

ENVIRONMENTAL THERMAL STRESSES AS A
FIRST PASSAGE PROBLEM

by

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Chapter 1

Introduction

The structure studied here is idealized as a layered circular cylinder. It is affected by environmental temperature changes that produce thermal stresses. These thermal stresses may reach levels at which the structure no longer performs its objective with a high level of reliability. The environmental temperature changes consist of a mean value due to geographic location, a yearly cycle due to seasonal changes, a diurnal cycle and random variations. Because the thermal environment is a statistically variable quantity, the resulting thermal stresses will also be stochastic.

Based on previous work [1-4], stochastic process theory is used to idealize the thermal environment of the system. The seasonal and daily cycles derived from narrow band processes are assumed to be deterministic with random variations attributed to noise, all superimposed on a mean value. The system is governed by the Fourier heat conduction equation which is a linear relation. Hence, the output process resembles the input. Thus, the same idealization that is made for the environmental temperature can be made for the thermal stresses: the thermal stresses consist of a mean value, two deterministic cyclic processes and noise, all superimposed to give the required stress at a certain hour.

A methodology to predict the service life for cylindrical structures has been under way for the last several years. In the early stages of this research, different simplifying assumptions have been made. For example, the statistical variability of the strength was considered without degradation and the statistical variability of the

elastic thermal stresses was assumed to be due to temperature alone [2,3]. The viscoelastic effects and the reduction of the strength due to cumulative damage and aging were incorporated in the analysis [5]. Later on, the statistical variability of the thermal stresses due to the statistical variabilities of Poisson's ratio, modulus of elasticity, and coefficient of thermal expansion was considered [6,7].

In all the above stages, the service life prediction was based on the stress-strength interference principle. This principle states that failure takes place whenever the strength is exceeded by the induced stress. Therefore, daily probability of survival is calculated by carrying out convolution integrations of the probability density functions for stress and strength. The probability of survival for n days is obtained by calculating the progressive product of the probabilities of survival for each day. This progressive probability is then compared with an allowable probability of survival to obtain the service life. This rationale implies complete, day to day independence and that the structure has to survive all n days in order to have a life expectancy greater than n days.

A more realistic approach that defines failure to be the instant at which the induced stress exceeds the strength of the system for the first time is the main objective of this study. This leads to the study of the first passage problem which has been extensively studied in the literature for the case of fixed barriers, that is, for constant, deterministic strength and strain capability. Because viscoelastic materials are time and temperature dependent, the strength and strain capability vary from day to day. Additionally the material is afflicted with

statistical variability and is subject to degradation by aging and cumulative damage. Hence the present study examines the first passage problem with statistically and time varying barriers.

Theoretically, this approach is less conservative than the progressive product method because it takes into account the behavior by which the stress reaches the strength. Hence, improved estimates on the probability values are expected.

Chapter 2

Literature Review

Before going into the details about the first passage problem, some of the random vibrations concepts that can be found in the literature are defined here [8]. The study of random vibrations is a study of structural dynamics and probability theory. The ultimate goal of this subject is to provide a sound basis for improving the reliability of structures that must withstand random loads. The key concepts from probability theory are those of random processes which are used to model the input and output time histories. A random phenomenon which develops in time and obeys probabilistic, rather than deterministic, laws is called a random process.

A random process is defined as an infinite population or ensemble whose samples are functions of time. To describe random processes, it is customary to deal with first and second order probability distributions or their statistical averages only. Processes that assume a sort of uniformity in time are known as stationary processes. A stationary process is defined as one whose probability distributions across the ensemble are invariant with respect to translations of time.

The familiar statistical averages are the mean and the mean square which depend on the first order probability distribution, and the correlation function which depends on the second order probability distribution. The correlation function, $R_x(\tau)$, for a random process, $x(t)$, is the average of the product $x(t_1) x(t_2)$. The correlation function depends on the time lag $\tau = t_2 - t_1$ if the process is stationary.

The Fourier transform of this function is known as the spectral density function, $G_x(\omega)$. The mean square of the stationary random process is given by the value of $R_x(\tau)$ when τ is zero, or by the total area under the spectral density function.

The importance of these two functions resides in the facts that they are closed with respect to linear time invariant operations in the sense that if these statistics are known for the input process then the corresponding statistics for the response of a linear system can be obtained. Additionally, this Fourier pair provides adequate information about the response so that engineering decisions concerning the severity of the excitation and the reliability of the system can be made.

One of the classifications of random processes is made according to the influence of the past on the present. There are processes with no memory at all. White noise, which is an idealization that cannot be realized, is an important example of this class. This process has a uniform spectral density for all frequencies and its correlation function is proportional to the Dirac delta function.

On the other hand, there are processes whose present distributions can be determined by the distributions at any single time in the past. An important example of this class is known as Markov processes. The distribution of a Markov process at any time is determined by the distribution at some initial time and by a transition probability density function. The importance of Markov processes stems from the fact that when the input of a linear or nonlinear dynamic system is ideal white noise, then the response is a Markov process.

Another classification of random processes is made according to the width of the band of its spectrum. A process is called a narrow band process if its spectral density function occupies only a narrow band of frequencies. A broad band process is one whose spectral density function occupies a broad band of frequencies. Normal random processes are a special class of random processes. They are important because they are closed with respect to linear operations and they satisfactorily describe many real phenomena.

Under random vibrations theory, structural failure can be characterized as either first passage failure or fatigue failure. When a particular response quantity oversteps a certain level, it is said that first passage failure has occurred. On the other hand, when the accumulation of small amount of damages inflicted throughout the life of the structure reaches a certain value, it is said that fatigue failure has occurred.

The interest of this study is in first passage failure. Among the first studies in this area were those of Rice [9]. His pioneering work developed in 1944 and 1945 has formed the basis for the most intensive continuing studies of stochastic process theory in general, and first passage problem in particular. Since then, this problem has been under investigation by many authors. Except for some simple cases, no exact solution has yet been found.

Because of the importance of this problem in the study of reliability of structures in random environment, considerable effort has been made to approach the problem in approximate sense both by analysis and simulation. The case that has been studied is that of a linear oscilla-

ter with constant deterministic barrier [10-17]. This problem can be posed under various kinds of safety regions and initial conditions.

Two different types of safety regions are usually looked at: the one sided barrier problem, or the double-sided barrier. In many cases the system can be assumed to be at rest in contrast to random initial conditions. It is observed that for random initial conditions the first passage probability density falls off significantly within the first few cycles of response. After the transient state has died out, the first passage probability becomes independent of the initial conditions. Another important point is that higher damping ratios or shorter correlation times yield higher first passage probabilities.

Some of the methods that are used to approach the first passage probability of the response of a linear oscillator are discussed now. A method that is used by many authors is to approximate the response process by a first order Markov process for which the first passage probability can in principle be derived. This approximation leads to the Fokker-Planck partial differential equation which governs the behavior of the first passage probability. Gray [10] used a different response variable to approximate the problem as a first order Markov process. He obtained upper bounds on the original problem by means of Laplace transforms. Ariaratnam and Pi [11] approximated the amplitude of the response by a one dimensional Markov process based on a method of stochastic averaging due to Stratonovich. Their argument is based on the assumption that the amplitude and the phase of the response change slowly with respect to time. Their results agree with those of Gray. A rigorous treatment of this problem is presented by Spanos [12]. He used

separation of variables to obtain a series solution of the Fokker-Planck equation.

In the context of the point process approach, Lin [13] considered level crossing rates to constitute a continuous random process. Assuming various models for its distribution and correlation structure, he derived approximate first passage probabilities. Using the same concept, Yang and Shinozuka [14,15] showed a number of useful approximations for the first passage problem of stationary narrow-band Gaussian processes, particularly those based on the concepts of the Markov process, the clump size, and the nonapproaching random points. They concluded that the Markov approximation is conservative and the clump size approximation is nonconservative. The approximation based on the nonapproaching random points is found to be the best among all because it falls between the other two approximations.

Crandall et al. [16] presented two numerical methods to examine the first passage probability. One method is to simulate a large number of sample functions of the system response on a computer, and to compute the sample distribution of the random time at which the barrier is crossed. Another procedure is to use the diffusion law of Markov processes to distribute the probability mass in the phase plane of the response process and its velocity process which has not passed out the safety region for every consecutive short time interval. Then the rate of decrease of the probability mass within the safety domain is the probability density of the first passage time. Vanmarcke [18] treated the more general problem of first passage, where he assumed all first passage probability estimates are of the exponential type.

In addition to the technical papers, there are some texts that deal with this subject. Crandall and Mark [19], Newland [20], and Lin [21] provided some introductory discussion to this subject. Cramer and Leadbetter [22], Vanmarke [23], and Nigam [24] gave deeper and more systematic description of the first passage problem.

Chapter 3

Response to Random Input

The structure is a long hollow, layered, viscoelastic cylinder with an axisymmetric configuration and is shown in Fig. 3.1. The physical and geometric parameters of this structure are given in table 3.1. To obtain the thermal stresses induced at certain points in the structure, one needs to know the temperature distribution at the corresponding points. Hence the first task is to determine the temperature at the instant at which the stresses are desired, and the second one is to compute the stresses themselves. To obtain the temperature distribution, hourly temperature records were obtained from the National Oceanic and Atmospheric Administration for different locations in the United States. Some of the limitations that have been assumed throughout this study include that the temperature is symmetric with respect to the axis of the cylinder, and the outside surface temperature is the same as the air temperature. These are not quite accurate because the surface temperature is a function of thermal radiation and wind. It is also a function of the angle of exposure to the sun.

The variations in the outside temperature have some significant cyclic trends. The spectrum of the outside temperature shows predominant peaks at the daily and yearly frequencies. At the arctic and antarctic sites, the daily peaks are absent altogether. To model such temperature records as a stationary random process, it is best to extract the harmonic trends from these records. The temperature noise which remains after the extraction of the mean and the cyclic components is used to obtain the spectral density function of the random process [7]. Spec-

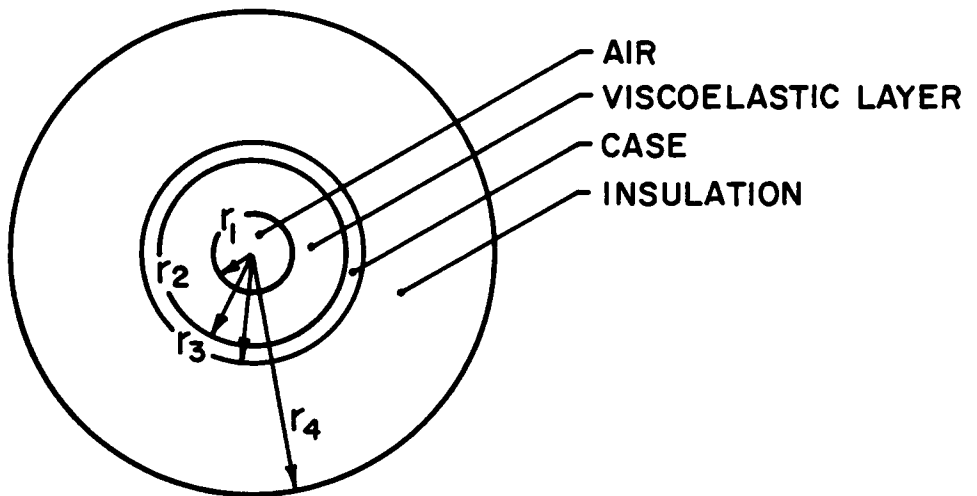


Figure 3.1 Configuration of the Structure.

TABLE 3.1
Geometrical and Physical Properties

	Layer 1	Layer 2	Layer 3	Layer 4
Radius, in (m)	1.875 (.048)	4.3 (.109)	4.4 (.112)	6.5 (.165)
Thermal conductivity, Btu/hr-ft-°F (Wat/m-°C)	.015 (.025)	.764 (1.322)	14.6 (25.25)	.016 (.027)
Diffusivity, ft ² /hr (m ² /sec)	.763 (.197 E-04)	.030 (.785 E-06)	.34 (.877 E-05)	8.575 E-03 (2.214 E-07)
Coeff. of Thermal Expansion, in/in/°F (m/m/°C)		5.78 E-05 (1.04 E-04)	6.5 E-05 (1.17 E-05)	
Modulus of Elasticity, Psi (N/m ²)		281.84 (1.943 E+06)	30 E+06 (2.068 E+11)	
Poisson's Ratio		.49	.25	

Stress Free Temperature,
°F (°C)

165 (74)

Damage Parameters,
Stress in Psi (N/m²)

B = 8.75
C = 1.421 E+16 (5.491 E+49)

Modulus Aging Parameters,
Temperature in °R (°K)

A = 4.1 E+05
B = 8.73 E+03 (4.85 E+03)

Strength Aging Parameters,
Temperature in °R (°K)

A = 1.15 E+10
B = 1.53 E+04 (8.53 E+03)

tral density functions obtained by this approach are shown in Figs. 3.2 and 3.3. Here, the probabilistic variation of the problem is attributed to the noise alone. In this case, the variance of the noise is obtained by integrating over the whole range of frequencies.

