UNIFORM L^1 BEHAVIOR FOR THE SOLUTION OF A VOLTERRA EQUATION WITH A PARAMETER

by

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1. Introduction

In this work we study the equation

$$(1.1) \qquad u'(t)+\lambda\int_0^t(d+a(t-\tau))u(\tau)d\tau=0,\ u(0)=1,\ t\geq 0,$$

where prime denotes differentiation with respect to t. The solution is $u=u(t)=u(t,\lambda)$. The convolution kernel satisfies $d\geq 0$ and a is nonnegative, nonincreasing and convex. Our purpose is to study the question: Under what additional conditions does

$$(1.2) \qquad \int_0^\infty \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$$

hold?

The organization of the paper is as follows. In Section 2 we discuss known results for (1.1) both with and without the parameter λ. In Section 3, we state our results and consider some examples. In Section 4 we give proofs. In Section 5 we give further examples.

The main results of this work are contained in Theorems 7 and 8. In Theorem 7 the hypotheses involve conditions on the Fourier transform of the kernel. The hypotheses in Theorem 8 are stated directly in terms of the kernel a.

2. History

Our primary interest concerning (1.1) is L^1 behavior, uniform in $\lambda \geq 1$, of the solution and its derivatives. We will show how (1.1) arises in the study of an abstract equation in Hilbert space. Uniform L^1 behavior of the solution to (1.1) and of its derivatives has consequences for the Hilbert space equation (regarding, in particular, asymptotic behavior of the solution).

We will begin with a look at asymptotic results for the scalar equation

(2.1) $\quad u'(t)+\int_0^t g(u(t))a(t-\tau)d\tau=f(t), \ t\geq 0, \ u(0)=u_0$

both in the linear case, $g(x)=x$, and the nonlinear case. The papers we will discuss have assumptions on the convolution kernel that are similar or related to the ones we will use in our study of (1.1). We will observe the progress made by several authors in weakening the assumptions made on the convolution kernel.

In the earlier papers we will discuss, the results were obtained by working directly on the equation and by using energy methods. We will observe that transform methods were later used to advantage even in the nonlinear case.

In the linear case more transform theory can be used. We will examine the methods in [31] that were used to obtain a crucial L^1 result for the linear version of (2.1). This result is quite useful in the study of (1.1).

After showing the above mentioned interplay between (1.1) and the Hilbert space problem we will consider results on the uniform L^1 behavior for (1.1) and discuss the techniques that are used. We will study the techniques in some detail as the methods used will have a considerable bearing on our study of (1.2).

In 1963 J. J. Levin [21] investigated the problem (2.1) with $f(t)\equiv 0$ under the assumptions $a\in C[0,\infty)$, $(-1)^k a^{(k)}(t)\geq 0$, $k=0,1,2,3$, a is nonconstant, $g\in C(-\infty,\infty)$, $xg(x)>0$ for $x\neq 0$ and $G(x)\equiv\int_0^x g(u)du\to\infty$ as $|x|\to\infty$. He proved the following:

__Theorem A. Under these hypotheses, if__ $u=u(t)$ __is any solution of__ (2.1) __that exists on__ $[0,\infty)$ __then__ $\lim_{t\to\infty} u^{(j)}(t)=0$, $j=0,1,2$.

(Note: In Theorems A through E, the proofs can easily be modified with at most minor addition hypotheses to establish global existence.)

To prove this Levin defined the nonnegative energy function

$$E(t)\equiv G(u(t))+\frac{a(t)}{2}\left[\int_0^t g(u(s))ds\right]^2-\frac{1}{2}\int_0^t a'(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau$$

and showed that $E'(t)\leq 0$, and $E''(t)$ is bounded. Then

$$G(u(t))\leq E(t)\leq E(0)=G(u_0) \tag{2.2}$$

and since $G(x)\to\infty$ as $|x|\to\infty$, u is bounded on $[0,\infty)$. The above facts for $E(t)$ imply $E'(t)\to 0$ as $t\to\infty$. Levin used this to prove his result. Notice that (2.2) implies that the solution $u(t,\lambda)$ of (1.1) satisfies

$$|u(t,\lambda)|\leq 1,\quad t\geq 0,\quad \lambda\geq 1. \tag{2.3}$$

Indeed (2.3) remains true for the equation (1.1) under the weaker hypothesis (2.12) below. Details can be worked out as in [8].

In 1971 [23] S-O Londen improved this Theorem using a somewhat different technique. Londen proved

<u>Theorem</u> <u>B.</u> <u>If</u> $a(t)\in L^1(0,1)$, $(-1)^k a^{(k)}(t)\geq 0$, $0<t<\infty$, $k=0,1,2$, a <u>not</u> <u>constant</u>, $g\in C(-\infty,\infty)$, $f\in C[0,\infty)\cap L^1(0,\infty)$, <u>then</u> <u>any</u> <u>solution</u> $u(t)$ <u>of</u> (2.1) <u>that</u> <u>satisfies</u> $\sup_{0\leq t<\infty} |u(t)|<\infty$ <u>must</u> <u>satisfy</u> $\lim_{t\to\infty} g(u(t))=0$. <u>If</u> <u>in</u> <u>addition</u> $\lim_{t\to 0} f(t)=0$ <u>then</u> $\lim_{t\to\infty} u'(t)=0$.

In particular, $a(0+)$ need not be finite. Instead of using an energy function, Londen wrote the equation (2.1) in the form

$$G(u(t))=G(u_0)+\int_0^t g(u(\tau))\int_0^\tau g(u(s))a(\tau-s)ds\,d\tau \tag{2.4}$$

$$+\int_0^t f(\tau)g(u(\tau))d\tau.$$

By rewriting the second term on the right in (2.4) one can immediately bring out the importance of the monotonicity conditions on a to the existence of $\lim_{t\to\infty} g(u(t))$.

Nohel and Shea, developing ideas introduced by A. Halanay [6] and R. C. MacCamy and J. S. Wong [25], use the same form (2.4), but they employ transform methods to analyze the key quadratic term.

Let us recall two definitions that are needed here. The function a is of <u>positive</u> <u>type</u> if $a\in L^1_{loc}[0,\infty)$ and

$$\int_0^T v(t)\int_0^t v(\tau)a(t-\tau)d\tau dt\geq 0$$

for all $v\in C[0,\infty)$, for all $T>0$. The function b is <u>strongly</u> <u>positive</u> if b

is of positive type and if there exists $\eta>0$ such that $b(t)-\eta e^{-t}$ is of positive type.

Nohel and Shea prove

<u>Theorem</u> <u>C</u>. (i) <u>Let</u> $u(t)$ <u>be a bounded solution of</u> (2.1). <u>Assume</u> $a(t)e^{-\sigma t}\in L^1(0,\infty)$ <u>for all</u> $\sigma>0$, $a\in BV[1,\infty)$, a <u>is strongly positive</u>, $f(t)\in L^1(0,\infty)$, $g(x)\in C(-\infty,\infty)$. <u>Then</u>

$$\lim_{t\to\infty} g(u(t))=0 \ \underline{\text{and}}\ \lim_{t\to\infty}[u'(t)-f(t)]=0.$$

(ii) <u>Let</u> $a(t)\in L^1_{loc}[0,\infty)$ <u>be not identically constant, nonnegative, nonincreasing and convex and such that</u> $da'(t)$ <u>is not a purely singular measure. Then</u> a <u>is strongly positive, so that the conclusions of</u> (i) <u>hold</u>.

This strengthens Theorem B of Londen.

We will now give a sketch of the proof of Theorem C (i) in [29]. Start with (2.4). Define

$$Q_a[v,T]\equiv\int_0^T v(t)\int_0^t v(\xi)a(t-\xi)d\xi dt,$$

where $v(t)\equiv g(u(t))$, and define

$$v_T(t)\equiv\begin{cases} v(t) & , 0\leqslant t\leqslant T \\ 0 & \text{elsewhere.}\end{cases}$$

Extend a by an even extension and define $a_\sigma(t)\equiv e^{-\sigma|t|}a(t),\sigma>0$. Then

$$Q_a[v,T]=\frac{1}{2}\int_0^T v(t)\int_0^T v(\xi)a(t-\xi)d\xi dt$$

$$=\frac{1}{2}\int_{-\infty}^{\infty} v_T(t)\int_{-\infty}^{\infty} v_T(\xi)a(t-\xi)d\xi dt$$

$$=\lim_{\sigma\to 0^+}\frac{1}{2}\int_{-\infty}^{\infty}v_T(t)\int_{-\infty}^{\infty}v_T(\xi)a_\sigma(t-\xi)d\xi dt,$$

where the last step uses the Lebesgue Dominated Convergence Theorem.

Let $\hat{f}(\tau)\equiv\int_{-\infty}^{\infty}f(t)e^{-it\tau}dt$, $\tilde{f}(s)=\int_0^{\infty}e^{-st}f(t)dt$. Then by evenness of a, $\hat{a}_\sigma(\tau)=2\mathrm{Re}\int_0^{\infty}e^{-(\sigma+i\tau)t}a_\sigma(t)dt=2\mathrm{Re}\tilde{a}(\sigma+i\tau)$. Note that $\mathrm{Re}\tilde{a}(\sigma+i\tau)\geqslant\frac{\eta}{1+\tau^2}$ by strong positivity. Since $a_\sigma\in L^1(-\infty,\infty)$ for $\sigma>0$ and v_T has compact support, the Parseval Theorem implies

$$Q_a[v,T]=\lim_{\sigma\to 0^+}\int_{-\infty}^{\infty}|\hat{v}_T(\tau)|^2\hat{a}_\sigma(\tau)d\tau\geqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{v}_T(\tau)|^2\frac{\eta}{1+\tau^2}d\tau$$

$$=Q_c(v,T) \text{ where } c=\eta e^{-|t|}.$$

Since u is bounded, $Q_a(v,T)\geqslant 0$, (2.4) and $f\in L^1(0,\infty)$, the above inequality shows $Q_c(v,T)$ is bounded. Further estimates and the Wiener tauberian Theorem or an elementary argument is then used to prove $v(t)\equiv g(u(t))\to 0$ as $t\to\infty$.

Staffans [32] generalizes the notion of strong positivity to strict positivity. A function a in $L^1_{loc}[0,\infty)$ is of <u>strictly positive type</u> if there exists a function $b\in L^1(0,\infty)$ such that $\int_0^{\infty}\cos wt\ b(t)dt>0, -\infty<w<\infty$ and a-b is of positive type. Note that strong positivity is the special case $b(t)=e^{-\eta t}$.

Staffans proves

<u>Theorem D.</u> <u>If</u> a <u>is of strictly positive type and if</u> $g\in C(-\infty,\infty)$,

$f\in L^1(0,\infty)$, *then for a bounded uniformly continuous locally absolutely continuous solution* $u(t)$ *of* (2.1), $g(u(t))\to 0$ *as* $t\to\infty$.

All nonnegative, nonincreasing, convex kernels satisfying

$$\int_0^\infty \cos wt\; a(t)dt>0, -\infty<w<\infty, \tag{2.5}$$

are strictly positive, and (2.5) excludes only certain piecewise linear kernels (see [7]). Staffans wrote a series of papers developing these ideas further. (See [34]-[38]).

Of course, transforms were used even earlier to study the linear case of (2.1) (e.g. [22]). The major landmark is [31], where Shea and Wainger prove

Theorem E. Let $g(x)=x$ *in* (2.1). *Let* $a(t)$ *satisfy* $a(t)=b(t)+\beta(t)$ *where* b *is nonnegative, nonincreasing and convex on* $(0,\infty)$, $b\in L^1(0,1)$ *and* $(1+t)\beta(t)\in L^1(0,\infty)$. *Let* $a(t)$ *satisfy* $-\tilde{a}(z)\neq z$, $\mathrm{Re}\, z\geq 0$. *Then in the particular case where* $f\equiv 0$, *and* $u_0=1$ *the solution* $r=r(t)$ *of* (2.1) *is in* $L^1(0,\infty)\cap C^1[0,\infty)$ *and its derivative tends to zero as* $t\to\infty$. *If* $f\in L^\infty(0,\infty)$, *the solution* $u=u(t)$ *to* (2.1) *is bounded*:

$$|u(t)|\leq|u_0|\,\|r\|_\infty+\|f\|_\infty\|r\|_1,\quad 0\leq t<\infty.$$

The main theorem in [31], which is used to prove Theorem E is a variant of the Wiener-Lévy Theorem.

Theorem F. Let $a(t)=b(t)+\beta(t)$ *with* b *nonnegative, nonincreasing, convex on* $(0,\infty)$ *and* $b\in L^1_{loc}[0,\infty)$ *and* $(1+t)\beta(t)\in L^1(0,\infty)$ *or* $a(t)=b+\beta(t)$ *where* b *is any constant*, $\beta(t)\in L^1(0,\infty)$. *Assume* $\phi(w,z)$ *is analytic on*

$S=\{(\tilde{a}(z),z):\operatorname{Re} z\geq 0\}$ _and at_ $(0,\infty)$, $(\infty,0)$ _and that_ $\phi(0,\infty)=0$. _Then there exists_ $r(t)\in L^1(0,\infty)$ _such that_ $\phi(\tilde{a}(z),z)=\tilde{r}(z)$, $\operatorname{Re} z\geq 0$.

Jordan and Wheeler weaken the hypothesis $(1+t)\beta(t)\in L^1(0,\infty)$ to $\beta(t)\in L^1(0,\infty)$ in [18]. For further developments along this line see Jordan, Staffans, Wheeler [17].

In the proof of Theorem F in the case that applies to Theorem E when $\beta=0$, the key step is to show $\int_{-\infty}^{\infty}\left|\frac{d}{dw}\hat{r}(w)\right|dw<\infty$ and then use the Hardy inequality $\int_0^{\infty}\left|\frac{\hat{f}(t)}{t}\right|dt\leq\pi\int_{-\infty}^{\infty}|f(x)|dx$ for $f\in H^1\{x+iy:\ y>0\}$ with

$$f(x)=(\check{r})'(x) \text{ where } \check{r}(x)\equiv\int_0^{\infty}e^{itx}\ r(x)dx.$$

Since $\hat{f}(t)=itr(t)$, this yields

$$\int_0^{\infty}|r(t)|dt\leq\pi\int_{-\infty}^{\infty}|\hat{r}'(x)|dx<\infty.$$

The difficult estimate in this step is showing $\int_{-1}^{1}|\hat{r}'(w)|dw<\infty$, that is, $\int_{-1}^{1}\left|\frac{1+\hat{a}'(w)}{(w+\hat{a}(w))^2}\right|dw<\infty$. Using the monotonicity condition on a, Shea and Wainger obtain the estimates

$$2^{-3/2}A\left(\frac{1}{|\tau|}\right)\leq|\hat{a}(\tau)|\leq 4A\left(\frac{1}{|\tau|}\right),\ \tau\neq 0, \tag{2.6}$$

and

$$|\hat{a}'(\tau)|\leq 40A_1\left(\frac{1}{|\tau|}\right),\ \tau\neq 0 \tag{2.7}$$

where

$$A(x)\equiv\int_0^x a(s)ds, A_1(x)\equiv\int_0^x sa(s)ds. \tag{2.8}$$

Inequality (2.6) reduces the needed estimate to $\int_{-1}^{1} \left|\frac{\hat{a}'(w)}{\hat{a}(w)^2}\right| dw < \infty$.

To show the latter inequality we use (2.6), (2.7), evenness of $|\hat{a}|$ and a change of variable to obtain

$$\int_{-1}^{1} \left|\frac{\hat{a}'(w)}{\hat{a}(w)^2}\right| dw$$

$$\leqslant 640 \int_0^1 \frac{A_1(\frac{1}{w})}{A^2(\frac{1}{w})} dw$$

$$= 640 \int_1^\infty \frac{A_1(y)}{A^2(y)} \frac{dy}{y^2}$$

$$\leqslant K + \int_1^\infty \frac{a(y)}{A^2(y)} dy = K + \frac{1}{A(\infty)} - \frac{1}{A(1)},$$

K is a constant and $\frac{1}{A(\infty)} = 0$ if $a \notin L^1(0,\infty)$. In the last inequality one uses integration by parts and monotonicity.

Recently Londen in [24] discovered that, for a certain subclass of functions of the form $a(t)=b(t)c(t)$ where $b(t)$ is completely monotone and $c(t)$ is of positive type, the solution u of (2.1), in the linear case with $f \equiv 0$, is in $L^1(0,\infty)$. For related work on (2.1) see [14], [19], [20].

The parameter problem (1.1) arises when one tries to obtain results like those above for the Hilbert space problem

(2.9) $\quad y'(t) + \int_0^t (d + a(t-\tau)) Ly(\tau) d\tau = f(t), \ t \geqslant 0, \ y(0) = y_0,$

where L is a self adjoint linear operator, defined on a dense domain D of a Hilbert space H, whose spectrum is contained in $[1,\infty)$. Letting u be the solution of (1.1), define $U(t)\equiv\int_1^\infty u(t,\lambda)dE_\lambda$ where (E_λ) is the spectral family (see [30]) corresponding to L.

Carr and Hannsgen establish the resolvent formula $y(t)=U(t)y_0+\int_0^t U(t-\tau)f(\tau)d\tau$ for (2.9). They also give sufficient conditions for

$$(2.10)\qquad \int_0^\infty \|U(t)\|dt<\infty,\ \int_0^\infty \|V(t)L^{-1/2}\|dt<\infty, \text{ where } V(t)\equiv\int_1^\infty u'(t,\lambda)dE_\lambda,$$

(see [2],[3]).

In particular, (2.10) holds if a is nonnegative, nonincreasing and convex with $-a'$ convex. The main work is to show that

$$(2.11)\qquad \int_0^\infty \sup_{\lambda\geq 1} |u(t,\lambda)|dt<\infty,\ \int_0^\infty \sup_{\lambda\geq 1} \left|\frac{u'(t,\lambda)}{\lambda^{1/2}}\right|dt<\infty$$

holds and then (2.10) follows by the functional calculus (see [30]). The techniques used by Carr and Hannsgen in proving (2.11) are crucial in our study of (1.2). We will consider them in some detail.

A helpful example is $a(t)=e^{-t}$. (1.1) reduces to the ordinary differential equation $u''(t)+u'(t)+\lambda u(t)=0$ with solution $u(t,\lambda)=e^{-t/2}(\cos\mu t+\frac{1}{2\mu}\sin\mu t)$ where $\mu=\left(\lambda-\frac{1}{4}\right)^{1/2}$. Differentiation shows that u' and u'' must be scaled by dividing by $\lambda^{1/2}$ and λ respectively if one expects to sup over $\lambda\geq 1$ and obtain a finite valued function of t.

