast in terms of the new coordinates most conveniently by using the second form of equation (2.37) and the expression for $\underline{B}'$ in (3.20):

$$(h^2)' = u^{2m+m}\underline{\xi}\cdot\left[2(\underline{P}\cdot\underline{\xi}')\underline{\xi} - (\underline{P}\cdot\underline{\xi})\underline{\xi}' - \left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right]mu^{-1}u'\right] \tag{3.27}$$

The unit vector identities (3.2) and (3.3) allow this expression to be reduced to the simple form

$$(h^2)' = 2u^{3m}\ \underline{P}\cdot\underline{\xi}' \tag{3.28}$$

Now the transformed governing equations can be summarized.

$$\underline{\xi}'' + 2mu^{-1}u'\underline{\xi}' + \left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= -nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.29}$$

$$u'' - 2EC^2\frac{n}{m}u^{2mn-2m+1} + \frac{(n-1)}{m}C^2h^2u^{2mn-4m+1} + (m-1)u^{-1}u'^2$$

$$= \frac{(2n-1)}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}\ (\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{3.30}$$

$$\underline{B}' = u^{2m}\left[2(\underline{P}\cdot\underline{\xi}')\underline{\xi} - (\underline{P}\cdot\underline{\xi})\underline{\xi}' - \left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right]mu^{-1}u'\right] \tag{3.31}$$

$$E' = u^m\left[mu^{-1}u'(\underline{P}\cdot\underline{\xi}) + (\underline{P}\cdot\underline{\xi}')\right] \tag{3.32}$$

$$(h^2)' = 2u^{3m}\ \underline{P}\cdot\underline{\xi}' \tag{3.33}$$

$$t' = C\ u^{mn} \tag{3.34}$$

It is worth noting that the mere form of the coordinate transformation (3.1) ensures that the differential equation (3.29) governing $\underline{\xi}$ will always be linear in $\underline{\xi}$, at least for unperturbed motion, regardless of the values of n, m and C. (As in the last chapter, C can be assumed, in light of later results, to be either strictly constant or else a constant of the unperturbed motion.) Hence, singularities and nonlinearities in this system arise only through u. The regularization of this system is still not a simple matter, though, since in general both positive and negative powers of u will appear.

Now in unperturbed motion equation (3.30) for u is not coupled to the rest of the system. Therefore, having solved that equation somehow, one has only to treat a linear (though variable-coefficient) equation for $\underline{\xi}$ and a quadrature for the time. However, this approach will not be analytically feasible for most values of m and n because of nonlinearities in the u equation. A more useful approach is to seek special values of m and n which will reduce the unperturbed equation (3.30) to a linear form. Notice that this linear form will necessarily be a constant-coefficient, and therefore regular, form. Unfortunately, it would seem, the choices of m and n for linearizing equation (3.30) as it stands are highly restricted. It is immediately obvious that the choice $m = 1$ is necessary for linearization regardless of the value of n. But then $u = r$ according to equation (3.1) and equation (3.30) merely reverts to the equation of the radius (2.36) of the last chapter. Linearization then requires the further choice $n = 1$, re-introducing the eccentric-anomaly regularization already discussed. No new results appear except the conclusion from the resulting complicated form of (3.29) that u and $\underline{\xi}$ are not the proper variables to describe position in the eccentric-anomaly domain; rather, the combination $\underline{r} = u\,\underline{\xi}$ is a better choice in that case. Further progress with this system of coordinate-transformed equations depends on the possibility mentioned before, namely, that E, $\underline{B}$ or h^2 might be eliminated algebraically.

Inspection of the system of equations (3.29) through (3.34) shows that many algebraic steps could be tried, eliminating first one of these quantities and then another, and so on. Furthermore, the coordinate-transformed integrals of unperturbed motion (3.10), (3.14) and (3.16) all contain terms of $\underline{\xi}' \cdot \underline{\xi}'$, making it feasible to substitute for one or more integrals in terms of the others. For example,

$$\underline{\xi}' \cdot \underline{\xi}' = 2C^2 u^{2mn-2m}(E + \mu u^{-m}) - m^2 u^{-2} u'^2 \tag{3.35}$$

or

$$\underline{\xi}' \cdot \underline{\xi}' = C^2 h^2 u^{2mn-4m} \tag{3.36}$$

or

$$\underline{\xi}' \cdot \underline{\xi}' = C^2 u^{2mn-3m} (\underline{B} \cdot \underline{\xi} + \mu) \tag{3.37}$$

or

$$\underline{\xi}' \cdot \underline{\xi}' = -\, m^{-1} C^2 u^{2mn-3m+1}\, u'^{-1}\, \underline{B} \cdot \underline{\xi}' \tag{3.38}$$

Equating various pairs of these four relations provides many more possible algebraic combinations. In searching for a strategy to follow at this juncture, let it be noticed that certain types of coupling ought to be avoided in modifying the the governing equations (3.29) through (3.34). Terms of $\underline{B} \cdot \underline{\xi}$ or $\underline{B} \cdot \underline{\xi}'$ cause the equations governing unperturbed motion to be coupled in the components of $\underline{\xi}$ and $\underline{\xi}'$. Therefore, the effect of using equations (3.37) and (3.38) would be to preclude any simple representation of the Keplerian motion. (Recall that the Keplerian solutions in the last chapter were easily obtained precisely because the vector differential equations were not only linear but also uncoupled in the vector components. It happens that in the special case $n = 2$, $m = -1$, $C = h^{-1}$ it is possible to derive a very useful set of linear equations which are coupled in the 4-vector Euler-parameter components of $\underline{\xi}$ rather than the 3-vector direction-cosine components, a situation which will be examined later. This linear set has been derived by Vitins (1978) and Junkins and Turner (1979). A related but less general set has been derived by Kamel (1983). The practical usefulness of this coupled system lies in the fact that a kinematical constraint among the 4-vector components permits the system order to be reduced to eight. As discussed later in Chapter 4 and Appendix B, this coupled system can be shown to be equivalent to an uncoupled (but higher order) system which appears for $n = 2$, $m = -1$, $C = h^{-1}$. Hence this coupled linear system does not represent a distinct type of linearization, and its existence does not affect the argument being pursued here.) Now if the use of equations (3.37) and (3.38) is avoided in order to prevent the above-mentioned coupling, then only the pair of relations (3.35) and (3.36) remains to be considered. Equating these two allows one to substitute for E in terms of h^2, or vice versa:

$$2C^2u^{2mn-2m}(E+\mu u^{-m}) - m^2u^{-2}u'^2 = C^2h^2u^{2mn-4m} \tag{3.39}$$

so that

$$E = \frac{1}{2}C^{-2}u^{2m-2mn}(C^2h^2u^{2mn-4m} + m^2u^{-2}u'^2) - \mu u^{-m} \tag{3.40}$$

or

$$h^2 = C^{-2}u^{4m-2mn}\left[2C^2u^{2mn-2m}(E+\mu u^{-m}) - m^2u^{-2}u'^2\right] \tag{3.41}$$

Of course, one might also substitute for $\underline{\xi}' \cdot \underline{\xi}'$ in terms of E or h^2 in equation (3.16) to obtain

$$\underline{B} = (2u^mE - C^{-2}m^2u^{3m-2mn-2}u'^2 + \mu)\underline{\xi} - mC^{-2}u^{3m-2mn-1}u'\underline{\xi}' \tag{3.42}$$

or

$$\underline{B} = (u^{-m}h^2 - \mu)\underline{\xi} - mC^{-2}u^{3m-2mn-1}u'\underline{\xi}' \tag{3.43}$$

The result of these considerations is that the strategy for modifying the system of governing equations (3.29) through (3.34) should be to eliminate E, $\underline{B}$ or h^2 by choosing among the four relations (3.40) through (3.43). At first sight it would appear that this approach still leaves many algebraic combinations to be tried. However, the choices can be narrowed considerably if one has in mind to linearize the system. In general, it would not be desirable merely to eliminate E, h^2 or $\underline{B}$ via equations (3.10), (3.14) or (3.16) because the nonlinear factor ($\underline{\xi}' \cdot \underline{\xi}'$) would be introduced. (An important special case which arises for $n = 2$ is discussed later.) Also, in the u equation (3.30) a prominent nonlinear term contains u'^2. The only possibility for cancelling this term, besides letting $m = 1$, lies in eliminating E or h^2 via, say, equation (3.40) or (3.41). In the $\underline{\xi}$ equation another prominent term contains $u'\underline{\xi}'$. Although this term is linear provided the solution for u is known, it may well be desirable to eliminate the occurrence of the first derivative $\underline{\xi}'$ in an attempt to arrive at a simple oscillator equation for $\underline{\xi}$. The only possibility for cancelling the product $u'\underline{\xi}'$ lies in eliminating $\underline{B}$ via, say, equation (3.42) or (3.43). These possibilities are now examined in turn, beginning with the u equation.

In equation (3.30) eliminate the occurrence of h^2 in favor of E using the relation (3.41). The result of that operation is

$$u'' - \frac{2}{m}EC^2u^{2mn-2m+1} + (n-m)u^{-1}u'^2$$

$$= \frac{1}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}(\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{3.44}$$

Evidently, a necessary condition for linearizing the unperturbed equation is

$$n = m \tag{3.45}$$

Subsequently, one can reason from the exponents of u that linearization requires n to take on values specified by any of the following pairs of relations:

$$2n^2 - 2n + 1 = 0 \quad \text{and} \quad 2n^2 - 3n + 1 = 0 \tag{3.46a}$$

or

$$2n^2 - 2n + 1 = 0 \quad \text{and} \quad 2n^2 - 3n + 1 = 1 \tag{3.46b}$$

or

$$2n^2 - 2n + 1 = 1 \quad \text{and} \quad 2n^2 - 3n + 1 = 0 \tag{3.46c}$$

or

$$2n^2 - 2n + 1 = 1 \quad \text{and} \quad 2n^2 - 3n + 1 = 1 \tag{3.46d}$$

Options (a) and (b) are not satisfied by any values of n. Option (c) is satisfied only by $n = 1$, while option (d) is satisfied only by $n = 0$. The value $n = 0$ means that no time transformation is made and furthermore this value is not permitted in equation (3.44) in view of the requirement (3.45). On the other hand, if $n = 1$ then equation (3.44) merely reverts to the equation of the radius (2.36) already discussed in the last chapter.

Now return to equation (3.30) and eliminate the occurrence of E in favor of h^2 using the relation (3.40). The result of that operation is

$$u'' - \frac{1}{m}C^2h^2u^{2mn-4m+1} + (m - mn - 1)u^{-1}u'^2$$

$$= -\frac{1}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}{}^2u^{2mn-m+1} \quad (\underline{P} \cdot \underline{\xi}) + C^{-1}C'u' \tag{3.47}$$

Evidently, a necessary condition for linearizing the unperturbed form of this equation is

$$m - mn - 1 = 0 \quad \text{or} \quad n = \frac{(m-1)}{m} \tag{3.48}$$

Substituting this value of n into equation (3-47) produces

$$u'' - \frac{1}{m}C^2h^2u^{-2m-1} = -\frac{1}{m}C^2\mu u^{-m-1} + \frac{1}{m}C^2u^{m-1}(\underline{P} \cdot \underline{\xi}) + C^{-1}C'u' \tag{3.49}$$

Then one can reason from the exponents of u that linearization requires m to take on values specified by any of the following pairs of relations:

$$-2m - 1 = 0 \quad \text{and} \quad -m - 1 = 0 \tag{3.50a}$$

or

$$-2m - 1 = 0 \quad \text{and} \quad -m - 1 = 1 \tag{3.50b}$$

or

$$-2m - 1 = 1 \quad \text{and} \quad -m - 1 = 0 \tag{3.50c}$$

or

$$-2m - 1 = 1 \quad \text{and} \quad -m - 1 = 1 \tag{3.50d}$$

Options (a), (b) and (d) are never satisfied. Option (c) is satisfied by $m = -1$, in which case $n = 2$. The resulting linearization is essentially that presented by Burdet (1969) who further used $C = 1$:

$$u'' + C^2 h^2 u = C^2 \mu - C^2 u^{-2}(\underline{P} \cdot \underline{\xi}) + C^{-1} C' u' \tag{3.51}$$

Actually, the linearization of the u equation corresponding to $m = -1$, $n = 2$, $C = h^{-1}$ has been known for several centuries. It is just the differential equation of the conic section in terms of the reciprocal radius and central angle. Burdet's innovation was to extend this type of linearization to the $\underline{\xi}$ equation and to present a complete set of equations governing the perturbed motion in these coordinates.

Now the $\underline{\xi}$ equation (3.29) will be examined retaining general values of m and n. Although that equation is already linear in $\underline{\xi}$ when $\underline{P} = \underline{0}$ and one might expect to be forced to use $m = -1$ and $n = 2$ in order to obtain a tractable u equation, it is interesting to see that these same values for m and n arise from other considerations about equation (3.29). For example, as noted before, it might be desirable to eliminate the occurrence of $\underline{\xi}'$ in the unperturbed equation in an attempt to obtain a simple oscillator-type equation. This step necessarily involves the elimination of $\underline{B}$ in favor of E or h^2. Also, for general values of m and n equation (3.29) will have variable coefficients. It would be desirable to choose special values for m and n to produce constant coefficients, if possible. Refer now to equation (3.29) and replace $\underline{B}$ in favor of E using the relation (3.42). The result is

$$\begin{aligned} &\underline{\xi}'' + (2m - mn)u^{-1}u'\underline{\xi}' \\ &+ \left[2nC^2 u^{2mn-2m} E - nm^2 u^{-2} u'^2 - 2\mu n C^2 u^{2mn-3m} + (1-n) C^2 h^2 u^{2mn-4m}\right]\underline{\xi} \\ &= C^2 u^{2mn-m}\left[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi}\right] + C^{-1} C' \underline{\xi}' \end{aligned} \tag{3.52}$$

On the other hand, if $\underline{B}$ had been replaced in favor of h^2 using the relation (3.43) the result would have been

$$\underline{\xi}'' + (2m - mn)u^{-1}u'\underline{\xi}' + \left[C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.53}$$

These two equations differ only in the coefficient of $\underline{\xi}$, the second equation being much simpler in form. In both equations the term containing $u'\underline{\xi}'$ is removed only by the choice $n = 2$ regardless of the value of m. Equation (3.53) is remarkable in that this same choice of n simultaneously reduces the unperturbed equation to a constant-coefficient form. Equation (3.52) cannot be so reduced as it stands, partly because of the term containing u'^2. In order to cancel this term it would be necessary at least to eliminate either E or h^2 in favor of the other. Eliminating E by means of the relation (3.40) merely reproduces equation (3.53), but eliminating h^2 by means of (3.41) produces

$$\underline{\xi}'' + (2m - nm)u^{-1}u'\underline{\xi}' + \left[2C^2u^{2mn-2m}E - m^2u^{-2}u'^2 - 2\mu C^2u^{2mn-3m}\right]\underline{\xi}$$

$$= C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.54}$$

However, this equation still cannot be reduced to a constant-coefficient form by any choices of n and m. (Recall that the choice $m = 0$ is excluded for these coordinates.)

In summary, the set of coordinate-transformed governing equations (3.29) through (3.34) can be reduced to regular equations which (except for the time quadrature) are linear with constant coefficients for Keplerian motion provided that $m = -1$ amd $n = 2$ and provided that the integrals of motion $\underline{B}$ and E are eliminated algebraically in terms of h^2. Actually, in collecting the pertinent formulae, it is easy to show that if both E and $\underline{B}$ are eliminated at once using equations (3.10) and (3.16) for general values of m and n then the same results appear straightforwardly. Once these substitutions for E and $\underline{B}$ are made in the governing set of equations, the differential equations for

E and $\underline{B}$ are superfluous. Then, after straightforward manipulations, the modified governing equations can be summarized as

$$\underline{\xi}'' + (2m - mn)u^{-1}u'\underline{\xi}' + \left[C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.55}$$

$$u'' - \frac{1}{m}C^2h^2u^{2mn-4m+1} + (m - mn - 1)u^{-1}u'^2$$

$$= -\frac{1}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}(\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{3.56}$$

$$(h^2)' = 2u^{3m}\,\underline{P}\cdot\underline{\xi}' \tag{3.57}$$

$$t' = C\,u^{mn} \tag{3.58}$$

Equation (3.55) is the same as (3.53) and equation (3.56) is the same as (3.47). This tenth-order system is a general version of Burdet's (1969) regularization, and Burdet's linear equations arise out of the above set in a very natural way. In order to reduce equations (3.55) and (3.56) to simple linear forms in the unperturbed case, it is first of all necessary that $n = 2$ regardless of the value of m. This is required in order to eliminate the second term of (3.55). It is subsequently necessary that $m = -1$ in order to eliminate the third term of (3.56). By themselves, these necessary conditions offer no assurance that all the various exponents of u found in other terms will be reduced to zero or unity, but in fact the system becomes

$$\underline{\xi}'' + C^2h^2\underline{\xi} = C^2u^{-3}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{3.59}$$

$$u'' + C^2h^2u = C^2\mu - C^2u^{-2}(\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{3.60}$$

$$(h^2)' = 2u^{-3}\,\underline{P}\cdot\underline{\xi}' \tag{3.61}$$

$$t' = C\,u^{-2} \tag{3.62}$$

The further choice $C = 1$ produces Burdet's (1969) equations:

$$\underline{\xi}'' + h^2\underline{\xi} = u^{-3}\left[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi}\right] \tag{3.63}$$

$$u'' + h^2 u = \mu - u^{-2}(\underline{P} \cdot \underline{\xi}) \tag{3.64}$$

$$(h^2)' = 2u^{-3}\, \underline{P} \cdot \underline{\xi}' \tag{3.65}$$

$$t' = u^{-2} \tag{3.66}$$

Here note is made that this tenth-order system can be reduced to ninth order without sacrificing either regularization or linearization. The expression for angular momentum magnitude has a special property when written in terms of the $(u, \underline{\xi})$ coordinates. From equation (3.14) or (3.36), it is apparent that in case, and only in case, $n = 2$, the scalar product $(\underline{\xi}' \cdot \underline{\xi}')$ is a constant of unperturbed motion:

$$\underline{\xi}' \cdot \underline{\xi}' = C^2 h^2 \tag{3.67}$$

This is true regardless of the value of m. Then since $C = 1$ in Burdet's case, his system of equations might as well be written as

$$\underline{\xi}'' + (\underline{\xi}' \cdot \underline{\xi}')\underline{\xi} = u^{-3}[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi}] \tag{3.68}$$

$$u'' + (\underline{\xi}' \cdot \underline{\xi}')u = \mu - u^{-2}(\underline{P} \cdot \underline{\xi}) \tag{3.69}$$

$$t' = u^{-2} \tag{3.70}$$

This system of equations preserves the constancy of $(\underline{\xi}' \cdot \underline{\xi}')$ in unperturbed motion, as can be proved by forming the scalar product of $\underline{\xi}'$ with equation (3.68). More importantly, the integration for h^2 is now superfluous even in perturbed motion. In other words, when the redundant coordinates $(u, \underline{\xi})$ are used to describe the motion, regularization, with linearization of the Keplerian motion, is possible without the introduction of any extra variables. However, this regular reduction in order is not available in these coordinates unless C is strictly constant. For a general value of C, substitute

$$C^2 = h^{-2}(\underline{\xi}' \cdot \underline{\xi}') \tag{3.71}$$

into equations (3.59) through (3.62) to obtain

$$\underline{\xi}'' + (\underline{\xi}' \cdot \underline{\xi}')\underline{\xi} = (\underline{\xi}' \cdot \underline{\xi}')h^{-2}u^{-3}[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi}] + C^{-1}C'\underline{\xi}' \tag{3.72}$$

$$u'' + (\underline{\xi}' \cdot \underline{\xi}')u = (\underline{\xi}' \cdot \underline{\xi}')h^{-2}\mu - (\underline{\xi}' \cdot \underline{\xi}')h^{-2}u^{-2}(\underline{P} \cdot \underline{\xi}) + C^{-1}C'u' \tag{3.73}$$

$$(h^2)' = 2u^{-3}(\underline{P} \cdot \underline{\xi}') \tag{3.74}$$

$$t' = \sqrt{(\underline{\xi}' \cdot \underline{\xi}')}\ h^{-1}u^{-2} \tag{3.75}$$

Here the integration for h^2 evidently cannot be disregarded since h appears elsewhere in the system, and this will be the case as long as C is not strictly constant. Notably, h now appears in denominators, so that a singularity may occur for rectilinear orbits. It is well to recall the remark in the last chapter to the effect that this regularization based on $n = 2$ is different in character from the regularization based on $n = 1$. Since the above equations cannot be used to describe an actual collision anyway, merely because $n = 2$, a possible restriction to nonrectilinear orbits should not be too alarming. Of course, it also remains to make a particular choice for C in these equations.

Solutions of the Linear Regular Equations

Burdet's description of Keplerian motion is

$$\underline{\xi}'' + h^2\underline{\xi} = \underline{0} \tag{3.76}$$

$$u'' + h^2 u = \mu \tag{3.77}$$

$$t' = u^{-2} \tag{3.78}$$

These equations are valid even in the rectilinear case $h = 0$, although the collision itself is not described: $u = r^{-1}$ in these coordinates so $u \to \infty$ as $r \to 0$. This set of equations should be compared with the eccentric-anomaly set (2.60) through (2.62). In the latter set, the type of orbit could be established by inspection according as the frequency of oscillations $\sqrt{-2E}$ was real, zero or imaginary. In the present case the frequency of oscillations h is always real (or zero) so one cannot distinguish elliptic, parabolic and hyperbolic orbits merely by inspection of the above equations. The type of orbit does become apparent when the time equation (3.78) is examined, but discussion of the time equation in terms of the present variables is reserved for a later chapter.

Now denoting the independent variable by σ, the complete σ -domain solution for Keplerian motion is

$$\underline{\xi}(\sigma) = \underline{\xi}(0)\cos(h\sigma) + \underline{\xi}'(0)\frac{\sin(h\sigma)}{h} \tag{3.79}$$

$$\underline{\xi}'(\sigma) = -\,\underline{\xi}(0)h\sin(h\sigma) + \underline{\xi}'(0)\cos(h\sigma) \tag{3.80}$$

$$u(\sigma) = u(0)\cos(h\sigma) + u'(0)\frac{\sin(h\sigma)}{h} + \mu\frac{[1-\cos(h\sigma)]}{h^2} \tag{3.81}$$

$$u'(\sigma) = -\,u(0)h\sin(h\sigma) + u'(0)\cos(h\sigma) + \mu\frac{\sin(h\sigma)}{h} \tag{3.82}$$

These formulae are valid for all types of orbits even though only circular trigonometric functions are used. The exceptional case $h = 0$ can also be handled by calculating

$$\frac{\sin h\sigma}{h} = \sigma - \frac{h^2\sigma^3}{3!} + \frac{h^4\sigma^5}{5!} - \frac{h^6\sigma^7}{7!} + \cdots \tag{3.83}$$

$$\frac{1-\cos h\sigma}{h^2}=\frac{\sigma^2}{2!}-\frac{h^2\sigma^4}{4!}+\frac{h^4\sigma^6}{6!}-\cdots \tag{3.84}$$

These expressions are analogous to the series developments made in the last chapter. After these series are substituted into the above equations many arrangements and groupings of terms are possible. Stumpff functions could have been introduced, or perhaps the following form of solution:

$$\underline{\xi}(\sigma)=\underline{\xi}(0)\left[1-h^2C(\sigma)\right]+\underline{\xi}'(0)\left[\sigma-h^2S(\sigma)\right] \tag{3.85}$$

$$\underline{\xi}(\sigma)=-\underline{\xi}(0)h^2\left[\sigma-h^2S(\sigma)\right]+\underline{\xi}(0)\left[1-h^2C(\sigma)\right] \tag{3.86}$$

$$u(\sigma)=u(0)\left[1-h^2C(\sigma)\right]+u'(0)\left[\sigma-h^2S(\sigma)\right]+\mu C(\sigma) \tag{3.87}$$

$$u'(\sigma)=-u(0)h^2\left[\sigma-h^2S(\sigma)\right]+u'(0)\left[1-h^2C(\sigma)\right]+\mu\left[\sigma-h^2S(\sigma)\right] \tag{3.88}$$

where the functions $S(\sigma)$ and $C(\sigma)$ are to be evaluated by means of

$$S(\sigma)=\frac{\sigma^3}{3!}-\frac{(h^2)\sigma^5}{5!}+\frac{(h^2)^2\sigma^7}{7!}-\cdots \tag{3.89}$$

$$C(\sigma)=\frac{\sigma^2}{2!}-\frac{(h^2)\sigma^4}{4!}+\frac{(h^2)^2\sigma^6}{6!}-\cdots \tag{3.90}$$

Since these are actually functions of the two parameters h^2 and σ , it may be convenient to express $S(\sigma)$ and $C(\sigma)$ further in terms of a single combined parameter z:

$$S(\sigma)=\sigma^3S^*(z) \tag{3.91}$$

$$C(\sigma)=\sigma^2C^*(z) \tag{3.92}$$

where

$$S^*(z)=\frac{1}{3!}-\frac{z}{5!}+\frac{z^2}{7!}-\frac{z^3}{9!}+\cdots \tag{3.93}$$

$$C^{*}(z) = \frac{1}{2!} - \frac{z}{4!} + \frac{z^2}{6!} - \frac{z^3}{8!} + \cdots \tag{3.94}$$

$$z = h^2 \sigma^2 \tag{3.95}$$

The above universal formulae should be compared with the eccentric-anomaly solution for Keplerian motion given in equations (2.94) through (2.103). In this case the power series developments are not needed to unify the formulae for all three kinds of orbits, but rather are needed only to accomodate the special case of rectilinear orbits ($h = 0$).

In developing initial conditions for $\underline{\xi}$, u, $\underline{\xi}'$ and u' in terms of time-domain data, the coordinates can be evaluated straightforwardly using equation (3.1), but the velocity relation requires more attention. First, the coordinate transformation implies

$$u = r^{-1} \quad \rightarrow \quad u(0) = r(0)^{-1} \tag{3.96}$$

and

$$\underline{\xi} = r^{-1}\underline{r} \quad \rightarrow \quad \underline{\xi}(0) = r(0)^{-1}\underline{r}(0) \tag{3.97}$$

The quantity h^2 follows from the definition of angular momentum. Now the velocity relation is developed using the time transformation

$$dt = C\, u^{-2}\, d\sigma \tag{3.98}$$

where the factor C is unity in Burdet's case. Then

$$\frac{d(\ldots)}{d\sigma} = C r^2 \frac{d(\ldots)}{dt} \tag{3.99}$$

so that

$$\underline{\xi}' = C r^2 \frac{d}{dt}(r^{-1}\underline{r}) \tag{3.100}$$

$$\underline{\xi}' = C(r\underline{\dot{r}} - \dot{r}\underline{r}) \tag{3.101}$$

$$\underline{\xi}' = Cr^{-1}[\underline{\dot{r}}(\underline{r} \cdot \underline{r}) - \underline{r}(\underline{r} \cdot \underline{\dot{r}})] \tag{3.102}$$

$$\underline{\xi}' = Cr^{-1}(\underline{r} \times \underline{\dot{r}}) \times \underline{r} = Cr^{-1}\underline{h} \times \underline{r} \tag{3.103}$$

from which

$$\underline{\xi}'(0) = Cr(0)^{-1}[\underline{\dot{r}}(0)[\underline{r}(0) \cdot \underline{r}(0)] - \underline{r}(0)[\underline{r}(0) \cdot \underline{\dot{r}}(0)]] \tag{3.104}$$

Likewise for the radial variable u,

$$u' = Cr^2\frac{d}{dt}(r^{-1}) \tag{3.105}$$

$$u' = Cr^2(-r^{-2}\dot{r}) \tag{3.106}$$

$$u' = -Cr^{-1}\underline{r} \cdot \underline{\dot{r}} \tag{3.107}$$

from which

$$u'(0) = -Cr(0)^{-1}\underline{r}(0) \cdot \underline{\dot{r}}(0) \tag{3.108}$$

Once $\underline{\xi}$, u, $\underline{\xi}'$ and u' are available for some value of σ, the time-domain position and velocity are computed by an inverse process.

$$\underline{r}(\sigma) = u^{-1}\underline{\xi} \tag{3.109}$$

From the time transformation, there results

$$\frac{d(\ldots)}{dt} = C^{-1}u^2\frac{d(\ldots)}{d\sigma} \tag{3.110}$$

so that

$$\dot{\underline{r}}(\sigma) = C^{-1}u^2(u^{-1}\underline{\xi})' \tag{3.111}$$

$$\dot{\underline{r}}(\sigma) = C^{-1}(u\underline{\xi}' - u'\underline{\xi}) \tag{3.112}$$

The remaining problem of computing time in terms of σ, or vice-versa, is addressed in a later chapter.

The universal σ -domain solution given above for Keplerian motion is in many ways analogous to the universal s-domain solution given in the last chapter. It is analogous also in its disadvantage that the frequency of oscillations, h, is a function of initial conditions so that a secular in-track position error will occur if h is inaccurately represented. Furthermore, it is evident that a σ-domain variation-of-parameters formulation will have mixed secular terms in the element rate equations. Fortunately, as in the s domain, a proper choice of the factor C will alleviate these difficuties.

Return to equations (3.59) through (3.62) and render the frequency of oscillations strictly constant by choosing

$$C = h^{-1} \tag{3.113}$$

This choice is valid for all three types of orbits and will exclude only rectilinear orbits from further consideration. The necessary derivative is obtained with the aid of equation (3.61) as

$$C' = -h^{-2}h' \tag{3.114}$$

$$C' = -h^{-3}u^{-3}\underline{P} \cdot \underline{\xi}' \tag{3.115}$$

$$C^{-1}C' = -h^{-2}u^{-3}\underline{P} \cdot \underline{\xi}' \tag{3.116}$$

Then the governing equations (3.59) through (3.62) become

$$\underline{\xi}'' + \underline{\xi} = \frac{1}{h^2u^3}\left[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi} - (\underline{P} \cdot \underline{\xi}')\underline{\xi}'\right] \tag{3.117}$$

$$u'' + u = \frac{\mu}{h^2} - \frac{1}{h^2 u^3}\left[(\underline{P} \cdot \underline{\xi})u + (\underline{P} \cdot \underline{\xi}')u'\right] \tag{3.118}$$

$$(h^2)' = \frac{2}{u^3}\underline{P} \cdot \underline{\xi}' \tag{3.119}$$

$$t' = \frac{1}{hu^2} \tag{3.120}$$

It is important to notice the effect of this particular choice of C on the angular momentum expression given in equation (3.67). That equation reduces to

$$\underline{\xi}' \cdot \underline{\xi}' = 1 \tag{3.121}$$

meaning that the magnitude scaling of $\underline{\xi}'$ introduced by the time transformation (3.120) is now such that $\underline{\xi}'$ is always precisely a unit vector. This is the geometrical aspect of the analytical fact that in equations (3.72) through (3.75) the integration for h^2 cannot be disregarded. Since now both $\underline{\xi}$ and $\underline{\xi}'$ are unit vectors, they can provide information about the direction of $\underline{h}$ but not about its magnitude. In fact, from equation (3.103) it is clear that now

$$\underline{\xi}' = \frac{1}{h}\underline{h} \times \underline{\xi} \quad \rightarrow \quad \underline{\xi} \times \underline{\xi}' = \frac{1}{h}\underline{h} \tag{3.122}$$

where $\frac{1}{h}\underline{h}$ is just the unit vector normal to the instantaneous orbit plane. This relation, together with

$$\underline{\xi} \cdot \underline{\xi} = 1, \quad \underline{\xi} \cdot \underline{\xi}' = 0, \quad \underline{\xi}' \cdot \underline{\xi}' = 1 \tag{3.123}$$

shows that the vectors $\underline{\xi}$, $\underline{\xi}'$ and $\underline{\xi} \times \underline{\xi}'$ form an orthonormal vector basis in which $\underline{\xi}$ and $\underline{\xi}'$ span the osculating plane with $\underline{\xi}'$ indicating the sense of motion. The rotating Cartesian coordinate frame so defined is of classical heritage in celestial mechanics, though it does not have a special name. For example, the Gauss form of Lagrange's planetary equations usually contains components of the perturbing force resolved in these directions, and it is found that this form provides much physical insight into the perturbed motion. In the present instance, this coordinate frame has arisen quite naturally in the quest for linear regular governing equations. It is worth noting that the $\underline{\xi}$ equation (3.117) contains only the normal component of $\underline{P}$, so this equation describes mainly changes in the orientation of the orbital plane. The u and h^2 equations (3.118) and (3.119) contain only in-plane components of $\underline{P}$, so these equations describe mainly changes in the size and shape of the orbit. The time equation (3.120) then describes the anomalistic motion along the orbital path. When it is realized that the independent variable in this set of governing equations is actually the orbital central angle (true anomaly) elapsed since the epoch, the unperturbed form of the $\underline{\xi}$ equation (3.117) is seen to be an almost obvious kinematical relation. Even in the general perturbed case, the kinematics of the orthonormal triad ($\underline{\xi}$, $\underline{\xi}'$, $\underline{\xi} \times \underline{\xi}'$) can be exploited to derive a useful analogy between the motion of the osculating plane and the rotation of a rigid body. This analogy has been developed mainly by Broucke, *et al.* (1971), by Vitins (1978) and by Junkins and Turner (1979), all for the case $n = 2$, $m = -1$ and $C = h^{-1}$; that is, when the independent variable is actually the true anomaly, so that the vectors $\underline{\xi}$, $\underline{\xi}'$, and $\underline{\xi} \times \underline{\xi}'$ are orthonormal. For general values of n and m these vectors will be orthogonal but not orthonormal since $\underline{\xi}'$ will have variable magnitude. But, as shown in Appendix B, the rigid-body analogy can still be carried through provided only that orthonormal vectors aligned with $\underline{\xi}$, $\underline{\xi}'$ and $\underline{\xi} \times \underline{\xi}'$ can be defined unambiguously. The basic requirement turns out to be $C = h^{-1}$; that is, the orbit must be nonrectilinear so that $\underline{\xi}'$ does not vanish.

Several authors have derived the set of equations (3.117) through (3.120), though not by the method used in this study. Vitins (1978) began with Burdet's (1969) equations of motion, derived the modified set, developed explicitly the analogy with rigid-body motion in terms of Euler parameters, introduced a time element for elliptical orbits, and indicated a first-order averaged solution for the J_2-perturbed motion. Kamel (1983) also developed equations of motion using true anomaly as the independent variable. His system of equations is of order eight and, due to the particular coordinate system used, happens to be singular when the inclination of the orbital plane is 180°. Even though his equations are not perfectly regular, they are linear in unperturbed motion, and even though he does not cite the work of Burdet or Vitins (or Junkins and Turner), his equations are similar to an eighth-order set derived by Vitins (1978) from Burdet's (1969) results. They are also similar to the set given in (3.117) through (3.120) above, with two main differences. The first difference is that Kamel uses components of the Laplace vector resolved in the radial and transverse directions in the osculating plane rather than a radial coordinate. However, inspection of his formulae shows that this amounts only to using $\left[\frac{h^2u}{\mu} - 1\right]$ rather than u. The second difference is that the direction cosine components of a certain unit vector are governed by first-order coupled equations rather than by second-order uncoupled equations. This feature is possible because of special kinematical properties which, as mentioned earlier, hold only when the three vectors $\underline{\xi}$, $\underline{\xi}'$, and $\underline{\xi} \times \underline{\xi}'$ are precisely orthonormal, or when unique corresponding orthonormal vectors exist (as when $\underline{\xi}'$ does not vanish). Essentially, Kamel shows that the angular velocity components of this unit vector triad have a special form, the $\underline{\xi}'$ component of angular velocity always vanishing. This kinematical fact has been noted earlier by Broucke, *et al.*, (1971), Vitins (1978) and Junkins and Turner (1979). It allows one to derive the first-order equations mentioned above, and leads to a coupled system of order eight instead of an uncoupled system of order ten. (Recall that coupling among vector components was specifically avoided in deriving equation (3.117) above.) Kamel calls the natural orthonormal vector basis ($\underline{\xi}$, $\underline{\xi}'$, $\underline{\xi} \times \underline{\xi}'$) the "Euler-Hill" frame and rightly observes that coordinates and elements related to this frame allow some decoupling of in-plane and out-of-plane perturbing effects. However, this fact is somewhat harder to appreciate when a particular coordinate system is used *a priori*, as in Kamel's case, than when the governing equations are cast in vector form, as in Vitins' (1978) article or in equations (3.117) through (3.119) above.

Now that the frequency of oscillations in the governing system has been rendered strictly constant, position errors in the transformed domain will not grow secularly, although such an effect is still to be expected in the time calculation. The price of obtaining this more favorable behavior has been to introduce a geometrical singularity at $h = 0$, which is analogous to the singularity at $E = 0$ in the eccentric-anomaly formulation. It is important to realize that a singularity at $h = 0$ is much less restrictive for analytical and numerical developments than a singularity at $E = 0$. The latter forbids treatment of parabolic orbits and hence forbids continuous transition between elliptic and hyperbolic conditions. The former forbids treatment only of rectilinear orbits, which in practice is usually not a serious limitation because of the finite size of the primary body.

The Keplerian solutions of equations (3.117) and (3.118) are readily obtained. Denoting the independent variable by η, the results are

$$\underline{\xi}(\eta) = \underline{\xi}(0) \cos \eta + \underline{\xi}'(0) \sin \eta \tag{3.124}$$

$$\underline{\xi}'(\eta) = -\, \underline{\xi}(0) \sin \eta + \underline{\xi}'(0) \cos \eta \tag{3.125}$$

$$u(\eta) = \frac{\mu}{h^2} + \left[u(0) - \frac{\mu}{h^2} \right] \cos \eta + u'(0) \sin \eta \tag{3.126}$$

$$u'(\eta) = -\left[u(0) - \frac{\mu}{h^2} \right] \sin \eta + u'(0) \cos \eta \tag{3.127}$$

The initial conditions are computed in terms of time-domain data by means of equations (3.96), (3.97), (3.104) and (3.108), letting $C = h^{-1}$ where necessary. Then time-domain position and velocity are calculated as functions of η from equations (3.109) and (3.112). Comparison of these formulae with more traditional two-body formulae shows that η is indeed the classical true anomaly to within an additive constant of unperturbed motion, being just the true anomaly elapsed since the epoch.

Variation of Parameters

A variation-of-parameters formulation in the η domain offers the interesting possibility of a universal description of perturbed motion in terms of regular elements, in which no mixed secular terms appear in the element rate equations. Being based on linear governing equations, the manipulations required are all elementary, much in analogy with the θ-domain formulation in the last chapter.

First, rewrite equations (3.124) through (3.127) in terms of the elements to be used.

$$\underline{\xi} = \underline{\xi}_0 \cos\eta + \underline{\zeta}_0 \sin\eta \tag{3.128}$$

$$\underline{\xi}' = -\,\underline{\xi}_0 \sin\eta \;+ \underline{\zeta}_0 \cos\eta \tag{3.129}$$

$$u = \frac{\mu}{h^2} + \left[u_0 - \frac{\mu}{h^2}\right]\cos\eta + w_0 \sin\eta \tag{3.130}$$

$$u' = -\left[u_0 - \frac{\mu}{h^2}\right]\sin\eta + w_0 \cos\eta \tag{3.131}$$

By correspondence with the previous formulae, one sees that in Keplerian motion the elements are $\underline{\xi}_0 = \underline{\xi}(0)$, $\underline{\zeta}_0 = \underline{\xi}'(0)$, $u_0 = u(0)$, $w_0 = u'(0)$, and h^2. These parameters are closely related to the so-called "focal elements" of Burdet (1969). In order to develop a solution of this form for the perturbed system (3.117) through (3.119), the elements $\underline{\xi}_0$, $\underline{\zeta}_0$, u_0 , w_0 and h^2 will be considered as functions of η. Upon differentiating equation (3.128), it is found that preserving the form of (3.129) will require that

$$\underline{0} = \underline{\xi}_0' \cos\eta + \underline{\zeta}_0' \sin\eta \tag{3.132}$$

Then differentiating equation (3.129) and substituting for $\underline{\xi}$, $\underline{\xi}'$ and $\underline{\xi}''$ in the governing equation (3.117) will produce a second relation involving $\underline{\xi}_0'$ and $\underline{\zeta}_0'$. Actually, some conciseness can be gained by modifying equation (3.117) as follows. Using the natural vector basis to resolve $\underline{P}$, there results

$$\underline{P} = (\underline{P}\cdot\underline{\xi})\underline{\xi} + (\underline{P}\cdot\underline{\xi}')\underline{\xi}' + \left[\underline{P}\cdot(\underline{\xi}\times\underline{\xi}')\right](\underline{\xi}\times\underline{\xi}') \tag{3.133}$$

Then equation (3.117) can be rewritten in the form

$$\underline{\xi}'' + \underline{\xi} = \frac{1}{h^2 u^3}\left[\underline{P} \cdot (\underline{\xi} \times \underline{\xi}')\right](\underline{\xi} \times \underline{\xi}') \tag{3.134}$$

Substituting for $\underline{\xi}$, $\underline{\xi}'$ and $\underline{\xi}''$ in terms of elements now yields the above-mentioned second relation

$$-\underline{\xi}_0{}' \sin \eta + \underline{\zeta}_0{}' \cos \eta = \frac{1}{h^2 u^3}\left[\underline{P} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0) \tag{3.135}$$

The form of the right-hand side could have been anticipated from equation (3.134) since $\underline{\xi} \times \underline{\xi}'$, the unit normal vector, is a constant of unperturbed motion and as such can be reckoned anywhere on the osculating orbit, including the epoch. Now equations (3.132) and (3.135) can be solved at once for the element rates:

$$\underline{\xi}_0{}' = -\frac{1}{h^2 u^3}\left[\underline{P} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0) \sin \eta \tag{3.136}$$

$$\underline{\zeta}_0{}' = +\frac{1}{h^2 u^3}\left[\underline{P} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0) \cos \eta \tag{3.137}$$

Rates for u_0 and w_0 are derived in an equally straightforward manner. Differentiating equation (3.130) leads to the requirement

$$0 = (h^{-2}\mu)' + (u_0 - h^{-2}\mu)' \cos \eta + w_0{}' \sin \eta \tag{3.138}$$

if the form of (3.131) is to be preserved. Then differentiating equation (3.131) and substituting for u and u'' in the governing equation (3.118) produces

$$-(u_0 - h^{-2}\mu)' \sin \eta + w_0{}' \cos \eta = -\frac{1}{h^2 u^3}\left[(\underline{P} \cdot \underline{\xi})u + (\underline{P} \cdot \underline{\xi}')u'\right] \tag{3.139}$$

Inverting equations (3.138) and (3.139) produces

$$(u_0 - h^{-2}\mu)' = -(h^{-2}\mu)' \cos \eta + \frac{1}{h^2 u^3}\left[(\underline{P} \cdot \underline{\xi})u + (\underline{P} \cdot \underline{\xi}')u'\right] \sin \eta \tag{3.140}$$

$$w_0' = -(h^{-2}\mu)' \sin\eta - \frac{1}{h^2u^3}\left[(\underline{P}\cdot\underline{\xi})u + (\underline{P}\cdot\underline{\xi}')u'\right]\cos\eta \tag{3.141}$$

or

$$u_0' = (h^{-2}\mu)'(1-\cos\eta) + \frac{1}{h^2u^3}\left[(\underline{P}\cdot\underline{\xi})u + (\underline{P}\cdot\underline{\xi}')u'\right]\sin\eta \tag{3.142}$$

$$w_0' = -(h^{-2}\mu)' \sin\eta - \frac{1}{h^2u^3}\left[(\underline{P}\cdot\underline{\xi})u + (\underline{P}\cdot\underline{\xi}')u'\right]\cos\eta \tag{3.143}$$

Of course, it remains to replace $(h^2)'$ wherever it occurs by means of equation (3.119) and then to replace $\underline{\xi}$, u, $\underline{\xi}'$ and u' in terms of elements and η. The results of that procedure, promising at first sight to be lengthy, are surprisingly short:

$$\begin{aligned} u_0' &= \left[u_0\sin\eta + \frac{\mu}{h^2}(1-\cos\eta)\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0 \\ &+\left[w_0\sin\eta + \frac{\mu}{h^2}(1-\cos\eta)^2\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0 \end{aligned} \tag{3.144}$$

$$\begin{aligned} w_0' &= \left[-u_0\cos\eta - \frac{\mu}{h^2}(1-\cos\eta+\sin^2\eta)\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0 \\ &+\left[-w_0\sin\eta + \frac{\mu}{h^2}\sin\eta\cos\eta\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0 \end{aligned} \tag{3.145}$$

(Recall that the corresponding results in the θ domain were quite lengthy and were valid only for elliptical orbits. See equations (2.135), (2.136), (2.144) and (2.145) in the last chapter.) Finally, the angular momentum equation (3.119) is easily put in terms of elements and η by substituting for $\underline{\xi}'$ from equation (3.108):

$$(h^2)' = 2h^2\left[-\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0\,\sin\eta + \frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0\,\cos\eta\right] \tag{3.146}$$

In all these equations the dimensionless quantity $\frac{1}{h^2u^3}\underset{\sim}{P}$ has been retained deliberately. The reason is that most expressions of geophysical perturbations $\underset{\sim}{P}$ are factored by terms of positive powers greater than 3 of $u = r^{-1}$. Hence a useful cancellation of factors of u can be made in these cases. It is also worth noting that attraction by a remote third body involves terms of negative powers of u, that is, positive powers of r. Therefore, typical geophysical perturbations can be expressed as finite Fourier series in the true anomaly η, while perturbations due to a remote third body can be expressed as finite Fourier series in the eccentric anomaly θ. Of course, this fact means only that the proper choice of independent variable in a particular problem can prevent needless complications and approximations, not that Fourier series are an especially desirable form to be handled. In the J_2-perturbed problems examined later, terms of powers and products of $\sin\eta$ and $\cos\eta$ prove to be no more cumbersome than Fourier terms to manage analytically and the results are more efficient computationally.

Summary

In Chapter 2 it was shown that if linear regularization of the equations of unperturbed motion is to be accomplished by means of the generalized Sundman time transformation $dt = Cr^n ds$, using also the integrals of Keplerian motion as extra dependent variables, then a true-anomaly regularization ($n = 2$) necessarily requires a transformation to non-Cartesian coordinates. Unfortunately, that conclusion does not indicate what type of coordinates should be used when true anomaly is taken as the independent variable. Furthermore, there is not as yet any general theory for choosing coordinates in problems of this type. Motivated by the work of Burdet (1969) and Szebehely and Bond (1983), this chapter examines the use of the coordinates $(u, \underset{\sim}{\xi})$, where $\underset{\sim}{\xi}$ is the unit radial vector and u is related to the radius r by a power law: $r = u^m$, $m \neq 0$. The principal result is that the values $n = 2$ and $m = -1$ are necessary to produce linear governing equations which are also uncoupled in the components of $\underset{\sim}{\xi}$. Furthermore, these linear equations are necessarily regular.

A closely related result was obtained by Bond (1985) at the same time that the present analysis was being done. Bond used a more general time transformation $dt = g(r)ds$ but a particular set of coordinates, namely, the radius, latitude and longitude. He showed that linear oscillator-type equations of motion are obtained only when $g(r) = r^2$, that is, when the independent variable is the true anomaly. In Bond's analysis, this result depends on the spherical coordinates themselves (together with an appropriate choice of coordinate transformations) and is not cast in vector form. The result presented here is less general with regard to the time transformation but does not depend on the particular choice of components of $\underline{\xi}$. In assessing these results, it should be borne in mind that the term "regularization" is used in two different senses according as $n = 1$ or $n \geq \frac{3}{2}$. The nature of the generalized Sundman time transformation is such that the actual collision of bodies is not described for $n \geq \frac{3}{2}$ even though the governing equations show no singularity in a finite interval of the independent variable.

An additional result, thought to be new, is that Burdet's (1969) tenth-order equations can be reduced to ninth-order without sacrificing regularization or linearization of the equations of unperturbed motion. In the ninth-order equations the explicit appearance of the integrals of unperturbed motion as extra dependent variables has been eliminated; only the coordinates $(u, \underline{\xi})$ and their derivatives appear.

Finally, a regular variation-of-parameters formulation is developed which is valid for all non-rectilinear orbits. The manipulations are based on linear governing equations and are thus quite straightforward. The resulting element equations are universally valid, unlike the analogous eccentric-anomaly results which were valid only for elliptic orbits. Importantly, the true-anomaly formulation permits each of the zonal terms in the geopotential function to be written in simple finite terms of the orbital elements. While this observation is not new, the fact does take on added significance when the elements to be used are also valid for orbits of all types and orientations. It makes possible the development of computational programs which are very concise as well as completely general. Analyses of the J_2-perturbed motion in terms of regular elements in the true-anomaly domain will be pursued in later chapters.

Chapter 4. Regularization Using KS-Type Coordinates

Introduction

The last chapter examined the consequences of introducing a particular set of coordinates into the time-transformed equations of motion. It was shown how the introduction of a scalar radial coordinate and a three-component unit radial vector permits a linear regular description of Keplerian motion different from that which appears when only a time transformation is used. Recall that, without a coordinate transformation, a time transformation based on $n = 1$ leads to linear governing equations having the eccentric anomaly (s or θ) as the independent variable. With the above-mentioned coordinate transformation, a time transformation based on $n = 2$ leads to linear governing equations having the true anomaly (σ or η) as the independent variable. The fact that these coordinates are redundant, requiring four scalars to represent three Cartesian coordinates, proved to be important, both theoretically, in view of Stiefel's (1971) result about redundant variables in regularization, and practically, in view of the algebraic steps required to derive the transformed governing equations.

In this chapter the introduction of another type of four-component redundant coordinates will be considered. Because of its importance in present-day applications, the set originally proposed by Kustaanheimo (1964) and later by Kustaanheimo and Stiefel (1965) will be of primary concern. These authors represent the three physical coordinates by four co-equal scalars u_j (not to be confused with the radial coordinate u in the last chapter) as follows. Let

$$\underline{r} = (x_1,\ x_2,\ x_3)^T \tag{4.1}$$

where superscript T signifies "matrix transposed". (In this chapter, vectors will be construed as column vectors.) Then the new coordinates are related to $\underline{r}$ by

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 \tag{4.2}$$

$$x_2 = 2u_1u_2 - 2u_3u_4 \tag{4.3}$$

$$x_3 = 2u_1u_3 + 2u_2u_4 \tag{4.4}$$

This is the so-called KS coordinate transformation, application of which has received its fullest exposition to date by Stiefel and Scheifele (1971). A concise notation results by considering the u_j as components of a 4-vector:

$$\underline{u} = (u_1,\ u_2,\ u_3,\ u_4)^T \tag{4.5}$$

In order to complete the four-dimensional formalism, 3-vectors in the physical space will, by convention, be augmented with a zero fourth component. For example,

$$\underline{r} = (x_1,\ x_2,\ x_3,\ 0)^T \tag{4.6}$$

Then the KS transformation can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{4.7}$$

and concisely as

$$\underline{r} = L(\underline{u})\, \underline{u} \tag{4.8}$$

Henceforth *in this chapter only*, the underbar symbol will denote 4-vectors. Matrices will be denoted by upper-case letters and defined in context as needed. Matrix multiplication will be indicated by simple juxtaposition, and the scalar product of two vectors will be indicated by the same dot symbol used with 3-vectors:

$$\underline{a} \cdot \underline{b} \equiv \underline{a}^T \underline{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \tag{4.9}$$

In this chapter occasion will not arise to define a 4-vector product analogous to the 3-vector "cross" product. With these conventions and notations established, the use of the KS transformation amounts to discovering and exploiting the properties of the matrix $L(\underline{u})$. It must be admitted that this notation, by its very conciseness, perhaps obscures more than it reveals. The properties of $L(\underline{u})$ depend on the precise form of the matrix found in equation (4.7), and the implications of this form are not very well conveyed merely by "$L(\underline{u})$". The essential steps required to make use of the KS transformation now have to be developed by carrying out many matrix operations in expanded scalar form. Thus the notation $L(\underline{u})$ does not reveal how several of the steps could have been discovered in the first place, but at least the final results of the manipulations can be put back into concise form using vectors and matrices. To make a formal discussion, one first builds up a small set of lemmas about $L(\underline{u})$ to be used in justifying the algebraic steps needed later. A more penetrating exposition of the KS transformation could perhaps be made using Kustaanheimo's original (1964) conception in terms of spinors. Some recent attempts in this direction have been made by Hestenes (1983a, 1983b), Hestenes and Lounesto (1983), and by Stiefel (1976). However, since spinor algebra is still not widely known in the field of celestial mechanics, the usual practice in the literature has been to present the transformation in terms of vector and matrix operations. This practice was begun by Kustaanheimo and Stiefel (1965) and continued by Stiefel and Sheifele (1971) so that it is now irrevocably established. Even Stumpff's (1968) exposition, the title of which advertises spinors, soon reverts to matrices and vectors. The discussion of the KS transformation in this chapter is designed to parallel the development given in Chapter 2 of Stiefel and Scheifele (1971), although several new details appear in the more general case treated here.

There is another four-component representation of 3-vectors which has been used more recently in celestial mechanics. Broucke and co-authors (1971), Vitins (1978) and Junkins and Turner (1979) use Euler parameters to represent unit vectors. For example, if ξ_1, ξ_2 and ξ_3 are the direction-cosine components of the unit radial vector then the Euler-parameter representation would be

$$\xi_1 = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \tag{4.10}$$

$$\xi_2 = 2\lambda_2\lambda_4 + 2\lambda_1\lambda_3 \tag{4.11}$$

$$\xi_3 = 2\lambda_1\lambda_3 - 2\lambda_2\lambda_4 \tag{4.12}$$

If the Euler parameters λ_j are considered as a four-component vector, then, by augmenting the unit 3-vector with a zero fourth component, the following form emerges:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_2 & -\lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & \lambda_4 & \lambda_3 \\ \lambda_3 & -\lambda_4 & \lambda_1 & -\lambda_2 \\ \lambda_4 & \lambda_3 & -\lambda_2 & -\lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \tag{4.13}$$

This is written concisely as

$$\underset{\sim}{\xi} = \Lambda(\underset{\sim}{\lambda})\, \underset{\sim}{\lambda} \tag{4.14}$$

It happens that the matrices $L(\underline{u})$ and $\Lambda(\underline{\lambda})$ have many properties in common. Vitins (1978) listed several such properties and suggested (but did not pursue the idea) that a matrix of the form

$$M(\underline{\lambda}) = \begin{bmatrix} \lambda_1 & -\lambda_2 & -\lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & \mp\lambda_4 & \mp\lambda_3 \\ \lambda_3 & \pm\lambda_4 & \lambda_1 & \pm\lambda_2 \\ \lambda_4 & \mp\lambda_3 & \pm\lambda_2 & -\lambda_1 \end{bmatrix} \tag{4.15a}$$

in the transformation

$$\underline{\xi} = M(\underline{\lambda})\,\underline{\lambda} \tag{4.15b}$$

would serve to develop other linear regular equations analogous to the Euler-parameter equations that he presents. In this chapter it will be shown how the matrix operator M can serve to develop both the KS and the Euler-parameter forms of the equations of motion from a coordinate transformation of the form

$$\underline{r} = M(\underline{u})\,\underline{u} \tag{4.16}$$

In using this new matrix M, it is understood that the ambiguous signs are not to be resolved independently. Rather, one chooses the upper signs throughout, reproducing the form of the KS matrix $L(\underline{u})$, or else one chooses the lower signs throughout, reproducing the form of the Euler parameter matrix $\Lambda(\underline{\lambda})$. Thus the matrix $M(\underline{u})$ represents a complementary pair of coordinate transformations, one of the pair producing the usual KS coordinates and the other an as-yet-unnamed set having similar properties. Likewise, if the coordinates are suitably normalized then the matrix $M(\underline{u})$ represents a complementary pair of Euler-parameter transformations, one of the pair producing the usual Euler parameters and the other an as-yet-unnamed set having similar properties.

No essentially new properties of the coordinate-transformed equations of motion appear merely through the use of M in place of L or Λ. However, for the sake of generality, the matrix M will be so used and its properties recorded. The important aspect of this chapter is the use of these four-parameter coordinate transformations in conjunction with the generalized Sundman time transformation. Recall that time transformations corresponding to $n = 1$ result in linear regular governing equations when no coordinate transformation is used. Furthermore, regular but non-linear equations appear for $n \geq \frac{3}{2}$, and some of these equations have been found useful for numerical work. In this chapter the question is asked, what values of n permit regularization and possibly linearization of the governing equations in terms of KS coordinates and the complementary coordinates implied by the form of the M matrix? It is, of course, known that $n = 1$ leads to linear regular equations for the KS coordinates (Kustaanheimo, 1964; Kustaanheimo and Stiefel, 1965; Stiefel, *et al.*, 1967; Stumpff, 1968; Stiefel and Scheifele, 1971; Stiefel, 1973; Bond, 1973, 1974). The main interest here is whether regular equations appear for other values of n. Likewise, it is known that $n = 2$ leads to linear regular equations for the Euler parameters (Vitins, 1978; Junkins and Turner, 1979). It would be interesting to know also whether other values of n can lead to regular equations for these parameters. Before the coordinate-transformed governing equations are derived, however, some properties of the M matrix in the coordinate transformation (4.16) will be presented.

The Use of 4-Space Coordinates

A straightforward matrix multiplication will verify that

$$M^T(\underset{\sim}{a})M(\underset{\sim}{a}) = (\underset{\sim}{a} \cdot \underset{\sim}{a})\, I \tag{4.17}$$

where I is the 4×4 identity matrix and $\underset{\sim}{a}$ is any 4-vector. That is, the matrix M is orthogonal in the sense of (4.17) regardless of whether one chooses upper signs throughout or lower signs throughout.

A well-known matrix transpose identity can be written in the form

$$[M(\underline{a})\,\underline{b}] \cdot \underline{c} = \underline{b} \cdot [M^T(\underline{a})\,\underline{c}] \tag{4.18}$$

where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any 4-vectors. This relation in turn allows one to reason from the coordinate transformation (4.16) as follows

$$\underline{r} \cdot \underline{r} = [M(\underline{u})\,\underline{u}] \cdot [M(\underline{u})\,\underline{u}] \tag{4.19}$$

$$r^2 = \underline{u} \cdot [M^T(\underline{u})\,M(\underline{u})\,\underline{u}] \tag{4.20}$$

$$r^2 = \underline{u} \cdot [(\underline{u} \cdot \underline{u})\,I\,\underline{u}] \tag{4.21}$$

$$r^2 = (\underline{u} \cdot \underline{u})^2 \tag{4.22}$$

$$r = \underline{u} \cdot \underline{u} \tag{4.23}$$

Thus, incidentally, the radius can be computed during numerical work without a square-root operation.

Now in order to introduce the coordinate transformation (4.16) into the governing equations, expressions for $\underline{r}'$ and $\underline{r}''$ must be developed. Differentiate (4.16) to obtain

$$\underline{r}' = M'(\underline{u})\underline{u} + M(\underline{u})\underline{u}' \tag{4.24}$$

$$\underline{r}' = M(\underline{u}')\underline{u} + M(\underline{u})\underline{u}' \tag{4.25}$$

The latter step follows since the elements of the matrix $M(\underline{u})$ are all linear and homogeneous in the components of $\underline{u}$. Writing the first term of (4.25) in scalar form shows that it can be factored as

$$M(\underline{u}')\underline{u} = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & \mp u_4 & \mp u_3 \\ u_3 & \pm u_4 & u_1 & \pm u_2 \\ -u_4 & \pm u_3 & \mp u_2 & u_1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} \tag{4.26}$$

Observe that the matrix on the right is merely $M(\underline{u})$ with the signs in the fourth row reversed. This means that when the two terms of (4.25) are combined the fourth component of $\underline{r}'$ will vanish as it should. Observe also that were it not for the presence of the fourth row of $M(\underline{u})$ it would be possible to write

$$\underline{r}' = 2M(\underline{u})\underline{u}' \tag{4.27}$$

This relation will actually be true if the condition is imposed that its fourth component vanish identically; that is, if

$$l(\underline{u}, \underline{u}') = u_4 u_1' \mp u_3 u_2' \pm u_2 u_3' - u_1 u_4' \equiv 0 \tag{4.28}$$

In matrix form, this is

$$l(\underline{u}, \underline{u}') = \underline{u}'^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \underline{u} \tag{4.29}$$

$$l(\underline{u}, \underline{u}') = \underline{u}'^T J \, \underline{u} = 0 \tag{4.30}$$

(Here again, the upper signs refer to the KS coordinates and the lower signs refer to the complementary set of coordinates.) In the KS theory, relation (4.30) is called the "bilinear" relation. The whole purpose in introducing it is to allow $\underline{r}'$ to be computed by the supposedly more elegant expression (4.27). At this stage of the discussion it is not at all obvious that imposing the bilinear relation represents progress, or even that such a relation could be rigorously enforced by means of the equations of motion yet to be derived. That some kind of constraint must exist among the components of $\underline{u}$ is clear since four scalars are being used to represent three physical coordinates, but whether a constraint such as (4.30), involving the velocity, can be met remains to be seen. In the following discussion, it will be assumed that the bilinear relation (4.30) does hold true and the consequences will be derived. The final results (that is, the equations of motion) will be vindicated only after it has been verified somehow that the bilinear relation does indeed hold true in all necessary generality. Obviously, some care will have to be exercised in order to avoid faulty logic, but this approach proves to be successful (see, for example, Stiefel and Scheifele, 1971). It should be noted that Stiefel (1973) did use an alternate approach: he considered the u_j as generalized coordinates and derived their Lagrangian equations of motion, which, of course, turn out to have the same form as the earlier (1965, 1967, 1971) equations. The bilinear relation was included as a constraint in the Lagrangian function, with the interesting result that its Lagrange multiplier is identically zero as long as the radius is not zero. In other words, the bilinear relation is not an essential constraint after all as regards the final form of the equations of motion. It does figure prominently in the derivation of those equations, however.

Supposing, then, that equation (4.30) holds true, equate the two alternate forms of $\underline{r}'$ given in (4.25) and (4.27) to obtain

$$M(\underline{u}')\underline{u} + M(\underline{u})\underline{u}' = 2M(\underline{u})\underline{u}' \tag{4.31}$$

$$M(\underline{u}')\underline{u} = M(\underline{u})\underline{u}' \tag{4.32}$$

The KS form of this equation appears as Theorem 1 in Section 9 of Stiefel and Scheifele (1971). A related KS result given in the same place as their Theorem 2 can be quoted now in the nomenclature of this chapter as

$$(\underline{u} \cdot \underline{u})M(\underline{u}')\underline{u}' - 2(\underline{u} \cdot \underline{u}')M(\underline{u})\underline{u}' + (\underline{u}' \cdot \underline{u}')M(\underline{u})\underline{u} = \underline{0} \tag{4.33}$$

An equivalent expression is obtained by premultiplying by $M^T(\underline{u})$ and dividing by $(\underline{u} \cdot \underline{u})$.

$$\left[M^T(\underline{u})M(\underline{u}') - 2(\underline{u} \cdot \underline{u}')\, I\right]\underline{u}' = -(\underline{u}' \cdot \underline{u}')\underline{u} \tag{4.34}$$

Stiefel and Scheifele verify this result merely by writing out each component in scalar form. However, a simple proof based on the property (4.32) can be offered. First examine the matrix product

$$M^T(\underline{u})M(\underline{u}') = \begin{bmatrix} (\underline{u} \cdot \underline{u}') & a & b & -l(\underline{u}, \underline{u}') \\ -a & (\underline{u} \cdot \underline{u}') & \pm\, l(\underline{u}, \underline{u}') & \pm\, b \\ -b & \mp l(\underline{u}, \underline{u}') & (\underline{u} \cdot \underline{u}') & \mp a \\ l(\underline{u}, \underline{u}') & \mp b & \pm\, a & (\underline{u} \cdot \underline{u}') \end{bmatrix} \tag{4.35}$$

where

$$a = -u_1 u_2' + u_2 u_1' \pm u_3 u_4' \mp u_4 u_3' \tag{4.36}$$

$$b = -u_1 u_3' \mp u_2 u_4' + u_3 u_1' \pm u_4 u_2' \tag{4.37}$$

Except for the non-zero main diagonal, the matrix $M^T(\underline{u})M(\underline{u}')$ would be skew-symmetric. Therefore, the following matrix is in fact skew-symmetric even regardless of whether the bilinear relation holds:

$$\left[M^T(\underline{u})M(\underline{u}') - (\underline{u} \cdot \underline{u}')\, I\right] = -\left[M^T(\underline{u})M(\underline{u}') - (\underline{u} \cdot \underline{u}')\, I\right]^T \tag{4.38}$$

An alternate (and more elegant) demonstration of this fact is to differentiate the identity $M^T(\underline{u})M(\underline{u}) - (\underline{u} \cdot \underline{u})I = 0$. In either case, one may continue with

$$\left[M^T(\underline{u})M(\underline{u}') - (\underline{u} \cdot \underline{u}')\, I\right] = -M^T(\underline{u}')M(\underline{u}) + (\underline{u} \cdot \underline{u}')\, I \tag{4.39}$$

Then

$$M^T(\underline{u})M(\underline{u}') - 2(\underline{u} \cdot \underline{u}')\, I = -\, M^T(\underline{u}')M(\underline{u}) \tag{4.40}$$

Postmultiply both sides by $\underline{u}'$ to obtain

$$\left[M^T(\underline{u})M(\underline{u}') - 2(\underline{u} \cdot \underline{u}')\, I\right]\underline{u}' = -\, M^T(\underline{u}')M(\underline{u})\underline{u}' \tag{4.41}$$

The left side now has the form of equation (4.34), and according to (4.32) the right side can be re-written as

$$\left[M^T(\underline{u})M(\underline{u}') - 2(\underline{u} \cdot \underline{u}')\, I\right]\underline{u}' = -\, M^T(\underline{u}')M(\underline{u}')\underline{u} \tag{4.42}$$

This step depends on the bilinear relation (4.30) holding true. Then the orthogonality of M insures that the right side can be further reduced:

$$\left[M^T(\underline{u})M(\underline{u}') - 2(\underline{u} \cdot \underline{u}')\, I\right]\underline{u}' = -\, (\underline{u}' \cdot \underline{u}')\underline{u} \tag{4.43}$$

Thus equation (4.34) is verified.

Finally, if $\underline{r}'$ is to be calculated according to equation (4.27) then $\underline{r}''$ will be

$$\underline{r}'' = \left[2M(\underline{u})\underline{u}'\right]' \tag{4.44}$$

$$\underline{r}'' = 2M(\underline{u}')\underline{u}' + 2M(\underline{u})\underline{u}'' \tag{4.45}$$

From Chapter 2, the time-transformed equations of motion are

$$\underline{r}'' - 2EC^2 n r^{2n-2}\underline{r} - (n-1)C^2 \mu r^{2n-3}\underline{r} - nC^2 r^{2n-2}\underline{B} + C^2 r^{2n}\underline{P} + C^{-1}C'\underline{r}' \tag{4.46}$$

$$\underline{B}' = 2\underline{r}(\underline{P} \cdot \underline{r}') - \underline{r}'(\underline{P} \cdot \underline{r}) - \underline{P}(\underline{r} \cdot \underline{r}') \tag{4.47}$$

$$E' = \underline{P} \cdot \underline{r}' \tag{4.48}$$

$$t' = Cr^n \tag{4.49}$$

$$r'' - 2EC^2 nr^{2n-1} = (2n-1)C^2 \mu r^{2n-2} + (1-n)C^2 h^2 r^{2n-3} + C^2 r^{2n-1}(\underline{P} \cdot \underline{r}) + C^{-1}C' r' \tag{4.50}$$

$$(h^2)' = 2(\underline{r} \cdot \underline{r})(\underline{P} \cdot \underline{r}') - 2(\underline{r} \cdot \underline{r}')(\underline{P} \cdot \underline{r}) = \underline{r} \cdot \underline{B}' \tag{4.51}$$

In these equations all vectors have become 4-vectors with zero fourth components. Now the $\underline{u}$ coordinates are to be introduced into these equations with the object of determining which values of n allow regularization and possibly linearization of the new equations. Substituting $\underline{r} = M(\underline{u})\underline{u}$ from (4.16) and $\underline{r}' = 2M(\underline{u})\underline{u}'$ from (4.27) into the position equation (4.46) produces

$$[2M(\underline{u})\underline{u}']' - 2EC^2 nr^{2n-2}M(\underline{u})\underline{u} - (n-1)C^2 \mu r^{2n-3}M(\underline{u})\underline{u}$$

$$= -nC^2 r^{2n-2}\underline{B} + C^2 r^{2n}\underline{P} + 2C^{-1}C' M(\underline{u})\underline{u}' \tag{4.52}$$

In this equation, r, wherever it may occur, can be computed as $r = \underline{u} \cdot \underline{u}$ from (4.23). Then

$$2M(\underline{u}')\underline{u}' + 2M(\underline{u})\,\underline{u}'' - 2EC^2 nr^{2n-2}M(\underline{u})\underline{u} - (n-1)C^2 \mu r^{2n-3}M(\underline{u})\underline{u}$$

$$= -nC^2 r^{2n-2}\underline{B} + C^2 r^{2n}\underline{P} + 2C^{-1}C' M(\underline{u})\underline{u}' \tag{4.53}$$

Premultiply by $\frac{1}{2}M^T(\underline{u})$ in order to isolate $\underline{u}''$.

$$M^T(\underline{u})M(\underline{u}')\underline{u}' + M^T(\underline{u})M(\underline{u})\underline{u}'' - EC^2 nr^{2n-2}M^T(\underline{u})M(\underline{u})\underline{u}$$

$$-\frac{1}{2}(n-1)C^2 \mu r^{2n-3}M^T(\underline{u})M(\underline{u})\underline{u}$$

$$-\frac{1}{2}nC^2 r^{2n-2}M^T(\underline{u})\underline{B} + \frac{1}{2}C^2 r^{2n}M^T(\underline{u})\underline{P} + C^{-1}C' M^T(\underline{u})M(\underline{u})\underline{u}' \tag{4.54}$$

Since $M(\underline{u})$ is orthogonal in the sense of (4.17) and $r = \underline{u} \cdot \underline{u}$ according to (4.23), the matrix products can be reduced as $M^T(\underline{u})M(\underline{u}) = r\,I$ to give

$$M^T(\underline{u})M(\underline{u}')\underline{u}' + r\underline{u}'' - EC^2 nr^{2n-2}r\underline{u} - \frac{1}{2}(n-1)C^2 \mu r^{2n-3}r\underline{u}$$

$$= -\frac{1}{2}nC^2r^{2n-2}M^T(\underline{u})\underline{B} + \frac{1}{2}C^2r^{2n}M^T(\underline{u})\underline{P} + C^{-1}C'r\underline{u}' \tag{4.55}$$

Then division by r produces

$$r^{-1}M^T(\underline{u})M(\underline{u}')\underline{u}' + \underline{u}'' - EC^2nr^{2n-2}\underline{u} - \frac{1}{2}(n-1)C^2\mu r^{2n-3}\underline{u}$$

$$= -\frac{1}{2}nC^2r^{2n-3}M^T(\underline{u})\underline{B} + \frac{1}{2}C^2r^{2n-1}M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \tag{4.56}$$

As it stands, this equation cannot be made regular because there is no choice of n which will remove the negative power of r in the first term. The only possibility for further progress with the $\underline{u}$ coordinates is to eliminate E or $\underline{B}$ (or both) algebraically in an attempt to cancel the troublesome first term with a similar term containing r^{-1}. To this end, rewrite the time-transformed definitions of E and $\underline{B}$ in terms of the $\underline{u}$ coordinates. From equation (2.8) of Chapter 2 obtain

$$E = \frac{1}{2}C^{-2}r^{-2n}(\underline{r}' \cdot \underline{r}') - \mu r^{-1} \tag{4.57}$$

$$E = \frac{1}{2}C^{-2}r^{-2n}[(2M(\underline{u})\underline{u}') \cdot (2M(\underline{u})\underline{u}')] - \mu r^{-1} \tag{4.58}$$

This reduces to

$$E = 2C^{-2}r^{-2n+1}(\underline{u}' \cdot \underline{u}') - \mu r^{-1} \tag{4.59}$$

From equation (2.14) obtain

$$\underline{B} = C^{-2}r^{-2n}[\underline{r}(\underline{r}' \cdot \underline{r}') - \underline{r}'(\underline{r} \cdot \underline{r}')] - \mu r^{-1}\underline{r} \tag{4.60}$$

$$\underline{B} = C^{-2}r^{-2n}[M(\underline{u})\underline{u}[(2M(\underline{u})\underline{u}') \cdot (2M(\underline{u})\underline{u}')] - 2M(\underline{u})\underline{u}'[(M(\underline{u})\underline{u}) \cdot (2M(\underline{u})\underline{u}')]]$$

$$- \mu r^{-1}M(\underline{u})\underline{u} \tag{4.61}$$

This reduces to

$$\underline{B} = 4C^{-2}r^{-2n+1}M(\underline{u})[\underline{u}(\underline{u}' \cdot \underline{u}') - \underline{u}'(\underline{u} \cdot \underline{u}')] - \mu r^{-1}M(\underline{u})\underline{u} \tag{4.62}$$

Actually, since $\underline{B}$ occurs in equation (4.56) only in the combination $M^T(\underline{u})\underline{B}$ it is just as convenient to form

$$M^T(\underline{u})\underline{B} = 4C^{-2}r^{-2n+2}[\underline{u}(\underline{u}' \cdot \underline{u}') - \underline{u}'(\underline{u} \cdot \underline{u}')] - \mu\underline{u} \tag{4.63}$$

Now note that in equations (4.59) and (4.63) the matrix factors of M have disappeared entirely from the right-hand sides. If the leading term of equation (4.56) is ever to be cancelled using these expressions then it must be rewritten so that it too has no factors of M. The algebraic device for accomplishing this is to use the relation (4.34) in the form

$$M^T(\underline{u})M(\underline{u}')\underline{u}' = 2(\underline{u} \cdot \underline{u}')\underline{u}' - (\underline{u}' \cdot \underline{u}')\underline{u} \tag{4.64}$$

so that

$$r^{-1}M^T(\underline{u})M(\underline{u}')\underline{u}' = 2r^{-1}(\underline{u} \cdot \underline{u}')\,\underline{u}' - r^{-1}(\underline{u}' \cdot \underline{u}')\,\underline{u} \tag{4.65}$$

Then the equation of motion (4.56) can be written as

$$\begin{aligned} &2r^{-1}(\underline{u} \cdot \underline{u}')\underline{u}' - r^{-1}(\underline{u}' \cdot \underline{u}')\underline{u} + \underline{u}'' - EC^2nr^{2n-2}\underline{u} - \frac{1}{2}(n-1)C^2\mu r^{2n-3}\underline{u} \\ &= -\frac{1}{2}nC^2r^{2n-3}M^T(\underline{u})\underline{B} + \frac{1}{2}C^2r^{2n-1}M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \end{aligned} \tag{4.66}$$

Before proceeding with this equation, observe that both leading terms of this equation must be cancelled in order for regularization to be possible. According to (4.59), $E\underline{u}$ contains a factor of $(\underline{u}' \cdot \underline{u}')\underline{u}$ but not of $(\underline{u} \cdot \underline{u}')\underline{u}'$. On the other hand, according to equation (4.63), $M^T(\underline{u})\underline{B}$ contains both of the quantities $(\underline{u}' \cdot \underline{u}')\underline{u}$ and $(\underline{u} \cdot \underline{u}')\underline{u}'$. Hence, regularization of equation (4.66) necessarily involves the elimination of $M^T(\underline{u})\underline{B}$ via (4.63), whatever is done with the quantity E. Initially, proceed by eliminating $M^T(\underline{u})\underline{B}$, keeping E as it stands.

$$2r^{-1}(\underline{u}\cdot\underline{u}')\underline{u}' - r^{-1}(\underline{u}'\cdot\underline{u}')\underline{u} + \underline{u}'' - EC^2nr^{2n-2}\underline{u} - \frac{1}{2}(n-1)C^2\mu r^{2n-3}\underline{u}$$

$$= -\frac{1}{2}nC^2r^{2n-3}\left[4C^{-2}r^{-2n+2}[\underline{u}(\underline{u}'\cdot\underline{u}') - \underline{u}'(\underline{u}\cdot\underline{u}')] - \mu\underline{u}\right]$$

$$+\frac{1}{2}C^2r^{2n-1}M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \tag{4.67}$$

This equation reduces to

$$(2-2n)r^{-1}(\underline{u}\cdot\underline{u}')\underline{u}' + (2n-1)r^{-1}(\underline{u}'\cdot\underline{u}')\underline{u} + \underline{u}''$$

$$-EC^2nr^{2n-2}\underline{u} - \frac{1}{2}(2n-1)C^2\mu r^{2n-3}\underline{u}$$

$$= \frac{1}{2}C^2r^{2n-1}M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \tag{4.68}$$

Evidently, there is still no choice of n which will cancel both of the first two terms. The possibility remains, however, of also eliminating E via equation (4.59). The result of that operation is

$$(2-2n)r^{-1}(\underline{u}\cdot\underline{u}')\underline{u}' + \underline{u}'' - \frac{1}{2}C^2\left[2C^{-2}r^{-1}(\underline{u}'\cdot\underline{u}') - \mu r^{2n-3}\right]\underline{u}$$

$$= \frac{1}{2}C^2r^{2n-1}M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \tag{4.69}$$

Here the choice $n = 1$ is necessary to eliminate the first term. Although the coefficient of $\underline{u}$ in the third term also contains r^{-1} regardless of the value of n, closer inspection reveals that in case, and only in case, $n = 1$, the expression in square brackets multiplying $\underline{u}$ is precisely the energy E according to equation (4.59) above. This result permits the regular coordinate-transformed equations to be summarized immediately as

$$\underline{u}'' - \frac{1}{2}C^2E\underline{u} = \frac{1}{2}C^2rM^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}' \tag{4.70}$$

$$E' = \underline{P} \cdot \underline{r}' = \underline{P} \cdot 2M(\underline{u})\underline{u}' = 2M^T(\underline{u})\underline{P} \cdot \underline{u}' \tag{4.71}$$

$$t' = Cr = C\,\underline{u} \cdot \underline{u} \tag{4.72}$$

$$r'' - 2EC^2 r = C^2\mu + C^2 r(M^T(\underline{u})\underline{P} \cdot \underline{u}) + C^{-1}C'r' \tag{4.73}$$

The integration for $\underline{B}$ is superfluous since $\underline{B}$ appears nowhere else in the system, and the equation of the radius (4.73) is optional, at least numerically, though its use simplifies the integration for the time. The rest of the system is of order ten, one lower than the Sperling-type regularization in spite of the use of 4-vectors. It is especially noteworthy that for the $\underline{u}$ coordinates the only value of n which produces regular equations, namely $n = 1$, also produces linear equations of unperturbed motion.

The development of initial conditions for the above set of equations requires some comment. While it is easy to establish initial conditions for E in terms of $\underline{r}(0)$ and $\dot{\underline{r}}(0)$, the initial conditions for $\underline{u}$ and $\underline{u}'$ are not self-evident. The original coordinate transformation (4.16) does not have a unique inverse since three physical coordinates are being represented by a set of four new coordinates u_j. Hence, the u_j are not all independent. Nevertheless, consistent values can always be assigned for $\underline{u}(0)$ in terms of $\underline{r}(0)$ and this can be done regardless of whether the bilinear relation holds true. The following procedure is essentially that given by Stiefel and Scheifele (1971) for the KS coordinates, but shown here with the choices of sign implied by the matrix operator M. From equations (4.16) and (4.23) obtain the scalar formulae

$$r = u_1^2 + u_2^2 + u_3^2 + u_4^2 \tag{4.74}$$

$$x_1 = u_1^2 - u_2^2 - u_3^3 + u_4^2 \tag{4.75}$$

$$x_2 = 2u_1u_2 \mp 2u_3u_4 \tag{4.76}$$

$$x_3 = 2u_1u_3 \pm 2u_2u_4 \tag{4.77}$$

which are now to be considered at the initial instant. If $x_1 \geq 0$, add (4.74) and (4.75) to get

$$r + x_1 = 2u_1^2 + 2u_4^2 \tag{4.78}$$

$$u_1^2 + u_4^2 = \frac{1}{2}(r + x_1) \tag{4.79}$$

Choose any values of u_1 and u_4 which satisfy this equation; for example, choose

$$u_1 = \sqrt{\frac{(r + x_1)}{2}} \quad \text{and} \quad u_4 = 0 \tag{4.80}$$

The freedom of choice here merely reflects the redundancy of the $\underline{u}$ coordinates. Then with u_1 and u_4 specified, solve equations (4.76) and (4.77) simultaneously for u_2 and u_3:

$$2\begin{bmatrix} u_1 & \mp u_4 \\ \pm u_4 & u_1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.81}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{2(u_1^2 + u_4^2)}\begin{bmatrix} u_1 & \pm u_4 \\ \mp u_4 & u_1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.82}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{(r + x_1)}\begin{bmatrix} u_1 & \pm u_4 \\ \mp u_4 & u_1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.83}$$

Actually, this scheme is valid as long as $x_1 \neq -r$. The restriction $x_1 \geq 0$ is designed to avoid the possible zero denominator but is not otherwise mandatory.

Similarly, if $x_1 < 0$, subtract (4.75) from (4.74) to get

$$r - x_1 = 2u_2^2 + 2u_3^2 \tag{4.84}$$

$$u_2^2 + u_3^2 = \frac{1}{2}(r - x_1) \tag{4.85}$$

Choose any values of u_2 and u_3 which satisfy this equation, such as

$$u_2 = \sqrt{\frac{(r - x_1)}{2}} \quad \text{and} \quad u_3 = 0 \tag{4.86}$$

With u_2 and u_3 specified, solve equations (4.76) and (4.77) simultaneously for u_1 and u_4:

$$2\begin{bmatrix} u_2 & \mp u_3 \\ u_3 & \pm u_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.87}$$

$$\begin{bmatrix} u_1 \\ u_4 \end{bmatrix} = \frac{1}{\pm 2(u_2^2 + u_3^2)}\begin{bmatrix} \pm u_2 & \pm u_3 \\ -u_3 & u_2 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.88}$$

$$\begin{bmatrix} u_1 \\ u_4 \end{bmatrix} = \frac{1}{(r - x_1)}\begin{bmatrix} u_2 & u_3 \\ \mp u_3 & \pm u_2 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{4.89}$$

(In all these equations, the upper signs correspond to the KS coordinates and the lower signs correspond to the complementary set of coordinates.) Therefore, even though there is no unique expression for $\underline{u}(0)$ in terms of $\underline{x}(0)$, consistent values for $\underline{u}(0)$ in terms of $\underline{x}(0)$ can always be assigned.

Now in order to establish the initial condition for $\underline{u}'$ consider equation (4.27) evaluated initially:

$$\underline{x}'(0) = 2M(\underline{u}(0))\underline{u}'(0) \tag{4.90}$$

As it stands, this equation depends on the bilinear relation holding true (at least initially), which has yet to be shown. However, since the problem is to assign the value of $\underline{u}'(0)$, not of $\underline{x}'(0)$, invert this equation to obtain

$$\underline{u}'(0) = \frac{1}{2}r(0)^{-1}M^T(\underline{u}(0))\underline{x}'(0) \tag{4.91}$$

In terms of the physical initial velocity vector $\dot{\underline{x}}(0)$, this would be

$$\underline{u}'(0) = \frac{1}{2} r(0)^{n-1} M^T(\underline{u}(0))\dot{\underline{r}}(0) \tag{4.92}$$

where $n = 1$ in all regular cases. Neither equation (4.91) nor (4.92) depends on the bilinear relation being true; rather, they serve to define $\underline{u}'(0)$ in a manner consistent with that relation once $\underline{u}(0)$ has been specified using equations (4.78) through (4.89).

It remains to show that the bilinear relation (4.30) holds true in general. The proof for the KS coordinates and $n = 1$ can be found in Stiefel and Scheifele (1971, section 9). The proof for both sets of coordinates and a general value of n is given here, supposing that $\underline{u}$ satisfies the differential equation (4.69) and also that $\underline{u}'$ is initialized according to equation (4.91) above. First, note that, for any 4-vector $\underline{a}$,

$$l(\underline{a}, \underline{a}) = \underline{a}^T J \underline{a} \equiv 0 \tag{4.93}$$

because of the form of J. Now differentiation of the bilinear relation (4.30) produces

$$l'(\underline{u}, \underline{u}') = (\underline{u}'^T J \underline{u})' = \underline{u}''^T J \underline{u} + \underline{u}'^T J \underline{u}' \tag{4.94}$$

The last term vanishes identically according to (4.93), leaving

$$l' = \underline{u}''^T J \underline{u} \tag{4.95}$$

Substituting for $\underline{u}''$ from equation (4.69) produces

$$l' = \Big[\frac{1}{2}C^2\big[2C^{-2}r^{-1}(\underline{u}'\cdot\underline{u}') - \mu r^{2n-3}\big]\underline{u} + 2(n-1)r^{-1}(\underline{u}\cdot\underline{u}')\underline{u}'$$

$$\frac{1}{2}C^2 r^{2n-1} M^T(\underline{u})\underline{P} + C^{-1}C'\underline{u}'\Big]^T J \underline{u} \tag{4.96}$$

which reduces straightforwardly to

$$l' = \big[2(n-1)r^{-1}(\underline{u}\cdot\underline{u}') + C^{-1}C'\big]\ l + \frac{1}{2}C^2 r^{2n-1}\underline{P}^T M(\underline{u}) J \underline{u} \tag{4.97}$$

Now by carrying out in scalar form the matrix product in the second term, it will be noticed that the first three components of $M(\underline{u})J\underline{u}$ vanish identically. Then, since the fourth component of $\underline{P}$ is zero by convention, the entire second term always vanishes identically. Therefore the bilinear quantity $l(\underline{u}, \underline{u}')$ obeys the differential equation

$$l' = \left[2(n-1)r^{-1}(\underline{u} \cdot \underline{u}') + C^{-1}C'\right]\ l \tag{4.98}$$

The initial condition for l is simply

$$l(0) = \underline{u}'(0)^T J\,\underline{u}(0) \tag{4.99}$$

Provided that $\underline{u}'(0)$ is to be calculated by (4.91), one can write

$$l(0) = \left[\frac{1}{2}r(0)^{-1}M^T(\underline{u}(0))\underline{r}'(0)\right]^T J\,\underline{u}(0) \tag{4.100}$$

$$l(0) = \frac{1}{2}r(0)^{-1}\underline{r}'(0)^T M(\underline{u}(0))J\,\underline{u}(0) \tag{4.101}$$

Here the same type of matrix product discussed above appears again. The fourth component of $\underline{r}'(0)$ is zero by convention, so that

$$l(0) \equiv 0 \tag{4.102}$$

Therefore the only solution of the differential equation (4.98) will be the trivial solution, and the bilinear relation holds in general.

The Use of Normalized Parameters

Earlier in this chapter it was noted that several authors (Vitins, 1978; Junkins and Turner, 1979) have used Euler parameters to represent unit vectors in the governing differential equations, with the result that linear regular equations of unperturbed motion appear when true anomaly is used as the independent variable, i.e., when $n = 2$. It was also mentioned that a suitable normalization of the $\underline{u}$ coordinates in the transformation (4.16) would introduce the Euler parameters in terms of the same matrix operator M used heretofore. These topics are now addressed. The approach will be to introduce the Euler parameters into the already coordinate-transformed governing equations in Chapter 3 with the object of determining whether the regularizing and linearizing choices for m and n are modified in any way.

The Euler parameters can be obtained from the coordinate transformation (4.16) as follows. Factor the position vector $\underline{r}$ as

$$\underline{r} = r\,\underline{\xi} = r\frac{1}{r}M(\underline{u})\,\underline{u} \tag{4.103}$$

where $\underline{\xi}$ is the unit radial vector augmented with a zero fourth component. Then, since M is linear and homogeneous in the components of $\underline{u}$, it is permissible to write

$$r\,\underline{\xi} = r\,M\!\left(\frac{1}{\sqrt{r}}\underline{u}\right)\frac{1}{\sqrt{r}}\underline{u} \tag{4.104}$$

Now divide both sides by r and denote

$$\underline{\lambda} = \frac{1}{\sqrt{r}}\underline{u} \tag{4.105}$$

so that

$$\underline{\xi} = M(\underline{\lambda})\,\underline{\lambda} \tag{4.106}$$

The normalization of $\underline{\lambda}$ is now evident if one recalls from equation (4.23) that $r = \underline{u}\cdot\underline{u}$, giving in this case

$$1 = \frac{1}{\sqrt{r}}\underline{u}\cdot\frac{1}{\sqrt{r}}\underline{u} = \underline{\lambda}\cdot\underline{\lambda} \tag{4.107}$$

Of course, differentiation of this equation leads to the corollary relation

$$\underline{\lambda} \cdot \underline{\lambda}' = 0 \tag{4.108}$$

Thus the vector $\underline{\lambda}$ is the four-component analog of the more familiar three-component unit radial vector. That the components of $\underline{\lambda}$ are actually the Euler parameters is clear from the form of M given in (4.15). When the lower signs of M are selected throughout, the Euler parameter matrix $\Lambda(\underline{\lambda})$ is obtained. When the upper signs are selected throughout, the components of $\underline{\lambda}$ must be interpreted as a different set of normalized parameters having practically identical properties.

The properties of the matrix $M(\underline{\lambda})$ follow readily from those of $M(\underline{u})$ given earlier, provided the normalization of $\underline{\lambda}$ is taken into account. For example,

$$M^T(\underline{\lambda})M(\underline{\lambda}) = (\underline{\lambda} \cdot \underline{\lambda})\, I = I \tag{4.109}$$

The matrix $M(\underline{\lambda})$ is strictly orthogonal. Likewise, when the rate of change of $\underline{\xi}$ is to be calculated, equation (4.106) leads to

$$\underline{\xi}' = M(\underline{\lambda}')\underline{\lambda} + M(\underline{\lambda})\underline{\lambda}' \tag{4.110}$$

since M is linear and homogeneous in the components of $\underline{\lambda}$. Then the first term of (4.110) can be factored in scalar form as

$$M(\underline{\lambda}')\underline{\lambda} = \begin{bmatrix} \lambda_1 & -\lambda_2 & -\lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & \mp\lambda_4 & \mp\lambda_3 \\ \lambda_3 & \pm\lambda_4 & \lambda_1 & \pm\lambda_2 \\ -\lambda_4 & \pm\lambda_3 & \mp\lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \\ \lambda_4' \end{bmatrix} \tag{4.111}$$

The matrix on the right is merely $M(\underline{\lambda})$ with the signs in the fourth row reversed, so that it is possible to write (4.110) in the form

$$\underline{\xi}' = 2M(\underline{\lambda})\, \underline{\lambda}' \tag{4.112}$$

provided that the bilinear relation is imposed as

$$l(\underline{\lambda}, \underline{\lambda}') = \underline{\lambda}'^T J \underline{\lambda} \equiv 0 \tag{4.113}$$

As before, the bilinear relation is merely the requirement that the fourth component of (4.112) vanish so that $\underline{\xi}'$ can be calculated from that formula rather than (4.110). The general validity of the bilinear relation (4.113) has yet to be established, but the proof can be deferred until after the equations governing $\underline{\lambda}$ have been derived.

Now if the bilinear relation (4.113) is supposed to hold true, then the two alternate forms of $\underline{\xi}'$ given in (4.112) and (4.110) can be equated, leading to the result

$$M(\underline{\lambda}')\underline{\lambda} = M(\underline{\lambda})\underline{\lambda}' \tag{4.114}$$

This is entirely analogous to equation (4.32) given earlier for the $\underline{u}$ coordinates. A related result, which also depends on the bilinear relation (4.113) holding true, might be surmised by analogy with equation (4.34):

$$\left[M^T(\underline{\lambda})M(\underline{\lambda}') - 2(\underline{\lambda} \cdot \underline{\lambda}')\, I\right] = -\,(\underline{\lambda}' \cdot \underline{\lambda}')\underline{\lambda} \tag{4.115}$$

Of course, since $\underline{\lambda} \cdot \underline{\lambda}' = 0$ this formula ought to reduce further to

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' = -\,(\underline{\lambda}' \cdot \underline{\lambda}')\underline{\lambda} \tag{4.116}$$

The proof of this result is easy to obtain. When the matrix product $M^T(\underline{\lambda})M(\underline{\lambda}')$ is examined in scalar form, an expression in every way analogous to equation (4.35) appears. The difference is that the main diagonal vanishes because $\underline{\lambda} \cdot \underline{\lambda}' = 0$. Therefore, this matrix product is skew-symmetric. An alternate demonstration of this fact is to differentiate the identity $M^T(\underline{\lambda})M(\underline{\lambda}) - (\underline{\lambda} \cdot \underline{\lambda})I = 0$. In either case, one may write

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' = -\left[M^T(\underline{\lambda})M(\underline{\lambda}')\right]^T \underline{\lambda}' \tag{4.117}$$

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' = -\,M^T(\underline{\lambda}')M(\underline{\lambda})\underline{\lambda}' \tag{4.118}$$

But, according to (4.114) above, the right-hand side can be rewritten as

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' = -\,M^T(\underline{\lambda}')M(\underline{\lambda}')\underline{\lambda} \tag{4.119}$$

Then

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' = -\,(\underline{\lambda}' \cdot \underline{\lambda}')\, I\, \underline{\lambda} \tag{4.120}$$

because of the orthogonality of M described by equation (4.17). Thus (4.116) is proved.

Now the equations of motion in which $\underline{\xi}$ appears explicitly are reproduced here from the last chapter for reference. The whole set is

$$\underline{\xi}'' + 2mu^{-1}u'\underline{\xi}' + \left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= -\,nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \tag{4.121}$$

$$u'' - 2EC^2\frac{n}{m}u^{2mn-2m+1} + \frac{(n-1)}{m}C^2h^2u^{2mn-4m+1} + (m-1)u^{-1}u'^2$$

$$= \frac{(2n-1)}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}(\underline{P}\cdot\underline{\xi}) + C^{-1}C'u' \tag{4.122}$$

$$\underline{B}' = u^{2m}\left[2(\underline{P}\cdot\underline{\xi}')\underline{\xi} - (\underline{P}\cdot\underline{\xi})\underline{\xi}' - \left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right]mu^{-1}u'\right] \tag{4.123}$$

$$E' = u^m\left[mu^{-1}u'(\underline{P}\cdot\underline{\xi}) + (\underline{P}\cdot\underline{\xi}')\right] \tag{4.124}$$

$$(h^2)' = 2u^{3m}\underline{P}\cdot\underline{\xi}' \tag{4.125}$$

$$t' = C\,u^{mn} \tag{4.126}$$

In these equations all vectors are considered to be augmented with zero fourth components. Also, the scalar quantity u is the radial variable in the Burdet-type coordinate transformation

$$\underline{r} = u^m \underline{\xi} \tag{4.127}$$

and should not be confused with the 4-vector $\underline{u}$ which appears earlier in this chapter.

The Euler parameters are introduced into this set of equations by substituting for $\underline{\xi}$ from (4.106) and for $\underline{\xi}'$ from (4.112). The first equation in the set, (4.121), becomes

$$[2M(\underline{\lambda})\underline{\lambda}']' + 2mu^{-1}u'[2M(\underline{\lambda})\underline{\lambda}'] + \left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]M(\underline{\lambda})\underline{\lambda}$$

$$= -nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}[\underline{P} - [\underline{P}\cdot(M(\underline{\lambda})\underline{\lambda})]M(\underline{\lambda})\underline{\lambda}] + C^{-1}C'[2M(\underline{\lambda})\underline{\lambda}'] \tag{4.128}$$

$$2M(\underline{\lambda}')\underline{\lambda}' + 2M(\underline{\lambda})\underline{\lambda}'' + 4mu^{-1}u'\, M(\underline{\lambda})\underline{\lambda}'$$

$$+\left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]M(\underline{\lambda})\underline{\lambda}$$

$$= -nC^2u^{2mn-3m}\underline{B} + C^2u^{2mn-m}[\underline{P} - [\underline{P}\cdot(M(\underline{\lambda})\underline{\lambda})]M(\underline{\lambda})\underline{\lambda}] + 2C^{-1}C'[M(\underline{\lambda})\underline{\lambda}'] \tag{4.129}$$

In order to isolate $\underline{\lambda}''$, premultiply by $\frac{1}{2}M^T(\underline{\lambda})$, remembering that $M(\underline{\lambda})$ is strictly orthogonal.

$$M^T(\underline{\lambda})M(\underline{\lambda}')\underline{\lambda}' + \underline{\lambda}'' + 2mu^{-1}u'\,\underline{\lambda}'$$

$$+\frac{1}{2}\left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]\underline{\lambda}$$

$$= -\frac{1}{2}nC^2u^{2mn-3m}M^T(\underline{\lambda})\underline{B} + C^2u^{2mn-m}\left[M^T(\underline{\lambda})\underline{P} - \left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\right]\underline{\lambda}\right] + C^{-1}C'\underline{\lambda}' \tag{4.130}$$

The u equation, (4.122), becomes

$$u'' - 2EC^2\frac{n}{m}u^{2mn-2m+1} + \frac{(n-1)}{m}C^2h^2u^{2mn-4m+1} + (m-1)u^{-1}u'^2$$

$$= \frac{(2n-1)}{m} C^2 \mu u^{2mn-3m+1} + \frac{1}{m} C^2 u^{2mn-m+1} (M^T(\underset{\sim}{\lambda})\underset{\sim}{P}) \cdot \underset{\sim}{\lambda} + C^{-1} C' u' \tag{4.131}$$

Even without considering the remaining equations of the set, a little inspection of these latter two equations reveals that much of the reasoning outlined in Chapter 3 can be carried over to the present case. For example, the u equation (4.131) is not coupled to the rest of the system during unperturbed motion. In that case, once the u equation is solved the solution can be substituted into the $\underset{\sim}{\lambda}$ equation (4.130) and that equation treated. However, for this approach to analytically feasible, the linearization of the u equation (4.131) must be attempted. Also, whereas the general regularization of these two equations is not easily addressed, a linear version of (4.131) will necessarily be a constant-coefficient, and therefore regular, form. Unfortunately, it would seem, the choices of m and n for linearizing (4.131) as it stands are highly restricted. The choice $m = 1$ is necessary for linearization regardless of the value of n. But then $u = r$ according to the coordinate transformation (4.127) and equation (4.131) reverts to the equation of the radius discussed in Chapter 2. Linearization then requires the further choice $n = 1$, which merely re-introduces the eccentric-anomaly regularization. Furthermore, these choices of m and n do not appreciably simplify the $\underset{\sim}{\lambda}$ equation, (4.130). In fact, unlike the situation in Chapter 3 in which the $\underset{\sim}{\xi}$ equation was linear in components of $\underset{\sim}{\xi}$, the first term of (4.130) is nonlinear in components of $\underset{\sim}{\lambda}$ and cannot be removed by any choices of m and n. Even rewriting that term using (4.116) will not remove the nonlinearity (a special case which arises for $n = 2$ is discussed later). Progress with this set of equations depends on the algebraic elimination of E, $\underset{\sim}{B}$ or h^2, and for this purpose these quantities must be defined in terms of the (u, $\underset{\sim}{\lambda}$) coordinates. Before that step is taken, though, let it be noted that the introduction of $\underset{\sim}{\lambda}$ in place of $\underset{\sim}{\xi}$ in the u equation has affected only the term containing $\underset{\sim}{P}$. Therefore the reasoning used in the last chapter to arrive at a linear u equation carries over to the present case entirely without modification. One is still led to eliminate $\underset{\sim}{B}$ and E in favor of h^2 and to select $m = -1$ and $n = 2$. The truth of this statement becomes evident below as E, $\underset{\sim}{B}$ and h^2 are recast in terms of $\underset{\sim}{\lambda}$ and it is found that replacing $\underset{\sim}{\xi}$ in terms of $\underset{\sim}{\lambda}$ cannot affect the exponents either of u or of any of its derivatives. On the other hand, the $\underset{\sim}{\lambda}$ equation (4.130) requires closer scrutiny because it is nonlinear in components of $\underline{\lambda}$ and thus is not exactly analogous to the $\underline{\xi}$ equation. The quantities E, $\underline{B}$ and h^2 are now recast in terms of u and $\underline{\lambda}$ with the object of reducing the $\underline{\lambda}$ equation to a linear form.