In previous work [1-5], the probabilistic variation was attributed to the daily and the yearly cycles, and the noise. The deterministic component was just due to the mean temperature. The spectral density function was considered to consist of two peaks at the daily and yearly cycles as shown in Figs. 3.4 and 3.5. The variance in this case is the sum of the daily and yearly variances. These are obtained by just integrating under these peaks. It is found that most of the area, 88%, of the power spectrum is under these two peaks [3]. Therefore this power spectrum was modeled as a superposition of two narrow band processes. In this study, the first passage analysis is applied to both approaches.

3.1 Frequency Response Function

The method of solution requires the determination of the frequency response function which is defined as the response to a sinusoidal input of unit amplitude. The Fourier heat conduction equation for a long axisymmetric layered cylinder is given by

$$\frac{\partial^2 T_j}{\partial r^2} + \frac{1}{r} \frac{\partial T_j}{\partial r} = \frac{1}{\alpha_j} \frac{\partial T_j}{\partial t} \quad (3.1)$$

The temperature of the j^{th} layer, $T_j(r,t)$, is a function of both the radial coordinate, r , and time t ; with α_j the thermal diffusivity of the j^{th} layer. The boundary conditions for a four layer cylinder are:

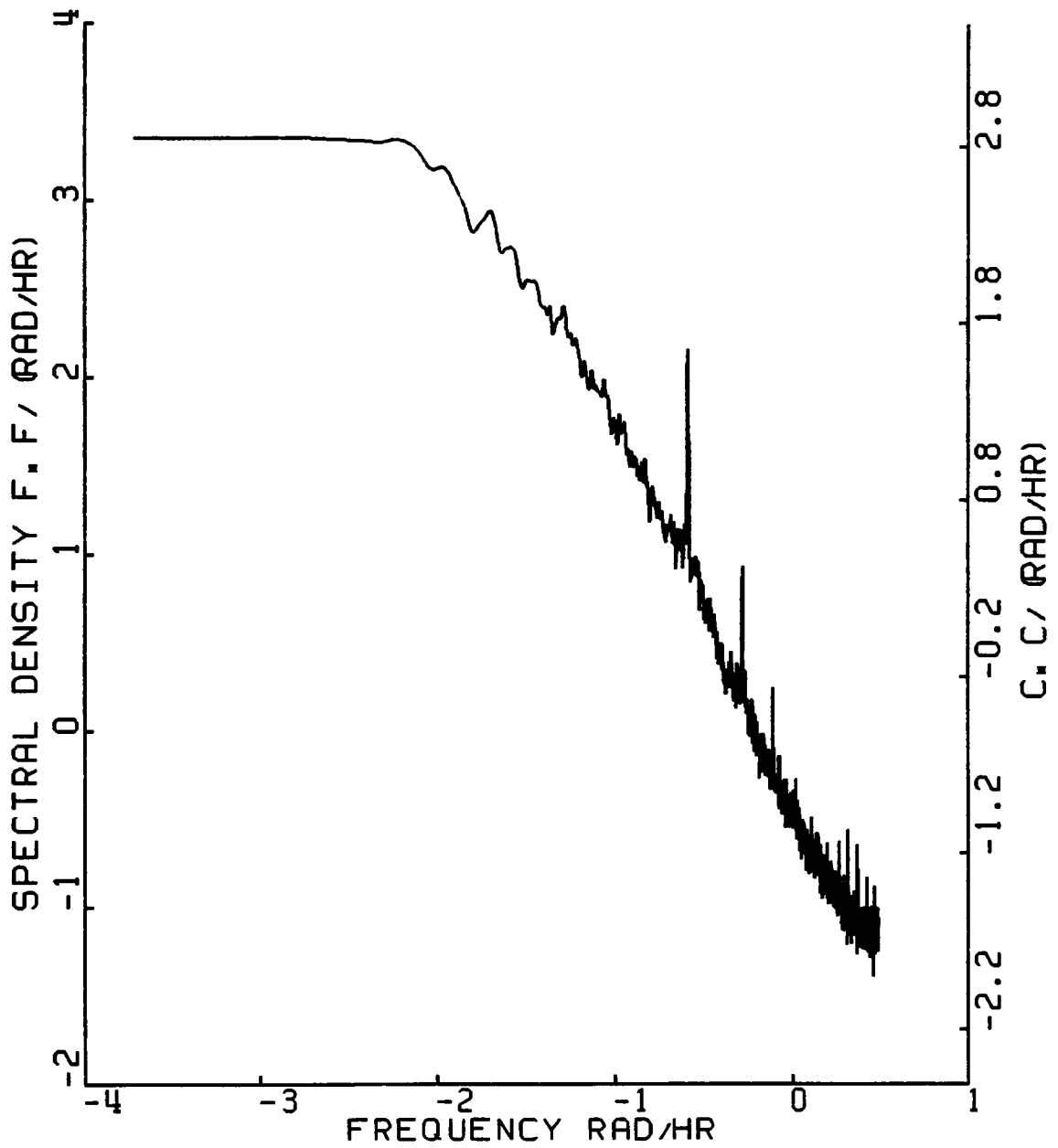


Figure 3.2 Power Spectrum for Barrow, AK, Approach I.

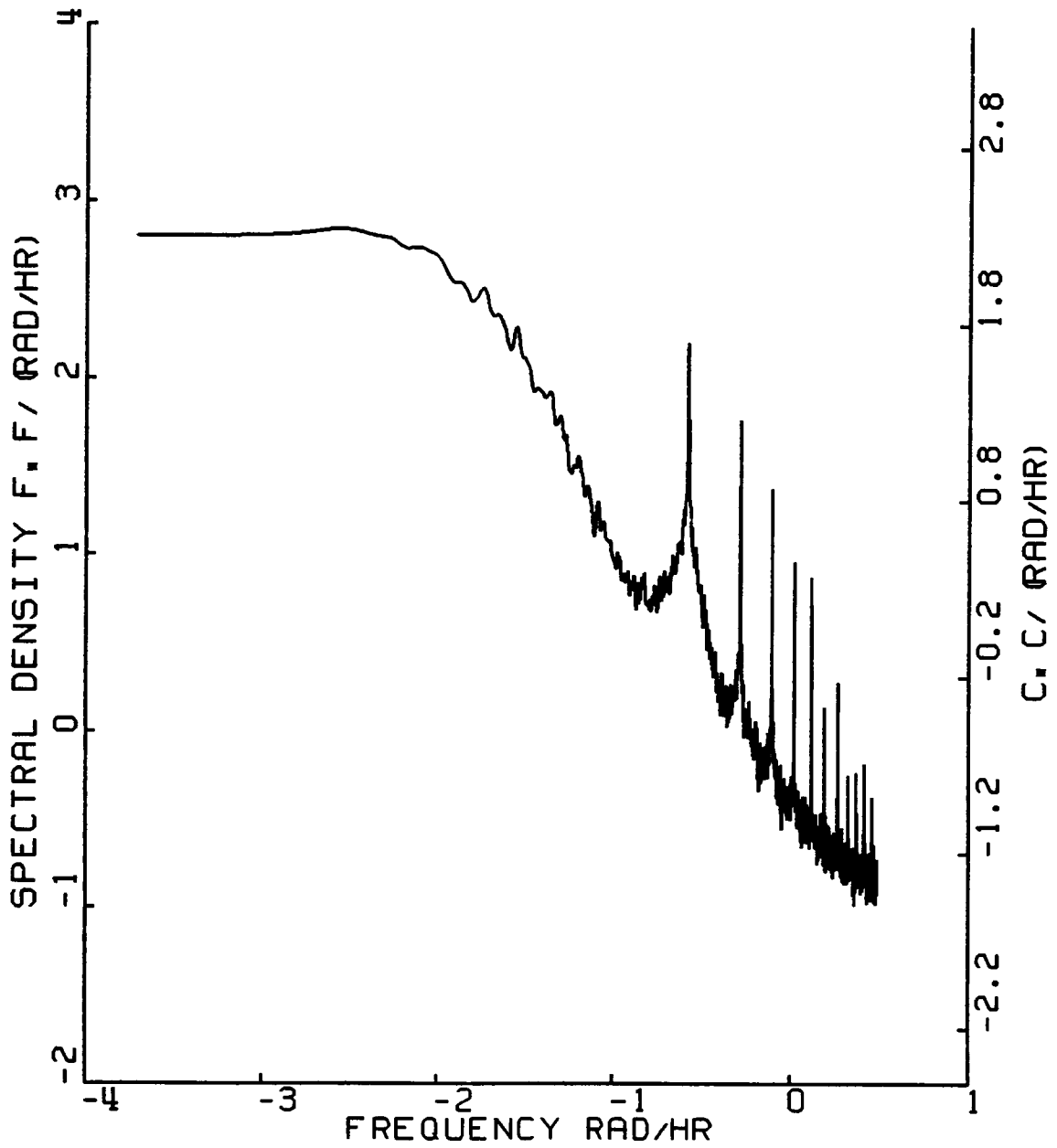


Figure 3.3 Power Spectrum for Yuma, AZ, Approach I.

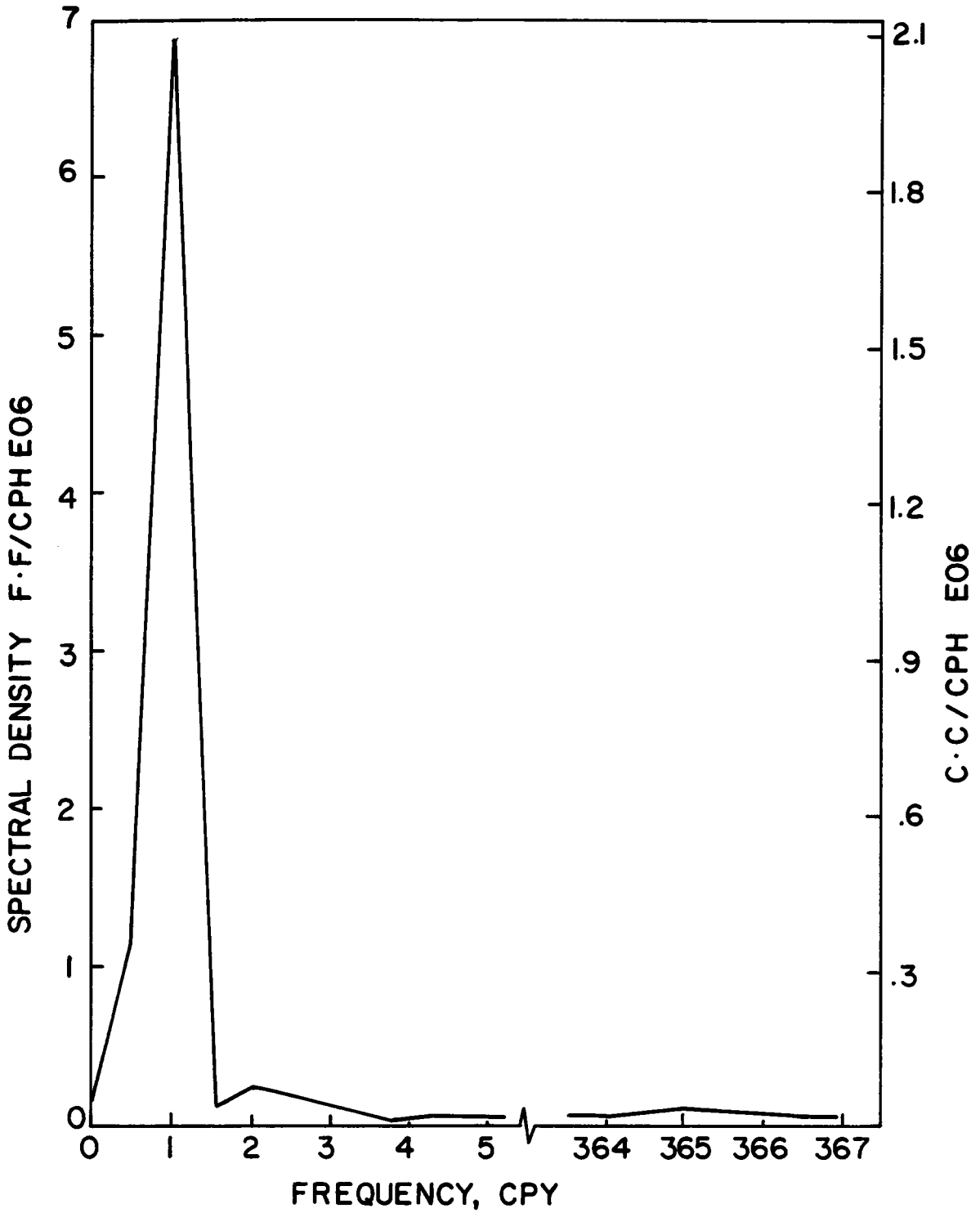


Figure 3.4 Power Spectrum for Barrow, AK, Approach II.