The assumptions used to prove (2.11) generally involve a sufficient transform condition which often implies a (generally stronger) direct sufficient condition. We illustrate this with the next two theorems from [2] and [3] respectively.

Theorem G. Assume $a \in C(0,\infty) \cap L^1(0,1)$,

(2.12) $\quad a$ *is nonnegative, nonincreasing and convex on* $(0,\infty)$,

$0 = a(\infty) < a(0+) \leq \infty$,

$d \geq 0$.

Assume moreover $a = b + c$ *where* b, c *satisfy* (2.12) *except that* $b(0+) = 0$ *or* $c(0+) = 0$ *is permitted*, $\int_1^\infty t^{-1} b(t)\,dt < \infty$ *and* $-c'$ *is convex on* $(0,\infty)$. *Assume*

$$\limsup_{\tau\to\infty} \left| \frac{\operatorname{Im}\hat{a}(\tau)}{\tau \operatorname{Re}\hat{a}(\tau)} \right| < \infty .$$

Then $\int_0^\infty \sup\limits_{\lambda \geq 1} |u(t,\lambda)|\,dt < \infty$.

Theorem H. Assume $a \in C(0,\infty) \cap L^1(0,1)$, (2.12) *and* $a = b + c$ *where* b, c *satisfy* (2.12) *except that* $b(0+) = 0$ *or* $c(0+) = 0$ *is permitted*, $\int_0^\infty t^{-1} b(t)\,dt < \infty$ *and* $-c'$ *is convex on* $(0,\infty)$. *Assume that for some* $\epsilon > 0$,

$$\limsup_{\tau\to\infty} \left| \frac{\operatorname{Im}\hat{a}(\tau)}{\operatorname{Re}\hat{a}(\tau)} \cdot \left[\frac{\operatorname{Im}\hat{a}(\tau)}{\tau} \right]^{\frac{1}{2}-\epsilon} \right| < \infty . \quad \textit{Then}$$

$$\int_0^\infty \sup_{\lambda \leq 1} \left| \frac{u'(t,\lambda)}{\lambda^{1/2}} \right| dt < \infty .$$

A direct condition on a that implies (2.11) is

(2.13) $\quad a$ is nonnegative, nonincreasing, convex and $-a'$ is convex, on $(0,\infty)$.

Now we will give a sketch of the proof of the first part of (2.11), under the assumption (2.13).

Using elementary transform theory the representation

$$\pi u(t,\lambda) = \frac{1}{\lambda} \int_0^\infty \operatorname{Re}\left(\frac{e^{i\tau t}}{D(\tau,\lambda)} \right) d\tau, \quad t \geq 0, \ \lambda > 0$$

is obtained for the solution $u=u(t,\lambda)$ of (1.1), where

$$D(\tau,\lambda)\equiv D(\tau)+i\tau\lambda^{-1}\equiv\hat{a}(\tau)-id\tau^{-1}+i\tau\lambda^{-1}.$$

This is integrated by parts, yielding

$$\text{(2.14)}\qquad \pi u(t,\lambda)=\mathrm{Re}\left(\frac{1}{it\lambda}\int_0^\infty e^{i\tau t}\frac{D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}d\tau\right),\ t\geqslant 0,\ \lambda>0.$$

Estimates (2.6) and (2.7) and lemma 2 of [31] ensure the absolute convergence of the integral as well as the vanishing of the boundary terms in the integration by parts.

The integrand is then written in the form

$$\frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}=\frac{D_\tau(\tau,\lambda)}{D(\tau)}\left[1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right]^2$$

$$=\frac{D'(\tau)+i\lambda^{-1}}{D^2(\tau)}\left[1-\frac{2i\tau\lambda^{-1}}{D(\tau)}\right]$$

$$-\frac{\tau^2D_\tau(\tau,\lambda)}{\lambda^2D^2(\tau)D(\tau,\lambda)}\left[\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right]$$

$$=\frac{D'(\tau)}{D^2(\tau)}+\frac{i}{\lambda D^2(\tau)}\left[1-\frac{2\tau D'(\tau)}{D(\tau)}\right]+\frac{1}{\lambda^2}\frac{2\tau}{D^3(\tau)}$$

$$-\frac{\tau^2D_\tau(\tau,\lambda)}{\lambda^2D^2(\tau)D(\tau,\lambda)}\left[\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right]$$

$$\equiv I_1+\frac{i}{\lambda}I_2+\frac{1}{\lambda^2}I_3+\frac{1}{\lambda^2}I_4.$$

Then (2.14) becomes

$$\pi u(t,\lambda)=\mathrm{Re}\left\{\frac{1}{it\lambda}\int_0^\rho e^{i\tau t}[I_1+\frac{i}{\lambda}I_2+\frac{1}{\lambda^2}I_3+\frac{1}{\lambda^2}I_4]d\tau+\frac{1}{it\lambda}\int_\rho^\infty e^{i\tau t}\frac{D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}d\tau\right\}$$

$$=\mathrm{Im}\left\{\lambda^{-1}u_1(t)+i\lambda^{-2}u_2(t)+\lambda^{-3}u_3(t)+u_4(t,\lambda)+u_5(t,\lambda)\right\}$$

where

$$u_1(t)\equiv\frac{1}{t}\int_0^\rho e^{i\tau t}I_1 d\tau,$$

$$u_2(t)\equiv\frac{1}{t}\int_0^\rho e^{i\tau t}I_2 d\tau,$$

$$u_3(t)\equiv\frac{1}{t}\int_0^\rho e^{i\tau t}I_3 d\tau,$$

$$u_4(t,\lambda)\equiv\frac{1}{\lambda^3 t}\int_0^\rho e^{i\tau t}I_4 d\tau,$$

$$u_5(t,\tau)\equiv\frac{1}{t\lambda}\int_\rho^\infty e^{i\tau t}\frac{D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}d\tau.$$

Note that u_1, u_2, u_3 are functions of t only. Carr and Hannsgen show that

$$|u_j(t,\lambda)|\leqslant\frac{M}{t^2}\in L^1(1,\infty),\ (1\leqslant\lambda<\infty,\ j=4,5)$$

for some constant M independent of λ. This with $u=\frac{1}{\pi}\mathrm{Im}[\lambda^{-1}u_1+i\lambda^{-2}u_2+\lambda^{-3}u_3+u_4+u_5]$ and $u(t,\lambda)\in L^1(0,\infty)$ for each $\lambda\geqslant 1$ (by Theorem F) implies $u_j\in L^1(1,\infty)$ for $j=1$, 2, 3. Thus,

$$\sup_{\lambda\geqslant 1}|u(t,\lambda)|\leqslant u_1(t)+u_2(t)+u_3(t)+\frac{2M}{t^2}\in L^1(1,\infty).$$

The inequality (2.3) then gives the first part of (2.11).

The bound $|u_j(t,\lambda)|\leqslant\frac{M}{t^2}$, $j=4,5$ used above is obtained by integrating the formula for u_j by parts. This brings another factor of t into the denominator. The coefficient of t^{-2} is estimated using (2.6), (2.7) and the inequality

(2.15) $$\frac{1}{5}A_1\left(\frac{1}{\tau}\right)\leqslant\theta(\tau)\leqslant C_1A_1\left(\frac{1}{\tau}\right),$$

proved in [2]. Here C_1 is a certain constant and ϕ, θ are defined as the real functions such that

(2.16) $\hat{a}(\tau)=\phi(\tau)-i\tau\theta(\tau)\ (\tau\in\mathbb{R},\tau\neq 0).$

The same methods are also used in [3] to show

(2.17) $\int_1^\infty \sup_{\lambda\geqslant 1} \left|\frac{u'(t,\lambda)}{\lambda^{1/2}}\right| dt<\infty$

under the assumption (2.13), and we will use them to show

$$\int_1^\infty \sup_{\lambda\geqslant 1} \left|\frac{u''(t,\lambda)}{\lambda}\right| dt<\infty.$$

Now consider the second part of (2.11) under (2.13). By (2.17), we need only look at $\int_0^1$. It turns out that, as opposed to $\sup_{\lambda\geqslant 1} |u(t,\lambda)|$, $\sup_{\lambda\geqslant 1} \left|\frac{u'(t,\lambda)}{\lambda^{1/2}}\right|$ is not a bounded function of t on (0,1). However, Carr and Hannsgen use the analogue of (2.14) for $u'(t,\lambda)$ to obtain an estimate of the form

$$\left|\frac{u'(t,\lambda)}{\lambda^{1/2}}\right|\leqslant\frac{M\lambda^{-\epsilon}}{t},$$

(M a constant), under the assumption $\sup_{\rho/2\leqslant\tau} \left(\frac{\tau\theta(\tau)^{\frac{3}{2}-\epsilon}}{\phi(\tau)}\right)<\infty$ for some ϵ, $0<\epsilon<\frac{1}{2}$. They also obtain the estimate

(2.18) $\sup_{t\geqslant 0} |u'(t,\lambda)|\leqslant K\sigma$

for some constant K under the assumption (2.12) where $\sigma=\sigma(\lambda)$ is defined by the formula

(2.19) $\frac{1}{\lambda} = \frac{1}{\sigma}\int_0^{1/\sigma} a(s)ds.$

Note that $\sigma\to\infty$ as $\lambda\to\infty$ and $\frac{\sigma}{\lambda}\to 0$ as $\lambda\to\infty$.

The above estimates on $u'(t,\lambda)$ are combined as follows: $\left|\frac{u'}{\lambda^{1/2}}\right|=\left|\frac{u'}{\lambda^{1/2}}\right|^{p}\left|\frac{u'}{\lambda^{1/2}}\right|^{1-p}\leqslant M_1 t^{-p}\lambda^{-p\epsilon}\left(\frac{\sigma}{\lambda^{1/2}}\right)^{1-p}\leqslant M_2 t^{-p}\lambda^{1/2-1/2p-p\epsilon}$ where M_1 and M_2 are constants and $\frac{1}{1+2\epsilon}\leqslant p<1$. Then $\left|\frac{u'}{\lambda^{1/2}}\right|\leqslant M_2 t^{-p}\epsilon L^1(0,1)$. This completes the proof of the second part of (2.11) under the assumptions mentioned above.

For related work see [8] - [13], [15], [16], [27] and [28].

3. New Results

We study the question of whether (1.2) holds for the solution of (1.1). By lemma 3 (i) in [7], $\hat{a}(\tau)$ is defined, finite and continuous for $\operatorname{Im}\tau \geq 0$, $\tau \neq 0$ when $a(t) \in C(0,\infty) \cap L^1(0,1)$ is nonnegative and nonincreasing with

$\lim_{t\to\infty} a(t)=0$ and $0<a(0+)\leq\infty$.

We will be using the auxiliary functions ω, ω^*, ϕ, θ, $D(\tau,\lambda)$, $D(\tau)$, A, σ, A_1 defined by (3.1), (3.2), (2.16), $D(\tau,\lambda)=D(\tau)+i\tau\lambda^{-1}$, $D(\tau)=\hat{a}(\tau)-id\tau^{-1}$, (2.8), (2.19) and (2.8) respectively. We also use inequalities of these functions which were established in [2] and [3]. In [3], assuming (2.12), it is shown that

$$\lambda^{-1}=\theta(\omega)+d\omega^{-2} \tag{3.1}$$

defines a continuous, strictly increasing function $\omega=\omega(\lambda)$ on some interval $[\lambda_0,\infty)$, where $\omega(\lambda_0)=\rho$ for some $\rho>0$. Extend ω to $[1,\infty)$ by defining $\omega(\lambda)=\rho$ on $[1,\lambda_0]$ (if $\lambda_0>1$). As in [3] define

$$\omega^*=\omega^*(\lambda) \tag{3.2}$$

to be any number in $[\frac{\omega}{2},2\omega]$ such that $\phi(\omega^*)=\min_{\frac{\omega}{2}\leq\tau\leq2\omega}\phi(\tau)$. Assuming (2.12), the following inequalities hold ([2] and [3]):

$$\phi^2(\tau)+\left(\frac{\tau-\omega}{\lambda}\right)^2\leq M\cdot|D(\tau,\lambda)|^2,\quad \tau\geq\frac{\rho}{2}. \tag{3.3}$$

$$A\left(\frac{1}{\tau}\right)\leq M\cdot|D(\tau,\lambda)|,\quad \tau\in\left[\frac{\rho}{2},\ \frac{\omega}{2}\right)\cup[2\omega,\infty). \tag{3.4}$$

(3.5) $\quad \omega \leqslant C_1\sigma;\ \lambda \leqslant C_2\sigma^2,\ \lambda \geqslant 1$ where $C_1, C_2 > 0, (C_1 > 12)$.

(3.6) $\quad \frac{1}{M} \leqslant \lambda\theta(\tau) \leqslant M,\ \frac{\omega}{2} \leqslant \tau \leqslant 2\omega$

(3.7) $\quad \tau\lambda^{-1} \leqslant M \cdot |D(\tau,\lambda)|,\ 2\omega \leqslant \tau < \infty$.

(3.8) $\quad MA\left(\frac{1}{\tau}\right) \leqslant \phi(\omega^*) + \omega^*\theta(\omega^*),\ \frac{\omega}{2} \leqslant \tau \leqslant 2\omega$.

(3.9) $\quad \lambda \leqslant M\omega^2$

(3.10) $\quad \frac{1}{5}A_1\left(\frac{1}{\omega}\right) \leqslant \frac{1}{\lambda} \leqslant C_1 A_1\left(\frac{1}{\omega}\right),\ \lambda \geqslant 1$.

Here M is a constant independent of τ and λ. We also will need the estimate

(3.11) $\quad \int_0^{kx} a(s)ds \geqslant k\int_0^x a(s)ds \ (0<k<1, 0<x<\infty)$

which holds for $a \in L^1_{loc}[0,\infty)$ satisfying a is nonnegative and nonincreasing. To see this define $F(k) \equiv \int_0^{kx} a(s)ds - k\int_0^x a(s)ds$ for fixed $x>0$. Then $F(0)=0=F(1)$. Moreover $F'(k) = xa(kx) - \int_0^x a(s)ds$ is a nonincreasing function of k because a is nonincreasing. Thus F is positive for $0<k<1$.

Theorem 1 gives representations for $u''(t,\lambda)$ in terms of the transform of the convolution kernel.

<u>Theorem 1.</u> <u>Assume</u> $a \in L^1(0,1)$, (2.12) <u>and</u> $\phi(\tau) > 0\ (\tau > 0)$. <u>Then</u>

(i) $$\pi u''(t,\lambda) = \lim_{R\to\infty} \mathrm{Re}\int_0^R e^{i\tau t}\left(\frac{-i\tau D(\tau)}{D(\tau,\lambda)}\right)d\tau$$

$$= \int_0^\infty \mathrm{Re}\left(e^{i\tau t}\left[\frac{-i\tau D(\tau)}{D(\tau,\lambda)}\right]\right)d\tau$$

and

(ii) $$\pi u''(t,\lambda)=\mathrm{Re}\frac{1}{t\lambda}\int_0^\infty e^{i\tau t}\left(\frac{i\tau^2 D'(\tau)+\lambda D(\tau)^2}{D(\tau,\lambda)^2}\right)d\tau.$$

(The integrals are Riemann or improper Riemann integrals).

Theorem 2 gives a necessary condition for (1.2).

Theorem 2. *Assume* (1.2) *and the hypotheses of Theorem 1. Then*

(3.12) $$\limsup_{\tau\to\infty} \frac{(\tau\theta(\tau))^2}{\phi(\tau)}<\infty.$$

The representation for $u''(t,\lambda)$ of Theorem 1 (ii) is used to obtain Theorem 3.

Theorem 3. *Assume* (3.12) *and the hypotheses of Theorem 1. Then*

$\sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|\leq\frac{M}{t}$ *for some constant* M *independent of* t *and* λ.

If we make a change of variable in the integrand of (1.1) and then differentiate the result we obtain

(3.13) $$\frac{-u''(t)}{\lambda}=a(t)+du(t)+\int_0^t a(\tau)u'(t-\tau)d\tau,\ t>0.$$

The next lemma gives an important estaimate on the integral term of (3.13) which yields another necessary condition for (1.2).

Lemma 1. *Under the assumptions of Theorem 1, there exist constants* $N_1,N_2>0$ *such that*

(3.14) $$N_1\frac{\sigma^2}{\lambda}\leq\sup_{t>0}\left|\int_0^t u'(t-\tau)a(\tau)d\tau\right|\leq N_2\frac{\sigma^2}{\lambda},\ \lambda\geq 1.$$

A consequence of this lemma is

Theorem 4. *For* (1.2) *to hold under the hypotheses of Theorem 1 it is necessary that*

(3.15) $(-\ln t)a(t)\in L^1(0,1)$.

We obtain a sufficient condition for (1.2). We first give two partial results which isolate transform conditions sufficient for integrability of $\sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|$ on $(0,1)$ and on $[1,\infty)$.

Theorem 5. In addition to the assumptions of Theorem 1, suppose (3.15) *and*

$$\sup_{\lambda\geq 1}\frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}<\infty \tag{3.16}$$

where

$$C(\lambda)\equiv\frac{\sigma}{\lambda}+\frac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)}. \tag{3.17}$$

Then

$$\int_0^1 \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty.$$

Theorem 6. In addition to the assumptions of Theorem 1, suppose (3.12), $a(t)=b(t)+c(t)$ *where* b,c *both satisfy* (2.12) *except that* $b(0+)=0$ *or* $c(0+)=0$ *is permitted. Assume*

$$\int_1^\infty \frac{b(t)}{t}dt<\infty \text{ and } -c' \text{ is convex}. \tag{3.18}$$

Then

$$\int_1^\infty \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty.$$

Thus in terms of the transform of a, we have the following answer to the main question of this study.

Theorem 7. a) *Under all the assumptions of Theorems* 5 *and* 6 *we have* (1.2). b) *If* (3.15) *and* (2.13) *hold, and if, for* some $q>1$, $\limsup_{\lambda\to\infty} C(\lambda)\left[\log\frac{\sigma^2}{\lambda}\right]^q<\infty$, then (1.2) holds.

Next we look for direct conditions on $a(t)$ that imply the transform conditions of Theorem 7. We have not found a satisfying "natural" sufficient condition, but some reasonable conditions which include wide classes of examples can be stated.

By [16], when (2.13) holds,

(3.19) $\frac{1}{5}B\left(\frac{1}{\tau}\right)\leqslant\phi(\tau)\leqslant 12B\left(\frac{1}{\tau}\right)$, $\tau>0$,

where

(3.20) $B(x)\equiv\int_0^x -sa'(s)ds$

and we also observe that the following holds:

Lemma 2. If $a\in L^1(0,1)$ and (2.13) holds then (3.12) holds.