As given by equation (3.10), the energy E is

$$E=\frac{1}{2}C^{-2}u^{2m-2mn}(m^2u^{-2}u'^2+\underline{\xi}'\cdot\underline{\xi}')-\mu u^{-m} \tag{4.132}$$

Replacing $\underline{\xi}'$ from (4.112) produces

$$E=\frac{1}{2}C^{-2}u^{2m-2mn}\left[m^2u^{-2}u'^2+(2M(\underline{\lambda})\underline{\lambda}')\cdot(2M(\underline{\lambda})\underline{\lambda}')\right]-\mu u^{-m} \tag{4.133}$$

$$E=\frac{1}{2}C^{-2}u^{2m-2mn}\left[m^2u^{-2}u'^2+4(M^T(\underline{\lambda})M(\underline{\lambda})\underline{\lambda}')\cdot\underline{\lambda}'\right]-\mu u^{-m} \tag{4.134}$$

$$E=\frac{1}{2}C^{-2}u^{2m-2mn}(m^2u^{-2}u'^2+4\underline{\lambda}'\cdot\underline{\lambda}')-\mu u^{-m} \tag{4.135}$$

The Laplace vector $\underline{B}$ is given by equation (3.16) as

$$\underline{B}=\left[C^{-2}u^{3m-2mn}(\underline{\xi}'\cdot\underline{\xi}')-\mu\right]\underline{\xi}-mC^{-2}u^{3m-2mn-1}u'\underline{\xi}' \tag{4.136}$$

Replacing $\underline{\xi}$ from (4.106) and $\underline{\xi}'$ from (4.112) produces

$$\underline{B}=\left[C^{-2}u^{3m-2mn}[(2M(\underline{\lambda})\underline{\lambda}')\cdot(2M(\underline{\lambda})\underline{\lambda}')]-\mu\right]M(\underline{\lambda})\underline{\lambda}$$

$$-mC^{-2}u^{3m-2mn-1}u'(2M(\underline{\lambda})\underline{\lambda}') \tag{4.137}$$

$$\underline{B}=\left[4C^{-2}u^{3m-2mn}(\underline{\lambda}'\cdot\underline{\lambda}')-\mu\right]M(\underline{\lambda})\underline{\lambda}-2mC^{-2}u^{3m-2mn-1}u'M(\underline{\lambda})\underline{\lambda}' \tag{4.138}$$

Since $\underline{B}$ occurs only in the combination $M^T(\underline{\lambda})\underline{B}$, it is just as convenient to form the product

$$M^T(\underline{\lambda})\underline{B}=\left[4C^{-2}u^{3m-2mn}(\underline{\lambda}'\cdot\underline{\lambda}')-\mu\right]\underline{\lambda}-2mC^{-2}u^{3m-2mn-1}u'\underline{\lambda}' \tag{4.139}$$

The angular momentum magnitude h^2 is given by equation (3.14) as

$$h^2 = C^{-2}u^{4m-2mn}(\underline{\xi}' \cdot \underline{\xi}') \tag{4.140}$$

When $\underline{\xi}'$ is replaced via (4.112), this becomes

$$h^2 = 4C^{-2}u^{4m-2mn}(\underline{\lambda}' \cdot \underline{\lambda}') \tag{4.141}$$

Now the factor $M(\underline{\lambda})$ has disappeared from the right-hand sides of (4.135), (4.139) and (4.141). If the nonlinear leading term of the $\underline{\lambda}$ equation (4.130) is ever to be cancelled by the use of one or more of these latter equations then that leading term must be rewritten so that it too contains no factor of $M(\underline{\lambda})$. The algebraic device for accomplishing this is to use the result given in equation (4.116), which depends on the bilinear relation holding true. Equation (4.130) becomes

$$-(\underline{\lambda}' \cdot \underline{\lambda}')\underline{\lambda} + \underline{\lambda}'' + 2mu^{-1}u'\,\underline{\lambda}' + \frac{1}{2}\left[nC^2\mu u^{2mn-3m} + (1-n)C^2h^2u^{2mn-4m}\right]\underline{\lambda}$$

$$= -\frac{1}{2}nC^2u^{2mn-3m}M^T(\underline{\lambda})\underline{B} + C^2u^{2mn-m}\left[M^T(\underline{\lambda})\underline{P} - \left[(M^T(\underline{\lambda})\underline{P}) \cdot \underline{\lambda}\right]\underline{\lambda}\right] + C^{-1}C'\underline{\lambda}' \tag{4.142}$$

The quantities $(\underline{\lambda}' \cdot \underline{\lambda}')$, E , $\underline{B}$ and h^2 are to be eliminated, or possibly recombined, in this equation in order to reduce it to a linear form in $\underline{\lambda}$. In searching for a strategy to follow in this regard, the following reasoning proves to be helpful. Suppose that all nonlinearities can indeed be eliminated by means of the manipulations now under consideration. In that case, one ought to attempt to obtain the simplest possible linear form. For example, if a term containing $M^T(\underline{\lambda})\underline{B}$ remains in the linear equation then that equation will be coupled in components of $\underline{\lambda}$. That is, even though the product $M^T(\underline{\lambda})\underline{B}$ is linear in components of $\underline{\lambda}$, its presence will preclude any simple representation of the Keplerian motion. (However, the same situation occurs here that was mentioned in the last chapter, namely, that for the special values $n = 2$, $m = -1$, $C = h^{-1}$ a useful set of coupled linear equations for the components of $\underline{\lambda}$ can be derived. This set has been derived by Vitins (1978) and others as mentioned before. But since that system is equivalent to an uncoupled (but higher order) system which appears for $n = 2$, $m = -1$, $C = h^{-1}$, no new type of linearization is involved. See

Appendix B.) Also in equation (4.142), if the term containing $u'\lambda'$ remains then it will not be possible to derive simple oscillator-type equations. In addition, it would be desirable to derive constant-coefficient equations, if possible, for which purpose special values of m and n will have to be sought. Although one might expect to be forced to use $m = -1$ and $n = 2$ in order to obtain a tractable u equation, there is no reason yet to think that these same values of m and n will appear out of an analysis of the λ equation.

The result of all these considerations is that, in the interest of obtaining the simplest possible linear equation for λ, the product $M^T(\lambda)B$ should be eliminated in favor of E or h^2 or possibly in favor of the coordinates as given in equation (4.139). Any of these eliminations is feasible since E, B and h^2 all contain the product $(\lambda' \cdot \lambda')$. One can rearrange (4.135) as

$$4(\lambda' \cdot \lambda') = 2C^2u^{2mn-2m}(E + \mu u^{-m}) - m^2u^{-2}u'^2 \tag{4.143}$$

so that (4.139) can be written as

$$M^T(\lambda)B = (2u^mE - C^{-2}m^2u^{3m-2mn-2}u'^2 + \mu)\lambda - 2mC^{-2}u^{3m-2mn-1}u'\lambda' \tag{4.144}$$

Likewise, one can rearrange (4.141) as

$$4(\lambda' \cdot \lambda') = C^2h^2u^{2mn-4m} \tag{4.145}$$

so that (4.139) can be written as

$$M^T(\lambda)B = (u^{-m}h^2 - \mu)\lambda - 2mC^{-2}u^{3m-2mn-1}u'\lambda' \tag{4.146}$$

Subsequently, of course, either of E or h^2 could be eliminated in favor of the other. Equate (4.143) with (4.145) to obtain

$$2C^2u^{2mn-2m}(E + \mu u^{-m}) - m^2u^{-2}u'^2 = C^2h^2u^{2mn-4m} \tag{4.147}$$

from which

$$E=\frac{1}{2}C^{-2}u^{2m-2mn}(C^2h^2u^{2mn-4m}+m^2u^{-2}u'^2)-\mu u^{-m} \tag{4.148}$$

or

$$h^2=C^{-2}u^{4m-2mn}\left[2C^2u^{2mn-2m}(E+\mu u^{-m})-m^2u^{-2}u'^2\right] \tag{4.149}$$

These latter three relations are the same as equations (3.39) through (3.41) of Chapter 3, and equations (4.144) and (4.146) correspond to equations (3.42) and (3.43) of that same chapter.

Besides eliminating the product $M^T(\underline{\lambda})\underline{B}$ in equation (4.142), one can revise the nonlinear first term of that equation by means of (4.143) or (4.145). Specifically,

$$-(\underline{\lambda}'\cdot\underline{\lambda}')\underline{\lambda}=-\frac{1}{4}\left[2C^2u^{2mn-2m}(E+\mu u^{-m})-m^2u^{-2}u'^2\right]\underline{\lambda} \tag{4.150}$$

or

$$-(\underline{\lambda}'\cdot\underline{\lambda}')\underline{\lambda}=-\frac{1}{4}\left[C^2h^2u^{2mn-4m}\right]\underline{\lambda} \tag{4.151}$$

Then it becomes clear that, since the nonlinear term can be exchanged in this manner for one linear in components of $\underline{\lambda}$, the product $M^T(\underline{\lambda})\underline{B}$ ought not to be eliminated merely in terms of the coordinates as in (4.139). Such an operation would re-introduce the unwanted nonlinear factor $(\underline{\lambda}'\cdot\underline{\lambda}')$. (A special case which arises in case $n=2$ is examined later.) Hence, the strategy for modifying the $\underline{\lambda}$ equation (4.142) should be to choose among the four relations (4.144), (4.146), (4.150) and (4.151) which, in effect, eliminate $\underline{B}$ and $(\underline{\lambda}'\cdot\underline{\lambda}')$ in favor of E and h^2. Additionally, E and h^2 can be exchanged as desired using (4.148) and (4.149). Each of the four basic possibilities is now examined in turn.

Substituting (4.144) and (4.150) into (4.142) produces

$$\underline{\lambda}''+(2m-nm)u^{-1}u'\underline{\lambda}'+\Big[\frac{1}{2}(2n-1)C^2u^{2mn-2m}E+\frac{1}{2}(2n-1)C^2\mu u^{2mn-3m}$$

$$+\frac{1}{2}(1-n)C^2h^2u^{2mn-4m}+\frac{1}{4}(1-2n)m^2u^{-2}u'^2\Big]\underline{\lambda}$$

$$=C^2u^{2mn-m}\Big[M^T(\underline{\lambda})\underline{P}-\big[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\big]\underline{\lambda}\Big]+C^{-1}C'\underline{\lambda}' \tag{4.152}$$

On the other hand, substituting (4.144) and (4.151) into (4.142) produces

$$\underline{\lambda}''+(2m-nm)u^{-1}u'\,\underline{\lambda}'$$

$$+\Big[nC^2u^{2mn-2m}E+nC^2\mu u^{2mn-3m}+\frac{1}{4}(1-2n)C^2h^2u^{2mn-4m}-\frac{1}{2}nm^2u^{-2}u'^2\Big]\underline{\lambda}$$

$$=C^2u^{2mn-m}\Big[M^T(\underline{\lambda})\underline{P}-\big[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\big]\underline{\lambda}\Big]+C^{-1}C'\underline{\lambda}' \tag{4.153}$$

Alternatively, substituting (4.146) and (4.150) into (4.142) produces

$$\underline{\lambda}''+(2m-nm)u^{-1}u'\,\underline{\lambda}'$$

$$+\Big[-\frac{1}{2}C^2u^{2mn-2m}E-\frac{1}{2}C^2\mu u^{2mn-3m}+\frac{1}{2}C^2h^2u^{2mn-4m}-\frac{1}{4}m^2u^{-2}u'^2\Big]\underline{\lambda}$$

$$=C^2u^{2mn-m}\Big[M^T(\underline{\lambda})\underline{P}-\big[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\big]\underline{\lambda}\Big]+C^{-1}C'\underline{\lambda}' \tag{4.154}$$

Finally, substituting (4.146) and (4.151) into (4.142) produces

$$\underline{\lambda}''+(2m-nm)u^{-1}u'\,\underline{\lambda}'+\Big[\frac{1}{4}C^2h^2u^{2mn-4m}\Big]\underline{\lambda}$$

$$=C^2u^{2mn-m}\Big[M^T(\underline{\lambda})\underline{P}-\big[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\big]\underline{\lambda}\Big]+C^{-1}C'\underline{\lambda}' \tag{4.155}$$

These four equations are linear in components of $\underline{\lambda}$ during unperturbed motion and differ in form only in the coefficient of $\underline{\lambda}$. In all four equations the value $n=2$ is necessary to eliminate the term containing $u'\underline{\lambda}'$, regardless of the value of m. In the first three equations the presence of the term containing u'^2 prevents the reduction of those equations to a constant-coefficient form. The fourth equation, which corresponds to equation (3.53) of the last chapter, lacks this complicating term and can be reduced to a constant-coefficient form by the choice $n = 2$ regardless of the value of m. In order to cancel the u'^2 term in the other three equations it would be necessary at least to eliminate E in favor of h^2 or else h^2 in favor of E using (4.148) or (4.149). Straightforward substitutions will show that if E is replaced in favor of h^2 via (4.148) then all three of the equations (4.152), (4.153) and (4.154) reduce to a common form identical with (4.155). On the other hand, if h^2 is replaced in favor of E via (4.149) then the three equations reduce to the common form

$$\underline{\lambda}'' + (2m - nm)u^{-1}u'\underline{\lambda}'$$

$$+ \left[\frac{1}{2}C^2u^{2mn-2m}E + \frac{1}{2}C^2\mu u^{2mn-3m} - \frac{1}{4}m^2u^{-2}u'^2\right]\underline{\lambda}$$

$$= C^2u^{2mn-m}\left[M^T(\underline{\lambda})\underline{P} - \left[(M^T(\underline{\lambda})\underline{P}) \cdot \underline{\lambda}\right]\underline{\lambda}\right] + C^{-1}C'\underline{\lambda}' \qquad (4.156)$$

which corresponds to equation (3.54) of Chapter 3. However, this equation still cannot be reduced to a constant-coefficient form by any choices of m and n. (Recall that the choice $m = 0$ is excluded for these coordinates.) Therefore, in order to obtain a linear constant-coefficient $\underline{\lambda}$ equation, the version given in (4.155) must be used. The necessary value $n = 2$ agrees with the values $n = 2$, $m = -1$ required to linearize the u equation (4.131). Returning now to the u equation, eliminate E in favor of h^2 as was done in Chapter 3 to obtain

$$u'' - \frac{1}{m}C^2h^2u^{2mn-4m+1} + (m - mn - 1)u^{-1}u'^2$$

$$= -\frac{1}{m}C^2\mu u^{2mn-3m+1} + \frac{1}{m}C^2u^{2mn-m+1}(M^T(\underline{\lambda})\underline{P}) \cdot \underline{\lambda} + C^{-1}C'u' \qquad (4.157)$$

This corresponds to equation (3.47) of that chapter. Then the values $n = 2$, $m = -1$ follow at once for the reasons given there.

Now the linear regular equations governing the $(u, \underline{\lambda})$ coordinates can be summarized.

$$\underset{\sim}{\lambda}'' + \frac{1}{4}C^2h^2\underset{\sim}{\lambda} = C^2u^{-3}\left[M^T(\underset{\sim}{\lambda})\underset{\sim}{P} - \left[(M^T(\underset{\sim}{\lambda})\underset{\sim}{P})\cdot\underset{\sim}{\lambda}\right]\underset{\sim}{\lambda}\right] + C^{-1}C'\underset{\sim}{\lambda}' \tag{4.158}$$

$$u'' + C^2h^2u = C^2\mu - C^2u^{-2}(M^T(\underset{\sim}{\lambda})\underset{\sim}{P})\cdot\underset{\sim}{\lambda} + C^{-1}C'u' \tag{4.159}$$

$$(h^2)' = 4u^{-3}(M^T(\underset{\sim}{\lambda})\underset{\sim}{P})\cdot\underset{\sim}{\lambda}' \tag{4.160}$$

$$t' = C\,u^{-2} \tag{4.161}$$

As it stands, this system is of order twelve, but the order can be reduced immediately to eleven without sacrificing either linearization or regularization. Equation (4.145) indicates that in case, and only in case, $n = 2$, the quantity $(\underset{\sim}{\lambda}'\cdot\underset{\sim}{\lambda}')$ is a constant of unperturbed motion:

$$(\underset{\sim}{\lambda}'\cdot\underset{\sim}{\lambda}') = \frac{1}{4}C^2h^2 \tag{4.162}$$

Substituting for C^2h^2 in the above equations makes the integration for h^2 superfluous even in perturbed motion as long as C is strictly constant. As pointed out in the discussion of the analogous equation (3.67) in Chapter 3, if C is variable in perturbed motion then the extra integration for h^2 cannot be avoided.

A different type of reduction in order for the above equations (4.158) through (4.161) was presented by Vitins (1978). In essence, Vitins made use of the kinematics of the orthonormal triad of 3-vectors $(\underset{\sim}{\xi},\ \underset{\sim}{\xi}',\ \underset{\sim}{\xi}\times\underset{\sim}{\xi}')$ expressed in terms of the 4-vector $\underset{\sim}{\lambda}$ to derive a first-order governing equation in place of the second-order equation (4.158). Since he uses $C = h^{-1}$, the integration for h^2 must be included, but the total system order is reduced to eight nevertheless. The resulting regular equations also happen to be linear, but coupled in the components of $\underset{\sim}{\lambda}$, for Keplerian motion. Actually, Vitins does not exhibit the second-order $\underset{\sim}{\lambda}$ equation (4.158) given above, but instead works directly from Burdet's (1969) second-order equation for the 3-vector $\underset{\sim}{\xi}$. He also retains only the lower signs throughout in the matrix $M(\underset{\sim}{\lambda})$. Several years earlier, Broucke, *et al.* (1971) had derived time-domain versions of Vitins' equations, including the first-order governing equation for

$\underline{\lambda}$ and a succinct geometrical and kinematical explanation of the bilinear relation for the Euler parameters. Broucke's Euler-parameter equations happen to be nonlinear even in Keplerian motion because time is used as the independent variable. A moment's inspection of his equations reveals that merely introducing η or σ as the independent variable reduces the Euler-parameter equations to linear forms in unperturbed motion, the results being identical with those of Vitins (1978). Some further comments on Vitins' results in light of the developments in this chapter are recorded in Appendix B.

Now for the governing equations (4.158) through (4.161) it remains to establish initial conditions for $\underline{\lambda}$ and $\underline{\lambda}'$ and to establish that the bilinear relation (4.113) holds true in all necessary generality. Initial conditions for the components of $\underline{\lambda}$ are determined in the same manner as were those for the components of $\underline{u}$. In scalar form, equations (4.107) and (4.106) are

$$1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \tag{4.163}$$

$$\xi_1 = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \tag{4.164}$$

$$\xi_2 = 2\lambda_1\lambda_2 \mp 2\lambda_3\lambda_4 \tag{4.165}$$

$$\xi_3 = 2\lambda_1\lambda_3 \pm 2\lambda_2\lambda_4 \tag{4.166}$$

If $\xi_1 \geq 0$, add (4.164) to (4.163) to obtain

$$\lambda_1^2 + \lambda_4^2 = \frac{1}{2}(1 + \xi_1) \tag{4.167}$$

Choose λ_1 and λ_4 to satisfy this relation. Then solve equations (4.165) and (4.166) simultaneously for λ_2 and λ_3:

$$\begin{bmatrix} \lambda_2 \\ \lambda_3 \end{bmatrix} = \frac{1}{1+\xi_1} \begin{bmatrix} \lambda_1 & \pm\lambda_4 \\ \mp\lambda_4 & \lambda_1 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix} \tag{4.168}$$

On the other hand, if $\xi_1 < 0$, subtract (4.164) from (4.163) to obtain

$$\lambda_2^2 + \lambda_3^2 = \frac{1}{2}(1 - \xi_1) \tag{4.169}$$

Choose λ_2 and λ_3 to satisfy this relation. Then solve equations (4.165) and (4.166) simultaneously for λ_1 and λ_4:

$$\begin{bmatrix} \lambda_1 \\ \lambda_4 \end{bmatrix} = \frac{1}{1 - \xi_1} \begin{bmatrix} \lambda_2 & \lambda_3 \\ \mp\lambda_3 & \pm\lambda_2 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix} \tag{4.170}$$

The choice of procedures depending on the sign of ξ_1 is designed merely to avoid possible zero values of $(1 + \xi_1)$ or $(1 - \xi_1)$ and is not otherwise mandatory.

Initial conditions for the components of $\underline{\lambda}'$ are developed by evaluating equation (4.112) initially:

$$\underline{\xi}'(0) = 2M(\underline{\lambda}(0))\,\underline{\lambda}'(0) \tag{4.171}$$

Although this equation depends on the bilinear relation holding true, which has yet to be demonstrated, it is permissible to define $\underline{\lambda}'(0)$ by the inverse relation

$$\underline{\lambda}'(0) = \frac{1}{2} M^T(\underline{\lambda}(0))\,\underline{\xi}'(0) \tag{4.172}$$

Then $\underline{\lambda}'(0)$ can be computed once $\underline{\lambda}(0)$ has been assigned using (4.167) through (4.170) above.

It is now shown that the bilinear relation (4.113) holds true for general values of n, m and C provided that $\underline{\lambda}$ satisfies the differential equation (4.155) and that $\underline{\lambda}'(0)$ is computed by (4.172). Differentiation of (4.113) produces

$$l'(\underline{\lambda}, \underline{\lambda}') = \underline{\lambda}''^T J\,\underline{\lambda} + \underline{\lambda}'^T J\,\underline{\lambda}' \tag{4.173}$$

The second term vanishes identically according to the property (4.93). Substituting for $\underline{\lambda}''$ from equation (4.155) gives

$$l' = \Big[-(2m-nm)u^{-1}u'\underline{\lambda}' - \frac{1}{4}C^2h^2u^{2mn-4m}\underline{\lambda}$$

$$+ C^2u^{2mn-m}\Big[M^T(\underline{\lambda})\underline{P} - \big[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}\big]\underline{\lambda}\Big] + C^{-1}C'\underline{\lambda}'\Big]^T J\,\underline{\lambda} \qquad (4.174)$$

which reduces straightforwardly to

$$l' = \big[C^{-1}C' - (2m-nm)u^{-1}u'\big]\,l + C^2u^{2mn-m}\underline{P}^T M(\underline{\lambda})\,J\,\underline{\lambda} \qquad (4.175)$$

Now by carrying out the matrix product $M(\underline{\lambda})\,J\,\underline{\lambda}$ in scalar form, it is found that the first three components vanish identically. Then, since the fourth component of $\underline{P}$ is zero by convention, the entire second term always vanishes identically. Hence l obeys the first-order differential equation

$$l' = \big[C^{-1}C' - (2m-nm)u^{-1}u'\big]\,l \qquad (4.176)$$

The initial condition for l is simply

$$l(0) = \underline{\lambda}'(0)^T J\,\underline{\lambda}(0) \qquad (4.177)$$

Provided that $\underline{\lambda}'(0)$ is to be calculated by (4.172), this is

$$l(0)\Big[\frac{1}{2}M^T(\underline{\lambda}(0))\underline{\xi}'(0)\Big]^T J\,\underline{\lambda}(0) \qquad (4.178)$$

$$l(0) = \frac{1}{2}\underline{\xi}'(0)^T M(\underline{\lambda}(0))\,J\,\underline{\lambda}(0) \qquad (4.179)$$

However, the first three components of $M(\underline{\lambda})\,J\,\underline{\lambda}$ always vanish identically. Since the fourth component of $\underline{\xi}'(0)$ is zero by convention, the initial condition for l is

$$l(0) \equiv 0 \qquad (4.180)$$

Therefore the only solution of the differential equation (4.176) will be the trivial solution, and the bilinear relation holds in general.

Summary

In Chapter 2 it was shown that if true-anomaly regularization is to lead to linear equations governing the unperturbed motion then some kind of coordinate transformation is required in addition to the time transformation. Without a general theory to aid in the selection of linearizing coordinates, one is forced to examine specific choices individually. Chapter 3 examined the use of coordinates of a type offered by Burdet (1969), namely, a radial variable and a three-component unit radial vector. The present chapter has examined the use of a type offered by Kustaanheimo (1964), Kustaanheimo and Stiefel (1965), Stiefel and Scheifele (1971) and others. In a result thought to be new, it is found that the use of these so-called "KS" coordinates, together with the generalized Sundman time transformation $dt = Cr^n ds$ and integrals of the Keplerian motion as extra dependent variables, leads to regular governing equations only in case $n = 1$. This contrasts with the use of Cartesian coordinates which leads to regular governing equations for $n = 1$ and $n \geq \frac{3}{2}$. However, like the Cartesian-coordinate equations, the equations governing the 4-parameter KS coordinates are also linear in unperturbed motion for $n = 1$. The subsequent conclusion is that the KS coordinates are not generally useful for either analytical or numerical work when true anomaly is the independent variable. There is a related set of coordinates which can be used with true anomaly, however. Enlarging on a suggestion of Vitins (1978), this chapter introduced the KS coordinates in a way that allows a set of Euler parameters to be derived by a simple normalization. The Euler parameters are then used to represent the unit radial vector, while the same radial variable used in Chapter 3 completes the specification of position in space. It is found that the value $n = 2$ in the generalized Sundman time transformation is necessary to produce linear equations for these 5-parameter coordinates if the equations are to be uncoupled in the Euler parameters. Moreover, these linear equations are necessarily regular. Analogously to the Burdet (1969) equations exhibited in Chapter 3, the order of the linear regular equations can, in certain circumstances, be reduced by one without sacrificing either linearization or regularization. The integrals of Keplerian motion do not appear explicitly in the reduced set; only the coordinates themselves and their derivatives appear.

Finally, Appendix B discusses the connection between the uncoupled linear equations derived here and a set of coupled linear equations derived by Vitins (1978). Vitins also used true anomaly as the independent variable and the Euler parameters and reciprocal radius as coordinates, but his equations, aside from being coupled, have lower order. It is shown in Appendix B that the two formulations are equivalent (as they must be), the different forms reflecting the different ways that the kinematical constraints among the Euler parameters are handled. In particular, no different type of linearization is involved.

Chapter 5. The True-Anomaly Time Equation

Introduction

Previous chapters have reviewed the use of true anomaly as the independent variable in universal treatments of the equations governing perturbed two-body motion. This chapter examines the explicit relationship between true anomaly and time in the case of unperturbed motion. Specifically, the problem is to integrate the η-domain differential equation of time given in equation (3.120) of Chapter 3:

$$\frac{dt}{d\eta} = \frac{1}{hu^2} \tag{5.1}$$

The corresponding σ-domain expression, which is valid for rectilinear orbits ($h = 0$), will be examined later. Now in Keplerian motion h is a constant and u is known as a simple explicit function of η from, say, equation (3.126):

$$u(\eta) = \frac{\mu}{h^2} + \left[u(0) - \frac{\mu}{h^2}\right] \cos\eta + u'(0) \sin\eta \tag{5.2}$$

where

$$u(0) = \frac{1}{r(0)} \tag{5.3a}$$

$$u'(0) = -\frac{1}{hr(0)}\underline{r}(0) \cdot \dot{\underline{r}}(0) \tag{5.3b}$$

The latter formulae are taken from equations (3.96) and (3.108). In short, the differential equation (5.1) reduces to a quadrature. For notational convenience in handling the quadrature, introduce the dimensionless parameters α and β as

$$\alpha = \frac{h^2 u(0)}{\mu} - 1 = \frac{h^2}{\mu r(0)} - 1 \tag{5.4}$$

$$\beta = \frac{h^2 u'(0)}{\mu} = -\frac{h}{\mu r(0)}\underline{r}(0) \cdot \dot{\underline{r}}(0) \tag{5.5}$$

so that $u(\eta)$ can be written as

$$u(\eta) = \frac{\mu}{h^2}(1 + \alpha \cos \eta + \beta \sin \eta) = \frac{1}{r(\eta)} \tag{5.6}$$

Comparison of equation (5.6) with the more traditional two-body formula will show that α and β are each proportional to the orbital eccentricity e in such a way that

$$e^2 = \alpha^2 + \beta^2 \tag{5.7}$$

(The two relations are given explicitly in a later section as equations (5.64) and (5.65).) Then the differential equation of time (5.1) can be cast in the form

$$\frac{\mu^2}{h^3}[t(\eta) - t(0)] = \int_0^\eta \frac{dx}{(1 + \alpha \cos x + \beta \sin x)^2} \equiv K(\eta; \alpha, \beta) \tag{5.8}$$

where $t(0)$ is the time at epoch. This expression is the η-domain version of Kepler's equation which is exhibited in Chapter 2 in terms of eccentric anomaly in equations (2.67) and (2.120). Recall that the integrations leading to those two equations were quite elementary. It is easy to see that the integration indicated in (5.8) will not be so straightforward, although the final results can be expressed in terms of elementary functions. In any case, the true-anomaly results are much more complicated algebraically than the corresponding eccentric-anomaly results. This may be the reason that most textbooks on celestial mechanics fail to mention the direct relation between time and true anomaly. When it is mentioned, it is usually given cursory treatment, even when special simplifying assumptions are made. For example, if the epoch is assumed to be at the pericenter then $\beta = 0$ and $\alpha = e$, so that the integrand is appreciably simplified. Even in that case, though, Battin (1964) concludes that the quadrature does not yield a useful expression unless consideration is further restricted to parabolic orbits, for which $\alpha = e = 1$, or to circular orbits, for which $\alpha = e = 0$. Stumpff (1959) gives explicit formulae for time in terms of true anomaly for elliptic, parabolic and hyperbolic orbits with the epoch at pericenter. Thomson (1961, section 4.13) gives similar formulae for elliptic and hyperbolic orbits, but not for parabolic ones. Geyling and Westerman (1971) give similar formulae for all three types of orbits and include several alternate forms for elliptic and hyperbolic orbits. They also use these true-anomaly time relations to develop a single Taylor's series expansion which is valid for all near-parabolic orbits having the epoch at pericenter. True-anomaly time formulae for which the epoch is located at an arbitrary point on the orbit apparently have not been published, though a development by Battin (1968) should be mentioned. As a by-product of his analysis of the Gauss/Lambert boundary value problem, a set of formulae are given which permit a treatment of the initial value problem directly in terms of orbital central angle, that is, true anomaly. The associated time equation, which is universal, is therefore the type of relation sought in this chapter. Battin's formulae require the evaluation of two hypergeometric functions; later analyses of his on the boundary value problem (1977, 1983) involve only a single transcendental function in the time equation, making the formulae much more efficient numerically, though modifications for the initial value problem were not presented with these latter versions. In general, because the boundary value problem inherently involves true anomaly (it is just the angle between the two given position vectors), the associated time equation is always a "true-anomaly time equation". If it proves possible to adapt the formulae of the boundary value problem to solve the initial value problem, as it usually should, then the resulting time equation will qualify as a direct true-anomaly time relation. However, this approach seems never to have been fully developed or applied. Moreover, the results do not necessarily come out in the most efficient form, as can be seen by comparing the formulae of Battin (1968) with results derived later in this chapter. The difficulties of such an approach are compounded when it is required to extend the formulae to treat the perturbed problem. The assumption behind all the developments in this chapter is that a "frontal attack" on the quadrature (5.8) leads most directly to the desired formulae. It will be seen that the working formulae so derived do turn out to have good numerical efficiency and they are cast in terms of parameters which arise naturally in the true-anomaly regularization of the equations of motion. In the analysis which follows in the remainder of this chapter, various forms of the Keplerian-motion true-anomaly time equation are presented. The intent is to develop universal two-body formulae which are computationally useful in their own right and which can be extended to the more complicated J_2-perturbed case in later chapters.

In undertaking to evaluate the definite integral (5.8) there are at least two alternatives. Initially, much effort can be saved by referring to a catalog of known integrals. Gradshteyn and Ryzhik (1980) have tabulated several special formulae which together allow (5.8) to be expressed in terms of elementary functions of η. Different expressions appear for elliptic, hyperbolic and parabolic orbits; however, by suitable algebraic devices these different forms can be brought into a single universal form. Another, more laborious, approach is to introduce a change of variable which converts the integrand into a rational algebraic expression. This technique from elementary calculus produces a denominator polynomial which can be be factored and leads to an integration by the method of partial fractions. Of course, the results are the same no matter which approach is taken, but the method of partial fractions has certain features which commend its use in the perturbed case considered later. Each approach is now described in detail.

The Three Elementary Quadratures

Gradshteyn and Ryzhik (1980) give in their table entry 2.558 the following formulae.

$$\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}dx = \frac{(Bc - Cb) + (Ac - Ca)\cos x - (Ab - Ba)\sin x}{(n-1)(a^2 - b^2 - c^2)(a + b\cos x + c\sin x)^{n-1}}$$

$$+ \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times$$

$$\int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba)\cos x - (Ac - Ca)\sin x]}{(a + b\cos x + c\sin x)^{n-1}}dx \tag{5.9}$$

where $a^2 \neq b^2 + c^2$ and the integer $n \neq 1$. On the other hand, if $a^2 = b^2 + c^2$ and the integer $n \neq 1$ then

$$\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}dx = \frac{(Cb - Bc) + Ca\cos x - Ba\sin x}{(n-1)a(a + b\cos x + c\sin x)^n}$$

$$+ \left[\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right]\frac{(-c\cos x + b\sin x)(n-1)!}{(2n-1)!!} \times$$

$$\sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k(a + b\cos x + c\sin x)^{n-k}} \tag{5.10a}$$

where the following conventions hold:

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1) \tag{5.10b}$$

$$(2n)!! = 2 \cdot 4 \cdot 6 \ldots (2n) \tag{5.10c}$$

$$0\,!! = 1 \tag{5.10d}$$

$$(-1)!! = 1 \tag{5.10e}$$

Two related formulae are also available in the same place. If $a^2 > b^2 + c^2$ then

$$\int \frac{dx}{a + b\cos x + c\sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1}\left[\frac{(a-b)\tan\frac{1}{2}x + c}{\sqrt{a^2 - b^2 - c^2}}\right] \tag{5.11}$$

On the other hand, if $a^2 < b^2 + c^2$ then

$$\int \frac{dx}{a + b\cos x + c\sin x} = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln\left[\frac{(a-b)\tan\frac{1}{2}x + c - \sqrt{b^2 + c^2 - a^2}}{(a-b)\tan\frac{1}{2}x + c + \sqrt{b^2 + c^2 - a^2}}\right] \tag{5.12}$$

Then straightforward substitutions give the following results for the time quadrature (5.8). For elliptic orbits $1 > \alpha^2 + \beta^2$ so that equations (5.9) and (5.11) apply:

$$K(\eta;\alpha,\beta) = \frac{(1+\alpha)(\beta\cos\eta - \alpha\sin\eta) - \beta(1 + \alpha\cos\eta + \beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)}$$

$$+ \frac{2}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}} \times$$

$$\left[\tan^{-1}\left[\frac{(1-\alpha)\tan\frac{1}{2}\eta+\beta}{\sqrt{1-\alpha^2-\beta^2}}\right]-\tan^{-1}\left[\frac{\beta}{\sqrt{1-\alpha^2-\beta^2}}\right]\right] \tag{5.13}$$

For hyperbolic orbits $1 < \alpha^2 + \beta^2$ so that equations (5.9) and (5.12) apply:

$$K(\eta;\alpha,\beta)=\frac{(1+\alpha)(\beta\cos\eta-\alpha\sin\eta)-\beta(1+\alpha\cos\eta+\beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)}$$

$$+\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\times$$

$$\left[\ln\left[\frac{(1-\alpha)\tan\frac{1}{2}\eta+\beta-\sqrt{\alpha^2+\beta^2-1}}{(1-\alpha)\tan\frac{1}{2}\eta+\beta+\sqrt{\alpha^2+\beta^2-1}}\right]-\ln\left[\frac{\beta-\sqrt{\alpha^2+\beta^2-1}}{\beta+\sqrt{\alpha^2+\beta^2-1}}\right]\right] \tag{5.14}$$

For parabolic orbits $1 = \alpha^2 + \beta^2$ so that equation (5.10) applies:

$$K(\eta;\alpha,\beta)=$$

$$\frac{(1+\alpha)^2(\alpha\sin\eta-\beta\cos\eta)(2+\alpha\cos\eta+\beta\sin\eta)+\beta(2+\alpha)(1+\alpha\cos\eta+\beta\sin\eta)^2}{3(1+\alpha)^2(1+\alpha\cos\eta+\beta\sin\eta)^2} \tag{5.15}$$

These three formulae are not yet in their most usable forms, but they already elicit some interest because the origin of time is taken at an arbitrary point on the orbit rather than only at the pericenter. Further simplification is desirable, however. Recall the trigonometric identity

$$\tan(\gamma \pm \delta)=\frac{\tan\gamma \pm \tan\delta}{1 \mp \tan\gamma\tan\delta} \tag{5.16}$$

in which one takes either the upper signs throughout or the lower signs throughout. By letting $\gamma = \tan^{-1}x$ and $\delta = \tan^{-1}y$, this identity can be written as

$$\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\frac{x \pm y}{1 \mp xy}\right] \tag{5.17}$$

Hence the difference of arctangents in the elliptic formula (5.13) can be computed by a single arctangent evaluation:

$$K(\eta;\alpha,\beta) = \frac{(1+\alpha)(\beta\cos\eta - \alpha\sin\eta) - \beta(1+\alpha\cos\eta+\beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{\sqrt{1-\alpha^2-\beta^2}\ \tan\frac{1}{2}\eta}{1+\alpha+\beta\tan\frac{1}{2}\eta}\right] \tag{5.18}$$

In the hyperbolic formula (5.14) the logarithm terms are readily combined to produce

$$K(\eta;\alpha,\beta) = \frac{(1+\alpha)(\beta\cos\eta - \alpha\sin\eta) - \beta(1+\alpha\cos\eta+\beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)}$$

$$+\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\ln\left[\frac{(\beta+\sqrt{\alpha^2+\beta^2-1}\,)\tan\frac{1}{2}\eta+(1+\alpha)}{(\beta-\sqrt{\alpha^2+\beta^2-1}\,)\tan\frac{1}{2}\eta+(1+\alpha)}\right] \tag{5.19}$$

This can be written in a form more suitable for the present purpose by recalling the identity

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right] \tag{5.20}$$

which holds for $-1 < x < +1$, the entire domain of the inverse hyperbolic tangent. Now let

$$y = \frac{1+x}{1-x} \tag{5.21}$$

so that

$$x = \frac{y-1}{y+1} \tag{5.22}$$

where the domain $0 < y < \infty$ corresponds to $-1 < x < +1$. The identity (5.20) becomes

$$\ln y = 2\tanh^{-1}\left[\frac{y-1}{y+1}\right] \tag{5.23}$$

so that the logarithm in equation (5.19) can be rewritten:

$$K(\eta;\alpha,\beta) = \frac{(1+\alpha)(\beta\cos\eta - \alpha\sin\eta) - \beta(1+\alpha\cos\eta + \beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{\sqrt{\alpha^2+\beta^2-1}\,\tan\frac{1}{2}\eta}{1+\alpha+\beta\tan\frac{1}{2}\eta}\right] \tag{5.24}$$

The elliptic formula (5.18) and the hyperbolic formula (5.24) resemble one another very closely. They have identical first terms and the arctangents are related by the complex identities

$$\sqrt{-1}\,\tan^{-1}x = \tanh^{-1}(\sqrt{-1}\,x) \tag{5.25a}$$

$$\sqrt{-1}\,\tanh^{-1}x = \tan^{-1}(\sqrt{-1}\,x) \tag{5.25b}$$

In a later section it will be shown that (5.18) and (5.24) can be reduced to a common real-valued, and therefore universal, form. In the remainder of this section other useful forms of the elliptic, hyperbolic and parabolic formulae will be given.

It proves to be especially convenient in some applications of these formulae to replace $\sin\eta$ and $\cos\eta$ in terms of $\tan\frac{1}{2}\eta$ by means of the identities

$$\sin\eta = \frac{2\tan\frac{1}{2}\eta}{1+\tan^2\frac{1}{2}\eta} \tag{5.26}$$

$$\cos\eta = \frac{1-\tan^2\frac{1}{2}\eta}{1+\tan^2\frac{1}{2}\eta} \tag{5.27}$$

For example, for the common first term of (5.18) and (5.24) one obtains, after straightforward manipulations,

$$\frac{(1+\alpha)(\beta\cos\eta-\alpha\sin\eta)-\beta(1+\alpha\cos\eta+\beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)} =$$

$$\frac{-2z[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)]}{(1-\alpha^2-\beta^2)[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)]} \tag{5.28}$$

where

$$z = \tan\frac{1}{2}\eta \tag{5.29}$$

Then the elliptic formulae (5.18) becomes

$$K(\eta;\alpha,\beta) = \frac{-2z[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)]}{(1-\alpha^2-\beta^2)[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)]}$$
$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] \tag{5.30}$$

The hyperbolic formula (5.24) becomes

$$K(\eta;\alpha,\beta) = \frac{-2z[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)]}{(1-\alpha^2-\beta^2)[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)]}$$
$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right] \tag{5.31}$$

When the parabolic equation (5.15) is examined, it is found to contain only terms of $\sin\eta$ and $\cos\eta$. Hence, it will be a rational expression in terms of z. Substituting (5.26) and (5.27) into (5.15) produces, after extensive algebra,

$$K(\eta;\alpha,\beta) = \frac{6z(1+\alpha+\beta z)+4z^3}{3(1+\alpha+\beta z)^3} \tag{5.32}$$

Notice that if, in this equation, the epoch is reckoned at the pericenter then $\beta = 0$ and $\alpha = 1$, and the formula reduces to a cubic polynomial form well known as Barker's equation.

The advantage of these latter three formulae over their preceding versions becomes apparent when one is given the time elapsed from epoch and is required to find the corresponding angle η. Ordinarily, an iterative solution procedure must be used. With the above three equations available, z can be chosen as the variable of iteration and afterwards, if necessary, the angle itself can be once computed as $\eta = 2\tan^{-1}z$. The computations then need involve only one transcendental function evaluation per iteration cycle. Of course, in the parabolic case only the solution of a cubic equation is required.

It is worth noting that this same improvement could be introduced into any of the forms of Kepler's equation presented in Chapter 2, such as (2.67), (2.74) or (2.120). The variable of iteration would be tangent of half the eccentric anomaly in the elliptic case, or, in the hyperbolic case, hyperbolic tangent of half the hyperbolic anomaly. The relevant θ-domain (elliptic orbit) formulae are collected in Appendix C and presented there in various forms suggested by analogy with other η-domain formulae yet to be developed. The corresponding universal s-domain formulation was developed by Shepperd (1985) at the same time that the present work was being done. The independent variable defined by his equation (31) is $u = \frac{1}{\sqrt{-2E}}\tan(\frac{1}{4}\sqrt{-2E}\,s)$, where E is the energy, and the resulting time equation has only a single transcendental function.

Integration by the Method of Partial Fractions

The above results can also be obtained by introducing a change of variable into the quadrature (5.8). The new integrand is a rational expression which can be integrated by the method of partial fractions. This method leads rather directly to a universal true-anomaly time formula, and was the means by which that formula was first derived. Other features of this integration method make it useful for handling the quadratures in the J_2-perturbed problem later. Let

$$z = \tan\frac{1}{2}\eta \tag{5.33}$$

so that

$$\eta = 2\tan^{-1}z \tag{5.34}$$

$$d\eta = \frac{2dz}{1+z^2} \tag{5.35}$$

$$\sin\eta = \frac{2z}{1+z^2} \tag{5.36}$$

$$\cos\eta = \frac{1-z^2}{1+z^2} \tag{5.37}$$

Substituting these expressions into (5.8) allows the quadrature to be recast as

$$\frac{1}{2}K(\eta;\alpha,\beta) = \int_0^{\tan\frac{1}{2}\eta} \frac{(1+z^2)dz}{[(1-\alpha)z^2 + 2\beta z + (1+\alpha)]^2} \tag{5.38}$$

Completing the square in the denominator brings the integrand into a form suitable for expansion into partial fractions.

$$\frac{1}{2}K(\eta;\alpha,\beta) = \int_0^{\tan\frac{1}{2}\eta} \frac{(1+z^2)dz}{(1-\alpha)^2(z+C_1)^2(z+C_2)^2} \tag{5.39}$$

where

$$C_1 = \frac{\beta - \sqrt{\alpha^2+\beta^2-1}}{1-\alpha} \tag{5.40}$$

and

$$C_2 = \frac{\beta + \sqrt{\alpha^2+\beta^2-1}}{1-\alpha} \tag{5.41}$$

The integrand is now to be written in the form

$$\frac{(1+z^2)}{(1-\alpha)^2(z+C_1)^2(z+C_2)^2} = \frac{N_1}{(z+C_1)} + \frac{N_2}{(z+C_1)^2} + \frac{N_3}{(z+C_2)} + \frac{N_4}{(z+C_2)^2} \tag{5.42}$$

The constant numerator factors are determined from

$$\frac{(1+z^2)}{(1-\alpha)^2} = N_1(z+C_1)(z+C_2)^2 + N_2(z+C_2)^2 + N_3(z+C_1)^2(z+C_2) + N_4(z+C_1)^2 \tag{5.43}$$

Evaluating this expression at $z = -C_1$ yields

$$N_2 = \frac{1+C_1^2}{(1-\alpha)^2(C_2-C_1)^2} \tag{5.44}$$

Evaluating the same expression at $z = -C_2$ yields

$$N_4 = \frac{1 + C_2^2}{(1-\alpha)^2 (C_2 - C_1)^2} \tag{5.45}$$

Values for N_1 and N_3 can be determined by equating coefficients of like powers of z on both sides of (5.43). Equating coefficients of z^3 produces

$$N_3 = -N_1 \tag{5.46}$$

Equating coefficients of z^2 produces

$$N_1 = \frac{\dfrac{1}{(1-\alpha)^2} - N_2 - N_4}{C_2 - C_1} \tag{5.47}$$

so that one has, after substitutions,

$$N_1 = \frac{-2(1 + C_1 C_2)}{(1-\alpha)^2 (C_2 - C_1)^3} \tag{5.48}$$

$$N_3 = \frac{+2(1 + C_1 C_2)}{(1-\alpha)^2 (C_2 - C_1)^3} \tag{5.49}$$

Now then the quadrature

$$\frac{1}{2}K(\eta; \alpha, \beta) = \int_0^{\tan\frac{1}{2}\eta} \left[\frac{N_1}{(z + C_1)} + \frac{N_2}{(z + C_1)^2} + \frac{N_3}{(z + C_2)} + \frac{N_4}{(z + C_2)^2} \right] dz \tag{5.50}$$

can be evaluated straightforwardly. With $N_3 = -N_1$, the result is

$$\frac{1}{2}K(\eta; \alpha, \beta) = N_3 \ln\left[\frac{C_1(z + C_2)}{C_2(z + C_1)} \right] + N_2\left[\frac{z}{C_1(z + C_1)} \right] + N_4\left[\frac{z}{C_2(z + C_2)} \right] \tag{5.51}$$

When C_1, C_2, N_2, N_3 and N_4 are replaced in terms of α and β extensive algebraic reductions must ensue, but the steps are straightforward and the result is

$$K(\eta;\alpha,\beta)=\frac{2z[(1+\alpha+\beta z)+(\alpha^2+\beta^2-1)]}{(\alpha^2+\beta^2-1)[(1+\alpha+\beta z)^2-(\alpha^2+\beta^2-1)z^2]}$$

$$+\frac{1}{(\alpha^2+\beta^2-1)\sqrt{\alpha^2+\beta^2-1}}\ln\left[\frac{(\beta-\sqrt{\alpha^2+\beta^2-1})z+(1+\alpha)}{(\beta+\sqrt{\alpha^2+\beta^2-1})z+(1+\alpha)}\right] \tag{5.52}$$

This equation is the same as the hyperbolic formula (5.19) when (5.28) is taken into account. At this stage a universal form for $K(\eta;\alpha,\beta)$ could be obtained by expanding the logarithm in the following power series:

$$\frac{1}{2}\ln(x)=\left[\frac{x-1}{x+1}\right]+\frac{1}{3}\left[\frac{x-1}{x+1}\right]^3+\frac{1}{5}\left[\frac{x-1}{x+1}\right]^5+\frac{1}{7}\left[\frac{x-1}{x+1}\right]^7+\quad\ldots \tag{5.53}$$

This series is valid for $x>0$, the entire domain of the logarithm function. It is easy to show that the argument of the series is

$$\left[\frac{x-1}{x+1}\right]=\frac{-z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z} \tag{5.54}$$

Then by combining the first term of the series with the first term of (5.52) it proves to be possible to cancel all occurrences of $\sqrt{\alpha^2+\beta^2-1}$ and all denominator occurrences of $(\alpha^2+\beta^2-1)$. It is just these occurrences which would forbid continuous real-valued transition between elliptic and hyperbolic conditions. The final result of introducing the series (5.53) is then a modified series expression which can be applied without change to all three types of orbits. Only rectilinear orbits are excluded from consideration, as they are from all η-domain formulations.

The details of this and other universal expressions will be examined in a later section. For now it should be noted that equation (5.53) is really just the series form of the identity (5.23). Hence the expression for $K(\eta;\alpha,\beta)$ might as well be written as

$$K(\eta; \alpha, \beta) = \frac{2z[(1+\alpha+\beta z) + (\alpha^2+\beta^2-1)]}{(\alpha^2+\beta^2-1)[(1+\alpha+\beta z)^2 - (\alpha^2+\beta^2-1)z^2]}$$

$$+ \frac{2}{(\alpha^2+\beta^2-1)\sqrt{\alpha^2+\beta^2-1}} \tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right] \tag{5.55}$$

This equation is the same as the hyperbolic formula (5.31). In order to obtain a formula which is real-valued for elliptic orbits, the complex identities (5.25) could be employed, resulting in an equation identical with (5.30). The parabolic formula (5.32) appears when the modified series expression mentioned above is examined in the special case $\alpha^2 + \beta^2 = 1$.

The Relationship Between the True and Eccentric Anomalies

Before the universal forms of $K(\eta; \alpha, \beta)$ are examined, it will be helpful to examine the explicit relationship between the anomalies η and θ. In this section only nonrectilinear elliptic orbits are considered in order to ensure that both anomalies are defined unambiguously. However, it will be clear how to develop analogous formulae for nonrectilinear hyperbolic orbits as well.

Now since only elliptical orbits are being discussed, it is convenient to introduce some special notations for this section. The semimajor axis of the ellipse, a, is related to the energy by the well-known formula

$$E = -\frac{\mu}{2a} \tag{5.56}$$

Since the Laplace vector has a magnitude equal to μe, where e is the eccentricity of the ellipse, equation (1.18) of Chapter 1 can be written in the form

$$h^2 = \mu a(1 - e^2) \tag{5.57}$$

The eccentric-anomaly time transformation given by

$$\frac{d\theta}{dt} = \frac{\sqrt{-2E}}{r} \tag{5.58}$$

leads to the following expression for the radius:

$$r(\theta) = a(1 + \alpha_1 \cos\theta + \beta_1 \sin\theta) \tag{5.59}$$

where

$$\alpha_1 = \frac{r_0}{a} - 1 = -\,e\cos\psi_0 \tag{5.60}$$

$$\beta_1 = \frac{1}{\sqrt{\mu a}}(\underline{r}_0 \cdot \dot{\underline{r}}_0) = +\,e\sin\psi_0 \tag{5.61}$$

In these latter two formulae, ψ_0 is the eccentric anomaly at epoch as measured from pericenter. Similarly, the true-anomaly time transformation given by

$$\frac{d\eta}{dt} = \frac{h}{r^2} \tag{5.62}$$

leads to the following expression for the radius:

$$r(\eta) = \frac{\left[\dfrac{h^2}{\mu}\right]}{1 + \alpha\cos\eta + \beta\sin\eta} \tag{5.63}$$

where

$$\alpha = \frac{h^2}{\mu r_0} - 1 = +e\cos v_0 \tag{5.64}$$

$$\beta = -\frac{h}{\mu r_0}(\underline{r}_0 \cdot \dot{\underline{r}}_0) = -e\sin v_0 \tag{5.65}$$

In these latter two formulae, v_0 is the true anomaly at epoch as measured from the pericenter. From the above relations it is clear that

$$\alpha^2 + \beta^2 = \alpha_1^2 + \beta_1^2 = e^2 \tag{5.66}$$

Now the relation between η and θ can be developed in two complementary forms. Consider first

$$\frac{d\theta}{d\eta} = \frac{d\theta}{dt}\frac{dt}{d\eta} = \frac{\sqrt{-2E}}{h}\, r \tag{5.67}$$

Inserting $r(\eta)$ from (5.63) produces

$$d\theta = \frac{h}{\mu}\sqrt{-2E}\,\frac{d\eta}{(1 + \alpha\cos\eta + \beta\sin\eta)} \tag{5.68}$$

Alternatively, consider

$$\frac{d\eta}{d\theta} = \frac{d\eta}{dt}\frac{dt}{d\theta} = \frac{h}{\sqrt{-2E}}\,\frac{1}{r} \tag{5.69}$$

Inserting $r(\theta)$ from (5.59) produces

$$d\eta = \frac{h}{a\sqrt{-2E}}\,\frac{d\theta}{(1 + \alpha_1\cos\theta + \beta_1\sin\theta)} \tag{5.70}$$

However, from the elementary two-body relations (5.56) and (5.57) it is easy to show that

$$\frac{h}{\mu}\sqrt{-2E} = \frac{h}{a\sqrt{-2E}} = \sqrt{1 - e^2} \tag{5.71}$$

Then, bearing in mind the relation (5.66), one has the nicely symmetrical pair of relations between $d\eta$ and $d\theta$:

$$d\theta = \frac{\sqrt{1-\alpha^2-\beta^2}}{1+\alpha\cos\eta+\beta\sin\eta}\, d\eta \tag{5.72}$$

$$d\eta = \frac{\sqrt{1-\alpha_1^2-\beta_1^2}}{1+\alpha_1\cos\theta+\beta_1\sin\theta}\, d\theta \tag{5.73}$$

The integrated form is available immediately from (5.11).

$$\frac{1}{2}\theta = \tan^{-1}\left[\frac{(1-\alpha)\tan\frac{1}{2}\eta+\beta}{\sqrt{1-\alpha^2-\beta^2}}\right] - \tan^{-1}\left[\frac{\beta}{\sqrt{1-\alpha^2-\beta^2}}\right] \tag{5.74}$$

$$\frac{1}{2}\eta = \tan^{-1}\left[\frac{(1-\alpha_1)\tan\frac{1}{2}\theta+\beta_1}{\sqrt{1-\alpha_1^2-\beta_1^2}}\right] - \tan^{-1}\left[\frac{\beta_1}{\sqrt{1-\alpha_1^2-\beta_1^2}}\right] \tag{5.75}$$

The arctangent identity (5.17) shortens these formulae to

$$\tan\frac{1}{2}\theta = \frac{\sqrt{1-\alpha^2-\beta^2}\,\tan\frac{1}{2}\eta}{1+\alpha+\beta\tan\frac{1}{2}\eta} \tag{5.76}$$

$$\tan\frac{1}{2}\eta = \frac{\sqrt{1-\alpha_1^2-\beta_1^2}\,\tan\frac{1}{2}\theta}{1+\alpha_1+\beta_1\tan\frac{1}{2}\theta} \tag{5.77}$$

The classical Gaussian relation

$$\tan\frac{1}{2}\psi = \sqrt{\frac{1-e}{1+e}}\;\tan\frac{1}{2}v \tag{5.78}$$

where v and ψ are the true and eccentric anomalies respectively, measured from pericenter, is a special case of the two complementary relations (5.76) and (5.77).

It can be seen that all of the above operations could be repeated for hyperbolic orbits if the hyperbolic anomaly ϕ were introduced by means of the time transformation

$$\frac{d\phi}{dt} = \frac{\sqrt{+2E}}{r} \tag{5.79}$$

For later purposes, it is sufficient to observe that θ and ϕ are related by

$$\theta = \sqrt{-1}\ \phi \tag{5.80}$$

Then the complex identities (5.25) allow the formula (5.76) to be rewritten as

$$\tanh\frac{1}{2}\phi = \frac{\sqrt{\alpha^2+\beta^2-1}\ \tan\frac{1}{2}\eta}{1+\alpha+\beta\tan\frac{1}{2}\eta} \tag{5.81}$$

Comparison of (5.76) with the elliptic formula (5.30) shows that the latter contains a term proportional to the eccentric anomaly θ . Likewise, the hyperbolic formula (5.31) contains a term proportional to the hyperbolic anomaly ϕ. As to be expected, the true-anomaly time equations given earlier are equivalent to the eccentric-anomaly Kepler's equations.

The Universal True-Anomaly Time Equation

Much of the utility of the true-anomaly solution of the differential equation of time derives from reducing the elliptic and hyperbolic formulae given earlier to a single, real-valued, universal form. This section presents the algebraic steps involved in such a reduction.

It was indicated in a previous section that a power series expansion of the natural logarithm does lead to a universal expression. That series was

$$\frac{1}{2}\ln(x) = \left[\frac{x-1}{x+1}\right] + \frac{1}{3}\left[\frac{x-1}{x+1}\right]^3 + \frac{1}{5}\left[\frac{x-1}{x+1}\right]^5 + \frac{1}{7}\left[\frac{x-1}{x+1}\right]^7 + \cdots \tag{5.82}$$

which is valid for $0 < x < \infty$, or for

$$-1 < \left[\frac{x-1}{x+1}\right] < +1 \tag{5.83}$$

By means of the identities (5.23) and (5.25), the following series are also available:

$$\tanh^{-1}(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots \tag{5.84}$$

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \tag{5.85}$$

These two series are both valid for $-1 < x < +1$, and are also a means by which the elliptic and hyperbolic forms given earlier can be brought into a common form. Specifically,

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] =$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\left[\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] - \frac{1}{3}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right]^3\right.$$

$$\left. + \frac{1}{5}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right]^5 - \frac{1}{7}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right]^7 + \cdots\right] \tag{5.86}$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] =$$

$$\frac{1}{(1-\alpha^2-\beta^2)}\left[\left[\frac{z}{1+\alpha+\beta z}\right] - \frac{1}{3}\left[\frac{z}{1+\alpha+\beta z}\right]^3(1-\alpha^2-\beta^2)\right.$$

$$+\frac{1}{5}\left[\frac{z}{1+\alpha+\beta z}\right]^5(1-\alpha^2-\beta^2)^2-\frac{1}{7}\left[\frac{z}{1+\alpha+\beta z}\right]^7(1-\alpha^2-\beta^2)^3+\cdots\Bigg] \tag{5.87}$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right]=$$

$$\frac{1}{(1-\alpha^2-\beta^2)}\left[\frac{z}{1+\alpha+\beta z}\right]-\left[\frac{z}{1+\alpha+\beta z}\right]^3\left[\frac{1}{3}-\frac{1}{5}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]\right.$$

$$\left.+\frac{1}{7}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^2-\frac{1}{9}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^3+\cdots\right] \tag{5.88}$$

Likewise, the hyperbolic arctangent can be expressed as

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]=$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\left[\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]+\frac{1}{3}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]^3\right.$$

$$\left.+\frac{1}{5}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]^5+\frac{1}{7}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]^7+\cdots\right] \tag{5.89}$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]=$$

$$\frac{1}{(1-\alpha^2-\beta^2)}\left[\left[\frac{z}{1+\alpha+\beta z}\right]+\frac{1}{3}\left[\frac{z}{1+\alpha+\beta z}\right]^3(\alpha^2+\beta^2-1)\right.$$

$$\left.+\frac{1}{5}\left[\frac{z}{1+\alpha+\beta z}\right]^5(\alpha^2+\beta^2-1)^2+\frac{1}{7}\left[\frac{z}{1+\alpha+\beta z}\right]^7(\alpha^2+\beta^2-1)^3+\cdots\right] \tag{5.90}$$

$$\frac{1}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right]=$$

$$\frac{1}{(1-\alpha^2-\beta^2)}\left[\frac{z}{1+\alpha+\beta z}\right]-\left[\frac{z}{1+\alpha+\beta z}\right]^3\left[\frac{1}{3}-\frac{1}{5}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]\right.$$

$$\left.+\frac{1}{7}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^2-\frac{1}{9}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^3+\cdots\right] \tag{5.91}$$

The right-hand sides of (5.91) and (5.88) are identical, meaning that the elliptic formula (5.30) and the hyperbolic formula (5.31) are identical when written as

$$K(\eta;\alpha,\beta)=\frac{-2z[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)]}{(1-\alpha^2-\beta^2)[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)]}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)}\left[\frac{z}{1+\alpha+\beta z}\right]-2\left[\frac{z}{1+\alpha+\beta z}\right]^3\left[\frac{1}{3}-\frac{1}{5}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]\right.$$

$$\left.+\frac{1}{7}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^2-\frac{1}{9}\left[\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}\right]^3+\cdots\right] \tag{5.92}$$

Collecting the first two terms over a common denominator produces

$$K(\eta;\alpha,\beta)=$$

$$\frac{2[z(1+\alpha+\beta z)+z^3]}{(1+\alpha+\beta z)^3(1+x)}-2\left[\frac{z}{1+\alpha+\beta z}\right]^3\left[\frac{1}{3}-\frac{1}{5}x+\frac{1}{7}x^2-\frac{1}{9}x^3+\cdots\right] \tag{5.93}$$

where

$$x=\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2} \tag{5.94}$$

Equation (5.93) can be used for all three kinds of orbits provided the series part converges and provided no denominator vanishes. The convergence of the series is easy to establish using the fact that an infinite series of non-zero terms, $a_1 + a_2 + a_3 + \cdots + a_k + \cdots$, converges absolutely if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1 \tag{5.95}$$

The general term of the series in question is evident from

$$\frac{1}{3}-\frac{1}{5}x+\frac{1}{7}x^2-\frac{1}{9}x^3+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{2k+3} \tag{5.96}$$

Then

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{k+1}}{2(k+1)+3}\cdot\frac{2k+3}{(-1)^k x^k}\right| = \lim_{k\to\infty}\left|\frac{2k+3}{2k+5}(-x)\right| = |x| \tag{5.97}$$

Hence the series converges for $-1 < x < +1$. For an elliptic orbit, one has $\alpha^2+\beta^2<1$ so that $x > 0$ always. Comparison of equation (5.94) with (5.76) shows that, in fact,

$$x = \tan^2\frac{1}{2}\theta \tag{5.98}$$

for an elliptic orbit. This means that equation (5.93) can be used for elliptic transfers provided $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$. The corresponding permissible interval of η can be calculated from equation (5.77) as

$$2\tan^{-1}\left[\frac{-\sqrt{1-\alpha_1^2-\beta_1^2}}{1+\alpha_1-\beta_1}\right] < \eta < 2\tan^{-1}\left[\frac{+\sqrt{1-\alpha_1^2-\beta_1^2}}{1+\alpha_1+\beta_1}\right] \tag{5.99}$$

or from equation (5.76) as

$$2\tan^{-1}\left[\frac{-(1+\alpha)}{\beta+\sqrt{1-\alpha^2-\beta^2}}\right]<\eta<2\tan^{-1}\left[\frac{(1+\alpha)}{-\beta+\sqrt{1-\alpha^2-\beta^2}}\right] \tag{5.100}$$

In evaluating the arctangents it may be necessary to add or subtract π radians to the arctangent at one end of the interval or the other in order to obtain the correct quadrants. Whether or not this will be necessary is determined by the fact that the interval must contain the point $\eta = 0$. The interval so defined by (5.100) is often appreciably less than the entire arc of the ellipse, as may be verified quickly by choosing the epoch at the apocenter of an eccentric orbit: $\alpha = -e$, $\beta = 0$. It is worth mentioning that, for an elliptic orbit, the series in equation (5.93) has alternating signs. In that case, the numerical error committed in truncating the series is less than the absolute value of the first neglected term. For a parabolic orbit, one has $\alpha^2 + \beta^2 = 1$ so that x vanishes identically. Equation (5.93) reduces to the rational algebraic form given in (5.32), which is valid for the entire arc of the parabola. For a hyperbolic orbit, one has $\alpha^2 + \beta^2 > 1$ so that $x < 0$ always and the series contains only positive terms. Comparison of equation (5.94) with equation (5.81) shows that, in fact,

$$x = -\tanh^2\frac{1}{2}\phi \tag{5.101}$$

for hyperbolic orbits. This means that the series in equation (5.93) converges absolutely for $-1 < \tanh\frac{1}{2}\phi < +1$, that is, for all real values of ϕ. The corresponding interval of η calculated from (5.81) as

$$2\tan^{-1}\left[\frac{-(1+\alpha)}{\beta+\sqrt{\alpha^2+\beta^2-1}}\right]<\eta<2\tan^{-1}\left[\frac{(1+\alpha)}{-\beta+\sqrt{\alpha^2+\beta^2-1}}\right] \tag{5.102}$$

therefore includes the entire arc of the hyperbola. The endpoints of this interval are the hyperbolic asymptotes. (As before, it may be necessary to add or subtract π radians to the arctangent at one end of the interval or the other.) Notice also that both of these endpoints correspond to $x = -1$ according to equation (5.101). Hence the denominator factor $(1 + x)$ is always positive and cannot vanish or become negative at any physically realizable point on any kind of nonrectilinear orbit. On the other hand, the denominator factor $(1 + \alpha + \beta z)$ in equation (5.93) may be either positive or negative, and may vanish or change sign under certain circumstances. For a hyperbolic orbit, one may conclude from equation (5.81) that $(1 + \alpha + \beta z)$ does not vanish, and hence does not change sign, anywhere on the entire arc of the hyperbola. If it did vanish, then $\tanh \frac{1}{2}\phi$ would go to infinity, which is impossible. In fact, one may conclude that $(1 + \alpha + \beta z)$ is always positive on a hyperbola and no smaller than $|z|\sqrt{\alpha^2 + \beta^2 - 1}$, because ϕ has the same sign as η and $|\tanh \frac{1}{2}\phi|$ never exceeds 1. The possibility of $(1 + \alpha + \beta z)$ vanishing together with z in such a way that $|\tanh \frac{1}{2}\phi|$ never exceeds 1 can be ruled out as not physically realizable since it requires $\alpha = -1$. Either of equations (5.4) or (5.64) shows that if $\alpha = -1$ then the epochal position is an infinite distance away and located precisely on an asymptote. For a parabolic orbit, one may conclude from equation (5.32) that the transfer time goes to infinity when the factor $(1 + \alpha + \beta z)$ vanishes. Hence $(1 + \alpha + \beta z)$ does not change sign anywhere on the whole arc of the parabola, and because it is positive at $z = 0$ it is always positive. For an elliptic orbit, equation (5.76) indicates that $\theta = \pm\pi$ when the factor $(1 + \alpha + \beta z)$ vanishes. Hence $(1 + \alpha + \beta z)$ changes sign once at some point on every ellipse, and this point is a singularity of the time equation (5.93). The singular point at $\theta = \pm\pi$ is itself of no direct concern in using equation (5.93) because the series part is valid only in the interval $-\frac{\pi}{2} < \theta < +\frac{\pi}{2}$. However, the fact that $(1 + \alpha + \beta z)$ changes sign can lead to a possible ambiguity in the calculation of time, namely, that negative elapsed times may be calculated for positive values of η corresponding to $\pi < \theta < 2\pi$. This ambiguity occurs only on elliptic orbits and is resolved by simply adding one time period of revolution to any negative elapsed time when η is positive. The period T calculated from Kepler's Third Law

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{5.103}$$

can be expressed directly in terms of η-domain quantities as

$$T = \frac{2\pi h^3}{\mu^2\sqrt{(1 - \alpha^2 - \beta^2)^3}} \tag{5.104}$$

using equations (5.57) and (5.66). These features of the time calculation based on equation (5.93) are illustrated by the example calculations recorded in Appendix D.

Another aspect of the formula (5.93), and all the formulae in this chapter which involve $\tan\frac{1}{2}\eta$, is that $z = \tan\frac{1}{2}\eta$ goes to infinity at $\eta = \pi$ (or $\eta = -\pi$). The presence of this function is a consequence of evaluating the quadrature (5.8) in closed form, but it effectively prevents the formulae given so far in this chapter from being used at or near $\eta = \pi$. Regardless of whether the series in (5.93) converges or not, for η sufficiently near π the value of z will exceed the arithmetic range of the computing device, rendering the formulae numerically useless in this region. This limitation is especially serious. Orbital transfer maneuvers involving 180° of true anomaly, say, from apse to apse, are important in engineering practice. Even if the series convergence problem could be overcome for such a case, the numerical "overflow" problem would remain.

The conclusion to be drawn from the above considerations is that equation (5.93) does qualify as a universally valid expression relating time and true anomaly, but that its practical application is nevertheless limited. In a simple time-of-flight calculation where α, β and the transfer angle η are given, the limitations can be overcome at the expense of a little extra programming. Excessively large values of z and nonconvergence of the series can be avoided, if necessary, by breaking the given angle η into smaller subarcs and summing the times for each subarc. Since equation (5.93) is referred to an arbitrary epoch, the programming for this procedure need not be complicated. For the initial-value problem, given α, β and time of flight to compute η (or z), the same limitations could be avoided by a similar procedure of splitting the time into smaller subintervals. However, for the boundary-value problem, given η and time of flight to compute α and β, the aforementioned limitations on equation (5.93) prove to be fatal. The reason is that one does not know whether the series will converge because α and β are unknown. Also, since an iterative solution is usually implemented, another difficulty arises. Even if the series converges at the solution, there is no guarantee that it will converge on every intervening iteration. Furthermore, equation (5.93) is not suitable for boundary-value problems with $\eta = \pi$ because z is infinite and there is no readily apparent way to split the angle or the time into smaller subintervals. The remedies for these limitations on equation (5.93) are the subject of the remainder of this chapter.

The Use of Continued Fractions

H. S. Wall (1948, p. 343) gives the following infinite continued-fraction representations.

$$\tan^{-1} y = \cfrac{y}{1 + \cfrac{1^2 y^2}{3 + \cfrac{2^2 y^2}{5 + \cfrac{3^2 y^2}{7 + \cfrac{4^2 y^2}{9 + \dots}}}}} \tag{5.105}$$

In contrast to the series representation (5.85), this continued fraction is valid for all real y, the entire domain of the arctangent function. Similarly,

$$\tanh^{-1} y = \cfrac{y}{1 - \cfrac{1^2 y^2}{3 - \cfrac{2^2 y^2}{5 - \cfrac{3^2 y^2}{7 - \cfrac{4^2 y^2}{9 - \dots}}}}} \tag{5.106}$$

This continued fraction, like the series (5.84), is valid for $-1 < y < +1$, the entire domain of the hyperbolic arctangent function. The fact that these two expressions differ only in the sign of y^2 means that the elliptic formula (5.30) and the hyperbolic formula (5.31) are identical when written as

$$K(\eta;\alpha,\beta)=\frac{-2z[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)]}{(1-\alpha^2-\beta^2)(1+\alpha+\beta z)^2(1+x)}$$

$$+\frac{2z}{(1-\alpha^2-\beta^2)(1+\alpha+\beta z)(1+C_1)} \tag{5.107}$$

where

$$C_1=\frac{x}{3+C_2} \tag{5.108}$$

$$C_2=\cfrac{2^2x}{5+\cfrac{3^2x}{7+\cfrac{4^2x}{9+\cfrac{5^2x}{11+\ldots}}}} \tag{5.109}$$

$$x=\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2} \tag{5.110}$$

Collecting the terms of equation (5.107) over a common denominator produces

$$K(\eta;\alpha,\beta)=$$

$$\frac{-2z(1+\alpha+\beta z)(1+C_1)+2z(1-\alpha^2-\beta^2)(1+C_1)+2z(1+\alpha+\beta z)(1+x)}{(1-\alpha^2-\beta^2)(1+\alpha+\beta z)^2(1+x)(1+C_1)} \tag{5.111}$$

After straightforward algebra, the factor $(1-\alpha^2-\beta^2)$ can be cancelled from numerator and denominator, leaving

$$K(\eta;\alpha,\beta)=\frac{2z\left[(1+\alpha+\beta z)(1+C_1)+z^2\left[1-\frac{1}{3+C_2}\right]\right]}{(1+\alpha+\beta z)^3(1+x)(1+C_1)} \tag{5.112}$$

This is the continued-fraction version of the universal formula (5.93). Like that formula, it can be used for all three kinds of orbits provided the continued fraction converges and provided no denominator vanishes. Observe that in the above manipulations only the top numerators of (5.105) and (5.106) were modified, as well as the top numerator of (5.108). Hence, it is evident that the continued fractions $1/(1+C_1)$, $x/(3+C_2)$ and $1/(3+C_2)$ all converge. It then follows that the denominator factors $(1+C_1)$ and $(3+C_2)$ do not vanish. The remarks already made about the factors $(1+x)$ and $(1+\alpha+\beta z)$ apply here as well.

Now considering the intervals over which equations (5.105) and (5.106) are valid, the interval of x over which (5.112) is valid should be $-1 < x < \infty$, which is (almost) the entire domain of interest for orbital problems. The value $x = -1$ corresponds to the asymptotes of a hyperbolic orbit. A value of x in the interval $-1 < x < 0$ corresponds to a point on the arc of a hyperbola. The value $x = 0$ holds at every point on a parabola. A value of x in the interval $0 < x < \infty$ corresponds to a point on the arc of an ellipse, and the case $x \to \infty$ corresponds to the point on the ellipse at which $\theta = \pi$ (or $\theta = -\pi$). This point is a singularity of equation (5.112), but the equation should remain valid at every other point of the ellipse. The interval of η corresponding to $-\pi < \theta < +\pi$ is, according to equation (5.76), determined from

$$1 + \alpha + \beta \tan \frac{1}{2}\eta = 0 \tag{5.113}$$

assuming temporarily that $\beta \neq 0$ and bearing in mind that the interval must contain the point $\eta = 0$. Thus, the value

$$\bar{\eta} = 2\tan^{-1}\left[\frac{1+\alpha}{-\beta}\right] \tag{5.114}$$

where $\bar{\eta}$ lies between $-\pi$ and $+\pi$, is one endpoint of the interval. If $\bar{\eta} < 0$ then the other endpoint will be $\bar{\eta} + 2\pi$. If $\bar{\eta} > 0$ then the other endpoint will be $\bar{\eta} - 2\pi$. The value $\bar{\eta} = 0$ is not physically realizable since it requires $\alpha = -1$ or $\beta \to \pm\infty$. Now since $(1+\alpha) > 0$ always, the sign of $\bar{\eta}$ is controlled by the sign of β. Therefore the permissible interval of η on the elliptic orbit is given by

$$2\tan^{-1}\left[\frac{1+\alpha}{-\beta}\right]-2\pi<\eta<2\tan^{-1}\left[\frac{1+\alpha}{-\beta}\right] \tag{5.115a}$$

if $\beta < 0$, and by

$$2\tan^{-1}\left[\frac{1+\alpha}{-\beta}\right]<\eta<2\tan^{-1}\left[\frac{1+\alpha}{-\beta}\right]+2\pi \tag{5.115b}$$

if $\beta > 0$. The limiting case of $\beta = 0$ corresponds to an epoch at any point of a circular orbit or at an apse of an elliptical orbit. Letting $\beta \to 0$ in either of the above two formulae produces the result

$$-\pi<\eta<+\pi \tag{5.115c}$$

when $\beta = 0$. Of course, as discussed before, the permissible intervals of η for parabolic and hyperbolic orbits include the entire arc of the orbit.

It should be remembered that the intervals just given for η are only those for which the continued fraction C_2 is mathematically convergent; nothing has been concluded about the rate of convergence. In practice, it is found that for both the series equation (5.93) and the continued-fraction equation (5.112) the convergence is rapid near $\eta = 0$ and quite slow near the endpoints of the interval. Some numerical examples which illustrate these features for equation (5.112) are given in Appendix D. Also, as was the case with equation (5.93), the factor $(1 + \alpha + \beta z)$ changes sign from positive to negative at the singular point $\theta = \pi$ on elliptic orbits. Because the formula is valid for $-\pi < \theta < +\pi$, values of η corresponding to $\pi < \theta < 2\pi$ lead to elapsed times calculated as if θ were in the interval $-\pi < \theta < 0$. If the elapsed time has a negative value when $\eta > 0$, one has only to add the orbital period to obtain the correct elapsed time. Finally, like equation (5.93), equation (5.112) becomes numerically unusable very near $\eta = \pm\pi$ due to the magnitude of z becoming excessively large.

An important note should be included here about evaluating continued fractions numerically. Suppose a convergent continued fraction of the following form is to be evaluated:

$$F = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}} \tag{5.116}$$

A certain approximation of the value of F is obtained by truncating the fraction after one of the partial denominators b_k and computing in a "bottom-up" fashion. Typically, a better approximation is obtained by truncating farther down in the fraction, in which case the bottom-up computation becomes more extensive. Unfortunately, it is not generally known in advance how many partial denominators must be retained in order to achieve a given accuracy in the value of F. The result is that several, and sometimes many, succesive bottom-up evaluations have to be made, a process which can become tedious and expensive. Fortunately, a very efficient alternative procedure has been developed by Gautschi (1967, pp. 28 - 30) based on a theorem given in Wall (1948, pp. 17 - 18). For values of an index $k = 1,\ 2,\ 3,\ \ldots\ \infty$ the following three formulae are to be evaluated in sequence:

$$u_{k+1} = \frac{1}{1 + \left[\frac{a_{k+1}}{b_k b_{k+1}}\right] u_k} \tag{5.117}$$

$$v_{k+1} = v_k(u_{k+1} - 1) \tag{5.118}$$

$$w_{k+1} = w_k + v_{k+1} \tag{5.119}$$

where

$$u_1 = 1 \tag{5.120}$$

$$v_1 = w_1 = \frac{a_1}{b_1} \tag{5.121}$$

If the continued fraction converges, the value of w_k approaches the value of the continued fraction as $k \to \infty$. The most striking feature of Gautschi's method is that the a_k and b_k enter the computation in "top-down" order, so that the continued fraction is evaluated recursively. Thus, in order to evaluate the continued fraction C_2 in equation (5.112), the partial numerators a_k and the partial denominators b_k would be computed from

$$a_k = (k+1)^2 x \tag{5.122}$$

$$b_k = 2k + 3 \tag{5.123}$$

with $k = 1, 2, 3, \ldots$

It is appropriate to mention here, though more detail will be given in a later chapter, that the use of continued fractions in place of series and the use of Gautschi's method to evaluate them were ideas introduced into orbital mechanics by Battin (1968, 1977). His purpose was to develop an improved solution procedure for the Gauss/Lambert problem. Actually, Gauss himself (1809) exhibited several continued-fraction expansions of functions required in the boundary-value problem, but he did not propose to use them for calculation, resorting instead to infinite series for numerical work. Likewise, Herget (1948, chapter 5) observes that the main computation in Gauss's approach can be reduced to the evaluation of a hypergeometric function, and that this function is represented by one of Gauss's continued fractions. But Herget follows Gauss in preferring to calculate with series. In retrospect, it seems that some of the practical difficulties which beset Gauss's approach can be traced to the limited radius of convergence of the series employed and could be overcome merely through the use of continued fractions (see, for example, the comments in Bate, *et al.*, (1971) near the end of their Section 5.6). Battin (1978, 1983) also developed major improvements of his own work, and in the latter (1983) paper the continued fraction $1/(1 + C_1)$ appears as the expansion of a hypergeometric function (which is the only transcendental function) in the time-of-flight equation. There is thus a close affinity between Battin's results and the results of this study even though the underlying reasoning and derivations are not at all similar. Battin's four articles cited above have dealt almost exclusively with the boundary-value problem and have made extensive use of manipulations peculiar to that problem. (The first (1968) paper did outline the initial-value problem briefly.) The present study aims at formulae which can be adapted equally well for time-of-flight calculations, initial-value problems, and boundary-value problems. Also, this study attempts to show explicitly how such results develop out of a regularization of the governing equations of motion.

The Use of Half-Angle Transformations

Now that the interval of convergence of the universal time equation has been enlarged through the use of a continued fraction, it is still desirable to eliminate, if possible, the singularity on elliptic orbits at the point $\theta = \pi$. The continued fraction does not converge at this point because the argument of the continued fraction, $x = \tan^2 \frac{1}{2}\theta$, goes to infinity. In the vicinity of $\theta = \pi$ the convergence may be too slow to be useful. In addition, as mentioned before, numerical difficulties occur in the neughborhood of $\eta = \pi$ because the value of $z = \tan \frac{1}{2}\eta$ goes to infinity at this point. Both of these difficulties can be alleviated through the following considerations.

In the elliptic formula (5.30) the eccentric anomaly θ occurs implicitly as

$$\theta = 2\tan^{-1}\left[\tan \frac{1}{2}\theta \right] \tag{5.124}$$

Any numerical evaluation of this arctangent breaks down near $\theta = \pi$. However, θ might as well be calculated by

$$\theta = 4\tan^{-1}\left[\tan \frac{1}{4}\theta \right] \tag{5.125}$$

or by

$$\theta = 8\tan^{-1}\left[\tan\frac{1}{8}\theta\right] \tag{5.126}$$

or by any like expression. In (5.125) the singularity in the evaluation of the arctangent has been moved to $\theta = 2\pi$; in (5.126) it has been moved to $\theta = 4\pi$. Thus the possibility exists of extending the interval of convergence of the continued fraction over one or more revolutions in the elliptic case. For this purpose one needs to be able to introduce $\tan\frac{1}{4}\theta$ in terms of $\tan\frac{1}{2}\theta$, the latter being already known in terms of η-domain quantities. The sum-of-angles identity (5.16) immediately gives

$$\tan\frac{1}{2}\theta = \frac{2\tan\frac{1}{4}\theta}{1-\tan^2\frac{1}{4}\theta} \tag{5.127}$$

Solving this quadratic equation for $\tan\frac{1}{4}\theta$ produces

$$\tan\frac{1}{4}\theta = \frac{1 \pm \sqrt{1+\tan^2\frac{1}{2}\theta}}{-\tan\frac{1}{2}\theta} \tag{5.128}$$

The negative sign should be selected on the radical because $\left|\tan\frac{1}{4}\theta\right| < \left|\tan\frac{1}{2}\theta\right|$ in the interval $-\pi < \theta < +\pi$. Then the following formula is readily obtained:

$$\tan\frac{1}{4}\theta = \frac{\tan\frac{1}{2}\theta}{1+\sqrt{1+\tan^2\frac{1}{2}\theta}} \tag{5.129}$$

This identity maps the range $(-\infty, +\infty)$ of $\tan\frac{1}{2}\theta$ into the range $(-1, +1)$ of $\tan\frac{1}{4}\theta$. Of course, one can also replace $\tan\frac{1}{2}\eta$ in terms of $\tan\frac{1}{4}\eta$ by

$$\tan\frac{1}{2}\eta = \frac{2\tan\frac{1}{4}\eta}{1-\tan^2\frac{1}{4}\eta} \tag{5.130}$$

Now in order for the final results of these substitutions to be universal, analogous operations must be possible with the hyperbolic formula (5.31). There the hyperbolic anomaly ϕ occurs implicitly as

$$\phi = 2\tanh^{-1}\left[\tanh\frac{1}{2}\phi\right] \tag{5.131}$$

Though the numerical evaluation of this arctangent breaks down near $\tanh\frac{1}{2}\phi = 1$ (that is, as $\phi \to \infty$, the asymptote of the orbit), ϕ might as well be calculated by

$$\phi = 4\tanh^{-1}\left[\tanh\frac{1}{4}\phi\right] \tag{5.132}$$

or by

$$\phi = 8\tanh^{-1}\left[\tanh\frac{1}{8}\phi\right] \tag{5.133}$$

or by any like expression. In these equations the singularity in the evaluation of the arctangent has been postponed to higher values of ϕ, although the magnitude of the argument itself can never be larger than 1. In order to obtain this improvement in the hyperbolic case, one needs to be able to introduce $\tanh\frac{1}{4}\phi$ in terms of $\tanh\frac{1}{2}\phi$. The hyperbolic analog of the identity (5.127) is

$$\tanh\frac{1}{2}\phi = \frac{2\tanh\frac{1}{4}\phi}{1+\tanh^2\frac{1}{4}\phi} \tag{5.134}$$

Solving this quadratic equation for $\tanh\frac{1}{4}\phi$ produces

$$\tanh\frac{1}{4}\phi = \frac{1 \pm \sqrt{1-\tanh^2\frac{1}{2}\phi}}{+\tanh\frac{1}{2}\phi} \tag{5.135}$$

The negative sign should be selected on the radical because both $\left|\tanh\frac{1}{4}\phi\right| < 1$ and $\left|\tanh\frac{1}{2}\phi\right| < 1$ for all real ϕ. Then the following formula is readily obtained:

$$\tanh\frac{1}{4}\phi = \frac{\tanh\frac{1}{2}\phi}{1+\sqrt{1-\tanh^2\frac{1}{2}\phi}} \tag{5.136}$$

Of course, one can also replace $\tan\frac{1}{2}\eta$ in terms of $\tan\frac{1}{4}\eta$ by means of equation (5.130) in the hyperbolic case.

A universal formula comparable to equation (5.112), but valid for $-2\pi < \theta < +2\pi$ on elliptic orbits, is now developed by means of the quarter-angle identities above. Substitute (5.130) into (5.76) to obtain

$$\tan\frac{1}{2}\theta = \frac{2\sqrt{1-\alpha^2-\beta^2}\,\tan\frac{1}{4}\eta}{(1+\alpha)\left[1-\tan^2\frac{1}{4}\eta\right]+2\beta\tan\frac{1}{4}\eta} \tag{5.137}$$

Substitute this formula into (5.129) to obtain $\tan\frac{1}{4}\theta$ in terms of $\tan\frac{1}{4}\eta$. The result is

$$\tan\frac{1}{4}\theta = \frac{2z\sqrt{1-\alpha^2-\beta^2}}{(1+\alpha)(1-z^2)+2\beta z+\sqrt{[(1+\alpha)(1-z^2)+2\beta z]^2+4z^2(1-\alpha^2-\beta^2)}} \tag{5.138}$$

where now $z = \tan\frac{1}{4}\eta$. Similar operations involving equations (5.130), (5.81) and (5.136) produce the hyperbolic result

$$\tanh\frac{1}{4}\phi = \frac{2z\sqrt{\alpha^2-\beta^2-1}}{(1+\alpha)(1-z^2)+2\beta z+\sqrt{[(1+\alpha)(1-z^2)+2\beta z]^2-4z^2(\alpha^2+\beta^2-1)}} \tag{5.139}$$

where again $z = \tan\frac{1}{4}\eta$. Notice that equations (5.138) and (5.139) differ only by an overall factor of $\sqrt{-1}$. The elliptic time formula (5.30) is put in terms of quarter angles using equations (5.130) and (5.125). Simple steps lead to

$$K(\eta;\alpha,\beta) = \frac{-4z\left[(1+\alpha)(1-z^2)+2\beta z-(1-z^2)(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[\left[(1+\alpha)(1-z^2)+2\beta z\right]^2+4z^2(1-\alpha^2-\beta^2)\right]} +$$

$$\frac{4}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\tan\frac{1}{4}\theta\right] \tag{5.140}$$

where $z=\tan\frac{1}{4}\eta$. Substituting for $\tan\frac{1}{4}\theta$ in terms of z, α and β from equation (5.138) and expanding the arctangent as a continued fraction via (5.105) produces

$$K(\eta;\alpha,\beta)=\frac{-4z\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[D^2+4z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{8z}{(1-\alpha^2-\beta^2)\left[D+\sqrt{D^2+4z^2(1-\alpha^2-\beta^2)}\,\right](1+C_1)} \tag{5.141}$$

where

$$D=\left[(1+\alpha)(1-z^2)+2\beta z\right] \tag{5.142}$$

$$C_1=\frac{y}{3+C_2} \tag{5.143}$$

$$C_2=\cfrac{2^2y}{5+\cfrac{3^2y}{7+\cfrac{4^2y}{9+\cfrac{5^2y}{11+\cdots}}}} \tag{5.144}$$

$$y=\frac{4z^2(1-\alpha^2-\beta^2)}{\left[D+\sqrt{D^2+4z^2(1-\alpha^2-\beta^2)}\,\right]^2}=\tan^2\frac{1}{4}\theta=-\tanh^2\frac{1}{4}\phi \tag{5.145}$$

and, of course, $z=\tan\frac{1}{4}\eta$. These formulae hold true for the hyperbolic case as well because equations (5.138) and (5.139) differ only by an overall factor of $\sqrt{-1}$, which factor has been cancelled in (5.141). In fact, equations (5.141) through (5.145) are just the quarter-angle versions of the half-angle formulae (5.107) through (5.110).

When one is trying to interpret these formulae, it is helpful to remember that x, which is given in terms of $\tan\frac{1}{2}\eta$ in equation (5.110), would be given in terms of $\tan\frac{1}{4}\eta$ as

$$x = \frac{4z^2(1-\alpha^2-\beta^2)}{\left[(1+\alpha)(1-z^2)+2\beta z\right]^2} = \frac{4z^2(1-\alpha^2-\beta^2)}{D^2} \tag{5.146}$$

where (5.130) has been used. Then, for comparison, equation (5.145) could be written as

$$y = \frac{x}{\left[1+\sqrt{1+x}\,\right]^2} \tag{5.147}$$

which is just the half-angle identity (5.129) squared. However, it is preferable not to introduce the quantity x into the formulae now being derived because x goes to infinity at $\theta = \pi$. On the other hand, the half-angle identity (5.147) maps the range $(-1, \infty)$ of x into the range $(-1, +1)$ of y, and y itself goes to infinity at $\theta = 2\pi$.

Now it remains to cancel the denominator factor $(1-\alpha^2-\beta^2)$ in (5.141). Collecting both terms of that equation over a common denominator produces

$$K(\eta;\alpha,\beta) = \frac{-4z\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right](D+\sqrt{F})(1+C_1)+8zF}{(1-\alpha^2-\beta^2)F(D+\sqrt{F})(1+C_1)} \tag{5.148}$$

Here the quantity F is defined for notational convenience as

$$F = D^2 + 4z^2(1-\alpha^2-\beta^2) \tag{5.149}$$

Clearly, the algebraic steps needed to cancel $(1-\alpha^2-\beta^2)$ from the denominator are going to be more complicated than in the half-angle case, equation (5.111) above. Nevertheless, one can rearrange the numerator of (5.148) so that

$$K(\eta;\alpha,\beta) =$$

$$\frac{\begin{array}{l}\left[4zD(D-\sqrt{F})+4z(1-z^2)(1-\alpha^2-\beta^2)(D+\sqrt{F})\right.\\ \qquad\left.-4zC_1\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right](D+\sqrt{F})+32z^3(1-\alpha^2-\beta^2)\right]\end{array}}{(1-\alpha^2-\beta^2)F(D+\sqrt{F})(1+C_1)} \tag{5.150}$$

Then by multiplying numerator and denominator by $(D+\sqrt{F})$ one can extract a common factor of $(1-\alpha^2-\beta^2)$ from the numerator. The final result is

$$K(\eta;\alpha,\beta)=$$

$$\frac{\begin{array}{l}4z\left[\left[8z^2+(1-z^2)(D+\sqrt{F})\right](D+\sqrt{F})\right.\\ \qquad\left.-4z^2\left[D+\dfrac{1}{(3+C_2)}\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right]\right]\right]\end{array}}{F(D+\sqrt{F})^2(1+C_1)}. \tag{5.151}$$

This is the quarter-angle version of the universal formula (5.112). It can be used for all three kinds of orbits provided the continued fraction converges and provided no denominator vanishes. Comparison of (5.143) and (5.144) with (5.105) and (5.106) shows that the continued fractions $1/(1+C_1)$ and $1/(3+C_2)$ both converge for $-1<y<\infty$, the entire domain of interest for orbital problems. Consequently, neither of the factors $(1+C_1)$ and $(3+C_2)$ can vanish. Comparison of (5.149) with (5.146) and (5.145) shows that the value $F=0$ corresponds to $x=-\tanh^2\frac{1}{2}\phi=-1$ and also to $y=-\tanh^2\frac{1}{4}\phi=-1$. As noted before, these values are attained only as $\phi\to\infty$, namely, on a hyperbolic asymptote. Therefore F is always positive and does not vanish at any physically realizable point on any kind of orbit. Lastly, equation (5.145) shows that $(D+\sqrt{F})$ vanishes only as $y\to\infty$, that is, at $\theta=\pm 2\pi$. Hence, equation (5.151) should be valid also in the interval $-2\pi<\eta<+2\pi$, although numerical difficulties are to be expected near the endpoints of this interval due to excessively large values of z and y and to slow convergence of the continued fraction. Example calculations using equation (5.151) are recorded in Appendix D.

Several important features of (5.151) ought to be noted here. First, because (5.151) is valid for $-2\pi < \theta < +2\pi$ on elliptic orbits and not merely for $-\pi < \theta < +\pi$, negative values of elapsed time should not occur for $\pi < \theta < 2\pi$. It should never be necessary to add the period in order to obtain the correct time. The example calculations in Appendix D confirm this expectation. Second, equation (5.151) is algebraically more complicated than the half-angle formula (5.112) but still requires only one transcendental function evaluation, namely, the evaluation of $C_2(y)$. However, (5.151) also requires a single square root evaluation in addition to the evaluation of $C_2(y)$. This requirement imposes a numerical burden nearly equivalent to a second transcendental function evaluation. Third, at least throughout the interval $-\pi < \theta < +\pi$ the continued fraction $C_2(y)$ converges in fewer recursions than does the continued fraction $C_2(x)$. The reason is that $y = \tan^2\frac{1}{4}\theta$ is smaller than $x = \tan^2\frac{1}{2}\theta$. In principle, one could proceed to develop a set of eighth-angle formulae in terms of $\tan\frac{1}{8}\eta$ and $\tan\frac{1}{8}\theta$. The resulting expression for $K(\eta; \alpha, \beta)$ would be very much more complicated algebraically than (5.151) but would be valid for $-4\pi < \eta < +4\pi$. The continued fraction C_2 would have an argument equal to $y/(1+\sqrt{1+y})^2$, which is smaller than y, so that the new version of C_2 would converge faster yet than $C_2(y)$. An explicit eighth-angle formula analogous to (5.151) will not be developed as part of the present study due to the amount of algebra involved; the job should be straightforward, however. Besides this, one can improve the convergence of the continued fraction without deriving an entirely new expression for $K(\eta; \alpha, \beta)$. By this approach one does not seek to enlarge the permissible interval of θ or η for the entire expression for $K(\eta; \alpha, \beta)$; instead, one works with the continued fraction alone in an effort to improve its convergence at a given permissible value of θ. The algebraic devices needed to improve the convergence are discussed in the next section.

Improving the Convergence of the Continued Fraction

As noted previously, the continued fraction expansion of the arctangent given in (5.105) converges in fewer recursions if the magnitude of the argument is smaller. The half-angle identity (5.129) can be used to exploit this convergence property in a useful and systematic way. Define the continued fractions of interest as

$$C_3(x) = \cfrac{3^2 x}{7 + \cfrac{4^2 x}{9 + \cfrac{5^2 x}{11 + \cdots}}} \tag{5.152}$$

$$C_2(x) = \frac{2^2 x}{5 + C_3(x)} \tag{5.153}$$

$$C_1(x) = \frac{x}{3 + C_2(x)} \tag{5.154}$$

These all converge for $-1 < x < \infty$. Now define a quantity γ such that

$$\sqrt{x} = \tan \gamma \tag{5.155}$$

Then according to (5.105) one may write

$$\gamma = \tan^{-1}(\sqrt{x}) = \frac{\sqrt{x}}{1 + C_1(x)} \tag{5.156}$$

Now the identity (5.129) implies

$$\tan \frac{1}{2}\gamma = \frac{\tan \gamma}{1 + \sqrt{1 + \tan^2 \gamma}} \tag{5.157}$$

so define a quantity y such that

$$\sqrt{y} = \tan\frac{1}{2}\gamma = \frac{\sqrt{x}}{1+\sqrt{1+x}} \tag{5.158}$$

or

$$y = \frac{x}{(1+\sqrt{1+x})^2} \tag{5.159}$$

This is the same form as equation (5.147), although in this section the quantities x and y need not have the same meanings as in previous sections. The analysis in this section is completely general regarding the continued fraction (5.105). Then, according to (5.158), one may now write

$$\gamma = 2\tan^{-1}(\sqrt{y}) = \frac{2\sqrt{y}}{1+C_1(y)} \tag{5.160}$$

Equating the two alternate forms of γ in (5.156) and (5.160), one has

$$\frac{\sqrt{x}}{1+C_1(x)} = \frac{2\sqrt{y}}{1+C_1(y)} \tag{5.161}$$

Rearranging and making use of (5.158) to replace $\sqrt{y}$, there results

$$C_1(x) = \frac{1}{2}(1+\sqrt{1+x})[1+C_1(y)] - 1 \tag{5.162}$$

The main feature to observe about this expression is that $|y| < |x|$ so $C_1(y)$ will converge in fewer iterations than will $C_1(x)$. For example, given some value of x (perhaps large), compute the corresponding value of y from (5.159), evaluate $C_1(y)$ by, say, Gautschi's method and compute $C_1(x)$ from (5.162). If x is large enough, the auxiliary calculations will save enough recursions of C_1 to reduce the overall computational load. Note that this procedure does require the evaluation of a square root. Note also that the transformation (5.159) can be applied repeatedly before the continued fraction is evaluated, with the result that the number of recursions may be very drastically reduced. Then, of course, equation (5.162) must be applied repeatedly, as many times as (5.159) was applied, with the penalty of another square root evaluation at every repetition. Eventually, one reaches a point of diminishing returns when the auxiliary computations become as extensive as the continued fraction evaluation itself.

The universal time formulae (5.112) and (5.151) require the evaluation of $C_2(x)$. To improve the convergence of this continued fraction, substitute (5.154) into both sides of (5.162) and solve for $C_2(x)$ in terms of $C_2(y)$:

$$\frac{x}{3+C_2(x)} = \frac{1}{2}(1+\sqrt{1+x})\left[1+\frac{y}{3+C_2(y)}\right] - 1 \tag{5.163}$$

Multiply both sides of this equation by $2(1+\sqrt{1+x})$. Cancelling terms on the right-hand side ultimately allows a common factor of x to be cancelled from both sides. Then

$$C_2(x) = 2(1+\sqrt{1+x})\left[\frac{3+C_2(y)}{4+C_2(y)}\right] - 3 \tag{5.164}$$

Here again y is defined by (5.159) and $C_2(y)$ will converge faster than $C_2(x)$.

The same procedure could be adopted if $C_3(x)$ were to be evaluated. Substitute (5.153) into both sides of (5.164) and solve for $C_3(x)$ in terms of $C_3(y)$:

$$\frac{4x}{5+C_3(x)} = \frac{2(1+\sqrt{1+x})\left[3+\dfrac{4y}{5+C_3(y)}\right] - 3\left[4+\dfrac{4y}{5+C_3(y)}\right]}{\left[4+\dfrac{4y}{5+C_3(y)}\right]} \tag{5.165}$$

Multiply both sides by $(1+\sqrt{1+x})$. Cancelling terms in the numerator of the right-hand side ultimately allows a common factor of x to be cancelled from both sides. Then

$$C_3(x) = \frac{8(1+\sqrt{1+x})^2[5+C_3(y)] + 8x}{3(1+\sqrt{1+x})[5+C_3(y)] + 4(1+\sqrt{1+x}) - 6} - 5 \tag{5.166}$$

Of course, many rearrangements of the above formulae are possible. For example, equation (5.166) should be equivalent to equation (53) of Battin (1983), though the notations are different. The form given in Battin's (1983) paper was derived by means of identities for hypergeometric functions, whereas only a trigonometric identity has been used here.

The Revised Series Implementation

The universal series formula (5.93) was abandoned for practical use partly because of the limited interval of convergence of the power series expansion of the arctangent function. The continued fraction expansion was preferred because it converged for all argument values of interest. In retrospect, now, it should be noted that the same ideas just set forth to improve the convergence of the continued fraction can also be used to improve the radius of convergence of the power series. Since a series is slightly easier than a continued fraction to implement for automatic computation and since rigorous estimates of truncation errors can sometimes be made for a series, the universal series calculation of time will now be reconsidered.

Recall that for the elliptic formula (5.30)

$$K(\eta;\alpha,\beta)=\frac{-2z\left[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] \tag{5.167}$$

and the hyperbolic formula (5.31)

$$K(\eta;\alpha,\beta)=\frac{-2z\left[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2+\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2+\beta^2-1}}{1+\alpha+\beta z}\right] \tag{5.168}$$

expansion of the arctangents via (5.84) and (5.85)

$$\tanh^{-1}(x) = x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\cdots \tag{5.169}$$

$$\tan^{-1}(x) = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots \tag{5.170}$$

produces the common universal form (5.93)

$$K(\eta;\alpha,\beta)=\frac{2z\left[(1+\alpha+\beta z)+z^2\right]}{(1+\alpha+\beta z)^3(1+x)}-2\left[\frac{z}{1+\alpha+\beta z}\right]^3 S_3(x) \tag{5.171}$$

where $z=\tan\frac{1}{2}\eta$ and

$$x=\frac{z^2(1-\alpha^2-\beta^2)}{(1+\alpha+\beta z)^2}=\tan^2\frac{1}{2}\theta=-\tanh^2\frac{1}{2}\phi \tag{5.172}$$

$$S_3(x)=\frac{1}{3}-\frac{1}{5}x+\frac{1}{7}x^2-\frac{1}{9}x^3+\cdots \tag{5.173}$$

The series $S_3(x)$ converges for $-1<x<+1$, corresponding to $-\infty<\phi<+\infty$ on a hyperbola but only $-\frac{\pi}{2}<\theta<+\frac{\pi}{2}$ on an ellipse. Near the point $\eta=\pi$ the universal formula (5.171) becomes numerically unusable, regardless of whether the series converges, because the magnitude of z becomes excessively large. In order to alleviate these difficulties a quarter-angle version of the half-angle formula (5.171) can be developed.

The quarter-angle version of the elliptic formula (5.167) is already given in (5.140):

$$K(\eta;\alpha,\beta)=\frac{-4z\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[D^2+4z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{4}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\tan\frac{1}{4}\theta\right] \tag{5.174}$$

where now $z=\tan\frac{1}{4}\eta$ and D is defined for notational convenience as in (5.142):

$$D=\left[(1+\alpha)(1-z^2)+2\beta z\right] \tag{5.175}$$

Equation (5.138) gives $\tan\frac{1}{4}\theta$ in terms of $z=\tan\frac{1}{4}\eta$, namely,

$$\tan\frac{1}{4}\theta=\frac{2z\sqrt{1-\alpha^2-\beta^2}}{D+\sqrt{D^2+4z^2(1-\alpha^2-\beta^2)}} \tag{5.176}$$

The analogous hyperbolic expression for $\tanh\frac{1}{4}\phi$, which differs from this one only by an overall factor of $\sqrt{-1}$, is given in equation (5.139). Now using (5.176) as the argument of the arctangent in (5.174) and expanding that function in series via (5.170) produces the common form for ellipses and hyperbolae

$$K(\eta;\alpha,\beta)=\frac{-4z\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)F}$$

$$+\frac{8z}{(1-\alpha^2-\beta^2)(D+\sqrt{F})}\left[1-\frac{1}{3}y+\frac{1}{5}y^2-\frac{1}{7}y^3+\frac{1}{9}y^4-\cdots\right] \tag{5.177}$$

where

$$y=\frac{4z^2(1-\alpha^2-\beta^2)}{(D+\sqrt{F})^2}=\tan^2\frac{1}{4}\theta=-\tanh^2\frac{1}{4}\phi \tag{5.178}$$

and F has been defined as before for notational convenience as

$$F=D^2+4z^2(1-\alpha^2-\beta^2) \tag{5.179}$$

These latter two expressions are the same as (5.145) and (5.149), respectively. The remarks already made about the relation between x and y as given in (5.147) apply here as well. Now by collecting the first two terms of (5.177) over a common denominator, one obtains

$$K(\eta;\alpha,\beta)=\frac{-4z\left[D-(1-z^2)(1-\alpha^2-\beta^2)\right](D+\sqrt{F})+8zF}{(1-\alpha^2-\beta^2)F(D+\sqrt{F})}$$

$$+\frac{8z}{(1-\alpha^2-\beta^2)(D+\sqrt{F})}\left[-\frac{1}{3}y+\frac{1}{5}y^2-\frac{1}{7}y^3+\frac{1}{9}y^4-\cdots\right] \tag{5.180}$$

which is comparable to equation (5.148) given before. The numerator of the first term can be re-written to produce

$$K(\eta;\alpha,\beta)=$$

$$\frac{\begin{aligned}\big[4zD(D-\sqrt{F})+4zD(1-z^2)(1-\alpha^2-\beta^2)\\ +4z(1-z^2)(1-\alpha^2-\beta^2)\sqrt{F}+32z^3(1-\alpha^2-\beta^2)\big]\end{aligned}}{(1-\alpha^2-\beta^2)F(D+\sqrt{F})}$$

$$+\frac{8z}{(1-\alpha^2-\beta^2)(D+\sqrt{F})}\left[-\frac{1}{3}y+\frac{1}{5}y^2-\frac{1}{7}y^3+\frac{1}{9}y^4-\cdots\right] \tag{5.181}$$

which is comparable to equation (5.150) given before. Cancelling the denominator factor $(1-\alpha^2-\beta^2)$ from the series part presents no difficulty. Cancelling the same factor from the first term requires multiplying numerator and denominator by $(D+\sqrt{F})$. The final result is

$$K(\eta;\alpha,\beta)=\frac{4z\left[\left[8z^2+(1-z^2)(D+\sqrt{F})\right](D+\sqrt{F})-4z^2D\right]}{F(D+\sqrt{F})^2}$$

$$-4\left[\frac{2z}{(D+\sqrt{F})}\right]^3 S_3(y) \tag{5.182}$$

which is comparable to equation (5.151) given before, and is the quarter-angle version of the universal series formula (5.171) or (5.93). The denominator factors F and $(D+\sqrt{F})$ vanish only as indicated before; that is, F vanishes only on a hyperbolic asymptote and is positive otherwise, while $(D+\sqrt{F})$ vanishes only as $y \to \infty$, corresponding to $\theta = \pm 2\pi$. The value of $z = \tan\frac{1}{4}\eta$ also goes to infinity only as η approaches 2π or -2π. However, the series $S_3(y)$ converges for $-1 < y < +1$, which, according to (5.178), corresponds to the interval $-\pi < \theta < +\pi$ on elliptic orbits. It would certainly be possible to derive an eighth-angle version of (5.182) in which the resulting series S_3 would converge for $-2\pi < \theta < +2\pi$. The formula would be much more complicated algebraically than (5.182), and for that reason will not be developed in the present study. The job should be straightforward, however. An alternative is to offer a transformation of the variable y which will improve the convergence of the series itself in (5.182) without affecting the rest of the formula.