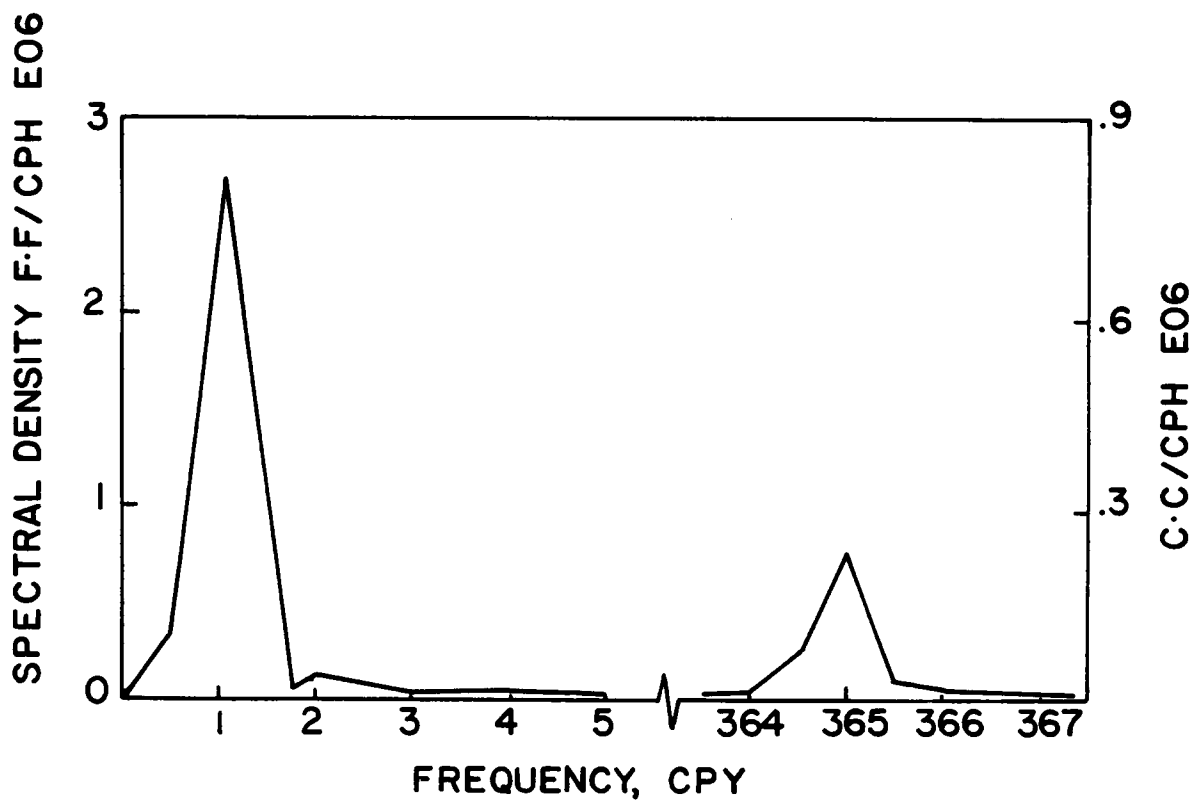


Figure 3.5 Power Spectrum for Yuma, AZ, Approach II.

B.C. 1 : $T_1(0,t)$ is finite.

B.C. 2,3,4 : $T_j(r,t) = T_{j+1}(r,t)$, temperatures are the same on both sides of an interface.

B.C. 5,6,7 : $k_j \frac{\partial T_j(r,t)}{\partial r} = k_{j+1} \frac{\partial T_{j+1}(r,t)}{\partial r}$, the heat flux

across an interface is continuous.

B.C. 8 : $T_4(r,t) = e^{i\omega t}$, the temperature on the surface of the fourth layer varies sinusoidally with a unit amplitude and frequency ω .

A detailed treatment of this B.V.P. can be found in Ref. 2. The solution is found to be

$$T_j(r,t) = |R_j(r,\omega)| \exp[i(\phi_j + \omega t)] \quad (3.2)$$

where ϕ_j is the phase angle and $R_j(r,\omega)$ is the frequency response function given in terms of complex Kelvin functions. It is observed that the temperature at an interior point in the structure is delayed and attenuated in comparison to the surface temperature. In addition, the slow variation of the yearly cycles affects the interior of the structure while the fast variation of the daily cycles affects only the surface layer.

Using the frequency response function, the output power spectral density, $G_{j0}(r,\omega)$, can be written in terms of the input power spectral density $G_{in}(\omega)$ as

$$G_{j0}(r, \omega) = |R(r, \omega)|^2 G_{in}(\omega) \quad (3.3)$$

The mean square value of the temperature at an internal point can be given as

$$E[T_j^2(r)] = \int_{-\infty}^{\infty} G_{j0}(r, \omega) d\omega \quad (3.4)$$

3.2 Thermal Stresses and Strains

The mean square values of the thermal stresses and strains may be given as

$$E[S_k^2(r, \omega)] = \int_{-\infty}^{\infty} |S_k(r, \omega)|^2 G_{in}(\omega) d\omega \quad (3.5)$$

$$E[\epsilon_k^2(r, \omega)] = \int_{-\infty}^{\infty} |\epsilon_k(r, \omega)|^2 G_{in}(\omega) d\omega \quad (3.6)$$

where $S_k(r, \omega)$ and $\epsilon_k(r, \omega)$ are the frequency response functions of radial or tangential stresses and strains. For completeness, the development of the frequency response functions is repeated in Appendix A. These functions can be written as

$$S_r(r, \omega) = -\frac{r_2^2 p'}{r_2^2 - r_1^2} \left(1 - \frac{r_1^2}{r^2}\right) + \frac{\bar{\alpha}_2 E_2}{(1 - \nu_2)(r_2^2 - r_1^2)}$$

$$\times \left(1 - \frac{r_1^2}{r^2}\right) \int_{r_1}^{r_2} R_2(r, \omega) r dr - \frac{\bar{\alpha}_2 E_2}{(1 - \nu_2) r^2} \int_{r_1}^r R_2(r, \omega) r dr \quad (3.7)$$

$$S_\theta(r, \omega) = - \frac{r_2^2 p'}{r_2^2 - r_1^2} \left(1 + \frac{r_1^2}{r^2}\right) + \frac{\bar{\alpha}_2 E_2}{(1 - \nu_2)(r_2^2 - r_1^2)}$$

$$\times \left(1 + \frac{r_1^2}{r^2}\right) \int_{r_1}^{r_2} R_2(r, \omega) r dr + \frac{\bar{\alpha}_2 E_2}{(1 - \nu_2) r^2} \int_{r_1}^r R_2(r, \omega) r dr$$

$$- \frac{\bar{\alpha}_2 E_2 R_2(r, \omega)}{1 - \nu_2} \quad (3.8)$$

$$\varepsilon_r(r, \omega) = - \frac{(1 + \nu_2) r_2^2 p'}{E_2 (r_2^2 - r_1^2)} \left[(1 - 2\nu_2) - \frac{r_1^2}{r^2} \right] + \frac{(1 + \nu_2) \bar{\alpha}_2}{(1 - \nu_2)(r_2^2 - r_1^2)}$$

$$\times \left[(1 - 2\nu_2) - \frac{r_1^2}{r^2} \right] \int_{r_1}^{r_2} R_2(r, \omega) r dr - \frac{\bar{\alpha}_2 (1 + \nu_2)}{(1 - \nu_2) r^2} \int_{r_1}^r R_2(r, \omega) r dr$$

$$+ \frac{1 + \nu_2}{1 - \nu_2} \bar{\alpha}_2 R_2(r, \omega) \quad (3.9)$$

$$\varepsilon_\theta(r, \omega) = - \frac{(1 + \nu_2) r_2^2 p'}{E_2 (r_2^2 - r_1^2)} \left[(1 - 2\nu_2) + \frac{r_1^2}{r^2} \right] + \frac{(1 + \nu_2) \bar{\alpha}_2}{(1 - \nu_2)(r_2^2 - r_1^2)}$$

$$\times \left[(1 - 2\nu_2) + \frac{r_1^2}{r^2} \right] \int_{r_1}^{r_2} R_2(r, \omega) r dr + \frac{(1 + \nu_2) \bar{\alpha}_2}{(1 - \nu_2) r^2} \int_{r_1}^r R_2(r, \omega) r dr \quad (3.10)$$

$$p' = \frac{E_2[2\bar{\alpha}_2(1+\nu_2) \int_{r_1}^{r_2} R_2(r, \omega) r dr - \bar{\alpha}_3(1 + \nu_3)(r_2^2 - r_1^2)R_2(r_2, \omega)]}{(1+\nu_2)[(1-2\nu_2)r_2^2 + r_1^2] + (E_2/E_3)(1-\nu_3^2)r_2[(r_2^2 - r_1^2)/(r_3 - r_2)]} \quad (3.11)$$

The frequency response function of the tangential stress at the bore of the hollow cylinder is shown in Fig. 3.6. Eq. 3.5 yields an important quantity, its square root is the standard deviation of the noise induced thermal stresses.

While the amplitudes of a wide band process (white noise) follow the Normal distribution, the amplitudes of a narrow band process follow the Rayleigh distribution with the following density function

$$f_A(a) = \frac{a}{\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2\right) \quad a > 0 \quad (3.12)$$

$$= 0 \quad a < 0$$

where a is a value of the amplitude A , and σ is either σ_D or σ_Y . The density functions of the Normal and Rayleigh distributions are shown in Figs. 3.7 and 3.8, respectively. The mean value of the Rayleigh distribution can be obtained by writing

$$E[a] = \int_0^{\infty} \frac{a^2}{\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2\right) da \quad (3.13)$$

Using the definite integral formula

$$\int_0^{\infty} x^n \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \frac{(\sqrt{2}\sigma)^{n+1}}{2} \Gamma\left(\frac{n+1}{2}\right) \quad (3.14)$$