This extends a result in [33]. (Lemma 2 is proved in [1] (Lemma 2 (iii)). For completeness we include the proof in Section 4.) Putting this lemma together with Theorem 3 yields the following corollary.

Corollary. If (2.13) and $a\in L^1(0,1)$, then the conclusion of Theorem 3 holds.

We next obtain a theorem ensuring (1.2) where the hypothesis are stated directly on $a(t)$. The four cases in part b of the Theorem say roughly this about $\hat{a}(\tau)$:

(i) $\mathrm{Re}\,\hat{a}(\tau)$, $\mathrm{Im}\,\hat{a}(\tau)$ have the same order of magnitude as $\tau\to\infty$; (ii) $\mathrm{Im}\,\hat{a}(\tau)$ is smaller than $\mathrm{Re}\,\hat{a}(\tau)$ as $\tau\to\infty$; (iii) and (iv) $\mathrm{Re}\,\hat{a}(\tau)$ is smaller than $\mathrm{Im}\,\hat{a}(\tau)$ as $\tau\to\infty$. This is shown in the discussion in Section 4.

Theorem 8. a) *Assume* (2.13) *and* $a(0+)<\infty$. *Then* (1.2) *holds*.

b) *Assume* (2.13), (3.15) *and one of the following:*

(i) *There exist constants* $c_1, c_2 > 0$ *such that*

$$c_1 \tau A_1\left(\frac{1}{\tau}\right) \leq B\left(\frac{1}{\tau}\right) \leq c_2 \tau A_1\left(\frac{1}{\tau}\right), \quad \frac{\rho}{2} < \tau, \tag{3.21}$$

or

(ii) $$\lim_{\tau\to\infty} \frac{\tau A_1\left(\frac{1}{\tau}\right)}{A\left(\frac{1}{\tau}\right)} = 0,$$

or

(iii) $$\lim_{\tau\to\infty} \frac{B\left(\frac{1}{\tau}\right)}{A\left(\frac{1}{\tau}\right)} = 0, \quad \frac{a^2(t)}{-a'(t)} \text{ is increasing for small } t \text{ and}$$

$$\frac{a^2(t)}{-ta'(t)} \in L^1(0,\epsilon) \text{ for some } \epsilon > 0,$$

or

(iv) $$\lim_{\tau\to\infty} \frac{B\left(\frac{1}{\tau}\right)}{A\left(\frac{1}{\tau}\right)} = 0 \text{ and } \frac{\tau A^3\left(\frac{1}{\tau}\right)}{B\left(\frac{1}{\tau}\right)} \leq M < \infty \text{ for } \tau \text{ in } [\tfrac{\rho}{2}, \infty).$$

Then (1.2) *holds*.

In Section 5 we apply this Theorem to examples. In particular if $a(t)=t^{-p}$, $0<p<1$ or $a(t)=-\ln t$ for t near 0 or $a(t)=t^{-1}(-\ln t)^{-q}$ near 0 $(q>2)$, then (1.2) holds. Note that $a(t)=t^{-1}(-\ln t)^{-q}$ does not satisfy the necessary condition (3.15) for $q\leq 2$.

After considering these examples we give an example that satisfies (2.13), (3.15) but not (3.16). However we still have (1.2) by

Theorem 7 b.

4. Proofs

The integrated version of (1.1) is

$u(t)+\lambda\int_0^t u(\tau)\int_0^{t-\tau}(d+a(s))ds\,d\tau=1$. The usual method of Picard successive approximations [26] ensures the local existence of a unique continuous solution. The a priori estimate $|u|\leqslant 1$ (2.3) ensures that the continuous solution exists on $[0,\infty)$. Now (1.1) shows that u' is continuous on $[0,\infty)$ and (3.13) shows u'' is locally integrable for $t\geqslant 0$ and continuous for $t>0$.

Let us prove Theorem 1, (i). By [7], $\lim_{t\to\infty} u(t)=0$ and by Theorem E, $\lim_{t\to\infty} u'(t)=0$. By (1.1), $u'(0)=0$. These facts and integration by parts yield

$$\widehat{u''}(\tau)=\int_0^\infty e^{-i\tau t}u''(t)dt$$

$$= \int_0^\infty i\tau e^{-i\tau t}u'(t)dt=i\tau(-1+i\tau\int_0^\infty e^{-i\tau t}u(t)dt)$$

$$= i\tau(-1+i\tau\hat{u}(\tau)).$$

Applying the Fourier transform to (1.1) yields

$\hat{u}(\tau)\equiv\hat{u}(\tau,\lambda)=\lambda^{-1}(\phi(\tau)+i\tau[\lambda^{-1}-\theta(\tau)-d\tau^{-2}])^{-1},(\tau>0,\lambda\geqslant 1)$; that is

$$\hat{u}(\tau)=\frac{1}{\lambda D(\tau,\lambda)}.$$

Thus,

$$\widehat{u''}(\tau)=i\tau(-1+\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)})=i\tau\left(\frac{-D(\tau)}{D(\tau,\lambda)}\right)(\tau>0,\lambda\geqslant 1).$$

By (2.6), $|\tau\hat{a}(\tau)|\leqslant 4\tau A\left(\frac{1}{\tau}\right)$ and by (2.12) $\lim_{\tau\to 0+}\tau\, A\left(\frac{1}{\tau}\right)=\lim_{x\to\infty}\frac{A(x)}{x}=$

0, so $\lim_{\tau\to 0+}|\tau\hat{a}(\tau)|=0$. Thus we write $\lim_{\tau\to 0+}|\widehat{u''}(\tau)|=\lim_{\tau\to 0+}\left|\frac{\tau D(\tau)}{D(\tau,\lambda)}\right|=$

$\lim_{\tau\to 0+}\left|\frac{i\tau(\hat{a}(\tau)-id\tau^{-1})}{\hat{a}(\tau)-id\tau^{-1}+i\tau\lambda^{-1}}\right| = \lim_{\tau\to 0+}\left|\frac{i\tau^2\hat{a}(\tau)+\tau d}{\tau\hat{a}(\tau)-id+i\tau^2\lambda^{-1}}\right|=0$ where we have used

(2.6). Also by (2.6),

$$\lim_{\tau\to\infty}|\widehat{u''}(\tau)|=\lim_{\tau\to\infty}\left|\frac{i\hat{a}(\tau)+d\tau^{-1}}{\hat{a}(\tau)/\tau-id/\tau^2+i\lambda^{-1}}\right|=0.$$

Theorem F can now be applied with $b(t)=a(t)+d$, $\beta\equiv 0$ and $\phi(w,z)=z\left(\frac{-w}{w+z\lambda^{-1}}\right)$. The conclusion is $\int_0^\infty|u''(t,\lambda)|dt<\infty$ for each λ. Thus the inversion formula

$$2\pi u''(t,\lambda)=\lim_{R\to\infty}\int_{-R}^{R}e^{i\tau t}\widehat{u''}(\tau)d\tau=\lim_{R\to\infty}\int_{-R}^{R}e^{i\tau t}\left(\frac{-i\tau D(\tau)}{D(\tau,\lambda)}\right)d\tau,\ t>0$$

holds (see p. 12,13[5]). Calling the integrand $I(\tau)$, we note that $I(-\tau)=\overline{I(\tau)}$ (where the bar denotes complex conjugate). Therefore,

$$\pi u''(t,\lambda)=\lim_{R\to\infty}\mathrm{Re}\int_0^R I(\tau)d\tau=\int_0^\infty \mathrm{Re}\,I(\tau)d\tau. \tag{4.1}$$

Throughout the remainder of the paper, M will denote a constant whose value may change each time it appears.

To prove Theorem 1 (ii) we integrate (4.1) by parts. This gives the correct integral term. We will see that the boundary term, $\frac{1}{t}\,\frac{\tau D(\tau)}{D(\tau,\lambda)}$, vanishes at $\tau=0$ and at $\tau=\infty$. We will complete the proof by showing that the integral term has an absolutely convergent integrand.

First we show the boundary term vanishes. If $d>0$ we follow Carr and Hannsgen [3] and use

(4.2) $$|D(\tau,\lambda)| \geqslant \max\left(\phi(\tau), \frac{d-\tau^2}{\tau}\right) \geqslant \frac{1}{M\tau}, \quad 0<\tau\leqslant\rho, d>0.$$

By (4.2) and (2.6),

$$\left|\frac{\tau D(\tau)}{D(\tau,\lambda)}\right| = \left|\frac{\tau\hat{a}(\tau)-id}{D(\tau,\lambda)}\right| \leqslant \left[\frac{4\tau A(\frac{1}{\tau})}{\frac{1}{M\tau}} + \frac{d}{\frac{1}{M\tau}}\right] \leqslant M\left[\tau^2 A\left(\frac{1}{\tau}\right)+\tau\right] \to 0 \text{ as } \tau\to 0.$$

For $d=0$, $|D(\tau,\lambda)| \geqslant |\hat{a}(\tau)|-\tau \geqslant 2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau$. Thus,

(4.3) $$|D(\tau,\lambda)| \geqslant \max\left(2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau, \phi(\tau)\right) \geqslant M \text{ for } d=0, \ 0\leqslant\tau<\rho.$$

With (2.6) we have $\frac{1}{t}\left|\frac{\tau\hat{a}(\tau)}{D(\tau,\lambda)}\right| \leqslant \frac{M}{t}\dfrac{\tau A\left(\frac{1}{\tau}\right)}{2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau} \to 0$ as $\tau\to 0$ $(d=0)$.

Also $\frac{1}{t}\left|\frac{\tau D(\tau)}{D(\tau,\lambda)}\right| = \frac{1}{t}\left|\frac{(\hat{a}(\tau)-id\tau^{-1})\tau}{\hat{a}(\tau)-id\tau^{-1}+i\tau\lambda^{-1}}\right| = \frac{1}{t}\left|\frac{\hat{a}(\tau)-id\tau^{-1}}{\tau^{-1}\hat{a}(\tau)-id\tau^{-2}+i\lambda^{-1}}\right| \to 0$

as $\tau\to\infty$ by (2.6) (for all d). To see that the integrand is absolutely integrable near the origin, when $d>0$, use (4.2), (2.7) and (2.6) to obtain

$$\int_0^\rho \left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2 d\tau = \int_0^\rho \left|1-\frac{i\tau}{\lambda D(\tau,\lambda)}\right|^2 d\tau$$

$$\leqslant \int_0^\rho 2+\frac{2\tau^2}{\lambda^2|D(\tau,\lambda)|^2}d\tau \leqslant 2\rho+\frac{M}{\lambda^2}\int_0^\rho \frac{\tau^2}{\frac{1}{\tau^2}}d\tau \leqslant M.$$

Also, $\int_0^\rho \left|\frac{\tau^2 D'(\tau)}{D(\tau,\lambda)^2}\right| d\tau \leqslant M\int_0^\rho \frac{\tau^2\left(\frac{1}{\tau^2}\right)}{\frac{1}{\tau^2}}d\tau = M\frac{\rho^3}{3}$.

When $d=0$, use (4.3) to obtain

$$\int_0^\rho \left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2 d\tau \leqslant \int_0^\rho 2+\frac{2\tau^2}{\lambda^2|D(\tau,\lambda)|^2}d\tau \leqslant 2\rho+\frac{M}{\lambda^2}\int_0^\rho \tau^2 d\tau \leqslant M.$$

Also,

$$\int_0^\rho\left|\frac{\tau^2D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau\leqslant M\int_0^\rho\frac{\tau A\left(\frac{1}{\tau}\right)d\tau}{\left(\max\{2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau,\phi(\tau)\}\right)^2}\leqslant M.$$

We have used that $\lim\limits_{\tau\to\infty}\tau A\left(\frac{1}{\tau}\right)=0$ and $\lim\limits_{\tau\to 0}\left(2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau\right)^2=\frac{1}{8}\left(\int_0^\infty a(s)ds\right)^2>0$.

Next we will see that the integrand is absolutely convergent at ∞.

Use (3.7), (2.19), (3.11), (2.6), (2.7) and the Fubini theorem to obtain

$$\int_{2C_1\sigma}^\infty\left|\frac{\tau^2D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau$$

$$\leqslant\lambda^2M\int_{2C_1\sigma}^\infty\left(\frac{d}{\tau^2}+\int_0^{1/\tau}sa(s)ds\right)d\tau$$

$$\leqslant\lambda^2M\left(\frac{1}{\sigma}+\int_{2C_1\sigma}^\infty\int_0^{1/\tau}sa(s)dsd\tau\right)$$

$$\leqslant M\lambda^2\left(\frac{\sigma}{\lambda}+\int_0^{1/2C_1\sigma}sa(s)\int_0^{1/s}d\tau ds\right)$$

$$=M\lambda^2\left(\frac{\sigma}{\lambda}+\int_0^{1/2C_1\sigma}a(s)ds\right)\leqslant M\lambda\sigma.$$

Also by (3.7), (3.11) and (2.6),

$$\int_{2C_1\sigma}^\infty\left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2d\tau\leqslant\lambda^2M\int_{2C_1\sigma}^\infty\frac{A\left(\frac{1}{\tau}\right)^2d\tau}{\tau^2}$$

$$\leqslant M\lambda^2\int_{2C_1\sigma}^\infty\frac{1}{\tau^2}d\tau\left(\int_0^{1/2C_1\sigma}a(s)ds\right)^2$$

$$\leqslant M\frac{\lambda^2}{\sigma}\left(\int_0^{1/\sigma}a(s)ds\right)^2=M\sigma.$$

This proves the theorem.

We note the argument contains the following estimate:

$$(4.4)\qquad \left(\int_0^\rho+\int_{2C_1\sigma}^\infty\right)\left|\frac{i\tau^2D'(\tau)+\lambda D^2(\tau)}{\lambda^2D(\tau,\lambda)^2}\right|d\tau\leqslant M\frac{\sigma}{\lambda}.$$

This will be used in the proof of Theorem 4.

To prove Theorem 2 use (2.16), (3.10), (3.1) and $\int_0^\infty|f(t)|dt\geqslant|\hat{f}(\tau)|$ for $\tau>0$ when $f\in L^1(0,\infty)$ to obtain

$$\infty>\int_0^\infty\sup_{\mu\geqslant 1}\left|\frac{u''(t,\mu)}{\mu}\right|dt\geqslant\frac{1}{\lambda}\int_0^\infty|u''(t,\lambda)|dt\geqslant\frac{1}{5}A_1\left(\frac{1}{\omega}\right)\left|\widehat{u''}(\omega)\right|$$

$$\geqslant\frac{\theta(\omega)}{5C_1}\left|i\omega\left(\frac{\hat{a}(\omega)-id\omega^{-1}}{\hat{a}(\omega)-id\omega^{-1}+\omega i\lambda^{-1}}\right)\right|=\frac{\theta(\omega)}{5C_1}\left|\omega\left(1-\frac{i\omega}{\lambda\phi(\omega)}\right)\right|$$

$$\geqslant\frac{\theta(\omega)\omega^2}{5C_1\lambda\phi(\omega)}\geqslant\left(\frac{1}{5C_1}\right)^2\ \frac{(\omega\theta(\omega))^2}{\phi(\omega)}\quad(1\leqslant\lambda<\infty).$$

By the properties of ω this proves Theorem 2. Theorem 3 follows directly from the next lemma.

<u>Lemma 3.</u> <u>Under the assumptions of Theorem 1</u>

$$(4.5)\qquad \left|\frac{u''(t,\lambda)}{\lambda}\right|\leqslant\frac{M}{t}\left[\frac{\sigma}{\lambda}+\frac{[\omega^*\theta(\omega^*)]^2}{\phi(\omega^*)}\right]\equiv\frac{MC(\lambda)}{t}\quad(\lambda\geqslant 1,t>0).$$

We begin the proof of (4.5) by using (3.4), (2.7), (3.5) the monotonicity of $\dfrac{\tau}{A\left(\frac{1}{\tau}\right)}$, (3.11) and (2.19) to obtain

$$\frac{1}{\lambda^2}\left[\int_\rho^{\omega/2}\left|\frac{\tau^2D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau+\int_{2\omega}^{2C_1\sigma}\left|\frac{\tau^2D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau\right]$$

$$\leqslant\frac{M}{\lambda^2}\int_\rho^{2C_1\sigma}\ \frac{\tau^2[\frac{d}{\tau^2}+\int_0^{1/\tau}sa(s)ds]}{[\int_0^{1/\tau}a(s)ds]^2}\ d\tau$$

$$\leqslant\frac{M}{\lambda^2}\int_\rho^{2C_1\sigma}\left[\frac{1}{A\left(\frac{1}{\tau}\right)^2}+\frac{\tau}{A\left(\frac{1}{\tau}\right)}\right]d\tau$$

$$\leqslant\frac{M}{\lambda^2}\left[\frac{\sigma}{A\left(\frac{1}{2C_1\sigma}\right)^2}+\frac{\sigma^2}{A\left(\frac{1}{2C_1\sigma}\right)}\right]$$

$$\leqslant\frac{M}{\lambda^2}\left[\frac{\sigma}{A\left(\frac{1}{\sigma}\right)^2}+\frac{\sigma^2}{A\left(\frac{1}{\sigma}\right)}\right]$$

$$=M\left[\frac{1}{\sigma}+\frac{\sigma}{\lambda}\right]\leqslant M\frac{\sigma}{\lambda}.$$

The last inequality is due to the fact $\lim\limits_{\sigma\to\infty}\frac{\sigma^2}{\lambda}=a(0+)>0$; hence, $0<\epsilon\leqslant\frac{\sigma^2}{\lambda}$ for some ϵ independent of λ, and $\frac{1}{\sigma}\leqslant\frac{1\sigma}{\epsilon\lambda}$.