Consider the relation between x and y already given in equation (5.147) as

$$y = \frac{x}{(1+\sqrt{1+x})^2} \tag{5.183}$$

This is just the square of the trigonometric identity

$$\tan\frac{1}{2}\gamma = \frac{\tan\gamma}{1+\sqrt{1+\tan^2\gamma}} \tag{5.184}$$

In effect, (5.183) maps the interval $-1 < x < +\infty$ into the interval $-1 < y < +1$. Now also, it is evident from (5.172) and (5.178) that

$$\tan^{-1}(\sqrt{x}) = 2\tan^{-1}(\sqrt{y}) \tag{5.185}$$

Then from the series (5.170) and (5.173) it is not difficult to show that

$$\tan^{-1}(\sqrt{x}) = \sqrt{x}\,[1 - xS_3(x)] \tag{5.186}$$

and that consequently

$$\sqrt{x}\,[1 - xS_3(x)] = 2\sqrt{y}\,[1 - yS_3(y)] \tag{5.187}$$

Solving this equation for $S_3(x)$ in terms of $S_3(y)$, one has, after several steps,

$$S_3(x) = \frac{1}{(1+\sqrt{1+x}\,)^2}\left[1 + \frac{2S_3(y)}{(1+\sqrt{1+x}\,)}\right] \tag{5.188}$$

The main interest in this formula is that, for $-1 < x < +1$, the series $S_3(y)$ converges in fewer terms than does $S_3(x)$ because $|y| < |x|$. In addition, $S_3(y)$ converges for all $x \geq +1$ because of the transformation (5.183) even though $S_3(x)$ diverges. Now it is clear that by defining

$$w = \frac{y}{(1+\sqrt{1+y}\,)^2} \tag{5.189}$$

and calculating $S_3(y)$ by means of

$$S_3(y) = \frac{1}{(1+\sqrt{1+y}\,)^2}\left[1 + \frac{2S_3(w)}{(1+\sqrt{1+y}\,)}\right] \tag{5.190}$$

the universal formula (5.182) can be used without difficulty for $-2\pi < \theta < +2\pi$ and consequently for $-2\pi < \eta < +2\pi$. Numerical difficulties due to large values of z and y and to slow convergence of the series are to be expected near the endpoints of this interval. Of course, the slow convergence can be partly alleviated by applying the transformation (5.189) repeatedly before S_3 is evaluated. Then (5.190) will have to be invoked a like number of times. Drastic reductions in the number of terms needed to converge S_3 can be attained in this manner, but at the cost of another square-root evaluation every time the transformation is used. Numerical examples using the universal series formula (5.182) are recorded in Appendix D.

The Sigma-Domain Formulation

The results presented so far in this chapter have been valid only for non-rectilinear orbits, that is, for $h \neq 0$. This is a consequence of adopting η as the independent variable and computing time from the time transformation (5.1). If rectilinear orbits are to be considered with $n = 2$ in the generalized Sundman time transformation then it is necessary to use σ as the independent variable. This section presents the details of representing time in the σ domain. Even if the orbits to be considered are not exactly rectilinear, but only nearly so (h small), the formulae given in this section are better adapted for time computations than are the previous η -domain formulae because now all divisions by h will be avoided. Of course, the σ-domain formulae developed below can be used for all values of h, not merely for small and zero values.

The differential equation of time to be integrated is given in equation (3.78) of Chapter 3:

$$\frac{dt}{d\sigma} = \frac{1}{u^2} \tag{5.191}$$

where $u(\sigma)$ is given in equation (3.82) as

$$u(\sigma) = u(0)\cos h\sigma + u'(0)\frac{\sin h\sigma}{h} + \mu\frac{1 - \cos h\sigma}{h^2} \tag{5.192}$$

Here the initial conditions are evaluated for all orbits from (3.96) and (3.108) as

$$u(0) = \frac{1}{r(0)} \tag{5.193}$$

$$u'(0) = -\frac{1}{r(0)}\underline{r}(0) \cdot \dot{\underline{r}}(0) \tag{5.194}$$

and divisions by h are avoided by means of the series expansions (3.83) and (3.84). Unfortunately, handling these series expansions inside the quadrature is very cumbersome. Also, this study has not developed the identities and differential relations for the special functions defined by those series which would permit closed-form evaluation of the quadrature in the σ domain. It happens to be easier in the present case to make use of some already-available η-domain results. For example, comparison of (5.1) and (5.191) shows that in Keplerian motion η and σ are related simply as

$$\eta = h\,\sigma \tag{5.195}$$

Then if the parameters α and β are defined as

$$\alpha = \frac{h^2 u(0)}{\mu} - 1 \tag{5.196}$$

$$\beta = \frac{h u'(0)}{\mu} \tag{5.197}$$

equation (5.192) can be rewritten as

$$u(\sigma) = \frac{\mu}{h^2}(1 + \alpha \cos h\sigma + \beta \sin h\sigma) \tag{5.198}$$

and the quadrature (5.8) can be rewritten as

$$t - t_0 = \frac{h^3}{\mu^2} \int_0^{h\sigma} \frac{dx}{(1 + \alpha \cos x + \beta \sin x)^2} \tag{5.199}$$

Formulae from previous sections of this chapter supply the integrated forms of (5.199) as long as $h \neq 0$. Then, if the coefficient (h^3/μ^2) can be used to cancel all denominator occurences of h inside the integral itself, the final results will be valid even for $h = 0$.

For nonrectilinear ellipses, equation (5.30) can be adopted directly:

$$\frac{\mu^2}{h^3}(t-t_0)=\frac{-2z\left[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{1-\alpha^2-\beta^2}}\tan^{-1}\left[\frac{z\sqrt{1-\alpha^2-\beta^2}}{1+\alpha+\beta z}\right] \tag{5.200}$$

where now $z=\tan\frac{1}{2}(h\sigma)$. Likewise, for nonrectilinear hyperbolae, equation (5.31) can be adopted:

$$\frac{\mu^2}{h^3}(t-t_0)=\frac{-2z\left[(1+\alpha+\beta z)-(1-\alpha^2-\beta^2)\right]}{(1-\alpha^2-\beta^2)\left[(1+\alpha+\beta z)^2+z^2(1-\alpha^2-\beta^2)\right]}$$

$$+\frac{2}{(1-\alpha^2-\beta^2)\sqrt{\alpha^2-\beta^2-1}}\tanh^{-1}\left[\frac{z\sqrt{\alpha^2-\beta^2-1}}{1+\alpha+\beta z}\right] \tag{5.201}$$

where again $z=\tan\frac{1}{2}(h\sigma)$. In the interest of making the above-mentioned cancellation of h, the following manipulations should be observed.

$$(1+\alpha+\beta z)=1+\frac{h^2u(0)}{\mu}-1+\frac{hu'(0)}{\mu}\tan\frac{1}{2}(h\sigma) \tag{5.202}$$

$$(1+\alpha+\beta z)=\frac{h^2}{\mu}\left[u(0)+u'(0)\frac{1}{h}\tan\frac{1}{2}(h\sigma)\right] \tag{5.203}$$

Defining Z as

$$Z=\frac{1}{h}\tan\frac{1}{2}(h\sigma)=\frac{z}{h} \tag{5.204}$$

this is

$$(1+\alpha+\beta z)=\frac{h^2}{\mu}[u(0)+u'(0)Z]=\frac{h^2}{\mu}H \tag{5.205}$$

where the quantity H is defined for notational convenience by this equation. Then also

$$z^2(1-\alpha^2-\beta^2)=\left[1-\left[\frac{h^2u(0)}{\mu}-1\right]^2-\left[\frac{hu'(0)}{\mu}\right]^2\right]\tan^2\frac{1}{2}(h\sigma) \tag{5.206}$$

$$z^2(1-\alpha^2-\beta^2)=\frac{h^4}{\mu^2}\left[2\mu u(0)-h^2u(0)^2-u'(0)^2\right]Z^2=\frac{h^4}{\mu^2}GZ^2 \tag{5.207}$$

where the quantity G is defined for notational convenience by this equation. Considering equation (5.204), one may also write

$$(1-\alpha^2-\beta^2)=\frac{h^2}{\mu^2}G \tag{5.208}$$

Evidently, G serves to distinguish elliptic, parabolic and hyperbolic conditions even on rectilinear orbits: $G>0$ for ellipses, $G=0$ for parabolae and $G<0$ for hyperbolae. In fact, by using equations (1.4), (3.96), (3.108), (3.112) and (3.67) it is easy to show that G is just twice the negative energy. Equation (5.208), like equation (5.57), is another form of the fundamental relation (1.18).

Now using the above notation, the elliptic formula (5.200) takes the form

$$\frac{\mu^2}{h^3}(t-t_0)=\frac{-2hZ\left[\frac{h^2}{\mu}H-\frac{h^2}{\mu^2}G\right]}{\frac{h^2}{\mu^2}G\left[\frac{h^4}{\mu^2}H^2+\frac{h^4}{\mu^2}GZ^2\right]}+\frac{2}{\frac{h^2}{\mu^2}G\frac{h}{\mu}\sqrt{G}}\tan^{-1}\left[\frac{\frac{h^2}{\mu}Z\sqrt{G}}{\frac{h^2}{\mu}H}\right] \tag{5.209}$$

which reduces to

$$(t-t_0)=\frac{-2Z(\mu H-G)}{G(H^2+GZ^2)}+\frac{2\mu}{G\sqrt{G}}\tan^{-1}\left[\frac{Z\sqrt{G}}{H}\right] \tag{5.210}$$

In this formula, the vanishing of h causes no difficulty, but G still vanishes on parabolic orbits. Expanding the arctangent in series as in (5.186),

$$(t - t_0) = \frac{-2Z(\mu H - G)}{G(H^2 + GZ^2)} + \frac{2\mu Z}{GH} - \frac{2\mu Z^3}{H^3} S_3(X) \tag{5.211}$$

where

$$X = \frac{Z^2 G}{H^2} \tag{5.212}$$

and $S_3(X)$ is the series in (5.173). Collecting the first two terms of (5.211) over a common denominator produces

$$t - t_0 = \frac{-2Z(\mu H - G)H + 2\mu Z(H^2 + GZ^2)}{GH(H^2 + GZ^2)} - \frac{2Z^3}{H^3} S_3(X) \tag{5.213}$$

After the numerator in the first term has been reduced, a common factor of G can be cancelled, leaving

$$t - t_0 = \frac{2Z(H + \mu Z^2)}{H^3(1 + X)} - \frac{2Z^3}{H^3} S_3(X) \tag{5.214}$$

This is the σ-domain version of the universal series formula (5.93). Comparing (5.205) and (5.207), it is readily verified that

$$X = x = \tan^2 \frac{1}{2}\theta = -\tanh^2 \frac{1}{2}\phi \tag{5.215}$$

so that all the remarks made previously about x apply in this case. The intervals of convergence obtained before also apply here, provided α and β are computed by (5.196) and (5.197) and η is replaced by $h\sigma$. The method previously used to improve the convergence of S_3 can be used here as well.

If a continued-fraction expansion of the arctangent in (5.210) had been used instead of a series expansion, the σ-domain version of equation (5.112) would have been obtained. First, there results

$$t - t_0 = \frac{-2Z(\mu H - G)}{G(H^2 + GZ^2)} + \frac{2\mu Z}{GH(1 + C_1)} \tag{5.216}$$

where

$$C_1 = \frac{X}{3 + C_2} \tag{5.217}$$

$$C_2 = \cfrac{2^2 X}{5 + \cfrac{3^2 X}{7 + \cfrac{4^2 X}{9 + \cfrac{5^2 X}{11 + \cdots}}}} \tag{5.218}$$

Then forming a common denominator in (5.216) produces

$$t - t_0 = \frac{-2Z(\mu H - G)H(1 + C_1) + 2\mu Z(H^2 + GZ^2)}{GH(H^2 + GZ^2)(1 + C_1)} \tag{5.219}$$

which can be reduced to the universal form

$$t - t_0 = \frac{2Z\left[H(1 + C_1) + \mu Z^2\left[1 - \frac{1}{(3 + C_2)}\right]\right]}{H^3(1 + X)(1 + C_1)} \tag{5.220}$$

Previous remarks about the analogous η-domain formula (5.112) can be adduced here, with α and β being computed by (5.196) and (5.197) and η being replaced by $h\sigma$. The previously discussed method of improving the convergence of C_2 applies here as well.

The above equations are all σ-domain half-angle formulae, being derived from (5.30) and (5.31) and based on the variable Z as given in (5.204). It is an easy matter to derive σ-domain quarter-angle formulae also, if Z is redefined as

$$Z = \frac{1}{h}\tan\frac{1}{4}(h\sigma) = \frac{z}{h} \tag{5.221}$$

The relevant η-domain formulae are modified as follows. Equation (5.142) can be recast as

$$D = \left[1 + \frac{h^2 u(0)}{\mu} - 1\right](1 - h^2 Z^2) + 2\frac{h^2 u'(0)}{\mu} Z \tag{5.222}$$

$$D = \frac{h^2}{\mu}\left[u(0)(1 - h^2 Z^2) + 2u'(0)Z\right] = \frac{h^2}{\mu} A \tag{5.223}$$

where A is defined for notational convenience by this equation. Then, using (5.145), equation (5.140) for elliptic orbits is rewritten as

$$\frac{\mu^2}{h^3}(t - t_0) = \frac{-4hZ\left[\frac{h^2}{\mu}A - (1 - h^2 Z^2)\frac{h^2}{\mu^2}G\right]}{\frac{h^2}{\mu^2}G\left[\frac{h^4}{\mu^2}A^2 + 4h^2 Z^2 \frac{h^2}{\mu^2}G\right]}$$

$$+ \frac{4}{\frac{h^2}{\mu^2}G\frac{h}{\mu}\sqrt{G}} \tan^{-1}\left[\frac{2\frac{h^2}{\mu}Z\sqrt{G}}{\frac{h^2}{\mu}A + \sqrt{\frac{h^4}{\mu^2}A^2 + 4h^2 Z^2 \frac{h^2}{\mu^2}G}}\right], \tag{5.224}$$

where also G has been introduced from equation (5.208). Straightforward cancellations produce

$$t - t_0 = \frac{-4Z\left[\mu A - (1 - h^2 Z^2)G\right]}{G(A^2 + 4GZ^2)} + \frac{4\mu}{G\sqrt{G}} \tan^{-1}(\sqrt{Y}) \tag{5.225}$$

where

$$Y = \frac{4Z^2 G}{\left[A + \sqrt{A^2 + 4GZ^2}\right]^2} = y = \tan^2\frac{1}{4}\theta = -\tanh^2\frac{1}{4}\phi \tag{5.226}$$

The universal series form of (5.225) is produced by expanding the arctangent via (5.186):

$$t - t_0 = \frac{-4Z\left[\mu A - (1 - h^2 Z^2)G\right]}{G(A^2 + 4GZ^2)} + \frac{4\mu}{G\sqrt{G}}\sqrt{Y}\left[1 - YS_3(Y)\right] \tag{5.227}$$

Defining Q for notational convenience as

$$Q = A^2 + 4GZ^2 \tag{5.228}$$

equation (5.227) becomes

$$t - t_0 = \frac{-4Z\left[\mu A - (1 - h^2 Z^2)G\right]}{GQ} + \frac{8\mu Z}{G(A + \sqrt{Q})} - \frac{32\mu Z^3}{(A + \sqrt{Q})^3} S_3(Y) \tag{5.229}$$

$$t - t_0 = \frac{-4Z\left[\mu A - (1 - h^2 Z^2)G\right](A + \sqrt{Q}) + 8\mu Z Q}{GQ(A + \sqrt{Q})} - \frac{32\mu Z^3}{(A + \sqrt{Q})^3} S_3(Y) \tag{5.230}$$

Paralleling the steps after equation (5.180), the numerator in the first term here can be rewritten to produce

$$t - t_0 =$$

$$\frac{\left[4\mu Z A(A - \sqrt{Q}) + 4Z(1 - h^2 Z^2)G(A + \sqrt{Q}) + 32\mu Z^3 G\right]}{GQ(A + \sqrt{Q})} - \frac{32\mu Z^3}{(A + \sqrt{Q})^3} S_3(Y) \tag{5.231}$$

Multiplying numerator and denominator in the first term by $(A + \sqrt{Q})$ allows a common factor of G to be cancelled, with the result

$$t - t_0 =$$

$$\frac{4Z\left[\left[8\mu Z^2 + (1 - h^2 Z^2)(A + \sqrt{Q})\right](A + \sqrt{Q}) - 4\mu Z^2 A\right]}{Q(A + \sqrt{Q})^2} - 4\mu\left[\frac{2Z}{(A + \sqrt{Q})}\right]^3 S_3(Y) \tag{5.232}$$

This is the σ-domain version of the universal η -domain equation (5.182).

The universal continued-fraction form of equation (5.225) is produced by expanding the arctangent via (5.105). First, there is obtained

$$t-t_0=\frac{-4Z\left[\mu A-(1-h^2Z^2)G\right]}{GQ}+\frac{8\mu Z}{G(A+\sqrt{Q})(1+C_1)} \tag{5.233}$$

where

$$C_1=\frac{Y}{3+C_2} \tag{5.234}$$

$$C_2=\cfrac{2^2Y}{5+\cfrac{3^2Y}{7+\cfrac{4^2Y}{9+\cfrac{5^2Y}{11+\cdots}}}} \tag{5.235}$$

in analogy with equation (5.141). Paralleling the steps after that equation, one obtains

$$t-t_0=\frac{-4Z\left[\mu A-(1-h^2Z^2)G\right](A+\sqrt{Q})(1+C_1)+8\mu ZQ}{GQ(A+\sqrt{Q})(1+C_1)} \tag{5.236}$$

$$t-t_0=$$

$$\frac{\begin{array}{l}\left[4\mu ZA(A-\sqrt{Q})+4Z(1-h^2Z^2)G(A+\sqrt{Q})\right.\\ \qquad\left.-4ZC_1\left[\mu A-(1-h^2Z^2)G\right](A+\sqrt{Q})+32\mu Z^3G\right]\end{array}}{GQ(A+\sqrt{Q})(1+C_1)} \tag{5.237}$$

Multiplying numerator and denominator by $(A+\sqrt{Q})$ allows the common factor of G to be cancelled. The final result is

$$t-t_0=$$

$$\frac{\begin{array}{l}4Z\left[\left[8\mu Z^2+(1-h^2Z^2)(A+\sqrt{Q})\right](A+\sqrt{Q})\right.\\ \qquad\left.-4Z^2\left[\mu A+\frac{1}{(3+C_2)}\left[\mu A-(1-h^2Z^2)G\right]\right]\right]\end{array}}{Q(A+\sqrt{Q})^2(1+C_1)} \tag{5.238}$$

This is the σ-domain version of the universal η -domain equation (5.151). By now it is clear that remarks previously made about the quarter-angle formulae in the η domain carry over to the present case, α and β being computed by (5.196) and (5.197) and η being replaced by $h\sigma$.

As a final note, observe that all the above σ-domain time formulae represent the dimensional elapsed time $(t - t_0)$. This is due ultimately to the fact that the independent variable σ is itself a dimensional quantity, having dimensions of (time)(length)$^{-2}$. In the η domain the independent variable is dimensionless so that time is expressed in the naturally occurring dimensionless combination $(\mu^2/h^3)(t - t_0)$. In the σ domain no such naturally occurring combination appears, and one is left to choose a convenient nondimensionalization of time. For most problems the choice $(\sqrt{\mu u(0)^3})(t - t_0)$ should be suitable. Of course, any constant reciprocal length could be used in place of $u(0)$.

Summary

This chapter has pursued the computation of orbital time of flight in explicit terms of the change in true anomaly elapsed from an arbitrary point on the orbit. The integration of the differential equation of time in terms of elementary functions poses no special difficulty since it is a quadrature of a simple rational function of sine and cosine of the true anomaly. Techniques of elementary calculus result in three different expressions according as the orbit is elliptic, parabolic or hyperbolic. Several versions of these formulae are given since comparable formulae, having the epoch reckoned at an arbitrary point on the orbit, seem never to have been published despite the uncomplicated derivations. The essential innovation of this chapter is to reduce these three different expressions to a common real-valued form valid for all orbits by means of power series expansions. Although this common form is the type of expression sought in this chapter, it is found that the change of variable originally introduced to permit the closed-form evaluation of the quadrature causes an unacceptably small radius of convergence in the universal series. Subsequent transformations of the time equation, involving trigonometric half-angle identities and the use of continued fractions in place of series, ultimately allow the radius of convergence to include, without undue complication, nearly one complete revolution. In fact, only the half-angle identities are needed to effect this improvement, but the evaluation of continued fractions offers some convergence improvement of its own even though it requires slightly more arithmetic per term than does the equivalent series evaluation. In principle, the same transformations could be used repeatedly to extend the convergence over several or many revolutions, though at the cost of greatly increased algebraic complexity in the formulae.

The final results of this analysis are new inasmuch as previously published true-anomaly time equations have always assumed the initial conditions to be given at the pericenter and, with the exception of some "near-parabolic" Taylor's series expansions, have never been presented in forms valid for all types of orbits. The apparently minor generalizations offered here lead to a new family of universal time equations, all members of which give the elapsed time in rational algebraic terms of a single transcendental function. Only the first two members of the family are exhibited in this study, those based on tangent of half the anomaly and on tangent of one-fourth the anomaly. The higher formulae, based on tangent of one-eighth the anomaly, and so forth, would be very much more complicated algebraically. It turns out that the same transcendental function occurs in all members of the family. The function is derived from the arctangent function and, as implied above, can be evaluated either as an infinite series or as an infinite continued fraction. The different members of the family are distinguished partly by different arguments of this one function, meaning that each time equation is valid up to a different maximum allowable transfer angle dictated by the radius of convergence of the infinite series or continued fraction. For example, the half-angle formula is valid only up to approximately half a revolution (really, π radians of change in *eccentric* anomaly), while the quarter-angle formula is valid up to (but not including) one complete revolution. The higher formulae not developed in this chapter would be valid up to two, four or more revolutions, though at the cost of greatly increased algebraic complexity.

Even though the physical transfer angle, η, proves to be a suitable independent variable for calculating time on elliptic, parabolic and hyperbolic orbits, it cannot be used if the orbit happens to be rectilinear (or nearly so). Based on the regularizations presented in previous chapters, it is possible to introduce a variable, σ, related to true anomaly, which is valid even for rectilinear orbits. The formulae needed to calculate time in terms of σ are obtained analogously to the η-domain formulae.

The results of this chapter are comparable to recent eccentric-anomaly (s-domain) formulae given by Shepperd (1985) and to a class of time equations developed for the boundary value problem by Battin (1968, 1977) and Battin and Vaughn (1983). Shepperd solves the initial value problem in terms of a single transcendental function whose argument is $u = \frac{1}{\sqrt{-2E}} \tan(\frac{1}{4}\sqrt{-2E}\, s)$ and which is evaluated as a continued fraction. In Appendix C are found the analogous θ-domain formulae which were first suggested by the true-anomaly results obtained here. Battin in several articles has given formulations of the boundary value problem in which only a single hypergeometric function appears in the time equation. The argument of that function is the square of the tangent of one-fourth the eccentric anomaly, similarly to the argument of the function in the quarter-angle true-anomaly formula given here. It was, in fact, Battin's success in evaluating this hypergeometric function as a continued fraction that prompted the use of continued fractions to evaluate the trigonometric functions (arctangents) which arose in this study. Likewise, the hypergeometric function identities used by Battin (1977) suggested the use of half-angle identities in this case. In a certain sense, Battin's time equations for the boundary value problem qualify as universal true-anomaly time relations because the true anomaly is just the angle between the terminal positions. Moreover, at least one version of his equations (from 1968) has been recast to solve the initial value problem. Formulations of this type derived from the boundary value problem are not exactly analogous to the calculation schemes proposed here, but, evidently, the results presented in this chapter are closely related to those of Battin, though the exact parameter-by-parameter correspondence has yet to be established. It would be interesting to know the precise connection because Battin's work proceeds from Lambert's time-of-flight theorem and from the ideas of Gauss (1809) on the boundary value problem (including several important extensions), whereas the present work proceeds from the idea of merely completing the regularized solutions of the two-body problem in the true-anomaly domain.

The new true-anomaly formulae obtained here naturally permit new formulations of the initial value and boundary value problems, as well as perturbed versions of these problems. These topics are examined in detail in the next several chapters. Numerical results for time of flight in unperturbed motion are given in Appendix D.

Chapter 6. The Keplerian Initial Value Problem

Introduction

With the results of previous chapters available, it is now mainly a matter of collecting the appropriate equations to formulate the orbital initial value problem of unperturbed motion: given position and velocity at some instant of time, calculate the position and velocity at some other instant of time, assuming only inverse-square attraction between the primary body and the satellite. This being one of the classical problems of celestial mechanics, there are many methods available for doing such a calculation. However, all these methods have as their main burden the inversion of the transcendental relation between time and eccentric anomaly, namely, the solution of Kepler's equation. It is a remarkable fact that the solution of Kepler's equation is still an active area of research more than three centuries after Kepler himself devised the first approximate solution procedure. An account of this activity is well beyond the scope of the present study, though for examples of recent effort to reduce the inversion of the eccentric-anomaly time equation to a truly routine process reference might be made to the articles by Siewert and Burniston (1972), Prussing (1977, 1979), Smith (1979), Ng (1979), Battin and Fill (1979), Broucke (1980), Sheela (1982), Danby and Burkardt (1983), Burkardt and Danby (1983), Ioakimidis and Papadakis (1985), Shepperd (1985), and Odell and Gooding (1986). This list is, of course, merely representative. All the while, as noted in the last chapter, the direct relation between time and true anomaly has been relatively neglected. Consequently, universal formulations of the initial value problem in the true-anomaly domain have not been available heretofore. The possible exception, namely the initial-value version of the Gauss/Lambert solution by Battin (1968) and similar developments which could be made from other Gauss/Lambert formulations, seems never to have received serious attention. The object of this chapter is to present universal formulae which can be used to solve the initial value problem in both the η domain and the σ domain, and to present the working formulae in terms of parameters which arise naturally in the true-anomaly regularization of the governing equations of motion. It will be seen that the true-anomaly formulation has at least two features of special interest when an iterative-type solution of the time equation is contemplated, namely, the need to evaluate only a single transcendental function per iteration cycle and the ease with which rigorous universal bounds on the solution can be computed before the iterations are started.

The pertinent formulae are now summarized for the initial value problem in the η domain. (The corresponding σ-domain formulation of the problem is described later.) Given a position vector $\underline{r}_0$ and a velocity vector $\underline{v}_0$ at some time t_0, compute the angular momentum as

$$\underline{h} = \underline{r}_0 \times \underline{v}_0 \quad \text{and} \quad h = \sqrt{\underline{h} \cdot \underline{h}} \tag{6.1}$$

If the angular momentum magnitude is zero or very small then the η -domain formulation will fail and recourse must be sought in the σ-domain formulation where divisions by h are avoided. The smallest acceptable value of h in the η-domain formulae is difficult to assign in general and must be decided for each particular application. If the η-domain formulation will suffice, then compute the radial and transverse unit vectors at time t_0 as

$$\underline{\xi}_0 = \frac{1}{r_0}\underline{r}_0 \tag{6.2}$$

$$\underline{\xi}'_0 = \frac{1}{h}\underline{h} \times \underline{\xi}_0 \tag{6.3a}$$

or

$$\underline{\xi}'_0 = \frac{1}{hr_0}\underline{h} \times \underline{r}_0 \tag{6.3b}$$

or

$$\underline{\xi}'_0 = \frac{1}{hr_0}[\underline{v}_0(\underline{r}_0 \cdot \underline{r}_0) - \underline{r}_0(\underline{r}_0 \cdot \underline{v}_0)] \tag{6.3c}$$

These formulae are taken from equations (3.97), (3.103) and (3.104) of Chapter 3. Then the reciprocal radius and its derivative at time t_0 are computed from equations (3.96) and (3.108) as

$$u_0 = \frac{1}{r_0} \tag{6.4}$$

$$u'_0 = \frac{-1}{hr_0}(\underline{r}_0 \cdot \underline{v}_0) \tag{6.5}$$

It is convenient to form the dimensionless parameters α and β as was done in Chapter 5 in equations (5.4) and (5.5):

$$\alpha = \frac{h^2 u_0}{\mu} - 1 = \frac{h^2}{\mu r_0} - 1 \tag{6.6}$$

$$\beta = \frac{h^2 u'_0}{\mu} = -\frac{h}{\mu r_0}(\underline{r}_0 \cdot \underline{v}_0) \tag{6.7}$$

Then for some value of the time t one computes from equations (3.124) through (3.127) to obtain

$$\underline{\xi}(\eta) = \underline{\xi}_0 \cos \eta + \underline{\xi}'_0 \sin \eta \tag{6.8}$$

$$\underline{\xi}'(\eta) = -\underline{\xi}_0 \sin \eta + \underline{\xi}'_0 \cos \eta \tag{6.9}$$

$$u(\eta) = \frac{\mu}{h^2}(1 + \alpha \cos \eta + \beta \sin \eta) \tag{6.10}$$

$$u'(\eta) = \frac{\mu}{h^2}(-\alpha \sin\eta + \beta \cos\eta) \tag{6.11}$$

Here the value of η corresponding to the time t must be determined from the true-anomaly time equation (5.8) of Chapter 5:

$$t - t_0 = \frac{h^3}{\mu^2} K(\eta; \alpha, \beta) \tag{6.12}$$

where some appropriate form of $K(\eta; \alpha, \beta)$ is to be used, such as that given in (5.151). Ordinarily, an iterative solution of (6.12) would be implemented. Finally, the position and velocity vectors at time t are computed according to equations (3.109) and (3.112) as

$$\underline{r}(t) = \frac{1}{u(\eta)} \underline{\xi}(\eta) \tag{6.13}$$

$$\underline{v}(t) = h\left[u(\eta)\underline{\xi}'(\eta) - u'(\eta)\underline{\xi}(\eta)\right] \tag{6.14}$$

The above calculation procedure, straightforward though it is, can be refined somewhat. As it stands, four transcendental functions must be evaluated. These are: $\sin\eta$ and $\cos\eta$, each to be evaluated once; $K(\eta; \alpha, \beta)$, which must be evaluated repeatedly during the iteration; and $\tan\frac{1}{2}\eta$ or $\tan\frac{1}{4}\eta$, to be evaluated once for each evaluation of K since η occurs in K only in this way. Previously, it was observed in Chapter 5 that if $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$, rather than η itself, were chosen as the variable of iteration in the solution of the time equation, then only one transcendental function evaluation per iteration cycle would be needed, namely that of K. Here it is noted further that the use of some trigonometric identities removes the need for explicit calculation of $\sin\eta$ and $\cos\eta$ as well. Suppose that $z = \tan\frac{1}{2}\eta$. The identities

$$\sin\eta = \frac{2\tan\frac{1}{2}\eta}{1 + \tan^2\frac{1}{2}\eta} = \frac{2z}{1+z^2} \tag{6.15}$$

$$\cos\eta = \frac{1-\tan^2\frac{1}{2}\eta}{1+\tan^2\frac{1}{2}\eta} = \frac{1-z^2}{1+z^2} \tag{6.16}$$

will replace $\sin\eta$ and $\cos\eta$ in equations (6.8) through (6.11) above, leaving z as the independent variable and K as the only transcendental function. Likewise, suppose that $z = \tan\frac{1}{4}\eta$. Then one can use the identity

$$\sin\eta = 2\sin\frac{1}{2}\eta\cos\frac{1}{2}\eta \tag{6.17}$$

to get

$$\sin\eta = 2\left[\frac{2\tan\frac{1}{4}\eta}{1+\tan^2\frac{1}{4}\eta}\right]\left[\frac{1-\tan^2\frac{1}{4}\eta}{1+\tan^2\frac{1}{4}\eta}\right] \tag{6.18}$$

from which

$$\sin\eta = 2\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right] \tag{6.19}$$

Also

$$\cos\eta = \cos^2\frac{1}{2}\eta - \sin^2\frac{1}{2}\eta \tag{6.20}$$

from which

$$\cos\eta = \left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2 \tag{6.21}$$

Equations (6.19) and (6.21) will replace $\sin\eta$ and $\cos\eta$ in equations (6.8) through (6.11), leaving z as the independent variable and K as the only transcendental function.

Regardless of these refinements, it is clear that most of the computational effort in the initial value problem will be expended in solving the time equation (6.12) for the unknown η or z. The following sections address this aspect of the problem.

Solution of the True-Anomaly Time Equation

Until recent times, the only approach for solving Kepler's equation (or any nonlinear or transcendental equation in a single variable) was by some numerical process of trial and error or successive approximation. Newton's method has been a favorite choice; derivative-free methods, such as *regula falsi* or fixed-point iteration have also found general application. To date, a large number of iterative schemes for finding a root of Kepler's equation (in one of its various forms) have been developed. Recently, though, Siewert and Burniston (1972) derived a finite formula for the exact solution of the classical form of Kepler's equation in which the anomalies are measured from pericenter. The value of the eccentric anomaly is given in terms of mean anomaly and eccentricity by a simple function of a definite integral taken on the interval 0 to 1. The integrand is quite complicated so that numerical quadrature is required for non-zero values of the eccentricity, but in any case the solution can be calculated to any desired accuracy by non-iterative means. Later, Burniston and Siewert (1973, 1974) extended their method to be able to treat the Gauss/Lambert boundary value problem as well. In view of the great theoretical importance of these results, it is interesting that the integral formulae of Burniston and Siewert have not replaced iterative solutions in practical applications, nor are they likely to. The reason is that the integral formulae are quite complicated. Obtaining sufficient accuracy in the numerical quadrature usually calls for several dozen evaluations of the integrand, each of which in turn calls for evaluation of several transcendental functions. In the end, the non-iterative exact solution requires significantly more computation to achieve a given accuracy than does a well-designed iterative approximation. More recently, an equally important and related theoretical advance was reported by Ioakimidis and

Papadakis (1985). These authors developed a family of integral formulae, each member of which by itself permits the non-iterative calculation of a real root of *any* nonlinear or transcendental equation $g(x) = 0$, including, of course, Kepler's equation in any of its forms. A feature of these formulae is that the limits of integration must be chosen for each particular problem such that one and only one root lies between them. Provided that the limits of integration can be specified rigorously, the formulae of Ioakimidis and Papadakis (1985) are directly applicable to solving the true-anomaly time equation whereas the formulae of Siewert and Burniston (1972) are not; the latter are explicitly eccentric-anomaly formulae. Unfortunately, the general formulae of Ioakimidis and Papadakis are no easier to evaluate than the previous formulae. First, it may not be easy to specify suitable limits of integration for a given problem. In the case of the true-anomaly time equation rigorous bounds on the one existing root can always be specified as described in a later section of this chapter; however, these bounds cannot always be used as limits of integration because they may lie beyond the asymptotes of a hyperbolic orbit. Second, the integrand always contains a transcendental function (an arctangent) besides extensive occurences of $g(x)$ itself and its derivatives. Third, the integrand of the simplest formula of the family is discontinuous at the solution value of x, though this difficulty can be avoided by using other, more complicated, formulae. In any case, several dozen evaluations of the integrand are usually necessary to obtain sufficient accuracy in the solution. These characteristics result in a significant computational burden, however great the theoretical advance embodied by the formulae. This chapter will formulate only an iterative method for the solution of the true-anomaly time equation, assuming that to be a more immediately practical approach, but realizing that exact non-iterative solution formulae are also available.

The goal now is to find a root of the equation

$$g(\eta) = K(z(\eta); \alpha, \beta) - \frac{\mu^2}{h^3}\Delta t = 0 \tag{6.22}$$

where h, α, β and $\Delta t = t - t_0$ are known, and $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$. For smoothly varying functions, such as this one turns out to be, the Newtonian iteration

$$\eta_{k+1} = \eta_k - \frac{g(\eta_k)}{g'(\eta_k)} \qquad \text{with} \qquad k = 0, 1, 2, 3, \ldots \tag{6.23}$$

is a generally successful method of approximation. Compared to other methods which do not require the evaluation of the derivative (such as *regula falsi*), Newton's method usually requires fewer iterations. Hence, if the derivative can be calculated easily and does not vanish or change sign during the iteration, Newton's method is usually the method of choice. Shortly, it will be proved that the derivative of the function in question is always positive and that the single root lies in a bounded interval, the endpoints of which are known *a priori*. This is an ideal setting for an application of Newton's method. The iteration in terms of η would be

$$\eta_{k+1} = \eta_k + \frac{\frac{\mu^2}{h^3}\Delta t - K(z(\eta_k); \alpha, \beta)}{K'(z(\eta_k); \alpha, \beta)} \tag{6.24}$$

where the prime mark denotes differentiation with respect to η . From equation (5.8) of Chapter 5, the derivative is available immediately as

$$\frac{dK}{d\eta} = \frac{1}{(1 + \alpha\cos\eta + \beta\sin\eta)^2} \tag{6.25}$$

Notice that this expression is always positive, so that one and only one root exists even in the general case. The iteration becomes

$$\eta_{k+1} = \eta_k + \left[\frac{\mu^2}{h^3}\Delta t - K(z(\eta_k); \alpha, \beta)\right](1 + \alpha\cos\eta_k + \beta\sin\eta_k)^2 \tag{6.26}$$

Of course, η is not the most efficient variable to use for the iteration. It has already been observed that some computational advantage accrues from using $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$ as the independent variable. The details of these modifications are now presented.

Half-Angle Formulae

Let

$$z = \tan \frac{1}{2}\eta \tag{6.27}$$

It follows that

$$d\eta = \frac{2dz}{1+z^2} \tag{6.28}$$

A root is now to be found for the equation

$$g(z) = K(z; \alpha, \beta) - \frac{\mu^2}{h^3}\Delta t = 0 \tag{6.29}$$

The Newtonian iteration will be

$$z_{k+1} = z_k + \frac{\frac{\mu^2}{h^3}\Delta t - K(z_k; \alpha, \beta)}{K'(z_k; \alpha, \beta)} \tag{6.30}$$

where the prime mark denotes differentiation with respect to z. The derivative is calculated as

$$\frac{dK}{dz} = \frac{dK}{d\eta}\,\frac{d\eta}{dz} \tag{6.31}$$

Using equations (6.25) and (6.28), this becomes

$$\frac{dK}{dz} = \frac{1}{(1 + \alpha \cos \eta + \beta \sin \eta)^2} \left[\frac{2}{1+z^2} \right] \tag{6.32}$$

Replacing $\sin \eta$ and $\cos \eta$ by means of the identities (6.15) and (6.16) produces, after some rearrangement,

$$\frac{dK}{dz} = \frac{2(1+z^2)}{\left[(1+\alpha) + 2\beta z + (1-\alpha)z^2\right]^2} \tag{6.33}$$

This expression can be put in a more convenient form by multiplying numerator and denominator by $(1+\alpha)^2$ and adding and subtracting $\beta^2 z^2$ inside the square brackets in the denominator. Regrouping terms then allows the derivative to calculated as

$$\frac{dK}{dz} = \frac{2(1+z^2)(1+\alpha)^2}{(1+\alpha+\beta z)^4(1+x)^2} \tag{6.34}$$

where

$$x = \frac{(1-\alpha^2-\beta^2)z^2}{(1+\alpha+\beta z)^2} \tag{6.35}$$

Thus the derivative $\frac{dK}{dz}$ can be formed out of quantities already used in the evaluation of K itself. The iteration finally takes the form

$$z_{k+1} = z_k + \left[\frac{\mu^2}{h^3}\Delta t - K(z_k; \alpha, \beta)\right]\frac{(1+\alpha+\beta z_k)^4(1+x_k)^2}{2(1+z_k^2)(1+\alpha)^2} \tag{6.36}$$

Here an appropriate half-angle form of $K(z; \alpha, \beta)$ from Chapter 5, such as (5.93) or (5.112), is to be introduced for actual calculations. Interestingly, a cancellation of the denominator factors $(1+\alpha+\beta z)$ and $(1+x)$ from the function K occurs in equation (6.36). Zero values of these factors correspond to singularities of the true-anomaly time equation as discussed in the last chapter. The effect of the cancellation is to render the iteration on z well-behaved near those singular points. Of course, as discussed in Chapter 5, no half-angle formulation can overcome the fundamental limitations on equations (5.93) or (5.112); the iterations in (6.36) will fail sufficiently near the singular points of the time equation due to poor convergence of the series or continued fraction calculation required for K. These limitations are probably not serious if one is dealing with only short-arc scenarios of the initial value problem, but otherwise are sufficient reason to consider the quarter-angle formulae presented next.

Quarter-Angle Formulae

Let

$$z = \tan \frac{1}{4}\eta \tag{6.37}$$

so that

$$d\eta = \frac{4dz}{1 + z^2} \tag{6.38}$$

Now a root is sought of an equation like (6.29) above using an iteration of the same form as (6.30). The derivative is still evaluated using the chain rule (6.31); in view of equation (6.38), it can be written as

$$\frac{dK}{dz} = \frac{1}{(1 + \alpha \cos \eta + \beta \sin \eta)^2} \left[\frac{4}{1 + z^2}\right] \tag{6.39}$$

Replacing $\sin \eta$ and $\cos \eta$ using the quarter-angle identities (6.19) and (6.21) produces, after some rearrangement,

$$\frac{dK}{dz} = \frac{4(1 + z^2)^3}{\left[(1 + z^2)^2 + \alpha\left[(1 - z^2)^2 - 4z^2\right] + 4\beta z(1 - z^2)\right]^2} \tag{6.40}$$

This expression can be put in a more convenient form by multiplying numerator and denominator by $(1 + \alpha)^2$. Then after adding and subtracting $4\beta^2 z^2$ inside the outer square brackets of the denominator, one can regroup terms to obtain

$$\frac{dK}{dz} = \frac{4(1+z^2)^3(1+\alpha)^2}{F^2} \tag{6.41}$$

where

$$F = \left[(1+\alpha)(1-z^2) + 2\beta z\right]^2 + 4z^2(1-\alpha^2-\beta^2) \tag{6.42}$$

The quantity F was defined in equations (5.149) and (5.142) of Chapter 5 and appears in the quarter-angle versions of the true-anomaly time equation. In short, the derivative $\frac{dK}{dz}$ can be formed out of quantities already used in the evaluation of K itself. The iteration will then take the form

$$z_{k+1} = z_k + \left[\frac{\mu^2}{h^3}\Delta t - K(z_k;\alpha,\beta)\right]\frac{F_k^2}{4(1+z_k^2)^3(1+\alpha)^2} \tag{6.43}$$

Here an appropriate quarter-angle version of $K(z;\alpha,\beta)$, such as that given in equation (5.151) or equation (5.182), is to be used for actual calculations. Notice that the factor F occurs in the denominators of those quarter-angle formulae and that it is cancelled in the iteration formula (6.43). In the last chapter, it was seen that F vanishes only on the asymptotes of a hyperbolic orbit and is positive otherwise. The effect of the cancellation is to make the iteration on z well-behaved near that singularity of the time equation. Sufficiently near an asymptote the time equation will fail due to poor convergence of the series or continued fraction calculation required for K, but, of course, an asymptote is approached very closely only for very large values of the time. Similar poor convergence is encountered as the transfer angle approaches 2π on an elliptical orbit, but this situation can be avoided in calculations by subtracting one orbital period from the given transfer time before iterations are begun.

Bounds on the Solution of the Time Equation

When Newton's method is to be used to solve the time equation, it is important to select a suitable starting value of the independent variable. Although the time equation is monotonic, so that the method is expected to converge no matter what the initial guess, many extra iterations may be required if the initial guess is very far from the solution. For the eccentric-anomaly Kepler's equation much attention has been given to finding efficient starting values for the iteration. Most of these analyses have dealt with the classical form of the equation in which the anomalies are measured from pericenter, so the results are not directly applicable to forms of the time equation in which the anomalies are measured from an arbitrary epoch. In this group, for recent examples, are the studies by Odell and Gooding (1986), Smith (1979), Ng (1979), Broucke (1980), Danby and Burkardt (1983), and Burkardt and Danby (1983). The latter also review the use of initial estimates for the solution the universal s-domain Kepler's equation, a subject of much less extensive study. Almost the only other analyses for the universal Kepler's equation are those by Bate, *et al.*, (1971), and Prussing (1979). Even in these latter two sources, the starting values proposed are not universal: different starting values are offered for elliptic and hyperbolic orbits. For the true-anomaly time equation, there seem to be no results in print relating to starting values for the iteration.

In order to derive a suitable starting value for the iterations indicated in (6.43) or (6.36) above, it is useful to review the work of Prussing (1979). That article presented rigorous upper and lower bounds on the solution of the universal s-domain Kepler's equation. The mean of the bounds serves as an efficient starting value for a Newtonian iteration of that equation, or else the two bounds can be used to start a *regula falsi* iteration which is guaranteed to converge to the solution. Evidently, the two bounds could also serve as the limits of integration in the exact non-iterative method given by Ioakimidis and Papadakis (1985). The method Prussing used to derive the upper and lower bounds is very simple and fundamental, and analogous reasoning can be applied to the true-anomaly time equation. It is slightly disappointing that Prussing's method does not lead to universal bounds in the s-domain problem; rather, different expressions must be evaluated for elliptic and hyperbolic orbits. When the same method is pursued for the true-anomaly time equation, however, universal expressions for the upper and lower bounds do appear. Here again, though, it is slightly disappointing that the universal bounds do not permit the derivation of a single universal starting value (other than zero) for the Newtonian iteration. Suitable starting values can be obtained if the orbit is identified as elliptical or non-elliptical. The development below is designed to parallel Prussing's derivation, and the notation is peculiar to this section.

From equations (5.6) and (5.8) of Chapter 5, time is given in terms of true anomaly by the quadrature

$$h\Delta t = \int_0^{\eta} r(x)^2 dx \equiv I(\eta) \tag{6.44}$$

Therefore, given a value of Δt, one seeks a root of the equation

$$F(\eta) = I(\eta) - h\Delta t = 0 \tag{6.45}$$

As noted previously, the derivative of this function is always positive:

$$F'(\eta) = I'(\eta) = r(\eta)^2 > 0 \tag{6.46}$$

If it were possible to specify minimum amd maximum values of the derivative $F'(\eta)$ on an interval containing the root, say,

$$F'_{\max} = r_{\max}^2 \quad \text{and} \quad F'_{\min} = r_{\min}^2 \quad , \tag{6.47}$$

then (6.46) could be written as

$$0 < r_{\min}^2 \le I'(\eta) \le r_{\max}^2 \tag{6.48}$$

By integration, assuming $\eta > 0$, one obtains

$$0 < \eta r_{\min}^2 \leq I(\eta) \leq \eta r_{\max}^2 \tag{6.49}$$

Then also

$$\eta r_{\min}^2 - h\Delta t \leq F(\eta) \leq \eta r_{\max}^2 - h\Delta t \tag{6.50}$$

Now introduce the positive quantities

$$U = \frac{h\Delta t}{r_{\min}^2} > 0 \quad \text{and} \quad L = \frac{h\Delta t}{r_{\max}^2} > 0 \tag{6.51}$$

Evaluating (6.50) at $\eta = U$ gives

$$0 \leq F(U) \leq \left[\frac{r_{\max}^2}{r_{\min}^2} - 1\right] h\Delta t \tag{6.52}$$

Likewise, evaluating (6.50) at $\eta = L$ gives

$$\left[\frac{r_{\min}^2}{r_{\max}^2} - 1\right] h\Delta t \leq F(L) \leq 0 \tag{6.53}$$

By comparing (6.52) and (6.53), it is easy to see that, since $F(L) \leq 0 \leq F(U)$ and F is continuous, a root must occur between $\eta = L$ and $\eta = U$. In other words, the root must be bounded as

$$0 < \frac{h\Delta t}{r_{\max}^2} \leq \eta \leq \frac{h\Delta t}{r_{\min}^2} \tag{6.54a}$$

Also, since $F'(\eta) > 0$ always, there can be only one root.

In retrospect, it is clear that Prussing's method of bounding the root can be applied to any time equation derived from the generalized Sundman time transformation $dt = Cr^n ds$. The integrated form analogous to (6.44) would be

$$\frac{1}{C}\Delta t = \int_0^s r(x)^n \, dx \equiv I(s) \tag{6.54b}$$

Prussing's method then leads to the inequality

$$0 < \frac{\Delta t}{C r_{\max}^n} \leq s \leq \frac{\Delta t}{C r_{\min}^n} \tag{6.54c}$$

and to proof that only one root exists. In other words, the solution value of the anomaly will always lie between those values which would be obtained on circular orbits of maximum and minimum radius respectively. The steps in Prussing's method merely give complete rigor to this intuitively appealing idea.

Now the problem of specifying bounds on η has been converted into the related problem of specifying minimum and maximum values of the radius. There is no special rule which must be followed; for example, on the outbound leg of a hyperbola one may choose the initial radius as the minimum radius. However, in order to avoid having to check for special configurations of the epochal point it is desirable to choose the most general values of minimum and maximum radius which could occur. For an elliptical orbit the obvious (and conservative) choices are the pericentral and apocentral radii. From equation (5.6) it is easy to find that

$$r_{\min} = \frac{\left[\frac{h^2}{\mu}\right]}{1 + \sqrt{\alpha^2 + \beta^2}} \tag{6.55}$$

where the radical will be recognized as the orbital eccentricity. Likewise,

$$r_{\max} = \frac{\left[\frac{h^2}{\mu}\right]}{1 - \sqrt{\alpha^2 + \beta^2}} \tag{6.56}$$

Then (6.54) takes the form

$$0<\frac{\mu^2}{h^3}\Delta t(1-\sqrt{\alpha^2+\beta^2})^2 \leq \eta \leq \frac{\mu^2}{h^3}\Delta t(1+\sqrt{\alpha^2+\beta^2})^2 \tag{6.57}$$

In the case of parabolic and hyperbolic orbits the minimum possible radius is still given by (6.55) but $r_{max} \to \infty$. Hence, for all orbits, it is true at least that

$$0 \leq \eta \leq \frac{\mu^2}{h^3}\Delta t(1+\sqrt{\alpha^2+\beta^2})^2 \tag{6.58}$$

Nevertheless, the tighter bounds given in (6.57) for elliptic orbits exhibit no numerical difficulty in passing from elliptic to parabolic to hyperbolic regimes. It might be thought that the mean of these bounds would provide a suitable starting value for the Newtonian iteration for all non-rectilinear orbits: $\eta_0 = \frac{\mu^2}{h^3}\Delta t(1+\alpha^2+\beta^2)$. However, this conclusion is oversimplified in several respects. First, no provision has been made to guarantee that this value of η_0 will be less than 2π as required for use with the true-anomaly time equations obtained in Chapter 5. In fact, examination of equation (5.104) shows that as Δt approaches the period of the orbit this value of η_0 will always exceed 2π. Second, no account has been taken of the positions of hyperbolic asymptotes, with the result that this value of η_0 may not be a physically realizable point on the orbit. Indeed, there is no guarantee that either of the bounds in (6.57) lies on the hyperbola, or that the upper bound in (6.58) lies on the hyperbola. These considerations leave one with only the trivial value $\eta_0 = 0$ as a single perfectly reliable universal starting value, even though arithmetically valid universal bounds are available which enclose other values. However, if one is willing to select different starting values for each type of orbit then some improvement can be made.

Consider the lower bound in (6.57). It is easy to show, using equation (5.104) for the period T, that as Δt approaches the period of the elliptical orbit this lower bound will always be less than 2π. If Δt exceeds one period then the initial value problem can still be solved correctly by replacing Δt by $(\Delta t \bmod T)$. Thus the lower bound can serve as a suitable non-zero starting value for an elliptical orbit. As the orbit becomes parabolic the period goes to infinity and the lower bound in (6.57) goes to zero, coinciding with the lower bound in (6.58). Zero is an acceptable starting value for parabolic orbits because convergence of the transcendental function in the time equation is trivial and a few extra Newton iterations are not costly. For a hyperbolic orbit one has the additional consideration that the starting value must lie on the physically realizable arc of the orbit. Hence, because even the lower bound in (6.57) may not lie on the arc for large Δt, one should choose $\eta_0 = 0$ regardless of the cost of extra Newton iterations. Actually, the numerical results recorded in Appendices D and E show that the transcendental function in the time equation tends to converge better on hyperbolic orbits (away from asymptotes) than on elliptic orbits, so it is appropriate to be more concerned with the starting value for the elliptic case. The following scheme is proposed. The given elapsed time is reduced "modulo period" for elliptic orbits:

$$\Delta t \leftarrow \Delta t \bmod T \quad \text{iff} \quad \Delta t > T \quad \text{and} \quad \alpha^2 + \beta^2 < 1 \tag{6.59a}$$

where the period T is given by (5.104):

$$T = \frac{2\pi h^3}{\mu^2 \sqrt{(1 - \alpha^2 - \beta^2)^3}} \tag{6.59b}$$

Then for all orbits the starting value of η is selected as

$$\eta_0 = \frac{\mu^2}{h^3} \Delta t \left\{ \max\left[0,\ (1 - \sqrt{\alpha^2 + \beta^2}) \right] \right\}^2 \tag{6.59c}$$

In short, use the lower bound in (6.57) as a starting value if the orbit is elliptical and use zero otherwise. While this procedure requires one to distinguish the type of orbit, it does produce a starting value which is a continuous function of the type of orbit.

The Sigma-Domain Formulation

As noted earlier, if the angular momentum magnitude h is very small or zero then the η-domain formulation of the initial value problem will fail because divisions by h are required at several places in the formulae. However, the whole problem can be successfully re-formulated in the σ domain. The relevant formulae, in which all divisions by h are avoided, are presented in this section. Of course, the σ-domain formulation can be used for any value of angular momentum magnitude, not just for small or zero values.

Given a position vector $\underline{r}_0$ and a velocity vector $\underline{v}_0$ at time t_0, compute angular momentum as

$$\underline{h} = \underline{r}_0 \times \underline{v}_0 \quad \text{and} \quad h = \sqrt{\underline{h} \cdot \underline{h}} \tag{6.60}$$

The unit radial vector at time t_0 is

$$\underline{\xi}_0 = \frac{1}{r_0}\underline{r}_0 \tag{6.61}$$

The derivative of the unit radial vector at time t_0 is transverse to $\underline{\xi}_0$ but is not a unit vector in general:

$$\underline{\xi}'_0 = \underline{h} \times \underline{\xi}_0 \tag{6.62a}$$

according to equation (3.103) of Chapter 3, or

$$\underline{\xi}'_0 = \frac{1}{r_0}[\underline{v}_0(\underline{r}_0 \cdot \underline{r}_0) - \underline{r}_0(\underline{r}_0 \cdot \underline{v}_0)] \tag{6.62b}$$

according to (3.104). The reciprocal radius and its derivative at time t_0 are calculated as in equations (3.96) and (3.108).

$$u_0 = \frac{1}{r_0} \tag{6.63}$$

$$u'_0 = \frac{-1}{r_0}(\underline{r}_0 \cdot \underline{v}_0) \tag{6.64}$$

Then for some value of time t one computes using the σ -domain solution of Keplerian motion given in, say, equations (3.79) through (3.82):

$$\underline{\xi}(\sigma) = \underline{\xi}_0 \cos(h\sigma) + \underline{\xi}'_0 \frac{\sin(h\sigma)}{h} \tag{6.65}$$

$$\underline{\xi}'(\sigma) = -\,\underline{\xi}_0 h \sin(h\sigma) + \underline{\xi}'_0 \cos(h\sigma) \tag{6.66}$$

$$u(\sigma) = u_0 \cos(h\sigma) + u'_0 \frac{\sin(h\sigma)}{h} + \mu \frac{1-\cos(h\sigma)}{h^2} \tag{6.67}$$

$$u'(\sigma) = -\,u_0 h \sin(h\sigma) + u'_0 \cos(h\sigma) + \mu \frac{\sin(h\sigma)}{h} \tag{6.68}$$

In these formulae the explicit divisions by h are avoided by expanding $\dfrac{\sin(h\sigma)}{h}$ and $\dfrac{1-\cos(h\sigma)}{h^2}$ in power series as in equations (3.83) and (3.84). Then the functions $\sin(h\sigma)$ and $\cos(h\sigma)$ can be computed as

$$\sin(h\sigma) = h\left[\frac{\sin(h\sigma)}{h}\right] \tag{6.69a}$$

and

$$\cos(h\sigma) = 1 - h^2\left[\frac{1-\cos(h\sigma)}{h^2}\right] \tag{6.69b}$$

Hence the above equations contain two separate transcendental functions. The value of σ corresponding to the time t must be determined from some form of the time equation given in Chapter 5, such as equation (5.232) or equation (5.238), represented functionally as

$$t - t_0 = F(\ Z(\sigma);\ \ u_0,\ \ u'_0\ ) \tag{6.70a}$$

In the time equation, σ occurs only in one of the forms

$$Z = \frac{1}{h}\tan\frac{1}{2}(h\sigma) \tag{6.70b}$$

or

$$Z = \frac{1}{h}\tan\frac{1}{4}(h\sigma) \tag{6.70c}$$

The functions F and Z are two more transcendental functions, each to be evaluated repeatedly during the iterative solution of (6.68), for a total of four different functions required. Finally, with the quantities in equations (6.65) through (6.68) available, position and velocity at time t are given by equations (3.109) and (3.112):

$$\underline{r}(t) = \frac{1}{u(\sigma)}\underline{\xi}(\sigma) \tag{6.71}$$

$$\underline{v}(t) = u(\sigma)\underline{\xi}'(\sigma) - u'(\sigma)\underline{\xi}(\sigma) \tag{6.72}$$

The computational efficiency of the above scheme can be improved considerably by adopting Z as the independent variable in place of σ. For example, let $Z = \frac{1}{h}\tan\frac{1}{2}(h\sigma)$. Then, making use of the identities (6.15) and (6.16) and remembering that $\eta = h\sigma$, it is easy to show that

$$\sin(h\sigma) = \frac{2hZ}{1 + h^2Z^2} \tag{6.73}$$

$$\cos(h\sigma) = \frac{1 - h^2Z^2}{1 + h^2Z^2} \tag{6.74}$$

$$\frac{\sin(h\sigma)}{h} = \frac{2Z}{1 + h^2Z^2} \tag{6.75}$$

$$\frac{1 - \cos(h\sigma)}{h^2} = \frac{2Z^2}{1 + h^2Z^2} \tag{6.76}$$

Substituting these expressions into the above calculation procedure leaves Z as the independent variable and F as the only transcendental function. Likewise, if $Z = \frac{1}{h}\tan\frac{1}{4}(h\sigma)$ then the further use of the identities (6.17) and (6.19) leads to

$$\sin(h\sigma) = 2\left[\frac{2hZ}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right] \tag{6.77}$$

$$\cos(h\sigma) = \left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]^2 - \left[\frac{2hZ}{1+h^2Z^2}\right]^2 \tag{6.78}$$

$$\frac{\sin(h\sigma)}{h} = 2\left[\frac{2Z}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right] \tag{6.79}$$

$$\frac{1-\cos(h\sigma)}{h^2} = 2\left[\frac{2Z}{1+h^2Z^2}\right]^2 \tag{6.80}$$

Once again, the use of these expressions leaves Z as the independent variable and F as the only transcendental function.

A Newtonian iteration method will now be developed for the solution of the time equation (6.69). The iteration in terms of σ would be

$$\sigma_{k+1} = \sigma_k + \frac{\Delta t - F(Z(\sigma_k); u_0, u'_0)}{F'(Z(\sigma_k); u_0, u'_0)} \tag{6.81}$$

where the prime mark denotes differentiation with respect to σ. The derivative is available immediately from equation (5.199) as

$$\frac{dF}{d\sigma} = \frac{1}{u(\sigma)^2} \tag{6.82}$$

where $u(\sigma)$ is given by equation (6.67) above. The iteration becomes

$$\sigma_{k+1} = \sigma_k + \left[\Delta t - F(Z(\sigma_k); u_0, u'_0)\right] u(\sigma_k)^2 \tag{6.83}$$

Of course, it is much more efficient computationally to use Z rather than σ as the variable of iteration. The appropriate modifications will now be presented.

Half-Angle Formulae

Let

$$Z = \frac{1}{h} \tan \frac{1}{2}(h\sigma) \tag{6.84}$$

so that

$$d\sigma = \frac{2dZ}{1 + h^2 Z^2} \tag{6.85}$$

Now the iteration will take the form

$$Z_{k+1} = Z_k + \frac{\Delta t - F(Z_k; u_0, u'_0)}{F'(Z_k; u_0, u'_0)} \tag{6.86}$$

where the prime mark denotes differentiation with respect to Z. The derivative is calculated as

$$\frac{dF}{dZ} = \frac{dF}{d\sigma} \frac{d\sigma}{dZ} \tag{6.87}$$

or

$$\frac{dF}{dZ} = \frac{1}{u(\sigma)^2} \left[\frac{2}{1 + h^2 Z^2} \right] \tag{6.88}$$

Substituting from (6.67) and then from (6.73) through (6.76) produces, after some rearrangement,

$$\frac{dF}{dZ} = \frac{2(1 + h^2 Z^2)}{\left[u_0(1 - h^2 Z^2) + 2u'_0 Z + 2\mu Z^2\right]^2} \tag{6.89}$$

This expression can be put in a more convenient form by multiplying numerator and denominator by u_0^2 and adding and subtracting $u_0'^2Z^2$ inside the square brackets in the denominator. Regrouping terms then allows the derivative to be calculated as

$$\frac{dF}{dZ} = \frac{2u_0^2(1+h^2Z^2)}{H^4(1+X)^2} \tag{6.90}$$

where

$$X = \frac{GZ^2}{H^2} \tag{6.91}$$

$$G = 2\mu u_0 - h^2u_0^2 - u_0'^2 \tag{6.92}$$

$$H = u_0 + u'_0 Z \tag{6.93}$$

All of these quantities are used in the evaluation of the half-angle versions of F given in Chapter 5, such as in equations (5.214) or (5.220). Hence, the derivative can be obtained with very little extra arithmetic. The iteration finally takes the form

$$Z_{k+1} = Z_k + [\Delta t - F(Z_k; u_0, u'_0)]\frac{H_k^4(1+X_k)^2}{2u_0^2(1+h^2Z_k^2)} \tag{6.94}$$

Notably, a cancellation of the factors H and $(1+X)$ from the denominators of the half-angle time formulae occurs in (6.94), with the result that the iteration should be well-behaved near the singularities of the time equation. Of course, the inherent limitations of the half-angle formulae may prove unacceptable in some applications. The major limitations can be overcome by using the quarter-angle formulae described next.

Quarter-Angle Formulae

Let

$$Z = \frac{1}{h}\tan\frac{1}{4}(h\sigma) \tag{6.95}$$

so that

$$d\sigma = \frac{4dZ}{1 + h^2Z^2} \tag{6.96}$$

Now an iteration of the same form as (6.86) is sought, where the derivative can be calculated using a chain rule like (6.87):

$$\frac{dF}{dZ} = \frac{1}{u(\sigma)^2}\left[\frac{4}{1 + h^2Z^2}\right] \tag{6.97}$$

Substituting from (6.67) and then from (6.77) through (6.80) produces, after some rearrangement,

$$\frac{dF}{dZ} = \frac{4(1 + h^2Z^2)^3}{\left[u_0\left[(1 - h^2Z^2)^2 - 4h^2Z^2\right] + 4u'_0Z(1 - h^2Z^2) + 2\mu(4Z^2)\right]^2} \tag{6.98}$$

This expression can be put in a more convenient form by multiplying numerator and denominator by u_0^2 and adding and subtracting $4u'^2_0Z^2$ inside the outer square brackets of the denominator. Regrouping terms then produces

$$\frac{dF}{dZ} = \frac{4u_0^2(1 + h^2Z^2)^3}{Q^2} \tag{6.99}$$

where

$$Q = \left[u_0(1 - h^2Z^2) + 2u'_0Z\right]^2 + 4Z^2(2\mu u_0 - h^2u_0^2 - u'^2_0) \tag{6.100}$$

The quantity Q was defined in equations (5.228), (5.223) and (5.207) of Chapter 5 and is used in the evaluation of the quarter-angle versions of the time equation. Hence, the derivative can be calculated with little extra effort. The iteration finally takes the form

$$Z_{k+1} = Z_k + [\Delta t - F(Z_k; u_0, u'_0)]\frac{Q_k^2}{4u_0^2(1 + h^2 Z_k^2)^3} \tag{6.101}$$

Here an appropriate quarter-angle version of F is to be used, such as that given in equation (5.232) or (5.238). Notice that the denominator factor Q is cancelled from these time equations when the iteration (6.101) is implemented, keeping the iteration well-behaved near a hyperbolic asymptote. Finally, if one wishes to implement starting values other than $\sigma_0 = 0$ for the iteration, the general result in (6.54c) is available:

$$0 < \frac{\Delta t}{r_{\max}^2} \le \sigma \le \frac{\Delta t}{r_{\min}^2} \tag{6.102}$$

Of course, the locations of hyperbolic asymptotes must be considered in the selection of σ_0, even though σ is not a physical angle. This consideration will complicate the detailed application of (6.102), just as it did the derivation of results in the η domain.

Summary

Previous chapters have developed a variety of formulae expressing the direct relation between time and true anomaly, as well as the closed solutions of two-body motion in the true-anomaly (η and σ) domain. This chapter has described the application of these formulae in solving the initial value problem of unperturbed motion. The results are new, inasmuch as the true-anomaly time equations have not been available until now, and consist of Newtonian iteration formulae which are universal and require only one transcendental function evaluation per iteration. One set of formulae, using η-domain variables, is valid for all non-rectilinear orbits, while a second set, using σ -domain variables, is valid for all orbits, including rectilinear ones. Additionally, simple reliable starting values for the iterations can obtained by adapting the ideas of Prussing (1979) to the true-anomaly domain. Numerical results are summarized in Appendix E.

Chapter 7. The Keplerian Boundary Value Problem

Introduction

The η-domain solution of unperturbed motion developed in Chapters 3 and 5 permits a very concise treatment of the Gauss/Lambert boundary value problem: given two position vectors $\underline{r}_0$ and $\underline{r}_1$ and the time of flight between them, $\Delta t = t_1 - t_0$, compute the velocity vector $\underline{v}_0$ at the initial time t_0 . Like the initial value problem, this problem is one of the classical exercises of celestial mechanics, and many approaches to its solution have been developed. Originally posed as part of the astronomical angles-only orbit determination problem, it is also fundamental in the engineering applications of spacecraft guidance and mission planning, and continues to be researched actively. A complete review of the modern work on the Gauss/Lambert problem is beyond the scope of this study, but several articles should be mentioned as especially pertinent to the present work. Wintner (1947, sections 247 and 248), in discussing Lambert's theorem of 1761, derived an integral equation expressing time of flight in terms of the two given radii, the chord distance between them, and the unknown semimajor axis. The classical Lambert time-of-flight equation, which is usually written in a form developed by Lagrange, takes different forms for elliptic, parabolic and hyperbolic orbits, whereas Wintner's integral equation holds for all orbits provided that the reciprocal semimajor axis (essentially, the Keplerian energy) is used as the unknown parameter. Much later, Sun (1981) showed that the solution of Wintner's integral equation also satisfies a hypergeometric differential equation. This leads immediately to a universal formulation of the boundary value problem in terms of hypergeometric functions. This connection can be seen even in more recent and apparently unrelated work by Taff and Randall (1985). These authors use a simple manipulation of the classical Lambert/Lagrange time-of-flight equation to derive a method for elliptic orbits in which the semimajor axis is the unknown parameter. Though Taff and Randall were not aiming at such a generalization, inspection of their equations shows that the reciprocal semimajor axis might as well be taken as the unknown. Two different evaluations of an inverse sine function are required in their equation, but if one were to compute the inverse sines in terms of hypergeometric functions (for example, see chapter 18 of Wall, 1948) then a universal time equation could be obtained. Meanwhile, other researchers have derived various universal formulations by means of *ad hoc* manipulations of Lagrange's form of the Lambert time-of-flight relation. Among these are Lancaster, *et al.*, (1966) and Lancaster and Blanchard (1969), who derived a universal formula by means of power series expansions of the inverse sine and inverse hyperbolic sine functions and by a careful selection of geometrical parameters. Their final results require the evaluation of two transcendental functions in the time-of-flight equation. Battin (1964) produced a universal formulation by expressing the Lambert/Lagrange time-of-flight equation in terms of parameters and transcendental functions which arise in the universal s -domain solution of unperturbed motion. His final calculations require consecutive iterations for two unknown parameters and the evaluation of two transcendental functions during the iterations. In the same place, several other methods are described which require iteration for only a single parameter. Battin (1968) also derived another universal Lambert-type formulation which requires the evaluation of two transcendental, in this case hypergeometric, functions and iteration for only one parameter. A notable feature of this formulation is that the hypergeometric functions are evaluated as continued fractions rather than as series, producing improved convergence characteristics. Later improvements were introduced by Battin (1977) and Battin, *et al.*, (1978) in which only a single hypergeometric function is required. In these two papers, Lambert's theorem is used together with the independent method invented by Gauss (1809) to obtain what might be described as a hybrid time equation. Later, Battin and Vaughn (1983) devised yet a different approach using, not Lambert's theorem, but a method which parallels and improves that of Gauss (1809). Once again only a single hypergeometric function occurs in the time equation. Still other methods have been developed which do not depend on Lambert's theorem. Pitkin (1968) presented a method based directly on the s-domain solution of unperturbed motion. His final calculations require the evaluation of several transcendental functions in the time equation and a two-dimensional Newton-Raphson iteration for two unknown parameters, so the method, while completely general, is less efficient computationally than some other schemes. Bate, *et al.* (1971) also present a method based directly on the s-domain two-body solution. They require the evaluation of two transcendental functions in the time equation, but the iteration involves only one parameter. Neither Pitkin nor Bate, *et al.*, take advantage of the linearity of the s-domain differential equations. Several authors have made use of the linear s-domain equations governing the KS coordinates to address the boundary value problem. Jezewski (1976) devised a method for iterating on a single parameter, but his method requires the evaluation of several transcendental functions and the results are valid only for elliptic orbits. Kriz (1976) developed a similar KS method which is valid for all orbits, though his formulae have singularities if the transfer eccentric anomaly (s-domain) is an integral multiple of π, a potentially serious limitation. Engels and Junkins (1981) also presented a universal KS method which, by means of a variation of parameters technique, they were able to generalize to handle the boundary value problem of perturbed motion. However, their Keplerian formulae are singular for physical transfer angles of 0° and 180° and for rectilinear orbits. A feature of all methods involving the KS coordinates is that extra boundary conditions have to be supplied in such a way that the bilinear relation is satisfied. This complicates the derivations which would otherwise be quite straightforward. It appears that no formulations of the boundary value problem have been made which utilize the linear s-domain equations governing the Cartesian coordinates. (Kriz (1976) did give explicit Cartesian-coordinate formulae which are derived from his KS solution.) However, the final results of such an approach would be equivalent to those of Battin (1964), Pitkin (1968), and Bate, *et al.*, (1971), and the derivations should be much easier to follow. Also, apparently, no formulations of the Gauss/Lambert problem have taken advantage of the linearity of the η-domain equations of unperturbed motion. Yet, in at least one respect, the η domain is the most natural setting for the boundary value problem: η is just the angle between the two position vectors. Recourse to Lambert's theorem, with its specialized relations and manipulations, is not required. Furthermore, the time equation already involves only a single transcendental function, and if the Burdet-type coordinates are used then no extra boundary conditions have to be supplied even though the coordinates are redundant. By collecting together results already obtained in this study, it proves to be possible to reduce the solution of the boundary value problem to an iteration on one parameter in which only one transcendental function is evaluated during each cycle. The σ-domain solution of unperturbed motion can also be used to formulate a universal method for the Gauss/Lambert problem. Such a method would be capable of handling the special case of rectilinear orbits, and its implementation will be described later.

In all methods, of course, the solution of the Gauss/Lambert problem finally requires the inversion of a transcendental equation for an unknown parameter. The previous chapter noted the fact that recent results by Ioakimidis and Papadakis (1985) make possible the non-iterative solution of such equations when the solution value lies between known bounds. Nevertheless, the extensive computation needed in their approach should make an efficient iterative method more practical in most cases. This chapter assumes the use of an iterative method for solving the time-of-flight equation, realizing that a non-iterative approach is possible.

The Eta-Domain Formulation

Position and velocity are related to η-domain quantities by equations (3.109) and (3.112).

$$\underline{r}(t) = \frac{1}{u(\eta)}\underline{\xi}(\eta) \tag{7.1}$$

$$\underline{v}(t) = h\left[u(\eta)\underline{\xi}'(\eta) - u'(\eta)\underline{\xi}(\eta)\right] \tag{7.2}$$

The time dependence indicated on the left-hand side implies that η is related to time through the true-anomaly time equation. The η-domain variables are given in terms of initial conditions by

$$\underline{\xi}(\eta) = \underline{\xi}_0 \cos\eta + \underline{\xi}'_0 \sin\eta \tag{7.3}$$

$$\underline{\xi}'(\eta) = -\,\underline{\xi}_0 \sin\eta + \underline{\xi}'_0 \cos\eta \tag{7.4}$$

$$u(\eta) = \frac{\mu}{h^2}(1 + \alpha\cos\eta + \beta\sin\eta) \tag{7.5}$$

$$u'(\eta) = \frac{\mu}{h^2}(-\,\alpha\sin\eta + \beta\cos\eta) \tag{7.6}$$

Substituting these expressions into the velocity relation (7.2) produces the succinct result

$$\underline{v}(t) = \frac{\mu}{h}\left[\underline{\xi}'_0(\alpha + \cos\eta) - \underline{\xi}_0(\beta + \sin\eta)\right] = \underline{v}_0 - \frac{\mu}{h}\left[\underline{\xi}'_0(1 - \cos\eta) + \underline{\xi}_0 \sin\eta\right] \tag{7.7}$$

In the boundary value problem η is a fixed parameter and this equation gives the velocity vector at the final time. The initial velocity vector is simply

$$\underline{v}_0 = \frac{\mu}{h}\left[\underline{\xi}'_0(1 + \alpha) - \beta\underline{\xi}_0\right] \tag{7.8}$$

To solve the boundary value problem, it is necessary to specify all the quantities on the right-hand side of this latter equation. The initial unit radial vector is available as

$$\underline{\xi}_0 = \frac{1}{r_0}\underline{r}_0 = u_0\underline{r}_0 \tag{7.9}$$

The transverse unit vector at initial time is found by using position information at the final time:

$$\underline{\xi}_1 = \frac{1}{r_1}\underline{r}_1 = u_1\underline{r}_1 \tag{7.10}$$

Then the vector defined by

$$\underline{N} = \underline{\xi}_0 \times \underline{\xi}_1 \tag{7.11}$$

is normal to the orbital plane and positive in the right-handed sense as the satellite travels from $\underline{r}_0$ to $\underline{r}_1$ the "short way", that is, via the transfer angle which is less than 180°. For transfers the "long way" (more than 180°), the vector normal to the orbital plane and positive in the right-handed sense as the satellite travels from $\underline{r}_0$ to $\underline{r}_1$ would be the negative of that given in (7.11). The vector $\underline{N}$ is not a unit vector unless the position vectors happen to be 90° apart, and its magnitude is given by

$$N = \sqrt{\underline{N} \cdot \underline{N}} \tag{7.12}$$

Substituting from equation (7.11), and using first a scalar triple-product identity and then a vector triple-product identity, one obtains

$$N = \sqrt{1 - (\underline{\xi}_0 \cdot \underline{\xi}_1)^2} \tag{7.13}$$

Then the transverse unit vector at the initial position is given by

$$\underline{\xi}'_0 = \frac{1}{N}\underline{N} \times \underline{\xi}_0 \tag{7.14}$$

Substituting from (7.11) and using a vector triple-product identity, one obtains

$$\underline{\xi}'_0 = \frac{1}{N}\left[\underline{\xi}_1 - \underline{\xi}_0(\underline{\xi}_1 \cdot \underline{\xi}_0)\right] \tag{7.15}$$

In case the position vectors are collinear, $\underline{N}$ is a zero vector and the orbit plane is not uniquely defined. As will be seen below, the time equation of the the boundary value problem can still be solved, and one is left free to choose a suitable orbital plane. In the exceptional case when not only are the position vectors collinear but also the transfer orbit turns out to be rectilinear, the η -domain formulation will fail and recourse must be sought in the σ-domain formulation described later.

It remains to specify h, α and β in equation (7.8). Recall that

$$\alpha = \frac{h^2 u_0}{\mu} - 1 = \frac{h^2}{\mu r_0} - 1 \tag{7.16}$$

Hence, the angular momentum magnitude can be replaced by

$$h = \sqrt{\frac{\mu(1+\alpha)}{u_0}} \tag{7.17}$$

so that

$$\underline{v}_0 = \sqrt{\frac{\mu u_0}{1+\alpha}}\left[(1+\alpha)\underline{\xi}'_0 - \beta\underline{\xi}_0\right] \tag{7.18}$$

The presence of only two unknown parameters, rather than three, in this equation corresponds to the fact that there are only two unknown components of initial velocity when that vector is resolved in the orbital plane. Now in order to relate α and β consider the two-body solution for $u(\eta)$ given in equation (7.5). Remembering that η is a fixed parameter in the boundary value problem, that values of u are known at both ends of the transfer arc, and that h can be replaced according to (7.17), one has the linear relationship between α and β

$$u_1(1+\alpha) = u_0(1 + \alpha\cos\eta + \beta\sin\eta) \tag{7.19}$$

Thus it is an easy matter to express $\underline{v}_0$ in terms of a single unknown parameter.

The fact that α and β are linearly related in the boundary value problem is significant and deserves some comment. From equation (1.13) of Chapter 1 the Laplace vector for the transfer orbit is

$$\underline{B} = \underline{r}(\underline{v}\cdot\underline{v}) - \underline{v}(\underline{r}\cdot\underline{v}) - \frac{\mu}{r}\underline{r} \tag{7.20}$$

Substituting for $\underline{r}$ from (7.1) and for $\underline{v}$ from (7.2), and using equations (7.3) through (7.6), there is obtained, after straightforward simplification,

$$\underline{B} = \mu(\alpha\underline{\xi}_0 + \beta\underline{\xi}'_0) \tag{7.21}$$

Hence, because α and β are linearly related, the locus of transfer-orbit Laplace vectors is a straight line in space, with each point on the locus corresponding to a different transfer time. This fact has been noted in many different contexts, such as in Battin, *et al.*, (1978), but the above demonstration of the fact is particularly simple. It also provides a clear reason why the boundary value problem can be solved by determining one scalar parameter rather than three independent components of the velocity vector. To see this explicitly, consider the linear locus of the Laplace vectors in connection with the so-called hodograph equation. The hodograph equation, sometimes known as Hamilton's integral even though it is not an independent integral of the unperturbed motion, is a relation between vector velocity and vector position which holds for all non-rectilinear orbits. It takes a particularly simple form in the present case, and can be derived as follows. From equations (1.7) and (1.13) of Chapter 1, the angular momentum and Laplace vector constants are defined as

$$\underline{h} = \underline{r} \times \underline{v} \tag{7.22}$$

$$\underline{B} = \underline{v} \times (\underline{r} \times \underline{v}) - \frac{\mu}{r}\underline{r} \tag{7.23}$$

Then

$$\underline{h} \times \underline{B} = \underline{h} \times (\underline{v} \times \underline{h}) - \underline{h} \times \frac{\mu}{r}\underline{r} \tag{7.24}$$

Carrying out the vector triple product,

$$\underline{h} \times \underline{B} = \underline{v}(\underline{h} \cdot \underline{h}) - \underline{h}(\underline{h} \cdot \underline{v}) - \underline{h} \times \frac{\mu}{r}\underline{r} \tag{7.25}$$

The second term on the right-hand side vanishes identically. Then, provided that the orbit is not rectilinear, one can solve for the velocity vector:

$$\underline{v} = \frac{\mu}{h}\left[\frac{1}{h}\underline{h} \times \frac{1}{\mu}\underline{B} + \frac{1}{h}\underline{h} \times \frac{1}{r}\underline{r}\right] \tag{7.26}$$

This is the hodograph equation. It holds at every point on the orbit, and provides a convenient "phase plane" representation of the unperturbed motion. It can be put into a slightly neater form in terms of the eccentricity vector $\underline{e} = \underline{B}/\mu$ and the unit normal vector $\underline{n} = \underline{h}/h$ as

$$\underline{v} = \frac{\mu}{h}\left[\underline{n} \times \underline{e} + \underline{n} \times \underline{\xi}\right] \tag{7.27}$$

The vector $\underline{n} \times \underline{e}$ has constant magnitude and direction normal to the major axis of the conic section, while the vector $\underline{n} \times \underline{\xi}$ is just the transverse unit vector in the direction of revolution. Now substituting for $\underline{e}$ from (7.21) and for $\underline{\xi}$ from (7.3), one obtains

$$\underline{v} = \frac{\mu}{h}\left[(\underline{n} \times \underline{\xi}_0)(\alpha + \cos\eta) + (\underline{n} \times \underline{\xi}'_0)(\beta + \sin\eta)\right] \tag{7.28}$$

Inspection of the vector products shows that this equation is really the same as equation (7.7) for the velocity developed above. Once h has been replaced in terms of α, the hodograph equation evaluated at initial time on the transfer orbit becomes the same as equation (7.18), having the two unknown parameters α and β. The linear relationship between α and β then makes it feasible to replace either of these in terms of the other so that the initial velocity vector is specified completely in terms of a single scalar parameter.

The one remaining link in the solution of the boundary value problem is to use the true-anomaly time equation

$$t_1 - t_0 = \frac{h^3}{\mu^2}K(\eta; \alpha, \beta) \tag{7.29}$$

to determine the unknown parameter, α or β. By replacing h according to (7.17), the time equation takes the form

$$\Delta t\sqrt{\mu u_0^3} = (1+\alpha)\sqrt{1+\alpha}\, K(\eta; \alpha, \beta) \tag{7.30}$$

In order to choose between α and β, consider the following rearrangements of (7.19).

$$\alpha = \frac{1 - \dfrac{u_1}{u_0} + \beta \sin\eta}{\dfrac{u_1}{u_0} - \cos\eta} \tag{7.31}$$

$$\beta = \frac{\left[\dfrac{u_1}{u_0} - \cos\eta\right]\alpha + \dfrac{u_1}{u_0} - 1}{\sin\eta} \tag{7.32}$$

Unfortunately, there are difficulties associated with either choice. If α is eliminated in favor of β by means of (7.31) then a singularity is introduced whenever $u_1 = u_0 \cos\eta$, that is, whenever $r_0 = r_1 \cos\eta$. This geometrical circumstance is entirely unexceptional, so it would be desirable to avoid introducing a singularity here unnecessarily. On the other hand, if β is eliminated in favor of α using (7.32) then a singularity is introduced for transfer angles of 0 and π. The remedy is to introduce a second linear relation between α and β using equation (7.6). When h has been replaced in terms of α, that equation becomes

$$u'_1(1+\alpha) = u_0(-\alpha\sin\eta + \beta\cos\eta) \tag{7.33}$$

The parameter u'_1 is essentially the radial rate at the final time and is the sole remaining unknown parameter in the η -domain boundary value problem. Then the two linear relations (7.33) and (7.19) can be solved simultaneously for α and β in terms of u'_1.

$$\alpha = \frac{(\rho - 1)\cos\eta - q\sin\eta}{1 - \rho\cos\eta + q\sin\eta} \tag{7.34}$$

$$\beta = \frac{(\rho - 1)(q + \sin\eta) - q(\rho - \cos\eta)}{1 - \rho\cos\eta + q\sin\eta} \tag{7.35}$$

where

$$\rho = \frac{u_1}{u_0} = \frac{r_0}{r_1} \tag{7.36}$$

$$q = \frac{u'_1}{u_0} \tag{7.37}$$

On this approach, the time equation (7.30) is to be solved for the unknown q, with α and β evaluated using (7.34) and (7.35).

Several comments about the above scheme should be made. First, it is does not appear that an efficient Newton-type iteration can be developed for this problem because the derivative of the time equation with respect to q is not readily available. Thus, in order to find a root of the equation

$$F(q) = (1 + \alpha(q))\sqrt{1 + \alpha(q)}\,K(\eta;\alpha(q),\beta(q)) - \Delta t\sqrt{\mu u_0^3} = 0 \tag{7.38a}$$

one might resort to a secant-type analog of Newton's method, such as

$$q_{k+1} = q_k - F(q_k)\left[\frac{q_k - q_{k-1}}{F(q_k) - F(q_{k-1})}\right] \qquad \text{with} \qquad k = 1, 2, 3, \ldots \tag{7.38b}$$

Of course, more sophisticated derivative-free methods are available. For completeness, let it be noted that Newton's method would require the repeated evaluation of

$$\frac{dF}{dq} = \frac{3}{2}\sqrt{1+\alpha}\,K\frac{d\alpha}{dq} + (1+\alpha)\sqrt{1+\alpha}\left[\frac{\partial K}{\partial \alpha}\frac{d\alpha}{dq} + \frac{\partial K}{\partial \beta}\frac{d\beta}{dq}\right] \tag{7.39}$$

Expressions for $\frac{d\alpha}{dq}$ and $\frac{d\beta}{dq}$ are readily obtained from (7.34) and (7.35). The chief obstacle to the calculation is in evaluating $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ since these are very extensive expressions. For this reason they are not quoted explicitly here. Later, in Chapter 9 in connection with the perturbed initial value problem, it will be necessary to calculate these partial derivatives. Explicit universal formulae are given there, but it will be seen that they would involve nearly as much computation as several extra iterations of a simpler scheme like (7.38b). Even though Newton's method often requires fewer iterations than a derivative-free method, the efficiency gain in this case is at least debatable.

Second, care must be taken to choose q during the iterations so that $\alpha > -1$. This condition will automatically be satisfied at the solution of the problem for nonrectilinear transfer orbits, but there is no guarantee that it will be satisfied on every intervening iteration by the numerical method one happens to be using. In order to decide whether to increase or decrease the value of q in case $\alpha \leq -1$, consider whether the function $\alpha(q)$ is increasing or decreasing: the value of q ought always to be modified so as to increase the value of α. The derivative of α with respect to q is

$$\frac{d\alpha}{dq} = \frac{(\cos\eta - 1)\sin\eta}{(1 - \rho\cos\eta + q\sin\eta)^2} \tag{7.40}$$

The sign of this fraction is controlled by the numerator. Therefore the rule for changing the value of q is independent of q : if $\alpha \leq -1$ and $(\cos\eta - 1)\sin\eta > 0$ then increase q ; if $\alpha \leq -1$ and $(\cos\eta - 1)\sin\eta < 0$ then decrease q . No checks are needed on β since the value of that parameter is unrestricted. Note that for a transfer angle of 180° the value of α is fixed and cannot be affected by the choice of q . However, no particular difficulty arises from the fact that equation (7.34) always gives $\alpha = (1-\rho)/(1+\rho)$ and only q and β are changing during the iteration. Note also that for a transfer angle of 0° the value of α is fixed once again and cannot be affected by the value of q. However, in this case $\alpha = -1$, which indicates that the 0° transfer orbit must be considered rectilinear. The η-domain formulation will fail due either to non-convergence of the transcendental function evaluation in the time equation or to excessively large values of q being generated in the attempt to increase the value of α. Naturally, similar numerical difficulties can be expected for near-zero transfer angles. This fact represents a fundamental limitation of the η-domain formulation: it is not well suited to handling very small transfer angles. Quantifying "very small" is difficult to do with generality since the allowable smallness depends on the available precision of the computing device and the required precision of the results. The small-angle difficulty can be avoided during the solution of the time equation by using the σ-domain formulation described later.

Third, the denominator of equations (7.34) and (7.35) for α and β will vanish if the value of q happens to be the critical value

$$\bar{q} = \frac{\rho \cos \eta - 1}{\sin \eta} \tag{7.41}$$

This value cannot occur at the solution because α and β are finite for all nonrectilinear orbits, but conceivably it could occur (or nearly so) during iterations. Hence, a rule is needed for modifying the guessed value of q in case q has approached $\bar{q}$ to within some pre-established tolerance. Actually, if avoiding the condition $q = \bar{q}$ were the only consideration then simply resetting q to any other value far away from $\bar{q}$ would be a sufficient corrective action. However, equation (7.40) indicates that as q approaches $\bar{q}$ the value of α becomes very sensitive to the value of q. Since it is required that $\alpha > -1$, the strategy for resetting q away from $\bar{q}$ should be the same as before, namely, that the value of q ought always to be modified so as to increase the value of α. Notice from (7.40) that the sign of $\frac{d\alpha}{dq}$ is unaffected by whether q approaches $\bar{q}$ from above or below. Therefore, the rule for modifying q is independent of q in this case also: if $(1 - \rho \cos \eta + q \sin \eta)^2 < \varepsilon$, where ε is some pre-established small positive tolerance, and $(\cos \eta - 1) \sin \eta > 0$ then increase q; if $(1 - \rho \cos \eta + q \sin \eta)^2 < \varepsilon$ and $(\cos \eta - 1) \sin \eta < 0$ then decrease q. For a transfer angle of 180° equation (7.41) indicates that $\bar{q} \rightarrow -\infty$ so that such modifications of the guessed value of q would never actually be effected; however, they are not needed since α is not changing and is in no danger of becoming less than -1.

Finally, it is recommended that the quarter-angle versions of the time equation, such as (5.151) or (5.182), rather than the half-angle versions, be used to address the boundary value problem. The half-angle formulae cannot be used for transfer angles approaching or exceeding 180°, or for those transfers for which the eccentric anomaly turns out to approach or exceed 180°, whereas the quarter-angle formulae are valid for almost one revolution. Of course, the eighth-angle and higher formulae referred to, but not developed, in this study could also be used without difficulty.

An additional remark might be inserted here about the possibility of deriving rigorous bounds on the solution value of q using the type of reasoning originally developed by Prussing (1979) for the initial value problem. It should be noted that Prussing's reasoning holds true regardless of which of the quantities in the formulae are known *a priori*. For example, in equation (6.57) or (6.58) of Chapter 6 one might regard η and time as given and the other parameters as unknown. Then h could be replaced in terms of α from (7.17), and α and β could be replaced in terms of q from (7.34) and (7.35). The result is an inequality which must be satisfied by the single unknown parameter q. However, the inequality turns out to be very complicated so that one cannot isolate q algebraically. Some numerical process (a simple marching scheme, say) might be used to find the upper and lower values of q which satisfy the inequality, but that computation would be almost as extensive as solving the time equation itself. Moreover, the inequality does not take account of the possible existence of hyperbolic asymptotes. Hence, the inequality is not immediately useful in the boundary value problem, nor does it indicate what other method might be used to obtain rigorous bounds on q.

The Sigma-Domain Formulation

As mentioned earlier, the η-domain formulation of the boundary value problem will fail if the angular momentum magnitude h is very small or zero. The σ-domain formulation suffers no such difficulty because all explicit divisions by h are avoided. Naturally, the σ-domain formulation can be used for all values of the angular momentum, not just for small or zero values. It proves to be more complicated to arrive at working σ-domain formulae because σ does not have a convenient interpretation as an angle. In fact, the transfer angle is

$$\eta = h\sigma \tag{7.42}$$

Thus, although the product $h\sigma$ is known, the separate values of h and σ are unknown. In general, h may be zero or nonzero but σ is always nonzero if Δt is nonzero. By handling h and σ separately one is able both to accomodate exactly rectilinear orbits and to overcome the small-angle difficulties in the solution of the time equation which arose in the η -domain formulation. The only other formulations of the boundary value problem which are capable of handling exactly rectilinear orbits are based on s -domain eccentric-anomaly formulae. An analogous situation occurs there in that the values of both s and the orbital energy are unknown. Battin (1964) uses a sequence of two one-parameter iterations to obtain the unknowns. Pitkin (1968) resorts to a two-dimensional Newton-Raphson iteration for the two parameters. Kriz (1976) cites a similar simultaneous two-parameter KS method, but then he is able to reduce the KS method to a one-parameter iteration, though at the cost of introducing singularities when s is an integral multiple of π. In the σ-domain it is possible to formulate a one-parameter iteration which has no singularities until the transfer angle approaches 2π . Even these could be avoided by using eighth-angle or higher formulae in place of the quarter-angle formulae used here. The main problem is to select a suitable parameter for the iteration. It should be mentioned that the methods of Battin (1968, 1977) and Battin and Vaughn (1983) also require only one-parameter iterations, are valid for transfer angles of up to 2π, and, when properly implemented, are valid for rectilinear transfers. These methods make explicit use of the eccentric anomaly though they are not based on regularized descriptions of two-body motion. The methods are singular for transfer angles of 2π, and, due to the way that certain geometric parameters are defined, it is not immediately obvious how this singularity could be avoided.

The satellite motion in the σ-domain is described by equations (3.109) and (3.112):

$$\underline{r}(t) = \frac{1}{u(\sigma)}\underline{\xi}(\sigma) \tag{7.43}$$

$$\underline{v}(t) = u(\sigma)\underline{\xi}'(\sigma) - u'(\sigma)\underline{\xi}(\sigma) \tag{7.44}$$

where the σ-domain variables are given by equations (3.79) through (3.82):

$$\underline{\xi}(\sigma) = \underline{\xi}_0 \cos(h\sigma) + \underline{\xi}'_0 \frac{\sin(h\sigma)}{h} \tag{7.45}$$

$$\underline{\xi}'(\sigma) = -\underline{\xi}_0 h \sin(h\sigma) + \underline{\xi}'_0 \cos(h\sigma) \tag{7.46}$$

$$u(\sigma) = u_0 \cos(h\sigma) + u'_0 \frac{\sin(h\sigma)}{h} + \mu \frac{1 - \cos(h\sigma)}{h^2} \tag{7.47}$$

$$u'(\sigma) = -u_0 h \sin(h\sigma) + u'_0 \cos(h\sigma) + \mu \frac{\sin(h\sigma)}{h} \tag{7.48}$$

In these formulae the explicit divisions by h are avoided by expanding $\dfrac{\sin(h\sigma)}{h}$ and $\dfrac{1-\cos(h\sigma)}{h^2}$ in power series as in equations (3.83) and (3.84). The value of σ corresponding to the given value of time t is determined from some form of the time equation given in Chapter 5, such as equation (5.232) or (5.238). That equation can be represented functionally as

$$\Delta t = F(\ Z(\sigma);\ u_0,\ u'_0\ ) \tag{7.49}$$

Only quarter-angle versions of the time equation are contemplated for use in the boundary value problem, so the variable Z is always to be taken as

$$Z = \frac{1}{h} \tan \frac{1}{4}(h\sigma) \tag{7.50}$$

for the purposes of this study. As described in the last chapter, the trigonometric functions in equations (7.45) through (7.48) can be replaced in terms of Z:

$$\sin(h\sigma) = 2\left[\frac{2hZ}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right] \tag{7.51}$$

$$\cos(h\sigma) = \left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]^2 - \left[\frac{2hZ}{1+h^2Z^2}\right]^2 \tag{7.52}$$

$$\frac{\sin(h\sigma)}{h} = 2\left[\frac{2Z}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right] \tag{7.53}$$

$$\frac{1-\cos(h\sigma)}{h^2} = 2\left[\frac{2Z}{1+h^2Z^2}\right]^2 \tag{7.54}$$

Now the velocity vector at the initial time is, from equation (7.44),

$$\underline{v}_0 = u_0\underline{\xi}'_0 - u'_0\underline{\xi}_0 \tag{7.55}$$

The unit radial vector and the reciprocal radius are known:

$$u_0 = \frac{1}{r_0} \tag{7.56}$$

$$\underline{\xi}_0 = \frac{1}{r_0}\underline{r}_0 = u_0\underline{r}_0 \tag{7.57}$$

The transverse vector $\underline{\xi}'_0$ is not a unit vector in general. From equation (3.103)

$$\underline{\xi}'_0 = \underline{h} \times \underline{\xi}_0 \tag{7.58}$$

It happens to be convenient to introduce the unit transverse vector

$$\underline{\zeta}_0 = \frac{1}{h}\underline{h} \times \underline{\xi}_0 \tag{7.59}$$

so that

$$\underline{v}_0 = hu_0\underline{\zeta}_0 - u'_0\underline{\xi}_0 \tag{7.60}$$

The reason is that in the boundary value problem the transverse unit vector is either determined from given data or else it can be chosen arbitrarily, whereas the magnitude of $\underline{\xi}'_0$ is really part of the solution of the boundary value problem itself. For example, equations (7.11) through (7.15) apply directly here:

$$\underline{\zeta}_0 = \frac{1}{N}\left[\underline{\xi}_1 - \underline{\xi}_0(\underline{\xi}_1 \cdot \underline{\xi}_0)\right] \tag{7.61}$$

where

$$N = \sqrt{1 - (\underline{\xi}_0 \cdot \underline{\xi}_1)^2} \tag{7.62}$$

and

$$\underline{\xi}_1 = \frac{1}{r_1}\underline{r}_1 \tag{7.63}$$

For transfer angles of 0° and 180° the orbit plane, and hence the vector $\underline{\zeta}_0$, is not uniquely defined, but $\underline{\zeta}_0$ can be chosen arbitrarily perpendicular to $\underline{\xi}_0$. Difficulties in computing the transverse unit vector occur in case the transfer angle is nearly, but not exactly, 0° or 180° on account of dividing small quantities in equation (7.61). In that case, unless h turns out to be small as well, errors in the direction of $\underline{\zeta}_0$ will compromise the accuracy of $\underline{v}_0$ as computed from equation (7.60). However, this type of inaccuracy is characteristic of the boundary value problem, occuring also in the η-domain formulation described earlier. It cannot be entirely eliminated, but at least it does not affect the accuracy of the main computation, the inversion of the time equation for the single unknown (and so far unselected) parameter.

With the transverse unit vector considered as known, one still has to specify h and u'_0 in equation (7.60). For this purpose, consider the radial equations (7.47) and (7.48), now written in terms of the variable Z.

$$u_1 = u_0\left[\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]^2 - \left[\frac{2hZ}{1+h^2Z^2}\right]^2\right] + u'_0\left[2\left[\frac{2Z}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]\right]$$
$$+\mu\left[2\left[\frac{2Z}{1+h^2Z^2}\right]^2\right] \tag{7.64}$$

$$u'_1 = -u_0\left[2h\left[\frac{2hZ}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]\right] + u'_0\left[\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]^2 - \left[\frac{2hZ}{1+h^2Z^2}\right]^2\right]$$
$$+\mu\left[2\left[\frac{2Z}{1+h^2Z^2}\right]\left[\frac{1-h^2Z^2}{1+h^2Z^2}\right]\right] \tag{7.65}$$

The unknowns in these two equations are h, Z, u'_0 and u'_1. A great simplification occurs because the product hZ is known even though h and Z are separately unknown. From equations (7.50) and (7.42) there is obtained

$$hZ = \frac{h}{h}\tan\frac{1}{4}(h\sigma) = \tan\frac{1}{4}\eta \tag{7.66}$$

This can be written as

$$hZ = \tan\frac{1}{4}\left[\cos^{-1}(\underline{\xi}_0 \cdot \underline{\xi}_1)\right] \tag{7.67}$$

which holds for "short way" trajectories. For "long way" trajectories, one would compute

$$hZ = \tan\frac{1}{4}\left[2\pi - \cos^{-1}(\underline{\xi}_0 \cdot \underline{\xi}_1)\right] \tag{7.68}$$

considering the value of the inverse cosine to lie between 0 and π. Now h could be computed in terms of Z as

$$h = \frac{(hZ)}{Z} \tag{7.69}$$

for use in the first term of (7.65), the only place h occurs alone in that equation. This calculation is always feasible, at least in principle, because the value of Z is nonzero as long as the value of Δt is nonzero, as can be proved by expanding the right side of equation (7.50) in power series. A difficulty arises in the direct evaluation of h by means of this equation when both the transfer angle and the time are small. In that case h would be computed as the ratio of small quantities, seriously degrading the accuracy of the formulae. Fortunately, the direct evaluation of h can be avoided in (7.65) simply by multiplying that equation by Z. Then (7.64) and (7.65) can be written as

$$u_1 = u_0 T_1 + u'_0 Z T_2 + \mu Z^2 T_3 \tag{7.70}$$

$$u'_1 Z = -u_0 T_4 + u'_0 Z T_1 + \mu Z^2 T_2 \tag{7.71}$$

where the coefficients T_j are just known factors:

$$T_1 = \left[\frac{1-(hZ)^2}{1+(hZ)^2}\right]^2 - \left[\frac{2(hZ)}{1+(hZ)^2}\right]^2 \tag{7.72}$$

$$T_2 = 2\left[\frac{2}{1+(hZ)^2}\right]\left[\frac{1-(hZ)^2}{1+(hZ)^2}\right] \tag{7.73}$$

$$T_3 = 2\left[\frac{2}{1+(hZ)^2}\right]^2 \tag{7.74}$$

$$T_4 = 2(hZ)\left[\frac{2(hZ)}{1+(hZ)^2}\right]\left[\frac{1-(hZ)^2}{1+(hZ)^2}\right] \tag{7.75}$$

The three unknowns in equations (7.70) and (7.71) are now Z, u'_0 and u'_1, and one can solve for any two in terms of the third. The evaluation of the time equation requires Z and u'_0 . Hence, one is led to solve for these two in terms of u'_1 , and the latter quantity becomes the iteration parameter in the solution of the time equation. The equations (7.70) and (7.71) are nonlinear in these variables. However, since u'_1 occurs only in a product with Z and Z is also unknown, one may as well choose the product itself as the iteration parameter. In fact, by letting

$$\rho = \frac{u_1}{u_0} \tag{7.76}$$

$$q = \frac{u'_1 Z}{u_0} \tag{7.77}$$

$$a = \frac{\mu Z^2}{u_0} \tag{7.78}$$

$$b = \frac{u'_0 Z}{u_0} \tag{7.79}$$

one is faced with the linear equations

$$\rho = T_1 + bT_2 + aT_3 \tag{7.80}$$

$$q = -T_4 + bT_1 + aT_2 \tag{7.81}$$

These can be inverted without difficulty to obtain a and b as functions of the single unknown parameter q:

$$a = \frac{(q + T_4)T_2 - (\rho - T_1)T_1}{T_2^2 - T_1T_3} \tag{7.82}$$

$$b = \frac{(\rho - T_1)T_2 - (q + T_4)T_3}{T_2^2 - T_1T_3} \tag{7.83}$$

Straightforward substitutions will show that the denominator is simply

$$T_2^2 - T_1T_3 = \frac{8}{\left[1 + (hZ)^2\right]^2} = T_3 \tag{7.84}$$

Then one has

$$a = \frac{1}{8}\left[1 + (hZ)^2\right]^2\left[(q + T_4)T_2 - (\rho - T_1)T_1\right] \tag{7.85}$$

$$b = \frac{1}{8}(\rho - T_1)T_2\left[1 + (hZ)^2\right]^2 - (q + T_4) \tag{7.86}$$

Once a and b are available for some guessed value of q , the parameters needed for evaluating the time equation are given by

$$Z = \sqrt{\frac{au_0}{\mu}} \tag{7.87}$$

$$u'_0 = \frac{bu_0}{Z} \tag{7.88}$$

This latter calculation becomes unreliable when the value of Z is small, that is, when both the transfer angle and time are small, and threatens to compromise the accuracy of the whole procedure. Fortunately, the direct evaluation of u'_0 is not really needed after all, since that quantity occurs only in product with Z in the quarter-angle versions of the σ-domain time equation. To see this clearly, consider equations (5.232) or (5.238), for which the following auxiliary quantities are used:

$$G = 2\mu u_0 - h^2 u_0^2 - u_0'^2 \tag{7.89}$$

$$A = u_0\left[1 - (hZ)^2\right] + 2(u'_0 Z) \tag{7.90}$$

$$Q = A^2 + 4GZ^2 \tag{7.91}$$

$$Y = \frac{4GZ^2}{(A + \sqrt{Q})^2} \tag{7.92}$$

These are taken from equations (5.207), (5.223), (5.228) and (5.226) respectively. The quantity Y is the argument of the transcendental function in the time equation. Now evidently

$$GZ^2 = 2\mu u_0 Z^2 - (hZ)^2 u_0^2 - (u'_0 Z)^2 \tag{7.93}$$

and since G occurs only in this combination in both of the time equations (5.232) and (5.238), it is sufficient merely to compute the product

$$(u'_0 Z) = b u_0 \tag{7.94}$$

for use in place of (7.88) in the solution of the time equation. This computation suffers no particular inaccuracy for small transfer angles. Notice also that the known product (hZ) occurs by itself at several places in the evaluation of the time equation.