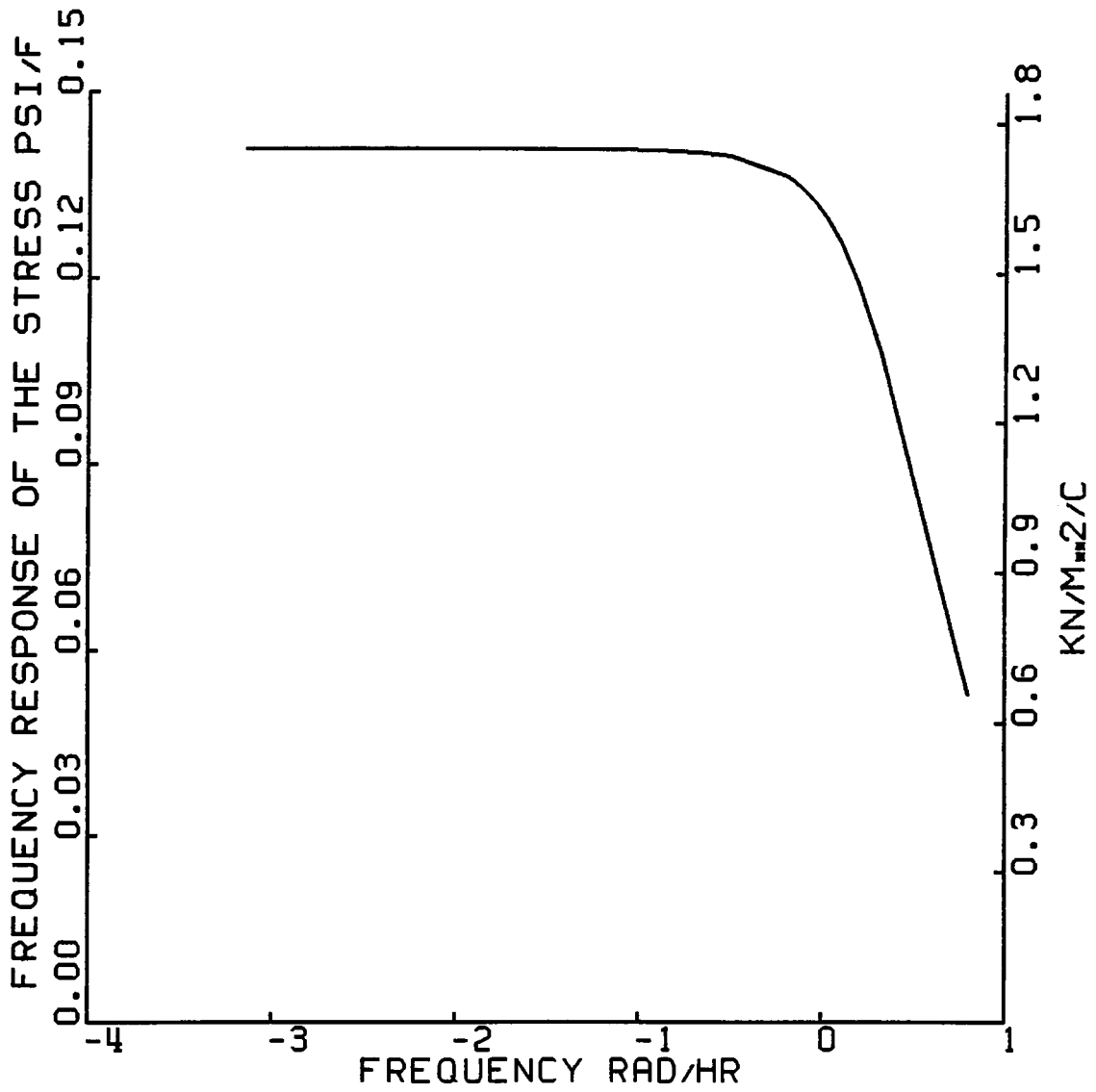


Figure 3.6 Frequency Response Function of the Tangential Stress at the Bore.

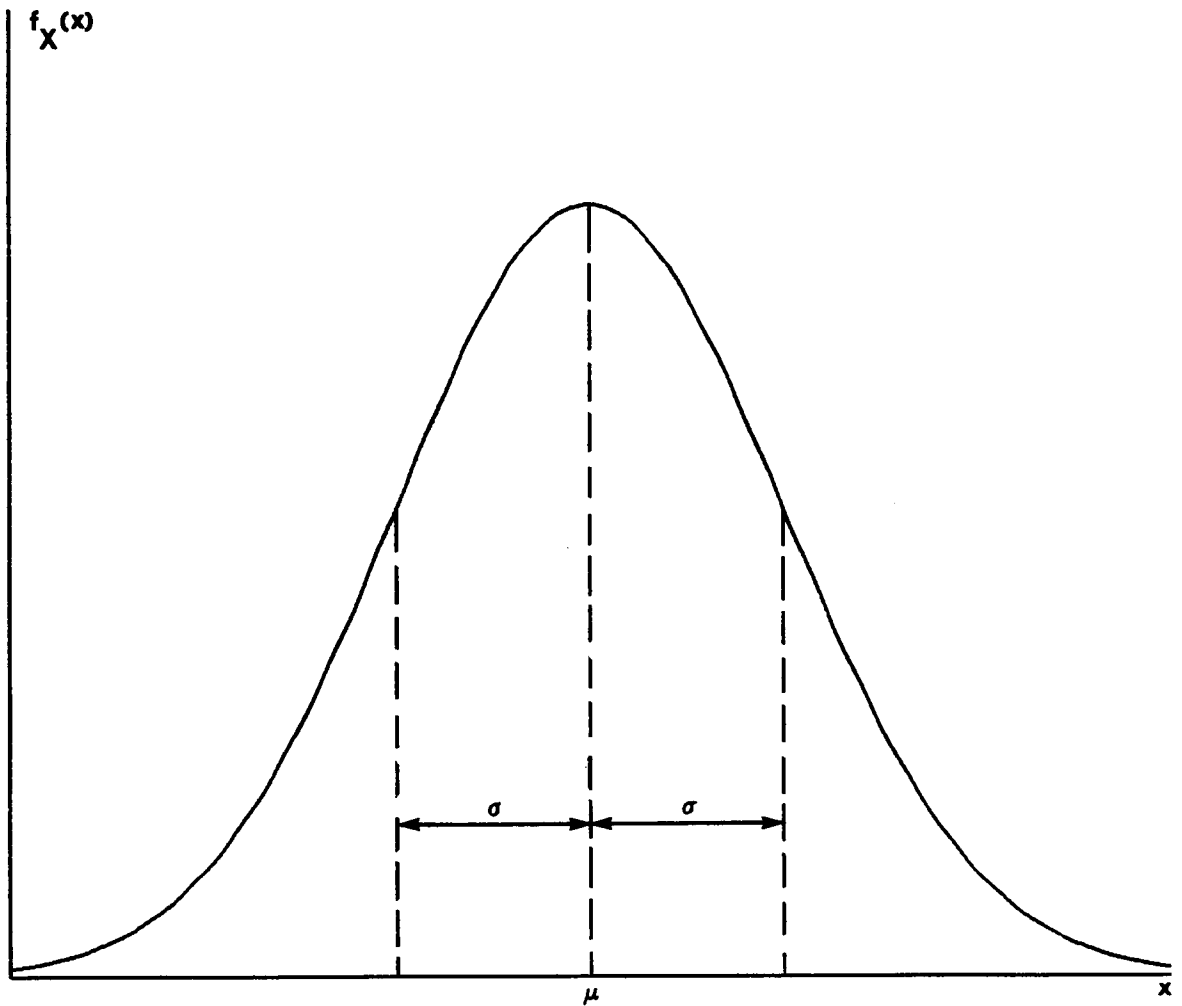


Figure 3.7 Probability Density Function of the Normal Distribution.

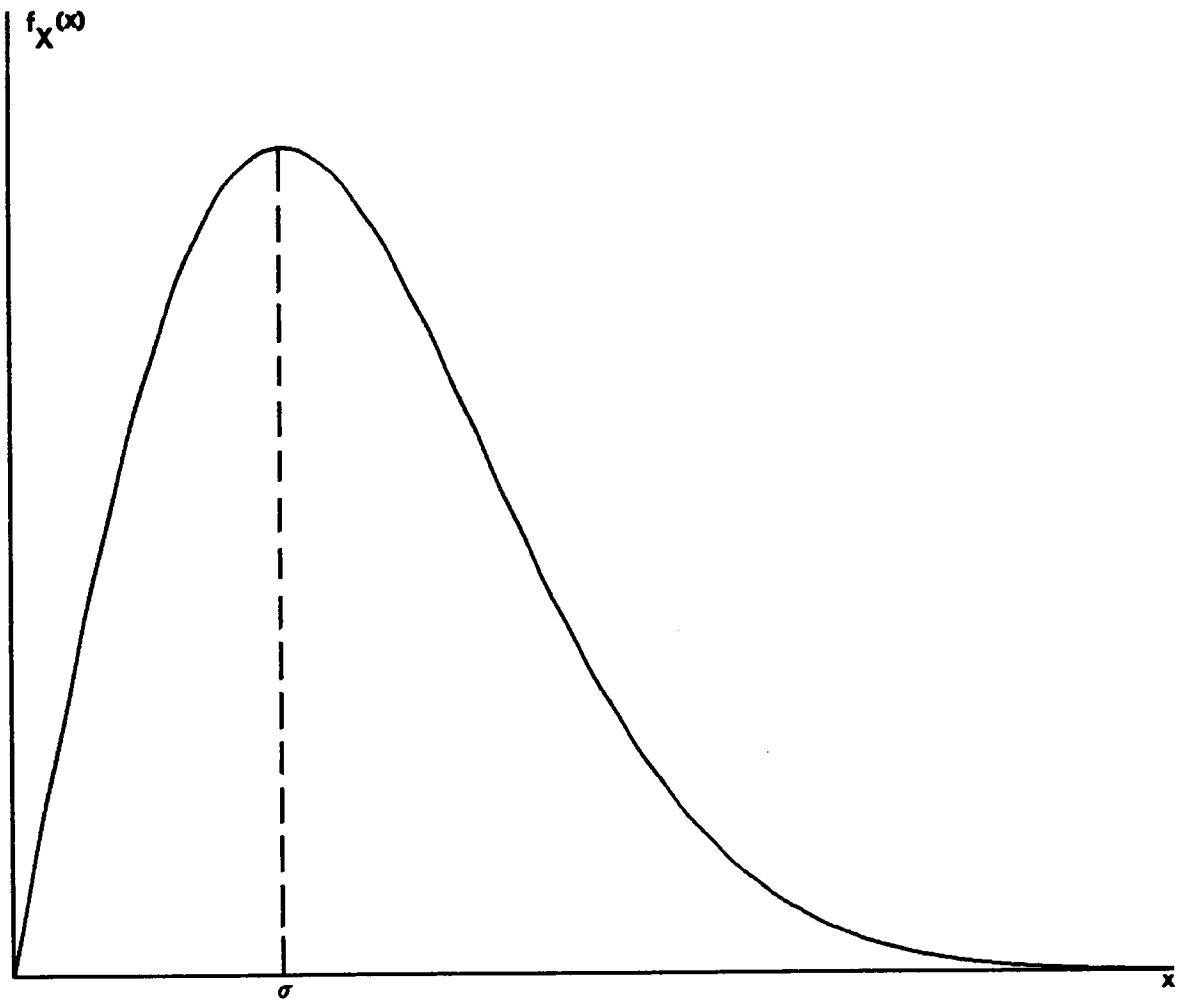


Figure 3.8 Probability Density Function of the Rayleigh Distribution.

The mean value of the Rayleigh distributed variable becomes

$$E[a] = \sqrt{\pi/2} \sigma \approx 1.25 \sigma \quad (3.15)$$

The mean square value of the Rayleigh distributed variable can be obtained by evaluating the integral

$$E[a^2] = \int_0^{\infty} \frac{a^3}{\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{a}{\sigma}\right)^2\right) da \quad (3.16)$$

Using Eq. 3.14, the integral in Eq. 3.16 is evaluated as

$$E[a^2] = 2\sigma^2 \quad (3.17)$$

The variance of a is written as

$$\sigma_a^2 = E[a^2] - E^2[a] \quad (3.18)$$

Substituting Eqs. 3.15 and 3.17 into Eq. 3.18, the variance of a can now be given as

$$\sigma_a^2 = \left(2 - \frac{\pi}{2}\right) \sigma^2 \approx 0.43 \sigma^2 \quad (3.19)$$

Thus the mean of the amplitudes of the daily and yearly cyclic stresses become

$$S_d \approx 1.25 \sigma_D \quad (3.20)$$

$$S_y \approx 1.25 \sigma_Y \quad (3.21)$$

while the standard deviations of the daily and yearly stresses are

$$\sigma_d \approx 0.65 \sigma_D \quad (3.22)$$

$$\sigma_y \approx 0.65 \sigma_Y \quad (3.23)$$

In addition to the above stresses and strains, there are the uniform ones that are produced by the difference between the stress free temperature, T_f , and the yearly mean temperature, μ_y . These are obtained by substituting

$$\mu_T = - (T_f - \mu_y) \quad (3.24)$$

in place of $R_2(r, \omega)$ in Eqs. 3.7-3.11. The uniform stresses and strains are written as

$$\mu'_{S_r}(r) = - \frac{r_2^2 \mu_p}{r_2^2 - r_1^2} \left(1 - \frac{r_1^2}{r^2}\right) \quad (3.25)$$

$$\mu'_{S_\theta}(r) = - \frac{r_2^2 \mu_p}{r_2^2 - r_1^2} \left(1 + \frac{r_1^2}{r^2}\right) \quad (3.26)$$

$$\mu'_{\epsilon_r}(r) = - \frac{(1 + \nu_2)r_2^2\mu_p}{E_2(r_2^2 - r_1^2)} \left[(1 - 2\nu_2) - \frac{r_1^2}{r^2} \right] + (1 + \nu_2)\bar{\alpha}_2\mu_T \quad (3.27)$$

$$\mu'_{\epsilon_\theta}(r) = - \frac{(1 + \nu_2)r_2^2\mu_p}{E_2(r_2^2 - r_1^2)} \left[(1 - 2\nu_2) + \frac{r_1^2}{r^2} \right] + (1 + \nu_2)\bar{\alpha}_2\mu_T \quad (3.28)$$

with

$$\mu_p = \frac{\{E_2(r_2^2 - r_1^2)[\bar{\alpha}_2(1 + \nu_2) - \bar{\alpha}_3(1 + \nu_3)]\mu_T\}}{(1 + \nu_2)[(1 - 2\nu_2)r_2^2 + r_1^2] + E_2/E_3(1 - \nu_3^2)r_2[(r_2^2 - r_1^2)/(r_3 = r_2)]} \quad (3.29)$$