This establishes

$$\frac{1}{\lambda^2}\left(\int_\rho^{\omega/2}+\int_{2\omega}^{2C_1\sigma}\left|\frac{\tau^2D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau\leqslant M\frac{\sigma}{\lambda}.\right. \tag{4.6}$$

Use (3.4), (3.5), (3.11) and the monotonicity of $\frac{x}{A\left(\frac{1}{x}\right)}$ to obtain

$$\frac{1}{\lambda}\int_\rho^{\omega/2}\left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2d\tau\leqslant\frac{1}{\lambda}\int_\rho^{\omega/2}2+\frac{2\tau^2}{\lambda^2|D(\tau,\lambda)|^2}d\tau$$

$$\leqslant\frac{\omega}{\lambda}+\frac{M}{\lambda^3}\int_\rho^{\omega/2}\frac{\tau^2}{A\left(\frac{1}{\tau}\right)^2}d\tau$$

$$\leqslant M\left(\frac{\sigma}{\lambda}+\frac{\omega}{\lambda^3}\left[\frac{\sigma}{A\left(\frac{1}{2C_1\sigma}\right)}\right]^2\right)$$

$$\leqslant M\left(\frac{\sigma}{\lambda}+\frac{\sigma}{\lambda^3}\left[\frac{\sigma}{A\left(\frac{1}{\sigma}\right)}\right]^2\right)=M\left(\frac{\sigma}{\lambda}+\frac{\sigma}{\lambda}\right)\leqslant M\frac{\sigma}{\lambda}.$$

Also, by (2.6), (3.4), (3.9), (3.11) it follows that

$$\frac{1}{\lambda}\int_{2\omega}^{2C_1\sigma}\left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2 d\tau\leqslant\frac{M}{\lambda}\int_{2\omega}^{2C_1\sigma}\frac{\left[A\left(\frac{1}{\tau}\right)+\frac{d}{\tau^2}\right]}{A\left(\frac{1}{\tau}\right)^2}d\tau$$

$$=\frac{M}{\lambda}\int_{2\omega}^{2C_1\sigma}\left[1+\frac{d}{\tau^2 A\left(\frac{1}{\tau}\right)}\right]^2 d\tau$$

$$\leqslant\frac{M}{\lambda}\int_{2\omega}^{2C_1\sigma}\left(1+\frac{d}{4\omega^2 A\left(\frac{1}{2C_1\sigma}\right)}\right)^2 d\tau$$

$$\leqslant M\frac{\sigma}{\lambda}\left(1+\frac{1}{\omega^2 A\left(\frac{1}{\sigma}\right)}\right)^2\leqslant M\frac{\sigma}{\lambda}\left(1+\frac{1}{\lambda A\left(\frac{1}{\sigma}\right)}\right)^2$$

$$=M\frac{\sigma}{\lambda}\left(1+\frac{1}{\sigma}\right)^2\leqslant M\frac{\sigma}{\lambda}.$$

The last two strings of inequalities give us

$$\frac{1}{\lambda}\left(\int_{\rho}^{\omega/2}+\int_{2\omega}^{2C_1\sigma}\right)\left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2 d\tau\leqslant M\frac{\sigma}{\lambda}. \tag{4.7}$$

Next we use (3.3), (2.7), (3.8), (3.11), (3.6) and (3.5) to obtain

$$\frac{1}{\lambda^2}\int_{\omega/2}^{2\omega}\left|\frac{\tau^2 D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau$$

$$\leqslant\frac{M}{\lambda^2}\int_{\omega/2}^{2\omega}\frac{\tau A\left(\frac{1}{\tau}\right)}{\phi^2(\tau)+\left(\frac{\tau-\omega}{\lambda}\right)^2}d\tau$$

$$\leqslant M\int_{\omega/2}^{2\omega}\frac{\tau}{(\phi(\omega^*)\lambda)^2+|\tau-\omega|^2}\,d\tau A\left(\frac{2}{\omega}\right)$$

$$\leqslant M\omega^*\int_{\omega/2}^{2w}\frac{1}{(\phi(\omega^*)\lambda)^2+(\tau-\omega)^2}d\tau(\phi(\omega^*)+\omega^*\theta(\omega^*))$$

$$\leqslant M\frac{\omega^*(\phi(\omega^*)+\omega^*\theta(\omega^*))}{\lambda\phi(\omega^*)} = M\left(\frac{\omega^*}{\lambda}+\omega\frac{^{*2}\theta(\omega^*)}{\lambda\phi(\omega^*)}\right)$$

$$\leqslant M\left(\frac{\sigma}{\lambda}+\frac{[\omega^*\phi(\omega^*)]^2}{\phi(\omega^*)}\right).$$

Thus

$$\frac{1}{\lambda^2}\int_{\omega/2}^{2\omega}\left|\frac{\tau^2 D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau\leqslant M\left(\frac{\sigma}{\lambda}+\frac{[\omega^*\theta(\omega^*)]^2}{\phi(\omega^*)}\right). \tag{4.8}$$

Also, by (3.3), (3.6), (3.5) we have

$$\frac{1}{\lambda}\int_{\omega/2}^{2\omega}\left|\frac{D(\tau)}{D(\tau,\lambda)}\right|^2 d\tau$$

$$=\frac{1}{\lambda}\int_{\omega/2}^{2\omega}\left|1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right|^2 d\tau$$

$$\leqslant\frac{2}{\lambda}\int_{\omega/2}^{2\omega}1+\frac{\tau^2\lambda^{-2}}{|D(\tau,\lambda)|^2}d\tau$$

$$\leqslant\frac{M}{\lambda}\int_{\omega/2}^{2\omega}1+\frac{\tau^2\lambda^{-2}}{\theta^2(\tau)+\left|\frac{\tau-\omega}{\lambda}\right|^2}\,d\tau$$

$$\leqslant\frac{M}{\lambda}\int_{\omega/2}^{2\omega}1+\frac{\tau^2}{(\lambda\phi(\omega^*))^2+|\tau-\omega|^2}d\tau$$

$$\leqslant\left(\frac{\omega}{\lambda}+\frac{\omega^2}{\phi(\omega^*)\lambda^2}\right)\leqslant M\left(\frac{\sigma}{\lambda}+\frac{[\omega^*\theta(\omega^*)]^2}{\phi(\omega^*)}\right).$$

Thus (4.8), (4.7), (4.6), (4.4) and Theorem 2, prove the lemma and hence the theorem.

Our proof of Lemma 1 depends on Theorem 2.2 in [3]. Namely, under the assumptions of Lemma 1 there exists a constant K so that

$$\frac{1}{K}\sigma\leqslant\sup_{t>0}|u'(t,\lambda)|\leqslant K\sigma,\ \lambda\geqslant 1. \tag{4.9}$$

The proof of Theorem 2.2 in [3] also contains the inequality

(4.10) $\quad u(t,\lambda)\geqslant 1/2$ for $0\leqslant t\leqslant \frac{1}{2\sigma(8+dC_2)}\equiv 2T.$

If $t\leqslant\frac{1}{\sigma}$, using (4.9) and (2.19) we get

$$\left|\int_0^t u'(t-\tau)a(\tau)d\tau\right|\leqslant K\sigma A(t)\leqslant K\sigma A\left(\frac{1}{\sigma}\right)=K\frac{\sigma^2}{\lambda}.$$

If $\frac{1}{\sigma}<t$, by (4.9) (2.19) and (2.3),

$$\left|\int_0^t u'(t-\tau)a(\tau)d\tau\right|\leqslant\left|\int_0^{1/\sigma}u'(t-\tau)a(\tau)d\tau\right|+\left|\int_{1/\sigma}^t u'(t-\tau)a(\tau)d\tau\right|\leqslant K\frac{\sigma^2}{\lambda}$$

$$+\left|\int_{1/\sigma}^t u'(t-\tau)a(\tau)d\tau\right|$$

$$=K\frac{\sigma^2}{\lambda}+\left|-a(t)+a\left(\frac{1}{\sigma}\right)u\left(t-\frac{1}{\sigma}\right)+\int_{1/\sigma}^t a'(\tau)u(t-\tau)d\tau\right|$$

$$\leqslant K\frac{\sigma^2}{\lambda}+a(t)+a\left(\frac{1}{\sigma}\right)+\left|a(t)-a\left(\frac{1}{\sigma}\right)\right|$$

$$=K\frac{\sigma^2}{\lambda}+2a\left(\frac{1}{\sigma}\right)\leqslant K\frac{\sigma^2}{\lambda}+2\sigma A\left(\frac{1}{\sigma}\right)=(K+2)\frac{\sigma^2}{\lambda}.$$

The second inequality in (3.3) is proved ($N_2=K+2$).

By (1.1), (4.10), and (3.11), for $T\leqslant t\leqslant 2T$ we have

$$\left|\int_0^t u'(t-\tau)a(\tau)d\tau\right|=\left|\int_0^t u'(\tau)a(t-\tau)d\tau\right|$$

$$=\lambda\left|\int_0^t\int_0^\tau u(\tau-s)a(s)ds\,a(t-\tau)d\tau\right|$$

$$\geqslant\frac{1}{2}\lambda\left|\int_0^t a(t-\tau)\int_0^\tau a(s)ds\,d\tau\right|$$

$$\geqslant\frac{\lambda}{2}\int_{t/2}^t a(t-\tau)\int_0^t a(s)ds\,d\tau$$

$$\geqslant\frac{\lambda}{2}\int_0^{t/2}a(s)ds\int_{t/2}^t a(t-\tau)d\tau$$

$$=\frac{\lambda}{2}\left(\int_0^{t/2}a(s)ds\right)^2 \geqslant \frac{\lambda}{2}\left(\int_0^{T/2}a(s)ds\right)^2$$

$$=\frac{\lambda}{2}\left(\int_0^{1/8(8+dC_2\sigma)} a(s)ds\right)^2$$

$$\geqslant M\lambda\left(\int_0^{1/\sigma}a(s)ds\right)^2=M\frac{\sigma^2}{\lambda}.$$

This is the first inequality of (3.3), so Lemma 1 is proved.

We will now prove Theorem 4. From the proof of Lemma 1,

$$(4.11) \qquad \left|\int_0^t u'(t-\tau)a(\tau)d\tau\right| \geqslant N_2\frac{\sigma^2}{\lambda} \text{ for } T\leqslant t\leqslant 2T.$$

Using $a\in L^1(0,1)$, $|u(t,\lambda)|\leqslant 1$ for $t\geqslant 0$, $\lambda\geqslant 1$ and (3.13) we see

$\int_0^1 \sup_{\lambda\geqslant 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$ if and only if $\int_0^1 \sup_{\lambda\geqslant 1}\left|\int_0^t a(\tau)u'(t-\tau)d\tau\right|dt<\infty$.

By (4.11) we have for $T\leqslant t\leqslant 2T$ that

$$\left|\int_0^t a(\tau)u'(t-\tau)d\tau\right| \geqslant N_2\frac{\sigma^2}{\lambda} = N_2\sigma\int_0^{1/\sigma}a(s)ds$$

$$\geqslant N_2 \frac{1}{4(8+dC_2)t}\int_0^{2t(8+dC_2)}a(s)ds \geqslant \frac{M}{t}\int_0^t a(s)ds.$$

By definition of T, $T\to 0$ as $\lambda\to\infty$ (as $\sigma\to\infty$ as $\lambda\to\infty$). Therefore for each t in $(0,\epsilon)$ for some $\epsilon>0$, there exists T with $T\leqslant t\leqslant 2T$.

Thus $\int_0^1 \sup_{\lambda\geqslant 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$ implies $\int_0^\epsilon \frac{1}{t}\int_0^t a(s)dsdt<\infty$

(hence $\int_0^1 \frac{1}{t}\int_0^t a(s)dsdt<\infty$). But $\int_0^1 \frac{1}{t}\int_0^t a(s)dsdt=\int_0^1 -\ln s\, a(s)ds$ finishing the proof.

We will now prove Theorem 5. Partition $S=\{(t,\lambda):t\geqslant 0,\lambda\geqslant 1\}$ into $S=S_1\cup S_2$ where $S_1\equiv S\cap\{(t,\lambda):\frac{\sigma^2}{\lambda}\leqslant a(t)\}$, $S_2\equiv S\cap\{(t,\lambda):\frac{\sigma^2}{\lambda}>a(t)\}$. On S_1,

$\left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant a(t)+d+N_2\frac{\sigma^2}{\lambda} \leqslant (1+N_2)a(t)+d \in L^1(0,1)$ by (3.13), Lemma 1 and (2.3).

On S_2, $\left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant a(t)+d+N_2\frac{\sigma^2}{\lambda} \leqslant (1+N_2)\frac{\sigma^2}{\lambda}+d \leqslant (1+N_2+C_2d)\frac{\sigma^2}{\lambda}=M\frac{\sigma^2}{\lambda}$

again by (3.13), Lemma 1 and (2.3) and also by (3.5).

Now partition S_2 into $S_2=S_3 \cup S_4$ where

$$S_3 \equiv \{(\tau,\lambda):\left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant \frac{1}{t^{1/2}}\} \cap S_2,\ S_4 \equiv \{(t,\lambda):\frac{1}{t^{1/2}} < \left|\frac{u''(t,\lambda)}{\lambda}\right|\} \cap S_2.$$

On S_3, $\left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant \frac{1}{t^{1/2}} \in L^1(0,1)$.

On S_4, $\frac{1}{t^{1/2}} < \left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant \frac{MC(\lambda)}{t}$ by Lemma 3. That is

$$(4.12) \qquad \frac{t}{C(\lambda)} \leqslant Mt^{1/2},\ \left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant M\frac{\sigma^2}{\lambda} \text{ on } S_4.$$

Now define $h(x)=x\int_0^{1/x}a(s)ds$, $g(x)=\frac{1}{\int_0^{1/x}a(s)ds}$. Clearly $g(x)$ is nondecreasing. To see that $h(x)$ is nondecreasing observe that , $h'(x)=\int_0^{1/x}a(s)ds-\frac{1}{x}a(\frac{1}{x}) \geqslant 0$. Thus on S_4,

$$\left|\frac{u''(t,\lambda)}{\lambda}\right|=h\left(\left|\frac{u''(t,\lambda)}{\lambda}\right|\right)g\left(\left|\frac{u''(t,\lambda)}{\lambda}\right|\right)$$

$$\leqslant h\left(\frac{C(\lambda)M}{t}\right)g\left(M\frac{\sigma^2}{\lambda}\right) \leqslant \frac{MC(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}\ \frac{1}{t}\int_0^{t^{1/2}} a(s)ds$$

$$\leqslant \frac{M}{t}\int_0^{t^{1/2}} a(s)ds \in L^1(0,1)$$

where the first inequality follows from Lemma 3, (4.12) and the monotonicity of f and g, the second inequality is a consequence of

(3.11) and (4.12), the last inequality is by (3.16) and the calculation

$$\int_0^1 \frac{1}{t}\int_0^{t^{1/2}} a(s)dsdt=\int_0^1 -2\ln s a(s)ds<\infty \text{ (by (3.15)) shows that}$$

$\frac{1}{t}\int_0^{t^{1/2}} a(s)ds\in L^1(0,1)$. Considering the estimates on S_1, S_3, and S_4 we see that

$$\sup_{\lambda\geqslant 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|\leqslant\max((1+N_2)a(t)+d,t^{-1/2},\frac{M}{t}\int_0^{t^{1/2}} a(s)ds)\in L^1(0,1).$$

The theorem is proved.

Except for minor details, the proof of Theorem 6 is the same as the corresponding proofs in [2] and [3].

At this point we introduce auxiliary functions and inequalities used in [2] and [3]. Let

$$J(u)=iu(1-e^{iu})-2(1-iu-e^{iu}); \text{ then } \hat{b}'(\tau)=\tau^{-3}\int_0^\infty J(-\tau s)db(s),\ \tau>0.$$

For τ, $t>0$ define

$$\beta^0(t,\tau)=\tau^{-3}\int_0^t J(-\tau s)db'(s); \tag{4.13}$$

$$\beta^\infty(t,\lambda)=\tau^{-3}\int_t^\infty J(-\tau s)db'(s), \tag{4.14}$$

$$\Delta(t,\tau)=\beta^0(t,\tau)+\hat{c}'(\tau)+id\tau^{-2}=D'(\tau)-\beta^\infty(t,\tau). \tag{4.15}$$

The following facts are proved in [2] and [3]. In particular see Lemma 5.1 of [2] and (5.42) and (5.44) of [3].

$$\hat{c}\in C^2(0,\infty),\frac{\partial\beta^0}{\partial\tau}\in C((0,\infty)\times(0,\infty))$$

$$|\hat{c}''(\tau)|\leqslant 6000\int_0^{1/\tau} s^2c(s)ds,\tau>0 \tag{4.16}$$

$$|\beta^\infty(t,\tau)|\leqslant 40\tau^{-2}(b(t)-tb'(t)),\ t>0,\tau>0 \tag{4.17}$$

(4.18) $\left|\frac{\partial\beta^0}{\partial\tau}(t,\tau)\right| \leqslant 500\tau^{-2}\int_0^t b(s)ds,\ t>0,\ \tau>0$

(4.19) $|\beta^0(t,\tau)| \leqslant 40\int_0^{1/\tau} sb(s)ds,\ t>0,\ \tau>0$

(4.20) $|\hat{c}'(\tau)| \leqslant 40\int_0^{1/\tau} sc(s)ds,\ \tau>0.$

With $q(t)=t^{-2}+t^{-2}\int_0^t b(s)ds+t^{-1}b(t)-b'(t)$, which is in $L^1(1,\infty)$ by [2], we also have

(4.21) (i) $|\Delta(t,\tau)|+|\tau\Delta_\tau(t,\tau)|+|D'(\tau)| \leqslant M[A_1\left(\frac{1}{\tau}\right)+t^2q(t)\tau^{-1}],\ t\geqslant 1,$

$\tau \geqslant \frac{\rho}{2}$

(ii) $|D_\tau(\tau,\lambda)| \leqslant MA_1\left(\frac{1}{t}\right) \leqslant M\frac{1}{\tau}A\left(\frac{1}{\tau}\right),\ \tau\geqslant\frac{\rho}{2}$

(iii) $\tau^2|\beta^\infty(t,\tau)| \leqslant Mtq(t),\ t\geqslant 1,\ \tau>0$

(iv) $\tau|\Delta(t,\tau)|+|D(\tau)| \leqslant MA\left(\frac{1}{t}\right),\ t\geqslant 1,\ \tau\geqslant\frac{\rho}{2}$

(v) $|\Delta(t,\tau)|+|\tau\Delta_\tau(t,\tau)|+|D'(\tau)| \leqslant M[\lambda^{-1}+\tau^{-1}t^2q(t)],\ t\geqslant 1,\ \tau\geqslant\omega/2$

(vi) $|D_\tau(\tau,\lambda)| \leqslant \frac{M}{\lambda},\ \tau\leqslant\ \omega/2$

(4.22) $\frac{\tau}{\lambda} \leqslant M\ |D(\tau,\lambda)|,\ 2\omega\leqslant\tau<\infty.$