Several comments should be made about the above σ-domain scheme. First, as was the case in the η-domain formulation, the derivative of the time equation with respect to the unknown parameter q does not appear to be readily accessible. Hence, it is not convenient to set up a Newton-type iteration for q ; rather, some derivative-free method is indicated. In principle, the derivative needed for Newton's method could be obtained from formulae already presented. The main complication would occur in the partial derivatives of the time equation with respect to a and b. The derivatives $\frac{da}{dq}$ and $\frac{db}{dq}$ can be obtained from (7.85) and (7.86) without difficulty.

Second, the value of q must always be chosen during the iteration so that $a \geq 0$. At the solution point this condition will be satisfied automatically because $Z > 0$ as long as $\Delta t > 0$, but a square-root operation is required for Z on every iteration and the numerical method used for the iteration may not guarantee that $a \geq 0$ during every cycle. The remedy is simply to reset the value of q whenever $a < 0$, and the rule for doing so can be found by examining whether the function $a(q)$ is increasing or decreasing. The derivative of a with respect to q is

$$\frac{da}{dq} = \frac{1}{8}\left[1 + (hZ)^2\right]^2 T_2 = \frac{1}{2}\left[1 - (hZ)^2\right] \tag{7.95}$$

which is independent of q. Hence, the rule for changing the value of q is as follows: if $a < 0$ and $(hZ)^2 < 1$ then increase q; if $a < 0$ and $(hZ)^2 > 1$ then decrease q. Note that for a transfer angle of 180° the value of (hZ) is precisely unity (see equation (7.67) above). In that case the value of a, and therefore Z, is fixed during the iterations and cannot be affected by the choice of q. Of course, it must be that $a > 0$ always in this case also. By comparing the expression for T_1 given in equation (7.72) with the expression for a given in equation (7.85), it can be seen that $a = (\rho + 1) > 0$ if $(hZ)^2 = 1$, so that no difficulty can arise.

Third, although care has been taken to avoid potential inaccuracies due to small transfer angles in the calculation of time, the inaccuracies return unavoidably in the final calculation of velocity components via (7.60). There explicit values of h and u_0' are required and so the division of potentially small numbers in (7.69) and (7.88) cannot be circumvented. However, as noted earlier, this type of indeterminateness is simply characteristic of the small-angle boundary value problem.

Finally, it should be emphasized that the σ-domain time calculation applies to rectilinear transfers without difficulty in case the transfer angle is 0°, that is, when $\underline{\xi}_0 \cdot \underline{\xi}_1 = +1$. The other possibility for a rectilinear transfer is when the transfer angle is 180°, that is, when $\underline{\xi}_0 \cdot \underline{\xi}_1 = -1$. In that case the satellite must pass through the attracting center at the origin on its way to the final position and, because the attracting center is not reached in a finite interval of σ , it might be expected that the present method would fail. However, when the position vectors are collinear and opposite the value of Z (essentially σ on a rectilinear orbit) is fixed during iterations because the parameter a is independent of q and therefore fixed. This can be seen by examining equations (7.87), (7.85), (7.73) and (7.66). Hence, no particular difficulty is encountered as long as neither position is too near the origin. Transfers to and from the origin can be handled only by some method based on $n = 1$ in the generalized Sundman time transformation, that is, only by an eccentric-anomaly method.

Summary

Previous chapters have developed a variety of formulae expressing the direct relation between time and true anomaly, as well as the closed solutions of two-body motion in the true-anomaly (η and σ) domain. This chapter has described the application of these formulae in solving the boundary value problem of unperturbed motion. The results are new, inasmuch as the true-anomaly time equations have not been available until now, and consist of iteration schemes which are universal and require only one transcendental function evaluation per iteration. Additionally, care has been taken to formulate the iterated equations to be insensitive to errors which are prone to occur for small transfer angles. Newton's root-finding method turns out to be cumbersome in this formulation because the necessary derivative is not readily available, but any derivative-free method can be used. During the iterations, certain checks on the values of some parameters are needed, and these are described in detail. One set of formulae, using η-domain variables, is valid for all non-rectilinear transfer orbits, while a second set, using σ -domain variables, is valid for all orbits, including rectilinear ones which do not originate or terminate at the attracting center. The formulae presented here are valid for any transfer angles up to $\pm 2\pi$. This upper limit is not inherent in the true-anomaly methods since the eighth-angle and higher formulae referred to in Chapter 5 enlarge the allowable transfer angle subject only to practical limitations of algebraic complexity in the formulae. Numerical results are summarized in Appendix F.

Chapter 8. Perturbed Satellite Motion

Introduction

It was noted at the beginning of this study that a particularly simple and general formulation of the zonal geopotential perturbations of a satellite orbit can be made in the true-anomaly domain. This chapter will discuss the so-called "main problem" of artificial satellite theory, namely, the first-order perturbations due to the J_2 zonal term in Earth's gravitational potential function. Of course, the analysis applies to any oblate primary body, but in the case of Earth the J_2 perturbing force is both three orders of magnitude smaller than the inverse-square attraction and three orders of magnitude larger than the next largest geopotential perturbing force. Thus the first-order J_2 problem is a necessary starting point in a more complete consideration of Earth satellite motion. Furthermore, the first-order solution is useful in its own right in the development of spacecraft guidance systems: it is a relatively easy correction to make to the unperturbed two-body solutions generally employed in guidance mechanizations. Such a correction might be required, for instance, when highly accurate intercept trajectories must be followed.

Usually there is a severe computational burden associated with including analytical expressions for any realistic type of perturbing effect in a trajectory problem, due mainly to the infinite series expansions and Fourier expansions used to develop the general theory of motion. It will be seen in this chapter that in the true-anomaly domain J_2 perturbations can be included in a very general form without excessive computational effort. As one might expect, the main complications occur in the calculation of time; this requires some special manipulations, but otherwise neither Fourier series nor power series expansions are required. Only the η-domain formulation will be considered in this study, since the σ-domain equations are not well suited for perturbation analysis, but the results will be valid for all non-rectilinear orbits. Variation of parameters will be used to derive perturbations of the regular elements introduced at the end of Chapter 3. The use of osculating elements is a convenient way to generalize the η -domain solutions of unperturbed motion already developed. In particular, computational programs for η-domain unperturbed motion can be generalized in a straightforward fashion. However, there is another reason for adopting the variation-of-parameters approach in this part of the study. The Burdet-type elements (those that Burdet (1969) calls "focal elements") have the interesting property that their first-order averaged differential equations are exactly solvable when only J_2 perturbations are considered. Thus the secular and long-period variations of the orbit can be described by very concise and general expressions of a type not usually obtainable in orbital perturbation analysis.

The first-order J_2 problem has become one of the neo-classical problems of celestial mechanics, and it has an extensive literature of its own. It has been formulated with a wide variety of coordinates, elements and independent variables, and the basic features of the motion are well known. There is even a solution exact to all orders of J_2 in terms of elliptic integrals when the problem is restricted to motion in the equatorial plane. That such a solution was available has been known for many years, yet it seems not to have been discussed in detail until comparatively recent times (Milnes, 1973; Jezewski, 1983b; Taff, 1985, chapter 6). The main interests in this study are fourfold:

1. the expression of the osculating regular elements in terms of powers and products of $\sin \eta$ and $\cos \eta$ rather than Fourier series;

2. the exact solution in the η domain of the first-order secular variations of the elements;
3. universal first-order perturbations of the Keplerian initial value problem;
4. universal first-order perturbations of the Keplerian boundary value problem.

The first of these is a routine development, though it is needed for producing computationally efficient working formulae. Perturbation corrections of the Keplerian time of flight will be considered, also. The exact secular perturbations have been considered either cursorily or not at all heretofore, at least in terms of the variables used in this study. The latter two items are discussed in chapters of their own after this one.

Several modern textbooks contain detailed treatment of Earth-gravity perturbed orbits; that by Geyling and Westerman (1971) makes the most extensive use of true anomaly in discussing J_2 -perturbed satellite motion. Their discussion of the perturbed true-anomaly time equation includes treatment of short-period, long-period and secular effects on a time element related to mean anomaly. However, their formulae are given in terms of the classical Keplerian elements and are neither universal nor regular. Many journal articles have appeared which use true anomaly as the independent variable in a description of the first-order J_2 perturbations of a satellite orbit, but again the direct true-anomaly time relation is relatively neglected. For example, Claus and Lubowe (1963) use true anomaly to derive perturbations of a set of orbital elements which are valid for nearly circular orbits, but they revert to a perturbed form of Kepler's (eccentric-anomaly) equation to calculate time. Morrison (1977) computes perturbations of the elements of an elliptical orbit in terms of true anomaly, and presents the true-anomaly quadrature for perturbed time. However, in order to evaluate the quadrature he introduces the eccentric anomaly. Vitins (1978) devised a regular variation-of-parameters solution in terms of true anomaly, but did not directly solve the differential equation of time. Rather, by "knowing" that the time quadrature must turn out to have the form of Kepler's eccentric-anomaly equation, he is able to introduce a time element which varies linearly with respect to true anomaly in unperturbed motion. He then shows that the perturbed rate of change of this time element in the true-anomaly domain has no secular or mixed secular part so that accurate calculations of perturbed time are feasible, at least by numerical means.

However, since Kepler's equation appears in the definition of the time element, that element can be used only with elliptical orbits. This is not a serious limitation in Vitins' case because he is discussing perturbation effects averaged over one or more revolutions. Vitins (1978) also obtained an exact first-order secular solution for a set of complex-valued elements related to the five-parameter coordinates he uses. He mentions the possiblity of a similar development in terms of the Burdet-type elements, but dismisses it as too cumbersome. He obtains a simple secular solution in terms of the special elements introduced for that purpose, but the transformation of that solution back to Burdet-type elements or classical elements is itself cumbersome. Apparently, the direct solution in terms of Burdet-type elements has not been published, though an unpublished thesis (Flury, 1969, cited by Vitins) may contain the solution. As will be seen later in this chapter, this solution does require some thoughtful manipulations, but it illustrates several basic features of the motion in a succinct way and it is not beyond physical or intuitive interpretation. Several other authors have introduced time elements which can be used in the true-anomaly domain (Nacozy, 1975, 1976, 1981; Kwok and Nacozy, 1981; Zare, 1983), but again only for elliptic orbits. Junkins and Turner (1979) outline a regular perturbed solution in true anomaly for Burdet's coordinates of the satellite rather than elements of the orbit, but they do not treat the differential equation of time. Kamel (1983) describes in a general way the derivation of perturbations of a set of regular variables in the true-anomaly domain. He presents solution procedures both by the method of multiple scales and by the method of averaging, but in both methods advocates the use of eccentric anomaly in the time equation. Jezewski (1983a) develops a regular J_2-perturbed solution based on true anomaly and a mixture of elements and special coordinates, but resorts to numerical quadrature for the time calculation.

In reflecting on the work cited above, and on many related papers which could have been cited as examples, two facts emerge. First, converting from true anomaly to eccentric anomaly does greatly simplify the analytical manipulation of the time equation. Second, this same conversion prevents the solution from being developed in a universally valid form. This chapter will derive a computationally useful first-order J_2 solution in terms of true anomaly, not by any new method but in terms of regular elements and in a finite form which is valid for all nonrectilinear orbits.

First-Order Perturbations of the Regular Elements

In order to represent the perturbing potential, it is convenient to assume that the direction of Earth's spin axis is fixed in space and to use nonrotating rectangular coordinates in which the (x, y) plane is the equatorial plane of Earth and $+z$ is the north polar axis. These axes are centered in the Earth and translate with it as the Earth moves relative to the center of mass of the two-body system. In all the discussion in this chapter it is understood that vectors are to be resolved in this reference frame. Effects due to precession and nutation of the frame enter the problem only at orders of J_2 higher than the first, unless the satellite is to be followed for extremely long times, and will be neglected here. The perturbing potential then takes the form (according to chapter 5 of Geyling and Westerman, 1971)

$$V = -\frac{1}{2}J_2\mu\frac{R^2}{r^3}\left[1 - 3\frac{z^2}{r^2}\right] \tag{8.1a}$$

where

$$r^2 = x^2 + y^2 + z^2 \tag{8.1b}$$

and R is the equatorial radius of Earth. The dimensionless parameter J_2 is of order 10^{-3}. Notice that, because of the rotational symmetry of this potential about the z axis, the direction of the $+x$ axis in the equatorial plane is arbitrary. When this potential is used to write the fundamental equation of motion (1.1) of Chapter 1, two well known first integrals can be found. These are the total energy, obtainable because the force is conservative, and the polar (z) component of angular momentum, obtainable because the perturbing field is rotationally symmetric. These conserved quantities will not figure prominently in the following analysis because the method of regularization used in this study has been based on integrals of the unperturbed, rather than the perturbed, motion. The two integrals of perturbed motion have found use, however. Stiefel and Schiefele (1971), Vitins (1978) and others before them describe how the total energy can be used to stabilize the numerical integration for the time and also to obtain a certain type of non-Keplerian intermediary orbit for analytical and numerical purposes. The polar component of angular momentum turns out to be important in the exact solution of the first-order secular variations of the elements as described in a later section. However, it does not have to introduced *a priori* in that analysis; rather, it arises in a natural way in the course of that solution.

Now with the perturbation parameter denoted as

$$\varepsilon = J_2 \mu R^2 \tag{8.2}$$

the potential can be written in terms of the vector position $\underline{r}$ as

$$V = -\frac{1}{2}\varepsilon(\underline{r}\cdot\underline{r})^{-\frac{3}{2}}\left[1 - 3(\underline{r}\cdot\underline{k})^2(\underline{r}\cdot\underline{r})^{-1}\right] \tag{8.3}$$

where $\underline{k}$ is the constant $+z$ unit vector. The perturbing force per unit mass is given by the negative gradient of the potential:

$$\underline{P} = -\frac{\partial V}{\partial \underline{r}} \tag{8.4}$$

$$\underline{P} = -\frac{3}{4}\varepsilon(\underline{r}\cdot\underline{r})^{-\frac{5}{2}}(2\underline{r})\left[1 - 3(\underline{r}\cdot\underline{k})^2(\underline{r}\cdot\underline{r})^{-1}\right]$$

$$+\frac{1}{2}\varepsilon(\underline{r}\cdot\underline{r})^{-\frac{3}{2}}\left[-3(2)(\underline{r}\cdot\underline{k})\underline{k}(\underline{r}\cdot\underline{r})^{-1} - 3(\underline{r}\cdot\underline{k})^2(-1)(\underline{r}\cdot\underline{r})^{-2}(2\underline{r})\right] \tag{8.5}$$

This reduces to

$$\underline{P} = \varepsilon\left[\left[\frac{15}{2}(\underline{r}\cdot\underline{k})^2 r^{-2} - \frac{3}{2}\right]r^{-5}\underline{r} - 3r^{-5}(\underline{r}\cdot\underline{k})\underline{k}\right] \tag{8.6}$$

In terms of the Burdet-type coordinates

$$\underline{\xi} = r^{-1}\underline{r} \tag{8.7a}$$

$$u = r^{-1} \tag{8.7b}$$

the perturbing force expression takes the form

$$\underline{P} = \varepsilon u^4\left[\left[\frac{15}{2}(\underline{\xi}\cdot\underline{k})^2 - \frac{3}{2}\right] - 3(\underline{\xi}\cdot\underline{k})\underline{k}\right] \tag{8.8}$$

This expression is to be inserted in the element equations given in Chapter 3, namely, (3.136), (3.137), (3.144), (3.145) and (3.146), reproduced here for easy reference:

$$\underline{\xi}_0{}' = -\frac{1}{h^2u^3}\left[\underline{P}\cdot(\underline{\xi}_0\times\underline{\zeta}_0)\right](\underline{\xi}_0\times\underline{\zeta}_0)\sin\eta \tag{8.9}$$

$$\underline{\zeta}_0{}' = +\frac{1}{h^2u^3}\left[\underline{P}\cdot(\underline{\xi}_0\times\underline{\zeta}_0)\right](\underline{\xi}_0\times\underline{\zeta}_0)\cos\eta \tag{8.10}$$

$$u_0{}' = \left[u_0\sin\eta + \frac{\mu}{h^2}\sin\eta(1-\cos\eta)\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0$$

$$+\left[w_0\sin\eta + \frac{\mu}{h^2}(1-\cos\eta)^2\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0 \tag{8.11}$$

$$w_0{}' = \left[-u_0\cos\eta - \frac{\mu}{h^2}(1-\cos\eta+\sin^2\eta)\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0$$

$$+\left[-w_0\cos\eta + \frac{\mu}{h^2}\sin\eta\cos\eta\right]\frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0 \tag{8.12}$$

$$(h^2)' = 2h^2\left[-\frac{1}{h^2u^3}\underline{P}\cdot\underline{\xi}_0\sin\eta + \frac{1}{h^2u^3}\underline{P}\cdot\underline{\zeta}_0\cos\eta\right] \tag{8.13a}$$

In some cases it is convenient to use the rate of change of h itself:

$$h' = \frac{1}{2h}(h^2)' \tag{8.13b}$$

In these equations the elements are intialized in terms of time-domain quantities at some epoch by means of formulae already given in Chapter 3. In particular, $\underline{\zeta}_0$ is initialized from (3.104) and w_0 is initialized from (3.108), in both cases letting $C = h^{-1}$. The dimensional elements u_0 and w_0 can be replaced, if desired, by the dimensionless parameters

$$\alpha = \frac{h^2 u_0}{\mu} - 1 \tag{8.14}$$

$$\beta = \frac{h^2 w_0}{\mu} \tag{8.15}$$

The rates of change are simply

$$\alpha' = \frac{1}{\mu}(h^2)' u_0 + \frac{h^2}{\mu} u_0' \tag{8.16}$$

$$\beta' = \frac{1}{\mu}(h^2)' w_0 + \frac{h^2}{\mu} w_0' \tag{8.17}$$

Substituting from (8.11) through (8.13) produces, after some regrouping of terms,

$$\alpha' = [-(1+\alpha)\sin\eta + \sin\eta(1-\cos\eta)]\frac{1}{h^2u^3}\mathcal{P}\cdot\underline{\xi}_0$$

$$+ \left[2(1+\alpha)\cos\eta + \beta\sin\eta + (1-\cos\eta)^2\right]\frac{1}{h^2u^3}\mathcal{P}\cdot\underline{\zeta}_0 \tag{8.18}$$

$$\beta' = \left[-(1+\alpha)\cos\eta - 2\beta\sin\eta - (1-\cos\eta+\sin^2\eta)\right]\frac{1}{h^2u^3}\mathcal{P}\cdot\underline{\xi}_0$$

$$+ [\beta\cos\eta + \sin\eta\cos\eta]\frac{1}{h^2u^3}\mathcal{P}\cdot\underline{\zeta}_0 \tag{8.19}$$

Now it remains to replace $\underline{\xi}$ and u in terms of elements and η according to

$$\underline{\xi} = \underline{\xi}_0 \cos \eta + \underline{\zeta}_0 \sin \eta \tag{8.20}$$

and

$$u = \frac{\mu}{h^2} + \left(u_0 - \frac{\mu}{h^2}\right) \cos \eta + w_0 \sin \eta \tag{8.21a}$$

or

$$u = \frac{\mu}{h^2}(1 + \alpha \cos \eta + \beta \sin \eta) \tag{8.21b}$$

and to substitute the force expression (8.8) into the element equations (8.9) through (8.13). It may be convenient to consider the α and β equations (8.18) and (8.19) as well. The steps involved in these substitutions are straightforward, though lengthy, and need not be recorded here. It is clear that only terms involving powers and products of $\sin \eta$ and $\cos \eta$ will result. Hence, all the quadratures necessary for the first-order solution can be obtained by integrating by parts:

$$\int \sin^n x \cos^m x \, dx = +\frac{\sin^{n+1} x \cos^{m-1} x}{(m+n)} + \frac{(m-1)}{(m+n)} \int \sin^n x \cos^{m-2} x \, dx \tag{8.22a}$$

or

$$\int \sin^n x \cos^m x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{(m+n)} + \frac{(n-1)}{(m+n)} \int \sin^{n-2} x \cos^m x \, dx \tag{8.22b}$$

This pair of reduction formulae indicates that secular terms appear in the solution only in case both m and n are even, with 0 being considered even. Otherwise the solution will contain only powers and products of $\sin \eta$ and $\cos \eta$; mixed secular terms will not appear. The results given below are quoted in a form which facilitates checking and verifying the formulae and which is not necessarily the most desirable form for calculation. In implementing these formulae, it will be obvious that some regrouping and even some cancellation of terms can be done. Since secular effects are often of primary concern in a first-order solution, the terms in the rate equations which lead to secular terms, as well as the secular terms themselves, are marked with an asterisk for easy reference in the following text. Of course, as in all first-order solutions, the elements which appear on the right-hand sides of equations (8.24), (8.26), (8.28), (8.30) and (8.32) below are considered as constants and are reckoned at the epoch. The extra notation "(0)" which should be attached to each element symbol has been omitted for brevity. On the other hand, the rate equations (8.23), (8.25), (8.27), (8.29) and (8.31) should be read exactly as they are written.

The angular momentum rate equation (8.13a) becomes

$$(h^2)' = 6\varepsilon\Big\{ + \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})^2 \cos\eta \sin\eta$$

$$+ \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \qquad *$$

$$- \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \cos^2\eta \qquad *$$

$$- \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})^2 \cos\eta \sin\eta$$

$$+ \left(u_0 - \frac{\mu}{h^2}\right)(\underline{\xi}_0 \cdot \underline{k})^2 \cos^2\eta \sin\eta$$

$$+ \left(u_0 - \frac{\mu}{h^2}\right)(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \cos\eta \sin^2\eta$$

$$- \left(u_0 - \frac{\mu}{h^2}\right)(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \cos^3\eta$$

$$- \left(u_0 - \frac{\mu}{h^2}\right)(\underline{\zeta}_0 \cdot \underline{k})^2 \cos^2\eta \sin\eta$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k})^2 \cos\eta \sin^2\eta$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta$$

$$- w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \cos^2\eta \sin\eta$$

$$- w_0(\underline{\zeta}_0 \cdot \underline{k})^2 \cos\eta \sin^2\eta \Big\} \qquad (8.23)$$

The first-order solution is given by

$$h^2(\eta) - h^2(0) = 6\varepsilon\Big\{ + \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\sin^2\eta\right]$$

$$+ \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\cos\eta \sin\eta\right]$$

$$- \frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta + \frac{1}{2}\cos\eta \sin\eta\right]$$

$$- \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{2}\sin^2\eta\right]$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$- \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin\eta \cos^2\eta + \frac{2}{3}\sin\eta\right]$$

$$- \left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$- w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$- w_0(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]\Bigg\} \tag{8.24a}$$

If the perturbations of h itself are required, equation (8.13b) can be integrated (to first order) to obtain

$$h(\eta) - h(0) = \frac{1}{2h}\left[h^2(\eta) - h^2(0)\right] \tag{8.24b}$$

Notice that the two secular terms cancel exactly. Hence, to first order in J_2 the angular momentum magnitude has only periodic variations.

The $\underline{\xi}_0$ rate equation becomes

$$\underline{\xi}_0' = \frac{3\varepsilon}{h^2}\left[\underline{k} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0)\Bigg\{\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})\cos\eta\sin\eta$$

$$+ \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})\sin^2\eta$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})\cos^2\eta\sin\eta$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})\cos\eta\sin^2\eta$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k}) \cos\eta \sin^2\eta$$

$$+ w_0(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta \Big\} \tag{8.25}$$

The first-order solution is

$$\begin{aligned}
\underline{\xi}_0(\eta) - \underline{\xi}_0(0) = \frac{3\varepsilon}{h^2}\left[\underline{k} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0)\Big\{ &\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{2}\sin^2\eta\right] \\
&+ \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta\right] \\
&+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right] \\
&+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right] \\
&+ w_0(\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right] \\
&+ w_0(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\right]\Big\}
\end{aligned} \tag{8.26}$$

The $\underline{\zeta}_0$ rate equation becomes

$$\begin{aligned}
\underline{\zeta}_0' = -\frac{3\varepsilon}{h^2}\left[\underline{k} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0)\Big\{ &\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})\cos^2\eta \\
&+ \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})\cos\eta\sin\eta \\
&+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})\cos^3\eta
\end{aligned}$$

$$+ \left[u_0 - \frac{\mu}{h^2} \right] (\underline{\zeta}_0 \cdot \underline{k}) \cos^2\eta \sin\eta$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k}) \cos^2\eta \sin\eta$$

$$+ w_0(\underline{\zeta}_0 \cdot \underline{k}) \cos\eta \sin^2\eta \Big\} \tag{8.27}$$

The first-order solution is

$$\underline{\zeta}_0(\eta) - \underline{\zeta}_0(0) = -\frac{3\varepsilon}{h^2}\left[\underline{k} \cdot (\underline{\xi}_0 \times \underline{\zeta}_0)\right](\underline{\xi}_0 \times \underline{\zeta}_0)\Big\{\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right]$$

$$+ \frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\sin^2\eta\right]$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right]$$

$$+ \left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ w_0(\underline{\xi}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ w_0(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right] \Big\} \tag{8.28}$$

The appearance of secular terms in the solutions for $\underline{\xi}_0$ and $\underline{\zeta}_0$ is unwelcome in view of the fact that these quantities are by definition orthogonal unit vectors which span the osculating plane. The growing amplitude of the secular terms will quickly obscure both the normalization and the orthogonality of these vectors. This shortcoming of the first-order solution is due to the truncated Picard-type integration method used to derive the above formulae. The significance of the secular terms will be made clear in a later section where the exact solution of the first-order averaged rate equations is presented. It will be found that the secular variations in the above solution are connected with the precession of the osculating plane of motion about the polar axis, and that the secular solution preserves both the normalization and orthogonality of the vectors defining that plane.

The u_0 rate equation is

$$u_0' = \frac{\varepsilon}{h^2}\left\{\frac{15}{2}\frac{\mu}{h^2}u_0(\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^3\eta\right.$$

$$+ 15\frac{\mu}{h^2}u_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos^2\eta$$

$$+ \frac{15}{2}\frac{\mu}{h^2}u_0(\underline{\zeta}_0 \cdot \underline{k})^2 \sin^3\eta \cos\eta$$

$$- \frac{3}{2}\frac{\mu}{h^2}u_0 \sin\eta \cos\eta$$

$$- 3\frac{\mu}{h^2}u_0(\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos\eta$$

$$- 3\frac{\mu}{h^2}u_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^3\eta(1 - \cos\eta)$$

$$+ 15\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos^2\eta(1 - \cos\eta)$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2 \sin^3\eta \cos\eta(1 - \cos\eta)$$

$$-\frac{3}{2}\frac{\mu^2}{h^4}\sin\eta\cos\eta(1-\cos\eta)$$

$$-3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos\eta(1-\cos\eta)$$

$$-3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta(1-\cos\eta)$$

$$+\frac{15}{2}u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^4\eta$$

$$+15u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^3\eta$$

$$+\frac{15}{2}u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos^2\eta$$

$$-\frac{3}{2}u_0\left[u_0-\frac{\mu}{h^2}\right]\sin\eta\cos^2\eta$$

$$-3u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^2\eta$$

$$-3u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos\eta$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^4\eta(1-\cos\eta)$$

$$+15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^3\eta(1-\cos\eta)$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos^2\eta(1-\cos\eta)$$

$$-\frac{3}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right]\sin\eta\cos^2\eta(1-\cos\eta)$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^2\eta(1-\cos\eta)$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos\eta(1-\cos\eta)$$

$$+\frac{15}{2}u_0w_0(\underline{\xi}_0\cdot\underline{k})^2\sin^2\eta\cos^3\eta$$

$$+15u_0w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^3\eta\cos^2\eta$$

$$+\frac{15}{2}u_0w_0(\underline{\zeta}_0\cdot\underline{k})^2\sin^4\eta\cos\eta$$

$$-\frac{3}{2}u_0w_0\sin^2\eta\cos\eta$$

$$-3u_0w_0(\underline{\xi}_0\cdot\underline{k})^2\sin^2\eta\cos\eta$$

$$-3u_0w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^3\eta$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})^2\sin^2\eta\cos^3\eta(1-\cos\eta)$$

$$+15\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^3\eta\cos^2\eta(1-\cos\eta)$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\sin^4\eta\cos\eta(1-\cos\eta)$$

$$-\frac{3}{2}\frac{\mu}{h^2}w_0\sin^2\eta\cos\eta(1-\cos\eta)$$

$$- 3\frac{\mu}{h^2} w_0 (\underline{\xi}_0 \cdot \underline{k})^2 \sin^2\eta \cos\eta (1 - \cos\eta)$$

$$- 3\frac{\mu}{h^2} w_0 (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta (1 - \cos\eta)$$

$$+ \frac{15}{2}\frac{\mu}{h^2} w_0 (\underline{\xi}_0 \cdot \underline{k})^2 \sin^2\eta \cos^2\eta$$

$$+ 15\frac{\mu}{h^2} w_0 (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta \cos\eta$$

$$+ \frac{15}{2}\frac{\mu}{h^2} w_0 (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^4\eta$$

$$- \frac{3}{2}\frac{\mu}{h^2} w_0 \sin^2\eta$$

$$- 3\frac{\mu}{h^2} w_0 (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos\eta$$

$$- 3\frac{\mu}{h^2} w_0 (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4} (\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^2\eta (1 - \cos\eta)^2$$

$$+ 15\frac{\mu^2}{h^4} (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos\eta (1 - \cos\eta)^2$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4} (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^3\eta (1 - \cos\eta)^2$$

$$- \frac{3}{2}\frac{\mu^2}{h^4} \sin\eta (1 - \cos\eta)^2$$

$$- 3\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \cos\eta(1-\cos\eta)^2$$

$$- 3\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2 \sin\eta(1-\cos\eta)^2$$

$$+ \frac{15}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2 \sin^2\eta \cos^3\eta$$

$$+ 15w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta \cos^2\eta$$

$$+ \frac{15}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2 \sin^4\eta \cos\eta$$

$$- \frac{3}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right] \sin^2\eta \cos\eta$$

$$- 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos^2\eta$$

$$- 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta \cos\eta$$

$$+ \frac{15}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^3\eta(1-\cos\eta)^2$$

$$+ 15\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos^2\eta(1-\cos\eta)^2$$

$$+ \frac{15}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2 \sin^3\eta \cos\eta(1-\cos\eta)^2$$

$$- \frac{3}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right] \sin\eta \cos\eta(1-\cos\eta)^2$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\cos^2\eta(1-\cos\eta)^2$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin\eta\cos\eta(1-\cos\eta)^2$$

$$+\frac{15}{2}w_0^2(\underline{\xi}_0\cdot\underline{k})^2\sin^3\eta\cos^2\eta$$

$$+15w_0^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^4\eta\cos\eta$$

$$+\frac{15}{2}w_0^2(\underline{\zeta}_0\cdot\underline{k})^2\sin^5\eta$$

$$-\frac{3}{2}w_0^2\sin^3\eta$$

$$-3w_0^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos\eta$$

$$-3w_0^2(\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})^2\sin^2\eta\cos^2\eta(1-\cos\eta)^2$$

$$+15\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^3\eta\cos\eta(1-\cos\eta)^2$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\sin^4\eta(1-\cos\eta)^2$$

$$-\frac{3}{2}\frac{\mu}{h^2}w_0\sin^2\eta(1-\cos\eta)^2$$

$$-3\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos\eta(1-\cos\eta)^2$$

$$- 3\frac{\mu}{h^2} w_0(\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta(1 - \cos\eta)^2 \Big\}$$

$$(8.29)$$

The first-order solution is

$$u_0(\eta) - u_0(0) = \frac{\varepsilon}{h^2}\Big\{\frac{15}{2}\frac{\mu}{h^2} u_0(\underline{\xi}_0 \cdot \underline{k})^2\Big[\frac{1}{4}(1 - \cos^4\eta)\Big]$$

$$+ 15\frac{\mu}{h^2} u_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\Big[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\Big]$$

$$+ \frac{15}{2}\frac{\mu}{h^2} u_0(\underline{\zeta}_0 \cdot \underline{k})^2\Big[\frac{1}{4}\sin^4\eta\Big]$$

$$- \frac{3}{2}\frac{\mu}{h^2} u_0\Big[\frac{1}{2}\sin^2\eta\Big]$$

$$- 3\frac{\mu}{h^2} u_0(\underline{\xi}_0 \cdot \underline{k})^2\Big[\frac{1}{2}\sin^2\eta\Big]$$

$$- 3\frac{\mu}{h^2} u_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\Big[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta\Big]$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})^2\Big[\frac{1}{4}(1 - \cos^4\eta) - \frac{1}{5}(1 - \cos^5\eta)\Big]$$

$$+ 15\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\Big[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta$$

$$+ \frac{1}{5}\sin\eta\cos^4\eta - \frac{1}{15}\sin\eta\cos^2\eta - \frac{2}{15}\sin\eta\Big]$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2\Big[\frac{1}{4}\sin^4\eta - \frac{1}{5}\sin^4\eta\cos\eta - \frac{2}{15}(1 - \cos\eta) + \frac{1}{15}\sin^2\eta\cos\eta\Big]$$

$$-\frac{3}{2}\frac{\mu^2}{h^4}\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta - \frac{1}{3}\sin^3\eta\right]$$

$$+\frac{15}{2}u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$+15u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+\frac{15}{2}u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$-\frac{3}{2}u_0\left[u_0-\frac{\mu}{h^2}\right]\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3u_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta) - \frac{1}{6}(1-\cos^6\eta)\right]$$

$$+15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta\right.$$

$$\left.+\frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right.$$

$$- \frac{1}{16}\eta + \frac{1}{16}\sin\eta\cos\eta - \frac{1}{8}\sin^3\eta\cos\eta - \frac{1}{6}\sin^3\eta\cos^3\eta \Big]$$

$$+ \frac{15}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\Big[\frac{1}{5}\sin^4\eta\cos\eta$$

$$+ \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta$$

$$- \frac{1}{6}\sin^4\eta\cos^2\eta - \frac{1}{12}\sin^4\eta\Big]$$

$$- \frac{3}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right]\left[\frac{1}{3}(1 - \cos^3\eta) - \frac{1}{4}(1 - \cos^4\eta)\right]$$

$$- 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1 - \cos^3\eta) - \frac{1}{4}(1 - \cos^4\eta)\right]$$

$$- 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\Big[\frac{1}{3}\sin^3\eta$$

$$- \frac{1}{8}\eta + \frac{1}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\Big]$$

$$+ \frac{15}{2}u_0 w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ 15u_0 w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}u_0 w_0(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$- \frac{3}{2}u_0 w_0\left[\frac{1}{3}\sin^3\eta\right]$$

$$- 3u_0 w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$- 3u_0 w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right.$$

$$\left. - \frac{1}{16}\eta + \frac{1}{16}\sin\eta\cos\eta - \frac{1}{8}\sin^3\eta\cos\eta - \frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$+ 15\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right.$$

$$\left. - \frac{1}{6}\sin^4\eta\cos^2\eta - \frac{1}{12}\sin^4\eta\right]$$

$$+ \frac{15}{2}\frac{\mu}{h^2} w_0(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right.$$

$$\left. - \frac{1}{16}\eta + \frac{1}{16}\sin\eta\cos\eta - \frac{1}{8}\sin^3\eta\cos\eta + \frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$- \frac{3}{2}\frac{\mu}{h^2} w_0\left[\frac{1}{3}\sin^3\eta - \frac{1}{8}\eta + \frac{1}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$- 3\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta - \frac{1}{8}\eta + \frac{1}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$- 3\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta - \frac{1}{4}\sin^4\eta\right]$$

$$+ \frac{15}{2}\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+ 15\frac{\mu}{h^2} w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{4}\sin^4\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{3}{8}\eta-\frac{3}{8}\sin\eta\cos\eta-\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$-\frac{3}{2}\frac{\mu}{h^2}w_0\left[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta\right]$$

$$-3\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\sin^2\eta\right]$$

$$-3\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta\right]$$

$$+\frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)-2\left(\frac{1}{4}(1-\cos^4\eta)\right)+\frac{1}{5}(1-\cos^5\eta)\right]$$

$$+15\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta-2\left(\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right)\right.$$

$$\left.-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$+\frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta-2\left(\frac{1}{4}\sin^4\eta\right)\right.$$

$$\left.+\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$-\frac{3}{2}\frac{\mu^2}{h^4}\left[(1-\cos\eta)-2\left(\frac{1}{2}\sin^2\eta\right)+\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\sin\eta-2\left(\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta\right)\right.$$

$$\left.+\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right]$$

$$- 3\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2\left[(1-\cos\eta) - 2\left(\frac{1}{2}\sin^2\eta\right) + \frac{1}{3}(1-\cos^3\eta)\right]$$

$$+ \frac{15}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ 15w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$- \frac{3}{2}w_0\left[u_0 - \frac{\mu}{h^2}\right]\left[\frac{1}{3}\sin^3\eta\right]$$

$$- 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ \frac{15}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta) - 2\left(\frac{1}{5}(1-\cos^5\eta)\right) + \frac{1}{6}(1-\cos^6\eta)\right]$$

$$+ 15\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$-2\left(-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right)$$

$$\left.+ \frac{1}{16}\eta - \frac{1}{16}\sin\eta\cos\eta + \frac{1}{8}\sin^3\eta\cos\eta + \frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$+ \frac{15}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{4}\sin^4\eta\right.$$

$$- 2\left(\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right)$$

$$+ \frac{1}{6}\sin^4\eta\cos^2\eta + \frac{1}{12}\sin^4\eta\bigg]$$

$$-\frac{3}{2}\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right]\left[\frac{1}{2}\sin^2\eta - 2\left(\frac{1}{3}(1-\cos^3\eta)\right) + \frac{1}{4}(1-\cos^4\eta)\right]$$

$$- 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\bigg[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta$$

$$- 2\left(\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right)$$

$$+ \frac{3}{8}\eta + \frac{3}{8}\sin\eta\cos\eta + \frac{1}{4}\sin\eta\cos^3\eta\bigg]$$

$$- 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{2}\sin^2\eta - 2\left(\frac{1}{3}(1-\cos^3\eta)\right) + \frac{1}{4}(1-\cos^4\eta)\right]$$

$$+ \frac{15}{2}w_0^2(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ 15w_0^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}\sin^5\eta\right]$$

$$+ \frac{15}{2}w_0^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{8}{15}(1-\cos\eta) - \frac{4}{15}\sin^2\eta\cos\eta - \frac{1}{5}\sin^4\eta\cos\eta\right]$$

$$- \frac{3}{2}w_0^2\left[\frac{2}{3}(1-\cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$- 3w_0^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$-3w_0^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{2}{3}(1-\cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$-2\left(-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right)$$

$$\left.+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$+15\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{4}\sin^4\eta\right.$$

$$-2\left(\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right)$$

$$\left.+\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{3}{8}\eta-\frac{3}{8}\sin\eta\cos\eta-\frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$\left.-2\left(\frac{1}{5}\sin^5\eta\right)+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$-\frac{3}{2}\frac{\mu}{h^2}w_0\left[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta-2\left(\frac{1}{3}\sin^3\eta\right)\right.$$

$$\left.+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$-3\frac{\mu}{h^2}w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\sin^2\eta-2\left(\frac{1}{3}(1-\cos^3\eta)\right)+\frac{1}{4}(1-\cos^4\eta)\right]$$

$$-3\frac{\mu}{h^2}w_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta-2\left(\frac{1}{3}\sin^3\eta\right)\right.$$

$$+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\Big]\Big\}$$

$$(8.30)$$

The w_0 rate equation becomes

$$w_0'=\frac{\varepsilon}{h^2}\Big\{-\frac{15}{2}u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\cos^4\eta$$

$$-15u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos^3\eta$$

$$-\frac{15}{2}u_0(\underline{\zeta}_0\cdot\underline{k})^2\sin^2\eta\cos^2\eta$$

$$+\frac{3}{2}u_0\frac{\mu}{h^2}\cos^2\eta$$

$$+3u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\cos^2\eta$$

$$+3u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos\eta$$

$$-\frac{15}{2}u_0\Big[u_0-\frac{\mu}{h^2}\Big](\underline{\xi}_0\cdot\underline{k})^2\cos^5\eta$$

$$-15u_0\Big[u_0-\frac{\mu}{h^2}\Big](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos^4\eta$$

$$-\frac{15}{2}u_0\Big[u_0-\frac{\mu}{h^2}\Big](\underline{\zeta}_0\cdot\underline{k})^2\sin^2\eta\cos^3\eta$$

$$+\frac{3}{2}u_0\Big[u_0-\frac{\mu}{h^2}\Big]\cos^3\eta$$

$$+ 3u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2 \cos^3\eta$$

$$+ 3u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos^2\eta$$

$$- \frac{15}{2}u_0 w_0(\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^4\eta$$

$$- 15u_0 w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos^3\eta$$

$$- \frac{15}{2}u_0 w_0(\underline{\zeta}_0 \cdot \underline{k})^2 \sin^3\eta \cos^2\eta$$

$$+ \frac{3}{2}u_0 w_0 \sin\eta \cos^2\eta$$

$$+ 3u_0 w_0(\underline{\xi}_0 \cdot \underline{k})^2 \sin\eta \cos^2\eta$$

$$+ 3u_0 w_0(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^2\eta \cos\eta$$

$$- \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})^2 \cos^3\eta(1 - \cos\eta + \sin^2\eta) \qquad *$$

$$- 15\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos^2\eta(1 - \cos\eta + \sin^2\eta)$$

$$- \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta \cos\eta(1 - \cos\eta + \sin^2\eta) \qquad *$$

$$+ \frac{3}{2}\frac{\mu^2}{h^4} \cos\eta(1 - \cos\eta + \sin^2\eta) \qquad *$$

$$+ 3\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})^2 \cos\eta(1 - \cos\eta + \sin^2\eta) \qquad *$$

$$+\,3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta(1-\cos\eta+\sin^2\eta)$$

$$-\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\cos^4\eta(1-\cos\eta+\sin^2\eta) \qquad *$$

$$-\,15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos^3\eta(1-\cos\eta+\sin^2\eta)$$

$$-\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^2\eta\cos^2\eta(1-\cos\eta+\sin^2\eta) \qquad *$$

$$+\frac{3}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right]\cos^2\eta(1-\cos\eta+\sin^2\eta) \qquad *$$

$$+\,3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\cos^2\eta(1-\cos\eta+\sin^2\eta) \qquad *$$

$$+\,3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos\eta(1-\cos\eta+\sin^2\eta)$$

$$-\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^3\eta(1-\cos\eta+\sin^2\eta)$$

$$-\,15w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^2\eta(1-\cos\eta+\sin^2\eta) \qquad *$$

$$-\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos\eta(1-\cos\eta+\sin^2\eta)$$

$$+\frac{3}{2}w_0\frac{\mu}{h^2}\sin\eta\cos\eta(1-\cos\eta+\sin^2\eta)$$

$$+\,3w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos\eta(1-\cos\eta+\sin^2\eta)$$

$$+ 3w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta(1-\cos\eta+\sin^2\eta)$$

$$-\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^3\eta$$

$$-15w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^2\eta$$

$$-\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos\eta$$

$$+\frac{3}{2}w_0\frac{\mu}{h^2}\sin\eta\cos\eta$$

$$+3w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\cos^2\eta$$

$$+3w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot\underline{k})^2\sin\eta\cos\eta$$

$$-\frac{15}{2}w_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin\eta\cos^4\eta$$

$$-15w_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^3\eta$$

$$-\frac{15}{2}w_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos^2\eta$$

$$+\frac{3}{2}w_0\left[u_0-\frac{\mu}{h^2}\right]\sin\eta\cos^2\eta$$

$$+3w_0\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\cos^3\eta$$

$$+ 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2 \sin\eta \cos^2\eta$$

$$- \frac{15}{2} w_0^2 (\underline{\xi}_0 \cdot \underline{k})^2 \sin^2\eta \cos^3\eta$$

$$- 15 w_0^2 (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta \cos^2\eta$$

$$- \frac{15}{2} w_0^2 (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^4\eta \cos\eta$$

$$+ \frac{3}{2} w_0^2 \sin^2\eta \cos\eta$$

$$+ 3w_0^2 (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos^2\eta$$

$$+ 3w_0^2 (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta \cos\eta$$

$$+ \frac{15}{2} \frac{\mu^2}{h^4} (\underline{\xi}_0 \cdot \underline{k})^2 \sin^2\eta \cos^3\eta$$

$$+ 15 \frac{\mu^2}{h^4} (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin^3\eta \cos^2\eta$$

$$+ \frac{15}{2} \frac{\mu^2}{h^4} (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^4\eta \cos\eta$$

$$- \frac{3}{2} \frac{\mu^2}{h^4} \sin^2\eta \cos\eta$$

$$- 3 \frac{\mu^2}{h^4} (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \sin\eta \cos^2\eta$$

$$- 3 \frac{\mu^2}{h^4} (\underline{\zeta}_0 \cdot \underline{k})^2 \sin^2\eta \cos\eta$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\sin^2\eta\cos^4\eta$$

$$+15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^3\eta\cos^3\eta$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^4\eta\cos^2\eta$$

$$-\frac{3}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right]\sin^2\eta\cos^2\eta$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin\eta\cos^3\eta$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\sin^2\eta\cos^2\eta$$

$$+\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\sin^3\eta\cos^3\eta$$

$$+15w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^4\eta\cos^2\eta$$

$$+\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot\underline{k})^2\sin^5\eta\cos\eta$$

$$-\frac{3}{2}w_0\frac{\mu}{h^2}\sin^3\eta\cos\eta$$

$$-3w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\sin^2\eta\cos^2\eta$$

$$-3w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot\underline{k})^2\sin^3\eta\cos\eta\Bigg\} \qquad (8.31)$$

The first-order solution is

$$
\begin{aligned}
w_0(\eta) - w_0(0) = \frac{\varepsilon}{h^2}\Bigg\{ & -\frac{15}{2}u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{3}{8}\eta + \frac{3}{8}\sin\eta\cos\eta + \frac{1}{4}\sin\eta\cos^3\eta\right] \\
& - 15u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{4}(1-\cos^4\eta)\right] \\
& - \frac{15}{2}u_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right] \\
& + \frac{3}{2}u_0\frac{\mu}{h^2}\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right] \\
& + 3u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right] \\
& + 3u_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\sin^2\eta\right] \\
& - \frac{15}{2}u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{4}{15}\sin\eta\cos^2\eta + \frac{8}{15}\sin\eta + \frac{1}{5}\sin\eta\cos^4\eta\right] \\
& - 15u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}(1-\cos^5\eta)\right] \\
& - \frac{15}{2}u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right] \\
& + \frac{3}{2}u_0\left[u_0 - \frac{\mu}{h^2}\right]\left[\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right] \\
& + 3u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right] \\
& + 3u_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]
\end{aligned}
$$

$$-\frac{15}{2}u_0w_0(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$-15u_0w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$-\frac{15}{2}u_0w_0(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+\frac{3}{2}u_0w_0\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3u_0w_0(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3u_0w_0(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$-\frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right.$$

$$-\left(\frac{3}{8}\eta+\frac{3}{8}\sin\eta\cos\eta+\frac{1}{4}\sin\eta\cos^3\eta\right)$$

$$\left.-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$-15\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}(1-\cos^3\eta)-\frac{1}{4}(1-\cos^4\eta)\right.$$

$$\left.+\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$-\frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta+\frac{1}{5}\sin^5\eta-\left(\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right)\right]$$

$$+\frac{3}{2}\frac{\mu^2}{h^4}\left[\sin\eta-\left(\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta\right)+\frac{1}{3}\sin^3\eta\right]$$

$$
\begin{aligned}
&+3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})^2\left[\sin\eta-\left(\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta\right)+\frac{1}{3}\sin^3\eta\right]\\
&+3\frac{\mu^2}{h^4}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[(1-\cos\eta)-\frac{1}{2}\sin^2\eta+\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta\right]\\
&-\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{3}{8}\eta+\frac{3}{8}\sin\eta\cos\eta+\frac{1}{4}\sin\eta\cos^3\eta\right.\\
&\qquad-\left(\frac{4}{15}\sin\eta\cos^2\eta+\frac{8}{15}\sin\eta+\frac{1}{5}\sin\eta\cos^4\eta\right)\\
&\qquad\left.+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\eta\cos^3\eta\right]\\
&-15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{4}(1-\cos^4\eta)-\frac{1}{5}(1-\cos^5\eta)\right.\\
&\qquad\left.+\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]\\
&-\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right.\\
&\qquad-\left(-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right)\\
&\qquad\left.+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\right]\\
&+\frac{3}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right]\left[\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta-\left(\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right)\right.\\
&\qquad\left.+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]
\end{aligned}
$$

$$+ 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta - \left(\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right)\right.$$

$$\left. + \frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+ 3\frac{\mu}{h^2}\left[u_0 - \frac{\mu}{h^2}\right](\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1 - \cos^3\eta) + \frac{1}{4}\sin^4\eta\right]$$

$$- \frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{4}(1 - \cos^4\eta) - \frac{1}{5}(1 - \cos^5\eta) + \frac{1}{6}\sin^4\eta\cos^2\eta + \frac{1}{12}\sin^4\eta\right]$$

$$- 15w_0\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$- \left(-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right)$$

$$\left. + \frac{1}{16}\eta - \frac{1}{16}\sin\eta\cos\eta + \frac{1}{8}\sin^3\eta\cos\eta - \frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$- \frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{4}\sin^4\eta + \frac{1}{6}\sin^6\eta\right.$$

$$\left. - \left(\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right)\right]$$

$$+ \frac{3}{2}w_0\frac{\mu}{h^2}\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1 - \cos^3\eta) + \frac{1}{4}\sin^4\eta\right]$$

$$+ 3w_0\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1 - \cos^3\eta) + \frac{1}{4}\sin^4\eta\right]$$

$$+ 3w_0\frac{\mu}{h^2}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta - \frac{1}{3}\sin^3\eta\right.$$

$$+ \frac{3}{8}\eta - \frac{3}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta \Big]$$

$$- \frac{15}{2} w_0 \frac{\mu}{h^2} (\underline{\xi}_0 \cdot \underline{k})^2 \left[\frac{1}{4}(1 - \cos^4\eta) \right]$$

$$- 15 w_0 \frac{\mu}{h^2} (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta \right]$$

$$- \frac{15}{2} w_0 \frac{\mu}{h^2} (\underline{\zeta}_0 \cdot \underline{k})^2 \left[\frac{1}{4}\sin^4\eta \right]$$

$$+ \frac{3}{2} w_0 \frac{\mu}{h^2} \left[\frac{1}{2}\sin^2\eta \right]$$

$$+ 3 w_0 \frac{\mu}{h^2} (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta \right]$$

$$+ 3 w_0 \frac{\mu}{h^2} (\underline{\zeta}_0 \cdot \underline{k})^2 \left[\frac{1}{2}\sin^2\eta \right]$$

$$- \frac{15}{2} w_0 \left[u_0 - \frac{\mu}{h^2} \right] (\underline{\xi}_0 \cdot \underline{k})^2 \left[\frac{1}{5}(1 - \cos^5\eta) \right]$$

$$- 15 w_0 \left[u_0 - \frac{\mu}{h^2} \right] (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta \right]$$

$$- \frac{15}{2} w_0 \left[u_0 - \frac{\mu}{h^2} \right] (\underline{\zeta}_0 \cdot \underline{k})^2 \left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta \right]$$

$$+ \frac{3}{2} w_0 \left[u_0 - \frac{\mu}{h^2} \right] \left[\frac{1}{3}(1 - \cos^3\eta) \right]$$

$$+ 3 w_0 \left[u_0 - \frac{\mu}{h^2} \right] (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta \right]$$

$$+ 3w_0\left[u_0 - \frac{\mu}{h^2}\right](\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$- \frac{15}{2}w_0^2(\underline{\xi}_0 \cdot \underline{k})^2\left[- \frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$- 15w_0^2(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$- \frac{15}{2}w_0^2(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$+ \frac{3}{2}w_0^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ 3w_0^2(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ 3w_0^2(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})^2\left[- \frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ 15\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}\frac{\mu^2}{h^4}(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$- \frac{3}{2}\frac{\mu^2}{h^4}\left[\frac{1}{3}\sin^3\eta\right]$$

$$- 3\frac{\mu^2}{h^4}(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$-3\frac{\mu^2}{h^4}(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{16}\eta\right.$$

$$\left.-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\cos^3\eta\right]$$

$$+15\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$+\frac{15}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{16}\eta\right.$$

$$\left.-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\cos^3\eta\right]$$

$$-\frac{3}{2}\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right]\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{4}(1-\cos^4\eta)\right]$$

$$-3\frac{\mu}{h^2}\left[u_0-\frac{\mu}{h^2}\right](\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$+15w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{16}\eta\right.$$

$$\left.-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$+\frac{15}{2}w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot \underline{k})^2\left[\frac{1}{6}\sin^6\eta\right]$$

$$-\frac{3}{2}w_0\frac{\mu}{h^2}\left[\frac{1}{4}\sin^4\eta\right]$$

$$-3w_0\frac{\mu}{h^2}(\underline{\xi}_0\cdot \underline{k})(\underline{\zeta}_0\cdot \underline{k})\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$-3w_0\frac{\mu}{h^2}(\underline{\zeta}_0\cdot \underline{k})^2\left[\frac{1}{4}\sin^4\eta\right]\Bigg\} \tag{8.32}$$

This completes the first-order solution for the elements in the η domain. Several comments should be made. First, the solutions for the dimensionless elements α and β could be obtained from the above formulae without developing their rate equations (8.18) and (8.19) in explicit terms. The right-hand sides of the above equations for the first-order solutions are put in terms of α and β by factoring out $\frac{\mu^2}{h(0)^4}$. Then the solutions for $\alpha(\eta)$ and $\beta(\eta)$ can be obtained by integrating (to first order) equations (8.16) and (8.17):

$$\alpha(\eta)-\alpha(0)=\frac{1}{\mu}\left[h^2(\eta)-h^2(0)\right]u_0(0)+\frac{h(0)^2}{\mu}\left[u_0(\eta)-u_0(0)\right] \tag{8.33}$$

$$\beta(\eta)-\beta(0)=\frac{1}{\mu}\left[h^2(\eta)-h^2(0)\right]w_0(0)+\frac{h(0)^2}{\mu}\left[w_0(\eta)-w_0(0)\right] \tag{8.34}$$

Second, these expressions for the elements are more efficient computationally than Fourier-type expressions in that only two trigonometric functions, $\sin\eta$ and $\cos\eta$, have to be evaluated. In the Fourier form of the solution, both functions of all the multiple arguments through 6η would be required. Of course, if one were to use $z=\tan\frac{1}{2}\eta$ or $z=\tan\frac{1}{4}\eta$ as the independent variable then only a single transcendental function occurs, namely, that for the secular terms: $\eta=2\tan^{-1}z$ or $\eta=4\tan^{-1}z$. All the other terms, which are 2π-periodic, become rational algebraic expressions in z.

Perturbations of the Time

In Chapter 5 the differential equation of time

$$\frac{dt}{d\eta}=\frac{1}{hu^2}=\frac{h^3}{\mu^2}\left[\frac{1}{(1+\alpha\cos\eta+\beta\sin\eta)^2}\right] \tag{8.35}$$

was solved in the case of unperturbed motion given that h was constant and u was known as a function of η. The resulting quadrature

$$t-t_0=\frac{h^3}{\mu^2}\int_0^\eta\frac{dx}{(1+\alpha\cos x+\beta\sin x)^2}=\frac{h^3}{\mu^2}K(\eta;\alpha,\beta) \tag{8.36}$$

was developed in explicit terms in a variety of forms. In the present case the differential equation of time

$$\frac{dt}{d\eta}=\frac{1}{h(\eta)u(\eta)^2}=\frac{h(\eta)^3}{\mu^2}\left[\frac{1}{(1+\alpha(\eta)\cos\eta+\beta(\eta)\sin\eta)^2}\right] \tag{8.37}$$

is still solved by a quadrature because time is not coupled to the other equations describing perturbed motion. (As noted earlier in this study, the time equation will be uncoupled from the rest of the system as long as the perturbing force does not depend explicitly on the time.) Now since first-order solutions for $h(\eta)$ and $u(\eta)$ have already been developed, it is, at least in principle, a straightforward job to insert these solutions into (8.37) and carry out a first-order solution for the time. Of course, an exact evaluation of the quadrature is not feasible even with only first-order expressions used for h and u. However, by expanding the right-hand side of (8.37) in power series in the perturbation parameter ε, a first-order quadrature is obtained which can be evaluated by the method of partial fractions. The steps in the literal evaluation of this quadrature are given sufficiently to show that the final algebraic reductions would be extremely complicated and that the intermediate (non-universal) results presented here are of doubtful computational value. While it may yet be possible to reduce the quadrature to useful finite terms, direct numerical treatment of equation (8.37) is at present a more practical method of computation. Interestingly, this difficulty in deriving perturbed time of flight does not impair the analytical treatment of the initial value and boundary value problems of perturbed motion. In these problems the time of flight is given and therefore suffers no perturbations; the consequent adjustments in the value of the independent variable η turn out to be relatively simple to obtain.

Before this straightforward solution method is carried out, the alternative perturbation method, namely variation of parameters, will be described. Since an exact Keplerian solution of the time equation is available, it is an easy matter to select a parameter to serve as a time element. Time elements have important numerical advantages since they vary either slowly or almost linearly when the perturbing force is small. They are useful analytically as well if the rate equations happen to be simple enough for the necessary quadratures to be carried out for a first-order solution. Unfortunately, the η-domain time element rate equations developed below are not simple enough for analytical treatment. They are presented here against the possibility that they may find some numerical applications.

Time Elements

By allowing the constants of Keplerian motion to vary with η the perturbed time on orbit can be expressed by

$$t - t_0(\eta) = \frac{h(\eta)^3}{\mu^2} K(\ \eta,\ \ \alpha(\eta),\ \ \beta(\eta)\ ) \tag{8.38}$$

Since the rates of change of h, α and β are all known, differentiation of this equation with respect to η should provide an expression for the rate of change of time at epoch, t_0, in terms of known quantities. One obtains

$$t' - t_0' = \frac{3h^2}{\mu^2}h'K + \frac{h^3}{\mu^2}\left[\frac{\partial K}{\partial \eta} + \frac{\partial K}{\partial \alpha}\alpha' + \frac{\partial K}{\partial \beta}\beta'\right] \tag{8.39}$$

According to the differential equation of time (8.37), it must be that

$$t' = \frac{h^3}{\mu^2}\frac{\partial K}{\partial \eta} \tag{8.40}$$

so that the desired element rate is

$$-t_0' = \frac{3}{2}\frac{h^3}{\mu^2}K\frac{(h^2)'}{h^2} + \frac{h^3}{\mu^2}\frac{\partial K}{\partial \alpha}\alpha' + \frac{h^3}{\mu^2}\frac{\partial K}{\partial \beta}\beta' \tag{8.41}$$

The rates $(h^2)'$, α' and β' are known from equations (8.13), (8.18) and (8.19) respectively. The quantity $K(\eta; \alpha, \beta)$ is given in several universal forms in Chapter 5, with the result that this element equation is valid for all nonrectilinear orbits. However, the complicated forms of K make this equation unsuited for analytical treatment. In the first place, the partial derivatives of K would be complicated. For this reason they will not be quoted here. Later, in Chapter 9 in connection with the perturbed initial value problem, it will be essential to calculate these derivatives and explicit formulae will be given there. But these formulae merely confirm the expectation that equation (8.41) is too complicated for even a first-order analytical treatment. In trying to avoid the direct differentiation of an expression like (5.151) of Chapter 5, one might differentiate the definite integral in equation (8.36) above:

$$\frac{\partial K}{\partial \alpha} = -2\int_0^{\eta} \frac{\cos x \; dx}{(1 + \alpha(\eta)\cos x + \beta(\eta)\sin x)^3} \tag{8.42}$$

$$\frac{\partial K}{\partial \beta} = -2\int_0^{\eta} \frac{\sin x \; dx}{(1 + \alpha(\eta)\cos x + \beta(\eta)\sin x)^3} \tag{8.43}$$

where it is understood that α and β are to be held constant inside these integrals since they are functions of η (and not of the dummy variable x) in perturbed motion. In other words, $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ are simply quadratures of the same type as the unperturbed time equation (8.36). Replacing the partial derivatives in (8.41) by these expressions produces a more compact integro-differential rate equation for t_0, but it is not clear that this step represents any simplification. The solution of (8.41) now involves three integrations instead of one. Of course, the quadratures can be evaluated in closed form using the method of partial fractions or the tabulated formulae of Gradshteyn and Ryzhik (1980) already quoted in Chapter 5 of this study. But the final forms would be equivalent to a direct differentiation of K, and just as extensive. The second analytical difficulty has to do with K itself, which, besides being complicated, is not periodic in η. To see this clearly, consider the case of elliptic orbits and compare equations (5.30) and (5.76) of Chapter 5. It is evident that K contains a term which is proportional to the eccentric anomaly θ. Since η differs from θ only by periodic terms, the first term of (8.41) suffers from mixed secular effects, hampering any analytical treatment and degrading even the numerical usefulness of this equation. Recall that this same difficulty arose in the θ domain when time at epoch was chosen as the time element. Because of the simpler form of the eccentric anomaly time equation, it was possible to exhibit the mixed secular terms more or less explicitly in equation (2.153) of Chapter 2. The remedy in that case was to introduce a different time element, one which varies linearly with respect to θ in unperturbed motion. In the present case it is natural to try to introduce an element which varies linearly with respect to true anomaly. Unfortunately, in effecting this modification in the η domain it will be necessary to abandon the universality of the formulae and restrict consideration to elliptic orbits only or to hyperbolic orbits only. The reason is that the universal forms of K were derived precisely by suppressing the explicit appearance of either the eccentric anomaly or the hyperbolic anomaly. These quantities appear only implicitly through the special transcendental function obtained by manipulating the arctangent expansions. If one must handle the eccentric anomaly itself, as will be necessary below, then the universal forms of K cannot be used. In the following analysis, a time element for elliptic orbits is introduced and its rate equation will be derived.

Using equations (5.18) and (5.76) of Chapter 5, the time equation for elliptic orbits can be expressed as

$$t - t_0 = \frac{h^3}{\mu^2}\left[K_1(\eta;\alpha,\beta) + \frac{\theta}{\sqrt{(1-\alpha^2-\beta^2)^3}}\right] = \frac{h^3}{\mu^2}K(\eta;\alpha,\beta) \tag{8.44}$$

where

$$K_1(\eta;\alpha,\beta) = \frac{(1+\alpha)(\beta\cos\eta - \alpha\sin\eta) - \beta(1+\alpha\cos\eta+\beta\sin\eta)}{(1+\alpha)(1-\alpha^2-\beta^2)(1+\alpha\cos\eta+\beta\sin\eta)} \tag{8.45}$$

and the eccentric anomaly is given by, say, equation (5.76):

$$\theta = 2\tan^{-1}\left[\frac{\sqrt{1-\alpha^2-\beta^2}\,\tan\frac{1}{2}\eta}{1+\alpha+\beta\tan\frac{1}{2}\eta}\right] \tag{8.46a}$$

Actually, some numerical difficulties near $\eta = \pi$ will be avoided if, instead of this half-angle formula, one uses the quarter-angle version given in equation (5.138):

$$\theta = 4\tan^{-1}\left[\frac{2z\sqrt{1-\alpha^2-\beta^2}}{D+\sqrt{D^2+4z^2(1-\alpha^2-\beta^2)}}\right] \tag{8.46b}$$

where

$$z = \tan\frac{1}{4}\eta \tag{8.46c}$$

and

$$D = (1 + \alpha)(1 - z^2) + 2\beta z \tag{8.46d}$$

Now in place of the time at epoch introduce the time element

$$\tau = t_0 + \frac{h^3}{\mu^2 \sqrt{(1 - \alpha^2 - \beta^2)^3}} \eta \tag{8.47}$$

so that time is given by

$$t = \tau + \frac{h^3}{\mu^2} K_1 - \frac{h^3}{\mu^2 \sqrt{(1 - \alpha^2 - \beta^2)^3}} (\eta - \theta) \tag{8.48}$$

The element τ contains the whole non-periodic part of $t(\eta)$. It can be seen from (8.45) that K_1 is 2π-periodic in η. The anomaly difference $(\eta - \theta)$ is also 2π -periodic in η as will now be shown. According to (8.46b), the difference can be written as

$$\eta - \theta = 4\tan^{-1} z - 4\tan^{-1}\left[\frac{2z\sqrt{1 - \alpha^2 - \beta^2}}{D + \sqrt{D^2 + 4z^2(1 - \alpha^2 - \beta^2)}}\right] \tag{8.49}$$

The arctangent identity (5.17) allows this to be written

$$\eta - \theta = 4\tan^{-1}\left[\frac{z\left[D + \sqrt{D^2 + 4z^2(1 - \alpha^2 - \beta^2)}\right] - 2z\sqrt{1 - \alpha^2 - \beta^2}}{D + \sqrt{D^2 + 4z^2(1 - \alpha^2 - \beta^2)} + 2z^2\sqrt{1 - \alpha^2 - \beta^2}}\right] \tag{8.50}$$

Notice that the numerator of the argument is factored by $z = \tan\frac{1}{4}\eta$, and hence the arctangent expression is 2π -periodic in η.

As an aside, the maximum amplitude of the anomaly difference is also of interest, partly since quadrant ambiguities would arise in the calculation of $(\eta - \theta)$ from (8.50) if the difference were to exceed 2π in magnitude. A quick inspection of the elliptical orbit shows that no such ambiguity should occur since $\theta = 0$ when $\eta = 0$ and $\theta = 2\pi$ when $\eta = 2\pi$. Notwithstanding this correct reasoning, a more detailed proof provides additional information about the limiting magnitudes of $(\eta - \theta)$ as a function of the position of the epochal point on the orbit.

If ν is the true anomaly measured from pericenter and ψ is the eccentric anomaly measured from pericenter then a simple geometric construction (as in Chapter 4 of Bate, *et al.*, 1971) will show that ν and ψ always lie in the same half-plane, that is,

$$|\nu - \psi| \leq \pi \tag{8.51}$$

The equality can be approached only on a nearly rectilinear orbit. Then, denoting epochal values with a subscript 0, it is clear that

$$|\nu_0 - \psi_0| \leq \pi \tag{8.52}$$

The same geometrical construction shows that

$$\nu \geq \psi \quad \text{if} \quad 0 \leq \nu \leq \pi \tag{8.53a}$$

and

$$\nu \leq \psi \quad \text{if} \quad \pi \leq \nu \leq 2\pi \tag{8.53b}$$

so that

$$0 \leq (\nu - \psi) \leq \pi \quad \text{if} \quad 0 \leq \nu \leq \pi \tag{8.54a}$$

and

$$0 \leq -(\nu - \psi) \leq \pi \quad \text{if} \quad \pi \leq \nu \leq 2\pi \tag{8.54b}$$

Now since

$$(\nu - \psi) = (\eta - \theta) + (\nu_0 - \psi_0) \tag{8.55}$$

it follows that

$$-(\nu_0-\psi_0)\leq(\eta-\theta)\leq\pi-(\nu_0-\psi_0)\quad \text{if}\quad 0\leq\nu\leq\pi \tag{8.56a}$$

and

$$+(\nu_0-\psi_0)\leq-(\eta-\theta)\leq\pi+(\nu_0-\psi_0)\quad \text{if}\quad \pi\leq\nu\leq 2\pi \tag{8.56b}$$

Of course, $(\nu_0-\psi_0)$ itself may be either positive or negative, depending on the value of ν_0. Analogously to (8.54) above, one may write

$$0\leq(\nu_0-\psi_0)\leq\pi\quad \text{if}\quad 0\leq\nu_0\leq\pi \tag{8.57a}$$

and

$$0\leq-(\nu_0-\psi_0)\leq\pi\quad \text{if}\quad \pi\leq\nu_0\leq 2\pi \tag{8.57b}$$

Then the following four possibilities present themselves. If $0\leq\nu\leq\pi$ and $0\leq\nu_0\leq\pi$ then

$$-\pi\leq-(\nu_0-\psi_0)\leq(\eta-\theta)\leq\pi-(\nu_0-\psi_0)\leq+\pi \tag{8.58}$$

If $\pi\leq\nu\leq 2\pi$ and $0\leq\nu_0\leq\pi$ then

$$0\leq+(\nu_0-\psi_0)\leq-(\eta-\theta)\leq\pi+(\nu_0-\psi_0)\leq+2\pi \tag{8.59}$$

If $0\leq\nu\leq\pi$ and $\pi\leq\nu_0\leq 2\pi$ then

$$+\pi\leq-(\nu_0-\psi_0)\leq(\eta-\theta)\leq\pi-(\nu_0-\psi_0)\leq+2\pi \tag{8.60}$$

If $\pi\leq\nu\leq 2\pi$ and $\pi\leq\nu_0\leq 2\pi$ then

$$-\pi\leq+(\nu_0-\psi_0)\leq-(\eta-\theta)\leq\pi+(\nu_0-\psi_0)\leq+\pi \tag{8.61}$$

If one needed to use the more restrictive inner inequalities above, ν_0 and ψ_0 could be calculated from dynamical initial conditions via equations (5.60), (5.61), (5.64) and (5.65) of Chapter 5. In any case, the net result is that for all ellipses the anomaly difference is bounded as

$$|\eta - \theta| \leq 2\pi \tag{8.62}$$

with the equality being approached only on nearly rectilinear orbits. This means that the quarter-angle formula (8.50) does suffice to evaluate $(\eta - \theta)$ unambiguously.

Now the differential equation for the time element τ can be obtained by differentiating the right-hand side of (8.47) with respect to η.

$$\tau' = t_0' + \left[\frac{\eta}{\sqrt{(1-\alpha^2-\beta^2)^3}}\right]\frac{d}{d\eta}\left[\frac{h^3}{\mu^2}\right] + \frac{h^3}{\mu^2}\frac{d}{d\eta}\left[\frac{\eta}{\sqrt{(1-\alpha^2-\beta^2)^3}}\right] \tag{8.63}$$

Using equation (8.41) and specializing the form of K to that used in equation (8.44), there is obtained, after several steps,

$$\begin{aligned}\tau' = {} & \frac{h^3}{\mu^2\sqrt{(1-\alpha^2-\beta^2)^3}} - \frac{3}{2}\frac{h^3}{\mu^2}K_1\frac{(h^2)'}{h^2} - \frac{h^3}{\mu^2}\left[\frac{\partial K_1}{\partial\alpha}\alpha' + \frac{\partial K_1}{\partial\beta}\beta'\right] \\ & + \frac{h^3}{\mu^2}\left[\frac{3}{2}\frac{1}{\sqrt{(1-\alpha^2-\beta^2)^3}}\frac{(h^2)'}{h^2} + \frac{3(\alpha\alpha'+\beta\beta')}{\sqrt{(1-\alpha^2-\beta^2)^5}}\right](\eta-\theta) \\ & + \frac{h^3}{\mu^2\sqrt{(1-\alpha^2-\beta^2)^3}}\left[\frac{\partial(\eta-\theta)}{\partial\alpha}\alpha' + \frac{\partial(\eta-\theta)}{\partial\beta}\beta'\right]\end{aligned} \tag{8.64}$$

In this equation the partial derivatives are to be calculated from equations (8.45) and (8.50), and rates on the right-hand side are to be replaced with their respective equations given earlier as equations (8.13a), (8.18) and (8.19). The final form of the equation will be lengthy. However, no mixed secular terms appear, much in analogy with equation (2.156) of Chapter 2. The first term, which is just the reciprocal of the mean motion (see equation (5.104) of Chapter 5), produces the unavoidable secular propagation of errors in the calculation of time, but all the other terms are periodic. It is almost needless to say that this equation is too cumbersome for developing even a first-order analytical solution for the element τ . In spite of its length and complicated terms it may find some numerical use, however.

Before the analysis turns elsewhere, it should be pointed out that several authors have developed time element equations suitable for use in the true-anomaly domain. At least the work of Nacozy (1975, 1976, 1981), Kwok and Nacozy (1985), Zare (1983) and Vitins (1978) should be cited in this connection. All of the time element equations in these articles are quite lengthy when put in terms of orbital elements. However, the common finding of these authors is that, when various factors in the rate equation are expressed in terms of instantaneous coordinates rather than orbital elements, the final expressions are considerably shorter. Stiefel and Scheifele (1971, section 18) also make a similar manipulation when introducing the time element in the eccentric-anomaly domain. The equations so derived are convenient to program for numerical integration though still not so useful for analytical purposes. The time element equation of Vitins (1978) is an exception to this rule; he is able to average his equation analytically to obtain the secular variation. In this study the time equations which serve as the basis for the variation of parameters are not readily put in terms of the coordinates. Hence all the development has been carried out in terms of the orbital elements only. It does not seem that the time element rate equations given above can be appreciably simplified. This is really the justification for returning to the differential equation of time (8.37) and attempting to treat it by a straightforward perturbation method.

Straightforward Solution for Time

Since only a first-order analytical solution for the time on orbit is being contemplated in this study, a simple perturbation approach to the differential equation of time (8.37) may suffice, especially since that equation can be treated as a quadrature. First-order solutions for the reciprocal radius u and the angular momentum magnitude h are available in the form

$$u(\eta) = u^{(0)}(\eta) + \varepsilon u^{(1)}(\eta) + \cdots O(\varepsilon^2) \tag{8.65}$$

$$h(\eta) = h^{(0)} + \varepsilon h^{(1)}(\eta) + \cdots O(\varepsilon^2) \tag{8.66}$$

For example, $h^{(0)} = h(0)$ is just the epochal value of angular momentum, a constant, and $\varepsilon h^{(1)}$ is given by the right-hand side of equation (8.24b). The quantity $u^{(0)}(\eta)$ is the unperturbed reciprocal radius. Its perturbation is found explicitly as follows. Accounting for variations of the elements, $u(\eta)$ is given exactly by

$$u(\eta) = \frac{\mu}{h^2(\eta)}[1 + \alpha(\eta)\cos\eta + \beta(\eta)\sin\eta] \tag{8.67}$$

The element variations are known to first order as

$$\alpha(\eta) = \alpha^{(0)} + \varepsilon\alpha^{(1)} + \cdots O(\varepsilon^2) \tag{8.68}$$

$$\beta(\eta) = \beta^{(0)} + \varepsilon\beta^{(1)} + \cdots O(\varepsilon^2) \tag{8.69}$$

Here $\alpha^{(0)} = \alpha(0)$ and $\beta^{(0)} = \beta(0)$ are the epochal values of α and β, constants, and $\varepsilon\alpha^{(1)}$ and $\varepsilon\beta^{(1)}$ are given by the right-hand sides of equations (8.33) and (8.34) respectively. Then

$$u^{(0)} + \varepsilon u^{(1)} + \cdots = \frac{\mu}{h^{(0)2}}\left[1 + \varepsilon\frac{h^{(1)}}{h^{(0)}} + \cdots\right]^{-2}\Big[1 + \left(\alpha^{(0)} + \varepsilon\alpha^{(1)} + \cdots\right)\cos\eta$$

$$+ \left(\beta^{(0)} + \varepsilon\beta^{(1)} + \cdots\right)\sin\eta\Big] \tag{8.70}$$

The first square bracket is to be expanded as a binomial series. Collecting like powers of ε, one finds

$$u^{(0)} = \frac{\mu}{h^{(0)2}}\left[1 + \alpha^{(0)}\cos\eta + \beta^{(0)}\sin\eta\right] \tag{8.71}$$

$$\varepsilon u^{(1)} = \varepsilon \frac{\mu}{h^{(0)2}}\left[\alpha^{(1)}\cos\eta + \beta^{(1)}\sin\eta\right] - 2\varepsilon \frac{\mu}{h^{(0)3}} h^{(1)}\left[1 + \alpha^{(0)}\cos\eta + \beta^{(0)}\sin\eta\right] \tag{8.72}$$

Notice that the second term of this latter equation contains $u^{(0)}$. Now using (8.65) and (8.66), the differential equation of time can be written as

$$\frac{dt}{d\eta} = \left[h^{(0)} + \varepsilon h^{(1)} + \cdots\right]^{-1}\left[u^{(0)} + \varepsilon u^{(1)} + \cdots\right]^{-2} \tag{8.73}$$

The solution is to be developed in the form

$$t(\eta) = t^{(0)}(\eta) + \varepsilon t^{(1)}(\eta) + \cdots O(\varepsilon^2) \tag{8.74}$$

Expanding the factors in (8.73), one has

$$\frac{dt^{(0)}}{d\eta} + \varepsilon\frac{dt^{(1)}}{d\eta} + \cdots = \frac{1}{h^{(0)}u^{(0)2}} - \frac{\varepsilon h^{(1)}}{h^{(0)2}u^{(0)2}} - \frac{2\varepsilon u^{(1)}}{h^{(0)}u^{(0)3}} + \cdots O(\varepsilon^2) \tag{8.75}$$

Collecting like powers of ε then produces

$$\frac{dt^{(0)}}{d\eta} = \frac{1}{h^{(0)}u^{(0)2}} \tag{8.76}$$

$$\frac{dt^{(1)}}{d\eta} = -\frac{h^{(1)}}{h^{(0)2}u^{(0)2}} - \frac{2u^{(1)}}{h^{(0)}u^{(0)3}} \tag{8.77}$$

The integration of equation (8.76) was the subject of Chapter 5. Several universal forms of $t^{(0)}(\eta)$ are available from that analysis. The integration of equation (8.77) will now be considered.

Substitute for $u^{(1)}$ from (8.72) to obtain

$$\frac{dt^{(1)}}{d\eta} = -\frac{h^{(1)}}{h^{(0)2}u^{(0)2}} - \frac{2}{h^{(0)}u^{(0)3}}\left[\frac{\mu}{h^{(0)2}}\left[\alpha^{(1)}\cos\eta + \beta^{(1)}\sin\eta\right] - \frac{2h^{(1)}u^{(0)}}{h^{(0)}}\right] \tag{8.78}$$

$$\frac{dt^{(1)}}{d\eta} = \frac{3h^{(1)}}{h^{(0)2}u^{(0)2}} - \frac{2\mu\left[\alpha^{(1)}\cos\eta + \beta^{(1)}\sin\eta\right]}{h^{(0)3}u^{(0)3}} \tag{8.79}$$

Substituting for $u^{(0)}$ from (8.71) and dropping all "(0)" superscripts for notational convenience, one can write

$$t^{(1)}(\eta) = \frac{3}{\mu h}I_1 - \frac{2}{\mu h}(I_2 + I_3) \tag{8.80}$$

where

$$I_1 = \frac{h^3}{\mu}\int_0^{\eta}\frac{h^{(1)}(x)}{(1+\alpha\cos x+\beta\sin x)^2}\,dx = \int_0^{\eta}\frac{H_1(x)}{(1+\alpha\cos x+\beta\sin x)^2}\,dx \tag{8.81}$$

$$I_2 = \frac{h^4}{\mu}\int_0^{\eta}\frac{\alpha^{(1)}(x)\cos x}{(1+\alpha\cos x+\beta\sin x)^3}\,dx = \int_0^{\eta}\frac{H_2(x)\cos x}{(1+\alpha\cos x+\beta\sin x)^3}\,dx \tag{8.82}$$

$$I_3 = \frac{h^4}{\mu}\int_0^{\eta}\frac{\beta^{(1)}(x)\sin x}{(1+\alpha\cos x+\beta\sin x)^3}\,dx = \int_0^{\eta}\frac{H_3(x)\sin x}{(1+\alpha\cos x+\beta\sin x)^3}\,dx \tag{8.83}$$

In these expressions all unadorned element symbols denote constants reckoned at the epoch and superscript "(1)" identifies quantities for which first-order η -domain solutions are required. The quantity $t^{(1)}$ has units of time when multiplied by ε, and ε itself has the same units as $\frac{h^4}{\mu}$. The factors of h in these latter four formulae have been distributed so as to render I_1, I_2 and I_3 dimensionless. The dimensionless functions H_1, H_2 and H_3 defined by the latter three equations contain the dimensionless elements α, β, $\underline{\xi}_0$ and $\underline{\zeta}_0$ as well as the independent variable η, but the elements enter only as constant parameters in this first-order solution. The initial condition on $t^{(1)}$ can be taken as zero since the time at epoch, $t_0(0)$, can be incorporated into the $t^{(0)}$ part of the solution. In fact, to first order, variations of $\varepsilon t^{(1)}$ correspond (to within a sign) to perturbations of the time at epoch. To see this clearly, consider the exact expression for elapsed time

$$\Delta t = t(\eta) - t_0(\eta) = t(\eta) - t_0(0) - \left[t_0(\eta) - t_0(0)\right] \tag{8.84}$$

where $t_0(\eta)$ is the element whose differential equation is given in (8.41). The first-order solution for elapsed time can be written as

$$\Delta t = t^{(0)}(\eta) - t^{(0)}(0) + \varepsilon t^{(1)}(\eta) - \varepsilon t^{(1)}(0) = t^{(0)}(\eta) - \left[t^{(0)}(0) + \varepsilon t^{(1)}(0)\right] + \varepsilon t^{(1)}(\eta) \tag{8.85}$$

These two equations agree term-by-term if one identifies $t^{(0)}(\eta) = t(\eta)$, $t^{(0)}(0) = t_0(0)$ and $t^{(1)}(0) = 0$. Additional insight into the first-order approximation being considered here could be obtained by a term-by-term comparison of equation (8.80) with the integro-differential equation (8.41). A detailed discussion of the mathematics is beyond the scope of this study, but in particular an interchange in the order of iterated integrations on the right-hand side is required when the integrated form of (8.41) is considered. This imposes no unusual restriction on the functions making up the perturbation solution.

The integrals I_1, I_2 and I_3 are of the same general type already treated in Chapter 5 beginning at equation (5.8). The main difference, and the main difficulty in their closed-form evaluation, is the very extensive numerator expressions which must be handled in the present case. The functions H_1, H_2 and H_3 are available from the first-order η -domain solutions already developed, and are quoted below in explicit terms for reference. They are taken from equations (8.24b), (8.33) and (8.34) respectively, appropriate substitutions being made.

The function H_1 is by definition from (8.81)

$$H_1(\eta) = \frac{h^3}{\mu} h^{(1)}(\eta) = \frac{h^3}{\mu} \frac{1}{\varepsilon} [h(\eta) - h(0)] \tag{8.86}$$

Then using equations (8.24b) and (8.24a), one has

$$H_1(\eta) = 3\frac{h^2}{\mu}\{ \text{... bracket from (8.24}a\text{) ...} \} \tag{8.87}$$

In explicit terms this is

$$H_1(\eta) = 3\left\{ + (\underline{\xi}_0 \cdot \underline{k})^2 \left[\frac{1}{2} \sin^2\eta\right]\right.$$

$$- (\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})[\cos\eta \sin\eta]$$

$$- (\underline{\zeta}_0 \cdot \underline{k})^2 \left[\frac{1}{2} \sin^2\eta\right]$$

$$+ \alpha(\underline{\xi}_0 \cdot \underline{k})^2 \left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ \alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{3} \sin^3\eta\right]$$

$$- \alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{3} \sin\eta \cos^2\eta + \frac{2}{3} \sin\eta\right]$$

$$- \alpha(\underline{\zeta}_0 \cdot \underline{k})^2 \left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$+ \beta(\underline{\xi}_0 \cdot \underline{k})^2 \left[\frac{1}{3} \sin^3\eta\right]$$

$$+ \beta(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3} \sin^2\eta \cos\eta\right]$$

$$- \beta(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \left[\frac{1}{3}(1 - \cos^3\eta)\right]$$

$$-\beta(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]\Bigg\} \tag{8.88}$$

The function H_2 is by definition from (8.82)

$$H_2(\eta) = \frac{h^4}{\mu}\alpha^{(1)}(\eta) = \frac{h^4}{\mu}\frac{1}{\varepsilon}\left[\alpha(\eta) - \alpha(0)\right] \tag{8.89}$$

Using equation (8.33), one obtains

$$H_2(\eta) = \frac{h^4}{\mu\varepsilon}\left[\frac{1}{\mu}\left[h^2(\eta) - h^2(0)\right]u_0 + \frac{h^2}{\mu}\left[u_0(\eta) - u_0(0)\right]\right] \tag{8.90}$$

Substituting from equations (8.24a) and (8.30) produces

$$H_2(\eta) = 6\frac{h^2}{\mu}\{\text{... bracket from (8.24a) ...}\}(1+\alpha)$$

$$+\frac{h^4}{\mu^2}\{\text{... bracket from (8.30) ...}\} \tag{8.91}$$

In explicit terms this is

$$H_2(\eta) = 2(1+\alpha)H_1(\eta) + \Bigg\{\frac{15}{2}(1+\alpha)(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta)\right]$$

$$+ 15(1+\alpha)(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+\frac{15}{2}(1+\alpha)(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{4}\sin^4\eta\right]$$

$$-\frac{3}{2}(1+\alpha)\left[\frac{1}{2}\sin^2\eta\right]$$

$$-3(1+\alpha)(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\sin^2\eta\right]$$

$$- 3\,(1+\alpha)(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta\right]$$

$$+ \frac{15}{2}\,(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta) - \frac{1}{5}(1-\cos^5\eta)\right]$$

$$+ 15\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$\left. + \frac{1}{5}\sin\eta\cos^4\eta - \frac{1}{15}\sin\eta\cos^2\eta - \frac{2}{15}\sin\eta\right]$$

$$+ \frac{15}{2}\,(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{4}\sin^4\eta - \frac{1}{5}\sin^4\eta\cos\eta - \frac{2}{15}(1-\cos\eta) + \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$- \frac{3}{2}\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3\,(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta - \frac{1}{3}\sin^3\eta\right]$$

$$+ \frac{15}{2}(1+\alpha)\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$+ 15(1+\alpha)\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ \frac{15}{2}(1+\alpha)\alpha(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$- \frac{3}{2}(1+\alpha)\alpha\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3(1+\alpha)\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3(1+\alpha)\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ \frac{15}{2}\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta) - \frac{1}{6}(1-\cos^6\eta)\right]$$

$$+ 15\,\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right.$$

$$\left. - \frac{1}{16}\eta + \frac{1}{16}\sin\eta\cos\eta - \frac{1}{8}\sin^3\eta\cos\eta - \frac{1}{6}\sin^3\eta\cos^3\eta\right] \qquad *$$

$$+ \frac{15}{2}\alpha(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right.$$

$$\left. - \frac{1}{6}\sin^4\eta\cos^2\eta - \frac{1}{12}\sin^4\eta\right]$$

$$- \frac{3}{2}\alpha\left[\frac{1}{3}(1-\cos^3\eta) - \frac{1}{4}(1-\cos^4\eta)\right]$$

$$- 3\,\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta) - \frac{1}{4}(1-\cos^4\eta)\right]$$

$$- 3\,\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta\right.$$

$$\left. - \frac{1}{8}\eta + \frac{1}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\right] \qquad *$$

$$+ \frac{15}{2}(1+\alpha)\beta(\underline{\xi}_0 \cdot \underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ 15(1+\alpha)\beta(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}(1+\alpha)\beta(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$
-\frac{3}{2}(1+\alpha)\beta\left[\frac{1}{3}\sin^3\eta\right]
$$

$$
-3(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]
$$

$$
-3(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta\right]
$$

$$
+\frac{15}{2}\beta(\underline{\xi}_0\cdot\underline{k})^2\Big[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta
$$

$$
-\frac{1}{16}\eta+\frac{1}{16}\sin\eta\cos\eta-\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\Big]
$$

$$
+15\,\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta
$$

$$
-\frac{1}{6}\sin^4\eta\cos^2\eta-\frac{1}{12}\sin^4\eta\Big]
$$

$$
+\frac{15}{2}\beta(\underline{\zeta}_0\cdot\underline{k})^2\Big[\frac{1}{5}\sin^5\eta
$$

$$
-\frac{1}{16}\eta+\frac{1}{16}\sin\eta\cos\eta-\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\eta\cos^3\eta\Big]
$$

$$
-\frac{3}{2}\beta\left[\frac{1}{3}\sin^3\eta-\frac{1}{8}\eta+\frac{1}{8}\sin\eta\cos\eta-\frac{1}{4}\sin^3\eta\cos\eta\right]
$$

$$
-3\,\beta(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta-\frac{1}{8}\eta+\frac{1}{8}\sin\eta\cos\eta-\frac{1}{4}\sin^3\eta\cos\eta\right]
$$

$$
-3\,\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta-\frac{1}{4}\sin^4\eta\right]
$$

$$
+\frac{15}{2}\beta(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]
$$

$$+ 15\,\beta(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{4}\sin^4\eta\right]$$

$$+ \frac{15}{2}\,\beta(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{3}{8}\eta - \frac{3}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$- \frac{3}{2}\,\beta\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta\right]$$

$$- 3\,\beta(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\sin^2\eta\right]$$

$$- 3\,\beta(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta\right]$$

$$+ \frac{15}{2}\,(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{3}(1 - \cos^3\eta) - 2\left(\frac{1}{4}(1 - \cos^4\eta)\right) + \frac{1}{5}(1 - \cos^5\eta)\right]$$

$$+ 15\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}\sin^3\eta - 2\left(\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right)\right.$$

$$\left. - \frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ \frac{15}{2}\,(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{2}{3}(1 - \cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta - 2\left(\frac{1}{4}\sin^4\eta\right)\right.$$

$$\left. + \frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1 - \cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$- \frac{3}{2}\left[(1 - \cos\eta) - 2\left(\frac{1}{2}\sin^2\eta\right) + \frac{1}{3}(1 - \cos^3\eta)\right]$$

$$- 3\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\sin\eta - 2\left(\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right)\right.$$

$$\left. + \frac{1}{3}\sin\eta\cos^2\eta + \frac{2}{3}\sin\eta\right]$$

$$- 3\,(\underline{\zeta}_0 \cdot \underline{k})^2\left[(1-\cos\eta) - 2\left(\frac{1}{2}\sin^2\eta\right) + \frac{1}{3}(1-\cos^3\eta)\right]$$

$$+ \frac{15}{2}\beta\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$+ 15\beta\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+ \frac{15}{2}\beta\alpha(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$- \frac{3}{2}\beta\alpha\left[\frac{1}{3}\sin^3\eta\right]$$

$$- 3\beta\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$- 3\beta\alpha(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+ \frac{15}{2}\,\alpha(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta) - 2\left(\frac{1}{5}(1-\cos^5\eta)\right) + \frac{1}{6}(1-\cos^6\eta)\right]$$

$$+ 15\,\alpha(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\Bigg[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta$$

$$-2\left(-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right)$$

$$+ \frac{1}{16}\eta - \frac{1}{16}\sin\eta\cos\eta + \frac{1}{8}\sin^3\eta\cos\eta + \frac{1}{6}\sin^3\eta\cos^3\eta\Bigg]$$

$$+ \frac{15}{2}\,\alpha(\underline{\zeta}_0 \cdot \underline{k})^2\Bigg[\frac{1}{4}\sin^4\eta$$

$$- 2\left(\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right)$$

$$+\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\Big]$$

$$-\frac{3}{2}\alpha\left[\frac{1}{2}\sin^2\eta-2\left(\frac{1}{3}(1-\cos^3\eta)\right)+\frac{1}{4}(1-\cos^4\eta)\right]$$

$$-3\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta$$

$$-2\left(\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right)$$

$$+\frac{3}{8}\eta+\frac{3}{8}\sin\eta\cos\eta+\frac{1}{4}\sin\eta\cos^3\eta\Big]$$

$$-3\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{2}\sin^2\eta-2\left(\frac{1}{3}(1-\cos^3\eta)\right)+\frac{1}{4}(1-\cos^4\eta)\right]$$

$$+\frac{15}{2}\beta^2(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+15\beta^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}\sin^5\eta\right]$$

$$+\frac{15}{2}\beta^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{8}{15}(1-\cos\eta)-\frac{4}{15}\sin^2\eta\cos\eta-\frac{1}{5}\sin^4\eta\cos\eta\right]$$

$$-\frac{3}{2}\beta^2\left[\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$-3\beta^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$-3\beta^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{2}{3}(1-\cos\eta)-\frac{1}{3}\sin^2\eta\cos\eta\right]$$

$$+\frac{15}{2}\beta(\underline{\xi}_0\cdot\underline{k})^2\Big[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta$$

$$-2\left(-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right)$$

$$+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\eta\cos^3\eta\Bigg]$$

$$+15\,\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Bigg[\frac{1}{4}\sin^4\eta$$

$$-2\left(\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right)$$

$$+\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\Bigg]$$

$$+\frac{15}{2}\beta(\underline{\zeta}_0\cdot\underline{k})^2\Bigg[\frac{3}{8}\eta-\frac{3}{8}\sin\eta\cos\eta-\frac{1}{4}\sin^3\eta\cos\eta$$

$$-2\left(\frac{1}{5}\sin^5\eta\right)+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\Bigg]$$

$$-\frac{3}{2}\beta\Bigg[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta-2\left(\frac{1}{3}\sin^3\eta\right)$$

$$+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\Bigg]$$

$$-3\,\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Bigg[\frac{1}{2}\sin^2\eta-2\left(\frac{1}{3}(1-\cos^3\eta)\right)+\frac{1}{4}(1-\cos^4\eta)\Bigg]$$

$$-3\,\beta(\underline{\zeta}_0\cdot\underline{k})^2\Bigg[\frac{1}{2}\eta-\frac{1}{2}\sin\eta\cos\eta-2\left(\frac{1}{3}\sin^3\eta\right)$$

$$+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\Bigg]\Bigg\}$$

$$(8.92)$$

The function H_3 is by definition from (8.83)

$$H_3(\eta) = \frac{h^4}{\mu}\beta^{(1)}(\eta) = \frac{h^4}{\mu}\frac{1}{\varepsilon}[\beta(\eta) - \beta(0)] \tag{8.93}$$

Using equation (8.34), one obtains

$$H_3(\eta) = \frac{h^4}{\mu\varepsilon}\left[\frac{1}{\mu}\left[h^2(\eta) - h^2(0)\right]w_0 + \frac{h^2}{\mu}\left[w_0(\eta) - w_0(0)\right]\right] \tag{8.94}$$

Substituting from equations (8.24a) and (8.32) produces

$$H_3(\eta) = 6\frac{h^2}{\mu}\{\text{ ... bracket from (8.24}a\text{) ... }\}\beta$$

$$+\frac{h^4}{\mu^2}\{\text{ ... bracket from (8.32) ... }\} \tag{8.95}$$

In explicit terms this is

$$H_3(\eta) = 2\beta H_1(\eta) + \left\{ -\frac{15}{2}(1+\alpha)\,(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{3}{8}\eta + \frac{3}{8}\sin\eta\cos\eta + \frac{1}{4}\sin\eta\cos^3\eta\right]\right.$$

$$-\,15(1+\alpha)\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{4}(1-\cos^4\eta)\right]$$

$$-\frac{15}{2}(1+\alpha)(\underline{\zeta}_0 \cdot \underline{k})^2\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+\frac{3}{2}(1+\alpha)\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right]$$

$$+\,3(1+\alpha)\,(\underline{\xi}_0 \cdot \underline{k})^2\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right]$$

$$+\,3(1+\alpha)\,(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\left[\frac{1}{2}\sin^2\eta\right]$$

$$-\frac{15}{2}(1+\alpha)\alpha(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{4}{15}\sin\eta\cos^2\eta+\frac{8}{15}\sin\eta+\frac{1}{5}\sin\eta\cos^4\eta\right]$$

$$-15(1+\alpha)\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$-\frac{15}{2}(1+\alpha)\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$+\frac{3}{2}(1+\alpha)\alpha\left[\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right]$$

$$+3(1+\alpha)\alpha(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right]$$

$$+3(1+\alpha)\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-\frac{15}{2}(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$-15(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$-\frac{15}{2}(1+\alpha)\beta(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+\frac{3}{2}(1+\alpha)\beta\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3(1+\alpha)\beta(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin^3\eta\right]$$

$$-\frac{15}{2}(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right.$$

$$-\left(\frac{3}{8}\eta + \frac{3}{8}\sin\eta\cos\eta + \frac{1}{4}\sin\eta\cos^3\eta\right)$$

$$-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\Big]$$

$$-15\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[\frac{1}{3}(1-\cos^3\eta) - \frac{1}{4}(1-\cos^4\eta)$$

$$+\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\Big]$$

$$-\frac{15}{2}(\underline{\zeta}_0\cdot\underline{k})^2\Big[\frac{1}{3}\sin^3\eta + \frac{1}{5}\sin^5\eta - \left(\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right)\Big]$$

$$+\frac{3}{2}\Big[\sin\eta - \left(\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right) + \frac{1}{3}\sin^3\eta\Big]$$

$$+3\,(\underline{\xi}_0\cdot\underline{k})^2\Big[\sin\eta - \left(\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right) + \frac{1}{3}\sin^3\eta\Big]$$

$$+3\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[(1-\cos\eta) - \frac{1}{2}\sin^2\eta + \frac{2}{3}(1-\cos\eta) - \frac{1}{3}\sin^2\eta\cos\eta\Big]$$

$$-\frac{15}{2}\alpha(\underline{\xi}_0\cdot\underline{k})^2\Big[\frac{3}{8}\eta + \frac{3}{8}\sin\eta\cos\eta + \frac{1}{4}\sin\eta\cos^3\eta$$

$$-\left(\frac{4}{15}\sin\eta\cos^2\eta + \frac{8}{15}\sin\eta + \frac{1}{5}\sin\eta\cos^4\eta\right)$$

$$+\frac{1}{16}\eta - \frac{1}{16}\sin\eta\cos\eta + \frac{1}{8}\sin^3\eta\cos\eta + \frac{1}{6}\sin^3\eta\cos^3\eta\Big]$$

$$-15\,\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[\frac{1}{4}(1-\cos^4\eta) - \frac{1}{5}(1-\cos^5\eta)$$

$$+\frac{1}{6}\sin^4\eta\cos^2\eta + \frac{1}{12}\sin^4\eta\Big]$$

$$-\frac{15}{2}\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$-\left(-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right)$$

$$\left.+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$+\frac{3}{2}\alpha\left[\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta-\left(\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right)\right.$$

$$\left.+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+3\,\alpha(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{2}\eta+\frac{1}{2}\sin\eta\cos\eta-\left(\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right)\right.$$

$$\left.+\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+3\,\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\sin^2\eta-\frac{1}{3}(1-\cos^3\eta)+\frac{1}{4}\sin^4\eta\right]$$

$$-\frac{15}{2}\beta\,(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta)-\frac{1}{5}(1-\cos^5\eta)+\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$-15\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right.$$

$$-\left(-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right)$$

$$\left.+\frac{1}{16}\eta-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\right]$$

$$-\frac{15}{2}\beta\,(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{4}\sin^4\eta+\frac{1}{6}\sin^6\eta\right.$$

$$- \left(\frac{1}{5}\sin^4\eta\cos\eta + \frac{2}{15}(1-\cos\eta) - \frac{1}{15}\sin^2\eta\cos\eta\right)\Bigg]$$

$$+ \frac{3}{2}\beta\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta) + \frac{1}{4}\sin^4\eta\right]$$

$$+ 3\beta\,(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{2}\sin^2\eta - \frac{1}{3}(1-\cos^3\eta) + \frac{1}{4}\sin^4\eta\right]$$

$$+ 3\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Bigg[\frac{1}{2}\eta - \frac{1}{2}\sin\eta\cos\eta - \frac{1}{3}\sin^3\eta \qquad *$$

$$+ \frac{3}{8}\eta - \frac{3}{8}\sin\eta\cos\eta - \frac{1}{4}\sin^3\eta\cos\eta\Bigg] \qquad *$$

$$- \frac{15}{2}\beta\,(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{4}(1-\cos^4\eta)\right]$$

$$- 15\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{8}\eta - \frac{1}{8}\sin\eta\cos\eta + \frac{1}{4}\sin^3\eta\cos\eta\right] \qquad *$$

$$- \frac{15}{2}\beta\,(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{4}\sin^4\eta\right]$$

$$+ \frac{3}{2}\beta\left[\frac{1}{2}\sin^2\eta\right]$$

$$+ 3\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{2}\eta + \frac{1}{2}\sin\eta\cos\eta\right] \qquad *$$

$$+ 3\beta\,(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{2}\sin^2\eta\right]$$

$$- \frac{15}{2}\beta\alpha(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{5}(1-\cos^5\eta)\right]$$

$$- 15\beta\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[-\frac{1}{5}\sin\eta\cos^4\eta + \frac{1}{15}\sin\eta\cos^2\eta + \frac{2}{15}\sin\eta\right]$$

$$-\frac{15}{2}\beta\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+\frac{3}{2}\beta\alpha\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3\beta\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}\sin\eta\cos^2\eta+\frac{2}{3}\sin\eta\right]$$

$$+3\beta\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-\frac{15}{2}\beta^2(\underline{\xi}_0\cdot\underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$-15\beta^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$-\frac{15}{2}\beta^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$+\frac{3}{2}\beta^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+3\beta^2(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$+3\beta^2(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+\frac{15}{2}(\underline{\xi}_0\cdot\underline{k})^2\left[-\frac{1}{5}\sin\eta\cos^4\eta+\frac{1}{15}\sin\eta\cos^2\eta+\frac{2}{15}\sin\eta\right]$$

$$+15(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{5}\sin^4\eta\cos\eta+\frac{2}{15}(1-\cos\eta)-\frac{1}{15}\sin^2\eta\cos\eta\right]$$

$$+\frac{15}{2}(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{5}\sin^5\eta\right]$$

$$-\frac{3}{2}\left[\frac{1}{3}\sin^3\eta\right]$$

$$-3\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{3}(1-\cos^3\eta)\right]$$

$$-3\,(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{3}\sin^3\eta\right]$$

$$+\frac{15}{2}\,\alpha(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{16}\eta\right.$$

$$\left.-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta+\frac{1}{6}\sin^3\cos^3\eta\right]$$

$$+15\,\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$+\frac{15}{2}\,\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{16}\eta\right.$$

$$\left.-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\cos^3\eta\right]$$

$$-\frac{3}{2}\,\alpha\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$-3\,\alpha(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{4}(1-\cos^4\eta)\right]$$

$$-3\,\alpha(\underline{\zeta}_0\cdot\underline{k})^2\left[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\right]$$

$$+\frac{15}{2}\beta\,(\underline{\xi}_0\cdot\underline{k})^2\left[\frac{1}{6}\sin^4\eta\cos^2\eta+\frac{1}{12}\sin^4\eta\right]$$

$$+15\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\left[\frac{1}{16}\eta\right.$$

$$-\frac{1}{16}\sin\eta\cos\eta+\frac{1}{8}\sin^3\eta\cos\eta-\frac{1}{6}\sin^3\eta\cos^3\eta\Big]$$

$$+\frac{15}{2}\beta\,(\underline{\zeta}_0\cdot\underline{k})^2\Big[\frac{1}{6}\sin^6\eta\Big]$$

$$-\frac{3}{2}\beta\Big[\frac{1}{4}\sin^4\eta\Big]$$

$$-3\beta\,(\underline{\xi}_0\cdot\underline{k})(\underline{\zeta}_0\cdot\underline{k})\Big[\frac{1}{8}\eta-\frac{1}{8}\sin\eta\cos\eta+\frac{1}{4}\sin^3\eta\cos\eta\Big] \qquad *$$

$$-3\beta\,(\underline{\zeta}_0\cdot\underline{k})^2\Big[\frac{1}{4}\sin^4\eta\Big]\Big\} \tag{8.96}$$

In the above equations terms linear in η have been marked with an asterisk for easy reference.