3.3 Induced Thermal Stresses

The induced thermal stresses possess a similar trend as that of the temperature. From the above analysis, the induced stress, $S(t)$, at any hour can be modeled as

$$S(t) = \mu(t) + S_y \cos \omega_y(t - t_y) + S_d \cos \omega_d(t - t_d) + w(t) \quad (3.30)$$

where ω_y and ω_d are yearly and daily frequencies in radians per hour, respectively; $t_y = 20^{\text{th}}$ day of the year and $t_d = 6^{\text{th}}$ hour of the day are the time at which the stress is expected to be a maximum; and $w(t)$ is the random noise. The subscript k is dropped from Eq. 3.30 because this study is concerned with the critical stress, the tangential component at the bore of the hollow cylinder. The first three terms on the right hand side of Eq. 3.30 are taken to be deterministic quantities with all randomness contributed by the last term. The terms of Eq. 3.30 can be obtained as in either one of the following two approaches:

Approach I:

Here the induced stress consists of the following components:

- a) mean stress produced by the difference of the stress free temperature and the mean temperature as shown in Eqs. 3.25-3.29,
- b) a deterministic cyclic stress with a one per year frequency,
- c) a deterministic cyclic stress with a one per day frequency, and
- d) a probabilistic component characterized by a variance obtained as in Eq. 3.5 and a power spectral density as in either Fig. 3.2 or Fig. 3.3.

The amplitudes of the deterministic cycles are obtained by multiplying the absolute values of the frequency response function at the yearly and daily frequencies with the amplitudes of the cyclic temperature at the yearly and daily frequencies, respectively, while the phase angles are calculated from the ratios of the imaginary and real parts of the frequency response functions

$$\tan \phi = \text{Im } S(r, \omega) / \text{Re } S(r, \omega) \quad (3.31)$$

Approach II:

In this approach, the induced stress consists of the following components:

- a) mean stress component, the same as approach I,

- b) a cyclic stress with a one per year frequency and a Rayleigh distributed amplitude (Eq. 3.15),
- c) a cyclic stress with a one per day frequency and a Rayleigh distributed amplitude, and
- d) a probabilistic component characterized by a variance given as the sum of variances of the amplitudes of the yearly and daily cycles. This is written as

$$\sigma_w^2 = .43(\sigma_D^2 + \sigma_Y^2) \quad (3.32)$$

The mean values and standard deviations of the daily and yearly amplitudes are given in Eqs. 3.20-3.23.

3.4 Viscoelastic Analysis

3.4.1 Viscoelastic Modulus

The viscoelastic material is sensitive to time and temperature changes. This is characterized by a reduced-time dependent relaxation modulus and reduced-time dependent strength. The time dependent relaxation modulus for the propellant material is written in terms of a Prony series as

$$E_r(t) = E_\infty + \sum_{i=1}^n E_i e^{-t/\tau_i a_T} \quad (3.33)$$

where E_∞ is the equilibrium modulus, E_i and τ_i are the moduli and relaxation times of parallel Maxwell elements given in table 3.2, n is the

TABLE 3.2

Parameters of the Relaxation Modulus

$$E_{\infty} = 281.84 \text{ psi } (1.943 \text{ E}+06 \text{ N/m}^2)$$

i	E_i , psi (N/m^2)	τ_i , hrs
1	1.979 E+04 (1.364 E+08)	3.333 E-12
2	7.990 E+03 (5.509 E+07)	3.333 E-10
3	2.522 E+03 (1.739 E+07)	3.333 E-08
4	1.153 E+03 (7.947 E+06)	3.333 E-06
5	7.346 E+02 (5.065 E+06)	3.333 E-04
6	2.041 E+02 (1.497 E+06)	3.333 E-02
7	2.718 E+02 (1.874 E+06)	3.333 E 00
8	8.643 E-15 (5.959 E-16)	3.333 E 02

number of terms in the Prony series, and a_T is the viscoelastic shift function which can be written for the material under investigation as

$$\log a_T = 1.42 \left\{ \frac{-8.86(T + 34)}{181.9 + (T + 34)} + 3.32 \right\} \quad (3.34)$$

where T is the temperature in °F. The shift function is a basic property of the material and is determined experimentally. The viscoelastic stresses are obtained by replacing the elastic modulus with the relaxation modulus.

3.4.2 Viscoelastic Strength

The mean viscoelastic strength is represented by

$$\log \bar{R} = 2.1430 - 0.0857 \log (t/a_T) \quad (3.35)$$

where \bar{R} is the mean strength in psi and t is the time in minutes in which the strain attains its maximum value. Because Eq. 3.35 is based on experimental data, there is a coefficient of variation, δ_R , associated with it.

3.4.3 Cumulative Damage

The linear cumulative damage rule proposed by Palmgren and Miner states that damage produced in a unit of time spent at a particular stress level, S_i , is inversely proportional to the time, t_{f_i} , required to produce failure in the material at that stress level

$$d_i = 1/t_{f_i} \quad (3.36)$$

The total damage produced by n stress levels is written as

$$D = \sum_{i=1}^n t_i/t_{f_i} \quad (3.37)$$

where t_i is the duration for each stress level. A relationship that relates the applied constant stress and reduced time to failure is given as [5]

$$t_f/a_T = CS^{-B} \quad (3.38)$$

where C and B are material parameters given in table 3.1. The cumulative damage becomes

$$D = \frac{1}{C} \sum_{i=1}^n \frac{S_i^B t_i}{a_{T_i}} \quad (3.39)$$

The statistical variability of the damage is influenced by the parameter C . Thus a coefficient of variation on this parameter is assumed.

3.4.4 Aging

Aging is defined here as the change in the physical and thermal parameters occurring in the material in an unloaded condition. Aging effects are higher in warm climates and during summer periods. The parameters that are affected by aging are the modulus and the strength of the material. Thus the hardening or softening of the viscoelastic

material is governed by aging. The aging factor for a parameter, modulus or strength, is written as [25]

$$\eta(T,t) = 1 - \rho \log t \quad (3.40)$$

where

$$\rho = Ae^{-B/T} \quad (3.41)$$

in which A and B are material parameters given in table 3.1, and T is the absolute temperature. The aging factor for a material that is aged at temperature T for a period t_1 is written as

$$\eta_1 = 1 - \rho_1 \log t_1 \quad (3.42)$$

where

$$\rho_1 = Ae^{-B/T_1} \quad (3.43)$$

The equivalent time during which the same aging is produced at a different temperature is given as

$$t_1' = t_1^{\rho_1/\rho_2} \quad (3.44)$$

in which

$$\rho_2 = Ae^{-B/T_2} \quad (3.45)$$

The total aging time, t_2 , when aging is continued at T_2 for an additional time, Δt , becomes

$$t_2 = t_1' + \Delta t \quad (3.46)$$

and the aging factor now becomes

$$\eta_2 = 1 - \rho_2 \log t_2 \quad (3.47)$$

3.4.5 Variance of Residual Strength

The mean residual strength of the material can be written as

$$\bar{R}' = \bar{\eta}_R \bar{R}_0 (1 - \bar{D}) = \bar{R} (1 - \bar{D}) \quad (3.48)$$

where \bar{R}_0 is the mean virgin strength adjusted for viscoelastic effects, $\bar{\eta}_R$ is the average strength aging factor. The standard deviation of the residual strength can be obtained by writing

$$\sigma_{R'}^2 = E[R'^2] - \bar{R}'^2 \quad (3.49)$$

in which the expected value of R'^2 is approximated as

$$\begin{aligned} E[R'^2] &= E[R^2] \quad E[(1 - D)^2] \\ &= (\bar{R}^2 + \sigma_R^2)(1 - 2\bar{D} + \bar{D}^2 + \sigma_D^2) \end{aligned} \quad (3.50)$$

and

$$\bar{R}'^2 = \bar{R}^2(1 - 2\bar{D} + \bar{D}^2) \quad (3.51)$$

Substituting Eqs. 3.50 and 3.51 into Eq. 3.49 yields

$$\sigma_{R'}^2 = \bar{R}^2 \sigma_D^2 + \sigma_R^2(1 - 2\bar{D} + \bar{D}^2 + \sigma_D^2) \quad (3.52)$$

Chapter 4

First Passage Problem

As any other phenomena, any crossing of the borders of past experience into new and unknown areas is always associated with a certain risk of failure, sometimes with minor consequences, sometimes with catastrophic ones. This risk is present in the problem at hand. It is there when the thermal stress exceeds the strength. To evaluate this risk, first passage concept is utilized as the failure criterion. It is necessary here to have some information about such things as the barrier and crossing rates. Crossing rates are then used to obtain probabilistic information about the first passage time which is a measure of the reliability or quality of performance of the system.

4.1 Analysis of the Barrier

The model suggested for the barrier which in this problem is the strength of the material is as follows

$$b(t) = \bar{R}'(t) + q(t) \quad (4.1)$$

where $b(t)$ is a probabilistic quantity. The first term on the right hand side of Eq. 4.1, $\bar{R}'(t)$, is its deterministic component which is given in Eq. 3.48. The second term, $q(t)$, is an idealization of the random variation of strength and damage. It is assumed to be of the form

$$q(t) = a \cos(\tilde{\omega}t + \theta) \quad (4.2)$$

the frequency, $\tilde{\omega}$, in this expression is chosen to be the yearly frequency, ω_y . The function, $q(t)$, is regarded as a zero-mean Normal stochastic narrow band process with a variance ($\sigma_b^2 = \sigma_R^2$) given by Eq. 3.52. The phase angle, θ , is uniformly distributed over $(0, 2\pi)$ and the amplitude, a , is Rayleigh distributed as given by Eq. 3.12. A proof that $q(t)$ follows the Normal distribution is shown next. Eq. 4.2 is considered as a product of a and $q_1 = \cos(\tilde{\omega}t + \theta)$. Introducing the density function of θ

$$\begin{aligned} f(\theta) &= \frac{1}{2\pi} & 0 < \theta < 2\pi \\ &= 0 & \text{otherwise} \end{aligned} \quad (4.3)$$

Solving q_1 for θ

$$\theta = \cos^{-1} q_1 - \tilde{\omega}t \quad (4.4)$$

Differentiating Eq. 4.4

$$\frac{d\theta}{dq_1} = \frac{-1}{\sqrt{1 - q_1^2}} \quad (4.5)$$

Thus, the density function of q_1 can be written as

$$f_{Q_1}(q_1) = \frac{1}{\pi\sqrt{1 - q_1^2}} \quad -1 < q_1 < 1 \quad (4.6)$$

The density function of q now becomes

$$f_Q(q) = \int_{-\infty}^{\infty} \frac{1}{a} f_A(a) f_{Q_1}\left(\frac{q}{a}\right) da \quad (4.7)$$

Substituting the appropriate expressions in Eq. 4.7, Eq. 4.7 becomes

$$f_Q(q) = \frac{1}{\sigma_b^2 \pi} \int_q^\infty \frac{a}{\sqrt{a^2 - q^2}} \exp\left\{-\frac{a^2}{2\sigma_b^2}\right\} da \quad (4.8)$$

Simplifying and rewriting Eq. 4.8

$$f_Q(q) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{q^2}{2\sigma_b^2}\right\} \int_0^\infty \frac{e^{-v}}{\sqrt{v}} dv \quad (4.9)$$

Carrying out the integration in Eq. 4.9, yields the result

$$f_Q(q) = \frac{1}{\sqrt{2\pi} \sigma_b} \exp\left\{-\frac{q^2}{2\sigma_b^2}\right\} \quad (4.10)$$