To prove Theorem 6 we start with

$$\frac{i\tau^2D'(\tau)}{D(\tau,\lambda)^2} = i\tau^2\left[\frac{D'(\tau)+i\lambda^{-1}}{D(\tau,\lambda)^2}-\frac{i\lambda^{-1}}{D(\tau,\lambda)^2}\right]$$

$$= \frac{i\tau^2(D'(\tau)+i\lambda^{-1})}{D(\tau)^2}\left[\frac{D(\tau)}{D(\tau,\lambda)}\right]^2+\frac{\tau^2\lambda^{-1}}{D(\tau,\lambda)^2}\equiv I_1+I_2.$$

$$I_1=\frac{i\tau^2(D'(\tau)+i\lambda^{-1})}{D(\tau)^2}\left[1-\frac{2i\tau\lambda^{-1}}{D(\tau,\lambda)}-\frac{\tau^2\lambda^{-2}}{D(\tau,\lambda)^2}\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D(\tau)^2}+\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^2D(\tau,\lambda)}\right.$$

$$\left.+\frac{2\tau\lambda^{-2}}{D(\tau)^2D(\tau,\lambda)}-\frac{(D'(\tau)+i\lambda^{-1})\tau^2\lambda^{-2}}{D(\tau)^2D(\tau,\lambda)^2}\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D(\tau)^2}+\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^3}\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)\right.$$

$$\left.+\frac{2\tau\lambda^{-2}}{D(\tau)^3}\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)-\frac{(D'(\tau)+i\lambda^{-1})\tau^2\lambda^{-2}}{D(\tau)^3D(\tau,\lambda)}\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D(\tau)^2}+\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^3}+\frac{2\tau\lambda^{-2}}{D(\tau)^3}\right.$$

$$\left.-\frac{2\tau^2\lambda^{-2}D'(\tau)}{D(\tau)^3D(\tau,\lambda)}-\frac{2i\tau^2\lambda^{-3}}{D(\tau)^3D(\tau,\lambda)}-\frac{(D'(\tau)+i\lambda^{-1})\tau^2\lambda^{-2}}{D(\tau)^3D(\tau,\lambda)}\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D(\tau)^2}+\left(\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^3}\right)+\frac{2\tau\lambda^{-2}}{D(\tau)^3}\right.$$

$$\left.-\frac{(D'(\tau)+i\lambda^{-1}2\tau^2\lambda^{-2}}{D(\tau)^3D(\tau,\lambda)}-\frac{(D'(\tau)+i\lambda^{-1})\tau^2\lambda^{-2}}{D(\tau)^3D(\tau,\lambda)}\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D(\tau)^2}+\left(\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^3}\right)+\frac{2\tau\lambda^{-2}}{D(\tau)^3}\right.$$

$$\left.-\frac{(D'(\tau)+i\lambda^{-1})\tau^2}{\lambda^2D^2(\tau)D(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{\left(1-\frac{i\tau\lambda^{-1}}{D(\tau,\lambda)}\right)}{D(\tau)}\right)\right]$$

$$= i\tau^2\left[\frac{D'(\tau)}{D^2(\tau)}+\left(\frac{i\lambda^{-1}}{D(\tau)^2}-\frac{2i\tau\lambda^{-1}D'(\tau)}{D(\tau)^3}\right)+\frac{2\tau\lambda^{-2}}{D(\tau)^3}\right.$$

$$-\frac{(D'(\tau)+i\lambda^{-1})\tau^2}{\lambda^2D^2(\tau)D(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right).$$

Using this expression for I_1 and referring to I_1+I_2, we may write

$$\frac{i\tau^2 D'(\tau)+\lambda D(\tau)^2}{D(\tau,\lambda)^2}=\frac{\tau^2\lambda^{-1}+\lambda D(\tau)^2}{D(\tau,\lambda)^2}+\frac{i\tau^2 D'(\tau)}{D(\tau)^2}$$

$$-\frac{\tau^2\lambda^{-1}}{D(\tau)^2}\left(1-\frac{2\tau D'(\tau)}{D(\tau)}\right)+\frac{2i\tau^3\lambda^{-2}}{D(\tau)^3}$$

$$-\frac{i\tau^4(D'(\tau)+i\lambda^{-1})}{\lambda^2 D(\tau)^2 D(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right).$$

Putting this into the representation for $\pi u''(t,\lambda)$ from Theorem 1 (ii) yields

(4.23) $$-\pi u''(t,\lambda)=\mathrm{Re}[i\lambda^{-1}q_1(t)+\lambda^{-2}q_2(t)+i\lambda^{-3}q_3(t)+q_4(t,\lambda)+q_5(t,\lambda)]$$

where

$$q_1(t)=\frac{1}{t}\int_0^\rho e^{i\tau t}\left(\frac{\tau^2 D'(\tau)}{D(\tau)^2}\right)d\tau,$$

$$q_2(t)=\frac{1}{t}\int_0^\rho e^{i\tau t}\frac{\tau^2}{D(\tau)^2}\left(1-\frac{2\tau D'(\tau)}{D(\tau)}\right)d\tau,$$

$$q_3(t)=\frac{1}{t}\int_0^\rho e^{i\tau t}\frac{2\tau^3}{D(\tau)^3}d\tau,$$

$$-q_4(t,\lambda)=\frac{1}{t\lambda}\int_0^\rho e^{i\tau t}\Big\{\frac{\tau^2\lambda^{-1}+\lambda D(\tau)^2}{D(\tau,\lambda)^2}$$

$$-\frac{i\tau^4(D'(\tau)+i\lambda^{-1})}{\lambda^2 D(\tau)^2 D(\tau,\lambda)}\left[\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right]\Big\}d\tau$$

and

$$q_5(t,\lambda)=\frac{-1}{t\lambda}\int_\rho^\infty e^{it\tau}\left(\frac{I\tau^2 D'(\tau)+\lambda D(\tau)^2}{D(\tau,\lambda)^2}\right)d\tau.$$

Let $-q_4=q_{41}+q_{42}$ where

$$q_{41}(t,\lambda)=\frac{1}{t\lambda}\int_0^\rho e^{i\tau t}[J_1-J_2]d\tau$$

and

$$J_1=J_1(\tau,\lambda)=\frac{\tau^2\lambda^{-1}+\lambda D(\tau)^2}{D(\tau,\lambda)^2},$$

$$J_2=J_2(\tau,\lambda)=\frac{i\tau^4(\Delta(t,\rho)+i\lambda^{-1})}{\lambda^2 D(\tau)^2 D(\tau,\lambda)}\left[\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right],$$

$$q_{42}(t,\lambda)=\frac{1}{t\lambda}\int_0^\rho e^{i\tau t}\left[\frac{-i\tau^4\beta^\infty(t,\tau)}{\lambda^2 D(\tau)^2 D(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right)\right]d\tau.$$

Note that the estimates (4.2) and (4.3) hold for $D(\tau)$ in place of $D(\tau,\lambda)$. Now we use (4.17), (4.3) and (4.2) to write

$$\left|\frac{q_{42}(t,\lambda)}{\lambda}\right|\leqslant\frac{M}{t\lambda^2}\int_0^\rho\frac{\tau^4(\tau^{-2})(b(t)-tb'(t))}{\lambda^2}d\tau\leqslant\frac{M}{t\lambda^2}(b(t)-tb'(t))$$

$$\leqslant Mq(t).$$

If we integrate $q_{412}(t,\lambda)\equiv\frac{1}{\lambda t}\int_0^\rho e^{i\tau t}J_2(\tau,\lambda)d\tau$ by parts we find

$$\lambda t^2 q_{412}(t,\lambda)$$

$$= e^{i\rho t}\left[\frac{\rho^4(\Delta(t,\rho)+i\lambda^{-1})}{\lambda^2 D^2(\rho)D(\rho,\lambda)}\left(\frac{2}{D(\rho)}+\frac{1}{D(\rho,\lambda)}\right)\right]$$

$$-\frac{1}{\lambda^2}\int_0^\rho e^{i\tau t}\left[\frac{4\tau^3(\Delta(t,\tau)+i\lambda^{-1})+\tau^4\Delta_\tau(t,\tau)}{D^2(\tau)D(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right)\right.$$

$$-\frac{\tau^4(\Delta(t,\tau)+i\lambda^{-1})(3D'(\tau)D(\tau,\lambda)+D(\tau)D_\tau(\tau,\lambda))}{D^3(\tau)D^2(\tau,\lambda)}\left(\frac{2}{D(\tau)}+\frac{1}{D(\tau,\lambda)}\right)$$

$$\left.-\frac{\tau^4(\Delta(t,\tau)+i\lambda^{-1})}{D^3(\tau)D(\tau,\lambda)}\left(\frac{2D'(\tau)}{D^2(\tau)}+\frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right)\right]d\tau$$

where the vanishing of the boundary term at $\tau=0$ is ensured by (4.19), (4.20), (2.6) and (4.15).

The estimates

$$(4.24)\qquad \int_0^\rho \left|\frac{D_\tau(\tau,\lambda)}{D^2(\tau,\lambda)}\right| d\tau \leqslant M\ (d>0);\ \int_0^1 \frac{A_1\left(\frac{1}{\tau}\right)}{A^2\left(\frac{1}{\tau}\right)} d\tau < \infty$$

follow from (2.7), (4.2) and the discussion of the proof of Theorem F respectively. It is straightforward to estimate $q_{412}(t,\lambda)$ by (2.6), (2.7), (4.2), (4.3), (4.16) through (4.21) and (4.23) to obtain

$$(4.25)\qquad \frac{1}{t\lambda}\left|\int_0^\rho e^{i\tau t} J_2(\tau,\lambda)\, d\tau\right| \leqslant Mq(t),\ t \geqslant 1.$$

Notice that

$$\frac{\tau^2\lambda^{-2}+D(\tau)^2}{D(\tau,\lambda)^2} = \frac{D(\tau,\lambda)^2-2i\tau\lambda^{-1}D(\tau)+2\tau^2\lambda^{-2}}{D(\tau,\lambda)^2}$$

$$= 1 + \frac{2\tau^2\lambda^{-2}-2i\tau\lambda^{-1}D(\tau)}{D(\tau,\lambda)^2},\ \text{so that}$$

$$\frac{1}{\lambda}\ \frac{\partial J_1(\tau,\lambda)}{\partial\tau} = \left(\frac{4\tau\lambda^{-2}-2i\lambda D(\tau)-2i\tau\lambda^{-1}D'(\tau)}{D(\tau,\lambda)^2}\right.$$

$$\left. - \frac{2D_\tau(\tau,\lambda)(2\tau^2\lambda^{-2}-2i\tau\lambda^{-1}D(\tau))}{D(\tau,\lambda)^3}\right) \equiv (G_1-G_2).$$

Hence integration by parts yields

$$\frac{it}{\lambda}\int_0^\rho e^{i\tau t} J_1(\tau,\lambda)\, d\tau = \frac{e^{i\tau t}J_1(\tau,\lambda)}{\lambda} + \int_0^\rho (G_2-G_1)e^{i\tau t}\, d\tau.$$

Let us estimate the boundary terms. We have

$$\left|e^{i\rho t}\left(1+\frac{2\rho^2\lambda^{-2}-2i\rho\lambda^{-1}D(\rho)}{D(\rho,\lambda)^2}\right)\right| \leqslant 1+\frac{2\rho^2+2\rho|D(\rho)|}{\gamma^2}$$

where $\gamma \equiv \inf\limits_{0<\tau<\rho,\ 1\leqslant\lambda} |D(\tau,\lambda)|$. $\gamma>0$ by (4.3), (4.2).

For the other boundary term, by (4.2) and (2.3)

$$\lim_{\epsilon\to 0}\left|\frac{2\epsilon i\lambda^{-1}D(\epsilon)+2\epsilon^2\lambda^{-2}}{D(\epsilon,\lambda)^2}\right|\leqslant\lim_{\epsilon\to 0}\left[\frac{M\epsilon(A\left(\frac{1}{\epsilon}\right)+\frac{1}{\epsilon})}{\lambda\frac{1}{\epsilon^2}}+\frac{2\epsilon^2}{\lambda^2\left(\frac{1}{\epsilon}\right)^2}\right]=0 \text{ for } d\neq 0.$$

For d=0 use (4.3) and (2.6) to obtain

$$\lim_{\epsilon\to 0}\frac{M\epsilon}{\lambda}\ \frac{(A\left(\frac{1}{\epsilon}\right)+\frac{1}{\epsilon})}{(A\left(\frac{1}{\epsilon}\right)-\epsilon)^2}+\frac{2\epsilon^2}{\lambda^2(A\left(\frac{1}{\epsilon}\right)-\epsilon)^2}=0.$$

Also,

$$\left|\int_0^\rho G_1 e^{i\tau t}d\tau\right|\leqslant\int_0^\rho\ \left|\frac{4\tau\lambda^{-2}-2i\lambda^{-1}D(\tau)-2i\tau\lambda^{-1}D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau$$

$$\leqslant M\int_0^\rho\left(\frac{1}{\tau}+\tau\right)\tau^2 d\tau\leqslant M$$

when d>0 by (2.6), (2.7) and (4.2). When d=0, by (2.6), (2.7) and (4.3),

$$\left|\int_0^\rho G_1 e^{i\tau t}d\tau\right|\leqslant\int_0^\rho\left|\frac{4\tau\lambda^{-2}-2i\lambda^{-1}D(\tau)-2i\tau\lambda^{-1}D'(\tau)}{D(\tau,\lambda)^2}\right|d\tau$$

$$\leqslant M\int_0^\rho\ \frac{\tau+A\left(\frac{1}{\tau}\right)}{[\max(\phi(\tau),2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau)]^2}d\tau\leqslant M.$$

To estimate the final term in q_{41} when d>0 use (4.2), (2.6) and (4.24);

$$\left|\int_0^\rho G_2 e^{i\tau t}d\tau\right|\leqslant\int_0^\rho\left|\frac{2D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}\right|\left|\frac{2\tau^2\lambda^{-2}-2i\tau\lambda^{-1}D(\tau)}{D(\tau,\lambda)}\right|d\tau$$

$$\leqslant M\int_0^\rho\left|\frac{D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}\right|(2\tau^2+2\tau\left(\frac{1}{\tau}\right))\tau^2 d\tau\leqslant M\int_0^\rho\left|\frac{D_\tau(\tau,\lambda)}{D(\tau,\lambda)^2}\right|d\tau\leqslant M.$$

When d=0, use the same first inequality as above and then observe that by (4.3) and (2.6)

$$\left|\frac{2\tau^2\lambda^{-2}-2i\tau\lambda^{-1}D(\tau)}{D(\tau,\lambda)}\right|\leqslant M\frac{2\tau^2+2\tau A\left(\frac{1}{\tau}\right)}{\max(2^{-3/2}A\left(\frac{1}{\tau}\right)-\tau,\phi(\tau))}\leqslant M.$$

Thus, by our estimates for q_{41} and q_{42},

$$(4.26)\qquad \left|\frac{q_4(t,\lambda)}{\lambda}\right|\leqslant Mq(t).$$

Next we write $-q_5=q_{51}+q_{52}$ where

$$\lambda tq_{51}(t,\lambda)=\int_\rho^\infty e^{i\tau t}\left(\frac{i\tau^2\Delta(t,\tau)+\lambda D(\tau)^2}{D(\tau,\lambda)^2}\right)d\tau,$$

$$\lambda tq_{52}(t,\lambda)=\int_\rho^\infty e^{i\tau t}\left(\frac{i\tau^2\beta^\infty(t,\tau)}{D(\tau,\lambda)^2}\right)d\tau.$$

Then $\left|\lambda tq_{52}(t,\lambda)\right|\leqslant\left[\int_\rho^{\omega/2}+\int_{\omega/2}^{2\omega}+\int_{2\omega}^\infty\right]\left|\frac{\tau^2\beta^\infty(t,\lambda)}{D(\tau,\lambda)^2}\right|d\tau.$

For the third term, we have

$$\int_{2\omega}^\infty\left|\frac{\tau^2\beta^\infty(t,\lambda)}{D(\tau,\lambda)^2}\right|d\tau\leqslant Mtq(t)\int_{2\omega}^\infty\frac{1}{\left(\frac{\tau}{\lambda}\right)^2}d\tau\leqslant\frac{Mtq(t)\lambda^2}{2\omega}\leqslant Mtq(t)\lambda^2,$$

by (4.22) and (4.21). By (3.4), (4.21), (3.11), (2.19) and (3.5) we have

$$\int_\rho^{\omega/2}\left|\frac{\tau^2\beta^\infty(t,\tau)}{D(\tau,\lambda)^2}\right|d\tau\leqslant Mtq(t)\int_\rho^{\omega/2}\frac{1}{A\left(\frac{1}{\tau}\right)^2}d\tau\leqslant\frac{Mtq(t)\omega}{A\left(\frac{2}{\omega}\right)^2}$$

$$\leqslant q\frac{Mtq(t)\omega}{\left(\frac{2}{C_1}\right)^2\left(\frac{\sigma}{\lambda}\right)^2}\leqslant Mtq(t)\lambda^2$$

where the next to the last inequality uses

$$(4.27)\qquad A\left(\frac{1}{\tau}\right)\geqslant A\left(\frac{2}{C_1\sigma}\right)\geqslant\frac{2}{C_1}A\left(\frac{1}{\sigma}\right)=\frac{2\sigma}{C_1\lambda}\ \text{ for }\ \tau\leqslant 2C_1\sigma$$

and the last inequality follows from (3.5). (Note that (4.27) is a consequence of (3.11).)

The next calculation uses (4.21), (3.3), (3.12), (3.9) and (3.6);

$$\int_{\omega/2}^{2\omega}\left|\frac{\tau^2\beta^{\infty}(t,\tau)}{D(\tau,\lambda)^2}\right|d\tau \leqslant Mtq(t)\lambda^2\int_{\omega/2}^{2\omega}\frac{d\tau}{(\lambda\phi(\omega^*))^2+|t-\omega|^2}$$

$$\leqslant\frac{Mtq(t)\lambda^2}{\lambda\phi(\omega^*)}=Mtq(t)\lambda^2\left(\frac{\theta(\omega^*)}{\lambda\theta(\omega^*)\phi(\omega^*)}\right)$$

$$\leqslant Mtq(t)\lambda^2\ \frac{\omega^{*2}\theta(\omega^*)^2}{\phi(\omega^*)}\ \frac{1}{\omega^{*2}\theta(\omega^*)}$$

$$\leqslant Mtq(t)\lambda^2\frac{1}{\lambda\theta(\omega^*)}\leqslant Mtq(t)\lambda^2.$$

Therefore, $\left|\frac{q_{52}(t,\lambda)}{\lambda}\right|\leqslant Mq(t)$.