Now in undertaking to evaluate I_1, I_2 and I_3, it is helpful to recall the integration of the differential equation of time in unperturbed motion as given in Chapter 5. That equation reduced to the quadrature

$$K(\eta;\alpha,\beta)=\int_0^\eta\frac{dx}{(1+\alpha\cos x+\beta\sin x)^2} \tag{8.97}$$

which is just equation (5.8). The change of variable from η to z according to

$$\eta=2\tan^{-1}z\ ,\qquad d\eta=\frac{2dz}{1+z^2}\ ,\qquad \sin\eta=\frac{2z}{1+z^2}\ ,\qquad \cos\eta=\frac{1-z^2}{1+z^2} \tag{8.98}$$

results in

$$K(\eta;\alpha,\beta)=2\int_0^{\tan\frac{1}{2}\eta}\frac{(1+z^2)dz}{\left[(1-\alpha)z^2+2\beta z+(1+\alpha)\right]^2} \tag{8.99}$$

Completing the square in the denominator brings the integrand into a form suitable for expansion into partial fractions:

$$K(\eta;\alpha,\beta)=2\int_0^{\tan\frac{1}{2}\eta}\frac{(1+z^2)dz}{(1-\alpha)^2(z+C_1)^2(z+C_2)^2} \tag{8.100}$$

where

$$C_1=\frac{\beta-\sqrt{\alpha^2+\beta^2-1}}{1-\alpha}\quad\text{and}\quad C_2=\frac{\beta+\sqrt{\alpha^2+\beta^2-1}}{1-\alpha} \tag{8.101}$$

In the same way, the quadrature

$$I_1=\int_0^{\eta}\frac{H_1(x)dx}{(1+\alpha\cos x+\beta\sin x)^2} \tag{8.102}$$

can be recast as

$$I_1=2\int_0^{\tan\frac{1}{2}\eta}\frac{H_1(\eta(z))\ (1+z^2)dz}{(1-\alpha)^2(z+C_1)^2(z+C_2)^2} \tag{8.103}$$

The notation $H_1(\eta(z))$ means that H_1 as defined by equation (8.88) is to be regarded as a function of z once η has been replaced according to (8.98) above. Since H_1 depends on η only through terms of powers and products of $\sin \eta$ and $\cos \eta$, it becomes a rational algebraic function of z.

Bearing this in mind, one procedes with the partial fraction expansion as follows.

$$\frac{H_1(\eta(z))(1+z^2)}{(1-\alpha)^2(z+C_1)^2(z+C_2)^2} = \frac{N_1}{(z+C_1)} + \frac{N_2}{(z+C_1)^2} + \frac{N_3}{(z+C_2)} + \frac{N_4}{(z+C_2)^2} \tag{8.104}$$

The constant numerator factors are to be determined by enforcing this identity in the form

$$\frac{H_1(\eta(z))(1+z^2)}{(1-\alpha)^2} = N_1(z+C_1)(z+C_2)^2 + N_2(z+C_2)^2$$

$$+ N_3(z+C_1)^2(z+C_2) + N_4(z+C_1)^2 \tag{8.105}$$

Evaluating this expression at $z = -C_1$ produces

$$N_2 = \frac{H_1(\eta(-C_1))(1+C_1^2)}{(1-\alpha)^2(C_2-C_1)^2} \tag{8.106}$$

Likewise, evaluating at $z = -C_2$ produces

$$N_4 = \frac{H_1(\eta(-C_2))(1+C_2^2)}{(1-\alpha)^2(C_2-C_1)^2} \tag{8.107}$$

The notation $H_1(\eta(-C))$ means that η has been replaced in terms of z in equation (8.88) and then z has been replaced by $-C$.

In order to determine N_1 and N_3, it is convenient to use the fact that the derivative of the identity (8.105) is also an identity.

$$\frac{d}{dz}\left[\frac{H_1(\eta(z))(1+z^2)}{(1-\alpha)^2}\right] = N_1\left[(z+C_2)^2 + 2(z+C_1)(z+C_2)\right] + 2N_2(z+C_2)$$

$$+ N_3\left[2(z+C_1)(z+C_2) + (z+C_1)^2\right] + 2N_4(z+C_1) \tag{8.108}$$

The derivative of H_1 with respect to z is calculated as

$$\frac{dH_1}{dz} = \frac{dH_1}{d\eta}\frac{d\eta}{dz} = \frac{dH_1}{d\eta}\frac{2}{1+z^2} \tag{8.109}$$

so that

$$\frac{2H_1'(\eta(z)) + 2zH_1(\eta(z))}{(1-\alpha)^2} = N_1\left[(z+C_2)^2 + 2(z+C_1)(z+C_2)\right] + 2N_2(z+C_2)$$

$$+ N_3\left[2(z+C_1)(z+C_2) + (z+C_1)^2\right] + 2N_4(z+C_1) \tag{8.110}$$

The notation $H_1'(\eta(z))$ means that (8.88) is to be differentiated with respect to η and then η is to be replaced in terms of z according to (8.98). Evidently, H_1', like H_1, will be a rational algebraic function of z. Then evaluating this expression at $z = -C_1$ produces

$$N_1 = \frac{2H_1'(\eta(-C_1)) - 2C_1H_1(\eta(-C_1))}{(1-\alpha)^2(C_2-C_1)^2} - \frac{2N_2}{(C_2-C_1)} \tag{8.111}$$

Evaluating the same expression at $z = -C_2$ produces

$$N_3 = \frac{2H_1'(\eta(-C_2)) - 2C_2H_1(\eta(-C_2))}{(1-\alpha)^2(C_2-C_1)^2} + \frac{2N_4}{(C_2-C_1)} \tag{8.112}$$

In these equations, the values of N_2 and N_4 are known from (8.106) and (8.107) above.

Now the integration for I_1 can proceed.

$$\frac{1}{2}I_1 = \int_0^{\tan\frac{1}{2}\eta}\left[\frac{N_1}{(z+C_1)} + \frac{N_2}{(z+C_1)^2} + \frac{N_3}{(z+C_2)} + \frac{N_4}{(z+C_2)^2}\right] dz \tag{8.113}$$

Simple steps lead to

$$\frac{1}{2}I_1 = N_1 \ln\left[\frac{(z+C_1)}{C_1}\right] + N_3 \ln\left[\frac{(z+C_2)}{C_2}\right] + N_2\left[\frac{z}{C_1(z+C_1)}\right] + N_4\left[\frac{z}{C_2(z+C_2)}\right] \tag{8.114}$$

The quadrature I_2 can be treated similarly. Using the same change of variable, the expression

$$I_2 = \int_0^{\eta} \frac{H_2(x)\cos x \; dx}{(1+\alpha\cos x+\beta\sin x)^3} \tag{8.115}$$

is converted into

$$I_2 = 2\int_0^{\tan\frac{1}{2}\eta} \frac{H_2(\eta(z))(1-z^2)(1+z^2)dz}{(1-\alpha)^3(z+C_1)^3(z+C_2)^3} \tag{8.116}$$

This quadrature is more complicated than was I_1, not only because of the larger exponent in the denominator of the integrand, but also because $H_2(\eta)$ contains terms linear in η. Under the change of variable from η to z the function H_2 proves to be transcendental in z, not merely rational algebraic. This fact does not impede the formal application of the method of partial fractions, but, as noted later, it may be an obstacle to the derivation of practical working formulae.

The partial fraction expansion has the form

$$\frac{H_2(\eta(z))(1-z^2)(1+z^2)}{(1-\alpha)^3(z+C_1)^3(z+C_2)^3}=\frac{N_5}{(z+C_1)}+\frac{N_6}{(z+C_1)^2}+\frac{N_7}{(z+C_1)^3}$$

$$+\frac{N_8}{(z+C_2)}+\frac{N_9}{(z+C_2)^2}+\frac{N_{10}}{(z+C_2)^3} \tag{8.117}$$

In order to determine the constant numerator factors, this identity will be enforced in the form

$$\frac{H_2(\eta(z))(1-z^2)(1+z^2)}{(1-\alpha)^3}=N_5(z+C_1)^2(z+C_2)^3+N_6(z+C_1)(z+C_2)^3+N_7(z+C_2)^3$$

$$+N_8(z+C_1)^3(z+C_2)^2+N_9(z+C_1)^3(z+C_2)+N_{10}(z+C_1)^3 \tag{8.118}$$

Evaluating this expression at $z=-C_1$ produces

$$N_7=+\frac{H_2(\eta(-C_1))(1-C_1^2)(1+C_1^2)}{(1-\alpha)^3(C_2-C_1)^3} \tag{8.119}$$

Evaluating at $z=-C_2$ produces

$$N_{10}=-\frac{H_2(\eta(-C_2))(1-C_2^2)(1+C_2^2)}{(1-\alpha)^3(C_2-C_1)^3} \tag{8.120}$$

In order to determine the factors N_6 and N_9, use the fact that the derivative of the identity (8.118) must also be an identity.

$$\frac{d}{dz}\left[\frac{H_2(\eta(z))(1-z^2)(1+z^2)}{(1-\alpha)^3}\right]=N_5\left[2(z+C_1)(z+C_2)^3+3(z+C_1)^2(z+C_2)^2\right]$$

$$+N_6\left[(z+C_2)^3+3(z+C_1)(z+C_2)^2\right]+3N_7(z+C_2)^2$$

$$+N_8\left[3(z+C_1)^2(z+C_2)^2+2(z+C_1)^3(z+C_2)\right]$$

$$+N_9\left[3(z+C_1)^2(z+C_2)+(z+C_1)^3\right]+3N_{10}(z+C_1)^2 \tag{8.121}$$

The derivative of H_2 with respect to z is most easily calculated as

$$\frac{dH_2}{dz}=\frac{dH_2}{d\eta}\,\frac{d\eta}{dz}=\frac{dH_2}{d\eta}\,\frac{2}{1+z^2} \tag{8.122}$$

so that

$$\frac{2H_2'(\eta(z))(1-z^2)-4z^3H_2(\eta(z))}{(1-\alpha)^3}=N_5\left[2(z+C_1)(z+C_2)^3+3(z+C_1)^2(z+C_2)^2\right]$$

$$+N_6\left[(z+C_2)^3+3(z+C_1)(z+C_2)^2\right]+3N_7(z+C_2)^2$$

$$+N_8\left[3(z+C_1)^2(z+C_2)^2+2(z+C_1)^3(z+C_2)\right]$$

$$+N_9\left[3(z+C_1)^2(z+C_2)+(z+C_1)^3\right]+3N_{10}(z+C_1)^2 \tag{8.123}$$

As before, the notation $H_2'(\eta(z))$ means that H_2 as given by (8.92) is to be differentiated with respect to η and then η is to be replaced in terms of z according to (8.98). Notice that the terms in (8.92) which are linear in η will be constants after differentiation and that all the other terms will still consist of powers and products of $\sin\eta$ and $\cos\eta$. Hence, $H_2'(\eta(z))$, unlike $H_2(\eta(z))$, will be a rational algebraic function of z. Now evaluating equation (8.123) at $z=-C_1$ produces

$$N_6=+\frac{2H_2'(\eta(-C_1))(1-C_1^2)+4C_1^3H_2(\eta(-C_1))}{(1-\alpha)^3(C_2-C_1)^3}-\frac{3N_7}{(C_2-C_1)} \tag{8.124}$$

Evaluating the same expression at $z=-C_2$ produces

$$N_9=-\frac{2H_2'(\eta(-C_2))(1-C_2^2)+4C_2^3H_2(\eta(-C_2))}{(1-\alpha)^3(C_2-C_1)^3}+\frac{3N_{10}}{(C_2-C_1)} \tag{8.125}$$

In order to determine the two remaining numerator factors, N_5 and N_8, differentiate the identity (8.123).

$$\frac{d}{dz}\left[\frac{2H_2'(\eta(z))(1-z^2)-4z^3H_2(\eta(z))}{(1-\alpha)^3}\right]=$$

$$N_5\big[2(z+C_2)^3+6(z+C_1)(z+C_2)^2+6(z+C_1)(z+C_2)^2+6(z+C_1)^2(z+C_2)\big]$$

$$+N_6\big[3(z+C_2)^2+3(z+C_2)^2+6(z+C_1)(z+C_2)\big]+6N_7(z+C_2)$$

$$+N_8\big[6(z+C_1)(z+C_2)^2+6(z+C_1)^2(z+C_2)+6(z+C_1)^2(z+C_2)+2(z+C_1)^3\big]$$

$$+N_9\big[6(z+C_1)(z+C_2)+3(z+C_1)^2+3(z+C_1)^2\big]+6N_{10}(z+C_1) \tag{8.126}$$

Using a chain rule similar to (8.122) to differentiate the function $H_2'(\eta(z))$, there results

$$\frac{1}{(1-\alpha)^3}\left[4\frac{(1-z^2)}{(1+z^2)}H_2''(\eta(z))-4zH_2'(\eta(z))-\frac{8z^3}{(1+z^2)}H_2'(\eta(z))-12z^2H_2(\eta(z))\right]=$$

$$N_5\big[2(z+C_2)^3+12(z+C_1)(z+C_2)^2+6(z+C_1)^2(z+C_2)\big]$$

$$+N_6\big[6(z+C_2)^2+6(z+C_1)(z+C_2)\big]+6N_7(z+C_2)$$

$$+N_8\big[6(z+C_1)(z+C_2)^2+12(z+C_1)^2(z+C_2)+2(z+C_1)^3\big]$$

$$+N_9\big[6(z+C_1)(z+C_2)+6(z+C_1)^2\big]+6N_{10}(z+C_1) \tag{8.127}$$

The notation $H_2''(\eta(z))$ means that H_2 as given by (8.92) is to be differentiated twice with respect to η and then η is to be replaced in terms of z according to (8.98). Clearly, $H_2''(\eta(z))$ will be a rational algebraic function of z. Now evaluating equation (8.127) at $z=-C_1$ produces

$$N_5=+\frac{1}{2(1-\alpha)^3(C_2-C_1)^3}\left[4\frac{(1-C_1^2)}{(1+C_1^2)}H_2''(\eta(-C_1))+4C_1H_2'(\eta(-C_1))\right.$$

$$+\frac{8C_1^3}{(1+C_1^2)}H_2'(\eta(-C_1)) - 12C_1^2 H_2(\eta(-C_1))\Bigg] - \frac{3N_6}{(C_2-C_1)} - \frac{3N_7}{(C_2-C_1)^2} \tag{8.128}$$

Evaluating the same expression at $z = -C_2$ produces

$$N_8 = -\frac{1}{2(1-\alpha)^3(C_2-C_1)^3}\Bigg[4\frac{(1-C_2^2)}{(1+C_2^2)}H_2''(\eta(-C_2)) + 4C_2 H_2'(\eta(-C_2))$$

$$+\frac{8C_2^3}{(1+C_2^2)}H_2'(\eta(-C_2)) - 12C_2^2 H_2(\eta(-C_2))\Bigg] + \frac{3N_9}{(C_2-C_1)} - \frac{3N_{10}}{(C_2-C_1)^2} \tag{8.129}$$

The quadrature I_2 can now be evaluated in the form

$$\frac{1}{2}I_2 = \int_0^{\tan\frac{1}{2}\eta}\Bigg[\frac{N_5}{(z+C_1)} + \frac{N_6}{(z+C_1)^2} + \frac{N_7}{(z+C_1)^3}$$

$$+\frac{N_8}{(z+C_2)} + \frac{N_9}{(z+C_2)^2} + \frac{N_{10}}{(z+C_2)^3}\Bigg]\,dz \tag{8.130}$$

Straightforward steps lead to

$$\frac{1}{2}I_2 = N_5 \ln\left[\frac{z+C_1}{C_1}\right] + N_8 \ln\left[\frac{z+C_2}{C_2}\right] + N_6\left[\frac{z}{C_1(z+C_1)}\right] + N_9\left[\frac{z}{C_2(z+C_2)}\right]$$

$$+\frac{N_7}{2}\left[\frac{z(z+2C_1)}{C_1^2(z+C_1)^2}\right] + \frac{N_{10}}{2}\left[\frac{z(z+2C_2)}{C_2^2(z+C_2)^2}\right] \tag{8.131}$$

The remaining quadrature

$$I_3 = \int_0^{\eta} \frac{H_3(x) \sin x dx}{(1 + \alpha \cos x + \beta \sin x)^3} \tag{8.132}$$

can be evaluated similarly to I_2. In fact, the form of the partial fraction expansion will be the same as for I_2; only the determination of the constants will be different since $H_3(\eta) \sin \eta$ occurs in place of $H_2(\eta) \cos \eta$ in the numerator of the integrand. The details are presented here for completeness. Remarks already made about notations and properties of the function $H_2(\eta(z))$ and its derivatives apply here equally well for $H_3(\eta(z))$.

Under the change of variable (8.98) the quadrature becomes

$$I_3 = 2\int_0^{\tan \frac{1}{2}\eta} \frac{2z(1 + z^2)H_3(\eta(z))\, dz}{(1 - \alpha)^3(z + C_1)^3(z + C_2)^3} \tag{8.133}$$

The integrand is expanded as

$$\frac{2z(1 + z^2)H_3(\eta(z))}{(1 - \alpha)^3(z + C_1)^3(z + C_2)^3} = \frac{N_{11}}{(z + C_1)} + \frac{N_{12}}{(z + C_1)^2} + \frac{N_{13}}{(z + C_1)^3}$$

$$+ \frac{N_{14}}{(z + C_2)} + \frac{N_{15}}{(z + C_2)^2} + \frac{N_{16}}{(z + C_2)^3} \tag{8.134}$$

The constant numerator factors are to be determined by enforcing this identity in the form

$$\frac{2z(1 + z^2)H_3(\eta(z))}{(1 - \alpha)^3} = N_{11}(z + C_1)^2(z + C_2)3 + N_{12}(z + C_1)(z + C_2)^3 + N_{13}(z + C_2)^3$$

$$+ N_{14}(z+C_1)^3(z+C_2)^2 + N_{15}(z+C_1)^3(z+C_2) + N_{16}(z+C_1)^3 \tag{8.135}$$

Evaluating this expression at $z = -C_1$ produces

$$N_{13} = -\frac{2C_1(1+C_1^2)H_3(\eta(-C_1))}{(1-\alpha)^3(C_2-C_1)^3} \tag{8.136}$$

Evaluating at $z = -C_2$ produces

$$N_{16} = +\frac{2C_2(1+C_2^2)H_3(\eta(-C_2))}{(1-\alpha)^3(C_2-C_1)^3} \tag{8.137}$$

Two more factors can be determined by differentiating the identity (8.135).

$$\frac{d}{dz}\left[\frac{2z(1+z^2)H_3(\eta(z))}{(1-\alpha)^3}\right] = N_{11}\left[2(z+C_1)(z+C_2)^3 + 3(z+C_1)^2(z+C_2)^2\right]$$

$$+ N_{12}\left[(z+C_2)^3 + 3(z+C_1)(z+C_2)^2\right] + 3N_{13}(z+C_2)^2$$

$$+ N_{14}\left[3(z+C_1)^2(z+C_2)^2 + 2(z+C_1)^3(z+C_2)\right]$$

$$+ N_{15}\left[3(z+C_1)^2(z+C_2) + (z+C_1)^3\right] + 3N_{16}(z+C_1)^2 \tag{8.138}$$

Using the same chain rule as before to differentiate H_3, one obtains

$$\frac{2(1+3z^2)H_3(\eta(z)) + 4zH_3'(\eta(z))}{(1-\alpha)^3} = N_{11}\left[2(z+C_1)(z+C_2)^3 + 3(z+C_1)^2(z+C_2)^2\right]$$

$$+ N_{12}\left[(z+C_2)^3 + 3(z+C_1)(z+C_2)^2\right] + 3N_{13}(z+C_2)^2$$

$$+ N_{14}\left[3(z+C_1)^2(z+C_2)^2 + 2(z+C_1)^3(z+C_2)\right]$$

$$+ N_{15}\left[3(z+C_1)^2(z+C_2) + (z+C_1)^3\right] + 3N_{16}(z+C_1)^2 \tag{8.139}$$

Evaluating this expression at $z = -C_1$ produces

$$N_{12} = + \frac{2(1+3C_1^2)H_3(\eta(-C_1)) - 4C_1H_3'(\eta(-C_1))}{(1-\alpha)^3(C_2-C_1)^3} - \frac{3N_{13}}{(C_2-C_1)} \tag{8.140}$$

Evaluating the same expression at $z = -C_2$ produces

$$N_{15} = - \frac{2(1+3C_2^2)H_3(\eta(-C_2)) - 4C_2H_3'(\eta(-C_2))}{(1-\alpha)^3(C_2-C_1)^3} + \frac{3N_{16}}{(C_2-C_1)} \tag{8.141}$$

The remaining two numerator factors can be determined by differentiating the identity (8.139).

$$\frac{d}{dz}\left[\frac{2(1+3z^2)H_3(\eta(z)) + 4zH_3'(\eta(z))}{(1-\alpha)^3}\right] =$$

$$N_{11}\left[2(z+C_2)^3 + 6(z+C_1)(z+C_2)^2 + 6(z+C_1)(z+C_2)^2 + 6(z+C_1)^2(z+C_2)\right]$$

$$+ N_{12}\left[3(z+C_2)^2 + 3(z+C_2)^2 + 6(z+C_1)(z+C_2)\right] + 6N_{13}(z+C_2)$$

$$+ N_{14}\left[6(z+C_1)(z+C_2)^2 + 6(z+C_1)^2(z+C_2) + 6(z+C_1)^2(z+C_2) + 2(z+C_1)^3\right]$$

$$+ N_{15}\left[6(z+C_1)(z+C_2) + 3(z+C_1)^2 + 3(z+C_1)^2\right] + 6N_{16}(z+C_1) \tag{8.142}$$

Once again applying the chain rule to differentiate H_3 and H_3', one obtains

$$\frac{1}{(1-\alpha)^3}\left[12zH_3(\eta(z)) + \frac{8(1+2z^2)}{(1+z^2)}H_3'(\eta(z)) + \frac{8z}{(1+z^2)}H_3''(\eta(z))\right] =$$

$$N_{11}\left[2(z+C_2)^3 + 12(z+C_1)(z+C_2)^2 + 6(z+C_1)^2(z+C_2)\right]$$

$$+ N_{12}\left[6(z+C_2)^2 + 6(z+C_1)(z+C_2)\right] + 6N_{13}(z+C_2)$$

$$+ N_{14}\left[6(z+C_1)(z+C_2)^2 + 12(z+C_1)^2(z+C_2) + 2(z+C_1)^3\right]$$

$$+ N_{15}\left[6(z+C_1)(z+C_2) + 6(z+C_1)^2\right] + 6N_{16}(z+C_1) \tag{8.143}$$

Evaluating this expression at $z = -C_1$ produces

$$N_{11} = \frac{+1}{2(1-\alpha)^3(C_2-C_1)^3}\left[-12C_1H_3(\eta(-C_1)) + \frac{8(1+2C_1^2)}{(1+C_1^2)}H_3'(\eta(-C_1))\right.$$

$$\left.-\frac{8C_1}{(1+C_1^2)}H_3''(\eta(-C_1))\right] - \frac{3N_{12}}{(C_2-C_1)} - \frac{3N_{13}}{(C_2-C_1)^2} \tag{8.144}$$

Evaluating at $z = -C_2$ produces

$$N_{14} = \frac{-1}{2(1-\alpha)^3(C_2-C_1)^3}\left[-12C_2H_3(\eta(-C_2)) + \frac{8(1+2C_2^2)}{(1+C_2^2)}H_3'(\eta(-C_2))\right.$$

$$\left.-\frac{8C_2}{(1+C_2^2)}H_3''(\eta(-C_2))\right] - \frac{3N_{15}}{(C_2-C_1)} - \frac{3N_{16}}{(C_2-C_1)^2} \tag{8.145}$$

Finally, the quadrature I_3 is easy to calculate in the form

$$\frac{1}{2}I_3 = \int_0^{\tan\frac{1}{2}\eta}\left[\frac{N_{11}}{(z+C_1)} + \frac{N_{12}}{(z+C_1)^2} + \frac{N_{13}}{(z+C_1)^3}\right.$$

$$\left.+\frac{N_{14}}{(z+C_2)} + \frac{N_{15}}{(z+C_2)^2} + \frac{N_{16}}{(z+C_2)^3}\right]dz \tag{8.146}$$

Straightforward steps lead to

$$\frac{1}{2}I_3 = N_{11}\ln\left[\frac{z+C_1}{C_1}\right] + N_{14}\ln\left[\frac{z+C_2}{C_2}\right] + N_{12}\left[\frac{z}{C_1(z+C_1)}\right] + N_{15}\left[\frac{z}{C_2(z+C_2)}\right]$$

$$+\frac{N_{13}}{2}\left[\frac{z(z+2C_1)}{C_1^2(z+C_1)^2}\right]+\frac{N_{16}}{2}\left[\frac{z(z+2C_2)}{C_2^2(z+C_2)^2}\right] \tag{8.147}$$

This completes the formal evaluation of the integrals needed to represent the first-order perturbation of time as given in equation (8.80). Unfortunately, these formulae are not suitable for practical calculations. First, the quantities C_1 and C_2, which appear in the process of completing the square in the integrand in order to permit the partial fraction expansions, are real-valued only for hyperbolic orbits. The same situation occurred in the evaluation of the time in unperturbed motion (see equation (5.51) of Chapter 5). There the problem resolved itself after C_1, C_2 and the numerator factors N_k were replaced in explicit terms of α and β. In the course of the algebraic reductions between equation (5.51) and the first universal time equation (5.93), all factors of $\sqrt{\alpha^2+\beta^2-1}$ were cancelled, as were all denominator factors of $(\alpha^2+\beta^2-1)$. In the present case performing the same type of reductions would be a daunting task, to say the least. Not only are the functions H_1, H_2 and H_3 (and their derivatives) very extensive functions of C_1 and C_2 , but H_2 and H_3 are transcendental in C_1 and C_2. That is, one is faced with reducing terms of $\tan^{-1}(-C_1)$ and $\tan^{-1}(-C_2)$ besides complicated rational algebraic terms of C_1 and C_2, and this is required merely to compute the numerator factors N_k. After that, there still remains the reduction of the final expressions (8.114), (8.131) and (8.147). A second obstacle to practical computation with these formulae is that they are all based on tangent of half the transfer angle and are therefore usable only for transfer angles up to 180° . Accomodating transfer angles up to 360° would necessitate introducing tangent of one-fourth the angle in (8.114), (8.131) and (8.147), complicating the algebra even more.

In view of these difficulties, the development of practical forms of equations (8.114), (8.131) and (8.147) requires a special study of its own, even though it involves only algebraic manipulations. It happens that the problems of main interest in this study, the initial value and boundary value problems of orbital motion, do not require the explicit calculation of perturbations of the time. If the given time of flight is used to generate a nominal Keplerian solution, then when perturbations of this solution are considered the perturbation of time is given as zero. This constraint results in a perturbation correction of the transfer angle (the independent variable η) which is much easier than time perturbations to compute.

The details of these problems will be considered in the next two chapters of this study. Before that, the present chapter will conclude by presenting the first-order secular perturbations of the regular elements in the η -domain.

Exact First-Order Secular Solutions

The first-order η-domain solutions for the regular elements given earlier in this chapter are not suitable for every type of perturbation analysis. If one is concerned with motion over only several revolutions then the approximations introduced so far should remain valid: the variations of the elements will be no larger than $O(\varepsilon)$. However, if one attempts to compute the variations over many revolutions then the secular terms, which are present in every element except the angular momentum h, will eventually grow larger than $O(\varepsilon)$, invalidating the results. The secular terms arose from the simple Picard-type integration method used to derive the formulae, so it should not be expected that the results would hold beyond some small neighborhood of $\eta = 0$. Typically, more Picard iterations are required to approximate the solution as η increases. But, since the labor involved in the Picard process increases dramatically with each succeeding iteration, it is usually not practical in celestial mechanics applications to pursue more than one iteration. It is clearly not practical in the present case. Moreover, the secular terms merely give rise to polynomials in η and "mixed-secular" terms on succeeding iterations and nothing is done to eliminate the unboundedness responsible for invalidating the results outside some neighborhood containing $\eta = 0$. Hence, if one wishes to examine the variations of the elements over large intervals of η then some different type of approximate integration method is needed.

In this section it is proposed to derive average rates of change of the elements and then to integrate the resulting differential equations in hopes of obtaining a larger domain of validity than is available with the Picard method. The averaging operation is prescribed as follows. If there is a function $q(x)$ whose rate of change $q'(x)$ is known, then the average rate of change between $x = a$ and $x = b$ is

$$< q'(x) > = \frac{1}{b-a}\int_a^b \frac{dq(x)}{dx}\,dx = \frac{q(b) - q(a)}{b-a} \tag{8.148}$$

Calculating the exact average rate of change requires knowledge of the function $q(x)$ itself, that is, of the solution of the differential equation giving the instantaneous rate of change $q'(x)$. Of course, the exact solutions of the orbital element differential equations are not available, but approximate solutions valid through $O(\varepsilon)$ are given in equations (8.24a), (8.26), (8.28), (8.30) and (8.32). If such a first-order solution is averaged according to (8.148) the result will be an approximate average rate which should differ from the exact average rate by no more than $O(\varepsilon^2)$. The approximate average rates derived in this manner are said to be "first-order averaged"; they depend on the first-order Picard-type solution. Higher order, more accurate, averages could be obtained if higher-order Picard iterations were available.

In calculating first-order average rates for the regular elements, it is necessary to select some interval of integration. It happens to be particularly natural and useful to select the interval from $a = 0$ to $b = 2\pi$, which corresponds to averaging over one revolution of the initial osculating orbit. This choice eliminates all the terms containing $\sin\eta$ and $\cos\eta$ in the above-mentioned rate equations, leaving only the secular terms themselves. Then simply collecting terms (the terms marked with an asterisk in preceding formulae) produces the following set of equations.

$$< (h^2)' > = 0 \tag{8.149}$$

$$<\underline{\xi}_0'> = +\frac{3\varepsilon\mu}{2h^4}\left[\underline{k}\cdot(\underline{\xi}_0\times\underline{\zeta}_0)\right](\underline{\zeta}_0\cdot\underline{k})(\underline{\xi}_0\times\underline{\zeta}_0) \tag{8.150}$$

$$<\underline{\zeta}_0'> = -\frac{3\varepsilon\mu}{2h^4}\left[\underline{k}\cdot(\underline{\xi}_0\times\underline{\zeta}_0)\right](\underline{\xi}_0\cdot\underline{k})(\underline{\xi}_0\times\underline{\zeta}_0) \tag{8.151}$$

$$<u_0'> = -\frac{3\varepsilon\mu}{2h^4}\left[1-\frac{3}{2}\left[(\underline{\xi}_0\cdot\underline{k})^2+(\underline{\zeta}_0\cdot\underline{k})^2\right]\right]w_0 \tag{8.152}$$

$$<w_0'> = +\frac{3\varepsilon\mu}{2h^4}\left[1-\frac{3}{2}\left[(\underline{\xi}_0\cdot\underline{k})^2+(\underline{\zeta}_0\cdot\underline{k})^2\right]\right]\left[u_0-\frac{\mu}{h^2}\right] \tag{8.153}$$

These equations are now to be treated as though they were ordinary differential equations: the left-hand sides are taken to be the instantaneous rates of change of the averaged elements rather than averaged rates of the instantaneous elements. This step requires some comment. As is always the case when averaging is employed, the new (non-Picard) approximation will be obtained at the price of neglecting all periodic phenomena, regardless of amplitude, whose periods are a submultiple of the averaging interval. In the present case, it so happens that *every* periodic term in the first-order Picard solution is 2π-periodic and so is neglected in the secular differential equations (8.149) through (8.153). This was the reason for choosing to average over one revolution, yet the approximation appears to be rather drastic. There are, in fact, great differences between the properties of the original element equations (8.23), (8.25), (8.27), (8.29) and (8.31) and the properties of the secular system (8.149) through (8.153). For example, the former are η-dependent while the latter are autonomous. But it should not be expected that the secular differential equations be traceable back to Newton's law of motion. Obviously, average values of a quantity need not obey the same "law of motion" that the instantaneous values of that quantity obey. In using the method of averaging, one hopes only that the solution of the secular system will remain close to the exact average value of the exact solution of the original element differential equations. The mathematical basis of this hope is still a matter of research in nonlinear mechanics (see Taff, 1985, chapter 9; and especially Arnol'd, 1978, sections 51 and 52). However, for the problem of J_2-perturbed satellite motion it is a matter of experience that averaging leads to correct and useful results for a variety of element sets. Most choices of elements lead to a rather complicated secular system of equations which can be solved only by numerical means. The above secular system for the regular Burdet-type elements is notable not only for its conciseness and generality but also because it can be solved exactly without further approximation. The details of the solution will now be discussed.

Equation (8.149) shows that the angular momentum magnitude is a constant in secular motion, a fact which greatly simplifies the remaining equations. Now note that the epochal unit vector equations (8.150) and (8.151) do not involve the elements u_0 and w_0. Thus the basic approach should be to solve (8.150) and (8.151) and then to insert that solution into the coefficients of (8.152) and (8.153). In order to solve the vector equations it is convenient to augment them with an additional equation for the epochal unit normal vector. Let

$$\underline{n}_0 = \underline{\xi}_0 \times \underline{\zeta}_0 \tag{8.154}$$

so that

$$\underline{n}_0' = \underline{\xi}_0' \times \underline{\zeta}_0 + \underline{\xi}_0 \times \underline{\zeta}_0' \tag{8.155}$$

Substituting from (8.150) and (8.151) then allows the set of vector equations to be written as

$$\underline{\xi}_0' = +\frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\, \underline{n}_0 \tag{8.156}$$

$$\underline{\zeta}_0' = -\frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\xi}_0 \cdot \underline{k})\, \underline{n}_0 \tag{8.157}$$

$$\underline{n}_0' = -\frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k})\, \underline{\xi}_0 + \frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\xi}_0 \cdot \underline{k})\, \underline{\zeta}_0 \tag{8.158}$$

Some information about the secular motion of the osculating plane can be obtained from these equations by inspection since they have the form of Poisson's kinematical equations. Equations of this type are discussed by Kane, *et al.* (1983, section 1.10) and in Appendix B of this study. Poisson's equations describe the rotational motion of an orthonormal vector triad in terms of its angular velocity components. The general form is usually quoted in the time domain and, for the unit vectors at hand, would be

$$\frac{d}{dt}\underline{\xi}_0 = \omega_3\ \underline{\zeta}_0 - \omega_2\ \underline{n}_0 \tag{8.159}$$

$$\frac{d}{dt}\underline{\zeta}_0 = \omega_1\ \underline{n}_0 - \omega_3\ \underline{\xi}_0 \tag{8.160}$$

$$\frac{d}{dt}\underline{n}_0 = \omega_2\ \underline{\xi}_0 - \omega_1\ \underline{\zeta}_0 \tag{8.161}$$

Here the coefficients ω_1, ω_2 and ω_3 are the components of the angular velocity of the $(\underline{\xi}_0, \underline{\zeta}_0, \underline{n}_0)$ triad resolved along $\underline{\xi}_0$, $\underline{\zeta}_0$ and $\underline{n}_0$ respectively. In the secular η domain the motion of the triad is governed by equations (8.156) through (8.159), and it is easy to see that the "angular velocities" are

$$\omega_1 = -\frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\xi}_0 \cdot \underline{k}) \ , \qquad \omega_2 = -\frac{3\varepsilon\mu}{2h^4}(\underline{n}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{k}) \ , \qquad \omega_3 = 0 \tag{8.162}$$

Thus the axis of secular rotation of the triad always lies in the osculating plane itself. There is no instantaneous secular rotation about the epochal unit normal vector. Also by inspection, it can be seen that for an initially polar orbit $(\underline{n}_0 \cdot \underline{k} = 0)$ the orientation of the triad does not change secularly. The same is true for an initially equatorial orbit $(\underline{\xi}_0 \cdot \underline{k} = 0,\ \underline{\zeta}_0 \cdot \underline{k} = 0)$. Finally, the Poisson form of equations (8.156) through (8.158) guarantees that the exact solution vectors will always form an orthonormal triad. Very seldom in practical problems of dynamics can the Poisson equations be solved in exact analytical terms, but they can be in this case, and one is thereby assured that no such limitations as were encountered with the Picard solution will occur in this analysis. This fact should be appreciated in advance because the form of the solution turns out to be such that a direct verification of orthonormality would be tedious.

Besides the angular momentum magnitude h, there is another constant of the secular motion which simplifies the equations. Forming the scalar product of equation (8.158) with the constant unit polar vector $\underline{k}$ produces

$$\underline{n}_0' \cdot \underline{k} = (\underline{n}_0 \cdot \underline{k})' = 0 \qquad \rightarrow \qquad (\underline{n}_0 \cdot \underline{k}) = K = \cos I \tag{8.163}$$

The constant K (not to be confused with the function $K(\eta; \alpha, \beta)$ used elsewhere in this study) is just the cosine of the orbital inclination I, demonstrating the well known fact that the inclination does not change secularly in J_2-perturbed motion. The constancy of h and K together implies that the projection of the angular momentum vector on the polar axis is constant. In turn, this fact is a partial check on the validity of the method of averaging in this problem because the polar component of angular momentum is actually conserved for J_2-perturbed motion, as can be verified by substituting the perturbing force expression (8.6) into the original equation of motion (1.1).

Now it is convenient to define the quantity

$$\omega = \frac{3\varepsilon\mu K}{2h^4} \tag{8.164}$$

and rewrite the unit vector equations as

$$\underline{\xi}_0' = +\,\omega(\underline{\zeta}_0 \cdot \underline{k})\,\underline{n}_0 \tag{8.165}$$

$$\underline{\zeta}_0' = -\,\omega(\underline{\xi}_0 \cdot \underline{k})\,\underline{n}_0 \tag{8.166}$$

$$\underline{n}_0' = -\,\omega(\underline{\zeta}_0 \cdot \underline{k})\,\underline{\xi}_0 + \omega(\underline{\xi}_0 \cdot \underline{k})\,\underline{\zeta}_0 \tag{8.167}$$

The constant factor ω has a physical interpretation which can be made as follows. Rewrite the latter equation as

$$\underline{n}_0' = -\,\omega\left[\underline{\xi}_0(\underline{k} \cdot \underline{\zeta}_0) - \underline{\zeta}_0(\underline{k} \cdot \underline{\xi}_0)\right] \tag{8.168}$$

A vector triple-product identity converts this into the form

$$\underline{n}_0' = -\omega\underline{k} \times \underline{n}_0 \tag{8.169}$$

The vector $(\underline{k} \times \underline{n}_0)$ is sometimes known as the nodal vector since it lies in the equatorial plane and points along the lines of nodes in the direction of the ascending node of the orbit. In secular motion it has constant magnitude, as can be seen by using first a scalar triple-product identity and then a vector triple-product identity:

$$(\underline{k} \times \underline{n}_0) \cdot (\underline{k} \times \underline{n}_0) = (\underline{k} \cdot \underline{k})(\underline{n}_0 \cdot \underline{n}_0) - (\underline{n}_0 \cdot \underline{k})^2 = 1 - K^2 \tag{8.170}$$

Hence the magnitude of the nodal vector is equal to the sine of the orbital inclination. Now by forming the vector product of $\underline{k}$ with (8.169), one obtains

$$\underline{k} \times \underline{n}_0' = (\underline{k} \times \underline{n}_0)' = -\omega\underline{k} \times (\underline{k} \times \underline{n}_0) \tag{8.171}$$

which reduces to

$$(\underline{k} \times \underline{n}_0)' = -\omega K\underline{k} + \omega\underline{n}_0 \tag{8.172}$$

Differentiate this expression, noting that the first term on the right is a constant, and then substitute for $\underline{n}_0'$ from (8.169). The result is the simple oscillator equation

$$(\underline{k} \times \underline{n}_0)'' + \omega^2(\underline{k} \times \underline{n}_0) = \underline{0} \tag{8.173}$$

Because the nodal vector has constant magnitude, this equation implies that it rotates in the equatorial plane with constant angular speed ω. During one orbital time period T of the initial osculating orbit, the satellite will travel through an arc in space corresponding to 2π radians of true anomaly in the initial osculating plane and the nodal vector will rotate through $2\pi\omega$ radians in the equatorial plane. Hence the time-domain average nodal rotation rate is

$$\frac{2\pi\omega}{T} = \frac{3}{2}\left(\frac{2\pi}{T}\right)\frac{\varepsilon\mu K}{2h^4} = \frac{3}{2}\left(\frac{2\pi}{T}\right)\frac{J_2R^2\cos I}{a^2(1-e^2)^2} \tag{8.174}$$

where a is the semimajor axis of the orbit and e is the eccentricity, and the fundamental relation (1.18) has been used in the form $h^2 = \mu a(1 - e^2)$. The right-hand side of (8.174) is precisely the magnitude of the average nodal regression rate as it is usually quoted in terms of classical elements. The sense of rotation of the nodal vector can be determined from (8.171). If $\omega > 0$ then the sense of rotation will be clockwise as seen from the north polar direction: the node regresses toward the west. This occurs for inclinations between 0° and 90°, that is, for direct orbits. If $\omega < 0$ then the sense of rotation will be counterclockwise as seen from the north polar direction: the node progresses toward the east. This occurs for inclinations between 90° and 180° , that is, for retrograde orbits. Interestingly, the nodal rotation rate ω is well defined even for equatorial orbits though the position of the node itself is then undefined. This befits the appearance of ω as a parameter in the regular element equations (8.165) through (8.167).

The solution now proceeds for the inertial components of the epochal unit vector triad. The $\underline{k}$-components of equations (8.165) and (8.166) are

$$\underline{\xi}_0' \cdot \underline{k} = +\,\omega(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{k}) \tag{8.175}$$

$$\underline{\zeta}_0' \cdot \underline{k} = -\,\omega(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{k}) \tag{8.176}$$

These equations reduce to the simple linear system

$$(\underline{\xi}_0 \cdot \underline{k})' = +\,\omega K(\underline{\zeta}_0 \cdot \underline{k}) \tag{8.177}$$

$$(\underline{\zeta}_0 \cdot \underline{k})' = -\,\omega K(\underline{\xi}_0 \cdot \underline{k}) \tag{8.178}$$

whose solution is easily found to be

$$(\underline{\xi}_0 \cdot \underline{k}) = +\,(\underline{\xi}_0(0) \cdot \underline{k})\cos(\omega K\eta) + (\underline{\zeta}_0(0) \cdot \underline{k})\sin(\omega K\eta) \tag{8.179}$$

$$(\underline{\zeta}_0 \cdot \underline{k}) = -\,(\underline{\xi}_0(0) \cdot \underline{k})\sin(\omega K\eta) + (\underline{\zeta}_0(0) \cdot \underline{k})\cos(\omega K\eta) \tag{8.180}$$

The equatorial components are found as follows. Let $\underline{i}$ and $\underline{j}$ be inertial orthogonal unit vectors in the equatorial plane. Their direction is arbitrary due to the symmetry of the J_2 force field, but typically $\underline{i}$ points toward the vernal equinox. In any case, the $(\underline{i}, \underline{j}, \underline{k})$ triad is to be right-handed in the sense $\underline{i} \times \underline{j} = \underline{k}$. Then the equatorial components of equations (8.165) and (8.166) can be written as

$$(\underline{\xi}_0 \cdot \underline{i})' = +\,\omega(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) \tag{8.181}$$

$$(\underline{\xi}_0 \cdot \underline{j})' = +\,\omega(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) \tag{8.182}$$

$$(\underline{\zeta}_0 \cdot \underline{i})' = -\,\omega(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) \tag{8.183}$$

$$(\underline{\zeta}_0 \cdot \underline{j})' = -\,\omega(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) \tag{8.184}$$

The $\underline{k}$ components of $\underline{\xi}_0$ and $\underline{\zeta}_0$ are already known. If the equatorial components of $\underline{n}_0$ were known then these equations could be solved as quadratures. Hence, an auxiliary solution for the epochal unit normal vector will developed first. The $\underline{i}$ component of equation (8.167) is

$$(\underline{n}_0 \cdot \underline{i})' = -\,\omega(\underline{\zeta}_0 \cdot \underline{k})(\underline{\xi}_0 \cdot \underline{i}) + \omega(\underline{\xi}_0 \cdot \underline{k})(\underline{\zeta}_0 \cdot \underline{i}) \tag{8.185}$$

Rewrite this equation as

$$(\underline{n}_0 \cdot \underline{i})' = \omega \underline{k} \cdot \left[\underline{\xi}_0(\underline{i} \cdot \underline{\zeta}_0) - \underline{\zeta}_0(\underline{i} \cdot \underline{\xi}_0)\right] \tag{8.186}$$

A vector triple-product identity brings this into the form

$$(\underline{n}_0 \cdot \underline{i})' = \omega \underline{k} \cdot (\underline{i} \times \underline{n}_0) \tag{8.187}$$

A scalar triple-product identity converts this to

$$(\underline{n}_0 \cdot \underline{i})' = \omega \underline{n}_0 \cdot (\underline{k} \times \underline{i}) \tag{8.188}$$

or simply

$$(\underline{n}_0 \cdot \underline{i})' = +\,\omega(\underline{n}_0 \cdot \underline{j}) \tag{8.189}$$

Analogous operations with the $\underline{j}$ component of equation (8.167) produce

$$(\underline{n}_0 \cdot \underline{j})' = -\,\omega(\underline{n}_0 \cdot \underline{i}) \tag{8.190}$$

These latter two equations are a simple linear system whose solution is easily found to be

$$(\underline{n}_0 \cdot \underline{i}) = +\,(\underline{n}_0(0) \cdot \underline{i}) \cos \omega\eta + (\underline{n}_0(0) \cdot \underline{j}) \sin \omega\eta \tag{8.191}$$

$$(\underline{n}_0 \cdot \underline{j}) = -\,(\underline{n}_0(0) \cdot \underline{i}) \sin \omega\eta + (\underline{n}_0(0) \cdot \underline{j}) \cos \omega\eta \tag{8.192}$$

Now the right-hand sides of (8.181) through (8.184) are known and those differential equations are reduced to quadratures. The algebra involved in substituting explicit terms into these equations is straightforward and need not be recorded. In the interest of simplifying the integrations, it proves to be convenient to use the following trigonometric identities and then to collect terms by sine and cosine.

$$\cos \omega K\eta \cos \omega\eta = \frac{1}{2}\cos(1-K)\omega\eta + \frac{1}{2}\cos(1+K)\omega\eta \tag{8.193}$$

$$\cos \omega K\eta \sin \omega\eta = \frac{1}{2}\sin(1+K)\omega\eta + \frac{1}{2}\sin(1-K)\omega\eta \tag{8.194}$$

$$\sin \omega K\eta \cos \omega\eta = \frac{1}{2}\sin(1+K)\omega\eta - \frac{1}{2}\sin(1-K)\omega\eta \tag{8.195}$$

$$\sin \omega K\eta \sin \omega\eta = \frac{1}{2}\cos(1-K)\omega\eta - \frac{1}{2}\cos(1+K)\omega\eta \tag{8.196}$$

Then equations (8.181) through (8.184) take the form

$$(\underline{\xi}_0 \cdot \underline{i})' = +\frac{1}{2}\omega\Big\{ + \big[(\underline{\zeta}_0(0) \cdot \underline{k})(\underline{n}_0(0) \cdot \underline{i}) - (\underline{\xi}_0(0) \cdot \underline{k})(\underline{n}_0(0) \cdot \underline{j})\big] \cos(1-K)\omega\eta$$

$$
\begin{aligned}
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\cos(1+K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\sin(1-K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\sin(1+K)\omega\eta\Big\}
\end{aligned}
\tag{8.197}
$$

$$
\begin{aligned}
(\underline{\xi}_0\cdot\underline{j})'=+\frac{1}{2}\omega\Big\{&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\cos(1-K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\cos(1+K)\omega\eta\\
&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\sin(1-K)\omega\eta\\
&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\sin(1+K)\omega\eta\Big\}
\end{aligned}
\tag{8.198}
$$

$$
\begin{aligned}
(\underline{\zeta}_0\cdot\underline{i})'=-\frac{1}{2}\omega\Big\{&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\cos(1-K)\omega\eta\\
&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\cos(1+K)\omega\eta\\
&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\sin(1-K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\sin(1+K)\omega\eta\Big\}
\end{aligned}
\tag{8.199}
$$

$$
\begin{aligned}
(\underline{\zeta}_0\cdot\underline{j})'=-\frac{1}{2}\omega\Big\{&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\cos(1-K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})\right]\cos(1+K)\omega\eta\\
&-\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})+(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\sin(1-K)\omega\eta\\
&+\left[(\underline{\zeta}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{j})-(\underline{\xi}_0(0)\cdot\underline{k})(\underline{n}_0(0)\cdot\underline{i})\right]\sin(1+K)\omega\eta\Big\}
\end{aligned}
\tag{8.200}
$$

These equations can be integrated immediately, completing the solution for the averaged elements $\underline{\xi}_0$ and $\underline{\zeta}_0$. However, the form of the results requires some discussion. For the special case $K = 0$ the above rates all vanish, as discussed before, and the vector elements are constant in secular motion. This is the case of a polar orbit, for which the inclination is 90°. For a general inclination, $-1 \le K \le +1$ and the denominators of the integrated forms of (8.197) through (8.200) will contain factors of $(1 - K)$ and $(1 + K)$. The formulae will contain apparent singularities at $K = +1$ (direct equatorial orbit) and $K = -1$ (retrograde equatorial orbit). However, the small denominators near $K = \pm 1$ will be compensated by vanishing values of the $\underline{k}$ components of $\underline{\xi}_0$ and $\underline{\zeta}_0$. As would be expected with these regular elements, the singularity is only apparent and can be avoided if the implicit dependence of the polar components on the value of K is properly taken into account. Nevertheless, the formulae as they stand would be numerically unusable in near-equatorial cases. It is necessary to rewrite the coefficients of sine and cosine in equations (8.197) through (8.200) in explicit terms of K to avoid introducing the apparent singularity. To this end, a digression must be made to obtain some identities among the vector components.

Notice that there are only four different coefficient factors in square brackets in the above equations: the coefficients listed in equation (8.197) reappear with only different signs and relative locations in (8.198) through (8.200). The following unit vector identities provide a means to recast the four coefficient factors in terms of K and equatorial components of $\underline{\xi}_0$ and $\underline{\zeta}_0$.

$$\underline{n}_0 \cdot \underline{n}_0 = 1 = (\underline{n}_0 \cdot \underline{i})^2 + (\underline{n}_0 \cdot \underline{j})^2 + (\underline{n}_0 \cdot \underline{k})^2 \tag{8.201}$$

from which

$$(\underline{n}_0 \cdot \underline{i})^2 + (\underline{n}_0 \cdot \underline{j})^2 = 1 - K^2 \tag{8.202}$$

Also

$$\underline{\xi}_0 \cdot \underline{n}_0 = 0 = (\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{k}) \tag{8.203}$$

from which

$$(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j}) = -\,K(\underline{\xi}_0 \cdot \underline{k}) \tag{8.204}$$

Also

$$\underline{\zeta}_0 \cdot \underline{n}_0 = 0 = (\underline{\zeta}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\zeta}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j}) + (\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{k}) \tag{8.205}$$

from which

$$(\underline{\zeta}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\zeta}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j}) = -\,K(\underline{\zeta}_0 \cdot \underline{k}) \tag{8.206}$$

The epochal vectors are also related to each other through vector products.

$$\underline{\xi}_0 \times \underline{n}_0 = -\,\underline{\zeta}_0 \tag{8.207}$$

The $\underline{k}$ component of this equation is

$$(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{j}) - (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{i}) = -\,(\underline{\zeta}_0 \cdot \underline{k}) \tag{8.208}$$

Also

$$\underline{\zeta}_0 \times \underline{n}_0 = +\,\underline{\xi}_0 \tag{8.209}$$

The $\underline{k}$ component is

$$(\underline{\zeta}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{j}) - (\underline{\zeta}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{i}) = (\underline{\xi}_0 \cdot \underline{k}) \tag{8.210}$$

The left-hand sides of (8.204), (8.206), (8.208) and (8.210) can be combined as follows to obtain four other identities.

$$\left[(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j})\right](\underline{n}_0 \cdot \underline{i}) + \left[(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{j}) - (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{i})\right](\underline{n}_0 \cdot \underline{j})$$

$$= (\underline{\xi}_0 \cdot \underline{i})\left[(\underline{n}_0 \cdot \underline{i})^2 + (\underline{n}_0 \cdot \underline{j})^2\right] = (\underline{\xi}_0 \cdot \underline{i})(1 - K^2) \tag{8.211}$$

$$\left[(\underline{\xi}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{i})+(\underline{\xi}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{j})\right](\underline{n}_0\cdot\underline{i})-\left[(\underline{\xi}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{i})\right](\underline{n}_0\cdot\underline{j})$$

$$=(\underline{\xi}_0\cdot\underline{j})\left[(\underline{n}_0\cdot\underline{j})^2+(\underline{n}_0\cdot\underline{i})^2\right]=(\underline{\xi}_0\cdot\underline{j})(1-K^2) \tag{8.212}$$

$$\left[(\underline{\zeta}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{i})+(\underline{\zeta}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{j})\right](\underline{n}_0\cdot\underline{i})+\left[(\underline{\zeta}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{j})-(\underline{\zeta}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{i})\right](\underline{n}_0\cdot\underline{j})$$

$$=(\underline{\zeta}_0\cdot\underline{i})\left[(\underline{n}_0\cdot\underline{i})^2+(\underline{n}_0\cdot\underline{j})^2\right]=(\underline{\zeta}_0\cdot\underline{i})(1-K^2) \tag{8.213}$$

$$\left[(\underline{\zeta}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{i})+(\underline{\zeta}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{j})\right](\underline{n}_0\cdot\underline{i})-\left[(\underline{\zeta}_0\cdot\underline{i})(\underline{n}_0\cdot\underline{j})-(\underline{\zeta}_0\cdot\underline{j})(\underline{n}_0\cdot\underline{i})\right](\underline{n}_0\cdot\underline{j})$$

$$=(\underline{\zeta}_0\cdot\underline{j})\left[(\underline{n}_0\cdot\underline{j})^2+(\underline{n}_0\cdot\underline{i})^2\right]=(\underline{\zeta}_0\cdot\underline{j})(1-K^2) \tag{8.214}$$

Then substituting the right-hand sides of (8.204), (8.206), (8.208) and (8.210) into these four identities produces the following forms.

$$-\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=(\underline{\xi}_0\cdot\underline{i})(1-K^2) \tag{8.215}$$

$$+\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=(\underline{\xi}_0\cdot\underline{j})(1-K^2) \tag{8.216}$$

$$-\left[K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=(\underline{\zeta}_0\cdot\underline{i})(1-K^2) \tag{8.217}$$

$$-\left[K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=(\underline{\zeta}_0\cdot\underline{j})(1-K^2) \tag{8.218}$$

It is these four equations which can be used to reconstruct the four coefficient factors in the rate equations (8.197) through (8.200). The method is simply to add and subtract equal terms in each coefficient and then to regroup terms to reproduce the left-hand sides of (8.215) through (8.218). For example, the first coefficient in (8.197) is rewritten according to

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})\right]=(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})$$

$$+K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})-K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j}) \tag{8.219}$$

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = -K\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right]$$

$$+\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - K(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] + \left[K(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] \tag{8.220}$$

Using equations (8.216) and (8.217), this can be written as

$$(1+K)\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = (\underline{\xi}_0 \cdot \underline{j})(1-K^2) - (\underline{\zeta}_0 \cdot \underline{i})(1-K^2) \tag{8.221}$$

and finally as

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = (1-K)\left[(\underline{\xi}_0 \cdot \underline{j}) - (\underline{\zeta}_0 \cdot \underline{i})\right] \tag{8.222}$$

The second coefficient in (8.197) is rewritten according to

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = (\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})$$

$$+K(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - K(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + K(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) - K(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) \tag{8.223}$$

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = +K\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right]$$

$$+\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - K(\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] - \left[K(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) - (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] \tag{8.224}$$

Again using equations (8.216) and (8.217), this can be written as

$$(1-K)\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = (\underline{\xi}_0 \cdot \underline{j})(1-K^2) + (\underline{\zeta}_0 \cdot \underline{i})(1-K^2) \tag{8.225}$$

and finally as

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j})\right] = (1+K)\left[(\underline{\xi}_0 \cdot \underline{j}) + (\underline{\zeta}_0 \cdot \underline{i})\right] \tag{8.226}$$

The third coefficient in (8.197) is rewritten according to

$$\left[(\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i})\right] = (\underline{\zeta}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{j}) + (\underline{\xi}_0 \cdot \underline{k})(\underline{n}_0 \cdot \underline{i})$$

$$+K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})-K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i}) \tag{8.227}$$

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=-K\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]$$

$$+\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]+\left[K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right] \tag{8.228}$$

Using equations (8.215) and (8.218), this can be written as

$$(1+K)\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=-(\underline{\xi}_0\cdot\underline{i})(1-K^2)-(\underline{\zeta}_0\cdot\underline{j})(1-K^2) \tag{8.229}$$

and finally as

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=-(1-K)\left[(\underline{\xi}_0\cdot\underline{i})+(\underline{\zeta}_0\cdot\underline{j})\right] \tag{8.230}$$

The fourth coefficient in (8.197) is rewritten according to

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})$$

$$+K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})-K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i}) \tag{8.231}$$

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=+K\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]$$

$$+\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+K(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]-\left[K(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})+(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right] \tag{8.232}$$

Again using equations (8.215) and (8.218), this can be written as

$$(1-K)\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=-(\underline{\xi}_0\cdot\underline{i})(1-K^2)+(\underline{\zeta}_0\cdot\underline{j})(1-K^2) \tag{8.233}$$

and finally as

$$\left[(\underline{\zeta}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{j})-(\underline{\xi}_0\cdot\underline{k})(\underline{n}_0\cdot\underline{i})\right]=-(1+K)\left[(\underline{\xi}_0\cdot\underline{i})-(\underline{\zeta}_0\cdot\underline{j})\right] \tag{8.234}$$

All of these expressions are identities which hold for a general value of η. Hence it is an easy matter to evaluate (8.222), (8.226), (8.230) and (8.234) initially and replace the four coefficients in the rate equations with the right-hand sides of these four equations. The results are

$$(\underline{\xi}_0 \cdot \underline{i})' = +\frac{1}{2}\omega\Big\{+(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) - (\underline{\zeta}_0(0)\cdot\underline{i})\big]\cos(1-K)\omega\eta$$

$$+(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) + (\underline{\zeta}_0(0)\cdot\underline{i})\big]\cos(1+K)\omega\eta$$

$$-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) + (\underline{\zeta}_0(0)\cdot\underline{j})\big]\sin(1-K)\omega\eta$$

$$-(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) - (\underline{\zeta}_0(0)\cdot\underline{j})\big]\sin(1+K)\omega\eta\Big\} \tag{8.235}$$

$$(\underline{\xi}_0 \cdot \underline{j})' = +\frac{1}{2}\omega\Big\{-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) + (\underline{\zeta}_0(0)\cdot\underline{j})\big]\cos(1-K)\omega\eta$$

$$-(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) - (\underline{\zeta}_0(0)\cdot\underline{j})\big]\cos(1+K)\omega\eta$$

$$-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) - (\underline{\zeta}_0(0)\cdot\underline{i})\big]\sin(1-K)\omega\eta$$

$$-(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) + (\underline{\zeta}_0(0)\cdot\underline{i})\big]\sin(1+K)\omega\eta\Big\} \tag{8.236}$$

$$(\underline{\zeta}_0 \cdot \underline{i})' = -\frac{1}{2}\omega\Big\{-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) + (\underline{\zeta}_0(0)\cdot\underline{j})\big]\cos(1-K)\omega\eta$$

$$+(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{i}) - (\underline{\zeta}_0(0)\cdot\underline{j})\big]\cos(1+K)\omega\eta$$

$$-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) - (\underline{\zeta}_0(0)\cdot\underline{i})\big]\sin(1-K)\omega\eta$$

$$+(1+K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) + (\underline{\zeta}_0(0)\cdot\underline{i})\big]\sin(1+K)\omega\eta\Big\} \tag{8.237}$$

$$(\underline{\zeta}_0 \cdot \underline{j})' = -\frac{1}{2}\omega\Big\{-(1-K)\big[(\underline{\xi}_0(0)\cdot\underline{j}) - (\underline{\zeta}_0(0)\cdot\underline{i})\big]\cos(1-K)\omega\eta$$

$$+ (1+K)\Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) + (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\cos(1+K)\omega\eta$$

$$+ (1-K)\Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) + (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1-K)\omega\eta$$

$$- (1+K)\Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) - (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1+K)\omega\eta\Big\} \tag{8.238}$$

These new rate equations can be integrated immediately to give results which are valid for orbits of any inclination, including equatorial orbits. The final results are

$$(\underline{\xi}_0\cdot\hat{\imath}) = (\underline{\xi}_0(0)\cdot\hat{\imath}) + \frac{1}{2}\Big\{ + \Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) - (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\sin(1-K)\omega\eta$$

$$+ \Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) + (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\sin(1+K)\omega\eta$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) + (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\big[1-\cos(1-K)\omega\eta\big]$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) - (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\big[1-\cos(1+K)\omega\eta\big]\Big\} \tag{8.239}$$

$$(\underline{\xi}_0\cdot\hat{\jmath}) = (\underline{\xi}_0(0)\cdot\hat{\jmath}) + \frac{1}{2}\Big\{ - \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) + (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1-K)\omega\eta$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) - (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1+K)\omega\eta$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) - (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\big[1-\cos(1-K)\omega\eta\big]$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) + (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\big[1-\cos(1+K)\omega\eta\big]\Big\} \tag{8.240}$$

$$(\underline{\zeta}_0\cdot\hat{\imath}) = (\underline{\zeta}_0(0)\cdot\hat{\imath}) - \frac{1}{2}\Big\{ - \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) + (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1-K)\omega\eta$$

$$+ \Big[(\underline{\xi}_0(0)\cdot\hat{\imath}) - (\underline{\zeta}_0(0)\cdot\hat{\jmath})\Big]\sin(1+K)\omega\eta$$

$$- \Big[(\underline{\xi}_0(0)\cdot\hat{\jmath}) - (\underline{\zeta}_0(0)\cdot\hat{\imath})\Big]\big[1-\cos(1-K)\omega\eta\big]$$

$$+\left[(\underline{\xi}_0(0)\cdot \underline{j}) + (\underline{\zeta}_0(0)\cdot \underline{i})\right]\left[1-\cos(1+K)\omega\eta\right]\Big\} \tag{8.241}$$

$$(\underline{\zeta}_0\cdot \underline{j}) = (\underline{\zeta}_0(0)\cdot \underline{j}) - \frac{1}{2}\Big\{-\left[(\underline{\xi}_0(0)\cdot \underline{j}) - (\underline{\zeta}_0(0)\cdot \underline{i})\right]\sin(1-K)\omega\eta$$

$$+\left[(\underline{\xi}_0(0)\cdot \underline{j}) + (\underline{\zeta}_0(0)\cdot \underline{i})\right]\sin(1+K)\omega\eta$$

$$+\left[(\underline{\xi}_0(0)\cdot \underline{i}) + (\underline{\zeta}_0(0)\cdot \underline{j})\right]\left[1-\cos(1-K)\omega\eta\right]$$

$$-\left[(\underline{\xi}_0(0)\cdot \underline{i}) - (\underline{\zeta}_0(0)\cdot \underline{j})\right]\left[1-\cos(1+K)\omega\eta\right]\Big\} \tag{8.242}$$

This completes the solution for the epochal unit vector triad in secular motion. Attention will now be given to the solution for the elements u_0 and w_0.

Because the angular momentum magnitude h is constant in secular motion, the rate equations (8.152) and (8.153) are conveniently rewritten as

$$\left[u_0 - \frac{\mu}{h^2}\right]' = -\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}\left[(\underline{\xi}_0\cdot \underline{k})^2 + (\underline{\zeta}_0\cdot \underline{k})^2\right]\right]w_0 \tag{8.243}$$

$$w_0' = +\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}\left[(\underline{\xi}_0\cdot \underline{k})^2 + (\underline{\zeta}_0\cdot \underline{k})^2\right]\right]\left[u_0 - \frac{\mu}{h^2}\right] \tag{8.244}$$

Actually, for the same reason, it is easy to obtain secular rate equations for the dimensionless elements α and β. Recall

$$\alpha = \frac{h^2 u_0}{\mu} - 1 \tag{8.245}$$

$$\beta = \frac{h^2 w_0}{\mu} \tag{8.246}$$

Then the rate equations are

$$\alpha' = -\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right]\right]\beta \tag{8.247}$$

$$\beta' = +\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right]\right]\alpha \tag{8.248}$$

Now the solutions for the $\underline{k}$ components of ξ_0 and ζ_0 can be inserted into these equations from (8.179) and (8.180). That operation turns out to simplify the equations considerably:

$$\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right] = (\underline{\xi}_0(0) \cdot \underline{k})^2 + (\underline{\zeta}_0(0) \cdot \underline{k})^2 = \text{ constant} \tag{8.249}$$

Therefore the linear equations (8.247) and (8.248) can be solved immediately. Before the solution is presented, it proves to be convenient to express the $\underline{k}$ components of $\underline{\xi}_0$ and $\underline{\zeta}_0$ in terms of K, much as was done in the solution of the epochal unit vector equations above. In this case there is no problem of eliminating a singularity, but the manipulations are useful nevertheless. First, square and add the four identities (8.204), (8.206), (8.208) and (8.210) to obtain

$$K^2(\underline{\xi}_0 \cdot \underline{k})^2 + K^2(\underline{\zeta}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2 + (\underline{\xi}_0 \cdot \underline{k})^2 =$$

$$\left[(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j})\right]^2 + \left[(\underline{\zeta}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{i}) + (\underline{\zeta}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{j})\right]^2$$

$$+ \left[(\underline{\xi}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{j}) - (\underline{\xi}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{i})\right]^2 + \left[(\underline{\zeta}_0 \cdot \underline{i})(\underline{n}_0 \cdot \underline{j}) - (\underline{\zeta}_0 \cdot \underline{j})(\underline{n}_0 \cdot \underline{i})\right]^2 \tag{8.250}$$

Upon expanding the right-hand side, one finds that all the cross terms cancel, leaving

$$(1 + K^2)\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right] =$$

$$\left[(\underline{n}_0 \cdot \underline{i})^2 + (\underline{n}_0 \cdot \underline{j})^2\right]\left[(\underline{\xi}_0 \cdot \underline{i})^2 + (\underline{\xi}_0 \cdot \underline{j})^2 + (\underline{\zeta}_0 \cdot \underline{i})^2 + (\underline{\zeta}_0 \cdot \underline{j})^2\right] \tag{8.251}$$

Now using the unit vector identities

$$\underline{\xi}_0 \cdot \underline{\xi}_0 = 1 \qquad \rightarrow \qquad (\underline{\xi}_0 \cdot \underline{i})^2 + (\underline{\xi}_0 \cdot \underline{j})^2 = 1 - (\underline{\xi}_0 \cdot \underline{k})^2 \tag{8.252}$$

$$\underline{\zeta}_0 \cdot \underline{\zeta}_0 = 1 \qquad \rightarrow \qquad (\underline{\zeta}_0 \cdot \underline{i})^2 + (\underline{\zeta}_0 \cdot \underline{j})^2 = 1 - (\underline{\zeta}_0 \cdot \underline{k})^2 \tag{8.253}$$

the relation (8.251) can be rewritten as

$$(1 + K^2)\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right] = (1 - K^2)\left[1 - (\underline{\xi}_0 \cdot \underline{k})^2 + 1 - (\underline{\zeta}_0 \cdot \underline{k})^2\right] \tag{8.254}$$

This reduces to the convenient expression

$$\left[(\underline{\xi}_0 \cdot \underline{k})^2 + (\underline{\zeta}_0 \cdot \underline{k})^2\right] = 1 - K^2 \tag{8.255}$$

The secular rate equations for α and β become

$$\alpha' = -\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}(1 - K^2)\right]\beta \tag{8.256}$$

$$\beta' = +\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}(1 - K^2)\right]\alpha \tag{8.257}$$

Letting

$$\omega_0 = +\frac{3\varepsilon\mu}{2h^4}\left[1 - \frac{3}{2}(1 - K^2)\right] \tag{8.259}$$

the solution is

$$\alpha = \alpha(0)\cos\omega_0\eta - \beta(0)\sin\omega_0\eta \tag{8.259}$$

$$\beta = \alpha(0)\sin\omega_0\eta + \beta(0)\cos\omega_0\eta \tag{8.260}$$

This completes the solution for the secular variations of the elements in the η domain. Some insight into the latter equations can be obtained by considering the Laplace vector expressed in the form given in equation (7.21) of Chapter 7:

$$\underline{B} = \mu(\alpha\underline{\xi}_0 + \beta\underline{\zeta}_0) \tag{8.261}$$

When secular variations of the classical elements are considered, the motion of the perigee point is usually reckoned in terms of the average rate of change of the argument of perigee. Here the motion of the perigee is given directly in the inertial coordinate system in terms of the secular variations of the regular elements on the right-hand side of (8.261). It is not particularly convenient to calculate the rate of change of the argument of perigee in terms of the present variables because the angle in question is the angle between the Laplace vector $\underset{\sim}{B}$ and the nodal vector $\underset{\sim}{k} \times \underset{\sim}{n}_0$, both of which are moving. It happens that the average time-domain rate of change of the argument of perigee is equal to

$$\frac{2\pi}{T}(\omega K + \omega_0) = \left(\frac{2\pi}{T}\right)\frac{3\varepsilon\mu}{2h^4}\left[K^2 + 1 - \frac{3}{2}(1 - K^2)\right] =$$

$$\left(\frac{2\pi}{T}\right)\frac{3\varepsilon\mu}{4h^4}\left[4 - 5(1 - K^2)\right] = \frac{3}{4}\left(\frac{2\pi}{T}\right)\frac{J_2R^2}{a^2(1 - e^2)^2}\left[4 - 5\sin^2 I\right] \tag{8.262}$$

The sign of the expression on the right-hand side is correct as it stands. The argument of perigee advances if $\sin^2 I < \frac{4}{5}$ and regresses if $\sin^2 I > \frac{4}{5}$, which is a well known classical result. Notice that both frequency terms on the left-hand side are related to rotations in the orbit plane itself: ωK is the projection of the nodal rotation rate on the orbit plane, while ω_0 is associated only with the in-plane elements α and β. Finally, an additional property of the secular motion can be deduced from the Laplace vector since the magnitude of that vector is just μe . Forming the scalar product of equation (8.261) with itself and substituting for α and β from (8.259) and (8.260) produces the result

$$e^2 = \alpha^2 + \beta^2 = \alpha(0)^2 + \beta(0)^2 \tag{8.263}$$

That is, the orbital eccentricity is constant in secular motion. The fundamental relation $h^2 = \mu a(1 - e^2)$ then shows that the length of semimajor axis is also constant in secular motion. The picture that emerges is that the size and shape of the orbit do not change secularly but the orbit plane precesses around the polar axis while the line of apsides (along which the Laplace vector lies) rotates within that plane. Of course, these are all well known effects, but the present analysis has presented the J_2 secular motion in terms of regular elements in a form that is especially convenient for calculation.

Summary

This chapter has presented a first-order solution of the J_2 -perturbed satellite motion by a variation-of-parameters approach. The particular advantage of using true anomaly η as the independent variable is evident because the complete perturbations are given in finite terms for all orbital eccentricities and orientations. In fact, only powers and products of $\sin\eta$ and $\cos\eta$, besides secular terms, are present in the final formulae, making the expressions well suited for computation. It is noteworthy that the calculation of the J_2 perturbations of the elements requires only a single transcendental function evaluation, namely $\tan^{-1}z$ for the secular (Picard) terms, if $\sin\eta$ and $\cos\eta$ are replaced in terms of $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$. Moreover, $\tan^{-1}z$ can be computed from the continued fraction C_2 according to equations (5.105), (5.108) and (5.109) of Chapter 5. One evaluation of the function $C_2(x)$ or $C_2(y)$ is needed to compute the Keplerian motion and then one evaluation of $C_2(z^2)$ is needed for the secular perturbations. Hence, only a single transcendental function routine needs to be programmed even in the perturbed case. All other calculations involve only square roots and rational algebraic expressions.

In the present formulation, time is a dependent variable and so perturbations of the time must be considered. The results available from the present analysis are not not satisfactory for analytical work due to the complicated form of the unperturbed true-anomaly time equation. Variation-of-parameters formulae are given for two time elements; one, the variation of time at epoch; and the other, the variation of an η-domain analog of mean anomaly. Neither of these differential equations is simple enough to invite even a first-order Picard-type treatment. However, there should be no particular difficulty in using these equations for numerical work. The equation for time at epoch can be used for all orbits, though it suffers from mixed secular effects as large values of η are pursued. The other equation is valid only for elliptic orbits, but contains no mixed secular terms. Besides the variation-of-parameters approach to calculating perturbed time, a straightforward perturbation technique is attempted. Instead of integrating approximately some exact expression for the time, the approach is to evaluate some approximate expression exactly. A quadrature accurate through first order in J_2 is developed and the integration is carried to formal completion by the method of expansion in partial fractions. The method of partial fractions is especially well suited for this problem because by it one is able to handle the complicated and extensive numerator expressions which occur in the integrands. Unfortunately, many of the intermediate parameters and coefficients required in this method do not appear in a form suitable for general computation. The reduction of the final formulae to useful working terms would be a large job in itself (if indeed it can be done) and the problem of the final algebraic reductions is left open.

A very useful property of the Burdet-type regular elements is exploited in the last section of this chapter. When the exact rate equations for the elements are averaged to first order, the resulting autonomous differential equations can be solved exactly, giving the secular and long-period variations of the orbit in periodic terms which are valid for indefinitely large values of η. (Of course, precession and nutation of Earth's rotation axis have been neglected.) This type of secular solution is not often obtainable in celestial mechanics, or in any type of perturbation analysis, so it has intrinsic interest for that reason besides its practical application. Vitins (1978) was able to obtain exact secular solutions for the J_2 problem using special variables designed for that purpose, but here the solution is given directly in terms of elements which arise naturally in the course of regularizing the the two-body problem. The results should be useful in a variety of problems where it is essential to know the behavior of the orbit over hundreds or thousands of revolutions. Especially, the readily computable form of the secular solution given here could be used to generate a realistically moving reference orbit for accurate long-time numerical computation of perturbations. Alfriend and Velez (1975) give an example of the usefulness of this approach based on the secular variations of a set of canonical elements. Their canonical elements are not regular so the present solution might lead to a more generally practical method. Finally, the secular variation of the time elements is left as an open problem in this study due to the difficulty of averaging the rate equations.

Chapter 9. The Perturbed Initial Value Problem

Introduction

The results of orbital perturbation analysis are usually presented in a form which is suitable for solving the initial value problem of perturbed motion. For example, all of the articles cited in the Introduction to Chapter 8 assume that a satellite's position and velocity (or its orbital elements) are known at some epoch and that one needs to compute its position and velocity (or its orbital elements) at some other instant of time. Likewise, most of the classical perturbation analyses of 17^{th} -, 18^{th}- and 19^{th} -century dynamical astronomy were directed toward predicting future observations of the natural bodies in the solar system. In contrast, explicit formulations of perturbed versions of the Gauss/Lambert boundary value problem seem to be recent developments, motivated by the need to target and guide artificial satellites very accurately. The boundary value problem of perturbed motion will be discussed in the next chapter. The initial value problem, which is simpler, is discussed in this chapter and the solution will be presented in a variation-of-parameters form in terms of the regular η-domain orbital elements. Besides being important in its own right, the solution of the perturbed initial value problem is needed (at least in part) in the procedure for solving the boundary value problem. Consistent with the scope of this study, only first-order perturbations will be considered. Moreover, it will be assumed that the unperturbed version of the problem has been solved. The goal of this chapter is then to present explicit corrections to the position and velocity vectors calculated for the nominal Keplerian orbit. Within the restrictions of a first-order analysis, direct, noniterative formulae for these corrections can be obtained.

First-Order Perturbation Theory

Before the solution procedure is developed, it will be helpful to discuss the perturbed initial value problem in general terms. The Burdet-type regularization presented in Chapter 3 of this study resulted in a solution of the Keplerian motion, which can be written in the form

$$\underline{r} = \underline{R}(\underline{r}_0, \underline{v}_0, \eta) \qquad (9.1)$$

$$\underline{v} = \underline{V}(\underline{r}_0, \underline{v}_0, \eta) \qquad (9.2)$$

$$t = H(\underline{r}_0, \underline{v}_0, \eta) \qquad (9.3)$$

The position, velocity and time are given parametrically in terms of the true anomaly η. One advantage of introducing η is that the functions $\underline{R}$, $\underline{V}$ and H are all closed expressions. The time equation (9.3) involves a special transcendental function if it is to be universally valid, but in Chapter 5 it was shown how that function can be evaluated efficiently. In the initial value problem, values for t, $\underline{r}_0$ and $\underline{v}_0$ are given in order to compute $\underline{r}$ and $\underline{v}$. Evidently, one has only to solve the time equation (9.3) for η in terms of t and insert that value into (9.1) and (9.2). The details of this procedure were given in Chapter 6. In the boundary value problem, values are given for $\underline{r}$, $\underline{r}_0$, t and η in order to compute $\underline{v}_0$. Since $\underline{v}_0$ occurs implicitly in all three of the above formulae, the formulation of a solution procedure is more complicated than for the initial value problem. The details were given in Chapter 7.

A second advantage of introducing the true anomaly η as a parameter in the problem becomes evident when perturbed motion is considered. It is possible to regard either t or η as the independent variable, providing a useful flexibility in the solution procedures. Consider the variation-of-parameters method, in which one adopts the form of the Keplerian solution to describe the perturbed motion:

$$\underline{r} = \underline{r}^{(0)} + \Delta\underline{r} = \underline{R}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{9.4}$$

$$\underline{v} = \underline{v}^{(0)} + \Delta\underline{v} = \underline{V}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{9.5}$$

$$t = t^{(0)} + \Delta t = H(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{9.6}$$

In these equations, quantities with superscript (0) are defined by

$$\underline{r}^{(0)} = \underline{R}(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{9.7}$$

$$\underline{v}^{(0)} = \underline{V}(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{9.8}$$

$$t^{(0)} = H(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{9.9}$$

That is, these quantities represent motion along the initial osculating orbit. This motion can be considered as known, since it involves no consideration of perturbations, and the perturbed motion is to be reckoned as a departure from this nominal solution. Quantities representing the departure from Keplerian motion are indicated with a Δ symbol in the above equations. The implication is that the epochal position and velocity vectors, $\underline{r}_0$ and $\underline{v}_0$, as they appear in the Keplerian formulae, are variable in perturbed motion. Their initial values, $\underline{r}_0(0)$ and $\underline{v}_0(0)$, define the initial osculating orbit via equations (9.7) through (9.9). Thus the true anomaly in unperturbed motion, $\eta^{(0)}$, is the angle between $\underline{r}_0(0)$ and $\underline{r}^{(0)}$ and is measured in the initial osculating plane. The true anomaly in perturbed motion, $\eta^{(0)} + \Delta\eta$, is the angle between $\underline{r}_0(0) + \Delta\underline{r}_0$ and $\underline{r}^{(0)} + \Delta\underline{r}$ and is measured in the instantaneous osculating plane.

The actual variations of the elements $\underline{r}_0$ and $\underline{v}_0$ are governed by differential equations in which the rates of change have the same order of magnitude as the perturbing force. It is here that the choice of independent variable afforded by the presence of η can simplify the analysis. The following discussion about the choice of independent variable is designed to complement related discussions by Geyling and Westerman (1971, mainly sections 5.7 and 6.2.3). These authors present what is certainly one of the very few discussions, and possibly the only basically complete discussion, of the use of independent variables other than time in calculating perturbations of the elements. They describe in detail the use of the classical true anomaly (measured from the pericenter of the orbit), and also indicate what steps would be necessary if the classical eccentric anomaly were used. The following discussion is made in terms of the true anomaly η but it will be plain that identical reasoning applies for any anomaly or parameter which allows the Keplerian motion to be described by equations of the form (9.1) through (9.3). Far from belaboring apparently obvious facts, as Geyling and Westerman remark at the end of their section 5.7, such a discussion establishes some results which are prerequisite to a rigorous implementation of a first-order perturbation solution.

Suppose time is taken as the independent variable. The differential equations for the elements have the form

$$\frac{d}{dt}\underline{r}_0 = \varepsilon \underline{f}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{9.10}$$

$$\frac{d}{dt}\underline{v}_0 = \varepsilon \underline{g}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{9.11}$$

where ε is the perturbation parameter. These equations are derived by the variation-of-parameters method applied to the time-domain governing equation of motion (1.1) of Chapter 1. When the perturbing force depends explicitly on the time then t must appear as an additional argument on the right-hand sides. In the case of interest in this study it does not appear. In either case, for every value of the time t the corresponding value of the true anomaly η is determined from the time equation

$$t = H(\ \underline{r}_0(t),\ \underline{v}_0(t),\ \eta\ ) \tag{9.12}$$

in order to evaluate the right-hand sides of the differential equations. If the equations are to be integrated analytically, even to first order, the explicit time dependence of η must be introduced by means of this time equation, a complicated manipulation which cannot be carried out in closed form. The other viewpoint, suggested by the Burdet-type regularization, is that true anomaly be considered as the independent variable. The time equation is then

$$t = H(\ \underline{r}_0(\eta),\ \underline{v}_0(\eta),\ \eta\ ) \tag{9.13}$$

and the differential equations have the form

$$\frac{d}{d\eta}\underline{r}_0 = \varepsilon\, \underline{F}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{9.14}$$

$$\frac{d}{d\eta}\underline{v}_0 = \varepsilon\, \underline{G}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{9.15}$$

These equations are derived by the variation-of-parameters method applied to the η-domain governing equations of motion (3.117) through (3.120) of Chapter 3. When the perturbing force does not depend on time explicitly, it is usually easier to solve equations (9.14) and (9.15) (to first order, say) than the time-domain equations (9.10) and (9.11) because the time equation itself now does not enter into the integrations. Even if the perturbing force were time-dependent the analytical inversion of the time equation would not be required, but only the substitution of (9.13) into the right-hand sides of (9.14) and (9.15). For example, for the time-independent case, the first-order variation of $\underline{r}_0$ could be found as follows.

$$\frac{d}{d\eta}\underline{r}_0 = \varepsilon\, \underline{F}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \underline{v}_0(0) + \Delta\underline{v}_0,\ \eta\ ) \tag{9.16}$$

$$\Delta \underline{r}_0 = \varepsilon \int \left[\underline{F}(\underline{r}_0(0), \underline{v}_0(0), \eta) + \frac{\partial \underline{F}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{F}}{\partial \underline{v}_0} \Delta \underline{v}_0 + \cdots O(\varepsilon^2) \right] d\eta \tag{9.17}$$

Since the Δ quantities are $O(\varepsilon)$, the first-order expression is simply

$$\Delta \underline{r}_0 = \varepsilon \int \underline{F}(\underline{r}_0(0), \underline{v}_0(0), \eta)\, d\eta \tag{9.18}$$

with the definite integral being taken from 0 to η. Likewise,

$$\Delta \underline{v}_0 = \varepsilon \int \underline{G}(\underline{r}_0(0), \underline{v}_0(0), \eta)\, d\eta \tag{9.19}$$

with the same limits of integration. These two formulae for the element variations are simply quadratures reckoned in terms of the true anomaly on the instantaneous osculating orbit. However, since η differs from $\eta^{(0)}$ by only an $O(\varepsilon)$ amount, one may as well proceed with

$$\Delta \underline{r}_0 = \varepsilon \int \left[\underline{F}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)}) + \frac{\partial \underline{F}}{\partial \eta} \Delta \eta + \cdots O(\varepsilon^2) \right] d(\eta^{(0)} + \Delta \eta) \tag{9.20}$$

After neglecting terms of $\varepsilon \Delta \eta$ and higher, the first order expression is

$$\Delta \underline{r}_0 = \varepsilon \int \underline{F}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\, d\eta^{(0)} \tag{9.21}$$

with the definite integral being taken from 0 to $\eta^{(0)}$. Likewise,

$$\Delta \underline{v}_0 = \varepsilon \int \underline{G}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\, d\eta^{(0)} \tag{9.22}$$

with the same limits of integration. These two formulae for the element variations are quadratures reckoned in terms of the true anomaly on the initial osculating orbit, and they are formally identical with (9.18) and (9.19) above. That is, the first-order perturbations of the elements can be reckoned either in terms of the "perturbed true anomaly" η or the "unperturbed true anomaly" $\eta^{(0)}$. As observed by Geyling and Westerman, this fact is basic to orbital perturbation theory but is often overlooked or simply taken for granted. The fact is important in the present study because the independent variable used to develop the regularized equations of motion and all the subsequent perturbation analysis is η, not $\eta^{(0)}$. Yet in the first-order solution of the perturbed initial value problem the element variations are most conveniently computed in terms of $\eta^{(0)}$, not η. The first-order equivalence just established allows this change of variable to be done trivially but with all rigor. It is important to note also that, while the "unperturbed true anomaly" $\eta^{(0)}$ suffices to calculate the first-order variations of the elements, the variation of true anomaly itself due to perturbations does contribute first-order terms to the position, velocity and time on the perturbed path according to equations (9.4) through (9.6). The perturbation of true anomaly will be developed explicitly later.

Other ideas basic to the η-domain perturbation theory can be brought out by considering the exact relations between the time-domain element rates and the η-domain element rates. From the differential equations (9.10), (9.11), (9.14) and (9.15), and the associated time equations, one can derive relations among the functions $\underline{f}$, $\underline{g}$, $\underline{F}$ and $\underline{G}$. These relations describe the effect of changing the independent variable from time to true anomaly or *vice versa*. Specifically, element rates are transformed from one domain to another according to

$$\frac{d}{dt}(\cdot) = \frac{d}{d\eta}(\cdot)\frac{d\eta}{dt} \qquad \text{and} \qquad \frac{d}{d\eta}(\cdot) = \frac{d}{dt}(\cdot)\frac{dt}{d\eta} \tag{9.23}$$

If time is the original independent variable and the differential equations (9.10) and (9.11) are to be transformed to the η domain, differentiate the time equation (9.12) to obtain

$$\frac{dt}{dt} = 1 = \frac{\partial H}{\partial \eta}\frac{d\eta}{dt} + \frac{\partial H}{\partial \underline{r}_0}\frac{d}{dt}\underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0}\frac{d}{dt}\underline{v}_0 \tag{9.24}$$

so that

$$\frac{d\eta}{dt} = \frac{1}{\left(\frac{\partial H}{\partial \eta}\right)}\left[1 - \varepsilon\frac{\partial H}{\partial \underline{r}_0}\underline{f} - \varepsilon\frac{\partial H}{\partial \underline{v}_0}\underline{g}\right] \tag{9.25}$$

and

$$\frac{dt}{d\eta} = \frac{\frac{\partial H}{\partial \eta}}{1 - \varepsilon\frac{\partial H}{\partial \underline{r}_0}\underline{f} - \varepsilon\frac{\partial H}{\partial \underline{v}_0}\underline{g}} \tag{9.26}$$

Here the functions $\underline{f}$, $\underline{g}$ and the derivatives of H are functions of $\underline{r}_0(t)$, $\underline{v}_0(t)$ and η, with η being related to time by the time equation (9.12). Equation (9.25) is a time-domain differential equation for η and equation (9.26) is an η-domain differential equation for time, but both of these are merely exact differentials, integrating immediately back to the expression (9.12) from which they come. The differential equations (9.10) and (9.11) for the elements become

$$\frac{d}{d\eta}\underline{r}_0 = \frac{d}{dt}\underline{r}_0\frac{dt}{d\eta} = \frac{\varepsilon\frac{\partial H}{\partial \eta}\underline{f}}{1 - \varepsilon\frac{\partial H}{\partial \underline{r}_0}\underline{f} - \varepsilon\frac{\partial H}{\partial \underline{v}_0}\underline{g}} \tag{9.27}$$

$$\frac{d}{d\eta}\underline{v}_0 = \frac{d}{dt}\underline{v}_0 \frac{dt}{d\eta} = \frac{\varepsilon \dfrac{\partial H}{\partial \eta} \underline{g}}{1 - \varepsilon \dfrac{\partial H}{\partial \underline{r}_0} \underline{f} - \varepsilon \dfrac{\partial H}{\partial \underline{v}_0} \underline{g}} \tag{9.28}$$

Comparison with (9.14) and (9.15) gives the relations

$$\underline{E} = \frac{\dfrac{\partial H}{\partial \eta} \underline{f}}{1 - \varepsilon \dfrac{\partial H}{\partial \underline{r}_0} \underline{f} - \varepsilon \dfrac{\partial H}{\partial \underline{v}_0} \underline{g}} \tag{9.29}$$

$$\underline{G} = \frac{\dfrac{\partial H}{\partial \eta} \underline{g}}{1 - \varepsilon \dfrac{\partial H}{\partial \underline{r}_0} \underline{f} - \varepsilon \dfrac{\partial H}{\partial \underline{v}_0} \underline{g}} \tag{9.30}$$

The differential equations (9.27) and (9.28) are η-domain rate equations equivalent to (9.14) and (9.15). The reason for the transformation from time to true anomaly is, of course, to avoid the inversion (analytical or numerical) of the time equation (9.12) in the integrations. This purpose has been accomplished. But it must be observed that equations (9.27) and (9.28) are vastly more complicated than the original equations (9.10) and (9.11), the chief complication arising from the denominator expressions. In fact, these equations are vastly more complicated than the equivalent η -domain equations (9.14) and (9.15), and this is one of the strongest reasons why true anomaly regularization is superior to merely introducing true anomaly into the time-domain element equations. If one plans to use true anomaly as the independent variable in the element differential equations, it is better to transform the original governing equations of motion to the η-domain before the variation-of-parameters method is applied, rather than to transform the time-domain variation-of-parameters formulae to the η-domain. The former procedure leads to equations (9.14) and (9.15); the latter procedure leads to equations (9.27) and (9.28). The articles by Geyling and Westerman (1971, sections 5.7 and 6.2.3) discuss manipulations of only the latter type. Of course, equations of the type (9.27) and (9.28) have been used for perturbation analysis, but typically at the first order only; it is likely that no complete second-order analysis has ever been done for a realistic perturbing force using these equations. At the first order, one can neglect the complicated denominator terms, clearing away the main obstacle:

$$\frac{d}{d\eta}\underline{r}_0 = \varepsilon \frac{\partial H}{\partial \eta}\underline{f} + \cdots O(\varepsilon^2) \tag{9.31}$$

$$\frac{d}{d\eta}\underline{r}_0 = \varepsilon\left[\frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta) + \frac{\partial^2 H}{\partial \underline{r}_0 \partial \eta}\Delta\underline{r}_0\right.$$

$$\left. + \frac{\partial^2 H}{\partial \underline{v}_0 \partial \eta}\Delta\underline{v}_0 + \cdots O(\varepsilon^2)\right]\left[\underline{f}(\underline{r}_0(0), \underline{v}_0(0), \eta) + \frac{\partial \underline{f}}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial \underline{f}}{\partial \underline{v}_0}\Delta\underline{v}_0 + \cdots O(\varepsilon^2)\right] \tag{9.32}$$

Then since the Δ terms are $O(\varepsilon)$ by hypothesis, the first-order variation must be

$$\Delta\underline{r}_0 = \varepsilon \int \frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta)\, \underline{f}(\underline{r}_0(0), \underline{v}_0(0), \eta)\ d\eta \tag{9.33}$$

with the definite integral being taken from 0 to η. Likewise,

$$\Delta\underline{v}_0 = \varepsilon \int \frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta)\, \underline{g}(\underline{r}_0(0), \underline{v}_0(0), \eta)\ d\eta \tag{9.34}$$

These expressions are quadratures for the first-order perturbations of the elements, reckoned in terms of the true anomaly on the instantaneous osculating orbit. They are equivalent to equations (9.18) and (9.19) given earlier. As with those equations, because η differs from $\eta^{(0)}$ only by $O(\varepsilon)$, one could as well proceed with

$$\Delta\underline{r}_0 = \varepsilon \int\left[\frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\right.$$

$$+\frac{\partial^2 H}{\partial \eta^2}\Delta\eta + \cdots O(\varepsilon^2)\Bigg]\Bigg[\underline{f}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)}) + \frac{\partial \underline{f}}{\partial \eta}\Delta\eta + \cdots O(\varepsilon^2)\Bigg]\ d(\eta^{(0)} + \Delta\eta) \tag{9.35}$$

and a similar expression for $\Delta\underline{v}_0$. After terms of $\varepsilon\Delta\eta$ and higher are omitted, there results

$$\Delta\underline{r}_0 = \varepsilon\int \frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\,\underline{f}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\ d\eta^{(0)} \tag{9.36}$$

with the definite integral being taken from 0 to $\eta^{(0)}$. Likewise,

$$\Delta\underline{v}_0 = \varepsilon\int \frac{\partial H}{\partial \eta}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\,\underline{g}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\ d\eta^{(0)} \tag{9.37}$$

These expressions are quadratures for the first-order perturbations of the elements, reckoned in terms of the true anomaly on the initial osculating orbit. They are formally identical to equations (9.33) and (9.34) above, and are equivalent to equations (9.21) and (9.22) given earlier.

The relations (9.29) and (9.30) show how the element rate equations transform when the independent variable is changed from time to true anomaly. The inverse relations, showing how the rate equations transform when the independent variable is changed from true anomaly to time, can be derived in the same way. Consider η as the independent variable and differentiate the time equation (9.13) to obtain

$$\frac{dt}{d\eta} = \frac{\partial H}{\partial \eta}\frac{d\eta}{d\eta} + \frac{\partial H}{\partial \underline{r}_0}\frac{d}{d\eta}\underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0}\frac{d}{d\eta}\underline{v}_0 \tag{9.38}$$

so that

$$\frac{dt}{d\eta} = \frac{\partial H}{\partial \eta} + \varepsilon\frac{\partial H}{\partial \underline{r}_0}\underline{F} + \varepsilon\frac{\partial H}{\partial \underline{v}_0}\underline{G} \tag{9.39}$$

and

$$\frac{d\eta}{dt}=\frac{1}{\dfrac{\partial H}{\partial \eta}+\varepsilon\dfrac{\partial H}{\partial \underline{r}_0}\underline{E}+\varepsilon\dfrac{\partial H}{\partial \underline{v}_0}\underline{G}} \tag{9.40}$$

Then the rate equations (9.14) and (9.15) are rewritten as

$$\frac{d}{dt}\underline{r}_0=\frac{d}{d\eta}\underline{r}_0\frac{d\eta}{dt}=\frac{\varepsilon\underline{E}}{\dfrac{\partial H}{\partial \eta}+\varepsilon\dfrac{\partial H}{\partial \underline{r}_0}\underline{E}+\varepsilon\dfrac{\partial H}{\partial \underline{v}_0}\underline{G}} \tag{9.41}$$

$$\frac{d}{dt}\underline{v}_0=\frac{d}{d\eta}\underline{v}_0\frac{d\eta}{dt}=\frac{\varepsilon\underline{G}}{\dfrac{\partial H}{\partial \eta}+\varepsilon\dfrac{\partial H}{\partial \underline{r}_0}\underline{E}+\varepsilon\dfrac{\partial H}{\partial \underline{v}_0}\underline{G}} \tag{9.42}$$

Comparison with (9.10) and (9.11) gives the relations

$$\underline{f}=\frac{\underline{E}}{\dfrac{\partial H}{\partial \eta}+\varepsilon\dfrac{\partial H}{\partial \underline{r}_0}\underline{E}+\varepsilon\dfrac{\partial H}{\partial \underline{v}_0}\underline{G}} \tag{9.43}$$

$$\underline{g}=\frac{\underline{G}}{\dfrac{\partial H}{\partial \eta}+\varepsilon\dfrac{\partial H}{\partial \underline{r}_0}\underline{E}+\varepsilon\dfrac{\partial H}{\partial \underline{v}_0}\underline{G}} \tag{9.44}$$

The differential equations (9.41) and (9.42) seem to have less practical utility than their counterparts (9.27) and (9.28) because the time equation (9.12) must be inverted, either analytically or numerically, during the integrations. This is true even when only first-order solutions are attempted and even for time-independent perturbing forces.

As applied to the solution of the initial value problem of perturbed motion, the above considerations confirm that equations (9.21) and (9.22) should be an especially attractive way to obtain the pertubations of the elements, compared to the other candidate equations. But while the "unperturbed true anomaly" $\eta^{(0)}$ suffices to calculate the first-order variations of the elements, the variation of true anomaly itself due to perturbing effects contributes first-order terms to the position, velocity and time according to equations (9.4) through (9.6). The quantity $\Delta\eta$ is obtained (to first order) from the time equation (9.6), taking the variations of the elements into account:

$$\Delta t = \frac{\partial H}{\partial \eta}\Delta\eta + \frac{\partial H}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0}\Delta\underline{v}_0 \tag{9.45}$$

Here the partial derivatives are to be evaluated at $\underline{r}_0(0)$, $\underline{v}_0(0)$ and $\eta^{(0)}$. Then if one is reckoning perturbations at a given instant of time, it must be that $\Delta t = 0$ and this formula determines the first-order difference $\Delta\eta$ between true anomaly on the perturbed orbit and true anomaly on the nominal orbit at that time. It is possible to imagine specifying a value of true anomaly instead of time. In that case, equal values of true anomaly on the perturbed and unperturbed orbits occur at different values of the time. The first-order difference Δt would be calculated by setting $\Delta\eta = 0$ in (9.45). Other situations can be imagined as well. One might be given some perturbation of a nominal time to calculate the corresponding perturbation of true anomaly, in which case Δt is not zero but some other fixed value. In any case, the true anomaly on the unperturbed orbit, $\eta^{(0)}$, used to calculate the perturbations of the elements via (9.21) and (9.22), is related to time on the unperturbed orbit by means of the Keplerian equation (9.9). Notice also that one does not necessarily have to use the first-order version (9.45) of the time equation (9.6). The original version (9.6) would be exact as it stands were the perturbations of the elements known exactly instead of only to first order. One could specify $\Delta t = 0$ in that equation and proceed to solve (iteratively) for $\eta = \eta^{(0)} + \Delta\eta$. One might expect that very few iterations would be required since η is close to $\eta^{(0)}$ if the perturbations are small enough that the first-order calculations (9.21) and (9.22) are valid. Finally, as a side note, the entire discussion so far holds not only for true anomaly but also for any anomaly or parameter which allows the Keplerian motion to be described by equations of the form (9.1) through (9.3).

The procedure for solving the perturbed initial value problem can now be outlined. One is given an initial state $\underline{r}_0(0)$, $\underline{v}_0(0)$ and a time of flight t to calculate the final state $\underline{r}$, $\underline{v}$, accounting for first-order perturbing effects. The deviations $\Delta\underline{r}$, $\Delta\underline{v}$ from the nominal final state $\underline{r}^{(0)}$, $\underline{v}^{(0)}$ may also be of interest.

1. Using $t^{(0)} = t$, solve the nominal time equation (9.9) for the true anomaly in unperturbed motion, $\eta^{(0)}$.
2. If desired, calculate the nominal final state from equations (9.4) and (9.5).
3. Calculate the perturbations of the elements using equations (9.21) and (9.22). The value of $\eta^{(0)}$ as determined in the first step is to be used for this calculation. It is permissible to use $\eta^{(0)}$ rather than η because of the equivalence of first-order results established in the previous discussion.
4. Calculate the perturbation of true anomaly using equation (9.45) or (9.6). Since the time of flight t is given, the perturbation of time Δt is zero.
5. The perturbed final state can be calculated directly from equations (9.4) and (9.5), whether or not the nominal final state has been calculated. The latter is needed only if explicit deviations from it are of interest, in which case the deviations are obtained by subtraction.

Because of the form of the functions $\underline{R}$ and $\underline{V}$, the direct evaluation of perturbed final state from (9.4) and (9.5) introduces corrections of higher than first order. These cannot be considered significant, but neither do they compromise the accuracy of the calculation of final state if the perturbations are small enough that a first-order treatment is valid. Nevertheless, to be completely consistent, one could calculate with first-order versions of (9.4) and (9.5) as follows:

$$\Delta \underline{r} = \frac{\partial \underline{R}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{R}}{\partial \underline{v}_0} \Delta \underline{v}_0 + \frac{\partial \underline{R}}{\partial \eta} \Delta \eta \tag{9.46}$$

$$\Delta \underline{v} = \frac{\partial \underline{V}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{V}}{\partial \underline{v}_0} \Delta \underline{v}_0 + \frac{\partial \underline{V}}{\partial \eta} \Delta \eta \tag{9.47}$$

The partial derivatives are evaluated at $\underline{r}_0(0)$, $\underline{v}_0(0)$ and $\eta^{(0)}$. Then the actual final state is found by adding these corrections to the nominal final state, so the calculation of the latter is no longer optional. When $\Delta \underline{r}$ and $\Delta \underline{v}$ themselves are needed, this method avoids the numerical imprecision which could occur in subtracting nearly equal quantities in the last step above.