Obviously, Eq. 4.10 is the Normal density function of q . The spectral density function of $q(t)$ consists of a single jump of size σ_b^2 at the frequency $\tilde{\omega}$. This is written as

$$G(\omega) = \delta(\omega - \tilde{\omega}) \sigma_b^2 \quad (4.11)$$

The reason for choosing the strength model of Eq. 4.2 is to take advantage of its velocity process as going to be apparent later. In light of Eqs. 3.30 and 4.1, a statement can be made about the probability of failure for the system based on the stress-strength interference principle:

$$P_f = P[S(t) > b(t)] \quad (4.12)$$

Eq. 4.12 is a probabilistic measure of the unreliability of the system. Substituting Eqs. 3.30 and 4.1 into Eq. 4.12, Eq. 4.12 can be written as

$$P_f = P[x(t) > y(t)] \quad (4.13)$$

where $y(t)$ is

$$y(t) = \bar{R}'(t) - \mu(t) - S_y \cos \omega_y (t-t_y) - S_d \cos \omega_d (t-t_d) \quad (4.14)$$

and $x(t)$ is

$$x(t) = w(t) - q(t) \quad (4.15)$$

Eq. 4.13 separates the deterministic components from the stochastic ones. In first passage terms, $y(t)$ can be regarded as the varying barrier (curve); and $x(t)$ as the stochastic process in question (see Fig. 4.1). This shows that the problem of crossing a random barrier by a stochastic process is reduced to that of crossing a curve by a different stochastic process. The variances of $x(t)$ and of its velocity process are given as

$$\sigma_x^2 = \sigma_w^2 + \sigma_b^2 \quad (4.16)$$

$$\sigma_{\dot{x}}^2 = \sigma_{\dot{w}}^2 + \omega_y^2 \sigma_b^2$$

These expressions are admissible because the noise and the strength are assumed to be independent, and so are their velocity processes. The

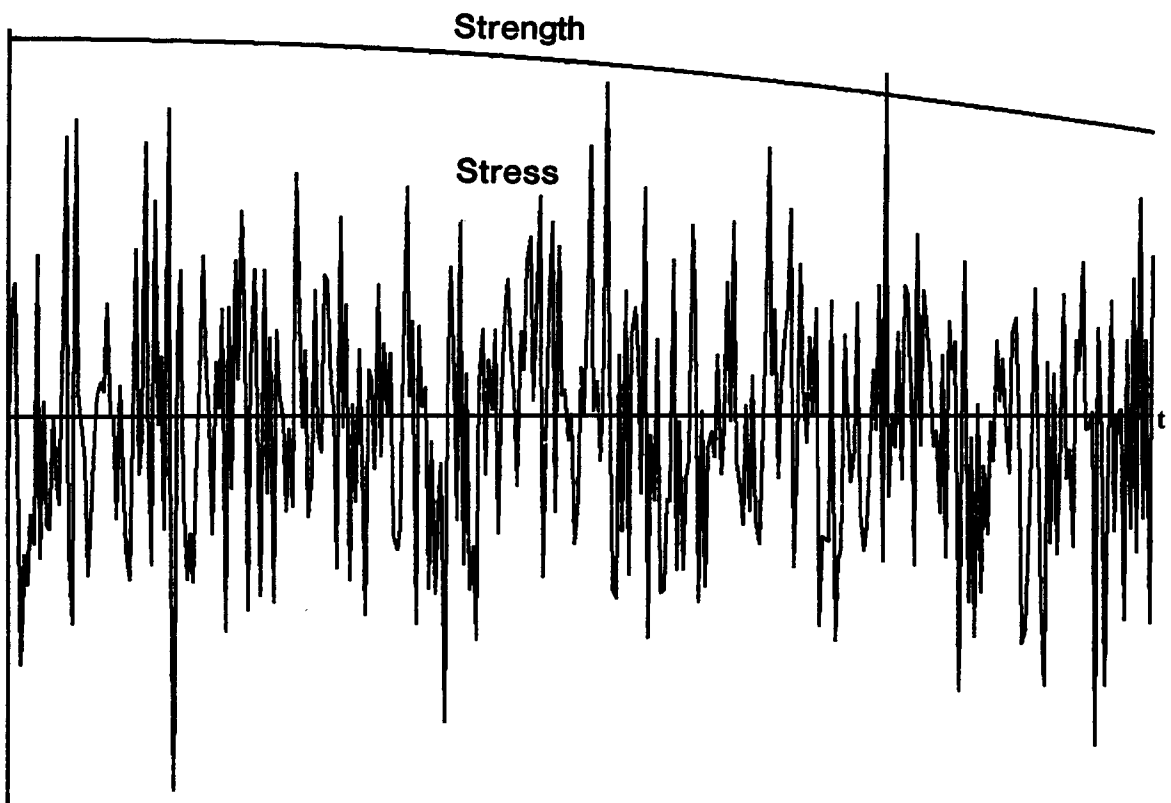


Figure 4.1 An Exaggeration of Eq. 4.12.

variance of the noise, σ_w^2 , can be written as

$$\sigma_w^2 = \int_{-\infty}^{\infty} |S_{\theta}(r_1, \omega)|^2 G_{in}(\omega) d\omega \quad (4.17)$$

where $S_{\theta}(r_1, \omega)$ is given in Eq. 3.8 and $G_{in}(\omega)$, for approach I, is as shown in either Figs. 3.2 or 3.3. For approach II, Eq. 4.17 reduces to that of Eq. 3.32. In this case, the Normal random noise is generated by means of a random generator with σ_w as its standard deviation. To obtain the variance of the velocity process of the noise, the autocorrelation function is given as

$$R_w(\tau) = E[w(t) w(t + \tau)] \quad (4.18)$$

Differentiating Eq. 4.18 twice with respect to τ , yields

$$\frac{d^2}{d\tau^2} R_w(\tau) = -R_w(\tau) \quad (4.19)$$

The autocorrelation function can also be written in terms of the spectral density function as

$$R_w(\tau) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega\tau} d\omega \quad (4.20)$$

Differentiating Eq. 4.20 twice with respect to τ to obtain

$$\frac{d^2}{d\tau^2} R_w(\tau) = - \int_{-\infty}^{\infty} \omega^2 G(\omega) e^{i\omega\tau} d\omega \quad (4.21)$$

Combining Eqs. 4.19 and 4.21, yields

$$R_{\dot{w}}(\tau) = \int_{-\infty}^{\infty} \omega^2 G(\omega) e^{i\omega\tau} d\omega \quad (4.22)$$

It is recalled here that the autocorrelation function of a zero mean process evaluated at $\tau = 0$ gives the variance of that process. Thus the variance of the velocity process of the noise is written as

$$\sigma_{\dot{w}}^2 = \int_{-\infty}^{\infty} \omega^2 G(\omega) d\omega \quad (4.23)$$

A general form of Eq. 4.23 is given as

$$\beta_k = \int_{-\infty}^{\infty} |\omega|^k G(\omega) d\omega \quad (4.24)$$

β_k is known as the k^{th} spectral moment of the random process. A parameter of interest that is associated with the spectral moments is written as

$$\beta = \left[1 - \frac{\beta_1^2}{\beta_0 \beta_2} \right]^{1/2} \quad (4.25)$$

The parameter, β , lies between zero and one, and it is an indication of the bandwidth of the spectral density. For approach I, the variance of the velocity process of the noise becomes

$$\sigma_{\dot{w}}^2 = \int_{-\infty}^{\infty} \omega^2 |S_{\theta}(r_1, \omega)|^2 G_{in}(\omega) d\omega \quad (4.26)$$

To obtain this variance for approach II, use is made of the fact that the noise is generated by computer as a Normal wide-band process. Hence, the slopes between each two points of $w(t)$ are written as

$$\dot{w}(t) = \frac{w(t+\Delta t) - w(t)}{\Delta t} \quad (4.27)$$

where $\Delta t = 1$ hour. Eq. 4.27 forms a new process, its variance is given by the area under its power spectrum. Another method of calculating this variance is obtained by writing

$$\begin{aligned} \sigma_{\dot{w}}^2 &= \frac{1}{(\Delta t)^2} \{2E[w^2(t)] - 2E[w(t)w(t+\Delta t)]\} \\ &= \frac{2}{(\Delta t)^2} \{\sigma_w^2 - R_w(\Delta t)\} \end{aligned} \quad (4.28)$$

For a wide band process and $\Delta t = 1$ hour, the autocorrelation function is concentrated around $t = 0$. Thus Eq. 4.28 can be approximated as

$$\sigma_{\dot{w}}^2 \approx 2\sigma_w^2 \quad (4.29)$$

The variance of the velocity process of $q(t)$ can be obtained from Eqs. 4.11 and 4.23 as

$$\begin{aligned} \sigma_{\dot{q}}^2 &= \int_{-\infty}^{\infty} \omega^2 \delta(\omega - \tilde{\omega}) \sigma_b^2 d\omega \\ &= \tilde{\omega}^2 \sigma_b^2 \end{aligned} \quad (4.30)$$

If $\tilde{\omega} = \omega_y$, Eq. 4.30 becomes as shown in the second of Eqs. 4.16.

4.2 Crossings of a Constant Barrier

The classical formula that was developed by Rice [24] for the mean rate of crossings of the level, $y(t) = c$, by a stationary stochastic process, $x(t)$, is given as

$$n(c) = \int_{-\infty}^{\infty} |\dot{x}| f_{x\dot{x}}(c, \dot{x}) d\dot{x} \quad (4.31)$$

where $f_{x\dot{x}}(x, \dot{x})$ is the joint density function of x and \dot{x} . If $x(t)$ is a stationary normal stochastic process, the crossing rate can be written as

$$n(c) = \frac{1}{\pi} \frac{\sigma_{\dot{x}}}{\sigma_x} \exp\left\{-\frac{c^2}{2\sigma_x^2}\right\} \quad (4.32)$$

where σ_x and $\sigma_{\dot{x}}$ are the standard deviations of $x(t)$ and of its velocity process, respectively.

A derivation of Eqs. 4.31 and 4.32 is shown next. A counting procedure for the number of crossings of the barrier, c , by the stochastic process, $x(t)$, can be obtained by means of the unit step function, $U[\cdot]$, [21]

$$Z_c = U[x(t)-c] \quad (4.33)$$

Eq. 4.33 assumes the values one and zero every time the level, c , is crossed with positive and negative slopes, respectively. Differentiat-

ing Eq. 4.33 with respect to t gives

$$Z_c(t) = \dot{x}(t)\delta[x(t)-c] = v(c,t) \quad (4.34)$$

where $\delta(\)$ is the Dirac delta function. Eq. 4.34 represents an impulse every time c is crossed. These relations are shown in Fig. 4.2. The expected total number of crossings in the interval T can be written as

$$\begin{aligned} N(c,T) &= \int_0^T E[v(c,t)] dt = \int_0^T E[|\dot{x}(t)|\delta(x(t)-c)] dt \\ &= \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{x}|\delta(x(t)-c)f_{x\dot{x}}(x,\dot{x},t) dx d\dot{x} dt \\ &= \int_0^T \int_{-\infty}^{\infty} |\dot{x}|f_{x\dot{x}}(c,\dot{x},t) d\dot{x} dt \\ &= \int_0^T n(c,t) dt \end{aligned} \quad (4.35)$$

where

$$n(c,t) = \int_{-\infty}^{\infty} |\dot{x}|f_{x\dot{x}}(c,\dot{x},t) d\dot{x} \quad (4.36)$$

Eq. 4.36 reduces to Eq. 4.31 if the process, $x(t)$, is stationary. To obtain Eq. 4.32, $x(t)$ and $\dot{x}(t)$ are uncorrelated because of stationarity. Moreover, $x(t)$ and $\dot{x}(t)$ are independent when $x(t)$ is Normal.