Let us write $q_{51}=q_{511}+q_{512}$ where

$$tq_{511}(t,\lambda)=\int_{\rho}^{\infty}e^{i\tau t}\left(\frac{D(\tau)^2}{D(\tau,\lambda)^2}\right)d\tau \text{ and}$$

$$t\lambda q_{512}(t,\lambda)=i\int_{\rho}^{\infty}e^{i\tau t}\left(\frac{\tau^2\Delta(t,\tau)}{D(\tau,\lambda)^2}\right)d\tau.$$

Integrate $tq_{511}(t,\lambda)$ by parts and obtain

$$it^2q_{511}(t,\lambda)=e^{i\tau t}\left(\frac{D(\tau)^2}{D(\tau,\lambda)^2}\right)\Bigg|_{\tau=\rho}^{\tau=\infty}$$

$$-2\int_{\rho}^{\infty}e^{i\tau t}\left(\frac{D(\tau)D'(\tau)}{D(\tau,\lambda)^2}-\frac{(D'(\tau)+i\lambda^{-1})D(\tau)^2}{D(\tau,\lambda)^3}\right)d\tau$$

$$=\lim_{x\to\infty}e^{ixt}\left(\frac{D(x)}{D(x,\lambda)}\right)^2-\left(\frac{D(\rho)}{D(\rho,\lambda)}\right)^2e^{i\rho t}-\frac{2i}{\lambda}\int_{\rho}^{\infty}e^{i\tau t}\left(\frac{D(\tau)(\tau D'(\tau)-D(\tau))}{D(\tau,\lambda)^3}\right)d\tau.$$

But $\lim_{x\to\infty}\left|\frac{D(x)}{D(x,\lambda)}\right|=0$ and by (3.4) $\left|\frac{D(\rho)}{D(\rho,\lambda)}\right|^2 \leqslant M$, so the boundary terms are bounded by a constant.

The integral term is bounded in absolute value by

$$\frac{1}{\lambda}\left[\int_{\rho}^{\omega/2}+\int_{\omega/2}^{2\omega}+\int_{2\omega}^{2C_1\sigma}+\int_{2C_1\sigma}^{\infty}\right]\left|\frac{D(\tau)(\tau D'(\tau)-D(\tau))}{D(\tau,\lambda)^3}\right|d\tau.$$

We use (3.5), (2.6), (2.7), (3.4), and (3.11) to show

$$\frac{1}{\lambda}\left(\int_{\rho/2}^{\omega/2}+\int_{2\omega}^{2C_1\sigma}\right)\left|\frac{D(\tau)(\tau D'(\tau)-D(\tau))}{D(\tau,\lambda)^3}\right|d\tau$$

$$\leqslant\frac{M}{\lambda}\int_{\rho/2}^{2C_1\sigma}\frac{A\left(\frac{1}{\tau}\right)^2}{A\left(\frac{1}{\tau}\right)^3}d\tau=\frac{M}{\lambda}\int_{\rho/2}^{2C_1\sigma}\frac{1}{A\left(\frac{1}{\tau}\right)}d\tau\leqslant M.$$

By (2.6), (2.7), and (3.7),

$$\frac{1}{\lambda}\int_{2C_1\sigma}^{\infty}\left|\frac{D(\tau)(\tau D'(\tau)-D(\tau))}{D(\tau,\lambda)^3}\right|d\tau\leqslant M\lambda^2\int_{2C_1\sigma}^{\infty}\frac{A\left(\frac{1}{\tau}\right)^2}{\tau^3}d\tau$$

$$\leqslant M\lambda^2A\left(\frac{1}{\sigma}\right)^2\int_{2C_1\sigma}^{\infty}\tau^{-3}d\tau\leqslant M.$$

By (3.3), (3.6), (2.6), (2.7), (3.12), (3.9), and (3.8),

$$\frac{1}{\lambda}\int_{\omega/2}^{2\omega}\left|\frac{D(\tau)(\tau D'(\tau)-D(\tau))}{D(\tau,\lambda)^3}\right|d\tau$$

$$\leqslant M\lambda^2A\left(\frac{2}{\omega}\right)^2\int_{\omega/2}^{2\omega}\frac{d\tau}{[(\phi(\tau)\lambda)^2+(\tau-\omega)^2]^{3/2}}$$

$$\leqslant M\lambda^2A\left(\frac{2}{\omega}\right)^2\int_{0}^{\omega}\frac{du}{[(\lambda\phi(\omega^*))^2+u^2]^{3/2}}$$

$$=M\lambda^2A\left(\frac{2}{\omega}\right)^2\left[\frac{u}{(\lambda\phi(\omega^*))^2\ ((\lambda\phi(\omega^*))^2+u^2)^{1/2}}\Bigg|_{u=0}^{u=\omega}\right]$$

$$\leqslant\frac{M\lambda^2A\left(\frac{2}{\omega}\right)^2}{(\lambda\phi(\omega^*))^2}\leqslant\frac{M\lambda^2(\phi(\omega^*)+\omega^*\theta(\omega^*))^2}{(\lambda\phi(\omega^*))^2}$$

$$=M\lambda\left[\frac{\phi(\omega^*)+\omega^*\theta(\omega^*)}{\lambda^{1/2}\phi(\omega^*)}\right]^2\leqslant M\lambda\left[1+\frac{\omega^*\theta(\omega^*)}{\lambda^{1/2}\phi(\omega^*)}\right]^2$$

$$\leqslant M\lambda\left[1+\frac{1}{\omega^*\theta(\omega^*)\lambda^{1/2}}\right]^2\leqslant M\lambda\left[1+\frac{1}{\omega^*\theta(\omega^*)^{1/2}}\right]^2$$

$$\leqslant M\lambda\left(1+\frac{1}{(\lambda\theta(\omega^*))^{1/2}}\right)^2\leqslant M\lambda.$$

Therefore $\left|\frac{q_{511}(t,\lambda)}{\lambda}\right|\leqslant\frac{M}{t^2}\leqslant Mq(t).$

Next we integrate $q_{512}(t,\lambda)$ by parts. Then

$$t^2\lambda q_{512}(t,\lambda)=\frac{-\rho^2\Delta(t,\rho)e^{i\rho t}}{D(\rho,\lambda)^2}-\int_\rho^\infty e^{i\tau t}(T_1-T_2)d\tau$$

where $T_1=\frac{2\tau\Delta(t,\tau)+\Delta_\tau(t,\tau)\tau^2}{D(\tau,\lambda)^2}$ and $T_2=\frac{2D_\tau(\tau,\lambda)\tau^2\Delta(t,\tau)}{D(\tau,\lambda)^3}$.

The vanishing boundary term at ∞ follows from (4.21) and (4.22).

By (4.21) and (3.3), $\left|\frac{\rho^2\Delta(t,\rho)e^{i\rho t}}{D(\rho,\lambda)^2}\right|\leqslant\frac{M\rho\int_0^{1/\rho}a(s)ds}{\phi(\rho)^2}$, a constant.

We write $\int_\rho^\infty|T_1|d\tau=\left(\int_\rho^{\omega/2}+\int_{\omega/2}^{2\omega}+\int_{2\omega}^\infty\right)|T_1|d\tau$. By (4.22) and (4.21),

$$\frac{1}{\lambda^2}\int_{2\omega}^\infty\left|\frac{\tau\Delta(t,\tau)+\tau^2\Delta_\tau(t,\tau)}{D(\tau,\lambda)^2}\right|d\tau$$

$$\leqslant\frac{M}{\lambda^2}\int_{2\omega}^\infty\frac{\tau A_1\left(\frac{1}{\tau}\right)+t^2q(t)}{\left(\frac{\tau}{\lambda}\right)^2}d\tau=M\int_{2\omega}^\infty\frac{1}{\tau}A_1\left(\frac{1}{\tau}\right)+\frac{t^2q(t)}{\tau^2}d\tau$$

$$\leqslant M\int_\rho^\infty\frac{1}{\tau^2}d\tau\int_0^M a(s)ds+M\int_\rho^\infty\frac{t^2q(t)}{\tau^2}d\tau\leqslant M+Mt^2q(t).\quad\text{Thus}$$

$$\frac{1}{\tau^2\lambda^2}\int_{2\omega}^\infty|T_1|d\tau\leqslant M\left(\frac{1}{t^2}+q(t)\right)\leqslant Mq(t).$$

By (4.21), (3.4), (3.11) and (2.19) it follows that

$$\frac{1}{\lambda^2}\int_{\rho/2}^{\omega/2}\left|\frac{\tau\Delta(t,\tau)+\tau^2\Delta_\tau(t,\tau)}{D(\tau,\lambda)^2}\right|d\tau\leqslant\frac{M}{\lambda^2}\int_{\rho/2}^{2C_1\sigma}\frac{\tau A_1\left(\frac{1}{\tau}\right)+t^2q(t)}{A\left(\frac{1}{\tau}\right)^2}d\tau$$

$$\leqslant\frac{M}{\lambda^2}\int_{\rho/2}^{2C_1\sigma}\frac{d\tau}{A\left(\frac{1}{\tau}\right)}+\frac{M}{\lambda^2}\int_{\rho/2}^{2C_1\sigma}\frac{t^2q(t)}{A\left(\frac{1}{\tau}\right)^2}d\tau$$

$$\leqslant\frac{M\sigma}{\lambda^2A\left(\frac{1}{2C_1\sigma}\right)}+\frac{M\ \sigma t^2q(t)}{\lambda^2A\left(\frac{1}{2C_1\sigma}\right)^2}$$

$$\leqslant\frac{M\sigma}{\lambda^2A\left(\frac{1}{\sigma}\right)}+\frac{M\sigma t^2q(t)}{(\lambda A\left(\frac{1}{\sigma}\right))^2}$$

$$=\frac{M}{\lambda}+\frac{Mt^2q(t)}{\sigma}\leqslant M+Mt^2q(t).$$

Therefore, $\frac{1}{t^2\lambda^2}\int_{\rho/2}^{\omega/2}|T_1|d\tau\leqslant Mq(t)$.

By (4.21) (v), (3.12), (3.6), (3.3), (3.9), (3.5) and the fact $\sigma\leqslant M\lambda$ (by (2.19)), we have

$$\frac{1}{\lambda^2}\int_{\omega/2}^{2\omega}|T_1|d\tau\leqslant\frac{M}{\lambda^2}\int_{\omega/2}^{2\omega}\frac{\tau\lambda^{-1}+t^2q(t)}{\phi(\tau)^2+\left|\frac{\tau-\omega}{\lambda}\right|^2}d\tau$$

$$\leqslant M\int_{\omega/2}^{2\omega}\frac{\tau\left(\frac{1}{\lambda}+\frac{t^2q(t)}{\tau}\right)}{(\lambda\phi(\omega^*))^2+|\tau-\omega|^2}\,d\tau$$

$$\leqslant M\left(\frac{1}{\lambda}+\frac{t^2q(t)}{\omega^*}\right)\int_{\omega/2}^{2\omega}\frac{\tau d\tau}{(\lambda\phi(\omega^*))^2+|\tau-\omega|^2}$$

$$\leqslant M\left(\frac{1}{\lambda}+\frac{t^2q(t)}{\omega^*}\right)\omega\int_0^{\omega}\frac{d\tau}{(\lambda\phi(\omega^*))^2+\tau^2}$$

$$\leqslant M\left(\frac{\omega}{\lambda}+t^2q(t)\right)\frac{1}{\lambda\phi(\omega^*)}\leqslant M(1+t^2q(t))\frac{\theta(\omega^*)}{\phi(\omega^*)}$$

$$=M(1+t^2q(t))\frac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)}\ \frac{1}{\omega^{*2}\theta(\omega^*)}$$

$$\leqslant M(1+t^2q(t))\frac{1}{\lambda\theta(\omega^*)}\leqslant M(1+t^2q(t)).$$

Therefore, $\frac{1}{t^2\lambda^2}\int_\rho^\infty|T_1|d\tau\leqslant Mq(t)$.

Similarly we write $\int_\rho^\infty|T_2|d\tau=\left(\int_\rho^{\omega/2}+\int_{\omega/2}^{2\omega}+\int_{2\omega}^\infty\right)|T_2|d\tau$. Then

$$\frac{1}{\lambda^2}\int_{2\omega}^\infty|T_2|d\tau=\frac{2}{\lambda^2}\int_{2\omega}^\infty\left|\frac{D_\tau(\tau,\lambda)\tau^2\Delta(t,\tau)}{D(\tau,\lambda)^3}\right|d\tau$$

$$\leqslant\frac{M}{\lambda^2}\int_{2\omega}^\infty\frac{\lambda^{-1}\tau A\left(\frac{1}{\tau}\right)}{\left(\frac{\tau}{\lambda}\right)^3}d\tau=M\int_{2\omega}^\infty\frac{A\left(\frac{1}{\tau}\right)}{\tau^2}d\tau$$

$$\leqslant M\int_0^M a(s)ds\int_\rho^\infty\frac{1}{\tau^2}d\tau,$$

a constant, where we have used (4.21) (iv), (vi) and (4.22). Thus

$$\frac{1}{t^2\lambda^2}\int_{2\omega}^\infty|T_2|d\tau\leqslant\frac{M}{t^2}\leqslant Mq(t).$$

By (4.21) (ii), (iv) and (3.4) it follows that

$$\frac{1}{\lambda^2}\int_{\rho/2}^{\omega/2}|T_2|d\tau\leqslant\frac{M}{\lambda^2}\int_{\rho/2}^{\omega/2}\frac{A\left(\frac{1}{\tau}\right)^2}{A\left(\frac{1}{\tau}\right)^3}d\tau$$

$$\leqslant\frac{M}{\lambda^2}\ \frac{\omega}{A\left(\frac{2}{\omega}\right)}\leqslant\frac{M\omega}{\lambda^2A\left(\frac{1}{\sigma}\right)}=\frac{M\omega}{\lambda\sigma}\leqslant M$$

where the 2nd to the last inequality is a consequence of (3.5) and (3.11). Hence,

$$\frac{1}{t^2\lambda^2}\int_{\rho/2}^{\omega/2}|T_2|d\tau\leqslant\frac{M}{t^2}\leqslant Mq(t).$$

Also,

$$\frac{1}{\lambda^2}\int_{\omega/2}^{2\omega}|T_2|d\tau\leqslant\frac{M}{\lambda^2}\int_{\omega/2}^{2\omega}\frac{\frac{1}{\lambda}\tau A\left(\frac{1}{\tau}\right)d\tau}{\left[\phi(\tau)^2+\left|\frac{\tau-\omega}{\lambda}\right|^2\right]^{3/2}}$$

$$\leqslant M(\phi(\omega^*)+\omega^*\theta(\omega^*))\int_{\omega/2}^{2\omega}\frac{\tau d\tau}{[(\lambda\phi(\omega^*))^2+|\tau-\omega|^2]^{3/2}}$$

$$\leqslant M(\phi(\omega^*)+\omega^*\theta(\omega^*))\omega^*\int_0^{\omega}\frac{d\tau}{[(\lambda\phi(\omega^*))^2+\tau^2]^{3/2}}$$

$$\leqslant\frac{M(\phi(\omega^*)+\omega^*\theta(\omega^*))\omega^{*2}}{(\lambda\phi(\omega^*))^2\ ((\lambda\phi(\omega^*))^2+\omega^{*2})^{1/2}}$$

$$\leqslant\frac{M(\phi(\omega^*)+\omega^*\theta(\omega^*))\omega^*\theta(\omega^*)^2}{\phi(\omega^*)^2}$$

$$=M\frac{\omega^*\theta(\omega^*)^2}{\phi(\omega^*)}+M\frac{\omega^{*2}\theta(\omega^*)^3}{\phi(\omega^*)^2}$$

$$\leqslant\frac{M}{\omega^*}+M\left(\frac{1}{\omega\theta(\omega^*)^{1/2}}\right)^2\leqslant M$$

where we have used (4.21) (iv), (vi), (3.9), (3.12), (3.3), (3.6) and (3.8).

Therefore

$$\frac{1}{t^2\lambda^2}\int_{\omega/2}^{2\omega}|T_2|d\tau\leqslant\frac{M}{t^2}\leqslant Mq(t).$$

Consequently, $\left|\frac{q_5(t,\lambda)}{\lambda}\right|\leqslant Mq(t)$ and hence

$$\left|\frac{q_4(t,\lambda)+q_5(t,\lambda)}{\lambda}\right|\leqslant Mq(t).$$

The representation

$-\pi u''(t,\lambda)=\mathrm{Re}(i\lambda^{-1}q_1+\lambda^{-2}q_2+i\lambda^{-3}q_3+q_4+q_5)$ together with $\int_0^\infty|u''(t,\lambda)|dt<\infty$ for each λ from the proof of Theorem 1 (i) now yields $\int_1^\infty \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$, proving the theorem.

Theorem 7 a is obtained by simply combining the hypothesis in Theorems 5 and 6.

For the proof Theorem 7 b we apply Theorem 6, (by (3.5) our hypothesis imply (3.12)), obtaining $\int_1^\infty \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$. To prove $\int_0^1 \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right|dt<\infty$, follow the proof of Theorem 5 exactly, but use different auxiliary functions to combine the estimates (4.11) and Lemma 1. The proof of Theorem 5 uses $h(x)=xA\left(\frac{1}{x}\right)$, and $g(x)=\frac{1}{A\left(\frac{1}{x}\right)}$. Here, use $h(x)=\frac{x}{\ln^q x}$ and $g(x)=\ln^q x$ instead, completing the proof.

Now consider Lemma 2. A result in [33] (Theorem 2 (iii)) is: if $a\in L^1(0,\infty)$ and (2.13) holds, then (3.12) holds. This extends to the case where $a\notin L^1(0,\infty)$. Define, as in [1],

$$a_1(t)=\begin{cases}\frac{1}{2}(t-t_1)^2a''(t_1)+(t-t_1)a'(t_1)+a(t_1), & 0<t<t_1\\ a(t), & t_1<t\end{cases}$$

and

$$a_2(t)=a(t)-a_1(t),\quad t>0$$

where t_1 is any fixed number with $t_1>0$ such that $a''(t_1)$ exists. Then

a_1, a_2 both satisfy (2.13) and $a_2 \in L^1(0,\infty)$. With $\hat{a}(\tau)=\phi(\tau)-i\tau\theta(\tau)$, $\hat{a}_1 \equiv \phi_1 - i\tau\theta_1$ and $\hat{a}_2 \equiv \phi_2 - i\tau\theta_2$, we have

$$\frac{\tau^2\theta(\tau)^2}{\phi(\tau)} = \frac{\tau^2(\theta_1(\tau)+\theta_2(\tau))^2}{\phi_1(\tau)+\phi_2(\tau)} \leqslant \frac{2\tau^2\theta_1^2(\tau)+2\tau^2\theta_2^2(\tau)}{\phi_2(\tau)}$$

$$\leqslant \frac{2\tau^2\theta_1^2(\tau)}{\phi_2(\tau)} + M$$

where we have used the result for $a_2 \in L^1(0,\infty)$. We will finish by showing that $\frac{2\tau^2\theta_1^2(\tau)}{\phi_2(\tau)}$ is bounded. By (2.15) and (3.19) we have

$$\frac{\tau^2\theta_1^2(\tau)}{\phi_2(\tau)} \leqslant \frac{M\tau^2\left(\int_0^{1/\tau} sa_1(s)ds\right)^2}{\phi_2(\tau)}$$

$$\leqslant \frac{M\left(\int_0^{1/\tau} a_1(s)ds\right)^2}{\phi_2(\tau)}$$

$$\leqslant \frac{Ma_1^2(0)}{\tau^2\phi_2(\tau)}$$

$$\leqslant \frac{M}{\tau^2\int_0^{1/\tau} -sa'_2(s)ds} \leqslant M$$

where the last inequality is a consequence of

$$\infty \geqslant \lim_{\tau\to\infty} \tau^2\int_0^{1/\tau} -sa'_2(s)ds = \lim_{x\to 0} \frac{\int_0^x -sa'_2(s)ds}{x^2}$$

$$\geqslant \lim_{x\to 0} \frac{a'_2(x)\frac{x^2}{2}}{x^2} > 0.$$

This result is Lemma 2.