Formulation in Terms of Regular Elements

The practical implementation of the above ideas is impeded if one is forced to work always in explicit terms of $\underline{r}_0$ and $\underline{v}_0$. It is better to take advantage of the way that the regular elements $\underline{\xi}_0$, $\underline{\zeta}_0$, h , α and β are interposed as parameters between the initial and final state vectors in Keplerian motion. This is especially true since η-domain perturbations of these regular elements have already been derived for the J_2 problem.

The formulae for Keplerian motion as derived in previous chapters can be summarized as follows. Given position $\underline{r}_0$ and velocity $\underline{v}_0$ at initial time, compute angular momentum as

$$\underline{h} = \underline{r}_0 \times \underline{v}_0 \qquad \text{and} \qquad h = \sqrt{\underline{h} \cdot \underline{h}} \tag{9.48}$$

The radial and transverse unit vectors at initial time are

$$\underline{\xi}_0 = \frac{1}{r_0} \underline{r}_0 \qquad \text{and} \qquad \underline{\zeta}_0 = \frac{1}{hr_0} \underline{h} \times \underline{r}_0 \tag{9.49}$$

The parameters α and β are calculated as

$$\alpha = \frac{h^2}{\mu r_0} - 1 \qquad \text{and} \qquad \beta = -\frac{h}{\mu r_0}(\underline{r}_0 \cdot \underline{v}_0) \tag{9.50}$$

The true anomaly η corresponding to the given time of flight t is found from the time equation

$$t = \frac{h^3}{\mu^2} K(\eta; \alpha, \beta) \tag{9.51}$$

Here some appropriate form of K from Chapter 5, such as that given in equation (5.151), is to be used. Then the following quantities are computed:

$$\underline{\xi} = \underline{\xi}_0 \cos \eta + \underline{\zeta}_0 \sin \eta \tag{9.52}$$

$$\underline{\zeta} = -\,\underline{\xi}_0 \sin \eta + \underline{\zeta}_0 \cos \eta \tag{9.53}$$

$$u = \frac{\mu}{h^2}(1 + \alpha \cos \eta + \beta \sin \eta) \tag{9.54}$$

$$w = \frac{\mu}{h^2}(-\,\alpha \sin \eta + \beta \cos \eta) \tag{9.55}$$

The position and velocity at time t follow from

$$\underline{r}(t) = \frac{1}{u}\,\underline{\xi} \qquad \text{and} \qquad \underline{v}(t) = h\,(u\,\underline{\zeta} - w\,\underline{\xi}) \tag{9.56}$$

The actual initial state in the perturbed problem is given as $\underline{r}_0(0)$, $\underline{v}_0(0)$. The initial osculating elements $\underline{\xi}_0(0)$, $\underline{\zeta}_0(0)$, $h(0)$, $\alpha(0)$ and $\beta(0)$ are calculated from equations (9.48) through (9.50). The nominal Keplerian solution is found by first solving the time equation

$$t = t^{(0)} = \frac{h(0)^3}{\mu} K(\eta^{(0)};\, \alpha(0), \beta(0)) \tag{9.57}$$

for $\eta^{(0)}$ and then evaluating equations (9.52) through (9.56), namely:

$$\underline{\xi}^{(0)} = \underline{\xi}_0(0) \cos \eta^{(0)} + \underline{\zeta}_0(0) \sin \eta^{(0)} \tag{9.58}$$

$$\underline{\zeta}^{(0)} = -\,\underline{\xi}_0(0) \sin \eta^{(0)} + \underline{\zeta}_0(0) \cos \eta^{(0)} \tag{9.59}$$

$$u^{(0)} = \frac{\mu}{h(0)^2}\left[1 + \alpha(0) \cos \eta^{(0)} + \beta(0) \sin \eta^{(0)}\right] \tag{9.60}$$

$$w^{(0)} = \frac{\mu}{h(0)^2}\left[-\,\alpha(0) \sin \eta^{(0)} + \beta(0) \cos \eta^{(0)}\right] \tag{9.61}$$

$$\underline{r}^{(0)} = \frac{1}{u^{(0)}}\,\underline{\xi}^{(0)} \qquad \text{and} \qquad \underline{v}^{(0)} = h(0)\left[u^{(0)}\underline{\zeta}^{(0)} - w^{(0)}\underline{\xi}^{(0)}\right] \tag{9.62}$$

The J_2 perturbations of the elements at time t are computed from formulae presented in Chapter 8. Specifically,

1. $\Delta\underline{\xi}_0$ is computed from (8.26).
2. $\Delta\underline{\zeta}_0$ is computed from (8.28).
3. Δh is computed from (8.24b) and (8.24a), or as $\Delta h = \frac{\varepsilon\mu}{h(0)^3} H_1$ from (8.81). The function H_1 is defined in equation (8.88).
4. $\Delta\alpha$ is computed from (8.33), (8.24a) and (8.30), or as $\Delta\alpha = \frac{\varepsilon\mu}{h(0)^4} H_2$ from (8.82). The function H_2 is defined in equation (8.92).
5. $\Delta\beta$ is computed from (8.34), (8.24a) and (8.32), or as $\Delta\beta = \frac{\varepsilon\mu}{h(0)^4} H_3$ from (8.83). The function H_3 is defined in equation (8.96).

With perturbations of the elements available, the time equation

$$t = t^{(0)} + \Delta t = \frac{(h(0) + \Delta h)^3}{\mu^2} K(\eta^{(0)} + \Delta\eta; \alpha(0) + \Delta\alpha, \beta(0) + \Delta\beta) \tag{9.63}$$

is solved for the true anomaly on the final osculating orbit, $\eta^{(0)} + \Delta\eta$, letting $\Delta t = 0$. Computationally, this operation is identical to solving the unperturbed time equation (9.57) above. Only the elements have been adjusted by $O(\varepsilon)$. An iterative solution is indicated, just as described in Chapter 6. However, not many iterations are expected since an excellent starting value is available, namely, $\eta^{(0)}$. Now the perturbed final state can be calculated directly as follows.

$$\underline{\xi} = \underline{\xi}^{(0)} + \Delta\underline{\xi} = (\underline{\xi}_0(0) + \Delta\underline{\xi}_0)\cos(\eta^{(0)} + \Delta\eta) + (\underline{\zeta}_0(0) + \Delta\underline{\zeta}_0)\sin(\eta^{(0)} + \Delta\eta) \tag{9.64}$$

$$\underline{\zeta} = (\underline{\zeta}^{(0)} + \Delta\underline{\zeta}) = -(\underline{\xi}_0(0) + \Delta\underline{\xi}_0)\sin(\eta^{(0)} + \Delta\eta) + (\underline{\zeta}_0(0) + \Delta\underline{\zeta}_0)\cos(\eta^{(0)} + \Delta\eta) \tag{9.65}$$

$$u = u^{(0)} + \Delta u =$$

$$\frac{\mu}{(h(0) + \Delta h)^2}\left[1 + (\alpha(0) + \Delta\alpha)\cos(\eta^{(0)} + \Delta\eta) + (\beta(0) + \Delta\beta)\sin(\eta^{(0)} + \Delta\eta)\right] \tag{9.66}$$

$$w = w^{(0)} + \Delta w =$$

$$\frac{\mu}{(h(0) + \Delta h)^2}\Big[-(\alpha(0) + \Delta\alpha)\sin(\eta^{(0)} + \Delta\eta) + (\beta(0) + \Delta\beta)\cos(\eta^{(0)} + \Delta\eta)\Big] \tag{9.67}$$

$$\underline{r} = \underline{r}^{(0)} + \Delta\underline{r} = \frac{1}{(u^{(0)} + \Delta u)}(\underline{\xi}^{(0)} + \Delta\underline{\xi}) \tag{9.68}$$

$$\underline{v} = \underline{v}^{(0)} + \Delta\underline{v} = (h(0) + \Delta h)\Big[(u^{(0)} + \Delta u)(\underline{\zeta}^{(0)} + \Delta\underline{\zeta}) - (w^{(0)} + \Delta w)(\underline{\xi}^{(0)} + \Delta\underline{\xi})\Big] \tag{9.69}$$

Again, this procedure is computationally identical to that for the nominal final state; only the elements and true anomaly have been adjusted by $O(\varepsilon)$. The expressions (9.63) through (9.69) are exact, though usually only first-order accuracy is present in the perturbations of the elements. It is easy to see that terms of $O(\varepsilon^2)$ and higher are introduced in equations (9.63) through (9.69). If the first-order calculation of the element variations is valid, these higher order corrections are superfluous and cannot be said to degrade the accuracy of the calculation. The advantage of this direct calculation of perturbed final state is convenience of programming for automatic computation. The very same routine is used for both the nominal and perturbed state vector computations; one needs only to include an additional routine to compute the perturbations of the elements and true anomaly. A possible disadvantage arises because an iteration is required for the true anomaly on the perturbed orbit. Since its value differs but little from that of the nominal true anomaly, roundoff errors may accumulate unacceptably during the iteration. Of course, only very few iterations, perhaps two or three, are expected. But it is a common situation to need greater numerical precision in the solution of the time equation than in the rest of the computations for final state.

If all terms of $O(\varepsilon^2)$ and higher are rigorously omitted in the above scheme, the iterative solution of the perturbed time equation can be avoided. More precisely, the solution is reduced to one single iteration which can be given explicitly. The results should exhaust the accuracy available from first-order perturbations of the elements without repeated evaluation of the transcendental function in the time equation. However, some additional computation is introduced in the evaluation of derivatives of the time equation, as will be seen.

On this approach, the procedure for the nominal solution and the calculation of element variations is the same as before. Retaining only first-order terms in the time equation (9.63) leaves

$$\Delta t = 3\frac{h(0)^2}{\mu^2}K(\eta^{(0)};\alpha(0),\beta(0))\Delta h + \frac{h(0)^3}{\mu^2}\left[\frac{\partial K}{\partial \eta}\Delta\eta + \frac{\partial K}{\partial \alpha}\Delta\alpha + \frac{\partial K}{\partial \beta}\Delta\beta\right] \tag{9.70}$$

The function K and its derivatives are to be evaluated at $\eta^{(0)}$, $\alpha(0)$ and $\beta(0)$. From equation (5.8) of Chapter 5 one has

$$\frac{\partial K}{\partial \eta} = \frac{1}{\left[1 + \alpha(0)\cos\eta^{(0)} + \beta(0)\sin\eta(0)\right]^2} = \frac{\mu^2}{h(0)^4 u^{(0)2}} \tag{9.71}$$

so that

$$\Delta\eta = \frac{h(0)^4 u^{(0)2}}{\mu^2}\left[\frac{\mu^2}{h(0)^3}\Delta t - \left[3\frac{K(\eta^{(0)};\alpha(0),\beta(0))}{h(0)}\Delta h + \frac{\partial K}{\partial \alpha}\Delta\alpha + \frac{\partial K}{\partial \beta}\Delta\beta\right]\right] \tag{9.72}$$

Typically, this equation is evaluated with $\Delta t = 0$ since time of flight is given. The first-order versions of (9.64) through (9.69) are found, after several steps, to be

$$\Delta\underline{\xi} = \Delta\underline{\xi}_0\cos\eta^{(0)} + \Delta\underline{\zeta}_0\sin\eta^{(0)} + \underline{\zeta}^{(0)}\Delta\eta \tag{9.73}$$

$$\Delta\underline{\zeta} = -\Delta\underline{\xi}_0\sin\eta^{(0)} + \Delta\underline{\zeta}_0\cos\eta^{(0)} - \underline{\xi}^{(0)}\Delta\eta \tag{9.74}$$

$$\Delta u = -\frac{2u^{(0)}}{h(0)}\Delta h + \frac{\mu}{h(0)^2}\left[\Delta\alpha\cos\eta^{(0)} + \Delta\beta\sin\eta^{(0)}\right] + w^{(0)}\Delta\eta \tag{9.75}$$

$$\Delta w = -\frac{2w^{(0)}}{h(0)}\Delta h + \frac{\mu}{h(0)^2}\left[-\Delta\alpha\sin\eta^{(0)} + \Delta\beta\cos\eta^{(0)}\right] - \left[u^{(0)} - \frac{\mu}{h(0)^2}\right]\Delta\eta \tag{9.76}$$

$$\Delta \underline{r} = -\frac{1}{u^{(0)2}}\,\underline{\xi}^{(0)}\Delta u + \frac{1}{u^{(0)}}\Delta\underline{\xi} \tag{9.77}$$

$$\Delta \underline{v} = \left[u^{(0)}\underline{\zeta}^{(0)} - w^{(0)}\underline{\xi}^{(0)}\right]\Delta h + h(0)\left[\underline{\zeta}^{(0)}\Delta u + u^{(0)}\Delta\underline{\zeta} - \underline{\xi}^{(0)}\Delta w - w^{(0)}\Delta\underline{\xi}\right] \tag{9.78}$$

Then the perturbed final state is simply

$$\underline{r} = \underline{r}^{(0)} + \Delta\underline{r} \qquad \text{and} \qquad \underline{v} = \underline{v}^{(0)} + \Delta\underline{v} \tag{9.79}$$

Now to be able to implement these first-order calculations, it remains only to develop suitable explicit formulae for the derivatives $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$. A separate discussion is in order because the working expressions turn out to be rather lengthy, though no fundamental difficulty is involved.

Derivatives of the Time Equation

The derivatives $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ have appeared in earlier chapters of this study, but the problem of actually evaluating them has been postponed until now. In Chapter 7 in the discussion of the Keplerian boundary value problem, it was noted that solving the time equation by Newton's method would require the repeated evaluation of these derivatives. Newton's method was not recommended due to the much additional computation expected in this evaluation. As will be seen in this section, the formulae for the derivatives are lengthy, so the amount of arithmetic for one iteration of Newton's method is much greater than for one iteration of some derivative-free method. Nevertheless, it turns out that, when the evaluation of the function K itself is properly implemented, no extra transcendental function evaluation is needed to compute the values of $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$. Hence it is feasible to implement Newton's method in the unperturbed boundary value problem using the results of this section, if one so desires. As noted in Chapter 7, the efficiency tradeoff between Newton's method and derivative-free methods is at least debatable in this case. In Chapter 8 the derivatives $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ appeared in the differential equation (8.41) for the time element t_0

(time at epoch). There it was suggested that direct differentiation of one of the universal forms of K given previously in Chapter 5 would be a cumbersome operation, and that differentiation of the definite integral (5.8) defining K provides more compact expressions:

$$\frac{\partial K}{\partial \alpha} = -2\int_0^{\eta} \frac{\cos x \; dx}{(1 + \alpha \cos x + \beta \sin x)^3} \tag{9.80}$$

$$\frac{\partial K}{\partial \beta} = -2\int_0^{\eta} \frac{\sin x \; dx}{(1 + \alpha \cos x + \beta \sin x)^3} \tag{9.81}$$

Of course, the numerical approximation of these quadratures poses no particular difficulty, but their closed-form analytical evaluation would be equivalent to direct differentiation of K, and just as cumbersome. The indefinite integrals can be obtained using the tabulated forms quoted earlier from Gradshteyn and Ryzhik (1980), or by the method of partial fractions. However, the subsequent algebraic reductions to universal working formulae would be extensive: recall that the quadrature (5.8) treated in Chapter 5 to obtain the universal forms of K was simpler than either of these two quadratures. Therefore, the direct differentiation of K will be pursued in this section.

Several universal forms of K are presented in Chapter 5. One that is especially well suited for general application is that in equation (5.151), a quarter-angle formula in which the transcendental function is evaluated by a continued fraction. It can be written as

$$K = \frac{4zN}{F(D + \sqrt{F})(1 + C_1)} \tag{9.82}$$

where $z = \tan \frac{1}{4}\eta$ and

$$D = (1 + \alpha)(1 - z^2) + 2\beta z \tag{9.83}$$

$$F = D^2 + 4z^2(1 - \alpha^2 - \beta^2) \tag{9.84}$$

$$N = \left[8z^2 + (1 - z^2)(D + \sqrt{F})\right](D + \sqrt{F})$$

$$+ 4z^2\left[D + \frac{1}{(3 + C_2)}\left[D - (1 - z^2)(1 - \alpha^2 - \beta^2)\right]\right] \tag{9.85}$$

$$y = \frac{4z^2(1 - \alpha^2 - \beta^2)}{(D + \sqrt{F})^2} \tag{9.86}$$

$$C_1 = \frac{y}{3 + C_2} \tag{9.87}$$

$$C_2 = \cfrac{2^2 y}{5 + \cfrac{3^2 y}{7 + \cfrac{4^2 y}{9 + \cfrac{5^2 y}{11 + \cdots}}}} \tag{9.88}$$

These formulae are taken from Chapter 5, equations (5.142) through (5.151). Application of the chain rule will produce reasonable, if necessarily lengthy, expressions for the needed derivatives.

Inspection shows that, in the process of developing $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$, eventually the derivative $\frac{dC_2}{dy}$ will be required and that finding this derivative is really the main complication in the direct differentiation of K. For example, differentiation of C_2 "term-by-term" via the quotient rule leads to a nearly useless proliferation of nested continued fractions. Fortunately, some special manipulations will alleviate the difficulty. At a similar juncture in his analyses, Battin (1977, 1983) uses the fact that his continued fraction can be expressed in terms of a hypergeometric function. Many identities among these functions and their derivatives are known, so it proves to be possible for him to obtain reasonable expressions for the derivative. A similar approach could be taken here, but for the present purpose it is not necessary to introduce manipulations of hypergeometric functions. According to equation (5.105), it is possible to write

$$\tan^{-1}\sqrt{y} = \frac{\sqrt{y}}{1+C_1} = \frac{\sqrt{y}}{1+\dfrac{y}{3+C_2}} \tag{9.89}$$

A rearrangement produces

$$3 + C_2 = \frac{y\left[\dfrac{\tan^{-1}\sqrt{y}}{\sqrt{y}}\right]}{1-\left[\dfrac{\tan^{-1}\sqrt{y}}{\sqrt{y}}\right]} \tag{9.90}$$

For completeness, and to facilitate comparison of this approach with other approaches, let it be noted that

$$\frac{\tan^{-1}\sqrt{y}}{\sqrt{y}} = F\left(\frac{1}{2},\ 1;\ \frac{3}{2};\ -y\right) \tag{9.91}$$

$$\frac{\tanh^{-1}\sqrt{y}}{\sqrt{y}} = F\left(\frac{1}{2},\ 1;\ \frac{3}{2};\ +y\right) \tag{9.92}$$

where the right-hand sides are hypergeometric functions in the usual notation (not to be confused with the unrelated quantity F in equation (9.84) above). These relations express the continued fraction C_2 in terms of either elementary functions or a hypergeometric function. The elementary form (9.90) is not suitable for computation since it is indeterminate at $y = 0$ (parabolic orbits), but it does provide an easy route for taking derivatives of C_2.

$$\frac{d}{dy}\left[\frac{\tan^{-1}\sqrt{y}}{\sqrt{y}}\right] = -\frac{1}{2}y^{-3/2}\tan^{-1}(y^{1/2}) + y^{-1/2}\left[\frac{1}{1+(y^{1/2})^2}\cdot\frac{1}{2}y^{-1/2}\right] \tag{9.93}$$

For notational convenience, temporarily denote the hypergeometric function as T. Then this equation can be written as

$$\frac{dT}{dy} = -\frac{T}{2y} + \frac{1}{2y(1+y)} \tag{9.94}$$

and (9.90) can be written as

$$3 + C_2 = \frac{yT}{1 - T} \tag{9.95}$$

It follows that

$$\frac{dC_2}{dy} = \frac{T}{1 - T} + \frac{y}{1 - T}\frac{dT}{dy} + \frac{yT}{(1 - T)^2}\frac{dT}{dy} \tag{9.96}$$

Substituting for $\frac{dT}{dy}$ from (9.94) and for T in terms of C_2 from (9.95) produces, after straightforward steps,

$$\frac{dC_2}{dy} = \frac{4y^2 - 3yC_2 + y^2C_2 - yC_2^2}{2y^2(1 + y)} \tag{9.97}$$

As already noted, $y = 0$ on parabolic orbits, but the possible zero denominator in this equation can be cancelled by introducing the continued fraction C_3 by means of equation (9.88):

$$C_2 = \frac{4y}{5 + C_3} \tag{9.98}$$

where

$$C_3 = \cfrac{3^2y}{7 + \cfrac{4^2y}{9 + \cfrac{5^2y}{11 + \cfrac{6^2y}{13 + \cdots}}}} \tag{9.99}$$

Substituting equation (9.98) into (9.97) produces, after some rearrangement,

$$\frac{dC_2}{dy} = \frac{1}{2(1 + y)}\left[(4 + C_2) - \frac{4}{(5 + C_3)}(3 + C_2)\right] \tag{9.100}$$

A little reflection on this formula and on the form of K given in (9.82) through (9.88) shows that C_3 should be evaluated by the continued fraction (9.99) and that C_2 and C_1 should be computed in terms of C_3. In that case, C_3 will be the only transcendental function, not only in K itself, but also in $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$. It is important to compute C_2 in terms of C_3 and not *vice versa*. Computing C_3 in terms of C_2 would always involve a division by y, rendering the formulae useless for parabolic orbits. Likewise, C_1 should be computed in terms of C_2 and not *vice versa*.

The remaining derivatives can be obtained more straightforwardly. From (9.83), one has

$$\frac{\partial D}{\partial \alpha} = 1 - z^2 \qquad \text{and} \qquad \frac{\partial D}{\partial \beta} = 2z \tag{9.101}$$

From (9.84), there is obtained

$$\frac{\partial F}{\partial \alpha} = 2D\frac{\partial D}{\partial \alpha} - 2(4z^2)\alpha = 2D(1 - z^2) - 8\alpha z^2 \tag{9.102}$$

and

$$\frac{\partial F}{\partial \beta} = 2D\frac{\partial D}{\partial \beta} - 2(4z^2)\beta = 4Dz - 8\beta z^2 \tag{9.103}$$

The argument y of the continued fraction is defined in (9.86) and is differentiated as follows:

$$\frac{\partial y}{\partial \alpha} = -\frac{8\alpha z^2}{(D + \sqrt{F})^2} - \frac{8z^2(1 - \alpha^2 - \beta^2)}{(D + \sqrt{F})^3}\left[\frac{\partial D}{\partial \alpha} + \frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \alpha}\right] \tag{9.104}$$

$$\frac{\partial y}{\partial \beta} = -\frac{8\beta z^2}{(D + \sqrt{F})^2} - \frac{8z^2(1 - \alpha^2 - \beta^2)}{(D + \sqrt{F})^3}\left[\frac{\partial D}{\partial \beta} + \frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \beta}\right] \tag{9.105}$$

The continued fraction C_1 as given by (9.87) can be differentiated to obtain

$$\frac{\partial C_1}{\partial \alpha} = \frac{1}{(3 + C_2)}\frac{\partial y}{\partial \alpha} - \frac{y}{(3 + C_2)^2}\frac{dC_2}{dy}\frac{\partial y}{\partial \alpha} \tag{9.106}$$

$$\frac{\partial C_1}{\partial \alpha} = \frac{1}{(3 + C_2)}\left[1 - C_1 \frac{dC_2}{dy}\right]\frac{\partial y}{\partial \alpha} \tag{9.107}$$

Likewise,

$$\frac{\partial C_1}{\partial \beta} = \frac{1}{(3 + C_2)}\left[1 - C_1 \frac{dC_2}{dy}\right]\frac{\partial y}{\partial \beta} \tag{9.108}$$

Of course, C_2 can be differentiated to obtain

$$\frac{\partial C_2}{\partial \alpha} = \frac{dC_2}{dy}\frac{\partial y}{\partial \alpha} \qquad \text{and} \qquad \frac{\partial C_2}{\partial \beta} = \frac{dC_2}{dy}\frac{\partial y}{\partial \beta} \tag{9.109}$$

The expression (9.82) for K also contains a numerator factor N, defined in (9.85), which is differentiated as follows.

$$\frac{\partial N}{\partial \alpha} = \left[8z^2 + 2(1 - z^2)(D + \sqrt{F})\right]\left[\frac{\partial D}{\partial \alpha} + \frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \alpha}\right]$$

$$- 4z^2\left[\frac{\partial D}{\partial \alpha} - \left[\frac{D - (1 - z^2)(1 - \alpha^2 - \beta^2)}{(3 + C_2)^2}\right]\frac{\partial C_2}{\partial \alpha} + \frac{1}{(3 + C_2)}\left[\frac{\partial D}{\partial \alpha} + 2\alpha(1 - z^2)\right]\right] \tag{9.110}$$

$$\frac{\partial N}{\partial \beta} = \left[8z^2 + 2(1 - z^2)(D + \sqrt{F})\right]\left[\frac{\partial D}{\partial \beta} + \frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \beta}\right]$$

$$- 4z^2\left[\frac{\partial D}{\partial \beta} - \left[\frac{D - (1 - z^2)(1 - \alpha^2 - \beta^2)}{(3 + C_2)^2}\right]\frac{\partial C_2}{\partial \beta} + \frac{1}{(3 + C_2)}\left[\frac{\partial D}{\partial \beta} + 2\beta(1 - z^2)\right]\right] \tag{9.111}$$

Finally, the chain rule as applied to K itself gives

$$\frac{\partial K}{\partial \alpha} = \frac{4z}{F(D + \sqrt{F})^2(1 + C_1)}\frac{\partial N}{\partial \alpha}$$

$$
-K\left[\frac{1}{F}\frac{\partial F}{\partial \alpha}+\frac{2}{(D+\sqrt{F})}\left[\frac{\partial D}{\partial \alpha}+\frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \alpha}\right]+\frac{1}{(1+C_1)}\frac{\partial C_1}{\partial \alpha}\right] \tag{9.112}
$$

$$
\frac{\partial K}{\partial \beta}=\frac{4z}{F(D+\sqrt{F})^2(1+C_1)}\frac{\partial N}{\partial \beta}
$$

$$
-K\left[\frac{1}{F}\frac{\partial F}{\partial \beta}+\frac{2}{(D+\sqrt{F})}\left[\frac{\partial D}{\partial \beta}+\frac{1}{2\sqrt{F}}\frac{\partial F}{\partial \beta}\right]+\frac{1}{(1+C_1)}\frac{\partial C_1}{\partial \beta}\right] \tag{9.113}
$$

The above collection of formulae is sufficient to evaluate (9.112) and (9.113) for all nonrectilinear orbits and for transfer angles up to (but not including) one revolution on the initial osculating orbit.

Summary

This chapter has presented a variation-of-parameters solution method, valid to first order in the perturbation parameter, for the perturbed initial value problem of satellite motion. The method is based on the universal solution of Keplerian motion using Burdet-type coordinates and on the universal true-anomaly time equations developed in Chapter 5. A complete discussion of the use of true anomaly instead of time in the first-order perturbation theory is given, and the resulting solution procedure is independent of how the perturbations arise. Of course, in each particular problem, values for the perturbations of the regular orbital elements at the final time must be supplied. For the purposes of this study, explicit formulae are available from the previous chapter for the first-order J_2 perturbations of the regular elements. The final results are valid for all nonrectilinear orbits and for transfer angles up to, but not including, one revolution (that is, $\pm 2\pi$) on the initial osculating orbit. The latter constraint arises only from the form of the time equation and could be enlarged to two, four or more revolutions if eighth-angle or higher time formulae were used in place of the quarter-angle form chosen here. In that case, the derivatives of the time equation would have to be obtained for the new form chosen. This would represent a lengthy algebraic exercise but one that should offer no particular difficulty if the special transcendental function is differentiated in the manner shown here.

It is especially noteworthy that the calculation of the J_2 perturbations involves only a single evaluation of a transcendental function, namely, $\tan^{-1}z$ for the secular terms, once $\sin\eta$ and $\cos\eta$ are replaced in terms of $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$. Moreover, $\tan^{-1}z$ can be computed from the continued fraction C_3 as described in the last section of this chapter. The nominal solution requires the repeated evaluation of $C_3(y)$ in the iteration; the perturbations then require only a single further evaluation of $C_3(z^2)$. That is, only one transcendental function subroutine needs to be programmed, even in the perturbed case. All other calculations involve only square roots and rational algebraic formulae.

Chapter 10. The Perturbed Boundary Value Problem

Introduction

It was noted at the beginning of the last chapter that the results of orbital perturbation analysis have usually been presented in a form which is suitable for solving the initial value problem of perturbed motion. In contrast, explicit formulations of perturbed versions of the Gauss/Lambert boundary value problem seem to be recent developments, motivated by the need to target and guide artificial satellites very accurately. The one area of astronomy that might have required perturbed Gauss/Lambert solutions before the advent of space flight is orbit determination from observations. But this problem is usually decomposed into a preliminary step, in which a best-fit Keplerian orbit is derived from the data, followed by a differential correction step, in which the orbital parameters can be adjusted to account for known perturbing effects. Of course, observational errors must be reckoned with in both steps and this complicates the evaluation of perturbations of the orbit. The main point is that, for those methods of orbit determination which are based on Gauss/Lambert solutions, the actual boundary value problem is solved only in the preliminary step, while perturbations of the orbit, being mathematically indistinguishable from observational errors, are accounted for implicitly in the differential correction process. The real need for an explicit perturbed

Gauss/Lambert solution appears in spacecraft navigation and guidance problems and so is relatively recent. For example, some scenarios of orbit-to-orbit intercept do not permit the interceptor to make mid-course corrections. The effect of perturbations must be anticipated with all necessary accuracy in the initial velocity increment. In fact, this particular example leads to the problem considered in this chapter: given two positions and the time of flight between them, what is the initial velocity vector required to effect the transfer? It will be assumed here that the unperturbed version of the problem has been solved. This chapter then attempts to answer the question, what correction to the initial velocity of the unperturbed problem is required to effect the transfer in the presence of perturbations? Consistent with the scope of this study, only first-order perturbations will be considered in the analysis. This restriction simplifies the manipulations considerably, as will be seen. It permits otherwise nonlinear relations to be handled in linearized fashion so that the required velocity correction can be computed finally by direct, noniterative formulae.

There seem to be few sources in the open literature which treat the perturbed boundary value problem in detail. The four articles about to be cited all discuss the J_2 perturbed motion as the main application of their methods. The first three use KS coordinates to describe position and velocity, and consequently use some form of the eccentric anomaly to compute the perturbations. The fourth uses special coordinates describing departures from a nominal Keplerian orbit. This nominal orbit is defined by the classical orbital elements, and the perturbations are computed in terms of true anomaly. As would be expected, the use of eccentric anomaly results in very complicated integrals for the perturbations, while the use of true anomaly results in simpler integrals for the perturbations, throwing most of the complication into the time equation.

The analysis by Andrus (1977) is a perturbed version of the Keplerian KS boundary value problem presented by Jezewski (1976). The solution is valid only for elliptical orbits and, like all KS formulations of the boundary value problem, requires the introduction of extra boundary conditions to satisfy the bilinear relation. The steps used by Andrus to satisfy this relation result in possible singularities for transfer angles of 180° . The complicated eccentric-anomaly quadratures for the perturbations can, in principle, be obtained in closed form, as Andrus asserts. However, he does not attempt these operations due to the amount of algebra involved (it would be greater even than that for the partial-fraction manipulations outlined in Chapter 8). Instead, the quadratures are simplified by assuming that the radius remains constant along the transfer arc. This assumption is offered as a reasonable one for those cases in which the altitude change of the satellite is small, such as the case of a near-Earth satellite. The resulting quadratures are about as complicated as the equivalent (but exact) true-anomaly quadratures would be, involving only powers and products of sine and cosine. However, the procedure is thereby restricted to near-circular nominal orbits. Engels and Junkins (1981) present a universal KS formulation of the Keplerian boundary value problem, which they proceed to generalize to the perturbed case by variation of parameters. As in the method of Andrus, the equations have singularities for a transfer angle of 180°, but direct noniterative formulae are given for the correction to the nominal initial velocity. In order to evaluate the quadratures for the perturbations, these authors introduce a change of variable which essentially replaces sine and cosine of the eccentric anomaly in terms of tangent of half the eccentric anomaly. The integrands become rational algebraic expressions which can be handled by the method of partial fractions. An analogous approach was used in Chapter 8 of this study to evaluate the perturbations of the time. Of course, the same difficulties with this approach encountered in the three quadratures in Chapter 8 also plague the formulation by Engels and Junkins in many dozens of quadratures. These authors do not discuss the final algebraic reductions which would be necessary to produce working formulae. Moreover, even those working formulae would be valid only for transfers of up to 180° of *eccentric* anomaly. This means that the whole method would be valid up to some physical transfer angle (true anomaly) which is not known in advance and which is different for every set of given data. In effect, one would not know in advance if the method would work if the quadratures were to be evaluated in closed form in this manner. The authors do not discuss this situation, but happen to avoid it in example calculations by evaluating all the perturbation quadratures numerically without the change of variable. The report by Hand (1982) adopts the same change of variable from the outset, accepting its limitations without discussion. However, by doing this, he is able to eliminate all but one transcendental function from the calculations. The function is evaluated for transfers up to 180° of eccentric anomaly by a series whose convergence is accelerated by a method of adding and subtracting terms from the simple geometric series. The result is a universal KS solution of the perturbed boundary value problem, valid up to an unknown transfer angle but requiring only a single transcendental function evaluation for the nominal trajectory. Perturbations of the KS coordinates are computed rather than perturbations of the elements, but the quadratures are not any simpler and are handled by numerical approximation. The report by Lutze and Goodhart (1983) gains considerable practical utility by computing perturbations of the classical elements in terms of the true anomaly. The perturbation expressions are valid for orbits of any eccentricity and also for any transfer angle, subject only to limitations arising from secular terms. The initial values of the elements represent the nominal Keplerian solution which is obtained by the method of Battin (1977). Battin's method happens to be singular for 360° transfers, but this is much less troublesome than restriction to less than 180° of transfer or to less than some unknown upper limit. Perturbations of the nominal final position are reckoned in terms of small departures of reciprocal radius, angular momentum and an angle measured out of the nominal plane. This choice of coordinates is motivated by the use of true anomaly as the independent variable in the equations of motion, and, as might be expected, the transformed equations of motion are regular. All formulae are closed expressions except the time equation where, of course, nearly all of the complications arise. Numerical approximation of the quadratures is used to compute the perturbation of time. The technique used to obtain the correction to nominal initial velocity is to guess the correction, evaluate the corresponding perturbations of time and final position, and then adjust the correction until these perturbations go to zero. This iterative procedure converges rapidly because the velocity correction is small when the perturbations are small. Strictly speaking, the iteration introduces second-order and higher terms into the solution, but these are not really significant since the perturbations of the elements are computed only to first order. A direct noniterative scheme is also described in which the conditions of zero perturbation of final position and zero perturbation of final time are enforced explicitly and a single "iteration" is performed by inverting a system of linear algebraic equations. The basic idea of satisfying first-order accuracy in a single step is implemented differently by Engels and Junkins (1981) and by Lutze and Goodhart (1983). The idea is used again in the developments in this chapter in yet a different implementation, this one arising out of a more general consideration of the first-order perturbation theory. By always requiring the perturbation of time to be zero (or, as it turns out, any *fixed* value), it is possible to avoid actually having to compute its value in a first-order solution. Moreover, two quadratures which were evaluated numerically by Lutze and Goodhart are available here in closed form as $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ from Chapter 9 (one special transcendental function is required). By incorporating these features, the first-order solution given here requires neither numerical quadrature nor iteration for the correction of nominal velocity.

First-Order Perturbation Theory

Before the solution procedure is developed, it will be helpful to discuss the perturbed boundary value problem in general terms. The η-domain description of Keplerian motion is available from Chapters 3 and 5 in the form

$$\underline{r} = \underline{R}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{10.1}$$

$$\underline{v} = \underline{V}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{10.2}$$

$$t = H(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{10.3}$$

The functions $\underline{R}$, $\underline{V}$ and H are all closed expressions, though a special transcendental function is required in the universal time equation (10.3). In the boundary value problem, values are given for $\underline{r}$, $\underline{r}_0$, t and η in order to calculate the initial velocity vector $\underline{v}_0$. Even though $\underline{v}_0$ occurs implicitly in all three of the above formulae, it can be computed by the iterative methods described in Chapter 7, or by any suitable methods which have been proposed, such as those of Battin (1977, 1978, 1983).

When perturbed motion is considered, it is often convenient to use the variation-of-parameters method in which one adopts the form of the Keplerian solution to describe the actual motion.

$$\underline{r} = \underline{r}^{(0)} + \Delta\underline{r} = \underline{R}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{10.4}$$

$$\underline{v} = \underline{v}^{(0)} + \Delta\underline{v} = \underline{V}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{10.5}$$

$$t = t^{(0)} + \Delta t = H(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0,\ \ \eta^{(0)} + \Delta\eta\ ) \tag{10.6}$$

In these equations, quantities with superscript (0) are defined by

$$\underline{r}^{(0)} = \underline{R}(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{10.7}$$

$$\underline{v}^{(0)} = \underline{V}(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{10.8}$$

$$t^{(0)} = H(\ \underline{r}_0(0),\ \ \underline{v}_0(0),\ \ \eta^{(0)}\ ) \tag{10.9}$$

The latter three equations represent Keplerian motion along the initial osculating orbit. This motion can be considered as known, since it involves no consideration of perturbations, and the perturbed motion can be reckoned as a departure from this nominal solution. In equations (10.4) through (10.6) quantities representing the departure from Keplerian motion are indicated with a Δ symbol. The epochal position and velocity vectors $\underline{r}_0$ and $\underline{v}_0$, as they appear in the Keplerian formulae, are variable in perturbed motion. Their initial values, $\underline{r}_0(0)$ and $\underline{v}_0(0)$, define the initial osculating orbit via equations (10.7) through (10.9). But in the boundary value problem the vector $\underline{v}_0(0)$ is not given; rather, it must be determined from the given data by the methods of Chapter 7 or some other methods. In the process of generating the nominal orbit the following considerations arise:

1. It is convenient, though not necessary in principle, that the nominal orbit pass through the given final position. Within the framework of a first-order perturbation theory, it would at

least have to pass close to the given position, so it is natural to require that the nominal final position be the same as the desired final position.

2. It is convenient, though not necessary in principle, that a satellite on the nominal orbit reach its final position in the given time of flight. Within the framework of a first-order perturbation theory, it would at least have to arrive with a time of flight close to the given one, so it is natural to require that the nominal time of flight be the same as the desired time of flight.

Then the nominal orbit is determined by solving the following equations for $\underline{v}_0(0)$ according to the methods given in Chapter 7 (or some other).

$$\underline{r} = \underline{r}^{(0)} = \underline{R}(\ \underline{r}_0(0),\ \underline{v}_0(0),\ \eta^{(0)}\ ) \tag{10.10}$$

$$\underline{v}^{(0)} = \underline{V}(\ \underline{r}_0(0),\ \underline{v}_0(0),\ \eta^{(0)}\ ) \tag{10.11}$$

$$t = t^{(0)} = H(\ \underline{r}_0(0),\ \underline{v}_0(0),\ \eta^{(0)}\ ) \tag{10.12}$$

In these formulae, the angle $\eta^{(0)}$ is just the angle between $\underline{r}_0(0)$ and $\underline{r}$.

Now if the satellite were launched in the presence of perturbing forces from position $\underline{r}_0(0)$ with velocity $\underline{v}_0(0)$ it would, in general, miss the target position due to perturbing effects. The deviations of the final state from the nominal final state could be calculated by solving the perturbed initial value problem as described in the last chapter. Briefly recalling, one has from (10.4) through (10.6), to first order,

$$\Delta\underline{r} = \frac{\partial\underline{R}}{\partial\underline{r}_0}\Delta\underline{r}_0 + \frac{\partial\underline{R}}{\partial\underline{v}_0}\Delta\underline{v}_0 + \frac{\partial\underline{R}}{\partial\eta}\Delta\eta \tag{10.13}$$

$$\Delta\underline{v} = \frac{\partial\underline{V}}{\partial\underline{r}_0}\Delta\underline{r}_0 + \frac{\partial\underline{V}}{\partial\underline{v}_0}\Delta\underline{v}_0 + \frac{\partial\underline{V}}{\partial\eta}\Delta\eta \tag{10.14}$$

$$\Delta t = \frac{\partial H}{\partial\underline{r}_0}\Delta\underline{r}_0 + \frac{\partial H}{\partial\underline{v}_0}\Delta\underline{v}_0 + \frac{\partial H}{\partial\eta}\Delta\eta \tag{10.15}$$

where the derivatives are evaluated at $\underline{r}_0(0)$, $\underline{v}_0(0)$ and $\eta^{(0)}$. The perturbations of the elements are calculated by expressions of the form

$$\Delta\underline{r}_0 = \varepsilon \int \underline{F}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\, d\eta^{(0)} \tag{10.16}$$

$$\Delta\underline{v}_0 = \varepsilon \int \underline{G}(\underline{r}_0(0), \underline{v}_0(0), \eta^{(0)})\, d\eta^{(0)} \tag{10.17}$$

with the definite integrals being taken from 0 to $\eta^{(0)}$. The perturbation of true anomaly is calculated from (10.15), setting $\Delta t = 0$ since the nominal time is the same as the given time:

$$\Delta\eta = -\frac{1}{\left(\dfrac{\partial H}{\partial \eta}\right)}\left[\frac{\partial H}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0}\Delta\underline{v}_0\right] \tag{10.18}$$

But in order to reach the given final position in the given time, the requirement $\Delta\underline{r} = \underline{0}$ must be imposed. This can be done by introducing an $O(\varepsilon)$ velocity offset at initial time. Equation (10.4) is modified to read

$$\underline{r} = \underline{r}^{(0)} + \Delta\underline{r} = \underline{R}(\ \underline{r}_0(0) + \Delta\underline{r}_0,\ \ \underline{v}_0(0) + \Delta\underline{v}_0 + \underline{\delta},\ \ \eta^{(0)} + \Delta\eta\ ) \tag{10.19}$$

Notice that $\underline{\delta}$, like $\underline{v}_0(0)$, is constant in perturbed motion, while the perturbation of epochal velocity $\Delta\underline{v}_0$ is variable along the transfer arc. The first-order version of this equation corresponding to (10.13) becomes

$$\Delta\underline{r} = \underline{0} = \frac{\partial \underline{R}}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial \underline{R}}{\partial \underline{v}_0}(\Delta\underline{v}_0 + \underline{\delta}) + \frac{\partial \underline{R}}{\partial \eta}\Delta\eta \tag{10.20}$$

The partial derivatives are still evaluated at $\underline{r}_0(0)$, $\underline{v}_0(0)$ and $\eta^{(0)}$, as in (10.13). The variations of the elements would be given by

$$\Delta \underline{r}_0 = \varepsilon \int \underline{F}(\underline{r}_0(0), \underline{v}_0(0) + \underline{\delta}, \eta^{(0)})\, d\eta^{(0)} \tag{10.21}$$

$$\Delta \underline{v}_0 = \varepsilon \int \underline{G}(\underline{r}_0(0), \underline{v}_0(0) + \underline{\delta}, \eta^{(0)})\, d\eta^{(0)} \tag{10.22}$$

but it is clear that these expressions differ from (10.16) and (10.17) only by $O(\varepsilon^2)$ terms if $\underline{\delta}$ is $O(\varepsilon)$ in magnitude. However, $\underline{\delta}$ contributes a first-order effect to the perturbation of true anomaly. When the velocity offset is included in the time equation (10.6) and its first-order version (10.15), setting $\Delta t = 0$ results in

$$\Delta \eta = -\frac{1}{\left(\dfrac{\partial H}{\partial \eta}\right)} \left[\frac{\partial H}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0} (\Delta \underline{v}_0 + \underline{\delta}) \right] \tag{10.23}$$

Substituting this equation for $\Delta \eta$ into (10.20) produces a linear equation for $\underline{\delta}$ which is easily inverted.

$$\underline{\delta} = -\Delta \underline{v}_0 - \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta} \left(\frac{\partial H}{\partial \eta} \right)^{-1} \frac{\partial H}{\partial \underline{v}_0} \right]^{-1} \left[\frac{\partial \underline{R}}{\partial \underline{r}_0} - \frac{\partial \underline{R}}{\partial \eta} \left(\frac{\partial H}{\partial \eta} \right)^{-1} \frac{\partial H}{\partial \underline{r}_0} \right] \Delta \underline{r}_0 \tag{10.24}$$

This equation represents the first-order solution of the boundary value problem of perturbed motion: adding the vector $\underline{\delta}$, which has $O(\varepsilon)$ magnitude, to the nominal initial velocity $\underline{v}_0(0)$ should reduce the perturbation of final position $\Delta \underline{r}$ to $O(\varepsilon^2)$ magnitude. Regarding the notation, the order of factors in this expression is important since matrix multiplication is implied. Derivatives with respect to vectors are to be conformed so that the corresponding differentials are found by matrix multiplication, considering vectors as column vectors. Thus, $\dfrac{\partial \underline{R}}{\partial \eta}$ is a 3×1 column vector, $\dfrac{\partial H}{\partial \underline{r}_0}$ is a 1×3 row vector, and $\dfrac{\partial \underline{R}}{\partial \underline{r}_0}$ is a 3×3 matrix. Equation (10.24) involves a single 3×3 matrix inversion (the first square bracket), besides matrix multiplications.

The procedure for solving the perturbed boundary value problem can now be outlined. One is given an initial position $\underline{r}_0(0)$, a final position $\underline{r}$ and a time of flight t to calculate the initial velocity vector $\underline{v}_0(0) + \underline{\delta}$ needed to effect the transfer.

1. Solve the nominal boundary value problem posed by equations (10.10) through (10.12). The result is the nominal initial velocity vector $\underline{v}_0(0)$. The Keplerian transfer arc so produced reaches the given final position in the given time.
2. Calculate the perturbations of the elements $\Delta\underline{r}_0$ and $\Delta\underline{v}_0$ using equations (10.16) and (10.17). The angle $\eta^{(0)}$ used in this calculation is just the angle between the given position vectors. If the initial velocity $\underline{v}_0(0)$ were not corrected then at the given time of flight the final position would differ from the desired final position by an $O(\varepsilon)$ amount calculated to first order in equation (10.13).
3. Calculate the initial velocity offset $\underline{\delta}$ using equation (10.24). Adding this offset to $\underline{v}_0(0)$ reduces the deviation of final position to $O(\varepsilon^2)$ magnitude.

Formulation in Terms of Regular Elements

The practical implementation of these ideas depends on the precise forms of the functions $\underline{R}$, $\underline{V}$ and H . Basically, the necessary partial derivatives have to obtained and evaluated on the nominal orbit, and the perturbations of the epochal state, $\Delta\underline{r}_0$ and $\Delta\underline{v}_0$, must be calculated. The regular elements associated with the η-domain solution of Keplerian motion are convenient parameters to use in these operations, especially since explicit formulae for the J_2 perturbations of the regular elements are already available from the last chapter.

The formulae for Keplerian motion as derived in previous chapters can be summarized as follows. Given position $\underline{r}_0$ and velocity $\underline{v}_0$ at initial time, the angular momentum is

$$\underline{h} = \underline{r}_0 \times \underline{v}_0 \qquad \text{and} \qquad h = \sqrt{\underline{h} \cdot \underline{h}} \tag{10.25}$$

The radial and transverse unit vectors at initial time are

$$\underline{\xi}_0 = \frac{1}{r_0}\underline{r}_0 \qquad \text{and} \qquad \underline{\zeta}_0 = \frac{1}{hr_0}\underline{h} \times \underline{r}_0 \tag{10.26}$$

The parameters α and β are calculated as

$$\alpha = \frac{h^2}{\mu r_0} - 1 \qquad \text{and} \qquad \beta = -\frac{h}{\mu r_0}(\underline{r}_0 \cdot \underline{v}_0) \tag{10.27}$$

The true anomaly η and the given time of flight t are related by the time equation

$$t = \frac{h^3}{\mu^2}K(\eta; \alpha, \beta) = H(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{10.28}$$

Here some appropriate form of K from Chapter 5, such as that given in equation (5.151), is to be used. In order to compute final position and velocity, first compute the following quantities:

$$\underline{\xi} = \underline{\xi}_0 \cos\eta + \underline{\zeta}_0 \sin\eta \tag{10.29}$$

$$\underline{\zeta} = -\ \underline{\xi}_0 \sin\eta + \underline{\zeta}_0 \cos\eta \tag{10.30}$$

$$u = \frac{\mu}{h^2}(1 + \alpha\cos\eta + \beta\sin\eta) \tag{10.31}$$

$$w = \frac{\mu}{h^2}(-\alpha\sin\eta + \beta\cos\eta) \tag{10.32}$$

Then the position and velocity at time t are computed as

$$\underline{r} = \frac{1}{u}\underline{\xi} = \underline{R}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \qquad \text{and} \qquad \underline{v} = h\,(u\underline{\zeta} - w\,\underline{\xi}) = \underline{V}(\ \underline{r}_0,\ \underline{v}_0,\ \eta\ ) \tag{10.33}$$

In the perturbed problem the actual initial position is $\underline{r}_0(0)$. The associated nominal initial velocity

$\underline{v}_0(0)$ is found as the solution to the Keplerian boundary value problem. Then the initial osculating elements $\underline{\xi}_0(0)$, $\underline{\zeta}_0(0)$, $h(0)$, $\alpha(0)$ and $\beta(0)$ are calculated from (10.25) through (10.27). If it were needed, the nominal final state would be obtained from (10.29) through (10.33).

The perturbations of epochal state at the final time are obtained by expressing $\underline{r}_0$ and $\underline{v}_0$ in terms of regular elements and then differentiating. First, substitute (10.29) and (10.31) into (10.33) and evaluate at initial time.

$$\underline{r}_0 = \frac{h^2}{\mu(1+\alpha)}\underline{\xi}_0 \tag{10.34}$$

The variation is

$$\Delta\underline{r}_0 = \left[\frac{2h}{\mu(1+\alpha)}\Delta h - \frac{h^2}{\mu(1+\alpha)^2}\Delta\alpha\right]\underline{\xi}_0 + \frac{h^2}{\mu(1+\alpha)}\Delta\underline{\xi}_0$$

$$= \left[\frac{2\Delta h}{h} - \frac{\Delta\alpha}{(1+\alpha)}\right]\underline{r}_0 + r_0\,\Delta\underline{\xi}_0 \tag{10.35}$$

Likewise, substituting (10.29) through (10.32) into (10.33) and evaluating at initial time produces

$$\underline{v}_0 = \frac{\mu}{h}\left[(1+\alpha)\underline{\zeta}_0 - \beta\underline{\xi}_0\right] \tag{10.36}$$

The variation is

$$\Delta\underline{v}_0 = -\frac{\mu}{h^2}\left[(1+\alpha)\underline{\zeta}_0 - \beta\underline{\xi}_0\right]\Delta h + \frac{\mu}{h}\left[\underline{\zeta}_0\Delta\alpha + (1+\alpha)\Delta\underline{\zeta}_0 - \underline{\xi}_0\Delta\beta - \beta\Delta\underline{\xi}_0\right]$$

$$= -\frac{\Delta h}{h}\underline{v}_0 + \frac{\mu}{h}\left[\underline{\zeta}_0\Delta\alpha + (1+\alpha)\Delta\underline{\zeta}_0 - \underline{\xi}_0\Delta\beta - \beta\Delta\underline{\xi}_0\right] \tag{10.37}$$

The J_2 perturbations of the regular elements at the final time are computed from formulae presented in Chapter 8. Specifically,

1. $\Delta\underline{\xi}_0$ is computed from (8.26).

2. $\Delta\underline{\zeta}_0$ is computed from (8.28).

3. Δh is computed from (8.24b) and (8.24a), or as $\Delta h = \dfrac{\varepsilon\mu}{h(0)^3} H_1$ from (8.81). The function H_1 is defined in equation (8.88).

4. $\Delta\alpha$ is computed from (8.33), (8.24a) and (8.30), or as $\Delta\alpha = \dfrac{\varepsilon\mu}{h(0)^4} H_2$ from (8.82). The function H_2 is defined in equation (8.92).

5. $\Delta\beta$ is computed from (8.34), (8.24a) and (8.32), or as $\Delta\beta = \dfrac{\varepsilon\mu}{h(0)^4} H_3$ from (8.83). The function H_3 is defined in equation (8.96).

Now the partial derivatives indicated in equation (10.24) have to be obtained. The derivative $\dfrac{\partial H}{\partial \eta}$ is easily found since

$$\frac{\partial H}{\partial \eta} = \frac{h^3}{\mu^2} \frac{\partial K}{\partial \eta} = \frac{h^3}{\mu^2} \frac{1}{(1 + \alpha \cos \eta + \beta \sin \eta)^2} \tag{10.38}$$

using equation (5.8) of Chapter 5. This equation is to be evaluated using the initial osculating elements $h(0)$, $\alpha(0)$ and $\beta(0)$ and the nominal true anomaly $\eta^{(0)}$. Comparison of the right-hand side of (10.38) with equation (10.31) shows that the former can be written as

$$\frac{\partial H}{\partial \eta} = \frac{1}{hu^2} = \frac{r^2}{h} \tag{10.39}$$

where r is the nominal final radius, the value of which is given. (Naturally, this is just the differential time transformation, evaluated at the final time.)

The derivative $\dfrac{\partial \underline{R}}{\partial \eta}$ can also be found easily. From (10.33), and using (10.29) through (10.32), there is obtained

$$\frac{\partial \underline{R}}{\partial \eta} = -\frac{1}{u^2} \frac{\partial u}{\partial \eta} \underline{\xi} + \frac{1}{u} \frac{\partial \underline{\xi}}{\partial \eta} = -\frac{w}{u^2} \underline{\xi} + \frac{1}{u} \underline{\zeta} \tag{10.40}$$

Again, this expression is to be evaluated using the initial osculating elements and the nominal true anomaly $\eta^{(0)}$. Since u is the reciprocal of the final radius, it is a given quantity. Also, $\underline{\xi}$ and $\underline{\zeta}$ are the radial and transverse unit vectors at the final time. Being evaluated on the nominal orbit, they can be deduced directly from the given position data, if desired; refer to equations (7.9) through (7.15) of Chapter 7.

The remaining derivatives require more extensive manipulations. Moreover, strict attention must now be given to the notation to ensure that all vectors and matrices are properly conformed for multiplication. It will be helpful to adopt the following conventions for derivatives taken with respect to vectors. If $s = s(\underline{a})$ is a scalar function of a vector $\underline{a}$, the differential of s is

$$ds = \frac{\partial s}{\partial \underline{a}} d\underline{a} \tag{10.41}$$

Considering vectors as column vectors, the partial derivative must be interpreted as a row vector if ordinary matrix multiplication is to be used. An example scalar function might be the scalar product of two vectors, $s = \underline{a} \cdot \underline{b}$. The two possible derivatives would be

$$\frac{\partial s}{\partial \underline{a}} = \underline{b}^T \qquad \text{and} \qquad \frac{\partial s}{\partial \underline{b}} = \underline{a}^T \tag{10.42}$$

where the superscript T signifies "transpose". The scalar product of a vector with itself, $s = \underline{a} \cdot \underline{a}$, would have the derivative

$$\frac{\partial s}{\partial \underline{a}} = 2\underline{a}^T \tag{10.43}$$

If $\underline{v} = \underline{v}(\underline{a})$ is a vector function of a vector $\underline{a}$ then the differential is written

$$d\underline{v} = \frac{\partial \underline{v}}{\partial \underline{a}} d\underline{a} \tag{10.44}$$

The partial derivative is the 3×3 Jacobian matrix. In this convention, the derivative of a vector with respect to itself is the identity matrix.

Now the derivatives of the time equation with respect to the initial position and velocity can be found. From equation (10.28), one has

$$\frac{\partial H}{\partial \underline{r}_0} = 3\frac{h^2}{\mu^2} K \frac{\partial h}{\partial \underline{r}_0} + \frac{h^3}{\mu^2}\left[\frac{\partial K}{\partial \alpha}\frac{\partial \alpha}{\partial \underline{r}_0} + \frac{\partial K}{\partial \beta}\frac{\partial \beta}{\partial \underline{r}_0} \right] \tag{10.45}$$

$$\frac{\partial H}{\partial \underline{v}_0} = 3\frac{h^2}{\mu^2} K \frac{\partial h}{\partial \underline{v}_0} + \frac{h^3}{\mu^2}\left[\frac{\partial K}{\partial \alpha}\frac{\partial \alpha}{\partial \underline{v}_0} + \frac{\partial K}{\partial \beta}\frac{\partial \beta}{\partial \underline{v}_0} \right] \tag{10.46}$$

The derivatives $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ can be calculated from formulae presented in the last chapter. They are somewhat lengthy and will not be repeated here. The derivatives of the elements with respect to the initial state are found as follows. From (10.25) form the product

$$\underline{h} \cdot \underline{h} = (\underline{r}_0 \times \underline{v}_0) \cdot (\underline{r}_0 \times \underline{v}_0) \tag{10.47}$$

Using first a scalar triple-product identity and then a vector triple-product identity, one obtains

$$h^2 = (\underline{r}_0 \cdot \underline{r}_0)(\underline{v}_0 \cdot \underline{v}_0) - (\underline{r}_0 \cdot \underline{v}_0)^2 \tag{10.48}$$

Then the derivatives are

$$\frac{\partial h^2}{\partial \underline{r}_0} = 2(\underline{v}_0 \cdot \underline{v}_0)\, \underline{r}_0^T - 2(\underline{r}_0 \cdot \underline{v}_0)\, \underline{v}_0^T \tag{10.49}$$

$$\frac{\partial h^2}{\partial \underline{v}_0} = 2(\underline{r}_0 \cdot \underline{r}_0)\, \underline{v}_0^T - 2(\underline{r}_0 \cdot \underline{v}_0)\, \underline{r}_0^T \tag{10.50}$$

and finally

$$\frac{\partial h}{\partial \underline{r}_0} = \frac{1}{2h}\frac{\partial h^2}{\partial \underline{r}_0} \qquad \text{and} \qquad \frac{\partial h}{\partial \underline{v}_0} = \frac{1}{2h}\frac{\partial h^2}{\partial \underline{v}_0} \tag{10.51}$$

Now according to (10.27), α is given by

$$\alpha = \frac{h^2}{\mu}(\underline{r}_0 \cdot \underline{r}_0)^{-1/2} - 1 \tag{10.52}$$

so that the derivatives are

$$\frac{\partial \alpha}{\partial \underline{r}_0} = \frac{1}{\mu r_0} \frac{\partial h^2}{\partial \underline{r}_0} - \frac{h^2}{\mu r_0^3} \underline{r}_0^T \tag{10.53}$$

$$\frac{\partial \alpha}{\partial \underline{v}_0} = \frac{1}{\mu r_0} \frac{\partial h^2}{\partial \underline{v}_0} \tag{10.54}$$

Also according to (10.27), β is given by

$$\beta = -\frac{h}{\mu}(\underline{r}_0 \cdot \underline{r}_0)^{-1/2}(\underline{r}_0 \cdot \underline{v}_0) \tag{10.55}$$

so that the derivatives are

$$\frac{\partial \beta}{\partial \underline{r}_0} = -\frac{1}{\mu r_0}(\underline{r}_0 \cdot \underline{v}_0) \frac{\partial h}{\partial \underline{r}_0} + \frac{h}{\mu r_0^3}(\underline{r}_0 \cdot \underline{v}_0)\, \underline{r}_0^T - \frac{h}{\mu r_0} \underline{v}_0^T \tag{10.56}$$

$$\frac{\partial \beta}{\partial \underline{v}_0} = -\frac{1}{\mu r_0}(\underline{r}_0 \cdot \underline{v}_0) \frac{\partial h}{\partial \underline{v}_0} - \frac{h}{\mu r_0} \underline{r}_0^T \tag{10.57}$$

This completes the set of formulae needed to evaluate the derivatives of H with respect to the initial state.

The derivatives of $\underline{R}$ with respect to initial state are found as follows. According to (10.33), using (10.29) through (10.32), one has

$$\frac{\partial \underline{R}}{\partial \underline{r}_0} = -\frac{1}{u^2} \underline{\xi} \left[\frac{\partial u}{\partial h} \frac{\partial h}{\partial \underline{r}_0} + \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial \underline{r}_0} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial \underline{r}_0} \right] + \frac{1}{u} \left[\frac{\partial \underline{\xi}}{\partial \underline{\xi}_0} \frac{\partial \underline{\xi}_0}{\partial \underline{r}_0} + \frac{\partial \underline{\xi}}{\partial \underline{\zeta}_0} \frac{\partial \underline{\zeta}_0}{\partial \underline{r}_0} \right] \tag{10.58}$$

Notice that the coefficients of the square brackets depend only on the final position and hence are available directly from given data. The factors inside the brackets are found as follows. From (10.31) it is easy to obtain

$$\frac{\partial u}{\partial h} = -\frac{2\mu}{h^3}(1 + \alpha\cos\eta + \beta\sin\eta) = -\frac{2u}{h} \tag{10.59}$$

$$\frac{\partial u}{\partial \alpha} = \frac{\mu}{h^2}\cos\eta \qquad \text{and} \qquad \frac{\partial u}{\partial \beta} = \frac{\mu}{h^2}\sin\eta \tag{10.60}$$

Two other derivatives are found using (10.29).

$$\frac{\partial \underline{\xi}}{\partial \underline{\xi}_0} = (\cos\eta)\ I \qquad \text{and} \qquad \frac{\partial \underline{\xi}}{\partial \underline{\zeta}_0} = (\sin\eta)\ I \tag{10.61}$$

where I is the identity matrix. The derivatives of the unit vector elements are found by first re-writing (10.26).

$$\underline{\xi}_0 = \underline{r}_0\,(\underline{r}_0 \cdot \underline{r}_0)^{-1/2} \tag{10.62}$$

so that

$$\frac{\partial \underline{\xi}_0}{\partial \underline{r}_0} = -\frac{1}{r_0^3}\underline{r}_0\,\underline{r}_0^T + \frac{1}{r_0} I = \frac{1}{r_0}\left[I - \underline{\xi}_0\,\underline{\xi}_0^T\right] \tag{10.63}$$

and

$$\underline{\zeta}_0 = \frac{1}{hr_0}(\underline{r}_0 \times \underline{v}_0) \times \underline{r}_0 = \frac{1}{h}(\underline{r}_0 \cdot \underline{r}_0)^{-1/2}[\underline{v}_0(\underline{r}_0 \cdot \underline{r}_0) - \underline{r}_0(\underline{r}_0 \cdot \underline{v}_0)] \tag{10.64}$$

so that

$$\frac{\partial \underline{\zeta}_0}{\partial \underline{r}_0} = -\frac{1}{h^2 r_0}(\underline{h} \times \underline{r}_0)\frac{\partial h}{\partial \underline{r}_0} - \frac{1}{hr_0^3}(\underline{h} \times \underline{r}_0)\,\underline{r}_0^T + \frac{1}{hr_0}\left[2\underline{v}_0\,\underline{r}_0^T - (\underline{r}_0 \cdot \underline{v}_0)\,I - \underline{r}_0\,\underline{v}_0^T\right]$$

$$= -\underline{\zeta}_0\left[\frac{1}{h}\frac{\partial h}{\partial \underline{r}_0} + \frac{1}{r_0}\underline{\xi}_0^T\right] + \frac{1}{hr_0}\left[2\underline{v}_0\,\underline{r}_0^T - (\underline{r}_0 \cdot \underline{v}_0)\,I - \underline{r}_0\,\underline{v}_0^T\right] \tag{10.65}$$

Now all the formulae are available to evaluate the derivative of $\underline{R}$ with respect to $\underline{r}_0$ in equation (10.58). The derivative of $\underline{R}$ with respect to $\underline{v}_0$ is found quite readily. Analogously to (10.58), one has

$$\frac{\partial \underline{R}}{\partial \underline{v}_0} = -\frac{1}{u^2}\,\underline{\xi}\left[\frac{\partial u}{\partial h}\frac{\partial h}{\partial \underline{v}_0} + \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial \underline{v}_0} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial \underline{v}_0}\right] + \frac{1}{u}\left[\frac{\partial \underline{\xi}}{\partial \underline{\zeta}_0}\frac{\partial \underline{\zeta}_0}{\partial \underline{v}_0}\right] \tag{10.66}$$

where it has been noted that $\underline{\xi}_0$ is independent of $\underline{v}_0$. In order to evaluate this expression, it remains only to calculate

$$\frac{\partial \underline{\zeta}_0}{\partial \underline{v}_0} = -\frac{1}{h^2 r_0}(\underline{h} \times \underline{r}_0)\frac{\partial h}{\partial \underline{v}_0} + \frac{1}{h r_0}\left[(\underline{r}_0 \cdot \underline{r}_0)\, I - \underline{r}_0\, \underline{r}_0^T\right]$$

$$= -\frac{1}{h}\,\underline{\zeta}_0\,\frac{\partial h}{\partial \underline{v}_0} + \frac{r_0}{h}\left[\, I - \underline{\xi}_0\, \underline{\xi}_0^T\right] \tag{10.67}$$

This completes the set of formulae needed to compute the velocity offset $\underline{\delta}$. Because regular orbital elements have been used both to construct the necessary matrices and to compute the perturbations of epochal state, the expression (10.24) for $\underline{\delta}$ is valid for all non-rectilinear orbits. Transfer angles up to, but not inluding, one revolution on the nominal orbit are permitted by the quarter-angle form of the time equation used here. Transfers up to two, four or more revolutions could be handled if an eighth-angle or higher form of the time equation were employed.

Related Results

Several of the equations developed so far in this chapter have important uses beyond their immediate application in the J_2 perturbed boundary value problem. Many problems in the guidance and navigation of space vehicles reduce to an analysis of motion in the neighborhood of a nominal

Keplerian orbit without particular reference to geophysical or other natural perturbing forces. From this point of view, deviations of initial and final state are considered to arise from deliberate midcourse corrections or from guidance system errors and uncertainties, as well as from unmodeled forces. Then the first-order perturbation analysis presented heretofore translates directly into an analysis of small departures from a given Keplerian motion; such departures are described by linear equations. Of course, in some problems, particularly in the multibody trajectory problems of lunar and interplanetary flight, one is forced to consider motion in the neighborhood of a non-Keplerian nominal orbit. But the linearized Keplerian motion warrants discussion, especially since the formulae offered here are universally valid, closed expressions given in explicit terms of the transfer angle. Moreover, it is possible to implement the formulae so that the special transcendental function used to relate the nominal time of flight to the transfer angle is the only transcendental function needed in the calculations. Two particular topics are mentioned here, namely, the derivation of the state transition matrix of Keplerian motion and a midcourse guidance method designed to steer a satellite along a nominal Keplerian orbit. Only the basic developments are presented since detailed discussion would lead too far beyond the scope of this study. Essentially complete discussions of these and related topics can be found in the text by Battin (1964), though not in terms of the variables used in this study.

Keplerian State Transition Matrix

The Taylor's series representation of motion in the neighborhood of a nominal Keplerian orbit, truncated after the terms linear in the departures, is given in equations (10.13) through (10.15), reproduced here for easy reference:

$$\Delta \underline{r} = \frac{\partial \underline{R}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{R}}{\partial \underline{v}_0} \Delta \underline{v}_0 + \frac{\partial \underline{R}}{\partial \eta} \Delta \eta \tag{10.68}$$

$$\Delta\underline{v} = \frac{\partial \underline{V}}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial \underline{V}}{\partial \underline{v}_0}\Delta\underline{v}_0 + \frac{\partial \underline{V}}{\partial \eta}\Delta\eta \tag{10.69}$$

$$\Delta t = \frac{\partial H}{\partial \underline{r}_0}\Delta\underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0}\Delta\underline{v}_0 + \frac{\partial H}{\partial \eta}\Delta\eta \tag{10.70}$$

The derivatives are evaluated using the nominal initial state $\underline{r}_0(0)$, $\underline{v}_0(0)$ and the nominal transfer angle $\eta^{(0)}$. Substituting for $\Delta\eta$ from (10.70) into the first two equations produces

$$\Delta\underline{r} = \left[\frac{\partial \underline{R}}{\partial \underline{r}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0 + \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right]\Delta\underline{v}_0$$

$$+\frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\Delta t \tag{10.71}$$

$$\Delta\underline{v} = \left[\frac{\partial \underline{V}}{\partial \underline{r}_0} - \frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0 + \left[\frac{\partial \underline{V}}{\partial \underline{v}_0} - \frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right]\Delta\underline{v}_0$$

$$+\frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\Delta t \tag{10.72}$$

Typically, one compares the departure state with the nominal state, each reckoned at the same instant of time, so that $\Delta t = 0$. Then the remaining terms relate the initial departure state to the final departure state though a 6×6 transition matrix, 3×3 partitions of which are enclosed in the square brackets.

Most of the derivatives needed to evaluate the transition matrix have already been obtained earlier in this chapter. It remains only to find the derivatives of $\underline{V}$ using the expression in (10.33). Actually, some work is saved by first substituting for u, w , $\underline{\xi}$ and $\underline{\zeta}$ from (10.29) through (10.32). Several steps lead to

$$\underline{v} = \underline{V}(\underline{r}_0, \underline{v}_0, \eta) = \frac{\mu}{h}\left[\underline{\zeta}_0(\alpha + \cos\eta) - \underline{\xi}_0(\beta + \sin\eta)\right] \tag{10.73}$$

Then evaluating at initial time to obtain $\underline{v}_0$ allows one to eliminate α and β.

$$\underline{v} = \underline{V}(\underline{r}_0, \underline{v}_0, \eta) = \underline{v}_0 - \frac{\mu}{h}\left[\underline{\zeta}_0(1 - \cos\eta) + \underline{\xi}_0 \sin\eta\right] \tag{10.74}$$

This latter step is not necessary, but it does simplify the taking of derivatives. It is now straightforward to obtain

$$\frac{\partial \underline{V}}{\partial \eta} = -\frac{\mu}{h}\left[\underline{\zeta}_0 \sin\eta + \underline{\xi}_0 \cos\eta\right] = -\frac{\mu}{h}\underline{\xi} \tag{10.75}$$

$$\begin{aligned}
\frac{\partial \underline{V}}{\partial \underline{r}_0} &= +\frac{\mu}{h^2}\left[\underline{\zeta}_0(1 - \cos\eta) + \underline{\xi}_0 \sin\eta\right]\frac{\partial h}{\partial \underline{r}_0} - \frac{\mu}{h}\left[(1 - \cos\eta)\frac{\partial \underline{\zeta}_0}{\partial \underline{r}_0} + (\sin\eta)\frac{\partial \underline{\xi}_0}{\partial \underline{r}_0}\right] \\
&= \frac{1}{h}(\underline{v}_0 - \underline{v})\frac{\partial h}{\partial \underline{r}_0} - \frac{\mu}{h}\left[(1 - \cos\eta)\frac{\partial \underline{\zeta}_0}{\partial \underline{r}_0} + (\sin\eta)\frac{\partial \underline{\xi}_0}{\partial \underline{r}_0}\right]
\end{aligned} \tag{10.76}$$

$$\begin{aligned}
\frac{\partial \underline{V}}{\partial \underline{v}_0} &= I + \frac{\mu}{h^2}\left[\underline{\zeta}_0(1 - \cos\eta) + \underline{\xi}_0 \sin\eta\right]\frac{\partial h}{\partial \underline{v}_0} - \frac{\mu}{h}(1 - \cos\eta)\frac{\partial \underline{\zeta}_0}{\partial \underline{v}_0} \\
&= I + \frac{1}{h}(\underline{v}_0 - \underline{v})\frac{\partial h}{\partial \underline{v}_0} - \frac{\mu}{h}(1 - \cos\eta)\frac{\partial \underline{\zeta}_0}{\partial \underline{v}_0}
\end{aligned} \tag{10.77}$$

The properties and applications of the transition matrix are treated in Battin (1964). That reference gives universal s-domain formulae for a Keplerian nominal orbit, and also gives procedures for obtaining the transition matrix for a general non-Keplerian trajectory. Other universal Keplerian s-domain formulae can be found in the articles by Herrick (1965) and Goodyear (1965a, 1966). Explicit η-domain formulae seem not to have been published. The present formulation has the additional feature that only a single transcendental function, the continued fraction C_3 mentioned in the last chapter, enters the calculations when $\sin\eta$ and $\cos\eta$ are replaced in terms of $\tan\frac{1}{2}\eta$ or $\tan\frac{1}{4}\eta$. Until recently, all s-domain formulations required the evaluation of two transcendental functions. An article by Shepperd (1985) shows how to compute the Keplerian state and the transition matrix with only a single transcendental function of the variable $u = \frac{1}{\sqrt{-2E}} \tan(\frac{1}{4}\sqrt{-2E}\, s)$, where E is the energy.

Midcourse Guidance

A different use of the linearized (small departure) Keplerian motion can be made in the context of a midcourse guidance problem. Consider the following scenario. A satellite is launched along a nominal Keplerian orbit which was predesigned to reach a given point in space at a given time. At some time prior to arrival, the satellite's state is found to differ from the nominal state due to launch errors at initial time and unmodeled forces acting in the meanwhile. The question arises, what velocity correction is required at that instant to ensure that the satellite reaches its intended destination, supposing that the actual current state differs only slightly from the nominal state? Taking the current time to be "initial" time, the subsequent departure state is described by the same truncated Taylor's series used before, but now including a velocity correction $\underline{\delta}$ at current time:

$$\Delta \underline{r} = \frac{\partial \underline{R}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{R}}{\partial \underline{v}_0} (\Delta \underline{v}_0 + \underline{\delta}) + \frac{\partial \underline{R}}{\partial \eta} \Delta \eta \tag{10.78}$$

$$\Delta \underline{v} = \frac{\partial \underline{V}}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial \underline{V}}{\partial \underline{v}_0} (\Delta \underline{v}_0 + \underline{\delta}) + \frac{\partial \underline{V}}{\partial \eta} \Delta \eta \tag{10.79}$$

$$\Delta t = \frac{\partial H}{\partial \underline{r}_0} \Delta \underline{r}_0 + \frac{\partial H}{\partial \underline{v}_0} (\Delta \underline{v}_0 + \underline{\delta}) + \frac{\partial H}{\partial \eta} \Delta \eta \tag{10.80}$$

As before, the derivatives are evaluated using the nominal initial state $\underline{r}_0(0)$, $\underline{v}_0(0)$ and the nominal transfer angle $\eta^{(0)}$. In the present interpretation, $\eta^{(0)}$ is the "true anomaly to go" before arrival, the associated nominal time of flight is the "time to go", and $\Delta \underline{r}_0$ and $\Delta \underline{v}_0$ are the measured deviations of the actual current state from the nominal state at the current time. The final position deviation $\Delta \underline{r}$ indicates by how much the target point will be missed. The guidance problem is to choose the value of $\underline{\delta}$ at the current time to compensate for this error being predicted for the final time. Evidently, by choosing to control final position error in this manner, one leaves the final velocity error uncontrolled, though it is predicted. Equation (10.80) shows that the actual arrival time differs from the nominal time due to deviations of the current state from nominal. Since the associated departure of "true anomaly to go" from its nominal value is of little direct concern in the guidance problem, it is convenient to eliminate its explicit occurence in these equations by substituting (10.80) into (10.78) and (10.79).

$$\Delta\underline{r} = \left[\frac{\partial \underline{R}}{\partial \underline{r}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0 + \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right](\Delta\underline{v}_0 + \underline{\delta})$$
$$+ \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\Delta t \qquad (10.81)$$

$$\Delta\underline{v} = \left[\frac{\partial \underline{V}}{\partial \underline{r}_0} - \frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0 + \left[\frac{\partial \underline{V}}{\partial \underline{v}_0} - \frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right](\Delta\underline{v}_0 + \underline{\delta})$$
$$+ \frac{\partial \underline{V}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\Delta t \qquad (10.82)$$

Since final position is being controlled, the velocity correction is determined by solving (10.81) for $\underline{\delta}$.

$$\underline{\delta} = -\Delta\underline{v}_0 + \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right]^{-1}\left[\Delta\underline{r} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\Delta t\right.$$
$$\left. + \left[\frac{\partial \underline{R}}{\partial \underline{r}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0\right] \qquad (10.83)$$

This equation presents several options for determining the velocity correction. Two such options are discussed by Battin (1964, chapter 8). If one requires the satellite to reach the nominal final position at the nominal time, then $\Delta\underline{r} = \underline{0}$ and $\Delta t = 0$. The velocity correction takes the form

$$\underline{\delta}_0 = -\Delta\underline{v}_0 - \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{v}_0}\right]^{-1}\left[\frac{\partial \underline{R}}{\partial \underline{r}_0} - \frac{\partial \underline{R}}{\partial \eta}\left(\frac{\partial H}{\partial \eta}\right)^{-1}\frac{\partial H}{\partial \underline{r}_0}\right]\Delta\underline{r}_0 \qquad (10.84)$$

This is the "fixed time of arrival" guidance method described by Battin. It is formally identical to equation (10.24) used earlier in the perturbed boundary value problem, though the interpretation of $\Delta \underline{r}_0$ and $\Delta \underline{v}_0$ is now different. Another option, not discussed by Battin, is to specify possibly nonzero fixed values of $\Delta \underline{r}$ and Δt; that is, one may re-target to a different fixed point in space and arrive at a different time using a single velocity correction. The only restriction is that the magnitudes of $\Delta \underline{r}$ and Δt must be small enough that the linearization is valid. This option essentially permits a new nominal trajectory to be followed without having to re-solve the Keplerian boundary value problem *ab initio*. Still another option is the "variable time of arrival" method discussed by Battin. In this approach, the constraint to arrive at a pre-specified time is relaxed, leaving the value of Δt open. It can be chosen freely as long as it is not so large as to violate the assumptions behind the linearization. The advantage of this extra freedom is that now Δt can be chosen to minimize the magnitude of the velocity correction. A slight complication arises because in practice one is rarely aiming merely at a fixed point in space; rather, the destination has some motion of its own. Hence, the required value of $\Delta \underline{r}$ depends on the value of Δt. But if Δt is so small that its square can be neglected then the motion of the destination is approximately along a straight line in space:

$$\Delta \underline{r} = \underline{v}_D \, \Delta t \tag{10.85}$$

where $\underline{v}_D$ is the velocity vector of the destination point at the nominal final time. Refinements of this approximation can easily be imagined, but the intent here is to handle only linear terms of Δt. Equation (10.83) can be written in the form

$$\underline{\delta} = \underline{\delta}_0 + \underline{c} \, \Delta t \tag{10.86}$$

where

$$\underline{c} = \left[\frac{\partial \underline{R}}{\partial \underline{v}_0} - \frac{\partial \underline{R}}{\partial \eta} \left(\frac{\partial H}{\partial \eta} \right)^{-1} \frac{\partial H}{\partial \underline{v}_0} \right]^{-1} \left[\underline{v}_D - \frac{\partial \underline{R}}{\partial \eta} \left(\frac{\partial H}{\partial \eta} \right)^{-1} \right] \tag{10.87}$$

In order to minimize the magnitude of the velocity correction, require that

$$\frac{d}{d\Delta t}(\underline{\delta} \cdot \underline{\delta}) = 0 = 2\underline{c}^T\underline{\delta} \tag{10.88}$$

Substituting for $\underline{\delta}$ in this equation produces

$$\underline{c}^T\underline{\delta} = 0 = \underline{c}^T\underline{\delta}_0 + \underline{c}^T\underline{c}\Delta t = \underline{c}^T\underline{\delta}_0 + c^2\Delta t \tag{10.89}$$

Hence, the variation of "time to go" should be chosen as

$$\Delta t = -\frac{1}{c^2}(\underline{c}^T\underline{\delta}_0) \tag{10.90}$$

and the corresponding velocity correction will be

$$\underline{\delta} = \left[I - \frac{1}{c^2}\underline{c}\,\underline{c}^T \right]\underline{\delta}_0 \tag{10.91}$$

Notice that these latter calculations fail only if the magnitude of the vector $\underline{c}$ is very small. Inspection of equation (10.87) shows that this can occur only if the difference of vectors in the second square bracket is small. The first vector is the velocity of the destination point at the nominal final time and the other vector is the nominal velocity of the satellite at the nominal final time. Hence, the magnitude of $\underline{c}$ is small only if the velocities of the satellite and destination are nearly matched in both magnitude and direction at the nominal final time, that is, when the two are close together. But this would mean the two had always been close together, both traveling near the same nominal orbit, an exceptional circumstance which does not really call for midcourse guidance at all.

Summary

This chapter has presented a variation-of-parameters solution method, valid to first order in the perturbation parameter, for the boundary value problem of perturbed satellite motion. The method is based on the universal solution of Keplerian motion using Burdet-type coordinates and on the universal true-anomaly time equations developed in Chapter 5. The procedure is independent of how the perturbations arise, a fact which makes it possible to adapt the formulae for other applications mentioned at the end of the chapter. For the purposes of this study, explicit formulae are available for the first-order J_2 perturbations of the regular elements. The final results are valid for all nonrectilinear orbits and for transfer angles up to, but not including, one revolution on the initial osculating orbit. The latter constraint arises only from the form of the time equation and could be enlarged to two, four or more revolutions if eighth-angle or higher time formulae were used in place of the quarter-angle form used here.

It is especially noteworthy that the calculation of the J_2 perturbations requires only a single transcendental function evaluation, namely, $\tan^{-1}z$ for the secular terms, when $\sin\eta$ and $\cos\eta$ are replaced in terms of $z = \tan\frac{1}{2}\eta$ or $z = \tan\frac{1}{4}\eta$. Moreover, $\tan^{-1}z$ can be computed from the continued fraction C_3 as described in the last chapter. The nominal solution requires repeated evaluation of $C_3(y)$; the perturbations require only a single further evaluation of $C_3(z^2)$. Hence, only a single transcendental function subroutine needs to be programmed, even in the perturbed case. All other calculations involve only square roots and rational algebraic formulae.

Once a first-order solution of the perturbed boundary value problem is available, results for certain other applications can be obtained immediately. Two topics which involve motion near a nominal Keplerian orbit are discussed briefly. First, complete formulae are given for the transition matrix of Keplerian motion. These are presented as separate formulae for each of the four 3×3 sensitivity matrices relating departures from the initial and final state vectors. The formulae are valid for all nonrectilinear orbits and require only one transcendental function evaluation to compute both the nominal final state and the entire transition matrix. Second, the basic developments of Battin (1964) for midcourse guidance near a Keplerian orbit are rederived in terms of the regular elements and the "true anomaly to go" to final position. Again, the results are valid for all nonrectilinear orbits and are readily put in a form that requires only one transcendental function evaluation to compute the velocity correction.

General Conclusions

The hypothesis offered in the General Introduction to this study has been confirmed: true-anomaly regularization of the differential equations governing perturbed two-body motion does lead to complete sets of analytical formulae permitting universal treatment of the orbital initial value and boundary value problems both in unperturbed cases and in perturbed cases arising from zonal (J_2) Earth-gravitational effects. There are at least three necessary ingredients in this result. First, the use of true anomaly as the independent variable in the governing equations allows those equations to be cast in a rigorously linear form when the motion is unperturbed and a suitable coordinate transformation is introduced. This property of the true anomaly has been known at least since Burdet's (1969) analysis. Since that time, several other analyses, including the present study, have added various details to the formulation of linear governing equations in the true-anomaly domain. Linear formulations are essential if solutions for the Keplerian motion are to be obtained. Second, the direct relation between time and true anomaly in unperturbed motion has been obtained in a universal form. This is the most important new result of the present study. Whereas universal formulae for the Keplerian motion in the true-anomaly domain have been known since Burdet's analysis, now formulae of this type can be used to solve the initial value and boundary value problems. Until now, the only universal solutions of these problems have used some form of the relation between time and the eccentric anomaly. A special feature of the true-anomaly time equation is that it contains only a single transcendental function, making it well adapted for computation. That function is, in fact, the only one needed to describe the Keplerian motion. All other calculations involve no more than square roots and rational algebriac operations. Third, the use of true anomaly as the independent variable has advantages when the equations governing perturbed motion are considered. The equations are in the form of linear oscillator equations having small nonlinear forcing terms and having unit (or constant) frequency in the linear part. Comparable regularized governing equations using eccentric anomaly as the independent variable turn out to be valid only for elliptical orbits, but the true anomaly equations are valid for all non-rectilinear orbits. Moreover, when a first order solution is pursued, the J_2 perturbing terms lead to simple quadratures in terms of true anomaly but complicated ones in terms of eccentric anomaly. The first-order solution of the J_2 perturbed motion is in finite terms of true anomaly, consisting only of secular terms and powers and products of sine and cosine, and is valid for all types of orbits except rectilinear ones. In fact, it proves to be possible to recast this solution in a form which requires only one evaluation of a transcendental function (for the secular terms), the function being the same one used in the description of Keplerian motion. Additionally, because a universal true-anomaly time equation is now available, it is possible to provide closed-form expressions for some perturbation quantities which heretofore had to be calculated by numerical quadrature. Taken together, these properties of the solution in terms of true anomaly point to extremely efficient methods for calculating satellite motion.

In summary, true-anomaly regularization offers the following practical results. The initial value and boundary value problems of unperturbed motion, typically requiring iterative solutions of the time equation, can be solved with only a single transcendental function evaluation per iteration cycle. Various means have been described which can accelerate the evaluation of this function. The same time equation applies uniformly to all types of orbits, and it is a well-behaved function so that its zero can be found reliably by Newton's method or other typical iteration methods. Once the time equation has been solved, the initial and final state vectors on the transfer arc can be related by rational algebraic formulae which, again, apply uniformly for all types of orbits. When these two problems are generalized by variation of parameters to the case of J_2-perturbed motion, it is found that, to first order, the corrections of the unperturbed solution can be obtained by direct, noniterative formulae which are valid for all types of orbits. Moreover, it is possible to compute these corrections with only a single extra evaluation of the same transcendental function used in the unperturbed problem.

In the course of this study some new details have been added to existing formulations of true-anomaly regularization, and some other results have appeared which are only corollary to the main purpose of the work. Hence, it will be useful to present a complete synopsis of the study before any recommendations are discussed.

Synopsis

Chapters 2 through 4 discuss transformations of the differential equations of motion. This is a vast subject with an extensive literature. In this study, because analytical results are being sought, the main interest is in those regularizing transformations which also produce rigorously linear equations governing the unperturbed motion. In Chapter 2, regularization based merely on a transformation of the independent variable (time) is discussed. Time is replaced by a "fictitious time" s according to the differential relation $dt = Cr^n ds$, a general version of Sundman's (1912) proposal $dt = rds$. The Keplerian energy and the Laplace vector are introduced as extra state variables to provide the redundancy needed for regularization, but the original Cartesian coordinates are retained to represent position. The discussion summarizes results available from the years between Sperling's (1961) regularization of the Keplerian motion for the special case $n = 1$ and Szebehely's (1976b) proof that only $n = 1$ produces linear regular governing equations. A significant detail added to this body of analysis by the present study is the inclusion of a separate differential equation for the radius. Regularization of the equation of the radius calls for the use of angular momentum magnitude as an extra state variable, and the equation is linear only for $n = 1$. It turns out that $n = 1$ corresponds to the use of eccentric anomaly as the independent variable. Regular, but nonlinear, governing equations appear for $n \geq \frac{3}{2}$. Because true-anomaly regularization corresponds to $n = 2$, the conclusion is that some kind of coordinate transformation must be considered if true anomaly is to be used as the independent variable. There is as yet no general theory to guide the selection of coordinates, so particular choices have to be examined one by one, a process taken up in Chapters 3 and 4.

In retrospect, it is remarkable that linearization is possible with only a time transformation, especially one of this simple type. A discussion by Szebehely (1976a) indicates that the conditions for transforming a general nonlinear equation to linear form are highly restrictive. In a system of equations linearization is not to be expected, even with transformations both of independent and dependent variables admitted. That a very important system of three nonlinear coupled equations, such as (1.1) of Chapter 1, can be not only uncoupled but linearized as well, and by only a time transformation, is surprising, to say the least. That other linearizations are within reach by simple coordinate transformations after the time transformation is more surprising still.

Chapter 3 begins with the time-transformed equations, retaining a general value of n, and considers the transformation of coordinates $\underline{r} = u^m \underline{\xi}$, where $\underline{\xi}$ is the unit radial vector. This particular choice is motivated mainly by Burdet's (1969) analysis for the special case $m = -1$. Having the equation of the radius already included in the time-transformed equations facilitates the introduction of these coordinates. The quest for linear governing equations leads eventually to the elimination of the energy and the Laplace vector as state variables, leaving the angular momentum. In the special case when C is a constant in the time transformation, the angular momentum can also be eliminated without destroying either the regularization or the linearization, a fact apparently not noted by Burdet or writers since. It is found that the values $n = 2$ and $m = -1$ are necessary to produce linear equations which are also uncoupled in the components of $\underline{\xi}$. Appendix B discusses a coupled linear system of unit-vector equations which arises for these same values of n and m.

Chapter 4 discusses two other choices of coordinates which have not been discussed together before, though they are related. A coordinate transformation of the form $\underline{r} = M(\underline{u})\underline{u}$, where M is a certain matrix linear and homogeneous in the components of $\underline{u}$, introduces the KS coordinates into the time-transformed equations. It is found that regularization of the equations governing $\underline{u}$ requires the elimination of the Laplace vector and angular momentum as extra state variables, leaving the energy. Moreover, the value $n = 1$ is necessary for regularization but in that case the equations happen to be linear as well. Then a simple normalization of the KS coordinates allows the unit radial vector to be represented by Euler parameters according to $\underline{\xi} = M(\underline{\lambda})\underline{\lambda}$. With the same power-law transformation of the radius used in Chapter 3, it is found that the energy and Laplace vector have to be eliminated as state variables. In case C is constant, the angular momentum can be eliminated as well. The values $n = 2$ and $m = -1$ are necessary to produce linear equations which are uncoupled in the Euler parameters. A system which is linear but coupled in the Euler parameters for the same values of n and m is discussed in Appendix B. This coupled linear system is a generalization of equations presented by Broucke, *et al.* (1971), Vitins (1978) and Junkins and Turner (1979).

The general conclusion of these three chapters is that the eccentric anomaly and the true anomaly should play central and almost equally important roles in applications of the equations of motion by virtue of their simple regularizing transformations. Of course, one does not need a theory of regularization to realize the basic importance of these two anomalies in understanding and calculating satellite motion. However, the discussion presented in this study, along with analyses by Szebehely (1976a,b), Szebehely and Bond (1983), Bond (1985), and others, leaves one with the conviction that simpler regularizations of the two-body problem are not soon, if ever, forthcoming. An extensive family of other linearizations of the unperturbed motion has been developed by Szebehely and Bond (1983), but these are based on more complicated transformations of the time and depend on first expressing the governing equations in plane polar coordinates. The angular momentum is used to eliminate derivatives of the angle from the equations, leaving only a scalar radial equation of motion. It is this equation which is transformed to a linear oscillator form.

Interestingly, the quadratures for the time equations in these more advanced formulations prove to be quite intractable unless the independent variable happens to be either the eccentric anomaly or the true anomaly. These authors do prove that if the time and coordinate transformations are limited to power-law expressions in the radius, the equation governing the radial variable is linear only if the independent variable is the eccentric anomaly or the true anomaly. But unless one is willing to use numerical quadrature routinely to calculate Keplerian motion, the higher linearizations do not lead to universal solution formulae. On the other hand, the simpler power-law Sundman time transformation used in this study suffices to obtain these two linear regularizations and the corresponding universal solution formulae. Moreover, the equations governing the motion in three dimensions can be handled directly in any convenient coordinate system; one does not have to select the orbital plane as the reference plane. In fact, the linearizations presented by Szebehely and Bond (1983) cannot be extended to the equations governing general perturbed motion since the perturbing force usually produces out-of-plane motion. This would prevent the elimination procedure which in the planar case reduces the governing equations to a single scalar equation. A sequel by Bond (1985) notes this fact and treats the three-dimensional unperturbed motion in terms of spherical coordinates, intending to handle the perturbed case by variation of parameters. It turns out that only the introduction of true anomaly as independent variable produces linear oscillator-type equations in the regular version of these coordinates. In short, it may well be that the only two "simple" universal solutions of Keplerian motion are those based on using eccentric anomaly or true anomaly in place of the time. Of these two, the eccentric-anomaly solution is the more thoroughly developed because the eccentric-anomaly time equation is simpler than the true-anomaly time equation. The object of Chapter 5 of this study is to examine the often-neglected direct relation between time and true anomaly.

Development of the true-anomaly time equation consists basically in integrating the general Sundman time transformation for the special case $n = 2$. Once the radius is expressed in terms of true anomaly (by solving the linear governing equation of radial motion), the integration is elementary. Different formulae appear according as the orbit is elliptical, parabolic or hyperbolic.

At most one transcendental function, either an arctangent or hyperbolic arctangent, appears in any formula, so the three different expressions can easily be unified into a common real-valued form valid on all three kinds of orbits by means of power series expansions. Previously, there have been two other types of expression which could qualify as "universal true-anomaly time equations". The first was some form of Taylor's series expansion valid for near-parabolic orbits. Classical expressions of this type are given, for example, by Moulton (1914, article 103) and Geyling and Westerman (1971, section 2.6). These expansions typically have a very limited radius of convergence. The second type of time equation was taken from a formulation of the boundary value problem. A famous theorem by Lambert expresses the time of flight between two positions as a function of the terminal radii, the chord distance between the terminals and the unknown semimajor axis of the orbit. Because the true anomaly is just the angle between the position vectors, Lambert's time equation is, in effect, a true anomaly time equation. Modern writers, such as Battin (1968, 1977, 1978), have developed universal forms of Lambert's time equation, and so universal true-anomaly time equations are available. The difficulty with expressions of this type is that they can be cumbersome to adapt to the initial value problem in which the final position is not given but rather is to be found. The idea pursued in this study is that a straightforward integration of the time transformation leads most directly to formulae which are equally suitable for time-of-flight calculations, initial value problems and boundary value problems. The final result of this direct approach is a family of true-anomaly time equations which indeed are well adapted for all computations.

The first universal series expression derived in Chapter 5 turns out to have a radius of convergence too small to be of general use. Moreover, the formula is in terms of tangent of half the true anomaly so it cannot be used for transfer angles near 180° . The basic remedy is to use a trigonometric half-angle identity to put the formula in terms of tangent of one-quarter of the true anomaly, so that the formula can be used up to (but not actually including) one revolution of transfer. The argument of the series is shown to be tangent of half the eccentric anomaly, so the same half-angle identity can be used to extend the radius of convergence of the series to include up to one revolution as well. In fact, repeated use of the identity can produce drastic improvements in the series convergence, though at the cost of repeated square-root calculations. It is also noted that the use of an infinite continued fraction in place of the series produces some convergence improvement of its own. There exists a very convenient recursion formula with which to evaluate this (or any) continued fraction, though the amount of arithmetic per term is slightly greater than for a series calculation. Of course, the convergence of the continued fraction can be improved similarly to that of the series. It turns out that repeated use of the half-angle identities would allow the time equation to used for up to two, four or more revolutions, though at the cost of greatly increased algebraic complexity in each succeding member of the family. The same single transcendental function occurs in each member of the family, with each member having a different argument of the function; the rest of each formula consists only of square roots and rational algebraic operations. For the purposes of this study, the quarter-angle versions of the time equations are accepted as working formulae.

With these universal time equations available, it is a straightforward job to formulate solution methods for the initial value and boundary value problems of unperturbed motion. In Chapter 6 Newton's method is used to solve the time equation of the initial value problem. The function is well-behaved so the starting value of zero transfer angle is always successful. A method proposed by Prussing (1979) to calculate upper and lower bounds on the solution of the eccentric-anomaly time equation is generalized to produce bounds on any Sundman-type time variable, including true anomaly. The intent is to generate more efficient starting values than zero. However, it is found that accounting for possible hyperbolic asymptotes interferes with the simple implementation of the idea. For elliptic orbits, a useful nonzero starting value is obtained.

The boundary value problem is presented in Chapter 7. This problem is fundamentally more complicated than the initial value problem and elsewhere it has been formulated with a wide variety of iteration parameters. In this study the variable of iteration is proportional to the radial rate at the final time. In the true-anomaly domain, this variable is related linearly to the components of the transfer orbit Laplace vector. Explicit use is also made of the fact that for given position vectors the locus of transfer-orbit Laplace vectors is a straight line in space, each point on the line corresponding to a different transfer time. It is found that this choice of iteration parameter avoids the singularity which occurs in some methods for a transfer angle of 180°, and permits the specification of orbits for transfer angles up to (but not including) one revolution. This limitation is a feature of the quarter-angle form of the time equation, not of the iteration parameter. Transfers up to two, four or more revolutions could be handled by other more complicated, members of the family of time equations. In both the initial value and boundary value problems, it proves to be possible, based on the regularization theory developed in earlier chapters, to adapt the true-anomaly solutions for rectilinear orbits. The only restriction here is that neither the initial nor the final position may lie at (or too near) the attracting center at the origin. Incidentally, the independent variable in this case is the same as one used by Waldvogel (1973) in a study of multibody collision singularities.

The generalization of these problems to account for J_2 perturbations of the orbit requires some preliminary developments taken up in Chapter 8. Here the first-order variations of the regular orbital elements are obtained. The formulae appear lengthy, but are valid for all nonrectilinear orbits and are in a form well-adapted for automatic computation. Only secular terms linear in the transfer angle and 2π-periodic terms of powers and products of sine and cosine of the transfer angle occur. It is shown how the formulae can be evaluated with only a single transcendental function, the same one used in the Keplerian time equation, once sine and cosine are replaced in terms of tangent of one-quarter of the transfer angle. This substitution renders the formulae usable for up to one revolution of transfer, but if tangent of the eighth-angle were introduced, the perturbation formulae would be valid up to two revolutions; and so forth. Perturbations of the time of flight are considered as well. The differential equations governing two different time elements are obtained but are found to be too complicated for even a first-order integration to be attempted, at least without further study. A simpler straightforward perturbation technique produces a quadrature for the time, valid to first order, and this quadrature is formally evaluated by the method of partial fractions. However, the intermediate constants which arise in this approach have such a complicated form in terms of the orbital elements that massive algebraic reductions would have to be done before real-valued working equations could be obtained. The formulae recorded in this study are unfortunately not suitable for computation as they stand. These difficulties in finding the perturbations of the time are due ultimately to the fact that time is a complicated function of true anomaly in unperturbed motion.

Also discussed in Chapter 8 is the fact that the first-order averaged differential equations for the elements can be solved exactly without further approximation (time elements excepted). Solutions of this type are not usually obtainable in orbital mechanics. Vitins (1978) and other coworkers have obtained such exact solutions by introducing special elements for the purpose. Here no special elements have to be introduced; it is sufficient to use the same parameters which appear naturally in the true-anomaly regularization of the governing equations of motion. The solution gives the secular variations of the elements in purely periodic terms of very long $O(1/J_2)$ period, the formulae being valid for all nonrectilinear orbits for hundreds or thousands of revolutions. Of course, only elliptical orbits actually make more than one revolution, but the formulae give the secular perturbations of parabolic and hyperbolic orbits over their fraction of a revolution. Naturally, over many thousands of revolutions, the secular effects of other than J_2 terms must be considered if the solution is to remain realistic. Also, the solution is only partly useful for orbit prediction until secular perturbations of the time are included.

With universal formulae available for calculating perturbations of the elements, it becomes possible to solve the initial value and boundary value problems of perturbed motion. Chapter 9 presents a complete review of the use of independent variables other than time in a first-order orbital perturbation theory, and then develops the procedure for the solution of the initial value problem. Within the framework of a first-order theory, the corrections of the corresponding unperturbed solution can be obtained by direct, non-iterative formulae. Because the time of flight is given, perturbations of the time are zero by definition and need not be calculated. Zero perturbation of the time implies some non-zero perturbation correction of the transfer angle but this calculation involves no difficulties.

Chapter 10 discusses perturbation corrections of the solution of the unperturbed boundary value problem. Here again, the time of flight is given so the perturbation of time is zero by definition. The associated perturbation of the transfer angle can be computed without difficulty. By virtue of having a universal true-anomaly time equation for the unperturbed problem, certain perturbation quantities which heretofore had to be calculated by numerical quadrature can now be evaluated by closed-form expressions. One special transcendental function is required in the evaluation, namely, the same one used in the Keplerian time equation. The solution of the perturbed boundary value problem as given here does not require any numerical quadrature. It turns out also that certain equations needed in the final calculations have general validity beyond this version of the boundary value problem. Two example applications are discussed. Relatively easy steps produce expressions for the Keplerian state transition matrix, and a midcourse guidance scheme, discussed originally by Battin (1964) and designed to steer a satellite along a nominal Keplerian orbit, is rederived in terms of new universal variables valid for all nonrectilinear orbits.

Recommendations for Further Study

Some topics broached in this study have been left incomplete, either because no particular difficulties are expected in the pursuit of the problem, or because great difficulties can be expected. In each of these topics, however, lengthy work is involved which could not be undertaken here. For example, there are several straightforward extensions of the present work. It may prove inconvenient for the true-anomaly time equation not to be valid for exactly one revolution. The remedy would be to derive eighth-angle formulae to use in place of the quarter-angle versions given here. These eighth-angle formulae would be much more complicated than the quarter-angle time equations, but a judicious grouping of terms should produce reasonable working expressions for computation. Throughout the derivation of the time equations in this study, mainly in Chapter 5, there are steps which are highly reminiscent of steps taken by Battin (1968, 1977, 1978, 1983) in his studies of the boundary value problem. The use of continued fractions in place of series was actually inspired by Battin's success in the same regard. But, for example, the use of trigonometric half-angle identities in this study corresponds to some hypergeometric function identities in the work of Battin, and some of his methods depend on Lambert's theorem which has not been used here. Hence, the underlying derivations and the final form of the results are quite different. It would be helpful to have a complete parameter-by-parameter correspondence between Battin's equations, which mostly are designed for the boundary value problem, and the equations in this study, which are proposed for the boundary value problem, the initial value problem and time-of-flight calculations. Of course, there is need to exercise the methods developed here for these problems in competition with various other existing methods. It is a truism in orbital mechanics that no one method is "best" across all examples, but it also appears that no complete numerical comparison of the different universal solutions for Keplerian motion has been made. A recent formulation by Shepperd (1985) of the Keplerian initial value problem has several features in common with the methods offered in this study (one transcendental function and use of continued fractions), but is based on the eccentric-anomaly time equation.