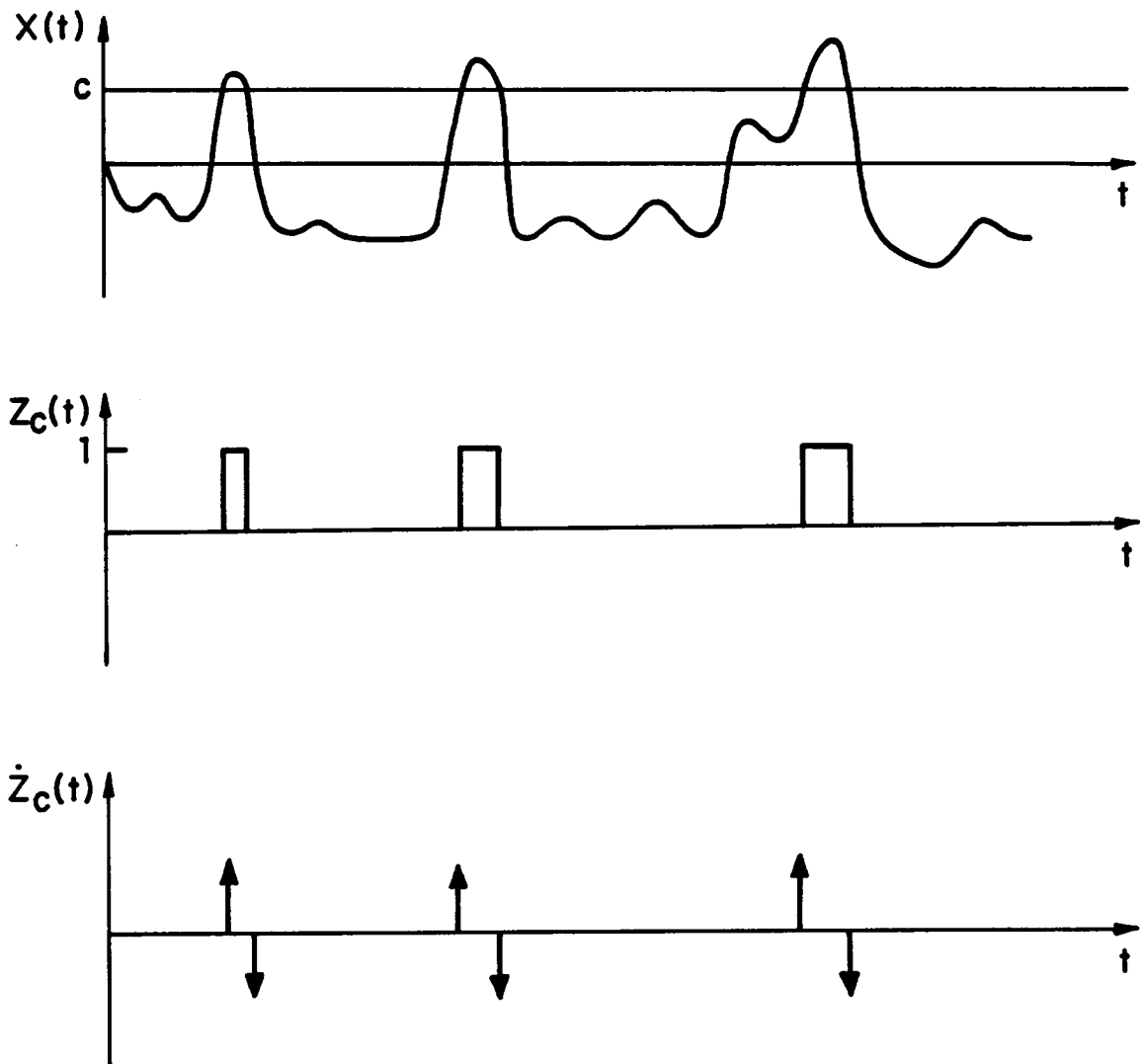


Figure 4.2 Crossings of a Constant Barrier.

Hence, Eq. 4.31 can be written as

$$n(c) = f_x(b)E[|\dot{x}|] \quad (4.37)$$

where $f_x(\cdot)$ is the normal density function given by

$$f_x(x) = \frac{1}{\sqrt{2\pi} \sigma_x} \exp\left\{-\frac{x^2}{2\sigma_x^2}\right\} \quad (4.38)$$

and

$$E[|\dot{x}|] = 2 \int_0^{\infty} \frac{\dot{x}}{\sqrt{2\pi} \sigma_{\dot{x}}} \exp\left\{-\frac{\dot{x}^2}{2\sigma_{\dot{x}}^2}\right\} = \sqrt{2/\pi} \sigma_{\dot{x}} \quad (4.39)$$

The result in Eq. 4.32 is obtained by substituting Eqs. 4.38 and 4.39 in Eq. 4.37.

Crossing rates calculated by Eq. 4.32 are acceptable as long as the barrier is constant. Whenever a time varying barrier is introduced, results obtained by Eq. 4.32 are, generally, not quite satisfactory. The reason is that Eq. 4.32 does not account for the rate of change of the barrier. A rigorous treatment of the crossing rate that accounts for this rate of change is shown next [22].

4.3 Crossing of a Curve

Based on the notation of Section 4.1, the crossing rate of a curve suggested here is written as

$$n(t) = \frac{1}{\sigma_x} \phi\left(\frac{y}{\sigma_x}\right) \left[2\sigma_{\dot{x}} \phi\left(\frac{\dot{y}}{\sigma_{\dot{x}}}\right) + \dot{y} \left\{ 2\Phi\left(\frac{\dot{y}}{\sigma_{\dot{x}}}\right) - 1 \right\} \right] \quad (4.40)$$

The validity of this relation will be examined in the following. In Eq. 4.40 $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cumulative functions of the normal distribution, respectively. A problem of curve crossings can often be regarded as zero crossings of a nonstationary process. Whether $x(t)$ is stationary or nonstationary, the number of times it crosses $y(t)$ is the same as the number of times the nonstationary process, $x^*(t) = x(t) - y(t)$, crosses the zero level. Hence to obtain Eq. 4.40, a nonstationary zero crossing problem is considered. Following the same procedure as in Section 4.2, Eq. 4.36 is written for $x^*(t) = 0$ as

$$n(0,t) = \int_{-\infty}^{\infty} |\dot{x}| f_{x\dot{x}}(0, \dot{x}, t) d\dot{x} \quad (4.41)$$

where $*$ is dropped for convenience. The function, $x(t)$, is now considered as a nonstationary random process with mean, $m(t)$, and $\dot{x}(t)$ is the velocity process with mean, $\dot{m}(t)$. The normal joint density function for $x(t)$ and $\dot{x}(t)$ in this case is given by

$$f_{x\dot{x}}(x, \dot{x}) = \frac{1}{2\pi\sigma_{\dot{x}}\sigma_x\sqrt{1-\rho^2}} \exp\left\{-\frac{\sigma_{\dot{x}}^2(x-m)^2 - 2\sigma_x\sigma_{\dot{x}}\rho(x-m)(\dot{x}-\dot{m}) + \sigma_x^2(\dot{x}-\dot{m})^2}{2\sigma_{\dot{x}}^2\sigma_x^2(1-\rho^2)}\right\} \quad (4.42)$$

where ρ is the correlation coefficient between x and \dot{x} . Substituting Eq. 4.42 into Eq. 4.41, Eq. 4.41 can be written as

$$n(0,t) = \frac{1}{2\pi\sigma_x\sigma_x\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} |x| \exp\left\{\frac{-1}{2\sigma_x^2(1-\rho^2)} \left(\frac{\sigma_x^2 m^2}{\sigma_x^2} + \frac{2\sigma_x \rho m(x-m)}{\sigma_x} + (x-m)^2\right)\right\} dx \quad (4.43)$$

Simplifying and rewriting Eq. 4.43

$$n(0,t) = \frac{e^{-m^2/2\sigma_x^2}}{2\pi\sigma_x\sigma_x\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} |x| \exp\left\{\frac{-1}{2\sigma_x^2(1-\rho^2)} (x^2 - 2g\sigma_x\sqrt{1-\rho^2} x + g^2\sigma_x^2(1-\rho^2))\right\} dx \quad (4.44)$$

results, where

$$g(t) = \frac{m - \sigma_x \rho m / \sigma_x}{\sigma_x \sqrt{1-\rho^2}} \quad (4.45)$$

Changing variables in Eq. 4.44, $n(0,t)$ becomes

$$n(0,t) = \frac{\sigma_x \sqrt{1-\rho^2}}{2\pi\sigma_x} e^{-m^2/2\sigma_x^2} \int_{-\infty}^{\infty} |u| \exp\left\{-\frac{1}{2} (u-g)^2\right\} du \quad (4.46)$$

with

$$u = \frac{x}{\sigma_x \sqrt{1-\rho^2}} \quad (4.47)$$

Carrying out the integration, Eq. 4.46 can be written as

$$n(0,t) = \frac{\sigma_x \sqrt{1-\rho^2}}{\sigma_x} \phi\left(\frac{m}{\sigma_x}\right) [2\phi(g) + g(2\phi(g)-1)] \quad (4.48)$$

Eq. 4.48 gives the rate of zero crossings of a nonstationary normal process. To obtain curve crossings from Eq. 4.48, $m(t)$ is replaced by $m(t)-y(t)$ and $\dot{m}(t)$ by $\dot{m}(t)-\dot{y}(t)$. The crossing rate of a constant barrier (Eq. 4.32) can also be obtained from Eq. 4.48 by substituting

$$\begin{aligned} m(t) &= -c \\ \rho(t) &= 0 \\ g(t) &= 0 \end{aligned} \quad (4.49)$$

Eq. 4.48 becomes

$$n(c) = \frac{2\sigma_x}{\sigma_x} \phi(0)\phi\left(\frac{c}{\sigma_x}\right) \quad (4.50)$$

which is identical with Eq. 4.32. The crossing rate of the curve, $y(t)$, by the process, $x(t)$, (Eq. 4.40) can be obtained from Eq. 4.48 by substituting

$$\begin{aligned} m(t) &= -y(t) \\ \rho(t) &= 0 \\ g(t) &= -\dot{y}(t)/\sigma_x \end{aligned} \quad (4.51)$$

Eqs. 4.32 and 4.40 give the total crossing rates. The up-crossing rates are obtained by dividing the expressions in these equations by 2. These equations account for the randomness of the barrier since this is embedded in the process $x(t)$. A different and more general treatment of the probabilistic variations of the barrier is shown in Section 4.6.

4.4 Poisson Process and First Passage Time

Poisson processes belong to the general class of counting processes which arise in problems concerning counting random events. A counting process is defined as an integer-valued process which counts the number of points distributed by some stochastic mechanism in an interval of time. In this study, crossings of the barrier are considered to occur according to a Poisson process. Other examples of Poisson processes are: incidence of earthquakes in a given region, number of vehicles crossing a bridge, number of customers arriving at a service station, etc. Poisson processes are of special interest because of their analytical simplicity and because the assumptions underlying their construction hold for many physical phenomena. These assumptions are:

1. Stationarity; the probability of an event in a short interval of time t to $t+\Delta t$ is approximately $n\Delta t$ for any t , where n is a positive constant known as the average rate.
2. Nonmultiplicity; the probability of more than one event in a short interval of time is negligible compared to $n\Delta t$.
3. Independence; the number of events in the future are independent of events in the past.

Under the above assumptions, the probability function for the Poisson Process is written as

$$P(m,T) = \frac{(nT)^m}{m!} \exp(-nT) \quad (4.52)$$

where m is the number of events in the interval $(0,T)$ and n is again the crossing rate. The probability of no events, or crossings, in the interval $(0,T)$ follows from Eq. 4.52 by letting $m = 0$. This becomes

$$P(0,T) = \exp(-nT) \quad (4.53)$$

If the first assumption is relaxed, Eq. 4.53 can be written as

$$P(0,T) = \exp(-\alpha_p(T)) \quad (4.54)$$

where

$$\alpha_p(T) = \int_0^T n(t) dt \quad (4.55)$$

Eqs. 4.53 and 4.54 are equivalent to the statement that the first passage time, T_f , is greater than T . The first passage time is defined here as the time at which $y(t)$ is crossed for the first time. The probability of first passage, which is the probability of at least one up crossing, in the interval $(0,T)$ is given by

$$F(T) = P[T_f < T] = 1 - \exp(-\alpha_p(T)) \quad (4.56)$$

The probability density of first passage time becomes

$$f(T) = n(T) \exp(-\alpha_p(T)) \quad (4.57)$$

This defines the Exponential density function. It describes the time to the first occurrence of a Poisson event. Though the Exponential distribution has the property of lack of memory, it is considered in the literature as a central approximation of the distribution of first passage time because many other solutions are measured by comparison to this approximation. Substituting Eq. 4.40 into Eq. 4.56, and replacing integration by summation $F(T)$ becomes

$$F(T) = 1 - \exp\left[- \sum_i^T \frac{1}{2\sigma_{x_i}} \phi\left(\frac{y_i}{\sigma_{x_i}}\right) \left[2\sigma_{x_i} \phi\left(\frac{y_i}{\sigma_{x_i}}\right) + y_i \left\{ 2\Phi\left(\frac{y_i}{\sigma_{x_i}}\right) - 1 \right\} \right] \right] \quad (4.58)$$

If Eq. 4.32 is substituted into Eq. 4.56

$$F(T) = 1 - \exp\left[- \sum_i^T \frac{1}{2\pi} \frac{\sigma_{x_i}}{\sigma_{x_i}} \exp\left(-\frac{y_i^2}{2\sigma_{x_i}}\right) \right] \quad (4.59)$$

results. Eqs. 4.58 and 4.59 give the first passage probabilities for varying and constant barriers, respectively.