Before proving Theorem 8 we give a preliminary estimate and make some comments. Using (2.6), (2.15) and (3.19) (under the assumption (2.13)) we obtain

$$\frac{1}{8}\, A^2\left(\frac{1}{\tau}\right) \leqslant |\hat{a}(\tau)|^2 = \phi^2(\tau) + \tau^2\theta^2(\tau)$$

$$\leqslant \left(12B\left(\frac{1}{\tau}\right)\right)^2 + \tau^2\left(12A_1\left(\frac{1}{\tau}\right)\right)^2$$

$$\leqslant 144\left(B\left(\frac{1}{\tau}\right) + \tau A_1\left(\frac{1}{\tau}\right)\right)^2.$$

Hence, $2^{-3/2}A\left(\frac{1}{\tau}\right) \leqslant 12\left(B\left(\frac{1}{\tau}\right) + \tau A_1\left(\frac{1}{\tau}\right)\right)$. Also

$$\left(4A\left(\frac{1}{\tau}\right)\right)^2 \geqslant |\hat{a}(t)|^2 = \phi^2(\tau) + \tau^2\theta^2(\tau)$$

$$\geqslant \left(\frac{1}{5}B\left(\frac{1}{\tau}\right)\right)^2 + \tau^2\left(\frac{1}{5}A_1\left(\frac{1}{\tau}\right)\right)^2$$

$$\geqslant \frac{1}{50}\left(B\left(\frac{1}{\tau}\right) + \tau A_1\left(\frac{1}{\tau}\right)\right)^2.$$

Therefore,

$$4A\left(\frac{1}{\tau}\right) \geqslant (50)^{1/2}\left(B\left(\frac{1}{\tau}\right) + \tau A_1\left(\frac{1}{\tau}\right)\right).$$

Combining these into one inequality yields

$$(4.28) \qquad A\left(\frac{1}{\tau}\right) \leqslant (1152)^{1/2}\left(B\left(\frac{1}{\tau}\right) + \tau A\left(\frac{1}{\tau}\right)\right) \leqslant 960A\left(\frac{1}{\tau}\right).$$

In view of (2.6), (2.15) and (3.19), the behavior of $|\hat{a}(\tau)|$, $\phi(\tau)=\mathrm{Re}\,\hat{a}(\tau)$ and $\tau\theta(\tau)=|\mathrm{Im}\,\hat{a}(\tau)|$ as $\tau\to\infty$ is like that of $A\left(\frac{1}{\tau}\right)$, $B\left(\frac{1}{\tau}\right)$ and $\tau A_1\left(\frac{1}{\tau}\right)$ respectively.

In view of (4.28), condition (i) in Theorem 8 corresponds to the case where $|\hat{a}(\tau)|$, $\mathrm{Re}\,\hat{a}(\tau)$ and $|\mathrm{Im}\,\hat{a}(\tau)|$ have the same order as $\tau\to\infty$. Theorem 8 (ii) corresponds to the case where $|\mathrm{Im}\,\hat{a}(\tau)|$ is small compared to $|\hat{a}(\tau)|$ as $\tau\to\infty$, $|\hat{a}(\tau)|$ and $\mathrm{Re}\,\hat{a}(\tau)$ having the same order as $\tau\to\infty$. Theorem 8 (iii) and (iv) are both in the case where $|\mathrm{Im}\,\hat{a}(\tau)|$ and $|\hat{a}(\tau)|$ have the same order as $\tau\to\infty$ and $\mathrm{Re}\,\hat{a}(\tau)$ is small by comparison, as $\tau\to\infty$. To treat this case the additional assumptions $\dfrac{a^2(t)}{-a'(t)}$ is increasing for small t and $\dfrac{a^2(t)}{-ta'(t)}\in L^1(0,\epsilon)$ for some ϵ are made in (iii), and in (iv) the extra assumption we use is $\dfrac{\omega A^3\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{\omega}\right)}<\infty$ for $\omega\in(p/2,\infty)$.

For the proof of Theorem 8 a we differentiate (1.1); thus

$$u''(t,\lambda)=-\lambda(d+a(0+))u(t,\lambda)-\lambda\int_0^t a'(t-\tau)u(\tau,\lambda)d\tau.$$

Therefore, $\left|\dfrac{u''(t,\lambda)}{\lambda}\right|\leqslant(d+a(0+))|u(t,\lambda)|+\int_0^t -a'(t-\tau)|u(\tau,\lambda)|d\tau$

$$\leqslant d+a(0+)+(a(0+)-a(t))\leqslant d+2a(0+),$$

where we have used (2.3). We use this uniform bound on (0,1) and Theorem 6 with Lemma 2 to complete the proof of Theorem 8 a.

Let us turn to the proof of (b). By (2.19), (3.10), (3.5), (3.11) we have for some constant K

$$\text{(4.29)}\qquad \frac{1}{\sigma}=\frac{1}{\lambda}\,\frac{1}{A\left(\frac{1}{\sigma}\right)}\geqslant\frac{1}{5}\,\frac{A_1\left(\frac{1}{\omega}\right)}{A\left(\frac{1}{\sigma}\right)}\geqslant K\frac{A_1\left(\frac{1}{\omega}\right)}{A\left(\frac{1}{\omega}\right)}.$$

In case (i), use (4.28), (4.29) and (3.22) to obtain

(4.30) $$\frac{1}{\sigma} \geqslant \frac{KA_1\left(\frac{1}{\omega}\right)}{A\left(\frac{1}{\omega}\right)} \geqslant \frac{MA_1\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{\omega}\right)+\omega A_1\left(\frac{1}{\omega}\right)} \geqslant \frac{MA_1\left(\frac{1}{\omega}\right)}{\omega A_1\left(\frac{1}{\omega}\right)} = \frac{M}{\omega}.$$

Partition $S \equiv \{(t,\lambda): 0 \leqslant t \leqslant 1,\ \lambda \geqslant 1\}$ into $S_1 \equiv \{(t,\lambda): t \leqslant \frac{1}{\sigma}\} \cap S$, $S_2 \equiv \{(t,\lambda): t > \frac{1}{\sigma}\} \cap S$.

For $(t,\lambda) \in S_1$, use (3.13), (2.18), $|u(t,\lambda)| \leqslant 1$ and (3.15) to make the estimate

$$\left|\frac{u''(t,\lambda)}{\lambda}\right| = \left|a(t)+du(t,\lambda)+\int_0^t a(\tau)u'(t-\tau)d\tau\right|$$

$$\leqslant a(t)+d+K\sigma\int_0^t a(\tau)d\tau \leqslant a(t)+d+K\frac{1}{t}\int_0^t a(\tau)d\tau \in L^1(0,1).$$

For $(t,\lambda) \in S_2$ we use Lemma 3, (2.12), (3.19) and (2.15); thus

$$\left|\frac{u''(t,\lambda)}{\lambda}\right| \leqslant \frac{M}{t}\left(\frac{\sigma}{\lambda}+\frac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)}\right)$$

$$\leqslant \frac{M}{t}\left(\int_0^{1/\sigma} a(s)ds + \frac{\left(\omega^* A_1\left(\frac{1}{\omega^*}\right)\right)^2}{B\left(\frac{1}{\omega^*}\right)}\right)$$

$$\equiv \frac{M}{t}(I_1+I_2).$$

By the definition of S_2 and by (3.15) it follows that

$$\frac{1}{t}I_1 \leqslant \frac{1}{t}\int_0^t a(s)ds \in L^1(0,1).$$

By (3.11), (3.21), (3.15) and (4.30),

$$\frac{1}{t}\omega^* A_1\left(\frac{1}{\omega^*}\right) \leqslant \frac{M}{t}A\left(\frac{2}{\omega}\right) \leqslant \frac{A\left(\frac{2}{M\sigma}\right)}{t} \leqslant \frac{A\left(\frac{2}{M}t\right)}{t} \leqslant \frac{MA(t)}{t} \in L^1(0,1).$$

This completes the proof of Theorem 8 b (i).

To prove (b) (ii) we will show $\frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}$ is bounded and use Theorem 7.

By (3.11), $\frac{A\left(\frac{1}{\sigma}\right)}{A\left(\frac{\lambda}{\sigma^2}\right)} = \frac{A\left(\frac{1}{\sigma}\right)}{A\left(\frac{1}{\sigma}\frac{1}{A\left(\frac{1}{\sigma}\right)}\right)} \leqslant \frac{A\left(\frac{1}{\sigma}\right)}{A\left(\frac{M}{\sigma}\right)} \leqslant M.$

Use (3.10) and (2.19) to make the estimate

$$\frac{\lambda}{\sigma^2} = \frac{1}{\lambda}\frac{1}{A^2\left(\frac{1}{\sigma}\right)} \geqslant \frac{MA_1\left(\frac{1}{\omega}\right)}{A^2\left(\frac{1}{\sigma}\right)} = \frac{1}{\omega}\frac{MA_1\left(\frac{1}{\omega}\right)}{\frac{1}{\omega}A^2\left(\frac{1}{\sigma}\right)} \equiv \frac{1}{\omega}L(\lambda).$$

There are two possibilities $L(\lambda) \geqslant 1$ or $L(\lambda) < 1$.

For $L(\lambda) \geqslant 1$, $A\left(\frac{\lambda}{\sigma^2}\right) \geqslant A\left(\frac{1}{\omega}\right)$ holds.

Now use (3.17), (3.19), (2.15) and (2.19) to obtain

$$\frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)} \leqslant \frac{M}{A\left(\frac{\lambda}{\sigma^2}\right)}\left(A\left(\frac{1}{\sigma}\right) + \frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)}\right).$$

Now,

$$\frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{A\left(\frac{\lambda}{\sigma^2}\right)B\left(\frac{1}{\omega^*}\right)} \leqslant \frac{M\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{A\left(\frac{1}{\omega}\right)B\left(\frac{1}{\omega^*}\right)} \leqslant M\left(\frac{\omega^*A_1\left(\frac{1}{\omega^*}\right)}{A\left(\frac{1}{\omega^*}\right)}\right)\left(\frac{\omega^*A\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)}\right) \leqslant M$$

where we have used (3.11) and the assumption (ii) together with (4.28).

We have shown that

$$\sup_{L(\lambda)\geqslant 1} \frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)} < \infty.$$

For $L(\lambda)<1$, $A\left(\frac{\lambda}{\sigma^2}\right)\geqslant A\left(\frac{1}{\omega}L(\lambda)\right)\geqslant L(\lambda)A\left(\frac{1}{\omega}\right)$ by (3.11). Finally, use (4.28), (3.11), (3.5) and the assumption (ii) to obtain

$$\frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{\lambda}{\sigma^2}\right)} \leqslant \frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{1}{\omega}\right)L(\lambda)} = \frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)\frac{1}{\omega}A^2\left(\frac{1}{\sigma}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{1}{\omega}\right)A_1\left(\frac{1}{\omega}\right)}$$

$$\leqslant \frac{M\omega A_1^2\left(\frac{1}{\omega^*}\right)A^2\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{\omega}\right)A\left(\frac{1}{\omega}\right)A_1\left(\frac{1}{\omega}\right)} \leqslant \frac{M\omega A_1^2\left(\frac{2}{\omega}\right)}{A_1\left(\frac{1}{\omega}\right)}.$$

To see this is bounded we use the inequality $A_1(2t)\leqslant 4A_1(t)$. This yields $\frac{\omega A_1^2\left(\frac{2}{\omega}\right)}{A_1\left(\frac{1}{\omega}\right)}\leqslant 16\omega A_1\left(\frac{1}{\omega}\right)\leqslant 16A\left(\frac{1}{\omega}\right)\leqslant M$. This shows that $\sup\limits_{L(\lambda)<1} \frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}<\infty$.

Thus,

$\frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}$ is bounded for $\lambda\geqslant 1$. The application of Theorem 7 finishes the proof.

To prove (iii) use (4.28), (4.29) and the assumption of (iii):

$$(4.31)\qquad \frac{1}{\sigma} \geqslant \frac{KA_1\left(\frac{1}{\omega}\right)}{A\left(\frac{1}{\omega}\right)} \geqslant \frac{M\frac{1}{\omega}A\left(\frac{1}{\omega}\right)}{A\left(\frac{1}{\omega}\right)} = \frac{M}{\omega}.$$

Thus, $\frac{1}{\sigma}\geqslant\frac{M}{\omega}$ as in (4.31) and the rest of the proof follows exactly as in the lines following (4.30) except for the term $\frac{1}{t}I_2$ which we treat next. By Lemma 1, (4.28), (3.11), assumption (iii) and integration by parts:

$$\frac{1}{t}I_2 \leq \frac{M}{t}\left(\frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)}\right)$$

$$\leq \frac{M}{t}\,\frac{A^2\left(\frac{2}{\omega}\right)}{B\left(\frac{1}{2\omega}\right)}$$

$$\leq \frac{M}{t}\,\frac{A^2\left(\frac{1}{2\omega}\right)}{B\left(\frac{1}{2\omega}\right)} \leq \frac{M}{t}\,\frac{(2\omega)^2A_1^2\left(\frac{1}{2\omega}\right)}{B\left(\frac{1}{2\omega}\right)}$$

$$\leq \frac{\omega^2 M}{t}\left(\frac{\left[\left(\frac{1}{2\omega}\right)^2 a\left(\frac{1}{2\omega}\right)+\int_0^{1/2\omega}-s^2a'(s)ds\right]^2}{B\left(\frac{1}{2\omega}\right)}\right)$$

$$\leq \frac{M}{t}\left(\frac{\frac{1}{\omega^2}a^2\left(\frac{1}{2\omega}\right)}{B\left(\frac{1}{2\omega}\right)}+\frac{\left[\int_0^{1/2\omega}-s^2a'(s)ds\right]^2}{\frac{1}{\omega^2}B\left(\frac{1}{2\omega}\right)}\right)$$

$$\equiv \frac{M}{t}[J_1+J_2].$$

By definition of S_2 and (4.31) and assumption (iii),

$$\frac{M}{t}J_1 \leq \frac{M\frac{1}{\omega^2}a^2\left(\frac{1}{2\omega}\right)}{t\left(-a'\left(\frac{1}{2\omega}\right)\frac{1}{\omega^2}\right)} = \frac{Ma^2\left(\frac{1}{2\omega}\right)}{-ta'\left(\frac{1}{2\omega}\right)}$$

$$\leq \frac{Ma^2(M_1t)}{-ta'(M_1t)} \in L^1\left(0,\frac{\epsilon}{M_1}\right) \text{ (some } M_1\text{)}.$$

Also,

$$\frac{M}{t}J_2 \leq \frac{M\int_0^{1/2\omega}-sa'(s)ds\int_0^{1/2\omega}-s^3a'(s)ds}{t\frac{1}{\omega^2}B\left(\frac{1}{2\omega}\right)}$$

$$=\frac{M\int_0^{1/2\omega} -s^3a'(s)ds}{t\frac{1}{\omega^2}}$$

$$\leqslant\frac{M}{t}\int_0^{1/2\omega} -sa'(s)ds\leqslant\frac{M}{t}\int_0^{t} -sa'(s)ds\leqslant\frac{M\int_0^t a(s)ds}{t}\in L^1(0,1)$$

where we have used the Cauchy-Schwarz inequality, the definition of S_2 and (3.15). Note, on $\left[\frac{\epsilon}{M_1},1\right]$, (3.13) implies $\left|\frac{u''(t,\lambda)}{\lambda}\right|<a\left(\frac{\epsilon}{M_1}\right)+d+2a\left(\frac{\epsilon}{M_1}\right)$, a constant. With Theorem 6, this finishes the proof of 8 b (iii).

To prove (iv), by Theorem 5, we only need to show that $\frac{C(\lambda)}{A\left(\frac{\lambda}{\sigma^2}\right)}$ is bounded. We showed in (ii) that $\frac{\frac{\sigma}{\lambda}}{A\left(\frac{\lambda}{\sigma^2}\right)}=\frac{A\left(\frac{1}{\sigma}\right)}{A\left(\frac{\lambda}{\sigma^2}\right)}$ is bounded. The last step is to use (2.15), (3.6), (3.19), (4.28), (3.11), (3.5) and assumption (iv) to obtain

$$\frac{\omega^{*2}\theta(\omega^*)^2}{\phi(\omega^*)A\left(\frac{\lambda}{\sigma^2}\right)}\leqslant\frac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{\lambda}{\sigma^2}\right)}\leqslant\frac{MA^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{\lambda}{\sigma^2}\right)}$$

$$\leqslant\frac{MA^2\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{2\omega}\right)\frac{\lambda}{\sigma^2}A(1)}\leqslant\frac{MA^2\left(\frac{1}{\omega}\right)A^2\left(\frac{1}{\sigma}\right)}{B\left(\frac{1}{2\omega}\right)A_1\left(\frac{1}{\omega}\right)}$$

$$\leqslant\frac{MA^4\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{2\omega}\right)A_1\left(\frac{1}{\omega}\right)}$$

$$\leqslant \frac{M\omega A^3\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{2\omega}\right)} \leqslant \frac{M2\omega A^3\left(\frac{1}{2\omega}\right)}{B\left(\frac{1}{2\omega}\right)} \leqslant M.$$

5. Examples

(a) Suppose $a(t)=t^{-p}$, $0<p<1$. Then an easy calculation yields

$$\tau A_1\left(\frac{1}{\tau}\right)=\frac{1}{2-p}\,\frac{1}{\tau^{1-p}} \text{ and } B\left(\frac{1}{\tau}\right)=\frac{1}{1-p}\,\frac{1}{\tau^{1-p}}.$$

Theorem 8 b (i) applies.