There are other, more difficult, problems left open by this study, having to do with calculating the time of flight in perturbed motion. The first-order J_2 perturbations of time were calculated in Chapter 8 by using partial fraction expansions in the integrands. Though the results are formally complete, they are not usable for computation without extensive algebraic reductions. Calculation of perturbed time using the time elements presented in this study is less satisfactory still. The rate equations for these elements are complicated, due to the complicated form of the Keplerian true-anomaly time equation, and do not invite even a first-order analysis. Special attention should be given to obtaining analytical results for the elliptic-orbit time element since it would be needed for long-time predictions. Even the secular perturbation alone of this element would be useful. Vitins (1978) was able to obtain the J_2 secular perturbation of a similar time element by averaging its rate equation in the true-anomaly domain, so there is hope for some progress here.

It has been remarked already that the Keplerian two-body problem has a surprising number of linearizing transformations. Even if it were true that eccentric anomaly and true anomaly are the only independent variables possible for a linearizing transformation, there still remains a variety of potentially useful coordinates. Also, in the linearizations already known the integrals of motion must be introduced in special ways. Szebehely's (1974) comments on this topic are still cogent: there is as yet no comprehensive theory of linearization of the equations of unperturbed motion which can guide the selection of coordinates and introduction of the integrals of motion. The later articles by Szebehely (1976a,b), Szebehely and Bond (1983), Stiefel (1976), and a few others, would seem to be the beginnings of such a theory. But particularly in the work to date with KS variables and Euler parameters, it is easy to see that the selection of suitable coordinates in orbital mechanics is far from systematic. It was remarked in Chapter 4 of this study that the matrix representation of spinor-type variables is not really satisfactory, yet on the other hand developments such as Kustaanheimo's (1964) original presentation in terms of the spinors themselves have often seemed obscure because of the special algebra involved. In a recent textbook, which became available too late to influence the developments in this study, Hestenes (1986) has advocated a generalization of ordinary vector algebra which includes within a common framework all operations with vectors, spinors and quaternions. With this approach, great economy of expression can be obtained in the equations of motion without too great abstraction. For example, the spinor kinematical equations describing the motion of a rotating coordinate system are "automatically" linear, similarly to the Poisson kinematical equations. Hestenes also shows how the KS variables can be introduced via spinors, and devotes a chapter to celestial mechanics applications. His new approach shows that, at very least, it should be possible to condense the J_2 secular perturbation theory in Chapter 8 of this study to very concise, yet revealing, form.

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Appendix A. Collision Properties of the Sundman-Type Time Variables

In Chapter 2 the generalized Sundman time transformation

$$dt = C\, r^n\, ds \tag{A.1}$$

was introduced and regularization of the governing equations of motion was demonstrated for $n = 1$ and $n \geq \frac{3}{2}$. A note of qualification must be included, though, regarding values of $n \geq \frac{3}{2}$. In these cases the regularized differential equations cannot be used to follow the motion through an actual collision because the elapsed value of s goes to infinity as the radius r goes to zero. This fact was proved by considering the regularized equation of the radius (2.36) in the special case of rectilinear orbits ($h = 0$) and separating variables as

$$C\, ds = \frac{dr}{\sqrt{2Er^{2n} + 2\mu r^{2n-1}}} \tag{A.2}$$

In Chapter 2 this expression was integrated for all real values of n but only for parabolic ($E = 0$) paths. Here the integration of equation (A.2) is undertaken for all real values of $E \neq 0$ and for positive integral and half-integral values of n in order to verify that s does indeed go to infinity as the radius r goes to zero on all kinds of orbits if $n \geq \frac{3}{2}$. Consideration is resticted to these discrete values of n in order to permit closed-form integrations in terms of elementary functions and because the most important values of n in current use happen to be just 1, $\frac{3}{2}$ and 2. It should be possible by means of continuity arguments to extend the results to all real values of n, but that generalization will not be pursued here.

Once n has been restricted to positive integral and half-integral values, the integrand in equation (A.2) can be reduced to forms tabulated by Gradshteyn and Ryzhik (1980) who will be referenced in this appendix simply as "GR". For example, if n is an integer not less than 1 then rewrite equation (A.2) as

$$C\,ds = \frac{dr}{r^{n-1}\sqrt{2Er^2 + 2\mu r}} \tag{A.3}$$

This form can be treated using GR formula 2.268, namely,

$$\int \frac{dx}{x^m\sqrt{bx + cx^2}} = \frac{-2\sqrt{bx + cx^2}}{(2m-1)bx^m} - \frac{(2m-2)c}{(2m-1)b}\int \frac{dx}{x^{m-1}\sqrt{bx + cx^2}} \tag{A.4}$$

Then the following reduction formula, valid for $n \geq 2$, is obtained:

$$\int \frac{dr}{r^{n-1}\sqrt{2Er^2 + 2\mu r}} = \frac{-\sqrt{2Er^2 + 2\mu r}}{(2n-3)\mu r^{n-1}} - \frac{(2n-4)E}{(2n-3)\mu}\int \frac{dr}{r^{n-2}\sqrt{2Er^2 + 2\mu r}} \tag{A.5}$$

Repeated application of this reduction formula leads to an evaluation of

$$Cs = \int_{r_0}^{r} \frac{dr}{\sqrt{2Er^2 + 2\mu r}} \qquad (A.6)$$

which corresponds to $n = 1$. In case $E > 0$, GR formula 2.261 produces

$$Cs = \frac{1}{\sqrt{2E}} \ln\left[\frac{\sqrt{2E(2Er^2 + 2\mu r)} + 2Er + \mu}{\sqrt{2E(2Er_0^2 + 2\mu r_0)} + 2Er_0 + \mu} \right] \qquad (A.7)$$

As was done in Chapter 2, it is convenient to visualize the ejection orbit and consider the limit $r_0 \to 0$.

$$\lim_{r_0 \to 0} Cs = \frac{1}{\sqrt{2E}} \ln\left[\frac{\sqrt{2E(2Er^2 + 2\mu r)} + 2Er + \mu}{\mu} \right] \qquad (A.8)$$

Hence the value of s is finite for all finite values of r, meaning the hyperbolic collision is described if $n = 1$. Likewise, if $E < 0$ then GR formula 2.261 produces

$$Cs = \frac{-1}{\sqrt{-2E}} \left[\sin^{-1}\left[\frac{2Er + \mu}{\mu} \right] - \sin^{-1}\left[\frac{2Er_0 + \mu}{\mu} \right] \right] \qquad (A.9)$$

Then for the ejection orbit one has

$$\lim_{r_0 \to 0} Cs = \frac{-1}{\sqrt{-2E}} \left[\sin^{-1}\left[\frac{2Er + \mu}{\mu} \right] - \frac{\pi}{2} \right] \qquad (A.10)$$

This expression is valid for all physically realizable values of r since r lies in the interval $0 \leq r \leq -\mu/E$ for an elliptic orbit. Hence the elliptic collision is described if $n = 1$. With these results available for $n = 1$, the reduction formula (A.5) permits closed-form integrations of (A.3) as long as n is a positive integer. However, inspection of (A.5) shows that the limit of Cs as $r_0 \to 0$ does not exist for integers $n \geq 2$, meaning the collision is not described for integers $n \geq 2$.

Now if n is an odd half-integer not less than $\frac{3}{2}$ then rewrite equation (A.3) as

$$C\,ds = \frac{dr}{r^{n-\frac{1}{2}}\sqrt{2Er + 2\mu}} \tag{A.11}$$

The GR formula 2.2241 applies, namely,

$$\int \frac{dx}{x^m\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{(m-1)ax^{m-1}} - \frac{(2m-3)b}{2(m-1)a}\int \frac{dx}{x^{m-1}\sqrt{a+bx}} \tag{A.12}$$

Then the following reduction formula, valid for $n > \frac{3}{2}$, is obtained:

$$\int \frac{dr}{r^{n-\frac{1}{2}}\sqrt{2Er+2\mu}} = \frac{-\sqrt{2Er+2\mu}}{(2n-3)\mu r^{n-\frac{3}{2}}} - \frac{(2n-4)E}{(2n-3)\mu}\int \frac{dr}{r^{n-\frac{3}{2}}\sqrt{2Er+2\mu}} \tag{A.13}$$

Repeated application of this reduction formula finally requires an evaluation of

$$Cs = \int_{r_0}^{r} \frac{dr}{r\sqrt{2Er+2\mu}} \tag{A.14}$$

which corresponds to $n = \frac{3}{2}$. As long as $E \neq 0$, the GR formula 2.2245 produces, after some rearrangement,

$$Cs = \frac{1}{\sqrt{2\mu}}\ln\left[\frac{1}{rr_0}\left[\frac{(\sqrt{2Er+2\mu} - \sqrt{2\mu})(\sqrt{2Er_0+2\mu} + \sqrt{2\mu})}{2E}\right]^2\right] \tag{A.15}$$

Here the limit of Cs as $r_0 \to 0$ does not exist. Hence neither the elliptic nor the hyperbolic collisions are described if $n = \frac{3}{2}$. Likewise, inspection of the reduction formula (A.13) shows that the limit does not exist for all odd half-integers $n \geq \frac{3}{2}$, so that the collision is not described in these cases either.

Appendix B. Remarks on Euler-Parameter Equations of Motion Obtained by M. Vitins

Although a complete discussion of Vitins' (1978) results is outside the scope of this study, several features of his Euler-parameter equations of motion require comment in light of the developments in Chapter 4. Vitins begins with the η -domain equation governing the motion of the unit radial vector $\underline{\xi}$:

$$\underline{\xi}'' + \underline{\xi} = \frac{1}{h^2 u^3}\left[\underline{P} - (\underline{P} \cdot \underline{\xi})\underline{\xi} - (\underline{P} \cdot \underline{\xi}')\underline{\xi}'\right] \tag{B.1}$$

This is the same as equation (3.117) of Chapter 3. The components of $\underline{\xi}$ are the direction cosines of the radius vector. It is immediately obvious that, in the η domain, the $\underline{\xi}$ motion is coupled to the radial (u) motion only through perturbing terms, and that in unperturbed motion $\underline{\xi}$ rotates uniformly in a fixed plane. Also, the fact that $\underline{\xi}$ is a unit vector is expressed by the identity

$$\underline{\xi} \cdot \underline{\xi} = 1 \tag{B.2}$$

and by differentiating this expression one obtains another identity

$$\underline{\xi} \cdot \underline{\xi}' = 0 \tag{B.3}$$

verifying the geometrical intuition that $\underset{\sim}{\xi}'$ must be normal to $\underset{\sim}{\xi}$. Less obvious from the above relations is the fact that $\underset{\sim}{\xi}'$ itself is a unit vector:

$$\underset{\sim}{\xi}' \cdot \underset{\sim}{\xi}' = 1 \tag{B.4}$$

This can be established by differentiating equation (B.3), forming the scalar product of the equation of motion (B.1) with the vector $\underset{\sim}{\xi}$ and comparing the two results. It is significant that if any independent variable other than η had been used to write the equation of motion, the resulting form of the equation of motion would have prevented $\underset{\sim}{\xi}'$ from being a unit vector. Some elaboration of this fact was given in Chapter 3 of this study. Hence, if and only if η is used as the independent variable, the vectors $\underset{\sim}{\xi}$ and $\underset{\sim}{\xi}'$ form an orthonormal basis spanning the osculating plane. This basis together with the unit normal vector $(\underset{\sim}{\xi} \times \underset{\sim}{\xi}')$ constitutes an orthonormal triad which Vitins calls simply the "orbital frame". It has other designations; for example, Kamel (1983), who independently rediscovered some of Vitins' results, calls it the "Euler-Hill frame". More often in the published literature, it is used without being given a special designation. Broucke, *et al.*, (1971) make extensive implicit use of this frame without once naming it, though they do remark that it is related to the so-called "ideal coordinates" used in the now-classic works of Hill and Hansen. It is worth noting also that most of Vitins' (1978) results concerning the kinematics of the orbital frame, including those involving the Euler parameters, were anticipated by Broucke and co-workers (1971). The latter authors present all their equations of orbital dynamics with time as the independent variable, so that all of their governing equations are nonlinear. Vitins, following the lead of Burdet (1969), uses true anomaly as the independent variable and so is able to deal with linear governing equations.

Now in Chapter 4 of the present study the manipulations were described which permit the introduction of Euler parameters into the equation of motion (B.1) by means of the transformation

$$
\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_2 & -\lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & \mp\lambda_4 & \mp\lambda_3 \\ \lambda_3 & \pm\lambda_4 & \lambda_1 & \pm\lambda_2 \\ \lambda_4 & \mp\lambda_3 & \pm\lambda_2 & -\lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \tag{B.5}
$$

abbreviated

$$
\underset{\sim}{\xi} = M(\underset{\sim}{\lambda})\underset{\sim}{\lambda} \tag{B.6}
$$

The result of this transformation is that the Euler parameters are governed by the differential equation (4.158):

$$
\underset{\sim}{\lambda}'' + \frac{1}{4}\underset{\sim}{\lambda} = \frac{1}{h^2u^3}\left[M^T(\underset{\sim}{\lambda})\underset{\sim}{P} - \left[(M^T(\underset{\sim}{\lambda})\underset{\sim}{P}) \cdot \underset{\sim}{\lambda}\right]\underset{\sim}{\lambda} - \left[2(M^T(\underset{\sim}{\lambda})\underset{\sim}{P}) \cdot \underset{\sim}{\lambda}'\right]\underset{\sim}{\lambda}'\right] \tag{B.7}
$$

where also the bilinear relation (4.113) must hold:

$$
\underset{\sim}{\lambda}'^T J \underset{\sim}{\lambda} = \underset{\sim}{\lambda}'^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\lambda} \equiv 0 \tag{B.8}
$$

The bilinear relation is merely the condition that the fourth row of

$$
\underset{\sim}{\xi}' = 2M(\underset{\sim}{\lambda})\underset{\sim}{\lambda}' \tag{B.9}
$$

vanish identically. While the step can be justified only by its successful result, imposing the bilinear relation permits a simplified evaluation of $\underset{\sim}{\lambda}''$ and ultimately the form of equation (B.7) to be obtained. That equation is linear and uncoupled in components of $\underset{\sim}{\lambda}$ in unperturbed motion. It turns out that the bilinear relation is satisfied for equation (B.7) provided merely that $\underset{\sim}{\lambda}'$ is initialized as

$$\underline{\lambda}'(0) = \frac{1}{2} M^T(\underline{\lambda}(0))\underline{\xi}'(0) \qquad (B.10)$$

There is another version of the 3-vector equation (B.1) which is useful. Because the vectors $\underline{\xi}$, $\underline{\xi}'$ and $\underline{\xi} \times \underline{\xi}'$ are orthonormal it is convenient to resolve the perturbing force as done in equation (3.133) of Chapter 3:

$$\underline{P} = (\underline{P} \cdot \underline{\xi})\underline{\xi} + (\underline{P} \cdot \underline{\xi}')\underline{\xi}' + \left[\underline{P} \cdot (\underline{\xi} \times \underline{\xi}')\right](\underline{\xi} \times \underline{\xi}') \qquad (B.11)$$

Then equation (B.1) can be written as

$$\underline{\xi}'' + \underline{\xi} = \frac{1}{h^2 u^3}\left[\underline{P} \cdot (\underline{\xi} \times \underline{\xi}')\right](\underline{\xi} \times \underline{\xi}') \qquad (B.12)$$

The unit normal vector $(\underline{\xi} \times \underline{\xi}')$ is a constant of unperturbed motion as can be verified by forming the vector product of this equation with $\underline{\xi}$. If one were to attempt to recast this equation in terms of Euler parameters an immediate difficulty is encountered, namely, that the vector product has been defined only for 3-vectors. No such problem arose with equation (B.1) because no vector-valued products appear there. It is not clear from the analysis in Chapter 4 how to define the vector product for 4-vectors. Stiefel and Scheifele (1971, section 43) propose a vector-valued product which reverts to the familiar "cross" product when the fourth components vanish. Their product is very similar to a product introduced by Hestenes (1983a, 1983b) and Hestenes and Lounesto (1983), and either of these proposals might enable one to put equation (B.12) in terms of Euler parameters (or KS coordinates). However, this mathematical complication can be avoided by re-writing (B.12) as a system of first order equations. Specifically, let

$$\underline{\zeta} = \underline{\xi}' \qquad \text{and} \qquad \underline{n} = \underline{\xi} \times \underline{\xi}' \qquad (B.13)$$

so that

$$\underline{\xi}' = \underline{\zeta} \qquad \text{and} \qquad \underline{\zeta}' = \underline{\xi}'' \qquad \text{and} \qquad \underline{n}' = \underline{\xi} \times \underline{\xi}'' \qquad (B.14)$$

Then substituting from (B.12) produces the first order equations

$$\underline{\xi}' = \underline{\zeta} \tag{B.15}$$

$$\underline{\zeta}' = -\underline{\xi} + \frac{1}{h^2 u^3}(\underline{P} \cdot \underline{n})\underline{n} \tag{B.16}$$

$$\underline{n}' = -\frac{1}{h^2 u^3}(\underline{P} \cdot \underline{n})\underline{\zeta} \tag{B.17}$$

These equations can be generalized to the 4-space of the Euler parameters readily since no vector-valued product appears. Also, the extra redundancy brought about by advancing the unit normal to state-vector status allows the kinematics of the orbital frame to be expressed in a transparent way. In fact, these ninth-order equations have the form of Poisson's kinematical equations (see Kane, *et al.*, 1983, section 1.10); the general form is usually quoted in the time domain:

$$\frac{d}{dt}\underline{\xi} = \omega_3 \underline{\zeta} - \omega_2 \underline{n} \tag{B.18}$$

$$\frac{d}{dt}\underline{\zeta} = \omega_1 \underline{n} - \omega_3 \underline{\xi} \tag{B.19}$$

$$\frac{d}{dt}\underline{n} = \omega_2 \underline{\xi} - \omega_1 \underline{\zeta} \tag{B.20}$$

Here ω_1, ω_2 and ω_3 are the components of the angular velocity of the orbital frame resolved along $\underline{\xi}$, $\underline{\zeta}$ and $\underline{n}$ respectively. In the η domain, the motion of the orbital frame is given by equations (B.15), (B.16) and (B.17), and it is easy to see that the "angular velocities" are

$$\omega_1 = \frac{1}{h^2 u^3}(\underline{P} \cdot \underline{n}) \ , \qquad \omega_2 = 0 \ , \qquad \omega_3 = 1 \tag{B.21}$$

Thus the orbital frame rotates instantaneously with unit speed about the unit normal vector and the entire effect of perturbations is to superimpose an instantaneous rotation about the unit radial vector. There is no instantaneous rotation about the unit transverse vector.

Notice that the first-order equations are linear in unperturbed motion but are coupled in the nine vector components; hence, they are excluded from the theory presented in Chapters 3 and 4. There it was required that the linear governing equations be uncoupled in the vector components. Nevertheless, no new type of linearization is involved here beyond that already present in equation (B.1). To prove this statement, it will be necessary to consider the second-order governing equation (3.55) for $\underline{\xi}$, where general values of C, n and m have been retained, and to rewrite that equation as a set of first-order equations in terms of the unit vectors defining the orbital frame.

Equation (3.55) is reproduced here for reference:

$$\underline{\xi}'' + (2m - mn)u^{-1}u'\underline{\xi}' + \left[C^2h^2u^{2mn-4m}\right]\underline{\xi}$$

$$= C^2u^{2mn-m}\left[\underline{P} - (\underline{P}\cdot\underline{\xi})\underline{\xi}\right] + C^{-1}C'\underline{\xi}' \qquad (B.22)$$

In this equation $\underline{\xi}'$ is perpendicular to $\underline{\xi}$ but is not necessarily a unit vector. Equation (3.36) gives its magnitude.

$$\underline{\xi}'\cdot\underline{\xi}' = C^2h^2u^{2mn-4m} \qquad (B.23)$$

Therefore the unit transverse vector is

$$\underline{\zeta} = C^{-1}h^{-1}u^{2m-mn}\underline{\xi}' \qquad (B.24)$$

so that

$$\underline{\xi}' = Chu^{mn-2m}\underline{\zeta} \qquad (B.25)$$

The unit normal vector is now

$$\underline{n} = \underline{\xi}\times\underline{\zeta} = C^{-1}h^{-1}u^{2m-mn}\underline{\xi}\times\underline{\xi}' \qquad (B.26)$$

Resolving $\underline{P}$ as

$$\underline{P} = (\underline{P}\cdot\underline{\xi})\underline{\xi} + (\underline{P}\cdot\underline{\zeta})\underline{\zeta} + (\underline{P}\cdot\underline{n})\underline{n} \tag{B.27}$$

allows equation (B.22) to be rewritten as

$$\underline{\xi}'' = -\left[C^2h^2u^{2mn-4m}\right]\underline{\xi} + Chu^{mn-2m}\left[C^{-1}C' + (mn-2m)u^{-1}u'\right]\underline{\zeta}$$

$$+ C^2u^{2mn-m}\left[(\underline{P}\cdot\underline{\zeta})\underline{\zeta} + (\underline{P}\cdot\underline{n})\underline{n}\right] \tag{B.28}$$

Then differentiating $\underline{\zeta}$ as given in (B.24) and $\underline{n}$ as given in (B.26) and substituting for $\underline{\xi}''$ produces

$$\underline{\zeta}' = -Chu^{mn-2m}\underline{\xi} + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{n})\underline{n}$$

$$+\left[Chu^{mn-2m}(C^{-1}h^{-1}u^{2m-mn})' + C^{-1}C' + (mn-2m)u^{-1}u' + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta})\right]\underline{\zeta} \tag{B.29}$$

$$\underline{n}' = -Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{n})\underline{\zeta}$$

$$+\left[Chu^{mn-2m}(C^{-1}h^{-1}u^{2m-mn})' + C^{-1}C' + (mn-2m)u^{-1}u' + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta})\right]\underline{n} \tag{B.30}$$

In order for these latter equations to have the Poisson form it is necessary (and sufficient) that $\underline{\zeta}'$ have no $\underline{\zeta}$ component and $\underline{n}'$ have no $\underline{n}$ component. Thus, one is led to investigate whether it is possible that

$$0 = Chu^{mn-2m}(C^{-1}h^{-1}u^{2m-mn})' + C^{-1}C' + (mn-2m)u^{-1}u' + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta}) \tag{B.31}$$

It is evident that .

$$0 = \frac{d}{ds}\ln(C^{-1}h^{-1}u^{2m-mn}) + \frac{d}{ds}\ln(C) + (mn-2m)\frac{d}{ds}\ln(u) + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta}) \tag{B.32}$$

$$0 = \frac{d}{ds}\ln\left[C^{-1}h^{-1}u^{2m-mn}Cu^{mn-2m}\right] + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta}) \tag{B.33}$$

$$0 = -\frac{d}{ds}\ln(h) + Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{\zeta}) \tag{B.34}$$

$$h' = Cu^{m+mn}(\underset{\sim}{P} \cdot \underset{\sim}{\zeta}) \tag{B.35}$$

According to equation (3.57) of Chapter 3, angular momentum is calculated from

$$(h^2)' = 2u^{3m}(\underset{\sim}{P} \cdot \underset{\sim}{\xi}') \tag{B.36}$$

Replacing $\underset{\sim}{\xi}'$ in terms of $\underset{\sim}{\zeta}$ from (B.25) produces exactly equation (B.35). Hence, the coefficient of $\underset{\sim}{\zeta}$ in $\underset{\sim}{\zeta}'$ vanishes identically, as does the coefficient of $\underset{\sim}{n}$ in $\underset{\sim}{n}'$. The first-order equations for the unit vectors defining the orbital frame then have, as expected, the Poisson form

$$\underset{\sim}{\xi}' = Chu^{mn-2m}\underset{\sim}{\zeta} \tag{B.37}$$

$$\underset{\sim}{\zeta}' = -Chu^{mn-2m}\underset{\sim}{\xi} + Ch^{-1}u^{m+mn}(\underset{\sim}{P} \cdot \underset{\sim}{n})\underset{\sim}{n} \tag{B.38}$$

$$\underset{\sim}{n}' = -Ch^{-1}u^{m+mn}(\underset{\sim}{P} \cdot \underset{\sim}{n})\underset{\sim}{\zeta} \tag{B.39}$$

These equations are obviously linear, but coupled, in unperturbed motion. A constant-coefficient linear system appears only for $n = 2$, regardless of the value of m, and the frequency is constant in perturbed motion only for $C = h^{-1}$. Thus the linearization is the same type already discussed in Chapter 3.

The present interest in these equations is that they can be recast in terms of Euler parameters without having to manipulate vector-valued products. In fact, Vitins' (1978) equations of motion are simply the the Euler-parameter versions of the ninth-order Poisson-form equations given above, together with governing equations for the reciprocal radius, angular momentum and time. The main importance of Vitins' equations is that, due to the redundancy present in the four-component representation of $\underset{\sim}{\xi}$, a system of only order 4 can be integrated, rather than one of order 6, 8 or 9, to obtain the motion of $\underset{\sim}{\xi}$. Like the Poisson-form equations, Vitins' equations are linear but coupled in unperturbed motion. They will now be derived explicitly. Although it is possible to rely on the analogy between the motion of the osculating plane and the kinematics of a rigid body, as done by Vitins, it is not necessary to do so. The following derivation is carried out entirely within the four-dimensional formalism used in Chapter 4. No special manipulations peculiar to 3-vectors are used, except that advantage is taken of certain identically vanishing fourth components. General values of C, n and m are retained where Vitins used only $C = h^{-1}$, $n = 2$ and $m = -1$. Also, the choices of sign implied by the form of the $M(\underset{\sim}{\lambda})$ matrix are retained since Vitins used only the lower signs throughout his analysis.

It is an easy matter to verify that the matrix $M(\underset{\sim}{\lambda})$ is orthogonal:

$$M^T(\underset{\sim}{\lambda})M(\underset{\sim}{\lambda}) = I \tag{B.40}$$

where I is the 4×4 identity matrix. The columns of $M(\underset{\sim}{\lambda})$ therefore are a set of mutually orthogonal 4-vectors, each of unit length. The first column is just $\underset{\sim}{\lambda}$. Denoting the remaining columns as

$$M(\underset{\sim}{\lambda}) = \left[\, \underset{\sim}{\lambda} \quad \underset{\sim}{\lambda}^{(2)} \quad \underset{\sim}{\lambda}^{(3)} \quad \underset{\sim}{\lambda}^{(4)} \,\right] \tag{B.41}$$

it is possible to construct each column in terms of $\underset{\sim}{\lambda}$ by simple operations.

$$\underset{\sim}{\lambda} = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \underset{\sim}{\lambda} = I\,\underset{\sim}{\lambda} \tag{B.42}$$

$$\underset{\sim}{\lambda}^{(2)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 0 & 0 & \mp 1 & 0 \end{bmatrix} \underset{\sim}{\lambda} = N^{(2)}\,\underset{\sim}{\lambda} \tag{B.43}$$

$$\underline{\lambda}^{(3)} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mp 1 \\ +1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \end{bmatrix} \underline{\lambda} = N^{(3)}\,\underline{\lambda} \tag{B.44}$$

$$\underline{\lambda}^{(4)} = \begin{bmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \underline{\lambda} = N^{(4)}\,\underline{\lambda} \tag{B.45}$$

These matrix operators serve merely to reorder the components of $\underline{\lambda}$ with appropriate changes of sign. Incidentally, the matrix $N^{(4)}$ is the same as the matrix J which appears in the bilinear relation.

Now consider the square of $M(\underline{\lambda})$:

$$D(\underline{\lambda}) = M(\underline{\lambda})M(\underline{\lambda}) \tag{B.46}$$

The columns of $D(\underline{\lambda})$ also form a set of mutually orthogonal vectors each having unit magnitude. Denote

$$D(\underline{\lambda}) = \begin{bmatrix} \underline{\xi} & \underline{\xi}^{(2)} & \underline{\xi}^{(3)} & \underline{\xi}^{(4)} \end{bmatrix} \tag{B.47}$$

where the notation is motivated by the fact that the first column of $D(\underline{\lambda})$ is indeed the unit radial vector $\underline{\xi}$. Then

$$\underline{\xi} = M(\underline{\lambda})\underline{\lambda} = \begin{bmatrix} (\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2) \\ (2\lambda_1\lambda_2 \mp 2\lambda_3\lambda_4) \\ (2\lambda_1\lambda_3 \pm 2\lambda_2\lambda_4) \\ 0 \end{bmatrix} \tag{B.48}$$

$$\underline{\xi}^{(2)} = M(\underline{\lambda})\underline{\lambda}^{(2)} = \begin{bmatrix} (-2\lambda_1\lambda_2 \mp 2\lambda_3\lambda_4) \\ (\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2) \\ (\pm 2\lambda_1\lambda_4 - 2\lambda_2\lambda_3) \\ 0 \end{bmatrix} \tag{B.49}$$

$$\underline{\xi}^{(3)} = M(\underline{\lambda})\underline{\lambda}^{(3)} = \begin{bmatrix} (-2\lambda_1\lambda_3 \pm 2\lambda_2\lambda_4) \\ (\mp 2\lambda_1\lambda_4 - 2\lambda_2\lambda_3) \\ (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) \\ 0 \end{bmatrix} \tag{B.50}$$

$$\underline{\xi}^{(4)} = M(\underline{\lambda})\underline{\lambda}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ +1 \end{bmatrix} \tag{B.51}$$

Since $\underline{\xi}$, $\underline{\xi}^{(2)}$ and $\underline{\xi}^{(3)}$ have identically vanishing fourth components they can be interpreted as orthogonal unit vectors in the physical 3-space. Of course, $\underline{\xi}^{(2)}$ and $\underline{\xi}^{(3)}$ are not the same as $\underline{\zeta}$ and $\underline{n}$ in general, but the ($\underline{\xi}$, $\underline{\xi}^{(2)}$, $\underline{\xi}^{(3)}$) frame differs from the orbital frame by at most a rotation about $\underline{\xi}$. Later, conditions will be imposed to ensure that these two frames coincide. For now, note that the upper left 3×3 partition of the D matrix

$$D(\underline{\lambda}) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 \\ D_{21} & D_{22} & D_{23} & 0 \\ D_{31} & D_{32} & D_{33} & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{B.52}$$

is just the direction cosine matrix for the transformation relating the fundamental reference frame and the ($\underline{\xi}$, $\underline{\xi}^{(2)}$, $\underline{\xi}^{(3)}$) frame. Naturally, $D(\underline{\lambda})$ is orthogonal:

$$D^T(\underline{\lambda})D(\underline{\lambda}) = M^T(\underline{\lambda})M^T(\underline{\lambda})M(\underline{\lambda})M(\underline{\lambda}) = M^T(\underline{\lambda})IM(\underline{\lambda}) = I \qquad (B.53)$$

By differentiating this identity, one obtains

$$\left[D^T(\underline{\lambda})D(\underline{\lambda})\right]' = 0 \text{ matrix} \qquad (B.54)$$

$$D'^T(\underline{\lambda})D(\underline{\lambda}) = -\,D^T(\underline{\lambda})D'(\underline{\lambda}) \qquad (B.55)$$

$$\left[D^T(\underline{\lambda})D'(\underline{\lambda})\right]^T = -\left[D^T(\underline{\lambda})D'(\underline{\lambda})\right] \qquad (B.56)$$

Hence the matrix $D^T(\underline{\lambda})D'(\underline{\lambda})$ is skew-symmetric. Of the sixteen elements in the matrix, there are at most six independent nonzero elements.

$$D^T(\underline{\lambda})D'(\underline{\lambda}) = \begin{bmatrix} 0 & -\gamma_1 & -\gamma_2 & -\gamma_4 \\ \gamma_1 & 0 & -\gamma_3 & -\gamma_5 \\ \gamma_2 & \gamma_3 & 0 & -\gamma_6 \\ \gamma_4 & \gamma_5 & \gamma_6 & 0 \end{bmatrix} = \Gamma \qquad (B.57)$$

Premultiplying by $D(\underline{\lambda})$, one obtains

$$D'(\underline{\lambda}) = D(\underline{\lambda})\,\Gamma \qquad (B.58)$$

In terms of the columns of $D(\underline{\lambda})$ this is

$$\underline{\xi}' = \gamma_1\underline{\xi}^{(2)} + \gamma_2\underline{\xi}^{(3)} + \gamma_4\underline{\xi}^{(4)} \qquad (B.59)$$

$$\underline{\xi}^{(2)\prime} = -\,\gamma_1\underline{\xi} + \gamma_3\underline{\xi}^{(3)} + \gamma_5\underline{\xi}^{(4)} \qquad (B.60)$$

$$\underline{\xi}^{(3)\prime} = -\,\gamma_2\underline{\xi} - \gamma_3\underline{\xi}^{(3)} + \gamma_6\underline{\xi}^{(4)} \qquad (B.61)$$

$$\underline{\xi}^{(4)\prime} = -\,\gamma_4\underline{\xi} - \gamma_5\underline{\xi}^{(2)} - \gamma_6\underline{\xi}^{(3)} \qquad (B.62)$$

Because $\underline{\xi}^{(4)}$ is a constant vector and $\underline{\xi}$, $\underline{\xi}^{(2)}$ and $\underline{\xi}^{(3)}$ have vanishing fourth components, it must be that

$$\gamma_4 = 0 \quad \text{and} \quad \gamma_5 = 0 \quad \text{and} \quad \gamma_6 = 0 \tag{B.63}$$

in general. Then one is left with the Poisson-form equations

$$\underline{\xi}' = \gamma_1 \underline{\xi}^{(2)} + \gamma_2 \underline{\xi}^{(3)} \tag{B.64}$$

$$\underline{\xi}^{(2)\prime} = -\gamma_1 \underline{\xi} + \gamma_3 \underline{\xi}^{(3)} \tag{B.65}$$

$$\underline{\xi}^{(3)\prime} = -\gamma_2 \underline{\xi} - \gamma_3 \underline{\xi}^{(2)} \tag{B.66}$$

$$\underline{\xi}^{(4)\prime} = \underline{0} \tag{B.67}$$

It is possible to recognize the γ coefficients as being proportional to the "angular velocity" components of the ($\underline{\xi}$, $\underline{\xi}^{(2)}$, $\underline{\xi}^{(3)}$) frame, though that is not necessary for the present purpose. It is enough to note that the ($\underline{\xi}$, $\underline{\xi}^{(2)}$, $\underline{\xi}^{(3)}$) frame coincides with the orbital frame if, by correspondence with (B.37), (B.38) and (B.39), the γ coefficients are chosen to be

$$\gamma_1 = Chu^{mn-2m} \tag{B.68}$$

$$\gamma_2 = 0 \tag{B.69}$$

$$\gamma_3 = Ch^{-1}u^{m+mn}(\underline{P} \cdot \underline{n}) \tag{B.70}$$

and initial conditions are assigned as $\underline{\xi}^{(2)}(0) = \underline{\zeta}(0)$ and $\underline{\xi}^{(3)}(0) = \underline{n}(0)$.

Similar equations can be written for the Euler parameters. Since the columns of $M(\underline{\lambda})$ are mutually orthogonal vectors of constant length, the rate of change of $\underline{\lambda}$ must be a linear combination of the remaining columns of $M(\underline{\lambda})$.

$$\underline{\lambda}' = C_2 \underline{\lambda}^{(2)} + C_3 \underline{\lambda}^{(3)} + C_4 \underline{\lambda}^{(4)} \tag{B.71}$$

Likewise, the rate of change of $\underline{\lambda}^{(2)}$ must be a linear combination of the other columns. It is convenient to compute this rate as

$$\underline{\lambda}^{(2)\prime} = N^{(2)}\underline{\lambda}' = C_2 N^{(2)}\underline{\lambda}^{(2)} + C_3 N^{(2)}\underline{\lambda}^{(3)} + C_4 N^{(2)}\underline{\lambda}^{(4)} \tag{B.72}$$

Simple multiplications produce

$$\underline{\lambda}^{(2)\prime} = -\, C_2\underline{\lambda} \pm C_3\underline{\lambda}^{(4)} \mp C_4\underline{\lambda}^{(3)} \tag{B.73}$$

The rate of change of $\underline{\lambda}^{(3)}$ is

$$\underline{\lambda}^{(3)\prime} = N^{(3)}\underline{\lambda}' = C_2 N^{(3)}\underline{\lambda}^{(2)} + C_3 N^{(3)}\underline{\lambda}^{(3)} + C_4 N^{(3)}\underline{\lambda}^{(4)} \tag{B.74}$$

which reduces to

$$\underline{\lambda}^{(3)\prime} = \mp C_2\underline{\lambda}^{(4)} - C_3\underline{\lambda} \pm C_4\underline{\lambda}^{(2)} \tag{B.75}$$

The rate of change of $\underline{\lambda}^{(4)}$ is

$$\underline{\lambda}^{(4)\prime} = N^{(4)}\underline{\lambda}' = C_2 N^{(4)}\underline{\lambda}^{(2)} + C_3 N^{(4)}\underline{\lambda}^{(3)} + C_4 N^{(4)}\underline{\lambda}^{(4)} \tag{B.76}$$

which reduces to

$$\underline{\lambda}^{(4)\prime} = \pm\, C_2\underline{\lambda}^{(3)} \mp C_3\underline{\lambda}^{(2)} - C_4\underline{\lambda} \tag{B.77}$$

In order to determine the coefficients C_2, C_3 and C_4, one can relate these equations to the Poisson-form equations (B.64), (B.65) and (B.66) given earlier. First, note that direct differentiation of (B.48) produces

$$\underline{\xi}' = \begin{bmatrix} (2\lambda_1\lambda_1' - 2\lambda_2\lambda_2' - 2\lambda_3\lambda_3' + 2\lambda_4\lambda_4') \\ (2\lambda_1'\lambda_2 + 2\lambda_1\lambda_2' \mp 2\lambda_3'\lambda_4 \mp 2\lambda_3\lambda_4') \\ (2\lambda_1'\lambda_3 + 2\lambda_1\lambda_3' \pm 2\lambda_2'\lambda_4 \pm 2\lambda_2\lambda_4') \\ 0 \end{bmatrix} \tag{B.78}$$

This can be factored in matrix form as

$$\underline{\xi}' = 2\begin{bmatrix} \lambda_1 & -\lambda_2 & -\lambda_3 & \lambda_4 \\ \lambda_2 & \lambda_1 & \mp\lambda_4 & \mp\lambda_3 \\ \lambda_3 & \pm\lambda_4 & \lambda_1 & \pm\lambda_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \\ \lambda_4' \end{bmatrix} = 2\overline{M}(\underline{\lambda})\underline{\lambda}' \tag{B.79}$$

The matrix $\overline{M}(\underline{\lambda})$ is just $M(\underline{\lambda})$ with a zero fourth row. For instance, one could have defined

$$\underline{\xi} = \overline{M}(\underline{\lambda})\underline{\lambda} \tag{B.80}$$

and obtained the above expression for $\underline{\xi}'$ without enforcing the bilinear relation as an auxiliary constraint. Similarly,

$$\underline{\xi}^{(2)\prime} = \begin{bmatrix} (-2\lambda_1'\lambda_2 - 2\lambda_1\lambda_2' \mp 2\lambda_3'\lambda_4 \mp 2\lambda_3\lambda_4') \\ (2\lambda_1\lambda_1' - 2\lambda_2\lambda_2' + 2\lambda_3\lambda_3' - 2\lambda_4\lambda_4') \\ (\pm 2\lambda_1'\lambda_4 \pm 2\lambda_1\lambda_4' - 2\lambda_2'\lambda_3 - 2\lambda_2\lambda_3') \\ 0 \end{bmatrix} \tag{B.81}$$

This can be factored as

$$\underline{\xi}^{(2)\prime} = 2\overline{M}(\underline{\lambda})\underline{\lambda}^{(2)\prime} \tag{B.82}$$

It is also true that

$$\underline{\xi}^{(2)} = \overline{M}(\underline{\lambda})\underline{\lambda}^{(2)} \tag{B.83}$$

Then also

$$\underset{\sim}{\xi}^{(3)\prime} = \begin{bmatrix} (-2\lambda_1'\lambda_3 - 2\lambda_1\lambda_3' \pm 2\lambda_2'\lambda_4 \pm 2\lambda_2\lambda_4') \\ (\mp 2\lambda_1'\lambda_4 \mp 2\lambda_1\lambda_4' - 2\lambda_2'\lambda_3 - 2\lambda_2\lambda_3') \\ (2\lambda_1\lambda_1' + 2\lambda_2\lambda_2' - 2\lambda_3\lambda_3' - 2\lambda_4\lambda_4') \\ 0 \end{bmatrix} \tag{B.84}$$

This can be factored as

$$\underset{\sim}{\xi}^{(3)\prime} = 2\overline{M}(\underset{\sim}{\lambda})\underset{\sim}{\lambda}^{(3)\prime} \tag{B.85}$$

It is also true that

$$\underset{\sim}{\xi}^{(3)} = \overline{M}(\underset{\sim}{\lambda})\underset{\sim}{\lambda}^{(3)} \tag{B.86}$$

Now as shown before

$$\underset{\sim}{\xi}^{(4)\prime} = \underset{\sim}{0} \tag{B.87}$$

identically. But it may be remarked that, because the first three components of $\underset{\sim}{\xi}^{(4)}$ vanish identically, it is true that

$$\overline{M}(\underset{\sim}{\lambda})\underset{\sim}{\lambda}^{(4)} = \underset{\sim}{0} \tag{B.88}$$

Now substitute for the Euler parameter rates in each of these three rate equations. First, there is obtained

$$\underset{\sim}{\xi}' = 2\overline{M}(\underset{\sim}{\lambda})\underset{\sim}{\lambda}' = 2\overline{M}(\underset{\sim}{\lambda})\left[C_2\underset{\sim}{\lambda}^{(2)} + C_3\underset{\sim}{\lambda}^{(3)} + C_4\underset{\sim}{\lambda}^{(4)}\right] \tag{B.89}$$

$$\underset{\sim}{\xi}' = 2C_2\underset{\sim}{\xi}^{(2)} + 2C_3\underset{\sim}{\xi}^{(3)} \tag{B.90}$$

Second, there is obtained

$$\underline{\xi}^{(2)\prime} = 2\overline{M}(\underline{\lambda})\underline{\lambda}^{(2)\prime} = 2\overline{M}(\underline{\lambda})\left[-C_2\underline{\lambda} \pm C_3\underline{\lambda}^{(4)} \mp C_4\underline{\lambda}^{(3)} \right] \tag{B.91}$$

$$\underline{\xi}^{(2)\prime} = -2C_2\underline{\xi} \mp 2C_4\underline{\xi}^{(3)} \tag{B.92}$$

Third, there is obtained

$$\underline{\xi}^{(3)\prime} = 2\overline{M}(\underline{\lambda})\underline{\lambda}^{(3)\prime} = 2\overline{M}(\underline{\lambda})\left[\mp C_2\underline{\lambda}^{(4)} - C_3\underline{\lambda} \pm C_4\underline{\lambda}^{(2)} \right] \tag{B.93}$$

$$\underline{\xi}^{(3)\prime} = -2C_3\underline{\xi} \pm 2C_4\underline{\xi}^{(2)} \tag{B.94}$$

Comparing these equations with the Poisson-form equations (B.64), (B.65) and (B.66), it is apparent that the C coefficients should be identified as

$$C_2 = \frac{1}{2}\gamma_1 \qquad \text{and} \qquad C_3 = \frac{1}{2}\gamma_2 \qquad \text{and} \qquad C_4 = \mp\frac{1}{2}\gamma_3 \tag{B.95}$$

This choice causes the first-order Euler-parameter equations to represent the motion of the ($\underline{\xi}$, $\underline{\xi}^{(2)}$, $\underline{\xi}^{(3)}$) frame. To represent the motion of the orbital frame ($\underline{\xi}$, $\underline{\zeta}$, $\underline{n}$) one has only to use the values of the γ coefficients given in (B.68), (B.69) and (B.70):

$$C_2 = \frac{1}{2}Chu^{mn-2m} \tag{B.96}$$

$$C_3 = 0 \tag{B.97}$$

$$C_4 = \mp\frac{1}{2}Ch^{-1}u^{m+mn}(\underline{P}\cdot\underline{n}) \tag{B.98}$$

Here $\underline{n}$ is calculated as

$$\underline{n} = \underline{\xi}^{(3)} = \overline{M}(\underline{\lambda})\underline{\lambda}^{(3)} = M(\underline{\lambda})\underline{\lambda}^{(3)} \tag{B.99}$$

so that

$$C_4 = \mp\frac{1}{2}Ch^{-1}u^{m+mn}\left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}^{(3)}\right] \tag{B.100}$$

The Poisson-form equations governing the Euler parameters of the orbital frame are then given by (B.71), (B.73), (B.75) and (B.77).

$$\underline{\lambda}' = +\frac{1}{2}Chu^{mn-2m}\underline{\lambda}^{(2)} \mp \frac{1}{2}Ch^{-1}u^{m+mn}\left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}^{(3)}\right]\underline{\lambda}^{(4)} \tag{B.101}$$

$$\underline{\lambda}^{(2)\prime} = -\frac{1}{2}Chu^{mn-2m}\underline{\lambda} + \frac{1}{2}Ch^{-1}u^{m+mn}\left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}^{(3)}\right]\underline{\lambda}^{(3)} \tag{B.102}$$

$$\underline{\lambda}^{(3)\prime} = \mp\frac{1}{2}Chu^{mn-2m}\underline{\lambda}^{(4)} - \frac{1}{2}Ch^{-1}u^{m+mn}\left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}^{(3)}\right]\underline{\lambda}^{(2)} \tag{B.103}$$

$$\underline{\lambda}^{(4)\prime} = \pm\frac{1}{2}Chu^{mn-2m}\underline{\lambda}^{(3)} \pm \frac{1}{2}Ch^{-1}u^{m+mn}\left[(M^T(\underline{\lambda})\underline{P})\cdot\underline{\lambda}^{(3)}\right]\underline{\lambda} \tag{B.104}$$

These are the generalizations of Vitins' (1978) Euler-parameter equations. Actually, this sixteenth-order set is greatly over-redundant. Because the four Euler-parameter vectors can be obtained from one another merely by interchanging components (with appropriate changes of sign), it suffices to integrate any one of these four equations. Vitins exhibited only equation (B.101), using $C = h^{-1}$, $n = 2$, $m = -1$ and lower signs throughout. The latter three equations above can be obtained from the first by operating with $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ respectively. Interestingly, this type of reduction in order is not possible in the Poisson-form equations governing the unit 3-vectors $(\underline{\xi}, \underline{\zeta}, \underline{n})$; those vectors differ by more than a simple re-ordering of components.

Some additional comments should be made. First, each of the four equations is linear, though coupled, in components of $\underset{\sim}{\lambda}$ in unperturbed motion. Constant-coefficient linear equations appear only for $n = 2$, regardless of the value of m, and fixed-frequency equations of perturbed motion appear only for the further choice $C = h^{-1}$. Second, the developments recorded here require that unique transverse and normal unit vectors, $\underset{\sim}{\zeta}$ and $\underset{\sim}{n}$ as given by equations (B.24) and (B.26), always exist. This requirement excludes rectilinear orbits ($h = 0$) from consideration *a priori*. In the general forms of the governing equations, (B.37) through (B.39) or (B.101) through (B.104), the singularity at $h = 0$ appears only in the perturbing terms but it is present nonetheless. Although these first-order equations correctly describe the behavior of the unit radial vector on an unperturbed rectilinear orbit, namely that it is a constant, this singular solution cannot be transformed by continuous processes into a near-rectilinear solution for either perturbed or unperturbed motion. Hence, these equations apply generally only to non-rectilinear orbits. Finally, it is notable that the bilinear relation (B.8) does not figure in the derivation of these first-order equations. In Chapter 4 the bilinear relation served to modify the calculation of $\underset{\sim}{\lambda}''$ in such a way that uncoupled linear second-order equations could be obtained. These second-order equations require initial conditions on both $\underset{\sim}{\lambda}$ and $\underset{\sim}{\lambda}'$, whereas the first-order coupled equations of Vitins' type require initial conditions only on $\underset{\sim}{\lambda}$. The question naturally arises, what is the significance of the extra boundary condition on $\underset{\sim}{\lambda}'$? The answer is that the bilinear relation needs to be enforced only because of insistence that the governing equations be second-order and uncoupled. The coupling present in the first-order equations insures automatically that the initial variation of $\underset{\sim}{\lambda}$ will be consistent with the actual kinematics of the orbital frame. This consistency is implied by the form of equation (B.79) for $\underset{\sim}{\xi}'$ in that its fourth component vanishes identically. In Chapter 4 the vanishing of the fourth component of $\underset{\sim}{\xi}'$ had to be secured by imposing the bilinear relation, an apparently artificial device. Yet the Euler parameters, being redundant with respect to the direction cosines of the unit radial vector, must be properly constrained in order to represent the motion of that vector. The uncoupled form of the second-order equations allows arbitrary initial rates of change of $\underset{\sim}{\lambda}$ without such a constraint. Recall from Chapter 4 that, in fact, the bilinear relation could be satisfied for the second-order equations only if the initial condition on $\underline{\lambda}'$ were specified not arbitrarily but in the particular way given by equation (4.172), namely, by

$$\underline{\lambda}'(0) = \frac{1}{2} M^T(\underline{\lambda}(0))\underline{\xi}'(0) \tag{B.105}$$

which should be compared with equation (B.79) above. This relation merely ensures that, for the uncoupled second-order equation, the initial variation of $\underline{\lambda}$ will be consistent with the actual kinematics of the orbital frame.

Appendix C. Alternate Forms of Kepler's Equation for Elliptic Orbits

The results obtained in Chapter 5 in the treatment of the true-anomaly time equation suggest some useful modifications of the eccentric-anomaly time equations presented in Chapter 2. The modifications are based on introducing tangent of half the anomaly in place of sine and cosine of the anomaly.

Consider the θ-domain time formula (2.120), rewritten here as

$$n[t - t(0)] = \theta + \alpha_1 \sin\theta + \beta_1(1 - \cos\theta) \equiv k(\theta; \alpha_1, \beta_1) \tag{C.1}$$

where

$$n = \frac{(-2E)\sqrt{-2E}}{\mu} \tag{C.2}$$

$$\alpha_1 = \frac{(-2E)r(0)}{\mu} - 1 \tag{C.3}$$

$$\beta_1 = \frac{(-2E)r'(0)}{\mu} \tag{C.4}$$

These formulae are valid only for elliptical orbits. Now when elapsed time is given and the angle θ is to be found, an iterative solution procedure is usually implemented (recent direct methods by Siewert and Buniston, 1972, and Ioakimidis and Papadakis, 1985, notwithstanding). During each iteration the two transcendental functions $\sin\theta$ and $\cos\theta$ must be evaluated. In some cases, it may be advantageous to consider the following modification. Introduce

$$z = \tan\frac{1}{2}\theta \tag{C.5}$$

and recall that

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta} = \frac{2z}{1+z^2} \tag{C.6}$$

$$\cos\theta = \frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta} = \frac{1-z^2}{1+z^2} \tag{C.7}$$

Then equation (C.1) can be recast as

$$k(\theta;\alpha_1,\beta_1) = 2\tan^{-1}z + \alpha_1\left[\frac{2z}{1+z^2}\right] + \beta_1\left[\frac{2z^2}{1+z^2}\right] \tag{C.8}$$

By iterating on the value of z in this equation, one needs only a single transcendental function evaluation per cycle.

A difficulty with this approach is that z goes to infinity at $\theta = \pm\pi$ and quadrant ambiguities arise for values of θ outside this interval. No such limitations arise with the original equation (C.1). The difficulty can be partly alleviated by redefining z as

$$z = \tan\frac{1}{4}\theta \tag{C.9}$$

Then $\sin\theta$ and $\cos\theta$ can be evaluated conveniently by means of the identities

$$\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta \tag{C.10}$$

$$\sin\theta = 2\left[\frac{2\tan\frac{1}{4}\theta}{1+\tan^2\frac{1}{4}\theta}\right]\left[\frac{1-\tan^2\frac{1}{4}\theta}{1+\tan^2\frac{1}{4}\theta}\right] \tag{C.11}$$

from which

$$\sin\theta = 2\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right] \tag{C.12}$$

and also

$$\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta \tag{C.13}$$

$$\cos\theta = \left[\frac{1-\tan^2\frac{1}{4}\theta}{1+\tan^2\frac{1}{4}\theta}\right]^2 - \left[\frac{2\tan\frac{1}{4}\theta}{1+\tan^2\frac{1}{4}\theta}\right]^2 \tag{C.14}$$

from which

$$\cos\theta = \left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2 \tag{C.15}$$

Then equation (C.1) can be recast as

$$k(\theta;\alpha_1,\beta_1) = 4\tan^{-1}z + \alpha_1\left[\frac{4z(1-z^2)}{(1+z^2)^2}\right] + \beta_1\left[\frac{8z^2}{(1+z^2)^2}\right] \tag{C.16}$$

In this formula z goes to infinity at $\theta = \pm 2\pi$. Since most applications of the elliptic-orbit Kepler's equation involve transfer angles of less than a revolution, equation (C.16) might be a useful alternate form in these cases. Nevertheless, computations involving more than one revolution of transfer can be made by further redefining z as

$$z = \tan\frac{1}{8}\theta \tag{C.17}$$

Then $\sin\theta$ and $\cos\theta$ would be evaluated by means of

$$\sin\theta = 2\left[2\sin\frac{1}{4}\theta\cos\frac{1}{4}\theta\right]\left[\cos^2\frac{1}{4}\theta\,\sin^2\frac{1}{4}\theta\right] \tag{C.18}$$

from which

$$\sin\theta = 4\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right]\left[\left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2\right] \tag{C.19}$$

and also

$$\cos\theta = \left[\cos^2\frac{1}{4}\theta - \sin^2\frac{1}{4}\theta\right]^2 - \left[2\sin\frac{1}{4}\theta\cos\frac{1}{4}\theta\right]^2 \tag{C.20}$$

from which

$$\cos\theta = \left[\left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2\right]^2 - 4\left[\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right]\right]^2 \tag{C.21}$$

Then equation (C.1) can be recast as

$$\begin{aligned} k(\theta;\alpha_1,\beta_1) = 8\tan^{-1}z + \alpha_1&\left[4\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right]\left[\left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2\right]\right] \\ + \beta_1&\left[1 - \left[\left[\frac{1-z^2}{1+z^2}\right]^2 - \left[\frac{2z}{1+z^2}\right]^2\right]^2 + 4\left[\left[\frac{2z}{1+z^2}\right]\left[\frac{1-z^2}{1+z^2}\right]\right]^2\right] \end{aligned} \tag{C.22}$$

In this formula z goes to infinity at $\theta = \pm 4\pi$ so that almost two full revolutions can be handled, though at some cost of increased algebraic complexity. It is clear that the useful interval of θ could be further enlarged by successively halving the angle in the definition of z. The resulting formulae become much more complicated algebraically, but still contain only the single transcendental function. Note that many algebraic combinations of z recur in the more complicated formulae so that judicious programming would allow even very extensive expressions to be evaluated efficiently.

A side benefit of these more complicated formulae (C.16) or (C.22) is that the arctangent function can usually be evaluated more efficiently because the magnitude of the argument is smaller. At least in the range $-\pi < \theta < +\pi$, it is true that

$$\tan\frac{1}{2}\theta > \tan\frac{1}{4}\theta > \tan\frac{1}{8}\theta > \cdots \tag{C.23}$$

so that the arctangent evaluation in (C.16) will require fewer series terms than the one in (C.8), and the arctangent evaluation in (C.22) will require fewer terms yet. Likewise, if a continued fraction method is used to evaluate the function then fewer recursions will be required. By means of the identity

$$\tan\frac{1}{2}\gamma = \frac{\tan\gamma}{1+\sqrt{1+\tan^2\gamma}} \tag{C.24}$$

it is possible to improve the convergence of the arctangent evaluations in any of the above formulae. Here the idea is not to enlarge the interval of θ for which a formula is valid but to improve the series or continued fraction convergence at a given value of θ. For example, let $z = \tan\gamma$ and $y = \tan\frac{1}{2}\gamma$. Then

$$y = \frac{z}{1+\sqrt{1+z^2}} \tag{C.25}$$

and

$$\tan^{-1}(z) = 2\tan^{-1}(y) \tag{C.26}$$

Now $\tan^{-1}(y)$ will converge faster than $\tan^{-1}(z)$ because $|y| < |z|$. In fact, equation (C.25) maps the interval $-\infty < z < +\infty$ into $-1 < y < +1$. Notice that, in principle, this transformation could be applied repeatedly to produce possibly drastic reductions in the number of terms or recursions needed to evaluate the arctangent. However, the price of this improvement is the evaluation of a square root every time (C.25) is invoked. Computationally, the square root evaluation may be just as costly as several more terms or recursions in the original arctangent evaluation.

Finally, it is worth mentioning here that Prussing (1979) has derived rigorous upper and lower bounds on the solution of the universal s-domain version of equation (C.1). Prussing's formulae are easily specialized to the present elliptic case; the result is that θ is bounded as

$$\frac{n[t - t(0)]}{1 + \sqrt{\alpha_1^2 + \beta_1^2}} \leq \theta \leq \frac{n[t - t(0)]}{1 - \sqrt{\alpha_1^2 + \beta_1^2}} \tag{C.27}$$

In this formula, the bounds may be adjustable inward in increments of 2π because the orbit is periodic in time and correspondingly 2π -periodic in θ. For example, since the time period T of the orbit is known then one may reason that

$$0 \leq t - t(0) \leq T \quad \rightarrow \quad 0 \leq \theta \leq 2\pi \tag{C.28a}$$

$$T \leq t - t(0) \leq 2T \quad \rightarrow \quad 2\pi \leq \theta \leq 4\pi \tag{C.28b}$$

and so forth. The most restrictive superposition of (C.27) and (C.28) applies. Then one immediately has rigorous bounds also on the value of z which satisfies one of the above alternate forms of equation (C.1). If a Newton-type iteration is to be used, the mean of the bounds is a reasonably efficient starting value for z. A *regula falsi* iteration method has merit also, since the use of these rigorous bounds to start the iteration will guarantee convergence to the solution.

APPENDIX D. NUMERICAL RESULTS FOR THE KEPLERIAN TIME OF FLIGHT

Included in this appendix is the FORTRAN computer program used to demonstrate various features of the true-anomaly time equations discussed in Chapter 5. Representative output is also included for thirty-eight sample cases. The program was compiled under VS FORTRAN Version 2 and run in double-precision on an IBM 3090 processor at the Virginia Tech Computing Center, as were all the programs used in this study.

The source code and output are largely self-explanatory. Zero values of the time mean that the calculation failed for some reason; return codes from each subroutine indicate the type of failure but were not recorded for these sample cases. A maximum of 101 terms or recursions is allowed in the evaluation of the transcendental function in the universal time equations. Both series and continued fraction calculations are implemented, and the effect of transforming the argument of the function by means of the half-angle transformation is demonstrated. It should be noted that part of the output from this program is a set of initial and final position and velocity vectors for each transfer arc considered. These vectors are not printed but are stored in a data file for use as input and reference data by later programs in the solution of the initial value and boundary value problems for these same thirty-eight transfer arcs.

```
C     COMPARISON OF KEPLERIAN TIME-OF-FLIGHT FORMULAE.
C        (APPENDIX  D  -  CONTINUED FRACTION METHODS)
C
C     FOR THE PURPOSES OF THIS STUDY, THE INITIAL PERICENTRAL
C     TRUE ANOMALY MUST LIE IN THE OPEN INTERVAL
C     - 180 DEGREES < ANGLE1 < + 180 DEGREES.
C     TRANSFER ANGLES WILL BE LIMITED TO LESS THAN ONE REVOLUTION
C     (POSITIVE OR NEGATIVE).
C
C  SETUP PROCEDURE AT VIRGINIA TECH COMPUTING CENTER:
C  COMPILE WITH:
C   FORTVS2 APPD1 ( SOURCE SRCFLG OPT(0) AUTODBL(DBL4)
C  THEN:
C   FILEDEF 1 DISK APPD1 ANGLES A1 ( RECFM F LRECL 80 BLKSIZE 80
C   FILEDEF 2 DISK APPD1 OUTPUT A1 ( RECFM F LRECL 80 BLKSIZE 80
C   FILEDEF 3 TERMINAL
C   FILEDEF 4 DISK APPD1 KPLRDATA A1 ( RECFM F LRECL 260 BLKSIZE 260
C   GLOBAL TXTLIB VSF2FORT CMSLIB
C   GLOBAL LOADLIB VSF2LOAD
C   LOAD APPD1 ( NOMAP
C   START
C
C     INPUT FROM UNIT 1 (DATA FILE), ECHO TO UNIT 3 (TERMINAL):
C        E = ORBITAL ECCENTRICITY
C        ANGLE1 = INTIAL TRUE ANOMALY FROM PERICENTER (DEGREES)
C        ANGLE2 = CHANGE IN TRUE ANOMALY FROM INITIAL (DEGREES)
C
C     OUTPUT TO UNIT 2 (PRINT FILE):
C        E = ORBITAL ECCENTRICITY
C        ANGLE1 = INTIAL TRUE ANOMALY FROM PERICENTER (DEGREES)
C        ANGLE2 = CHANGE IN TRUE ANOMALY FROM INITIAL (DEGREES)
C        TOF = TIME OF FLIGHT
C        TOFA = EITHER 0.0 OR UNADJUSTED TIME + ONE ORBITAL PERIOD
C        N = NUMBER OF TERMS OR RECURSIONS IN TRANSCENDENTAL FUNCTION
C
C     OUTPUT TO UNIT 4 (TO BE DATA FOR KEPLERIAN I.V.P. AND B.V.P.):
C        E = ORBITAL ECCENTRICITY
C        ANGLE1 = INTIAL TRUE ANOMALY FROM PERICENTER (DEGREES)
C        ANGLE2 = CHANGE IN TRUE ANOMALY FROM INITIAL (DEGREES)
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTF(I) = FINAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        TOF = TIME OF FLIGHT
C
C     CALLS: TOFS, TOFU2S, TOFU2, TOFU4
C
C     LOCAL PARAMETERS:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTF(I) = FINAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        GNUI = INITIAL PERICENTRAL TRUE ANOMALY (RADIANS)
C        GNUF = FINAL PERICENTRAL TRUE ANOMALY (RADIANS)
C        ETA = TRANSFER ANGLE (RADIANS)
C        NTA = 1 FOR TRANSFORMATION OF SPECIAL FUNCTION; =0 IF NO
C        LINES = OUTPUT PAGING PARAMETER, NUMBER OF LINES WRITTEN
C        LPAGE = OUTPUT PAGING PARAMETER, NUMBER OF LINES PER PAGE
C
      DIMENSION XI(3), XDOTI(3), XF(3), XDOTF(3)
      LPAGE = 60
      LINES = 99
C              FOR THIS STUDY, GM IS ALWAYS SET EQUAL TO 1.0 IN ORDER
C              TO NORMALIZE THE COORDINATES AND TIME.
      GM = 1.0
      TWOPI = 2. * ACOS(-1.)
```

```
      DEGRAD = 360./TWOPI
100   READ(1,102,END=900) E, ANGLE1, ANGLE2
102   FORMAT(F9.3,F10.2,F10.2)
      IF ( ANGLE1 .LE. -180. .OR. ANGLE1 .GE. +180. ) GO TO 100
      IF ( ANGLE2 .LE. -360. .OR. ANGLE2 .GE. +360. ) GO TO 100
      IF ( E .LT. 0.0 ) GO TO 900
         WRITE(3,102) E, ANGLE1, ANGLE2
         GNUI = ANGLE1 / DEGRAD
         ETA = ANGLE2 / DEGRAD
         GNUF = GNUI + ETA
         IF ( E .LT. 1.0 ) GO TO 110
C
C           CHECK THAT HYPERBOLIC TRANSFER IS PHYSICALLY POSSIBLE:
C           INITIAL AND FINAL POSITIONS MUST LIE BETWEEN ASYMPTOTES.
C
            ASYMP1 = ACOS ( -1. / E )
            ASYMP2 = - ASYMP1
            IF ( GNUI .LE. ASYMP2 .OR. GNUI .GE. ASYMP1 ) GO TO 100
            IF ( GNUF .LE. ASYMP2 .OR. GNUF .GE. ASYMP1 ) GO TO 100
110      CONTINUE
C
C        COMPUTE INITIAL AND FINAL POSITION AND VELOCITY VECTORS FROM
C        ECCENTRICITY AND CORRESPONDING PERICENTRAL TRUE ANOMALIES.
C        THESE VECTORS WILL BE NORMALIZED TO UNIT GRAVITATIONAL
C        PARAMETER AND UNIT ANGULAR MOMENTUM, AND WILL BE RESOLVED IN
C        THE PERIFOCAL FRAME. HENCE, ALL Z-COMPONENTS ARE SET TO ZERO.
C
         XI(1) = ( COS(GNUI) ) / ( 1. + E * COS(GNUI) )
         XI(2) = ( SIN(GNUI) ) / ( 1. + E * COS(GNUI) )
         XI(3) = 0.0
         XDOTI(1) = - SIN(GNUI)
         XDOTI(2) = E + COS(GNUI)
         XDOTI(3) = 0.0
         XF(1) = ( COS(GNUF) ) / ( 1. + E * COS(GNUF) )
         XF(2) = ( SIN(GNUF) ) / ( 1. + E * COS(GNUF) )
         XF(3) = 0.0
         XDOTF(1) = - SIN(GNUF)
         XDOTF(2) = E + COS(GNUF)
         XDOTF(3) = 0.0
C
C        ORBITAL PERIOD MAY BE NEEDED FOR TIME ON ELLIPTIC ORBITS.
C        THIS PERIOD IS NORMALIZED TO UNIT GRAVITATIONAL PARAMETER
C        AND UNIT ANGULAR MOMENTUM, USING THESIS EQUATION (5.104).
C
         PERIOD = 0.
         IF ( E .LT. 1.0 ) PERIOD = TWOPI / SQRT ( ( 1. - E**2 )**3 )
C
      IF ( LINES .LT. LPAGE ) GO TO 510
         WRITE(2,502)
502      FORMAT('1'/' ',
     $          T2,'ECCEN-',T17,'INITIAL',T29,'TRANSFER',
     $          T43,'TIME OF FLIGHT'/' ',
     $          T5,'TRICITY',T15,'TRUE ANOMALY',T29,'ANGLE'/' ',
     $          T17,'(DEG.)',T29,'(DEG.)'/' ',
     $          T2,55('-'))
         LINES = 5
510   CONTINUE
C
      TOF = 0.0
      CALL TOFS ( GM, XI, XDOTI, ETA, TOF, IFLAG )
      WRITE(2,512) E, ANGLE1, ANGLE2, TOF
512   FORMAT(' '/' ',T2,'E = ',F6.3,T15,'FROM ',F6.1,T28,'FOR ',F7.2,
     $       T42,F15.9,T59,'STANDARD TIME')
      LINES = LINES + 2
      TOFA = 0.0
      IF (E.LT.1. .AND. TOF.LT.0. .AND. ETA.GT.0.) TOFA=TOF+PERIOD
      IF (TOFA .LE. 0.0 ) GO TO 515
```

```
         WRITE(2,514) TOFA
514      FORMAT(' ',T42,F15.9,T59,'S. T. + PERIOD')
         LINES = LINES + 1
515   CONTINUE
C
      TOF = 0.
      NTA = 0
      CALL TOFU2S ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,522) TOF, N
522   FORMAT(' '/' ',T2,'1/2 ANGLE SERIES',
     $        T42,F15.9,', ',I3,T63,'TERMS')
      LINES = LINES + 2
      TOF = 0.
      NTA = 1
      CALL TOFU2S ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,524) TOF, N
524   FORMAT(' ',T2,'1/2 ANGLE SERIES, 1 TRANSFORMATION',
     $        T42,F15.9,', ',I3,T63,'TERMS')
      LINES = LINES + 1
      TOFA = 0.0
      IF (E.LT.1. .AND. TOF.LT.0. .AND. ETA.GT.0.) TOFA=TOF+PERIOD
      IF (TOFA .LE. 0.0 ) GO TO 527
         WRITE(2,526) TOFA
526      FORMAT(' ',T2,'1/2 ANGLE SERIES + PERIOD',T42,F15.9)
         LINES = LINES + 1
527   CONTINUE
C
      TOF = 0.
      NTA = 0
      CALL TOFU2 ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,532) TOF, N
532   FORMAT(' ',T2,'1/2 ANGLE CONTINUED FRACTION',
     $        T42,F15.9,', ',I3,T63,'RECURSIONS')
      LINES = LINES + 1
      TOF = 0.
      NTA = 1
      CALL TOFU2 ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,534) TOF, N
534   FORMAT(' ',T2,'1/2 ANGLE C. F., 1 TRANSFORMATION',
     $        T42,F15.9,', ',I3,T63,'RECURSIONS')
      LINES = LINES + 1
      TOFA = 0.0
      IF (E.LT.1. .AND. TOF.LT.0. .AND. ETA.GT.0.) TOFA=TOF+PERIOD
      IF (TOFA .LE. 0.0 ) GO TO 537
         WRITE(2,536) TOFA
536      FORMAT(' ',T2,'1/2 ANGLE C. F. + PERIOD',T42,F15.9)
         LINES = LINES + 1
537   CONTINUE
C
      TOF = 0.
      NTA = 0
      CALL TOFU4 ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,542) TOF, N
542   FORMAT(' ',T2,'1/4 ANGLE CONTINUED FRACTION',
     $        T42,F15.9,', ',I3,T63,'RECURSIONS')
      LINES = LINES + 1
      TOF = 0.
      NTA = 1
      CALL TOFU4 ( GM, XI, XDOTI, ETA, NTA, TOF, N, IFLAG )
      WRITE(2,544) TOF, N
544   FORMAT(' ',T2,'1/4 ANGLE C. F., 1 TRANSFORMATION',
     $        T42,F15.9,', ',I3,T63,'RECURSIONS')
      LINES = LINES + 1
      TOFA = 0.0
      IF (E.LT.1. .AND. TOF.LT.0. .AND. ETA.GT.0.) TOFA=TOF+PERIOD
      IF (TOFA .LE. 0.0 ) GO TO 547
```

```
      WRITE(2,546) TOFA
546      FORMAT(' ',T2,'1/4 ANGLE C. F. + PERIOD',T42,F15.9)
         LINES = LINES + 1
547   CONTINUE
C
      IF ( TOF .LE. 0.0 ) TOF = TOFA
      WRITE(4,556) E, ANGLE1, ANGLE2, (XI(I),I=1,3), (XDOTI(J),J=1,3),
     $             (XF(K),K=1,3), (XDOTF(L),L=1,3), TOF
556   FORMAT(F6.3,1X,F6.1,1X,F7.2,13(1X,F15.9))
      GO TO 100
900   CONTINUE
      STOP
      END
```

```
      SUBROUTINE TOFS ( GM, X, XDOT, ANGLE, TOF, IFLAG )
C
C     TIME OF FLIGHT CALCULATED BY STANDARD CLASSICAL FORMULAE.
C        (GEYLING AND WESTERMAN, 1971, CHAPTER 2.)
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        X(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOT(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        ANGLE = TRANSFER ANGLE = CHANGE IN TRUE ANOMALY (RADIANS)
C
C     OUTPUT:
C        TOF = TIME OF FLIGHT
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C               -1 = CALCULATION INCOMPLETE.
C                0 = CALCULATION COMPLETED NORMALLY.
C                1 = TRANSFER IS PHYSICALLY IMPOSSIBLE (ON HYPERBOLA).
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        E = ORBITAL ECCENTRICITY
C        EPSLN = TOLERANCE USED TO DETECT PARABOLIC AND
C                                              CIRCULAR ORBITS
C        A = LENGTH OF SEMIMAJOR AXIS
C        P = LENGTH OF SEMILATUS RECTUM
C        XN = GENERALIZED MEAN MOTION, DIFFERENT FOR EACH ORBIT
C        GNU(I) = TRUE ANOMALY FROM PERICENTER (1,2 = INITIAL,FINAL)
C        ALPHA = +E * COS (GNU(1))
C        BETA = -E * SIN (GNU(1))
C        TIME(I) = NORMALIZED TIME FROM PERICENTER (1,2 = INITIAL,FINAL)
C        ASYMP(I) = TRUE ANOMALY OF ASYMPTOTE FROM PERICENTER (1,2 = +,-)
C
      DIMENSION X(3), XDOT(3), HVEC(3), GNU(2), TIME(2), ASYMP(2)
C
C     FIXED PARAMETERS, SPECIFIED BY USER
C
      EPSLN = 1.0E-10
C
C     FIXED PARAMETERS, CALCULATED:
C
      PI = ACOS(-1.)
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
      TOF = 0.
C
      HVEC(1) = X(2) * XDOT(3) - X(3) * XDOT(2)
      HVEC(2) = X(3) * XDOT(1) - X(1) * XDOT(3)
      HVEC(3) = X(1) * XDOT(2) - X(2) * XDOT(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = X(1)**2 + X(2)**2 + X(3)**2
      R = SQRT ( RSQRD )
      ALPHA = ( HSQRD / ( GM * R ) ) - 1.
      D = X(1) * XDOT(1) + X(2) * XDOT(2) + X(3) * XDOT(3)
      BETA = - ( H * D ) / ( GM * R )
      ESQRD = ALPHA**2 + BETA**2
      E = SQRT ( ESQRD )
      CHECK = ABS ( 1. - E )
      IF ( CHECK .LE. EPSLN ) E = 1.0
      GNU(1) = 0.
      IF ( E .GT. EPSLN ) GNU(1) = ACOS ( ALPHA / E )
      IF ( BETA .GT. 0. ) GNU(1) = - GNU(1)
```

```
      GNU(2) = GNU(1) + ANGLE
      IF ( E .GE. 1.0 ) GO TO 200
C
C        STANDARD ELLIPTIC FORMULA: G&W EQUATION (2.2.37)
C
         F = SQRT ( ( 1. - E ) / ( 1. + E ) )
         QSQRD = 1. - ESQRD
         Q = SQRT ( QSQRD )
         A = HSQRD / ( GM * QSQRD )
         XN = SQRT ( GM / A**3 )
         DO 150 I=1,2
            TERM2 = ( E * Q * SIN(GNU(I)) ) / ( 1. + E * COS(GNU(I)) )
            TIME(I)= 2. * ATAN ( F * TAN ( GNU(I)/2. ) ) - TERM2
150      CONTINUE
         TOF = ( TIME(2) - TIME(1) ) / XN
         IFLAG = 0
      GO TO 400
200   CONTINUE
      IF ( E .GT. 1.0 ) GO TO 300
C
C        STANDARD PARABOLIC FORMULA: G&W EQUATION (2.3.8)
C
         P = HSQRD / GM
         XN = 2. * SQRT ( P**3 / GM )
         DO 250 I=1,2
            T = TAN ( GNU(I) / 2. )
            TIME(I) = T + ( T**3 / 3. )
250      CONTINUE
         TOF = ( TIME(2) - TIME(1) ) / XN
         IFLAG = 0
      GO TO 400
300   CONTINUE
C
C        STANDARD HYPERBOLIC FORMULA: G&W EQUATION (2.4.12)
C
         ASYMP(1) = ACOS ( -1. / E )
         ASYMP(2) = - ASYMP(1)
         IF ( GNU(1).LT.ASYMP(1) .AND. GNU(1).GT.ASYMP(2) ) GO TO 310
            IFLAG = 1
            RETURN
310      CONTINUE
         IF ( GNU(2).LT.ASYMP(1) .AND. GNU(2).GT.ASYMP(2) ) GO TO 320
            IFLAG = 1
            RETURN
320      CONTINUE
         F = SQRT ( ( E - 1. ) / ( E + 1. ) )
         QSQRD = ESQRD - 1.
         Q = SQRT ( QSQRD )
         A = HSQRD / ( GM * QSQRD )
         XN = SQRT ( GM / A**3 )
         DO 350 I=1,2
            Y = F * TAN ( GNU(I) / 2. )
            TERM1 = ( E * Q * SIN(GNU(I)) ) / ( 1. + E * COS(GNU(I)) )
            TIME(I) = TERM1 - ALOG ( ( 1. + Y ) / ( 1. - Y ) )
350      CONTINUE
         TOF = ( TIME(2) - TIME(1) ) / XN
         IFLAG = 0
400   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TOFU2S ( GM, X, XDOT, ANGLE, NTA, TOF, N, JFLAG )
C
C     TIME OF FLIGHT USING THESIS EQUATION (5.93).
C       THIS IS A UNIVERSAL ETA-DOMAIN HALF-ANGLE FORMULA, USING
C       AN INFINITE SERIES TO CALCULATE THE ONE TRANSCENDENTAL
C       FUNCTION.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        X(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOT(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        ANGLE = TRANSFER ANGLE = CHANGE IN TRUE ANOMALY (RADIANS)
C        NTA = 1 FOR TRANSFORMATION OF SFUN ARGUMENT; =0 FOR NO TRANS.
C
C     OUTPUT:
C        TOF = TIME OF FLIGHT
C        N = NUMBER OF TERMS USED IN SFUN SUBROUTINE
C        JFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C               -2 = CALCULATION NOT COMPLETE, SFUN CALL INCOMPLETE
C               -1 = CALCULATION NOT COMPLETE, LOCAL
C                0 = CALCULATION COMPLETE
C                1 = TRANSFER ANGLE IS PHYSICALLY IMPOSSIBLE
C                2 = TRANSCENDENTAL FUNCTION FAILED TO CONVERGE
C
C     CALLS:  SFUN
C
C     LOCAL PARAMETERS:
C        Z = TANGENT OF HALF THE TRANSFER ANGLE
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        ALPHA, BETA = ECCENTRICITY PARAMETERS (THESIS (5.64), (5.65))
C        ESQRD = SQUARE OF THE ORBITAL ECCENTRICITY
C
      DIMENSION X(3), XDOT(3), HVEC(3)
C
C     FIXED PARAMETERS, CALCULATED:
C
      TWOPI = 2. * ACOS ( -1. )
C
C     VARIABLE PARAMETERS:
C
      TOF = 0.
      JFLAG = -1
C
C     CALCULATE REGULAR ELEMENTS ALPHA AND BETA
C
      HVEC(1) = X(2) * XDOT(3) - X(3) * XDOT(2)
      HVEC(2) = X(3) * XDOT(1) - X(1) * XDOT(3)
      HVEC(3) = X(1) * XDOT(2) - X(2) * XDOT(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = X(1)**2 + X(2)**2 + X(3)**2
      R = SQRT ( RSQRD )
      ALPHA = ( HSQRD / ( GM * R ) ) - 1.
      D = X(1) * XDOT(1) + X(2) * XDOT(2) + X(3) * XDOT(3)
      BETA = - ( H * D ) / ( GM * R )
      ESQRD = ALPHA**2 + BETA**2
C      IF ( ESQRD .LT. 1.0 ) GO TO 100
C
C        CHECK THAT HYPERBOLIC TRANSFER IS PHYSICALLY POSSIBLE:
C        FINAL POSITION MUST LIE BETWEEN ASYMPTOTES CALCULATED
C        FROM THESIS EQUATION (5.102).
C
C         ARG1 = - ( 1. + ALPHA ) / ( + BETA + SQRT ( ESQRD - 1. ) )
C         ARG2 = + ( 1. + ALPHA ) / ( - BETA + SQRT ( ESQRD - 1. ) )
C         ASYMP1 = 2. * ATAN ( ARG1 )
C         ASYMP2 = 2. * ATAN ( ARG2 )
```

```
C         IF ( ASYMP1.LT.0.0 .AND. ASYMP2.LT.0.0 ) ASYMP2=ASYMP2+TWOPI
C         IF ( ASYMP1.GT.0.0 .AND. ASYMP2.GT.0.0 ) ASYMP1=ASYMP1-TWOPI
C         IF ( ASYMP1 .LT. ANGLE .AND. ANGLE .LT. ASYMP2 ) GO TO 100
C            JFLAG = 1
C            RETURN
C100   CONTINUE
C
C     IMPLEMENT THESIS EQUATION (5.93)
C
      Z = TAN ( ANGLE / 2. )
      FACTOR = 1 + ALPHA + BETA * Z
      XX = Z**2 * ( 1 - ALPHA**2 - BETA**2 ) / FACTOR**2
      CALL SFUN ( S3, XX, NTA, N, IFLAG )
      IF ( IFLAG .EQ. 0 ) GO TO 110
         IF ( IFLAG .EQ. -1 ) JFLAG = -2
         IF ( IFLAG .EQ. 1 ) JFLAG = 2
         RETURN
110   CONTINUE
      TERM1 = 2. * ( Z * FACTOR + Z**3 ) / ( FACTOR**3 * ( 1. + XX ) )
      TERM2 = 2. * ( Z / FACTOR )**3 * S3
      T = TERM1 - TERM2
      TOF = T * H**3 / GM**2
      JFLAG = 0
      RETURN
      END
```

```
      SUBROUTINE SFUN ( S3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTION WHICH OCCURS IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATION (5.93).
C        THE VALUE OF THE FUNCTION S3 IS COMPUTED USING
C        AN INFINITE SERIES.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1. + X * S3(X)
C
C        TRANSFORMATION OF THE ARGUMENT X TO IMPROVE CONVERGENCE
C        IS DONE, IF DESIRED, USING THESIS EQNS (5.183) AND (5.188).
C
C     INPUT:
C        X = ANY REAL VALUE BETWEEN -1. AND +1.
C        NTA = 1 FOR TRANSFORMATION OF ARGUMENT X; = 0 FOR NO TRANS.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        S3 = TRANSCENDENTAL FUNCTION RELATED TO THE ARCTANGENT.
C        N = NUMBER OF TERMS NEEDED IN THE SERIES
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -1 = CALCULATION NOT COMPLETE
C                 0 = CALCULATION CONVERGED AND COMPLETED.
C                 1 = SERIES NOT CONVERGED AFTER MAX TERMS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF TERMS ALLOWED FOR SERIES
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE SUMMATION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        S3LP = PREVIOUS ("LAST PASS") ESTIMATE OF S3
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQUATION (5.183)
C
      IF ( NTA .EQ. 0 ) GO TO 95
         SAVEX = X
         X = X / ( 1. + SQRT ( 1. + X ) )**2
95    CONTINUE
C
C     IMPLEMENT THESIS EQUATION (5.96).
C
      IF ( X .GT. -1. .AND. X .LT. 1. ) GO TO 100
         S3 = 0.
         IFLAG = 1
         RETURN
100   CONTINUE
      S3LP = 0.
      TERM = 1. / 3.
      N = 0
110   S3 = S3LP + TERM
         CHECK = ABS ( S3 - S3LP )
         IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 200
         S3LP = S3
```

```
         TERM = - TERM * X * ( 2. * N + 3. ) / ( 2. * N + 5. )
         N = N + 1
      GO TO 110
200                                                       CONTINUE
      IF ( N .LE. MAX ) IFLAG = 0
      IF ( N .GT. MAX ) IFLAG = 1
C
C     INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.188)
C
      IF ( NTA .EQ. 0 ) GO TO 210
         X = SAVEX
         XTERM = 1. + SQRT ( 1. + X )
         S3 = ( 1. + 2. * S3 / XTERM ) / XTERM**2
210   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TOFU2 ( GM, X, XDOT, ANGLE, NTA, TOF, N, JFLAG )
C
C     TIME OF FLIGHT USING THESIS EQUATION (5.112).
C       THIS IS A UNIVERSAL ETA-DOMAIN HALF-ANGLE FORMULA, USING
C       A CONTINUED FRACTION TO CALCULATE THE ONE TRANSCENDENTAL
C       FUNCTION.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        X(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOT(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        ANGLE = TRANSFER ANGLE = CHANGE IN TRUE ANOMALY (RADIANS)
C        NTA = 1 FOR TRANSFORMATION OF CFUN ARGUMENT; =0 FOR NO TRANS.
C
C     OUTPUT:
C        TOF = TIME OF FLIGHT
C        N = NUMBER OF RECURSIONS USED IN CFUN SUBROUTINE
C        JFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -2 = CALCULATION NOT COMPLETE, CFUN CALL INCOMPLETE
C                -1 = CALCULATION NOT COMPLETE, LOCAL
C                 0 = CALCULATION COMPLETE
C                 1 = TRANSFER ANGLE IS PHYSICALLY IMPOSSIBLE
C                 2 = TRANSCENDENTAL FUNCTION FAILED TO CONVERGE
C
C     CALLS:  CFUN
C
C     LOCAL PARAMETERS:
C        Z = TANGENT OF HALF THE TRANSFER ANGLE
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        ALPHA, BETA = ECCENTRICITY PARAMETERS (THESIS (5.64), (5.65))
C        ESQRD = SQUARE OF THE ORBITAL ECCENTRICITY
C
      DIMENSION X(3), XDOT(3), HVEC(3)
C
C     FIXED PARAMETERS, CALCULATED:
C
      TWOPI = 2. * ACOS ( -1. )
C
C     VARIABLE PARAMETERS:
C
      TOF = 0.
      JFLAG = -1
C
C     CALCULATE REGULAR ELEMENTS ALPHA AND BETA
C
      HVEC(1) = X(2) * XDOT(3) - X(3) * XDOT(2)
      HVEC(2) = X(3) * XDOT(1) - X(1) * XDOT(3)
      HVEC(3) = X(1) * XDOT(2) - X(2) * XDOT(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = X(1)**2 + X(2)**2 + X(3)**2
      R = SQRT ( RSQRD )
      ALPHA = ( HSQRD / ( GM * R ) ) - 1.
      D = X(1) * XDOT(1) + X(2) * XDOT(2) + X(3) * XDOT(3)
      BETA = - ( H * D ) / ( GM * R )
      ESQRD = ALPHA**2 + BETA**2
C      IF ( ESQRD .LT. 1.0 ) GO TO 100
C
C        CHECK THAT HYPERBOLIC TRANSFER IS PHYSICALLY POSSIBLE:
C        FINAL POSITION MUST LIE BETWEEN ASYMPTOTES CALCULATED
C        FROM THESIS EQUATION (5.102).
C
C         ARG1 = - ( 1. + ALPHA ) / ( + BETA + SQRT ( ESQRD - 1. ) )
C         ARG2 = + ( 1. + ALPHA ) / ( - BETA + SQRT ( ESQRD - 1. ) )
C         ASYMP1 = 2. * ATAN ( ARG1 )
C         ASYMP2 = 2. * ATAN ( ARG2 )
```

```
C         IF ( ASYMP1.LT.0.0 .AND. ASYMP2.LT.0.0 ) ASYMP2=ASYMP2+TWOPI
C         IF ( ASYMP1.GT.0.0 .AND. ASYMP2.GT.0.0 ) ASYMP1=ASYMP1-TWOPI
C         IF ( ASYMP1 .LT. ANGLE .AND. ANGLE .LT. ASYMP2 ) GO TO 100
C            JFLAG = 1
C            RETURN
C100   CONTINUE
C
C     IMPLEMENT THESIS EQUATION (5.112)
C
      Z = TAN ( ANGLE / 2. )
      FACTOR = 1 + ALPHA + BETA * Z
      XX = Z**2 * ( 1 - ALPHA**2 - BETA**2 ) / FACTOR**2
      CALL CFUN ( C1, C2, C3, XX, NTA, N, IFLAG )
      IF ( IFLAG .EQ. 0 ) GO TO 110
         IF ( IFLAG .EQ. -1 ) JFLAG = -2
         IF ( IFLAG .EQ. 1 ) JFLAG = 2
         RETURN
110   CONTINUE
      TERM1 = FACTOR * ( 1. + C1 )
      TERM2 = Z**2 * ( 1. - ( 1. / ( 3. + C2 ) ) )
      XNUMER = 2. * Z * ( TERM1 + TERM2 )
      T = XNUMER / ( FACTOR**3 * ( 1. + XX ) * ( 1. + C1 ) )
      TOF = T * H**3 / GM**2
      JFLAG = 0
      RETURN
      END
```

```
      SUBROUTINE TOFU4 ( GM, X, XDOT, ANGLE, NTA, TOF, N, JFLAG )
C
C     TIME OF FLIGHT USING THESIS EQUATION (5.151).
C       THIS IS A UNIVERSAL ETA-DOMAIN QUARTER-ANGLE FORMULA, USING
C       A CONTINUED FRACTION TO CALCULATE THE ONE TRANSCENDENTAL
C       FUNCTION.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        X(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOT(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        ANGLE = TRANSFER ANGLE = CHANGE IN TRUE ANOMALY (RADIANS)
C        NTA = 1 FOR TRANSFORMATION OF CFUN ARGUMENT; =0 FOR NO TRANS.
C
C     OUTPUT:
C        TOF = TIME OF FLIGHT
C        N = NUMBER OF RECURSIONS USED IN CFUN SUBROUTINE
C        JFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -2 = CALCULATION NOT COMPLETE, CFUN CALL INCOMPLETE
C                -1 = CALCULATION NOT COMPLETE, LOCAL
C                 0 = CALCULATION COMPLETE
C                 1 = TRANSFER ANGLE IS PHYSICALLY IMPOSSIBLE
C                 2 = TRANSCENDENTAL FUNCTION FAILED TO CONVERGE
C
C     CALLS:  CFUN
C
C     LOCAL PARAMETERS:
C        Z = TANGENT OF ONE-FOURTH THE TRANSFER ANGLE
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        ALPHA, BETA = ECCENTRICITY PARAMETERS (THESIS (5.64), (5.65))
C        ESQRD = SQUARE OF THE ORBITAL ECCENTRICITY
C
      DIMENSION X(3), XDOT(3), HVEC(3)
C
C     VARIABLE PARAMETERS:
C
      TOF = 0.
      JFLAG = -1
C
C     CALCULATE REGULAR ELEMENTS ALPHA AND BETA
C
      HVEC(1) = X(2) * XDOT(3) - X(3) * XDOT(2)
      HVEC(2) = X(3) * XDOT(1) - X(1) * XDOT(3)
      HVEC(3) = X(1) * XDOT(2) - X(2) * XDOT(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = X(1)**2 + X(2)**2 + X(3)**2
      R = SQRT ( RSQRD )
      ALPHA = ( HSQRD / ( GM * R ) ) - 1.
      D = X(1) * XDOT(1) + X(2) * XDOT(2) + X(3) * XDOT(3)
      BETA = - ( H * D ) / ( GM * R )
      ESQRD = ALPHA**2 + BETA**2
C      IF ( ESQRD .LT. 1.0 ) GO TO 100
C
C        CHECK THAT HYPERBOLIC TRANSFER IS PHYSICALLY POSSIBLE:
C        FINAL POSITION MUST LIE BETWEEN ASYMPTOTES CALCULATED
C        BY THESIS EQUATIONS (5.102) AND (5.129).
C        THE IDENTITY (5.129) ELIMINATES EXTRA QUADRANT RESOLUTIONS.
C
C         ARG1 = - ( 1. + ALPHA ) / ( + BETA + SQRT ( ESQRD - 1. ) )
C         ARG2 = + ( 1. + ALPHA ) / ( - BETA + SQRT ( ESQRD - 1. ) )
C         ASYMP1 = 4. * ATAN ( ARG1 / ( 1. + SQRT ( 1. + ARG1**2 ) ) )
C         ASYMP2 = 4. * ATAN ( ARG2 / ( 1. + SQRT ( 1. + ARG2**2 ) ) )
C         IF ( ASYMP1 .LT. ANGLE .AND. ANGLE .LT. ASYMP1 ) GO TO 100
C            JFLAG = 1
C            RETURN
```

```
C100   CONTINUE
C
C      IMPLEMENT THESIS EQUATION (5.151)
C
       Z = TAN ( ANGLE / 4. )
       D = ( 1. + ALPHA ) * ( 1. - Z**2 ) + 2. * BETA * Z
       XNUMER = 4. * Z**2 * ( 1. - ALPHA**2 - BETA**2 )
       F = D**2 + XNUMER
       DROOTF = D + SQRT ( F )
       Y = XNUMER / DROOTF**2
       CALL CFUN ( C1, C2, C3, Y, NTA, N, IFLAG )
       IF ( IFLAG .EQ. 0 ) GO TO 110
          IF ( IFLAG .EQ. -1 ) JFLAG = -2
          IF ( IFLAG .EQ. 1 ) JFLAG = 2
          RETURN
110    CONTINUE
       TERM1 = ( 8. * Z**2 + ( 1. - Z**2 ) * DROOTF ) * DROOTF
       TERM2 = ( D - ( 1. - Z**2 ) * ( 1. - ALPHA**2 - BETA**2 ) )
       T = 4. * Z * ( TERM1 - 4. * Z**2 * ( D + TERM2 / ( 3. + C2 ) ) )
       TOF = H**3 * T / ( GM**2 * F * DROOTF**2 * ( 1. + C1 ) )
       JFLAG = 0
       RETURN
       END
```

```
      SUBROUTINE CFUN ( C1, C2, C3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTIONS WHICH OCCUR IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATIONS. THE VALUE OF THE FUNCTION C3 IS COMPUTED USING
C        GAUTSCHI'S (1967) METHOD TO APPROXIMATE AN INFINITE
C        CONTINUED FRACTION. THE VALUES OF C1 AND C2 ARE COMPUTED
C        IN RATIONAL ALGEBRAIC TERMS OF C3.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1./(1.+C1(X))
C
C        TRANSFORMATION OF THE ARGUMENT OF THE CONTINUED FRACTIONS
C        TO IMPROVE CONVERGENCE IS IMPLEMENTED, IF DESIRED, BY
C        THESIS EQUATIONS (5.159) AND (5.166).
C
C     INPUT:
C        X = ANY REAL VALUE GREATER THAN -1.
C        NTA = 1 FOR TRANSFORMATION OF X; = 0 FOR NO TRANSFORMATION.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        C1,C2,C3 = TRANSCENDENTAL FUNCTIONS RELATED TO THE ARCTANGENT.
C        N = NUMBER OF RECURSIONS NEEDED IN GAUTSCHI'S METHOD FOR C3.
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -1 = CALCULATION NOT COMPLETE
C                 0 = CALCULATION CONVERGED AND COMPLETED.
C                 1 = CONTINUED FRAC. NOT CONVERGED AFTER MAX RECURSIONS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF RECURSIONS ALLOWED FOR CONTINUED FRAC.
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE RECURSION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        A = NTH PARTIAL NUMERATOR
C        B = NTH PARTIAL DENOMINATOR
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQN (5.159)
C
      IF ( NTA .EQ. 0 ) GO TO 105
         SAVEX = X
         X = X / ( 1. + SQRT( 1. + X ) )**2
105   CONTINUE
C
C     IMPLEMENT THESIS EQNS (5.105), (5.108), (5.109), (9.98), (9.99).
C
      N = 1
      A = 9. * X
      B = 7.
      U = 1.
      V = A / B
      W = V
110   CHECK = ABS ( V )
C                          NOTE:  V = W - WLP, "LP" MEANING "LAST PASS"
      IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 120
```

```
            BLP = B
            ULP = U
            VLP = V
            WLP = W
            N = N + 1
            A = ( N + 2 ) ** 2 * X
            B = 2 * N + 5
            U = 1. / ( 1. + ( A / ( B * BLP ) ) * ULP )
            V = VLP * ( U - 1. )
            W = WLP + V
         GO TO 110
120      CONTINUE
         IF ( N .LE. MAX ) IFLAG = 0
         IF ( N .GT. MAX ) IFLAG = 1
         C3 = W
         C2 = 4. * X / ( 5. + C3 )
         C1 = X / ( 3. + C2 )
C
C        INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.166)
C
         IF ( NTA .EQ. 0 ) GO TO 125
            X = SAVEX
            XTERM = 1. + SQRT ( 1. + X )
            TERM1 = 8. * XTERM**2 * ( 5. + C3 ) + 8. * X
            TERM2 = ( 3. * ( 5. + C3 ) + 4. ) * XTERM - 6.
            C3 = TERM1 / TERM2 - 5.
            C2 = 4. * X / ( 5. + C3 )
            C1 = X / ( 3. + C2 )
125      CONTINUE
         RETURN
         END
```

ECCEN-TRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	
E = 0.000	FROM 0.0	FOR 90.00	1.570796327	STANDARD TIME
1/2 ANGLE SERIES			0.000000000,	101 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			1.570796327,	12 TERMS
1/2 ANGLE CONTINUED FRACTION			1.570796327,	15 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			1.570796327,	8 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			1.570796327,	8 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			1.570796327,	6 RECURSIONS
E = 0.500	FROM 0.0	FOR 90.00	0.945599435	STANDARD TIME
1/2 ANGLE SERIES			0.945599435,	18 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.945599435,	8 TERMS
1/2 ANGLE CONTINUED FRACTION			0.945599435,	10 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.945599435,	7 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.945599435,	7 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.945599435,	5 RECURSIONS
E = 0.990	FROM 0.0	FOR 90.00	0.670689651	STANDARD TIME
1/2 ANGLE SERIES			0.670689651,	4 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.670689651,	4 TERMS
1/2 ANGLE CONTINUED FRACTION			0.670689651,	4 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.670689651,	4 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.670689651,	4 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.670689651,	3 RECURSIONS
E = 1.000	FROM 0.0	FOR 90.00	0.666666667	STANDARD TIME
1/2 ANGLE SERIES			0.666666667,	1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.666666667,	1 TERMS
1/2 ANGLE CONTINUED FRACTION			0.666666667,	1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.666666667,	1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.666666667,	1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.666666667,	1 RECURSIONS
E = 1.010	FROM 0.0	FOR 90.00	0.662689398	STANDARD TIME
1/2 ANGLE SERIES			0.662689398,	4 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.662689398,	4 TERMS
1/2 ANGLE CONTINUED FRACTION			0.662689398,	4 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.662689398,	4 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.662689398,	4 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.662689398,	3 RECURSIONS
E = 2.000	FROM 0.0	FOR 90.00	0.413218001	STANDARD TIME
1/2 ANGLE SERIES			0.413218001,	18 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.413218001,	9 TERMS
1/2 ANGLE CONTINUED FRACTION			0.413218001,	11 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.413218001,	7 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.413218001,	7 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.413218001,	5 RECURSIONS
E = 10.000	FROM 0.0	FOR 90.00	0.097971412	STANDARD TIME
1/2 ANGLE SERIES			0.097971412,	89 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.097971412,	22 TERMS
1/2 ANGLE CONTINUED FRACTION			0.097971412,	26 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.097971412,	12 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.097971412,	12 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.097971412,	8 RECURSIONS

ECCEN- TRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	
E = 0.000	FROM -45.0	FOR 135.00	2.356194490	STANDARD TIME
1/2 ANGLE SERIES			0.000000000,	8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			2.356194490,	24 TERMS
1/2 ANGLE CONTINUED FRACTION			2.356194490,	33 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			2.356194490,	11 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			2.356194490,	11 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			2.356194490,	7 RECURSIONS
E = 0.500	FROM -45.0	FOR 135.00	1.320132051	STANDARD TIME
1/2 ANGLE SERIES			0.000000000,	101 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			1.320132051,	11 TERMS
1/2 ANGLE CONTINUED FRACTION			1.320132051,	14 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			1.320132051,	8 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			1.320132051,	8 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			1.320132051,	6 RECURSIONS
E = 0.990	FROM -45.0	FOR 135.00	0.891714599	STANDARD TIME
1/2 ANGLE SERIES			0.891714599,	5 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.891714599,	4 TERMS
1/2 ANGLE CONTINUED FRACTION			0.891714599,	5 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.891714599,	4 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.891714599,	4 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.891714599,	3 RECURSIONS
E = 1.000	FROM -45.0	FOR 135.00	0.885618083	STANDARD TIME
1/2 ANGLE SERIES			0.885618083,	1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.885618083,	1 TERMS
1/2 ANGLE CONTINUED FRACTION			0.885618083,	1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.885618083,	1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.885618083,	1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.885618083,	1 RECURSIONS
E = 1.010	FROM -45.0	FOR 135.00	0.879596413	STANDARD TIME
1/2 ANGLE SERIES			0.879596413,	5 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.879596413,	4 TERMS
1/2 ANGLE CONTINUED FRACTION			0.879596413,	5 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.879596413,	4 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.879596413,	4 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.879596413,	3 RECURSIONS
E = 2.000	FROM -45.0	FOR 135.00	0.514615140	STANDARD TIME
1/2 ANGLE SERIES			0.514615140,	29 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.514615140,	12 TERMS
1/2 ANGLE CONTINUED FRACTION			0.514615140,	15 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.514615140,	9 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.514615140,	9 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.514615140,	6 RECURSIONS
E = 10.000	FROM -45.0	FOR 135.00	0.106021259	STANDARD TIME
1/2 ANGLE SERIES			0.000000000,	101 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION			0.106021259,	31 TERMS
1/2 ANGLE CONTINUED FRACTION			0.106021259,	38 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION			0.106021259,	15 RECURSIONS
1/4 ANGLE CONTINUED FRACTION			0.106021259,	15 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION			0.106021259,	9 RECURSIONS

```
ECCEN-          INITIAL     TRANSFER      TIME OF FLIGHT
   TRICITY   TRUE ANOMALY  ANGLE
                (DEG.)     (DEG.)
-----------------------------------------------------------------

E =  0.000    FROM  -45.0  FOR  179.00        3.124139361  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   9 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            0.000000000, 101 TERMS
1/2 ANGLE CONTINUED FRACTION                  0.000000000, 101 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.124139361,  15 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.124139361,  15 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.124139361,   8 RECURSIONS

E =  0.500    FROM  -45.0  FOR  179.00        2.524446185  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            2.524446185,  24 TERMS
1/2 ANGLE CONTINUED FRACTION                  2.524446185,  32 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             2.524446185,  11 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  2.524446185,  11 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             2.524446185,   7 RECURSIONS

E =  0.990    FROM  -45.0  FOR  179.00        3.518846604  STANDARD TIME

1/2 ANGLE SERIES                              3.518846604,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.518846604,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.518846604,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.518846604,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.518846604,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.518846604,   4 RECURSIONS

E =  1.000    FROM  -45.0  FOR  179.00        3.576056889  STANDARD TIME

1/2 ANGLE SERIES                              3.576056889,   1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.576056889,   1 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.576056889,   1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.576056889,   1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.576056889,   1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.576056889,   1 RECURSIONS

E =  1.010    FROM  -45.0  FOR  179.00        3.636374408  STANDARD TIME

1/2 ANGLE SERIES                              3.636374408,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.636374408,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.636374408,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.636374408,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.636374408,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.636374408,   4 RECURSIONS

E =  0.000    FROM  -45.0  FOR  180.00        3.141592654  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   4 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            0.000000000, 101 TERMS
1/2 ANGLE CONTINUED FRACTION                  0.000000000, 101 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.141592654,  15 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.141592654,  15 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.141592654,   8 RECURSIONS

E =  0.500    FROM  -45.0  FOR  180.00        2.565813983  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            2.565813983,  25 TERMS
1/2 ANGLE CONTINUED FRACTION                  2.565813983,  33 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             2.565813983,  11 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  2.565813983,  11 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             2.565813983,   7 RECURSIONS
```

```
ECCEN-          INITIAL      TRANSFER      TIME OF FLIGHT
   TRICITY   TRUE ANOMALY   ANGLE
               (DEG.)       (DEG.)
-------------------------------------------------------

E =  0.990    FROM  -45.0  FOR  180.00        3.705184939  STANDARD TIME

1/2 ANGLE SERIES                              3.705184939,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.705184939,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.705184939,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.705184939,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.705184939,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.705184939,   4 RECURSIONS

E =  1.000    FROM  -45.0  FOR  180.00        3.771236166  STANDARD TIME

1/2 ANGLE SERIES                              3.771236166,   1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.771236166,   1 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.771236166,   1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.771236166,   1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.771236166,   1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.771236166,   1 RECURSIONS

E =  1.010    FROM  -45.0  FOR  180.00        3.841039286  STANDARD TIME

1/2 ANGLE SERIES                              3.841039286,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.841039286,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.841039286,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.841039286,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.841039286,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.841039286,   4 RECURSIONS

E =  0.000    FROM  -45.0  FOR  181.00       -3.124139361  STANDARD TIME
                                              3.159045946  S. T. + PERIOD

1/2 ANGLE SERIES                              0.000000000,   4 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            0.000000000, 101 TERMS
1/2 ANGLE CONTINUED FRACTION                  0.000000000, 101 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION            -3.124139361,  15 RECURSIONS
1/2 ANGLE C. F. + PERIOD                      3.159045946
1/4 ANGLE CONTINUED FRACTION                  3.159045946,  15 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.159045946,   8 RECURSIONS

E =  0.500    FROM  -45.0  FOR  181.00        2.607979071  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            2.607979071,  25 TERMS
1/2 ANGLE CONTINUED FRACTION                  2.607979071,  34 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             2.607979071,  11 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  2.607979071,  11 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             2.607979071,   7 RECURSIONS

E =  0.990    FROM  -45.0  FOR  181.00        3.907342103  STANDARD TIME

1/2 ANGLE SERIES                              3.907342103,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.907342103,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.907342103,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.907342103,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.907342103,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.907342103,   4 RECURSIONS
```

```
ECCEN-          INITIAL      TRANSFER       TIME OF FLIGHT
  TRICITY   TRUE ANOMALY  ANGLE
               (DEG.)     (DEG.)
-------------------------------------------------------

E =  1.000    FROM  -45.0  FOR  181.00        3.983581181  STANDARD TIME

1/2 ANGLE SERIES                              3.983581181,   1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.983581181,   1 TERMS
1/2 ANGLE CONTINUED FRACTION                  3.983581181,   1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.983581181,   1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.983581181,   1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.983581181,   1 RECURSIONS

E =  1.010    FROM  -45.0  FOR  181.00        4.064362404  STANDARD TIME

1/2 ANGLE SERIES                              4.064362404,   7 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            4.064362404,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  4.064362404,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             4.064362404,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  4.064362404,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             4.064362404,   4 RECURSIONS

E =  0.000    FROM  -90.0  FOR  225.00       -2.356194490  STANDARD TIME
                                              3.926990817  S. T. + PERIOD

1/2 ANGLE SERIES                              0.000000000,   4 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION           -2.356194490,  24 TERMS
1/2 ANGLE SERIES + PERIOD                     3.926990817
1/2 ANGLE CONTINUED FRACTION                 -2.356194490,  33 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION            -2.356194490,  11 RECURSIONS
1/2 ANGLE C. F. + PERIOD                      3.926990817
1/4 ANGLE CONTINUED FRACTION                  3.926990817,  21 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.926990817,   9 RECURSIONS

E =  0.500    FROM  -90.0  FOR  225.00        3.136880802  STANDARD TIME

1/2 ANGLE SERIES                              0.000000000,   9 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            3.136880802,  91 TERMS
1/2 ANGLE CONTINUED FRACTION                  0.000000000, 101 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             3.136880802,  14 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  3.136880802,  14 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             3.136880802,   8 RECURSIONS

E =  0.990    FROM  -90.0  FOR  225.00        4.154849643  STANDARD TIME

1/2 ANGLE SERIES                              4.154849643,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            4.154849643,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                  4.154849643,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             4.154849643,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  4.154849643,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             4.154849643,   4 RECURSIONS

E =  1.000    FROM  -90.0  FOR  225.00        4.218951416  STANDARD TIME

1/2 ANGLE SERIES                              4.218951416,   1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION            4.218951416,   1 TERMS
1/2 ANGLE CONTINUED FRACTION                  4.218951416,   1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION             4.218951416,   1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                  4.218951416,   1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION             4.218951416,   1 RECURSIONS
```

```
ECCEN-          INITIAL      TRANSFER      TIME OF FLIGHT
   TRICITY   TRUE ANOMALY  ANGLE
               (DEG.)      (DEG.)
-------------------------------------------------------

E =  1.010   FROM  -90.0  FOR  225.00       4.286821668  STANDARD TIME

1/2 ANGLE SERIES                            4.286821668,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION          4.286821668,   5 TERMS
1/2 ANGLE CONTINUED FRACTION                4.286821668,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION           4.286821668,   5 RECURSIONS
1/4 ANGLE CONTINUED FRACTION                4.286821668,   5 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION           4.286821668,   4 RECURSIONS

E =  0.000   FROM -179.0  FOR  358.00      -0.034906585  STANDARD TIME
                                            6.248278722  S. T. + PERIOD

1/2 ANGLE SERIES                           -0.034906585,   3 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION         -0.034906585,   3 TERMS
1/2 ANGLE SERIES + PERIOD                   6.248278722
1/2 ANGLE CONTINUED FRACTION               -0.034906585,   3 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION          -0.034906585,   3 RECURSIONS
1/2 ANGLE C. F. + PERIOD                    6.248278722
1/4 ANGLE CONTINUED FRACTION                0.000000000, 101 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION           6.248278722,  15 RECURSIONS

E =  0.500   FROM -179.0  FOR  358.00       9.533984444  STANDARD TIME

1/2 ANGLE SERIES                           -0.139612165,   3 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION         -0.139612165,   3 TERMS
1/2 ANGLE SERIES + PERIOD                   9.533984444
1/2 ANGLE CONTINUED FRACTION               -0.139612165,   3 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION          -0.139612165,   3 RECURSIONS
1/2 ANGLE C. F. + PERIOD                    9.533984444
1/4 ANGLE CONTINUED FRACTION                0.000000000, 101 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION           9.533984444,  15 RECURSIONS

E =  0.990   FROM -179.0  FOR  358.00    1892.603114797  STANDARD TIME

1/2 ANGLE SERIES                         -345.603906230,   8 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION       -345.603906230,   5 TERMS
1/2 ANGLE SERIES + PERIOD                1892.603114797
1/2 ANGLE CONTINUED FRACTION             -345.603906230,   6 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION        -345.603906230,   5 RECURSIONS
1/2 ANGLE C. F. + PERIOD                 1892.603114797
1/4 ANGLE CONTINUED FRACTION                0.000000000, 101 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION        1892.603114797,  13 RECURSIONS

E =  1.000   FROM -179.0  FOR    0.01    7376.014175001  STANDARD TIME

1/2 ANGLE SERIES                         7376.014175005,   1 TERMS
1/2 ANGLE SERIES, 1 TRANSFORMATION       7376.014175005,   1 TERMS
1/2 ANGLE CONTINUED FRACTION             7376.014175005,   1 RECURSIONS
1/2 ANGLE C. F., 1 TRANSFORMATION        7376.014175005,   1 RECURSIONS
1/4 ANGLE CONTINUED FRACTION             7376.014175005,   1 RECURSIONS
1/4 ANGLE C. F., 1 TRANSFORMATION        7376.014175005,   1 RECURSIONS
```

APPENDIX E. NUMERICAL RESULTS FOR THE KEPLERIAN INITIAL VALUE PROBLEM

Included in this appendix are the FORTRAN computer programs used to solve the initial value problem discussed in Chapter 6. The first program implements the η-domain method and the second program implements the σ-domain method. Representative output is also included for thirty-eight sample cases for both programs. The input data for these programs was generated from the time-of-flight program in Appendix D.