(b) Suppose $a(t)=-\ln t$ for small t and (2.13). Then $B(x)=x$, $A(x)=x-x\ln x$ for small x and $\lim\limits_{\tau\to\infty}\frac{B\left(\frac{1}{\tau}\right)}{A\left(\frac{1}{\tau}\right)}=0$. Part (iii) and (iv) in Theorem 8 b can both be applied. For instance, $\frac{\omega A^3\left(\frac{1}{\omega}\right)}{B\left(\frac{1}{\omega}\right)}$ $=\omega^2\left(\frac{1}{\omega}+\frac{\ln\omega}{\omega}\right)^3$ which is bounded on $(p/2,\infty)$, so (iv) applies.

(c) Suppose $a(t)=t^{-1}(-\ln t)^{-q}$ for small t and (2.13) holds. The necessary condition (3.15) holds for $q>2$. For these q we see that Theorem 8 b (ii) applies. We note $A(x)=(q-1)^{-1}(-\ln x)^{1-q}$. Use L'Hospital's rule in the following calculation:

$$\lim_{\tau\to\infty}\frac{\tau A_1\left(\frac{1}{\tau}\right)}{A\left(\frac{1}{\tau}\right)}$$

$$=\lim_{x\to 0^+}\frac{A_1(x)(q-1)}{x(-\ln x)^{1-q}}$$

$$=\lim_{x\to 0^+}\frac{(-\ln x)^{-q}(q-1)}{(-\ln x)^{1-q}+(q-1)(-\ln x)^{-q}}$$

$$=\lim_{x\to 0^+}\frac{q-1}{(-\ln x)+q-1}=0.$$ Now (ii) applies.

(d) Let $c''(t)=\sum_{k=0}^{\infty} a_k \chi_{[0,x_k)}(t)$ where $a_k \geq 0$, $k=0,1,2,\ldots$ and $0<x_{k+1}<x_k\leq 1/2$, $k=0,1,2,\ldots$.

$$-c'(t)=c'(1)-c'(t)=\int_t^1 c''(s)ds=\sum_{k=0}^{\infty} a_k(x_k-t)\chi_{[0,x_k)}(t).$$

(5.1) $$c(t)=\int_t^1 -c'(s)ds=1/2\sum_{k=0}^{\infty} a_k(x_k-t)^2\chi_{[0,x_k)}(t).$$

We will show that for $d+a(t)=c(t)$ with $\{a_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ appropriately chosen, (2.13) and (3.15) hold but not (3.16). (2.13) holds by definition. To show (3.16) does not hold we will show $\dfrac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)A\left(\frac{\lambda}{\sigma^2}\right)}$ is not bounded.

Let

(5.2) $$a_n=2^{11\cdot 2^{2^n}},\ x_n=2^{-4\cdot 2^{2^n}},\ n=0,1,2,\ldots$$

Note

$$c(0+)=\sum_{k=0}^{\infty} a_k x_k^2=\sum_{k=0}^{\infty} 2^{3\cdot 2^{2^k}}=\infty.$$

Also,

$$\int_0^1(-\ln t)c(t)dt=\sum_{k=0}^{\infty} 1/2\int_0^{x_k}(-\ln t)(x_k^2-2x_k t+t^2)dt$$

$$=\sum_{k=0}^{\infty}\left(a_k x_k^3\,\frac{11}{36}-a_k x_k^3\,\frac{1}{6}\ln x_k\right)$$

$$=\frac{11}{36}\sum_{k=0}^{\infty}2^{-2^{2^k}}+\frac{1}{\delta}\sum_{k=0}^{\infty}4(\ln 2)2^{2^n}2^{-2^{2^n}}<\infty.$$

Thus (3.15) holds. To see $\frac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)A\left(\frac{\lambda}{\sigma^2}\right)}$ is not bounded, first we note

$$\frac{\lambda}{\sigma^2}\leqslant\frac{M}{\omega^*A\left(\frac{1}{\omega^*}\right)} \tag{5.3}$$

To see this, observe that $\frac{\sigma^2}{\lambda}=\sigma\int_0^{1/\sigma}a(s)ds$ is increasing with σ so

$$\frac{\sigma^2}{\lambda}\geqslant\frac{\omega}{C_1}\int_0^{C_1/\omega}a(s)ds\geqslant M\omega^*\int_0^{1/\omega^*}a(s)ds=M\omega^*A\left(\frac{1}{\omega^*}\right)$$

where we have used (3.5) and (3.11).

By (5.3), (3.19), (2.15), and (3.11) we have

$$\frac{1}{A\left(\frac{\lambda}{\sigma^2}\right)}\geqslant\frac{1}{A\left(\frac{M}{\omega^*A\left(\frac{1}{\omega^*}\right)}\right)}\geqslant\frac{M}{A\left(\frac{2}{\omega A\left(\frac{2}{\omega}\right)}\right)}.$$

Thus,

$$\frac{(\omega^*\theta(\omega^*))^2}{\phi(\omega^*)A\left(\frac{\lambda}{\sigma^2}\right)}\geqslant\frac{M\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)A\left(\frac{2}{\omega A\left(\frac{2}{\omega}\right)}\right)}\geqslant\frac{M\left(\frac{\omega}{2}\right)^2A_1^2\left(\frac{2}{\omega}\right)}{B\left(\frac{2}{\omega}\right)A\left(\frac{2}{\omega A\left(\frac{2}{\omega}\right)}\right)},$$

where we have used $A_1(2x)\leqslant 4A_1(x)$, $x>0$. Let $t=\frac{2}{\omega}$. Then $t\to 0$ as $\omega\to\infty$, so by the above inequality, we only have to show that there is a sequence $\{t_n\}$, $t_n\to 0$ such that

$$\frac{A_1^2(t_n)}{t_n^2B(t_n)A\left(\frac{t_n}{A(t_n)}\right)}\to\infty \text{ as } n\to\infty. \tag{5.4}$$

We integrate $A_1(t)$ by parts twice and use the definition of $c(t)$, $c'(t)$ and $c''(t)$ to obtain

$$A_1(t)=\frac{t^2c(t)}{2}+\frac{t^3(-c'(t))}{6}+\frac{1}{6}\int_0^t s^3c''(s)ds$$

$$=\frac{t^2}{2}\sum_{k=0}^{n}a_k\left(\frac{x_k^2}{2}-x_kt+\frac{t^2}{2}\right)+\frac{t^3}{6}\sum_{k=0}^{n}a_k(x_k-t)$$

$$+\frac{t^4}{6}\sum_{k=0}^{n}\frac{a_k}{4}+\frac{1}{6}\sum_{k=n+1}^{\infty}\frac{a_kx_k^4}{4}\text{ for }x_{n+1}\leqslant t<x_n.$$

Thus,

$$A_1(t)=t^2\sum_{k=0}^{n}a_k\left(\frac{1}{4}x_k^2-\frac{1}{3}x_kt+\frac{1}{8}t^2\right)+\frac{1}{24}\sum_{k=n+1}^{\infty}a_kx_k^4\text{ for }x_{n+1}\leqslant t<x_n. \tag{5.5}$$

Similarly,

$$A(t)=\int_0^t c(s)ds=\sum_{k=0}^{\infty}\frac{a_k}{2}\int_0^t(x_k-s)^2\chi_{[0,x_k)}(s)ds$$

$$=\sum_{k=0}^{n}\frac{a_k}{2}\int_0^t(x_k-s)^2ds+\sum_{k=n+1}^{\infty}\frac{a_k}{2}\int_0^{x_k}(x_k-s)^2ds\text{ for }x_{n+1}\leqslant t<x_n.$$

Thus,

$$A(t)=\frac{1}{6}\sum_{k=0}^{n}a_k(3x_k^2t-3x_kt^2+t^3)+\frac{1}{6}\sum_{k=n+1}^{\infty}a_kx_k^3,\ x_{n+1}\leqslant t<x_n. \tag{5.6}$$

Also,

$$B(t)=\frac{1}{2}t^2\sum_{k=0}^{n}(a_kx_k-\frac{2}{3}a_kt)+\frac{1}{6}\sum_{k=n+1}^{\infty}a_kx_k^3\text{ for }x_{n+1}\leqslant t<x_n. \tag{5.7}$$

To see this, observe

$$B(t)=\int_0^t -sc'(s)ds=\frac{t^2}{2}(-c'(t))+\frac{1}{2}\int_0^t s^2c''(s)ds$$

$$=\frac{t^2}{2}\sum_{k=0}^{n} a_k(x_k-t)+\frac{1}{2}\sum_{k=0}^{n}\frac{a_k t^3}{3}+\frac{1}{2}\sum_{k=n+1}^{\infty}\frac{a_k x_k^3}{3} \text{ for } x_{n+1}\leqslant t<x_n.$$

To show (5.4) let $t_n=x_{n+1}\left(\frac{n+1}{n}\right)=2^{-4\cdot 2^{2^n}}\left(\frac{n+1}{n}\right)$, then $x_{n+1}<t_n<x_n$.

For this choice of t_n, we observe from (5.5) and (5.7) that $A_1(t_n)\sim\frac{1}{4}t_n^2a_nx_n^2$ as $n\to\infty$ and $B(t_n)\sim\frac{1}{2}t_n^2a_nx_n$ as $n\to\infty$, where $G_n\sim F_n$ as $n\to\infty$ means $\lim\limits_{n\to\infty}\frac{G_n}{F_n}=1$.

Also (5.6) implies $A(t_n)\sim\frac{1}{2}a_nx_n^2t_n$ as $n\to\infty$.

Therefore, as $n\to\infty$, the expression in (5.4) is asymptotic to $\dfrac{\left(t_n^2a_nx_n^2\right)^2}{8t_n^2t_n^2a_nx_nA\left(\frac{2t_n}{a_nx_n^2t_n}\right)}\geqslant\dfrac{Ma_nx_n^3}{A\left(\frac{1}{a_nx_n^2}\right)}$ by (3.11). But for large n,

$$x_n=2^{-4\cdot 2^{2^n}}<2^{-3\cdot 2^{2^n}}=\frac{1}{a_nx_n^2}<2^{-42^{2^{n-1}}}=x_{n-1}.$$

By (5.6),

$$A\left(\frac{1}{a_nx_n^2}\right)=\frac{1}{6}\sum_{k=0}^{n-1}a_k\left(\frac{3x_k^2}{a_nx_n^2}-\frac{3x_k}{\left(a_nx_n^2\right)^2}+\frac{1}{\left(a_nx_n^2\right)^3}\right)+\frac{1}{6}\sum_{k=n}^{\infty}a_kx_k^3\sim\frac{1}{2}\frac{a_{n-1}x_{n-1}^2}{a_nx_n^2}.$$

Therefore,

$$\frac{a_nx_n^3}{A\left(\frac{1}{a_nx_n^2}\right)}\sim\frac{2a_nx_n^3a_nx_n^2}{a_{n-1}x_{n-1}^2}=\frac{2a_n^2x_n^5}{a_{n-1}x_{n-1}^2}=\frac{2^{2\cdot 2^{2^n}}}{2^{3\cdot 2^{2^{n-1}}}}=2^{2\cdot 2^{2^n}-3\cdot 2^{2^{n-1}}}\to\infty \text{ as}$$

$n\to\infty$.

To see this we will show that $\left(2\cdot 2^{2^n}-3\cdot 2^{2^{n-1}}\right)\to\infty$. Thus,

$2\cdot 2^{2^n}-32^{2^{n-1}}=2^{2^n}(2-3\cdot 2^{2^{n-1}-2^n})=2^{2^n}\left(2-3\cdot 2^{-2^{n-1}}\right)\to\infty$.

We show however that Theorem 7 b applies yielding (1.2). We have

$C(\lambda)=\dfrac{\omega^{*2}\theta(\omega^*)^2}{\phi(\omega^*)}+A\left(\frac{1}{\sigma}\right)\leqslant M\left(\dfrac{\omega^{*2}A_1^2\left(\frac{1}{\omega^*}\right)}{B\left(\frac{1}{\omega^*}\right)}+A\left(\frac{1}{\omega^*}\right)\right)$ by (2.15), (3.19) and (3.11).

Also,

$$\frac{\sigma^2}{\lambda}=\lambda A^2\left(\frac{1}{\sigma}\right)\leqslant M\ \frac{A^2\left(\frac{1}{\omega^*}\right)}{A_1\left(\frac{1}{\omega^*}\right)}$$

by (3.10), (3.5), (3.11) and the fact $A_1(2x)\leqslant 4A_1(x)$.

Thus, for any $q>1$,

$$C(\lambda)\ln^q\left(\frac{\sigma^2}{\lambda}\right)\leqslant M\left(\frac{A_1^2\left(\frac{1}{\omega^*}\right)}{\left(\frac{1}{\omega^*}\right)^2B\left(\frac{1}{\omega^*}\right)}+A\left(\frac{1}{\omega^*}\right)\right)\ln^q\left(\frac{A^2\left(\frac{1}{\omega^*}\right)}{A_1\left(\frac{1}{\omega^*}\right)}\right).$$

To show this is bounded we only need to show that

$D(t)\equiv\left(\dfrac{A_1^2(t)}{t^2B(t)}+A(t)\right)\ln^q\left(\dfrac{A^2(t)}{A_1(t)}\right)$ is bounded for small t.

By (5.5), (5.6) and (5.7) we have for $x_{n+1}\leqslant t<x_n$ that

$A_1(t)\sim t^2\left(\frac{1}{4}a_nx_n^2-\frac{1}{3}a_nx_nt+\frac{1}{8}a_nt^2\right)$ as $t\to 0$

$A(t)\sim\frac{1}{6}\left(3a_nx_n^2t-3a_nx_nt^2+a_nt^3\right)$ as $t\to 0$

$$B(t)\sim\frac{1}{2}t^2\left(a_nx_n-\frac{2}{3}\ a_nt\right)\ \text{as}\ t\to0,$$

respectively.

Also,

$$t^2\left(\frac{1}{4}a_nx_n^2-\frac{1}{3}a_nx_nt+\frac{1}{8}a_nt^2\right)\leqslant\frac{3}{8}t^2t^2a_nx_n^2,$$

$$t^2\left(\frac{1}{4}a_nx_n^2-\frac{1}{3}a_nx_nt+\frac{1}{8}a_nt^2\right)\geqslant t^2\frac{1}{24}a_nx_n^2,$$

$$\frac{1}{6}(3a_nx_n^2t-3a_nx_nt^2+a_nt^3)\leqslant\frac{2}{3}a_nx_n^2t,$$

and

$$\frac{1}{2}t^2\left(a_nx_n-\frac{2}{3}a_nt\right)\geqslant\frac{1}{6}t^2a_nx_n$$

all for $x_{n+1}\leqslant t<x_n$. Thus, $D(t)$ is asymptotic to a function that is bounded by

$$\left(\frac{\left(3/8t^2a_nx_n^2\right)^2}{t^2\left(1/6t^2a_nx_n\right)}+\frac{2}{3}a_nx_n^2t\right)\ln^q\left(\frac{\left(2/3a_nx_n^2t\right)^2}{t^21/24a_nx_n^2}\right)\leqslant M\left(a_nx_n^3\right)\ln^q\left(a_nx_n^2\right)$$

$$=M2^{-2^{2^n}}\ln^q(2^{32^{2^n}})\leqslant M2^{-2^{2^n}}2^{q2^n}\to0\quad\text{as}\quad t\to0,\quad\text{since}\quad n\to\infty\quad\text{as}\quad t\to0.$$

Theorem 7 b shows (1.2) holds.

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UNIFORM L^1 BEHAVIOR FOR THE SOLUTION OF A

VOLTERRA EQUATION WITH A PARAMETER

by

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Mathematics

(ABSTRACT)

The solution $u=u(t)=u(t,\lambda)$ of

(E) $$u'(t)+\lambda\int_0^t u(t-\tau)(d+a(\tau))d\tau=0,\ u(0)=1,\ t\geq 0,\ \lambda\geq 1$$

where $d\geq 0$, a is nonnegative, nonincreasing, convex and $\infty > a(0+) > a(\infty)=0$ is studied. In particular the question asked is: When is

(F) $$\int_0^\infty \sup_{\lambda\geq 1}\left|\frac{u''(t,\lambda)}{\lambda}\right| dt<\infty?$$

We obtain two necessary conditions for (F). For (F) to hold, it is necessary that $(-\ln t)a(\tau)\in L^1(0,1)$ and $\limsup_{\tau\to\infty} \frac{(\tau\theta(\tau))^2}{\phi(\tau)}<\infty$ where $\hat{a}(\tau)\equiv\int_0^\infty e^{-i\tau t}a(t)dt=\phi(\tau)-i\tau\theta(\tau)$ (ϕ,θ both real).

We obtain sufficient conditions for (F) to hold which involve ϕ and θ (See Theorem 7). Then we look for direct conditions on a which imply (F). With the addition assumption $-a'$ is convex, we prove that (F) holds provided any one of the following hold:

(i) $a(0+)<\infty$,

(ii) $0<\liminf_{\tau\to\infty} \dfrac{\tau\int_0^{1/\tau} sa(s)ds}{\int_0^{1/\tau} -sa'(s)ds} \leqslant \limsup_{\tau\to\infty} \dfrac{\tau\int_0^{1/\tau} sa(s)ds}{\int_0^{1/\tau} -sa'(s)ds} <\infty$,

(iii) $\lim_{\tau\to\infty} \dfrac{\tau\int_0^{1/\tau} sa(s)ds}{\int_0^{1/\tau} a(s)ds} =0$,

(iv) $\lim_{\tau\to\infty} \dfrac{\int_0^{1/\tau} -sa'(s)ds}{\int_0^{1/\tau} a(s)ds} =0$, $\dfrac{a^2(t)}{-a'(t)}$ is increasing for small t and

$\dfrac{a^2(t)}{-ta'(t)}\in L^1(0,\epsilon)$ for some $\epsilon>0$,

(v) $\lim_{\tau\to\infty} \dfrac{\int_0^{1/\tau} -sa'(s)ds}{\int_0^{1/\tau} a(s)ds} =0$ and $\dfrac{\tau(\int_0^{1/\tau} a(s)ds)^3}{\int_0^{1/\tau} -sa'(s)ds} \leqslant M<\infty$ for $\delta\leqslant\tau<\infty$ (some

$\delta>0$).

Thus (F) holds for wide classes of examples. In particular, (F) holds when $d+a(t)=t^{-p}, 0<p<1$; $a(t)+d=-\ln t$ (small t); $a(t)+d=t^{-1}(-\ln t)^{-q}$, $q>2$ (small t).