The source code and output are largely self-explanatory. It should be recalled that Newton's method was proposed for solving the time equation of the initial value problem. For some of the example cases, the increment in the independent variable z called for by Newton's formula (equation (6.23) of Chapter 6) occasionally turned out to be excessively large and had to be limited by a "stepsize coefficient" *CONV*, $0 < CONV < 1$. A rather small value of this coefficient was used to generate all the solutions recorded here, and the value was halved during iterations if necessary. The extra logic needed to allow the stepsize to increase was not implemented and no attempt was made to find the largest permissible value. This accounts for the apparently large number of iterations required in most cases. A maximum of 999 Newtonian iterations was allowed in all cases. It will be seen that the η-domain method fails for some transfers near one revolution and for the last example case which is nearly rectilinear. The σ-domain method succeeds in all cases considered, though appreciable position error occurs in the near-rectilinear example case.

```
C     KEPLERIAN INITIAL VALUE PROBLEM
C        (APPENDIX  E  -  ETA-DOMAIN METHOD)
C
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE) IF THE ORBIT
C     IS ELLIPTICAL.
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.151) AND
C     IMPLEMENTS THE ETA-DOMAIN QUARTER-ANGLE EQUATIONS OF THESIS
C     CHAPTER 6.
C
C  SETUP PROCEDURE AT VIRGINIA TECH COMPUTING CENTER:
C  COMPILE WITH:
C   FORTVS2 APPE1 ( SOURCE SRCFLG OPT(0) AUTODBL(DBL4)
C  THEN:
C   FILEDEF 1 DISK APPD1 KPLRDATA A1 ( RECFM F LRECL 260 BLKSIZE 260
C   FILEDEF 2 DISK APPE1 OUTPUT A1 ( RECFM F LRECL 121 BLKSIZE 121
C   FILEDEF 3 TERMINAL
C   GLOBAL TXTLIB VSF2FORT CMSLIB
C   GLOBAL LOADLIB VSF2LOAD
C   LOAD APPE1 ( NOMAP
C   START
C
C     INPUT FROM UNIT 1:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION DETERMINED OTHERWISE (1,2,3 = X,Y,Z)
C        XDOTF(I) = FINAL VELOCITY DETERMINED OTHERWISE (1,2,3 = X,Y,Z)
C
C     OUTPUT TO UNIT 2:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        DR = ERROR IN FINAL POSITION MAGNITUDE
C        DV = ERROR IN FINAL VELOCITY MAGNITUDE
C        NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C
C     LOCAL PARAMETERS:
C        XFC(I) = FINAL POSITION COMPUTED HERE (1,2,3 = X,Y,Z)
C        XDOTFC(I) = FINAL VELOCITY COMPUTED HERE (1,2,3 = X,Y,Z)
C        SUMR = SUM-SQUARED ERROR IN FINAL POSITION COMPONENTS
C        SUMV = SUM-SQUARED ERROR IN FINAL VELOCITY COMPONENTS
C        LINES = OUTPUT PAGING PARAMETER, NO. OF LINES WRITTEN
C        LPAGE = OUTPUT PAGING PARAMETER, NO. OF LINES PER PAGE
C
C     CALLS: KIVP1
C
      DIMENSION XI(3),XDOTI(3),XF(3),XDOTF(3),XFC(3),XDOTFC(3)
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C              FOR THE PURPOSES OF THIS STUDY, GM IS SET EQUAL TO 1.0
C              IN ORDER TO NORMALIZE THE COORDINATES AND TIME.
      GM = 1.0
      LPAGE = 55
C
C     VARIABLE PARAMETERS:
C
      LINES = 99
C
100   READ(1,112,END=900) E,ANGLE1,ANGLE2,(XI(I),I=1,3),
     $             (XDOTI(J),J=1,3), (XF(K),K=1,3),(XDOTF(L),L=1,3),TOF
112   FORMAT(F6.3,1X,F6.1,1X,F7.2,13(1X,F15.9))
```

```
      WRITE(3,122) E,ANGLE1,ANGLE2
122   FORMAT(F9.3,2F9.1)
      CALL KIVP1( GM, TOF, XI, XDOTI, XFC, XDOTFC, NIT, IFLAG )
      SUMR = 0.
      SUMV = 0.
      DO 210 I=1,3
         SUMR = SUMR + ( XFC(I) - XF(I) ) ** 2
         SUMV = SUMV + ( XDOTFC(I) - XDOTF(I) ) ** 2
210   CONTINUE
      DR = SQRT ( SUMR )
      DV = SQRT ( SUMV )
      IF ( LINES .LT. LPAGE ) GO TO 220
         WRITE(2,212)
212      FORMAT('1',' '/' ',
     $          T15,'KEPLERIAN INITIAL VALUE PROBLEM',
     $          T46,'  -  ETA DOMAIN',3(' '/),
     $          T2,'ECCEN-',T14,'INITIAL',T26,'TRANSFER',
     $          T36,'TIME OF',T45,'FINAL',T56,'FINAL',
     $          T64,'NUMBER OF'/' ',
     $          T3,'TRICITY',T12,'TRUE ANOMALY',T26,'ANGLE',
     $          T36,'FLIGHT',T44,'POSITION',T55,'VELOCITY',
     $          T64,'NEWTONIAN'/' ',
     $          T14,'(DEG.)',T26,'(DEG.)',T45,'ERROR',T56,'ERROR',
     $          T65,'ITER-'/' ',T67,'ATIONS'/
     $          ' ',T2,71('-'))
         LINES = 9
220   CONTINUE
      WRITE(2,224) E,ANGLE1,ANGLE2,TOF,DR,DV,NIT
224   FORMAT(' ',T2,F6.3,T14,F6.1,T25,F7.2,T34,F8.3,T43,E9.2,
     $        T54,E9.2,T67,I3)
      LINES = LINES + 1
    GO TO 100
900 CONTINUE
    STOP
    END
```

```
      SUBROUTINE KIVP1( GM, TOF, XI, XDOTI, XF, XDOTF, NIT, IFLAG )
C
C     KEPLERIAN INITIAL VALUE PROBLEM  -  UNIVERSAL SOLUTION.
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE) IF THE ORBIT
C     IS ELLIPTICAL.
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.151) AND
C     IMPLEMENTS THE ETA-DOMAIN QUARTER-ANGLE EQUATIONS OF THESIS
C     CHAPTER 6.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C
C     OUTPUT:
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTF(I) = FINAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C               -2 = CALCULATION NOT COMPLETE, CFUN CALL NOT COMPLETE
C               -1 = CALCULATION NOT COMPLETE, LOCAL
C                0 = CALCULATION COMLETED NORMALLY
C                1 = NUMBER OF NEWTON ITERATIONS EXCEEDED MAXIMUM
C                2 = NUMBER OF CFUN RECURSIONS EXCEEDED MAXIMUM
C                3 = TRANSFER TIME EXCEEDS RANGE OF STUDY ( < ONE REV.)
C
C     LOCAL PARAMETERS:
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        RI = INITIAL RADIUS
C        ALPHA = ECCENTRICITY PARAMETER
C        BETA = ECCENTRICITY PARAMETER
C        ESQRD = SQUARE OF ORBITAL ECCENTRICITY
C        PERIOD = TIME PERIOD OF ELLIPTICAL ORBIT
C        UNITXI(I) = INITIAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETAI(I) = INITIAL UNIT TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        UNITX(I) = FINAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETA(I) = FINAL UNIT TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        Z = TANGENT OF ONE-QUARTER OF THE TRANSFER ANGLE
C        DELTAZ = NEWTONIAN CORRECTION TO ESTIMATED VALUE OF Z
C        CONV = CONVERGENCE FACTOR USED IN NEWTON'S METHOD
C        MAX = MAXIMUM NUMBER OF NEWTON ITERATIONS PERMITTED
C        EPSLN = TOLERANCE USED TO DETECT CONVERGENCE OF ITERATIONS
C
C     CALLS:  CFUN
C
      DIMENSION XI(3),XDOTI(3),XF(3),XDOTF(3),HVEC(3)
      DIMENSION UNITXI(3), UNITX(3), ZETAI(3), ZETA(3)
      TWOPI = 2. * ACOS( -1. )
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 998
      CONV = 0.2
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
      NIT = 0
      DO 110 I=1,3
         XF(I) = 0.
         XDOTF(I) = 0.
110   CONTINUE
```

```
C
C     COMPUTE REGULAR ELEMENTS USING THESIS EQNS (6.1) - (6.7)
C
      HVEC(1) = XI(2) * XDOTI(3) - XI(3) * XDOTI(2)
      HVEC(2) = XI(3) * XDOTI(1) - XI(1) * XDOTI(3)
      HVEC(3) = XI(1) * XDOTI(2) - XI(2) * XDOTI(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = XI(1)**2 + XI(2)**2 + XI(3)**2
      RI = SQRT ( RSQRD )
      D = XI(1) * XDOTI(1) + XI(2) * XDOTI(2) + XI(3) * XDOTI(3)
      ALPHA = HSQRD / ( GM * RI ) - 1.
      BETA = - ( H * D ) / ( GM * RI )
      ESQRD = ALPHA**2 + BETA**2
      IF ( ESQRD .GE. 1.0 ) GO TO 120
C
C        COMPUTE PERIOD OF ELLIPTICAL ORBIT USING THESIS EQN (5.104).
C        REJECT ELLIPTICAL TRANSFERS LONGER THAN ONE PERIOD
C        AS BEING OUTSIDE THE RANGE OF THIS STUDY.
C
         PERIOD = TWOPI * H * HSQRD / (GM**2 * SQRT((1.-ESQRD)**3) )
         IF ( TOF .LT. PERIOD ) GO TO 115
            IFLAG = 3
            RETURN
115      CONTINUE
120   CONTINUE
C
C     THESIS EQUATIONS (6.2) AND (6.3C) FOR INITIAL UNIT VECTORS
C
      DO 125 I=1,3
         UNITXI(I) = XI(I) / RI
         ZETAI(I) = ( XDOTI(I) * RI - UNITXI(I) * D ) / H
125   CONTINUE
C
C     SOLVE TIME EQUATION (5.151) FOR Z = TAN (1/4 ETA), WHERE ETA
C     IS THE TRANSFER ANGLE.
C     NEWTONIAN ITERATION IS IMPLEMENTED BY THESIS EQNS (6.42) - (6.43).
C
C        COMPUTE FIRST GUESS FOR Z = TAN (1/4 ETA); THESIS EQN (6.59)
C
      Z = 0.
      IF ( ESQRD .GE. 1.0 ) GO TO 140
         ETA0 = GM**2 * TOF * ( 1. - SQRT ( ESQRD ) )**2 / H**3
         Z = TAN ( ETA0 / 4. )
140   CONTINUE
      D = ( 1. + ALPHA ) * ( 1. - Z**2 ) + 2. * BETA * Z
      XNUMER = 4. * Z**2 * ( 1. - ALPHA**2 - BETA**2 )
      F = D**2 + XNUMER
      DROOTF = D + SQRT ( F )
      Y = XNUMER / DROOTF**2
      DELTAZ = 999.
150   CHECK = ABS ( DELTAZ )
      IF ( CHECK .LE. EPSLN .OR. NIT .GT. MAX ) GO TO 200
C
C        EVALUATE THESIS TIME EQUATION (5.151) * F
C        (DENOMINATOR FACTOR F HAS BEEN CANCELLED)
C
         NTA = 1
         CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
         IF ( JFLAG .EQ. 0 ) GO TO 160
            IF ( JFLAG .EQ. -1 ) IFLAG = -2
            IF ( JFLAG .EQ. 1 ) IFLAG = 2
            RETURN
160      CONTINUE
         TERM1 = ( 8. * Z**2 + ( 1. - Z**2 ) * DROOTF ) * DROOTF
         TERM2 = ( D - ( 1. - Z**2 ) * ( 1. - ALPHA**2 - BETA**2 ) )
```

```
          T = 4. * Z * ( TERM1 - 4. * Z**2 * ( D + TERM2 / ( 3. + C2 )))
          FK = T / ( DROOTF**2 * ( 1. + C1 ) )
C
C         COMPUTE NEWTONIAN CORRECTION USING THESIS EQUATION (6.43)
C         AND MODIFYING CONVERGENCE FACTOR AS NEEDED TO KEEP ALL
C         VARIABLES IN PROPER RANGE ( F > 0, Y > -1 ).
C
          ZTERM = GM**2 * TOF * F**2 / H**3 - FK * F
          DELTAZ = ZTERM / ( 4. * ( 1. + Z**2 )**3 * ( 1. + ALPHA ) )
165       Z = Z + CONV * DELTAZ
          D = ( 1. + ALPHA ) * ( 1. - Z**2 ) + 2. * BETA * Z
          XNUM = 4. * Z**2 * ( 1. - ALPHA**2 - BETA**2 )
          F = D**2 + XNUM
          IF ( F .GT. 0.0 )          GO TO 170
             Z = Z - CONV * DELTAZ
             CONV = CONV / 2.
          GO TO 165
170                                  CONTINUE
          DROOTF = D + SQRT ( F )
          Y = XNUM / DROOTF**2
          IF ( Y .GT. -1.0 )         GO TO 175
             Z = Z - CONV * DELTAZ
             CONV = CONV / 2.
          GO TO 165
175                                  CONTINUE
          NIT = NIT + 1
       GO TO 150
200    CONTINUE
       IF ( NIT .LE. MAX ) GO TO 210
          IFLAG = 1
          RETURN
210    CONTINUE
C
C      USE Z = TAN (1/4 ETA) TO COMPUTE FINAL POSITION AND VELOCITY
C      BY MEANS OF THESIS EQUATIONS (6.8) - (6.21).
C
       SINETA = 2. * ( 2. * Z ) * ( 1. - Z**2 ) / ( 1. + Z**2 )**2
       COSETA = ( ( 1. - Z**2 )**2 - ( 2. * Z )**2 ) / ( 1. + Z**2 )**2
       U = GM * ( 1 + ALPHA * COSETA + BETA * SINETA ) / H**2
       UPRIME = GM * ( - ALPHA * SINETA + BETA * COSETA ) / H**2
       DO 300 I=1,3
          UNITX(I) = UNITXI(I) * COSETA + ZETAI(I) * SINETA
          ZETA(I) = - UNITXI(I) * SINETA + ZETAI(I) * COSETA
          XF(I) = UNITX(I) / U
          XDOTF(I) = H * ( U * ZETA(I) - UPRIME * UNITX(I) )
300    CONTINUE
       IFLAG = 0
       RETURN
       END
```

```
      SUBROUTINE CFUN ( C1, C2, C3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTIONS WHICH OCCUR IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATIONS. THE VALUE OF THE FUNCTION C3 IS COMPUTED USING
C        GAUTSCHI'S (1967) METHOD TO APPROXIMATE AN INFINITE
C        CONTINUED FRACTION. THE VALUES OF C1 AND C2 ARE COMPUTED
C        IN RATIONAL ALGEBRAIC TERMS OF C3.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1./(1.+C1(X))
C
C        TRANSFORMATION OF THE ARGUMENT OF THE CONTINUED FRACTIONS
C        TO IMPROVE CONVERGENCE IS IMPLEMENTED, IF DESIRED, BY
C        THESIS EQUATIONS (5.159) AND (5.166).
C
C     INPUT:
C        X = ANY REAL VALUE GREATER THAN -1.
C        NTA = 1 FOR TRANSFORMATION OF X; = 0 FOR NO TRANSFORMATION.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        C1,C2,C3 = TRANSCENDENTAL FUNCTIONS RELATED TO THE ARCTANGENT.
C        N = NUMBER OF RECURSIONS NEEDED IN GAUTSCHI'S METHOD FOR C3.
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -1 = CALCULATION NOT COMPLETE
C                 0 = CALCULATION CONVERGED AND COMPLETED.
C                 1 = CONTINUED FRAC. NOT CONVERGED AFTER MAX RECURSIONS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF RECURSIONS ALLOWED FOR CONTINUED FRAC.
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE RECURSION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        A = NTH PARTIAL NUMERATOR
C        B = NTH PARTIAL DENOMINATOR
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQN (5.159)
C
      IF ( NTA .EQ. 0 ) GO TO 105
         SAVEX = X
         X = X / ( 1. + SQRT( 1. + X ) )**2
105   CONTINUE
C
C     IMPLEMENT THESIS EQNS (5.105), (5.108), (5.109), (9.98), (9.99).
C
      N = 1
      A = 9. * X
      B = 7.
      U = 1.
      V = A / B
      W = V
110   CHECK = ABS ( V )
C                          NOTE:  V = W - WLP, "LP" MEANING "LAST PASS"
```

```
      IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 120
         BLP = B
         ULP = U
         VLP = V
         WLP = W
         N = N + 1
         A = ( N + 2 ) ** 2 * X
         B = 2 * N + 5
         U = 1. / ( 1. + ( A / ( B * BLP ) ) * ULP )
         V = VLP * ( U - 1. )
         W = WLP + V
      GO TO 110
120   CONTINUE
      IF ( N .LE. MAX ) IFLAG = 0
      IF ( N .GT. MAX ) IFLAG = 1
      C3 = W
      C2 = 4. * X / ( 5. + C3 )
      C1 = X / ( 3. + C2 )
C
C     INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.166)
C
      IF ( NTA .EQ. 0 ) GO TO 125
         X = SAVEX
         XTERM = 1. + SQRT ( 1. + X )
         TERM1 = 8. * XTERM**2 * ( 5. + C3 ) + 8. * X
         TERM2 = ( 3. * ( 5. + C3 ) + 4. ) * XTERM - 6.
         C3 = TERM1 / TERM2 - 5.
         C2 = 4. * X / ( 5. + C3 )
         C1 = X / ( 3. + C2 )
125   CONTINUE
      RETURN
      END
```

KEPLERIAN INITIAL VALUE PROBLEM - ETA DOMAIN

ECCEN-TRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	FINAL POSITION ERROR	FINAL VELOCITY ERROR	NUMBER OF NEWTONIAN ITER-ATIONS
0.000	0.0	90.00	1.571	0.21E-09	0.21E-09	1
0.500	0.0	90.00	0.946	0.10E-08	0.14E-08	63
0.990	0.0	90.00	0.671	0.11E-08	0.14E-08	44
1.000	0.0	90.00	0.667	0.36E-09	0.26E-09	44
1.010	0.0	90.00	0.663	0.73E-09	0.56E-09	44
2.000	0.0	90.00	0.413	0.11E-08	0.18E-08	32
10.000	0.0	90.00	0.098	0.24E-09	0.15E-08	139
0.000	-45.0	135.00	2.356	0.28E-08	0.30E-08	12
0.500	-45.0	135.00	1.320	0.32E-09	0.54E-09	72
0.990	-45.0	135.00	0.892	0.27E-08	0.41E-08	55
1.000	-45.0	135.00	0.886	0.22E-08	0.35E-08	55
1.010	-45.0	135.00	0.880	0.10E-08	0.17E-08	54
2.000	-45.0	135.00	0.515	0.97E-09	0.79E-09	34
10.000	-45.0	135.00	0.106	0.33E-08	0.65E-08	49
0.000	-45.0	179.00	3.124	0.50E-08	0.52E-08	16
0.500	-45.0	179.00	2.524	0.14E-08	0.12E-08	72
0.990	-45.0	179.00	3.519	0.14E-07	0.50E-08	52
1.000	-45.0	179.00	3.576	0.11E-07	0.43E-08	52
1.010	-45.0	179.00	3.636	0.52E-08	0.20E-08	51
0.000	-45.0	180.00	3.142	0.55E-08	0.55E-08	16
0.500	-45.0	180.00	2.566	0.65E-09	0.76E-09	72
0.990	-45.0	180.00	3.705	0.14E-07	0.48E-08	52
1.000	-45.0	180.00	3.771	0.12E-07	0.40E-08	51
1.010	-45.0	180.00	3.841	0.48E-08	0.17E-08	50
0.000	-45.0	181.00	3.159	0.49E-08	0.50E-08	16
0.500	-45.0	181.00	2.608	0.11E-08	0.10E-08	72
0.990	-45.0	181.00	3.907	0.15E-07	0.49E-08	51
1.000	-45.0	181.00	3.984	0.12E-07	0.42E-08	49
1.010	-45.0	181.00	4.064	0.56E-08	0.20E-08	49
0.000	-90.0	225.00	3.927	0.26E-09	0.26E-09	1
0.500	-90.0	225.00	3.137	0.44E-09	0.29E-09	104
0.990	-90.0	225.00	4.155	0.37E-09	0.27E-09	98
1.000	-90.0	225.00	4.219	0.80E-09	0.30E-09	97
1.010	-90.0	225.00	4.287	0.77E-09	0.29E-09	97
0.000	-179.0	358.00	6.248	0.52E-08	0.52E-08	83
0.500	-179.0	358.00	9.534	0.43E-08	0.77E-09	296
0.990	-179.0	358.00	1892.603	0.99E+02	0.20E-01	999
1.000	-179.0	0.01	7376.014	0.64E+04	0.18E-01	999

```
C       KEPLERIAN INITIAL VALUE PROBLEM
C          (APPENDIX  E  -  SIGMA-DOMAIN METHOD)
C
C       FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C       LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE) IF THE ORBIT
C       IS ELLIPTICAL.
C       THE METHOD IS BASED ON THESIS TIME EQUATION (5.238) AND
C       IMPLEMENTS THE SIGMA-DOMAIN QUARTER-ANGLE EQUATIONS OF THESIS
C       CHAPTER 6.
C       RECTILINEAR TRANSFERS ARE PERMITTED.
C
C    SETUP PROCEDURE AT VIRGINIA TECH COMPUTING CENTER:
C    COMPILE WITH:
C     FORTVS2 APPE2 ( SOURCE SRCFLG OPT(0) AUTODBL(DBL4)
C    THEN:
C     FILEDEF 1 DISK APPD1 KPLRDATA A1 ( RECFM F LRECL 260 BLKSIZE 260
C     FILEDEF 2 DISK APPE2 OUTPUT A1 ( RECFM F LRECL 121 BLKSIZE 121
C     FILEDEF 3 TERMINAL
C     GLOBAL TXTLIB VSF2FORT CMSLIB
C     GLOBAL LOADLIB VSF2LOAD
C     LOAD APPE2 ( NOMAP
C     START
C
C       INPUT FROM UNIT 1:
C          E = ECCENTRICITY; USED TO IDENTIFY CASE
C          ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C          ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C          TOF = TIME OF FLIGHT
C          XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C          XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C          XF(I) = FINAL POSITION DETERMINED OTHERWISE (1,2,3 = X,Y,Z)
C          XDOTF(I) = FINAL VELOCITY DETERMINED OTHERWISE (1,2,3 = X,Y,Z)
C
C       OUTPUT TO UNIT 2:
C          E = ECCENTRICITY; USED TO IDENTIFY CASE
C          ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C          ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C          TOF = TIME OF FLIGHT
C          DR = ERROR IN FINAL POSITION MAGNITUDE
C          DV = ERROR IN FINAL VELOCITY MAGNITUDE
C          NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C
C       LOCAL PARAMETERS:
C          XFC(I) = FINAL POSITION COMPUTED HERE (1,2,3 = X,Y,Z)
C          XDOTFC(I) = FINAL VELOCITY COMPUTED HERE (1,2,3 = X,Y,Z)
C          SUMR = SUM-SQUARED ERROR IN FINAL POSITION COMPONENTS
C          SUMV = SUM-SQUARED ERROR IN FINAL VELOCITY COMPONENTS
C          LINES = OUTPUT PAGING PARAMETER, NO. OF LINES WRITTEN
C          LPAGE = OUTPUT PAGING PARAMETER, NO. OF LINES PER PAGE
C
C       CALLS: KIVP2
C
        DIMENSION XI(3),XDOTI(3),XF(3),XDOTF(3),XFC(3),XDOTFC(3)
C
C       FIXED PARAMETERS, SPECIFIED BY USER:
C                FOR THE PURPOSES OF THIS STUDY, GM IS SET EQUAL TO 1.0
C                IN ORDER TO NORMALIZE THE COORDINATES AND TIME.
        GM = 1.0
        LPAGE = 55
C
C       VARIABLE PARAMETERS:
C
        LINES = 99
C
100     READ(1,112,END=900) E,ANGLE1,ANGLE2,(XI(I),I=1,3),
```

```
     $              (XDOTI(J),J=1,3), (XF(K),K=1,3),(XDOTF(L),L=1,3),TOF
112   FORMAT(F6.3,1X,F6.1,1X,F7.2,13(1X,F15.9))
         WRITE(3,122) E,ANGLE1,ANGLE2
122      FORMAT(F9.3,2F9.1)
         CALL KIVP2( GM, TOF, XI, XDOTI, XFC, XDOTFC, NIT, IFLAG )
         SUMR = 0.
         SUMV = 0.
         DO 210 I=1,3
            SUMR = SUMR + ( XFC(I) - XF(I) ) ** 2
            SUMV = SUMV + ( XDOTFC(I) - XDOTF(I) ) ** 2
210      CONTINUE
         DR = SQRT ( SUMR )
         DV = SQRT ( SUMV )
         IF ( LINES .LT. LPAGE ) GO TO 220
            WRITE(2,212)
212         FORMAT('1',' '/' ',
     $              T15,'KEPLERIAN INITIAL VALUE PROBLEM',
     $              T46,'  -  SIGMA DOMAIN',3(' '/),
     $              T2,'ECCEN-',T14,'INITIAL',T26,'TRANSFER',
     $              T36,'TIME OF',T45,'FINAL',T56,'FINAL',
     $              T64,'NUMBER OF'/' ',
     $              T3,'TRICITY',T12,'TRUE ANOMALY',T26,'ANGLE',
     $              T36,'FLIGHT',T44,'POSITION',T55,'VELOCITY',
     $              T64,'NEWTONIAN'/' ',
     $              T14,'(DEG.)',T26,'(DEG.)',T45,'ERROR',T56,'ERROR',
     $              T65,'ITER-'/' ',T67,'ATIONS'/
     $              ' ',T2,71('-'))
            LINES = 9
220      CONTINUE
         WRITE(2,224) E,ANGLE1,ANGLE2,TOF,DR,DV,NIT
224      FORMAT(' ',T2,F6.3,T14,F6.1,T25,F7.2,T34,F8.3,T43,E9.2,
     $           T54,E9.2,T67,I3)
         LINES = LINES + 1
      GO TO 100
900   CONTINUE
      STOP
      END
```

```
      SUBROUTINE KIVP2 (GM, TOF, XI, XDOTI, XF, XDOTF, NIT, IFLAG )
C
C     KEPLERIAN INITIAL VALUE PROBLEM  -  UNIVERSAL SOLUTION.
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE) IF THE ORBIT
C     IS ELLIPTICAL.
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.238) AND
C     IMPLEMENTS THE SIGMA-DOMAIN QUARTER-ANGLE EQUATIONS OF THESIS
C     CHAPTER 6.
C     RECTILINEAR TRANSFERS ARE PERMITTED.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C
C     OUTPUT:
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTF(I) = FINAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -2 = CALCULATION NOT COMPLETE, CFUN CALL NOT COMPLETE
C                -1 = CALCULATION NOT COMPLETE, LOCAL
C                 0 = CALCULATION COMLETED NORMALLY
C                 1 = NUMBER OF NEWTON ITERATIONS EXCEEDED MAXIMUM
C                 2 = NUMBER OF CFUN RECURSIONS EXCEEDED MAXIMUM
C                 3 = TRANSFER TIME EXCEEDS RANGE OF STUDY ( < ONE REV.)
C
C     LOCAL PARAMETERS:
C        HVEC(I) = ANGULAR MOMENTUM VECTOR (1,2,3 = X,Y,Z)
C        H = ANGULAR MOMENTUM MAGNITUDE
C        RI = INITIAL RADIUS
C        G = TWICE THE NEGATIVE ENERGY
C        PERIOD = TIME PERIOD OF ELLIPTICAL ORBIT
C        UI = INITIAL RECIPROCAL RADIUS
C        UPI = INITIAL RECIPROCAL RADIUS RATE (SIGMA-DOMAIN)
C        UNITXI(I) = INITIAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETAI(I) = INITIAL TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        U = FINAL RECIPROCAL RADIUS
C        UP = FINAL RECIPROCAL RADIUS RATE (SIGMA-DOMAIN)
C        UNITX(I) = FINAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETA(I) = FINAL TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        Z = 1/H TAN (1/4 H * SIGMA), THESIS EQN (6.95)
C        DELTAZ = NEWTONIAN CORRECTION TO ESTIMATED VALUE OF Z
C        MAX = MAXIMUM NUMBER OF NEWTON ITERATIONS PERMITTED
C        CONV = CONVERGENCE FACTOR USED IN NEWTON'S METHOD
C
C     CALLS:  CFUN
C
      DIMENSION XI(3),XDOTI(3),XF(3),XDOTF(3),HVEC(3)
      DIMENSION UNITXI(3), UNITX(3), ZETAI(3), ZETA(3)
      TWOPI = 2. * ACOS( -1. )
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 998
      CONV = 0.2
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
      NIT = 0
      DO 110 I=1,3
         XF(I) = 0.
```

```
         XDOTF(I) = 0.
110   CONTINUE
C
C     COMPUTE REGULAR ELEMENTS USING THESIS EQNS (6.60) - (6.64).
C
      HVEC(1) = XI(2) * XDOTI(3) - XI(3) * XDOTI(2)
      HVEC(2) = XI(3) * XDOTI(1) - XI(1) * XDOTI(3)
      HVEC(3) = XI(1) * XDOTI(2) - XI(2) * XDOTI(1)
      HSQRD = HVEC(1)**2 + HVEC(2)**2 + HVEC(3)**2
      H = SQRT ( HSQRD )
      RSQRD = XI(1)**2 + XI(2)**2 + XI(3)**2
      RI = SQRT ( RSQRD )
      D = XI(1) * XDOTI(1) + XI(2) * XDOTI(2) + XI(3) * XDOTI(3)
      UI = 1. / RI
      UPI = - D / RI
C        THESIS EQN (6.92). G = -2 * ENERGY, FROM THESIS EQN (5.208).
      G = 2. * GM * UI - H**2 * UI**2 - UPI**2
      IF ( G .LE. 0.0 ) GO TO 120
C
C        COMPUTE PERIOD OF ELLIPTICAL ORBIT USING
C        THESIS EQNS (5.104) AND (5.208).
C        REJECT ELLIPTICAL TRANSFERS LONGER THAN ONE PERIOD
C        AS BEING OUTSIDE THE RANGE OF THIS STUDY.
C
         PERIOD = TWOPI * SQRT ( GM**2 / G**3 )
         IF ( TOF .LT. PERIOD ) GO TO 115
            IFLAG = 3
            RETURN
115      CONTINUE
120   CONTINUE
C
C     THESIS EQUATIONS (6.61) AND (6.62B) FOR INITIAL VECTORS
C
      DO 125 I=1,3
         UNITXI(I) = XI(I) / RI
         ZETAI(I) = ( XDOTI(I) * RI - UNITXI(I) * D )
125   CONTINUE
C
C     SOLVE TIME EQUATION (5.238) FOR  Z.
C     NEWTON ITERATION IS IMPLEMENTED BY THESIS EQNS (6.100) - (6.101).
C
      Z = 0.
      A = UI * ( 1. - H**2 * Z**2 ) + 2. * UPI * Z
      XNUM = 4. * Z**2 * G
      Q = A**2 + XNUM
      AROOTQ = A + SQRT ( Q )
      Y = XNUM / AROOTQ**2
      DELTAZ = 999.
150   CHECK = ABS ( DELTAZ )
      IF ( CHECK .LE. EPSLN .OR. NIT .GT. MAX ) GO TO 200
C
C        EVALUATE THESIS TIME EQUATION (5.238) * Q
C        (DENOMINATOR FACTOR Q HAS BEEN CANCELLED)
C
         NTA = 1
         CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
         IF ( JFLAG .EQ. 0 ) GO TO 160
            IF ( JFLAG .EQ. -1 ) IFLAG = -2
            IF ( JFLAG .EQ. 1 ) IFLAG = 2
            RETURN
160      CONTINUE
         HZ = H * Z
         GMA = GM * A
         TERM1 = ( 8.* GM * Z**2 + ( 1. - HZ**2 ) * AROOTQ ) * AROOTQ
         TERM2 = ( GMA - ( 1. - HZ**2 ) * G )
         T = 4.* Z * ( TERM1 - 4. * Z**2 * ( GMA + TERM2 / ( 3. + C2 )))
```

```
          QF = T / ( AROOTQ**2 * ( 1. + C1 ) )
C
C         COMPUTE NEWTONIAN CORRECTION USING THESIS EQUATION (6.101)
C         AND MODIFYING CONVERGENCE FACTOR AS NEEDED TO KEEP ALL
C         VARIABLES IN PROPER RANGE ( Q > 0, Y > -1 ).
C
          ZTERM = TOF * Q**2 - QF * Q
          DELTAZ = ZTERM / ( 4. * UI**2 * ( 1. + HZ**2 )**3 )
165       Z = Z + CONV * DELTAZ
          A = UI * ( 1. - H**2 * Z**2 ) + 2. * UPI * Z
          XNUM = 4. * Z**2 * G
          Q = A**2 + XNUM
          IF ( Q .GT. 0.0 )        GO TO 170
             Z = Z - CONV * DELTAZ
             CONV = CONV / 2.
          GO TO 165
170                                CONTINUE
          AROOTQ = A + SQRT ( Q )
          Y = XNUM / AROOTQ**2
          IF ( Y .GT. -1.0 )       GO TO 175
             Z = Z - CONV * DELTAZ
             CONV = CONV / 2.
          GO TO 165
175                                CONTINUE
          NIT = NIT + 1
       GO TO 150
200    CONTINUE
       IF ( NIT .LE. MAX ) GO TO 210
          IFLAG = 1
          RETURN
210    CONTINUE
C
C      USE  Z  TO COMPUTE FINAL POSITION AND VELOCITY
C      BY MEANS OF THESIS EQUATIONS (6.65) - (6.80).
C
       SOH = 2. * ( 2. * Z ) * ( 1. - HZ**2 ) / ( 1. + HZ**2 )**2
       COH2 = 2. * ( ( 2. * Z ) / ( 1. + HZ**2 ) )**2
       S = H * SOH
       C = 1. - H**2 * COH2
       U = UI * C + UPI * SOH + GM * COH2
       UP = - UI * H * S + UPI * C + GM * SOH
       DO 300 I=1,3
          UNITX(I) = UNITXI(I) * C + ZETAI(I) * SOH
          ZETA(I) = - UNITXI(I) * H * S + ZETAI(I) * C
          XF(I) = UNITX(I) / U
          XDOTF(I) = U * ZETA(I) - UP * UNITX(I)
300    CONTINUE
       IFLAG = 0
       RETURN
       END
```

```
      SUBROUTINE CFUN ( C1, C2, C3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTIONS WHICH OCCUR IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATIONS. THE VALUE OF THE FUNCTION C3 IS COMPUTED USING
C        GAUTSCHI'S (1967) METHOD TO APPROXIMATE AN INFINITE
C        CONTINUED FRACTION. THE VALUES OF C1 AND C2 ARE COMPUTED
C        IN RATIONAL ALGEBRAIC TERMS OF C3.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1./(1.+C1(X))
C
C        TRANSFORMATION OF THE ARGUMENT OF THE CONTINUED FRACTIONS
C        TO IMPROVE CONVERGENCE IS IMPLEMENTED, IF DESIRED, BY
C        THESIS EQUATIONS (5.159) AND (5.166).
C
C     INPUT:
C        X = ANY REAL VALUE GREATER THAN -1.
C        NTA = 1 FOR TRANSFORMATION OF X; = 0 FOR NO TRANSFORMATION.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        C1,C2,C3 = TRANSCENDENTAL FUNCTIONS RELATED TO THE ARCTANGENT.
C        N = NUMBER OF RECURSIONS NEEDED IN GAUTSCHI'S METHOD FOR C3.
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -1 = CALCULATION NOT COMPLETE
C                 0 = CALCULATION CONVERGED AND COMPLETED.
C                 1 = CONTINUED FRAC. NOT CONVERGED AFTER MAX RECURSIONS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF RECURSIONS ALLOWED FOR CONTINUED FRAC.
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE RECURSION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        A = NTH PARTIAL NUMERATOR
C        B = NTH PARTIAL DENOMINATOR
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQN (5.159)
C
      IF ( NTA .EQ. 0 ) GO TO 105
         SAVEX = X
         X = X / ( 1. + SQRT( 1. + X ) )**2
105   CONTINUE
C
C     IMPLEMENT THESIS EQNS (5.105), (5.108), (5.109), (9.98), (9.99).
C
      N = 1
      A = 9. * X
      B = 7.
      U = 1.
      V = A / B
      W = V
110   CHECK = ABS ( V )
C                          NOTE:  V = W - WLP, "LP" MEANING "LAST PASS"
```

```
      IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 120
         BLP = B
         ULP = U
         VLP = V
         WLP = W
         N = N + 1
         A = ( N + 2 ) ** 2 * X
         B = 2 * N + 5
         U = 1. / ( 1. + ( A / ( B * BLP ) ) * ULP )
         V = VLP * ( U - 1. )
         W = WLP + V
      GO TO 110
120   CONTINUE
      IF ( N .LE. MAX ) IFLAG = 0
      IF ( N .GT. MAX ) IFLAG = 1
      C3 = W
      C2 = 4. * X / ( 5. + C3 )
      C1 = X / ( 3. + C2 )
C
C     INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.166)
C
      IF ( NTA .EQ. 0 ) GO TO 125
         X = SAVEX
         XTERM = 1. + SQRT ( 1. + X )
         TERM1 = 8. * XTERM**2 * ( 5. + C3 ) + 8. * X
         TERM2 = ( 3. * ( 5. + C3 ) + 4. ) * XTERM - 6.
         C3 = TERM1 / TERM2 - 5.
         C2 = 4. * X / ( 5. + C3 )
         C1 = X / ( 3. + C2 )
125   CONTINUE
      RETURN
      END
```

KEPLERIAN INITIAL VALUE PROBLEM - SIGMA DOMAIN

ECCENTRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	FINAL POSITION ERROR	FINAL VELOCITY ERROR	NUMBER OF NEWTONIAN ITERATIONS
0.000	0.0	90.00	1.571	0.54E-10	0.54E-10	101
0.500	0.0	90.00	0.946	0.11E-08	0.15E-08	99
0.990	0.0	90.00	0.671	0.12E-08	0.16E-08	97
1.000	0.0	90.00	0.667	0.21E-09	0.63E-10	97
1.010	0.0	90.00	0.663	0.65E-09	0.72E-09	97
2.000	0.0	90.00	0.413	0.14E-08	0.15E-08	94
10.000	0.0	90.00	0.098	0.23E-08	0.21E-08	184
0.000	-45.0	135.00	2.356	0.27E-08	0.30E-08	104
0.500	-45.0	135.00	1.320	0.24E-09	0.48E-09	101
0.990	-45.0	135.00	0.892	0.26E-08	0.40E-08	99
1.000	-45.0	135.00	0.886	0.22E-08	0.33E-08	99
1.010	-45.0	135.00	0.880	0.95E-09	0.15E-08	99
2.000	-45.0	135.00	0.515	0.66E-09	0.11E-08	96
10.000	-45.0	135.00	0.106	0.12E-08	0.70E-08	186
0.000	-45.0	179.00	3.124	0.49E-08	0.52E-08	107
0.500	-45.0	179.00	2.524	0.12E-08	0.11E-08	101
0.990	-45.0	179.00	3.519	0.13E-07	0.48E-08	94
1.000	-45.0	179.00	3.576	0.11E-07	0.41E-08	94
1.010	-45.0	179.00	3.636	0.49E-08	0.18E-08	94
0.000	-45.0	180.00	3.142	0.54E-08	0.55E-08	107
0.500	-45.0	180.00	2.566	0.49E-09	0.71E-09	101
0.990	-45.0	180.00	3.705	0.14E-07	0.46E-08	94
1.000	-45.0	180.00	3.771	0.12E-07	0.38E-08	94
1.010	-45.0	180.00	3.841	0.46E-08	0.16E-08	94
0.000	-45.0	181.00	3.159	0.48E-08	0.50E-08	107
0.500	-45.0	181.00	2.608	0.10E-08	0.99E-09	102
0.990	-45.0	181.00	3.907	0.15E-07	0.48E-08	94
1.000	-45.0	181.00	3.984	0.12E-07	0.40E-08	94
1.010	-45.0	181.00	4.064	0.55E-08	0.18E-08	94
0.000	-90.0	225.00	3.927	0.33E-09	0.33E-09	110
0.500	-90.0	225.00	3.137	0.52E-09	0.34E-09	105
0.990	-90.0	225.00	4.155	0.44E-09	0.29E-09	98
1.000	-90.0	225.00	4.219	0.89E-09	0.35E-09	97
1.010	-90.0	225.00	4.287	0.86E-09	0.34E-09	97
0.000	-179.0	358.00	6.248	0.52E-08	0.52E-08	154
0.500	-179.0	358.00	9.534	0.43E-08	0.77E-09	136
0.990	-179.0	358.00	1892.603	0.13E-06	0.61E-09	130
1.000	-179.0	0.01	7376.014	0.20E-03	0.12E-08	60

APPENDIX F. NUMERICAL RESULTS FOR THE KEPLERIAN BOUNDARY VALUE PROBLEM

Included in this appendix are the FORTRAN computer programs used to solve the boundary value problem discussed in Chapter 7. The first program implements the η-domain method and the second program implements the σ-domain method. Representative output is also included for thirty-eight example cases for both programs. The input data for these programs was generated from the time-of-flight program in Appendix D.

The source code and output are largely self-explanatory. It should be recalled that Newton's method for solving the time equation of the boundary value problem turned out to require extensive calculations. For the example cases recorded here a secant-type iteration was programmed. Two starting values of the variable of iteration q are needed in this method and there is no general way of choosing them. In this study the first starting value was always chosen as $q = 0$, which, incidentally, is the solution for circular orbits, and the second starting value was always chosen from the interval $-1 < q < +1$. No attempt was made to insure that the two starting values bounded the solution value. Many different second starting values were tried; the example cases recorded here are representative of the degree of success which may be expected with such arbitrary starting values. It will be seen that the η-domain method occasionally encounters difficulty for transfer arcs near one revolution. Both methods failed on the last, near-rectilinear case, though the σ -domain method seems to fail only through slow convergence.

```
C     KEPLERIAN BOUNDARY VALUE PROBLEM
C        (APPENDIX  F  -  ETA-DOMAIN METHOD)
C
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE).
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.151) AND
C     IMPLEMENTS THE ETA-DOMAIN EQUATIONS OF THESIS CHAPTER 7.
C
C  SETUP PROCEDURE AT VIRGINIA TECH COMPUTING CENTER:
C  COMPILE WITH:
C   FORTVS2 APPF1 ( SOURCE SRCFLG OPT(0) AUTODBL(DBL4)
C  THEN:
C   FILEDEF 1 DISK APPD1 KPLRDATA A1 ( RECFM F LRECL 260 BLKSIZE 260
C   FILEDEF 2 DISK APPF1 OUTPUT A1 ( RECFM F LRECL 121 BLKSIZE 121
C   FILEDEF 3 TERMINAL
C   GLOBAL TXTLIB VSF2FORT CMSLIB
C   GLOBAL LOADLIB VSF2LOAD
C   LOAD APPF1 ( NOMAP
C   START
C
C     INPUT FROM UNIT 1:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY DETERMINED OTHERWISE(1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C
C     OUTPUT TO UNIT 2:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        DV = ERROR IN INITIAL VELOCITY MAGNITUDE
C        NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C
C     LOCAL PARAMETERS:
C        XDOTIC(I) = INITIAL VELOCITY COMPUTED HERE (1,2,3 = X,Y,Z)
C        SUMV = SUM-SQUARED ERROR IN INITIAL VELOCITY COMPONENTS
C        LINES = OUTPUT PAGING PARAMETER, NO. OF LINES WRITTEN
C        LPAGE = OUTPUT PAGING PARAMETER, NO. OF LINES PER PAGE
C
C     CALLS: KBVP1
C
      LOGICAL LONG
      DIMENSION XI(3),XDOTI(3),XF(3),XDOTIC(3)
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C              FOR THE PURPOSES OF THIS STUDY, GM IS SET EQUAL TO 1.0
C              IN ORDER TO NORMALIZE THE COORDINATES AND TIME.
      GM = 1.0
      LPAGE = 55
C
C     VARIABLE PARAMETERS:
C
      LINES = 99
C
100   READ(1,112,END=900) E,ANGLE1,ANGLE2,(XI(I),I=1,3),
     $             (XDOTI(J),J=1,3), (XF(K),K=1,3),TOF
112   FORMAT(F6.3,1X,F6.1,1X,F7.2,9(1X,F15.9),3(16X),1X,F15.9)
         WRITE(3,122) E,ANGLE1,ANGLE2
122      FORMAT(F9.3,2F9.1)
         IF ( ANGLE2 .GT. 180. ) LONG = .TRUE.
         IF ( ANGLE2 .LE. 180. ) LONG = .FALSE.
         CALL KBVP1( GM, TOF, XI, XF, LONG, XDOTIC, NIT, IFLAG )
```

```
        SUMV = 0.
        DO 210 I=1,3
           SUMV = SUMV + ( XDOTIC(I) - XDOTI(I) ) ** 2
210     CONTINUE
        DV = SQRT ( SUMV )
        IF ( LINES .LT. LPAGE ) GO TO 220
           WRITE(2,212)
212        FORMAT('1',' '/' ',
     $            T15,'KEPLERIAN BOUNDARY VALUE PROBLEM',
     $            T47,'  -  ETA DOMAIN',3(' '/),
     $            T2,'ECCEN-',T14,'INITIAL',T26,'TRANSFER',
     $            T36,'TIME OF',T50,'INITIAL',
     $            T64,'NUMBER OF'/' ',
     $            T3,'TRICITY',T12,'TRUE ANOMALY',T26,'ANGLE',
     $            T36,'FLIGHT',T50,'VELOCITY',
     $            T64,'SECANT'/' ',
     $            T14,'(DEG.)',T26,'(DEG.)',T51,'ERROR',
     $            T65,'ITER-'/' ',T67,'ATIONS'/
     $            ' ',T2,71('-'))
           LINES = 9
220     CONTINUE
        WRITE(2,224) E,ANGLE1,ANGLE2,TOF,DV,NIT
224     FORMAT(' ',T2,F6.3,T14,F6.1,T25,F7.2,T34,F8.3,
     $        T49,E9.2,T67,I3)
      GO TO 100
900   CONTINUE
      STOP
      END
```

```
      SUBROUTINE KBVP1( GM, TOF, XI, XF, LONG, XDOTI, NIT, IFLAG )
C
C     KEPLERIAN BOUNDARY VALUE PROBLEM  -  UNIVERSAL SOLUTION.
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE).
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.151) AND
C     IMPLEMENTS THE ETA-DOMAIN EQUATIONS OF THESIS CHAPTER 7.
C     A DERIVATIVE-FREE SECANT METHOD (THESIS EQN (7.38B))
C     IS USED TO SOLVE THE TIME EQUATION.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        LONG = LOGICAL VARIABLE DENOTING SENSE OF TRANSFER:
C                  .TRUE. = TRANSFER ANGLE > 180 DEGREES
C                  .FALSE. = TRANSFER ANGLE <= 180 DEGREES
C
C     OUTPUT:
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        NIT = NUMBER OF ITERATIONS TO SOLVE TIME EQUATION
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -3 = CALCULATION NOT COMPLETE, SETQ1 CALL NOT COMPLETE
C                -2 = CALCULATION NOT COMPLETE, CFUN CALL NOT COMPLETE
C                -1 = CALCULATION NOT COMPLETE, LOCAL
C                 0 = CALCULATION COMLETED NORMALLY
C                 1 = NUMBER OF ITERATIONS EXCEEDED MAXIMUM
C                 2 = NUMBER OF CFUN RECURSIONS EXCEEDED MAXIMUM
C                 3 = NUMBER OF ATTEMPTED Q-RESETS EXCEEDED MAXIMUM
C
C     LOCAL PARAMETERS:
C        H = ANGULAR MOMENTUM MAGNITUDE
C        RI = INITIAL RADIUS
C        TIME = NONDIMENSIONAL TIME OF FLIGHT BASED ON RI
C        RF = FINAL RADIUS
C        RHO = RATIO OF INITIAL TO FINAL RADIUS
C        ALPHA = ECCENTRICITY PARAMETER, FROM THESIS EQN (7.34)
C        BETA = ECCENTRICITY PARAMETER, FROM THESIS EQN (7.35)
C        UNITXI(I) = INITIAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETAI(I) = INITIAL UNIT TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        UNITXF(I) = FINAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ETA = TRANSFER ANGLE
C        Z = TANGENT OF ONE-QUARTER OF THE TRANSFER ANGLE
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQN (7.37)
C        MAX = MAXIMUM NUMBER OF SECANT-METHOD ITERATIONS PERMITTED
C        EPSLN = TOLERANCE USED TO DETECT CONVERGENCE OF ITERATION
C        XEPSLN = TOLERANCE USED TO DETECT ILL-DEFINED ORBIT PLANE
C
C     CALLS:  SETQ1, CFUN
C
      LOGICAL LONG
      DIMENSION XI(3),XDOTI(3),XF(3)
      DIMENSION UNITXI(3), UNITXF(3), ZETAI(3)
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 998
      EPSLN = 1.0E-10
      XEPSLN = 1.0E-06
C
C     FIXED PARAMETERS, CALCULATED:
C
      PI = ACOS ( -1. )
      TWOPI = 2. * PI
C
```

```
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
      NIT = 0
      DO 110 I=1,3
         XDOTI(I) = 0.
110   CONTINUE
C
C     COMPUTE TRANSFER PARAMETERS USING THESIS EQNS (7.9) - (7.15) AND
C     (7.66) - (7.68).
C
      RSQRD = XI(1)**2 + XI(2)**2 + XI(3)**2
      RI = SQRT ( RSQRD )
      TIME = TOF * SQRT ( GM / RI**3 )
      RSQRD = XF(1)**2 + XF(2)**2 + XF(3)**2
      RF = SQRT ( RSQRD )
      RHO = RI / RF
      DO 113 I=1,3
         UNITXI(I) = XI(I) / RI
         UNITXF(I) = XF(I) / RF
113   CONTINUE
C
C        COMPUTE TRANSFER ANGLE, AND CHECK FOR NEAR-ZERO MAGNITUDE
C        OF INITIAL POSITION VECTOR  X  FINAL POSITION VECTOR.
C        IF TRANSFER PLANE IS ILL-DEFINED THEN ASSIGN ARBITRARY
C        DIRECTION TO THE INITIAL TRANSVERSE VECTOR SUCH THAT IT IS
C        PERPENDICULAR TO INITIAL POSITION VECTOR;
C        OTHERWISE, CALCULATE ITS COMPONENTS.
C        THE ASSIGNMENT USED HERE IS DESIGNED FOR CONVENIENCE FOR THE
C        PURPOSES OF THIS STUDY AND IS NOT A GENERAL PROCEDURE:
C        IT ASSUMES THE Z-COMPONENTS OF ALL POSITION VECTORS VANISH
C        AND ASSIGNS  ZETAI  =  UNITZ  X  UNITXI  (CCW SENSE OF MOTION).
C
      D = ( XI(1) * XF(1) + XI(2) * XF(2) + XI(3) * XF(3) ) / ( RI*RF )
      ETA = ACOS ( D )
      IF ( LONG ) ETA = TWOPI - ETA
      Z = TAN ( ETA / 4. )
      SINETA = 2. * ( 2. * Z ) * ( 1. - Z**2 ) / ( 1. + Z**2 )**2
      COSETA = ( ( 1. - Z**2 )**2 - ( 2. * Z )**2 ) / ( 1. + Z**2 )**2
      XN = SQRT ( 1. - D**2 )
      IF ( XN .GT. XEPSLN ) GO TO 115
         ZETAI(1) = - UNITXI(2)
         ZETAI(2) = + UNITXI(1)
         ZETAI(3) = 0.
      GO TO 125
115                          CONTINUE
         DO 117 I=1,3
            ZETAI(I) = ( UNITXF(I) - UNITXI(I) * D ) / XN
            IF ( LONG ) ZETAI(I) = - ZETAI(I)
117      CONTINUE
125   CONTINUE
C
C     SOLVE THESIS EQUATION (7.38A) FOR Q.
C     SECANT-METHOD ITERATION IS IMPLEMENTED BY THESIS EQN (7.38B)
C     USING ALSO EQNS (7.34) AND (7.35).
C     TWO VALUES OF Q ARE REQUIRED TO START THE ITERATION.
C     THE VALUE OF Q IS RESET AS NEEDED TO GUARANTEE ALPHA > -1,
C     F > 0, Y > -1.
C
C     FIRST STARTING VALUE OF Q AND ASSOCIATED F(Q):
C
      QLP = 0.0
      CALL SETQ1( QLP, ALPHA, BETA, SINETA, COSETA, RHO, JFLAG,
     $            Z, D, F, DROOTF, Y)
      IF ( JFLAG .EQ. 0 ) GO TO 130
         IF ( JFLAG .EQ. -1 ) IFLAG = -3
```

```
         IF ( JFLAG .EQ. 1  ) IFLAG = 3
         RETURN
130   CONTINUE
C
C     EVALUATE THESIS EQUATION (7.38A) FOR F(Q)
C
      NTA = 1
      CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
      IF ( JFLAG .EQ. 0 ) GO TO 135
         IF ( JFLAG .EQ. -1 ) IFLAG = -2
         IF ( JFLAG .EQ. 1 ) IFLAG = 2
         RETURN
135   CONTINUE
      TERM1 = ( 8. * Z**2 + ( 1. - Z**2 ) * DROOTF ) * DROOTF
      TERM2 = ( D - ( 1. - Z**2 ) * ( 1. - ALPHA**2 - BETA**2 ) )
      T = 4. * Z * ( TERM1 - 4. * Z**2 * ( D + TERM2 / ( 3. + C2 )))
      FK = T / ( F * DROOTF**2 * ( 1. + C1 ) )
      FOFQLP = ( 1. + ALPHA ) * SQRT ( 1. + ALPHA ) * FK - TIME
      CHECK = ABS ( FOFQLP )
      IF ( CHECK .LE. EPSLN ) GO TO 200
C
C     SECOND STARTING VALUE OF Q AND ASSOCIATED F(Q)
C
      Q = QLP - 0.5
      CALL SETQ1( Q, ALPHA, BETA, SINETA, COSETA, RHO, JFLAG,
     $            Z, D, F, DROOTF, Y )
      IF ( JFLAG .EQ. 0 ) GO TO 140
         IF ( JFLAG .EQ. -1 ) IFLAG = -3
         IF ( JFLAG .EQ. 1  ) IFLAG = 3
         RETURN
140   CONTINUE
C
C     EVALUATE THESIS EQUATION (7.38A) FOR F(Q)
C
      NTA = 1
      CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
      IF ( JFLAG .EQ. 0 ) GO TO 145
         IF ( JFLAG .EQ. -1 ) IFLAG = -2
         IF ( JFLAG .EQ. 1 ) IFLAG = 2
         RETURN
145   CONTINUE
      TERM1 = ( 8. * Z**2 + ( 1. - Z**2 ) * DROOTF ) * DROOTF
      TERM2 = ( D - ( 1. - Z**2 ) * ( 1. - ALPHA**2 - BETA**2 ) )
      T = 4. * Z * ( TERM1 - 4. * Z**2 * ( D + TERM2 / ( 3. + C2 )))
      FK = T / ( F * DROOTF**2 * ( 1. + C1 ) )
      FOFQ = ( 1. + ALPHA ) * SQRT ( 1. + ALPHA ) * FK - TIME
C
C     BEGIN SECANT-METHOD ITERATIONS BASED ON THESIS EQUATION (7.38B)
C
150   CHECK = ABS ( FOFQ )
      IF ( CHECK .LE. EPSLN .OR. NIT .GT. MAX ) GO TO 200
         SAVEQ = Q
         Q = Q - FOFQ * ( ( Q - QLP ) / ( FOFQ - FOFQLP ) )
         QLP = SAVEQ
         CALL SETQ1( Q, ALPHA, BETA, SINETA, COSETA, RHO, JFLAG,
     $               Z, D, F, DROOTF, Y )
         IF ( JFLAG .EQ. 0 ) GO TO 155
            IF ( JFLAG .EQ. -1 ) IFLAG = -3
            IF ( JFLAG .EQ. 1 ) IFLAG = 3
            RETURN
155      CONTINUE
C
C        EVALUATE THESIS EQUATION (7.38A)
C
         NTA = 1
         CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
```

```
         IF ( JFLAG .EQ. 0 ) GO TO 160
            IF ( JFLAG .EQ. -1 ) IFLAG = -2
            IF ( JFLAG .EQ. 1 ) IFLAG = 2
            RETURN
160      CONTINUE
         TERM1 = ( 8. * Z**2 + ( 1. - Z**2 ) * DROOTF ) * DROOTF
         TERM2 = ( D - ( 1. - Z**2 ) * ( 1. - ALPHA**2 - BETA**2 ) )
         T = 4. * Z * ( TERM1 - 4. * Z**2 * ( D + TERM2 / ( 3. + C2 )))
         FK = T / ( F * DROOTF**2 * ( 1. + C1 ) )
         FOFQLP = FOFQ
         FOFQ = ( 1. + ALPHA ) * SQRT ( 1. + ALPHA ) * FK - TIME
         NIT = NIT + 1
      GO TO 150
200   CONTINUE
      IF ( NIT .LE. MAX ) GO TO 210
         IFLAG = 1
         RETURN
210   CONTINUE
C
C     COMPUTE INITIAL VELOCITY VECTOR
C     BY MEANS OF THESIS EQUATIONS (7.17) AND (7.8).
C
      H = SQRT ( GM * ( 1. + ALPHA ) * RI )
      DO 300 I=1,3
         XDOTI(I) = GM * ( ZETAI(I)*( 1.+ALPHA ) - BETA*UNITXI(I) ) / H
300   CONTINUE
      IFLAG = 0
      RETURN
      END
```

```
      SUBROUTINE SETQ1( Q,ALPHA,BETA,SINETA,COSETA,RHO,JFLAG,
     $                  Z,D,F,DROOTF,Y)
C
C     RESET THE VALUE OF Q, IF NECESSARY,
C     USING THESIS EQUATIONS (7.40) - (7.41) AND RELATED DISCUSSION.
C     Q MUST BE CHOSEN SO THAT ALPHA > -1, F > 0, Y > -1 AND A
C     CERTAIN DENOMINATOR DOES NOT VANISH.
C
C     INPUT:
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQUATION (7.37)
C        SINETA = SINE OF TRANSFER ANGLE
C        COSETA = COSINE OF TRANSFER ANGLE
C        Z = TANGENT OF 1/4 OF TRANSFER ANGLE
C        RHO = RATIO OF INITIAL RADIUS TO FINAL RADIUS
C
C     OUTPUT:
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQUATION (7.37)
C        ALPHA = ECCENTRICITY PARAMETER IN THESIS EQUATION (7.34)
C        BETA = ECCENTRICITY PARAMETER IN THESIS EQUATION (7.35)
C        D, F, DROOTF, Y = PARAMETERS USED IN TIME EQUATION
C        JFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C               -1 = CALCULATION NOT COMPLETE
C                0 = CALCULATION COMPLETED NORMALLY
C                1 = Q NOT RESET AFTER MAX ATTEMPTS
C
C     LOCAL VARIABLES:
C        QSTEP = POSITIVE AMOUNT BY WHICH Q IS RESET ON ONE ATTEMPT
C        MAX = MAXIMUM PERMITTED NUMBER OF ATTEMPTS TO RESET Q
C        EPSLN = SMALLNESS TOLERANCE USED TO AVOID ZERO DENOMINATOR
C
C     CALLS: NONE
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      QSTEP = 0.05
      MAX = 1000
      EPSLN = 1.0E-08
C
C     VARIABLE PARAMETERS:
C
      JFLAG = -1
C
      N = 0
      XNUMER = ( COSETA - 1. ) * SINETA
      IF ( XNUMER .GE. 0.0 ) SIGN = +1.
      IF ( XNUMER .LT. 0.0 ) SIGN = -1.
100   CONTINUE
      IF ( N .LE. MAX ) GO TO 105
         JFLAG = 1
         RETURN
105   CONTINUE
      XDENOM = 1. - RHO * COSETA + Q * SINETA
      ADENOM = ABS ( XDENOM )
      IF ( ADENOM .GT. EPSLN ) GO TO 110
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
110                           CONTINUE
200   ALPHA = ( ( RHO - 1. ) * COSETA - Q * SINETA ) / XDENOM
      IF ( ALPHA .GT. -1.0 ) GO TO 210
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
210                         CONTINUE
      BETA = ( (RHO - 1.) * (Q + SINETA) - Q * (RHO - COSETA) ) / XDENOM
      D = ( 1. + ALPHA ) * ( 1. - Z**2 ) + 2. * BETA * Z
```

```
      XNUM = 4. * Z**2 * ( 1. - ALPHA**2 - BETA**2 )
      F = D**2 + XNUM
      IF ( F .GT. 0.0 ) GO TO 310
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
310                         CONTINUE
      DROOTF = D + SQRT ( F )
      Y = XNUM / DROOTF**2
      IF ( Y .GT. -1.0 ) GO TO 410
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
410                          CONTINUE
      JFLAG = 0
      RETURN
      END
```

```
      SUBROUTINE CFUN ( C1, C2, C3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTIONS WHICH OCCUR IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATIONS. THE VALUE OF THE FUNCTION C3 IS COMPUTED USING
C        GAUTSCHI'S (1967) METHOD TO APPROXIMATE AN INFINITE
C        CONTINUED FRACTION. THE VALUES OF C1 AND C2 ARE COMPUTED
C        IN RATIONAL ALGEBRAIC TERMS OF C3.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1./(1.+C1(X))
C
C        TRANSFORMATION OF THE ARGUMENT OF THE CONTINUED FRACTIONS
C        TO IMPROVE CONVERGENCE IS IMPLEMENTED, IF DESIRED, BY
C        THESIS EQUATIONS (5.159) AND (5.166).
C
C     INPUT:
C        X = ANY REAL VALUE GREATER THAN -1.
C        NTA = 1 FOR TRANSFORMATION OF X; = 0 FOR NO TRANSFORMATION.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        C1,C2,C3 = TRANSCENDENTAL FUNCTIONS RELATED TO THE ARCTANGENT.
C        N = NUMBER OF RECURSIONS NEEDED IN GAUTSCHI'S METHOD FOR C3.
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -1 = CALCULATION NOT COMPLETE
C                 0 = CALCULATION CONVERGED AND COMPLETED.
C                 1 = CONTINUED FRAC. NOT CONVERGED AFTER MAX RECURSIONS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF RECURSIONS ALLOWED FOR CONTINUED FRAC.
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE RECURSION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        A = NTH PARTIAL NUMERATOR
C        B = NTH PARTIAL DENOMINATOR
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQN (5.159)
C
      IF ( NTA .EQ. 0 ) GO TO 105
         SAVEX = X
         X = X / ( 1. + SQRT( 1. + X ) )**2
105   CONTINUE
C
C     IMPLEMENT THESIS EQNS (5.105), (5.108), (5.109), (9.98), (9.99).
C
      N = 1
      A = 9. * X
      B = 7.
      U = 1.
      V = A / B
      W = V
110   CHECK = ABS ( V )
C                       NOTE:  V = W - WLP, "LP" MEANING "LAST PASS"
```

```
      IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 120
         BLP = B
         ULP = U
         VLP = V
         WLP = W
         N = N + 1
         A = ( N + 2 ) ** 2 * X
         B = 2 * N + 5
         U = 1. / ( 1. + ( A / ( B * BLP ) ) * ULP )
         V = VLP * ( U - 1. )
         W = WLP + V
      GO TO 110
120   CONTINUE
      IF ( N .LE. MAX ) IFLAG = 0
      IF ( N .GT. MAX ) IFLAG = 1
      C3 = W
      C2 = 4. * X / ( 5. + C3 )
      C1 = X / ( 3. + C2 )
C
C     INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.166)
C
      IF ( NTA .EQ. 0 ) GO TO 125
         X = SAVEX
         XTERM = 1. + SQRT ( 1. + X )
         TERM1 = 8. * XTERM**2 * ( 5. + C3 ) + 8. * X
         TERM2 = ( 3. * ( 5. + C3 ) + 4. ) * XTERM - 6.
         C3 = TERM1 / TERM2 - 5.
         C2 = 4. * X / ( 5. + C3 )
         C1 = X / ( 3. + C2 )
125   CONTINUE
      RETURN
      END
```

KEPLERIAN BOUNDARY VALUE PROBLEM - ETA DOMAIN

ECCEN- TRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	INITIAL VELOCITY ERROR	NUMBER OF SECANT ITER- ATIONS
0.000	0.0	90.00	1.571	0.16E-09	1
0.500	0.0	90.00	0.946	0.64E-09	5
0.990	0.0	90.00	0.671	0.16E-08	4
1.000	0.0	90.00	0.667	0.86E-09	2
1.010	0.0	90.00	0.663	0.12E-08	4
2.000	0.0	90.00	0.413	0.20E-08	5
10.000	0.0	90.00	0.098	0.40E-08	7
0.000	-45.0	135.00	2.356	0.47E-09	1
0.500	-45.0	135.00	1.320	0.16E-09	5
0.990	-45.0	135.00	0.892	0.15E-08	5
1.000	-45.0	135.00	0.886	0.13E-08	5
1.010	-45.0	135.00	0.880	0.10E-08	5
2.000	-45.0	135.00	0.515	0.17E-08	6
10.000	-45.0	135.00	0.106	0.29E-07	7
0.000	-45.0	179.00	3.124	0.59E-09	3
0.500	-45.0	179.00	2.524	0.23E-09	6
0.990	-45.0	179.00	3.519	0.14E-08	6
1.000	-45.0	179.00	3.576	0.12E-08	6
1.010	-45.0	179.00	3.636	0.51E-09	6
0.000	-45.0	180.00	3.142	0.57E-09	3
0.500	-45.0	180.00	2.566	0.20E-09	6
0.990	-45.0	180.00	3.705	0.14E-08	6
1.000	-45.0	180.00	3.771	0.12E-08	6
1.010	-45.0	180.00	3.841	0.12E-07	6
0.000	-45.0	181.00	3.159	0.54E-09	3
0.500	-45.0	181.00	2.608	0.18E-09	6
0.990	-45.0	181.00	3.907	0.14E-08	6
1.000	-45.0	181.00	3.984	0.11E-08	6
1.010	-45.0	181.00	4.064	0.49E-09	6
0.000	-90.0	225.00	3.927	0.18E-09	3
0.500	-90.0	225.00	3.137	0.26E-09	6
0.990	-90.0	225.00	4.155	0.12E-09	6
1.000	-90.0	225.00	4.219	0.55E-10	6
1.010	-90.0	225.00	4.287	0.71E-10	6
0.000	-179.0	358.00	6.248	0.27E-09	3
0.500	-179.0	358.00	9.534	0.50E+00	1
0.990	-179.0	358.00	1892.603	0.46E-09	8
1.000	-179.0	0.01	7376.014	0.17E-01	0

```
C     KEPLERIAN BOUNDARY VALUE PROBLEM
C        (APPENDIX  F  -  SIGMA-DOMAIN METHOD)
C
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE).
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.238) AND
C     IMPLEMENTS THE SIGMA-DOMAIN EQUATIONS OF THESIS CHAPTER 7.
C     RECTILINEAR TRANSFERS ARE PERMITTED.
C
C  SETUP PROCEDURE AT VIRGINIA TECH COMPUTING CENTER:
C  COMPILE WITH:
C   FORTVS2 APPF2 ( SOURCE SRCFLG OPT(0) AUTODBL(DBL4)
C  THEN:
C   FILEDEF 1 DISK APPD1 KPLRDATA A1 ( RECFM F LRECL 260 BLKSIZE 260
C   FILEDEF 2 DISK APPF2 OUTPUT A1 ( RECFM F LRECL 121 BLKSIZE 121
C   FILEDEF 3 TERMINAL
C   GLOBAL TXTLIB VSF2FORT CMSLIB
C   GLOBAL LOADLIB VSF2LOAD
C   LOAD APPF2 ( NOMAP
C   START
C
C     INPUT FROM UNIT 1:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XDOTI(I) = INITIAL VELOCITY DETERMINED OTHERWISE(1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C
C     OUTPUT TO UNIT 2:
C        E = ECCENTRICITY; USED TO IDENTIFY CASE
C        ANGLE1 = INITIAL PERICENTRAL TRUE ANOMALY (DEG.); "
C        ANGLE2 = TRANSFER ANGLE ( DEG.); USED TO IDENTIFY CASE
C        TOF = TIME OF FLIGHT
C        DV = ERROR IN INITIAL VELOCITY MAGNITUDE
C        NIT = NUMBER OF NEWTONIAN ITERATIONS TO SOLVE TIME EQUATION
C
C     LOCAL PARAMETERS:
C        XDOTIC(I) = INITIAL VELOCITY COMPUTED HERE (1,2,3 = X,Y,Z)
C        SUMV = SUM-SQUARED ERROR IN INITIAL VELOCITY COMPONENTS
C        LINES = OUTPUT PAGING PARAMETER, NO. OF LINES WRITTEN
C        LPAGE = OUTPUT PAGING PARAMETER, NO. OF LINES PER PAGE
C
C     CALLS: KBVP2
C
      LOGICAL LONG
      DIMENSION XI(3),XDOTI(3),XF(3),XDOTIC(3)
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C              FOR THE PURPOSES OF THIS STUDY, GM IS SET EQUAL TO 1.0
C              IN ORDER TO NORMALIZE THE COORDINATES AND TIME.
      GM = 1.0
      LPAGE = 55
C
C     VARIABLE PARAMETERS:
C
      LINES = 99
C
100   READ(1,112,END=900) E,ANGLE1,ANGLE2,(XI(I),I=1,3),
     $             (XDOTI(J),J=1,3), (XF(K),K=1,3),TOF
112   FORMAT(F6.3,1X,F6.1,1X,F7.2,9(1X,F15.9),3(16X),1X,F15.9)
         WRITE(3,122) E,ANGLE1,ANGLE2
122      FORMAT(F9.3,2F9.1)
         IF ( ANGLE2 .GT. 180. ) LONG = .TRUE.
         IF ( ANGLE2 .LE. 180. ) LONG = .FALSE.
```

```
      CALL KBVP2( GM, TOF, XI, XF, LONG, XDOTIC, NIT, IFLAG )
      SUMV = 0.
      DO 210 I=1,3
         SUMV = SUMV + ( XDOTIC(I) - XDOTI(I) ) ** 2
210   CONTINUE
      DV = SQRT ( SUMV )
      IF ( LINES .LT. LPAGE ) GO TO 220
         WRITE(2,212)
212      FORMAT('1',' '/' ',
     $          T15,'KEPLERIAN BOUNDARY VALUE PROBLEM',
     $          T47,'  -  SIGMA DOMAIN',3(' '/),
     $          T2,'ECCEN-',T14,'INITIAL',T26,'TRANSFER',
     $          T36,'TIME OF',T50,'INITIAL',
     $          T64,'NUMBER OF'/' ',
     $          T3,'TRICITY',T12,'TRUE ANOMALY',T26,'ANGLE',
     $          T36,'FLIGHT',T50,'VELOCITY',
     $          T64,'SECANT'/' ',
     $          T14,'(DEG.)',T26,'(DEG.)',T51,'ERROR',
     $          T65,'ITER-'/' ',T67,'ATIONS'/
     $          ' ',T2,71('-'))
         LINES = 9
220   CONTINUE
      WRITE(2,224) E,ANGLE1,ANGLE2,TOF,DV,NIT
224   FORMAT(' ',T2,F6.3,T14,F6.1,T25,F7.2,T34,F8.3,
     $       T49,E9.2,T67,I3)
      LINES = LINES + 1
    GO TO 100
900 CONTINUE
    STOP
    END
```

```
      SUBROUTINE KBVP2( GM, TOF, XI, XF, LONG, XDOTI, NIT, IFLAG )
C
C     KEPLERIAN BOUNDARY VALUE PROBLEM  -  UNIVERSAL SOLUTION.
C     FOR THE PURPOSES OF THIS STUDY, TRANSFERS WILL BE LIMITED TO
C     LESS THAN ONE REVOLUTION (POSITIVE OR NEGATIVE).
C     THE METHOD IS BASED ON THESIS TIME EQUATION (5.238) AND
C     IMPLEMENTS THE SIGMA-DOMAIN EQUATIONS OF THESIS CHAPTER 7.
C     A DERIVATIVE-FREE SECANT METHOD (SIMILAR TO THESIS EQN (7.38B))
C     IS USED TO SOLVE THE TIME EQUATION.
C     RECTILINEAR TRANSFERS ARE PERMITTED, AS LONG AS NEITHER THE
C     INITIAL OR FINAL POSITION IS TOO NEAR THE ORIGIN.
C
C     INPUT:
C        GM = GRAVITATIONAL PARAMETER (G TIMES SUM OF MASSES)
C        TOF = TIME OF FLIGHT
C        XI(I) = INITIAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        XF(I) = FINAL POSITION VECTOR (1,2,3 = X,Y,Z)
C        LONG = LOGICAL VARIABLE DENOTING SENSE OF TRANSFER:
C                  .TRUE. = TRANSFER ANGLE > 180 DEGREES
C                  .FALSE. = TRANSFER ANGLE <= 180 DEGREES
C
C     OUTPUT:
C        XDOTI(I) = INITIAL VELOCITY VECTOR (1,2,3 = X,Y,Z)
C        NIT = NUMBER OF ITERATIONS TO SOLVE TIME EQUATION
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                -3 = CALCULATION NOT COMPLETE, SETQ2 CALL NOT COMPLETE
C                -2 = CALCULATION NOT COMPLETE, CFUN CALL NOT COMPLETE
C                -1 = CALCULATION NOT COMPLETE, LOCAL
C                 0 = CALCULATION COMLETED NORMALLY
C                 1 = NUMBER OF ITERATIONS EXCEEDED MAXIMUM
C                 2 = NUMBER OF CFUN RECURSIONS EXCEEDED MAXIMUM
C                 3 = NUMBER OF ATTEMPTED Q-RESETS EXCEEDED MAXIMUM
C                 4 = TRANSFER ENDPOINT TOO NEAR ORIGIN
C
C     LOCAL PARAMETERS:
C        H = ANGULAR MOMENTUM MAGNITUDE
C        RI = INITIAL RADIUS
C        UI = RECIPROCAL INITIAL RADIUS
C        RF = FINAL RADIUS
C        XN = MAGNITUDE OF INITIAL X FINAL POSITION VECTORS
C        RHO = RATIO OF INITIAL TO FINAL RADIUS
C        A = AUXILIARY PARAMETER, FROM THESIS EQN (7.85)
C        B = AUXILIARY PARAMETER, FROM THESIS EQN (7.86)
C        UNITXI(I) = INITIAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ZETAI(I) = INITIAL UNIT TRANSVERSE VECTOR (1,2,3 = X,Y,Z)
C        UNITXF(I) = FINAL UNIT RADIAL VECTOR (1,2,3 = X,Y,Z)
C        ETA = TRANSFER ANGLE
C        HZ = TANGENT OF 1/4 TRANSFER ANGLE, THESIS EQN (7.66)
C        Z = 1/H TIMES TANGENT 1/4 TRANSFER ANGLE, THESIS EQN (7.66)
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQN (7.77)
C        MAX = MAXIMUM NUMBER OF SECANT-METHOD ITERATIONS PERMITTED
C        EPSLN = TOLERANCE USED TO DETECT CONVERGENCE OF ITERATION
C        XEPSLN = TOLERANCE USED TO DETECT ILL-DEFINED ORBIT PLANE
C
C     CALLS:  SETQ2, CFUN
C
      LOGICAL LONG
      DIMENSION XI(3),XDOTI(3),XF(3)
      DIMENSION UNITXI(3), UNITXF(3), ZETAI(3)
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 998
      EPSLN = 1.0E-10
      XEPSLN = 1.0E-06
C
```

```
C     FIXED PARAMETERS, CALCULATED:
C
      PI = ACOS ( -1. )
      TWOPI = 2. * PI
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
      NIT = 0
      DO 110 I=1,3
         XDOTI(I) = 0.
110   CONTINUE
C
C     COMPUTE TRANSFER PARAMETERS USING THESIS EQNS (7.9) - (7.15) AND
C     (7.67) - (7.75).
C
      RSQRD = XI(1)**2 + XI(2)**2 + XI(3)**2
      RI = SQRT ( RSQRD )
      RSQRD = XF(1)**2 + XF(2)**2 + XF(3)**2
      RF = SQRT ( RSQRD )
      IF ( RI .GT. XEPSLN .AND. RF .GT. XEPSLN ) GO TO 105
         IFLAG = 4
         RETURN
105   UI = 1. / RI
      RHO = RI / RF
      DO 113 I=1,3
         UNITXI(I) = XI(I) / RI
         UNITXF(I) = XF(I) / RF
113   CONTINUE
C
C        COMPUTE TRANSFER ANGLE, AND CHECK FOR NEAR-ZERO MAGNITUDE
C        OF XN  =  INITIAL POSITION VECTOR  X  FINAL POSITION VECTOR.
C        IF TRANSFER PLANE IS ILL-DEFINED THEN ASSIGN ARBITRARY
C        DIRECTION TO THE INITIAL TRANSVERSE VECTOR SUCH THAT IT IS
C        PERPENDICULAR TO INITIAL POSITION VECTOR;
C        OTHERWISE, CALCULATE ITS COMPONENTS.
C        THE ASSIGNMENT USED HERE IS DESIGNED FOR CONVENIENCE FOR THE
C        PURPOSES OF THIS STUDY AND IS NOT A GENERAL PROCEDURE:
C        IT ASSUMES THE Z-COMPONENTS OF ALL POSITION VECTORS VANISH
C        AND ASSIGNS  ZETAI  =  UNITZ  X  UNITXI  (CCW SENSE OF MOTION).
C
      D = ( XI(1) * XF(1) + XI(2) * XF(2) + XI(3) * XF(3) ) / ( RI*RF )
      ETA = ACOS ( D )
      IF ( LONG ) ETA = TWOPI - ETA
      HZ = TAN ( ETA / 4. )
      T1 = ( ( 1. - HZ**2 )**2 - ( 2. * HZ )**2 ) / ( 1. + HZ**2 )**2
      T2 = 2. * ( 2. * ( 1. - HZ**2 ) ) / ( 1. + HZ**2 )**2
      T4 = 2. * HZ * ( 2. * HZ ) * ( 1. - HZ**2 ) / ( 1. + HZ**2 )**2
      XN = SQRT ( 1. - D**2 )
      IF ( XN .GT. XEPSLN ) GO TO 115
C        ARBITRARILY ASSIGN:
         ZETAI(1) = - UNITXI(2)
         ZETAI(2) = + UNITXI(1)
         ZETAI(3) = 0.
      GO TO 120
115                       CONTINUE
         DO 117 I=1,3
            ZETAI(I) = ( UNITXF(I) - UNITXI(I) * D ) / XN
            IF ( LONG ) ZETAI(I) = - ZETAI(I)
117      CONTINUE
120   CONTINUE
C
C     SOLVE THESIS EQUATION (7.38A) FOR Q, WHERE ESTIMATED TIME
C     CORRESPONDING TO Q IS EVALUATED FROM THESIS EQN (5.238)
C     USING ALSO EQNS (7.85) - (7.94).
C     SECANT-METHOD ITERATION IS IMPLEMENTED BY THESIS EQN (7.38B).
```

```
C       TWO VALUES OF Q ARE REQUIRED TO START THE ITERATION.
C       THE VALUE OF Q IS RESET AS NEEDED TO GUARANTEE A > 0,
C       QQ > 0, Y > -1.
C
C       FIRST STARTING VALUE OF Q AND ASSOCIATED F(Q):
C
        QLP = 0.
        CALL SETQ2( QLP, A, B, HZ, T1, T2, T4, RHO, JFLAG,
     $              Z, QQ, AA, AROOTQ, Y, GM, UI, UPIZ, GZSQRD )
        IF ( JFLAG .EQ. 0 ) GO TO 130
           IF ( JFLAG .EQ. -1 ) IFLAG = -3
           IF ( JFLAG .EQ. 1  ) IFLAG = 3
           RETURN
130     CONTINUE
        NTA = 1
        CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
        IF ( JFLAG .EQ. 0 ) GO TO 135
           IF ( JFLAG .EQ. -1 ) IFLAG = -2
           IF ( JFLAG .EQ. 1 ) IFLAG = 2
           RETURN
135     CONTINUE
        TERM1 = ( 8. * GM * Z**2 + ( 1. - HZ**2 ) * AROOTQ ) * AROOTQ
        TERM2 = 4. * Z**2 * GM * AA * ( 1. + 1. / ( 3. + C2 ) )
        TERM3 = 4. * GZSQRD * ( 1. - HZ**2 ) / ( 3. + C2 )
        F = 4.*Z*( TERM1-TERM2+TERM3 ) / ( QQ * AROOTQ**2 * ( 1. + C1 ) )
        FOFQLP = F - TOF
C
C       SECOND STARTING VALUE OF Q AND ASSOCIATED F(Q)
C
        Q = QLP - 1.0
        CALL SETQ2( Q, A, B, HZ, T1, T2, T4, RHO, JFLAG,
     $              Z, QQ, AA, AROOTQ, Y, GM, UI, UPIZ, GZSQRD )
        IF ( JFLAG .EQ. 0 ) GO TO 140
           IF ( JFLAG .EQ. -1 ) IFLAG = -3
           IF ( JFLAG .EQ. 1  ) IFLAG = 3
           RETURN
140     CONTINUE
        NTA = 1
        CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
        IF ( JFLAG .EQ. 0 ) GO TO 145
           IF ( JFLAG .EQ. -1 ) IFLAG = -2
           IF ( JFLAG .EQ. 1 ) IFLAG = 2
           RETURN
145     CONTINUE
        TERM1 = ( 8. * GM * Z**2 + ( 1. - HZ**2 ) * AROOTQ ) * AROOTQ
        TERM2 = 4. * Z**2 * GM * AA * ( 1. + 1. / ( 3. + C2 ) )
        TERM3 = 4. * GZSQRD * ( 1. - HZ**2 ) / ( 3. + C2 )
        F = 4.*Z*( TERM1-TERM2+TERM3 ) / ( QQ * AROOTQ**2 * ( 1. + C1 ) )
        FOFQ = F - TOF
C
C       BEGIN SECANT-METHOD ITERATIONS BASED ON THESIS EQUATION (7.38B)
C
150     CHECK = ABS ( FOFQ )
        IF ( CHECK .LE. EPSLN .OR. NIT .GT. MAX ) GO TO 200
           SAVEQ = Q
           Q = Q - FOFQ * ( ( Q - QLP ) / ( FOFQ - FOFQLP ) )
           QLP = SAVEQ
           CALL SETQ2( Q, A, B, HZ, T1, T2, T4, RHO, JFLAG,
     $                 Z, QQ, AA, AROOTQ, Y, GM, UI, UPIZ, GZSQRD )
           IF ( JFLAG .EQ. 0 ) GO TO 155
              IF ( JFLAG .EQ. -1 ) IFLAG = -3
              IF ( JFLAG .EQ. 1 ) IFLAG = 3
              RETURN
155        CONTINUE
           NTA = 1
           CALL CFUN ( C1, C2, C3, Y, NTA, N, JFLAG )
```

```
          IF ( JFLAG .EQ. 0 ) GO TO 160
             IF ( JFLAG .EQ. -1 ) IFLAG = -2
             IF ( JFLAG .EQ. 1 ) IFLAG = 2
             RETURN
160       CONTINUE
          TERM1 = ( 8. * GM * Z**2 + ( 1. - HZ**2 ) * AROOTQ ) * AROOTQ
          TERM2 = 4. * Z**2 * GM * AA * ( 1. + 1. / ( 3. + C2 ) )
          TERM3 = 4. * GZSQRD * ( 1. - HZ**2 ) / ( 3. + C2 )
          F = 4.*Z*( TERM1-TERM2+TERM3 ) / ( QQ * AROOTQ**2 * ( 1.+C1 ) )
          FOFQLP = FOFQ
          FOFQ = F - TOF
          NIT = NIT + 1
       GO TO 150
200    CONTINUE
       IF ( NIT .LE. MAX ) GO TO 210
          IFLAG = 1
          RETURN
210    CONTINUE
C
C      COMPUTE INITIAL VELOCITY VECTOR
C      BY MEANS OF THESIS EQUATIONS (7.69), (7.88), AND (7.60).
C
       H = HZ / Z
       UPI = UPIZ / Z
       DO 300 I=1,3
          XDOTI(I) =  H * UI * ZETAI(I) - UPI * UNITXI(I)
300    CONTINUE
       IFLAG = 0
       RETURN
       END
```

```
      SUBROUTINE SETQ2( Q,A,B,HZ,T1,T2,T4,RHO,JFLAG,
     $                  Z, QQ, AA, AROOTQ, Y, GM, UI, UPIZ, GZSQRD )
C
C     RESET THE VALUE OF Q, IF NECESSARY,
C     USING THESIS EQUATION (7.95) AND RELATED DISCUSSION.
C     Q MUST BE CHOSEN SO THAT A > 0, QQ > 0, AND Y > -1.
C
C     INPUT:
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQUATION (7.77)
C        HZ = TANGENT OF ONE-QUARTER OF THE TRANSFER ANGLE
C        T1,T2,T4 = FUNCTIONS OF THE TRANSFER ANGLE, EQNS (7.72)-(7.75)
C        RHO = RATIO OF INITIAL RADIUS TO FINAL RADIUS
C        UI = RECIPROCAL INITIAL RADIUS
C        GM = GRAVITATONAL PARAMETER
C
C     OUTPUT:
C        Q = ITERATION VARIABLE DEFINED IN THESIS EQUATION (7.77)
C        A = AUXILIARY PARAMETER IN THESIS EQUATION (7.85)
C        B = AUXILIARY PARAMETER IN THESIS EQUATION (7.86)
C        Z, QQ, AA, AROOTQ, Y = PARAMETERS USED IN TIME EQUATION (5.238)
C        UPIZ, GZSQRD = PARAMETERS USED IN TIME EQUATION (5.238)
C        JFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C                   -1 = CALCULATION NOT COMPLETE
C                    0 = CALCULATION COMPLETED NORMALLY
C                    1 = Q NOT RESET AFTER MAX ATTEMPTS
C
C     LOCAL VARIABLES:
C        QSTEP = POSITIVE AMOUNT BY WHICH Q IS RESET ON ONE ATTEMPT
C        MAX = MAXIMUM PERMITTED NUMBER OF ATTEMPTS TO RESET Q
C
C
C     CALLS: NONE
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      QSTEP = 0.05
      MAX = 1000
C
C     VARIABLE PARAMETERS:
C
      JFLAG = -1
C
      N = 0
      HZSQRD = HZ**2
      IF ( HZSQRD .GT. 1. ) SIGN = -1.
      IF ( HZSQRD .LE. 1. ) SIGN = +1.
100   CONTINUE
      IF ( N .LE. MAX ) GO TO 105
         JFLAG = 1
         RETURN
105   CONTINUE
      A = ( 1. + HZSQRD )**2 * ( ( Q+T4 ) * T2 - ( RHO-T1 ) * T1 ) / 8.
         IF ( A .GT. 0.0 ) GO TO 210
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
210                             CONTINUE
      B = ( ( RHO - T1 ) * T2 * ( 1. + HZSQRD )**2 ) / 8. - ( Q + T4 )
      Z = SQRT ( A * UI / GM )
      UPIZ = B * UI
      AA = UI * ( 1. - HZ**2 ) + 2. * UPIZ
      GZSQRD = 2. * GM * UI * Z**2 - HZ**2 * UI**2 - UPIZ**2
      QQ = AA**2 + 4. * GZSQRD
      IF ( QQ .GT. 0.0 ) GO TO 310
         Q = Q + SIGN * QSTEP
         N = N + 1
```

```
      GO TO 100
310                           CONTINUE
      AROOTQ = AA + SQRT ( QQ )
      Y = 4. * GZSQRD / AROOTQ**2
      IF ( Y .GT. -1.0 ) GO TO 410
         Q = Q + SIGN * QSTEP
         N = N + 1
      GO TO 100
410                           CONTINUE
      JFLAG = 0
      RETURN
      END
```

```
      SUBROUTINE CFUN ( C1, C2, C3, X, NTA, N, IFLAG )
C
C     EVALUATION OF THE SPECIAL ARCTANGENT-DERIVED TRANSCENDENTAL
C        FUNCTIONS WHICH OCCUR IN THE UNIVERSAL TRUE-ANOMALY TIME
C        EQUATIONS. THE VALUE OF THE FUNCTION C3 IS COMPUTED USING
C        GAUTSCHI'S (1967) METHOD TO APPROXIMATE AN INFINITE
C        CONTINUED FRACTION. THE VALUES OF C1 AND C2 ARE COMPUTED
C        IN RATIONAL ALGEBRAIC TERMS OF C3.
C        THE RELATIONS WITH ELEMENTARY AND HYPERGEOMETRIC FUNCTIONS ARE:
C
C        (ARCTAN(SQRT(X)))/SQRT(X) = F(0.5,1.0;1.5;-X) = 1./(1.+C1(X))
C
C        TRANSFORMATION OF THE ARGUMENT OF THE CONTINUED FRACTIONS
C        TO IMPROVE CONVERGENCE IS IMPLEMENTED, IF DESIRED, BY
C        THESIS EQUATIONS (5.159) AND (5.166).
C
C     INPUT:
C        X = ANY REAL VALUE GREATER THAN -1.
C        NTA = 1 FOR TRANSFORMATION OF X; = 0 FOR NO TRANSFORMATION.
C              (NOTE: TRANSFORMATION IS LOCAL ONLY;
C                     ORIGINAL VALUE OF X IS ALWAYS RESTORED.)
C
C     OUTPUT:
C        C1,C2,C3 = TRANSCENDENTAL FUNCTIONS RELATED TO THE ARCTANGENT.
C        N = NUMBER OF RECURSIONS NEEDED IN GAUTSCHI'S METHOD FOR C3.
C        IFLAG = RETURN CODE, HAVING VALUES AS FOLLOWS:
C               -1 = CALCULATION NOT COMPLETE
C                0 = CALCULATION CONVERGED AND COMPLETED.
C                1 = CONTINUED FRAC. NOT CONVERGED AFTER MAX RECURSIONS
C
C     CALLS:  NONE.
C
C     LOCAL PARAMETERS:
C
C        MAX = MAXIMUM NUMBER OF RECURSIONS ALLOWED FOR CONTINUED FRAC.
C        EPSLN = CONVERGENCE TOLERANCE USED TO TERMINATE RECURSION
C        SAVEX = ORIGINAL UNTRANSFORMED VALUE OF ARGUMENT X
C        A = NTH PARTIAL NUMERATOR
C        B = NTH PARTIAL DENOMINATOR
C
C     FIXED PARAMETERS, SPECIFIED BY USER:
C
      MAX = 100
      EPSLN = 1.0E-10
C
C     VARIABLE PARAMETERS:
C
      IFLAG = -1
C
C     TRANSFORM X IF DESIRED, USING THESIS EQN (5.159)
C
      IF ( NTA .EQ. 0 ) GO TO 105
         SAVEX = X
         X = X / ( 1. + SQRT( 1. + X ) )**2
105   CONTINUE
C
C     IMPLEMENT THESIS EQNS (5.105), (5.108), (5.109), (9.98), (9.99).
C
      N = 1
      A = 9. * X
      B = 7.
      U = 1.
      V = A / B
      W = V
110   CHECK = ABS ( V )
C                          NOTE:  V = W - WLP, "LP" MEANING "LAST PASS"
```

```
      IF ( CHECK .LE. EPSLN .OR. N .GT. MAX ) GO TO 120
         BLP = B
         ULP = U
         VLP = V
         WLP = W
         N = N + 1
         A = ( N + 2 ) ** 2 * X
         B = 2 * N + 5
         U = 1. / ( 1. + ( A / ( B * BLP ) ) * ULP )
         V = VLP * ( U - 1. )
         W = WLP + V
      GO TO 110
120   CONTINUE
      IF ( N .LE. MAX ) IFLAG = 0
      IF ( N .GT. MAX ) IFLAG = 1
      C3 = W
      C2 = 4. * X / ( 5. + C3 )
      C1 = X / ( 3. + C2 )
C
C     INVERT TRANSFORMATION IF IT HAS BEEN INVOKED; THESIS EQN (5.166)
C
      IF ( NTA .EQ. 0 ) GO TO 125
         X = SAVEX
         XTERM = 1. + SQRT ( 1. + X )
         TERM1 = 8. * XTERM**2 * ( 5. + C3 ) + 8. * X
         TERM2 = ( 3. * ( 5. + C3 ) + 4. ) * XTERM - 6.
         C3 = TERM1 / TERM2 - 5.
         C2 = 4. * X / ( 5. + C3 )
         C1 = X / ( 3. + C2 )
125   CONTINUE
      RETURN
      END
```

KEPLERIAN BOUNDARY VALUE PROBLEM - SIGMA DOMAIN

ECCEN- TRICITY	INITIAL TRUE ANOMALY (DEG.)	TRANSFER ANGLE (DEG.)	TIME OF FLIGHT	INITIAL VELOCITY ERROR	NUMBER OF SECANT ITER- ATIONS
0.000	0.0	90.00	1.571	0.16E-09	1
0.500	0.0	90.00	0.946	0.64E-09	5
0.990	0.0	90.00	0.671	0.16E-08	5
1.000	0.0	90.00	0.667	0.85E-09	5
1.010	0.0	90.00	0.663	0.12E-08	5
2.000	0.0	90.00	0.413	0.20E-08	5
10.000	0.0	90.00	0.098	0.41E-08	5
0.000	-45.0	135.00	2.356	0.47E-09	3
0.500	-45.0	135.00	1.320	0.16E-09	8
0.990	-45.0	135.00	0.892	0.15E-08	7
1.000	-45.0	135.00	0.886	0.13E-08	7
1.010	-45.0	135.00	0.880	0.96E-09	7
2.000	-45.0	135.00	0.515	0.17E-08	7
10.000	-45.0	135.00	0.106	0.21E-07	5
0.000	-45.0	179.00	3.124	0.59E-09	3
0.500	-45.0	179.00	2.524	0.24E-09	8
0.990	-45.0	179.00	3.519	0.14E-08	9
1.000	-45.0	179.00	3.576	0.12E-08	9
1.010	-45.0	179.00	3.636	0.51E-09	9
0.000	-45.0	180.00	3.142	0.57E-09	3
0.500	-45.0	180.00	2.566	0.22E-09	8
0.990	-45.0	180.00	3.705	0.14E-08	9
1.000	-45.0	180.00	3.771	0.12E-08	9
1.010	-45.0	180.00	3.841	0.12E-07	9
0.000	-45.0	181.00	3.159	0.54E-09	3
0.500	-45.0	181.00	2.608	0.19E-09	8
0.990	-45.0	181.00	3.907	0.14E-08	9
1.000	-45.0	181.00	3.984	0.11E-08	9
1.010	-45.0	181.00	4.064	0.49E-09	9
0.000	-90.0	225.00	3.927	0.18E-09	3
0.500	-90.0	225.00	3.137	0.26E-09	7
0.990	-90.0	225.00	4.155	0.12E-09	5
1.000	-90.0	225.00	4.219	0.46E-10	5
1.010	-90.0	225.00	4.287	0.57E-10	5
0.000	-179.0	358.00	6.248	0.27E-09	3
0.500	-179.0	358.00	9.534	0.80E-09	7
0.990	-179.0	358.00	1892.603	0.46E-09	14
1.000	-179.0	0.01	7376.014	0.17E-01	999

The vita has been removed from the scanned document