4.5 Markov Process and First Passage Time

The crossing rates obtained so far become more accurate as the barrier increases [22]. This is true because crossings are rare events for high barriers. These crossing rates are utilized in this section as reference values with which some improvement on the first passage proba-

bilities is obtained. Vanmarcke [18] visualizes the process as passing randomly back and forth from state 0, below the barrier, to state 1, above the barrier. The successive time intervals T_0 and T_1 that the process spends below and above the barrier are assumed to be independent and exponentially distributed with rates ζ_0 and ζ_1 , respectively. The time $(T_0 + T_1)$ is the time between successive up-crossings. Its expected value is written in terms of either Eq. 4.32 or Eq. 4.40 as

$$E[T_0 + T_1] = \frac{1}{\bar{n}(\tau)} \quad (4.60)$$

$E[T_0]$ and $E[T_1]$ are found by applying the continuous Markov process theory. The probability of going from state k to state j is approximately given as

$$p_{k,j}(\Delta t) \approx \zeta_{k,j} \cdot (\Delta t) \quad k \neq j \quad (4.61)$$

and the probability that a process remains at state k is

$$p_{k,k}(\Delta t) = 1 - \zeta_k \cdot (\Delta t) \quad (4.62)$$

The probability that the process remains at state j at time $(t+\Delta t)$ can be written as

$$p_j(t+\Delta t) = \sum p_k(t) p_{k,j}(\Delta t) \quad (4.63)$$

Substituting Eqs. 4.61 and 4.62 into Eq. 4.63, yields

$$\frac{p_j(t+\Delta t) - p_j(t)}{\Delta t} = \sum_{k \neq j} p_k(t) \zeta_{k,j} - p_j(t) \zeta_j \quad (4.64)$$

Taking the limit as $\Delta t \rightarrow 0$, Eq. 4.64 becomes

$$\frac{dp_j(t)}{dt} = \sum_{k \neq j} p_k(t) \zeta_{k,j} - p_j(t) \zeta_j \quad (4.65)$$

The steady state probabilities, p_j^* , are obtained as

$$p_j^* \zeta_j = \sum_{k \neq j} p_k^* \zeta_{k,j} \quad (4.66)$$

For a two state process, Eq. 4.66 becomes

$$p_0^* \zeta_0 = p_1^* \zeta_{1,0} \quad (4.67)$$

and since

$$p_0^* + p_1^* = 1 \quad (4.68)$$

The steady state probabilities are written as

$$p_0^* = \frac{\zeta_1}{\zeta_0 + \zeta_1} \quad (4.69)$$

$$p_1^* = \frac{\zeta_0}{\zeta_0 + \zeta_1} \quad (4.70)$$

Eqs. 4.69 and 4.70 can be rewritten as

$$p_0^* = \frac{E[T_0]}{E[T_0 + T_1]} \quad (4.71)$$

$$p_1^* = \frac{E[T_1]}{E[T_0 + T_1]} \quad (4.72)$$

Eq. 4.72 expresses the long-run probability that the process spends above the barrier. This can also be expressed as

$$\begin{aligned} p_1^* &= \int_y^{\infty} f_x(x) dx \\ &= 1 - \Phi\left(\frac{y}{\sigma_x}\right) \end{aligned} \quad (4.73)$$

where $\Phi(\cdot)$ is the Normal distribution function. Eqs. 4.60, 4.72 and 4.73 yield

$$E[T_1] = \frac{1}{n(\tau)} \left(1 - \Phi\left(\frac{y}{\sigma_x}\right)\right) \quad (4.74)$$

$$E[T_0] = \frac{1}{n(\tau)} \Phi\left(\frac{y}{\sigma_x}\right) \quad (4.75)$$

In light of the above analysis, a modified expression for the first passage probability can be written as [18]

$$F(T) = 1 - L(0) \exp(-\alpha_m(T)) \quad (4.76)$$

where $L(0)$ is the probability that the process starts in the safe region. It is written as

$$L(0) = \frac{E[T_0]}{E[T_0 + T_1]} = \Phi\left(\frac{y}{\sigma_x}\right) \quad (4.77)$$

$\alpha_m(T)$ is given as

$$\alpha_m(T) = \int_0^T \zeta_0(t) dt \quad (4.78)$$

in which $\zeta_0(t)$ is the reciprocal of Eq. 4.75. It is noted here that both $L(0)$ and $\zeta_0(t)$ approach their exact values when the barrier increases to infinity. These are:

$$L(0) = 1 \quad (4.79)$$

$$\zeta_0(t) = n(t)$$

4.6 Probabilistic Barrier

To this end, the probabilistic variation of the strength is treated by superimposing a Normal narrow band process on the deterministic mean value of the strength. Then, the variance of this process is added to the variance of the noise of the thermal stresses; thus forming a new process to which crossing analysis is applied. This argument presents difficulties when distributions other than the Normal need to be considered. One way to deal with this problem is to consider the strength as a random variable, $b(t)$, with mean, $\bar{R}'(t)$ given in Eq. 3.48, and standard deviation, $\sigma_b(t)$ given in Eq. 3.52. Therefore, the stochastic process, $x(t)$ given in Eq. 4.15, becomes just that of the noise of the thermal stresses, $w(t)$. The discussion below is based on the conditional probability theory.

Because the following equality holds, the first passage probability obtained in the previous section for the process $x(t) = w(t)$ is in fact conditioned on the distribution of b .

$$p(T_f > T) = P(\max_{(0,T)} w(t) < b(t) | b(t) = b_0(t)) \quad (4.80)$$

Thus Eq. 4.56 can be written as

$$F_{w|B}(T) = 1 - \exp(-\alpha_p(T)) \quad (4.81)$$

The total first passage probability for a probabilistic barrier becomes

$$F_w(T) = \int_{-\infty}^{\infty} F_{w|B}(T) f_B(b) db \quad (4.82)$$

where $f_B(b)$ is the density function of the strength. A more explicit expression for $F_w(T)$ can be given as

$$F_w(T) = 1 - \int_{-\infty}^{\infty} \exp\left[-\sum_i^T \frac{1}{2\sigma_{w_i}} \phi\left(\frac{y_i}{\sigma_{w_i}}\right) \left[2\sigma_{w_i} \phi\left(\frac{y_i}{\sigma_{w_i}}\right) + y_i \left\{2\phi\left(\frac{y_i}{\sigma_{w_i}}\right) - 1\right\}\right]\right] f_B(b) db \quad (4.83)$$

For a normally distributed strength, Eq. 4.83 holds with y and $f_U(u)$, in place of $f_B(b)$, which have the forms

$$y(t) = \sqrt{2} u \sigma_b + \bar{R}'(t) - \mu(t) - S_y \cos \omega_y (t - t_y) - S_d \cos \omega_d (t - t_d) \quad (4.84)$$

$$f_U(u) = \frac{1}{\sqrt{\pi}} e^{-u^2} \quad (4.85)$$

where

$$u = \frac{1}{\sqrt{2}} \left(\frac{b - \bar{R}'(t)}{\sigma_b} \right) \quad (4.86)$$

Similarly for a log-normal strength, y and $f_U(u)$ can be written as

$$y(t) = \exp(\sqrt{2} u \zeta(t) + \lambda(t)) - \mu(t) - S_y \cos \omega_y (t - t_y) - S_d \cos \omega_d (t - t_d) \quad (4.87)$$

$$f_U(u) = \frac{1}{\sqrt{\pi}} e^{-u^2} \quad (4.88)$$

where

$$u = \frac{1}{\sqrt{2}} \left(\frac{\ln b - \lambda(t)}{\zeta(t)} \right) \quad (4.89)$$

$$\lambda(t) = \ln \bar{R}'(t) - \frac{1}{2} \zeta^2(t) \quad (4.90)$$

$$\zeta^2(t) = \ln \left(1 + \frac{\sigma_b^2}{\bar{R}'^2(t)} \right) \quad (4.91)$$

For a Weibull strength distribution, y and $f_U(u)$ become

$$y(t) = r_c(t) u^{1/\bar{R}'} - \mu(t) - S_y \cos \omega_y (t - t_y) - S_d \cos \omega_d (t - t_d) \quad (4.92)$$

$$f_U(u) = e^{-u} \quad (4.93)$$

where

$$u = \left(\frac{b}{r_c(t)}\right)^\gamma \quad (4.94)$$

$$r_c(t) = \frac{\bar{R}'(t)}{\Gamma(1+1/\gamma)} \quad (4.95)$$

and the value of γ can be calculated from

$$\sigma_b^2 = r_c^2(t) [\Gamma(1+2/\gamma) - \Gamma^2(1+1/\gamma)] \quad (4.96)$$

or it may be approximated as

$$\gamma \approx \frac{1.2\bar{R}'(t)}{\sigma_b} \quad (4.97)$$

The integration in Eq. 4.83 is carried out numerically for the three distributions.

A special case of Eq. 4.83 is the case of a constant barrier and Normal strength, rewriting Eq. 4.83

$$F_w(T) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\pi} \frac{\sigma_w}{\sigma_b} e^{-1/2\left(\frac{\sqrt{2} u \sigma_b + \bar{R}'(t)}{\sigma_w}\right)^2} T\right] e^{-u^2} du \quad (4.98)$$

where u is given in Eq. 4.86. For small crossing rate, Eq. 4.98 can be written as

$$F_w(T) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[1 - \frac{1}{2\pi} \frac{\sigma_w}{\sigma_b} e^{-\frac{1}{2} \left(\frac{\sqrt{2} u \sigma_b + \bar{R}'(t)}{\sigma_w} \right)^2} T \right] e^{-u^2} du \quad (4.99)$$

the integration in Eq. 4.99 yields

$$F_w(T) = \frac{1}{2\pi} \frac{\sigma_w}{(\sigma_b^2 + \sigma_w^2)^2} \exp\left(\frac{-b^2}{2(\sigma_b^2 + \sigma_w^2)}\right) \quad (4.100)$$

An interesting resemblance between Eq. 4.100 and Eq. 4.32 is clear.

Chapter 5

Results and Conclusions

To demonstrate, the first passage analysis presented here is applied to the structure of Fig. 3.1 whose physical and geometric properties are given in Table 3.1.

Numerical calculations are carried out for different periods of time at two sites; namely, Barrow (AK) and Yuma (AZ). These sites represent cold and warm climates, respectively. Table 4.1 shows the parameters used in the first passage calculations.

Fig. 5.1 a,b shows the variation of the thermal stresses and strengths for twelve years at Barrow. The oscillatory nature of the thermal stresses and strengths is due to the viscoelastic effects. When the temperature decreases in the winter season, the modulus becomes higher and thus higher thermal stresses occur. The reverse is true when the temperature increases. The reduction in the strength due to cumulative damage is clear.

The corresponding crossing rates, Eq. 4.40, and first passage probabilities, Eq. 4.58, are shown in Fig. 5.2 a,b. In this figure the coefficient of variation of the strength, δ_R , is equal to ten percent and the coefficient of variation of damage, δ_D , is equal to twenty percent. The effects of different combinations of these coefficients are shown in Figs. 5.3-5.8. The designations a,b,c stand for 0.0,0.1,0.2 coefficients of variations, respectively. Higher coefficients of variations yield higher first passage probabilities. While approach I is used in all the above figures, approach II is used in Figs. 5.9 a,b and 5.10 a,b to produce the equivalent to Figs. 5.1 a,b

TABLE 4.1

Temperature and Stress Parameters at the Two Sites

	<u>Barrow (AK)</u>	<u>Yuma (AZ)</u>
<u>Approach I:</u>		
μ_y , °F (°C)	9.35 (-12.58)	73.49 (23.05)
A_y , °F (°C)	30.68 (17.04)	19.23 (10.68)
A_d , °F (°C)	1.60 (0.89)	11.53 (6.41)
σ_{temp} , °F (°C) see Figs. 3.2 and 3.3	10.18 (5.66)	6.28 (3.48)
σ_w , Psi (N/m ²)	13.76 (9.49 E04)	1.30 (8.96 E03)
$\sigma_{\dot{w}}$, Psi/hr (N/m ² /hr)	0.82 (5.65 E03)	0.25 (1.72 E03)
<u>Approach II:</u>		
μ_y , °F (°C)	8.01 (-13.33)	73.90 (23.28)
σ_Y , °F (°C) see Figs. 3.4 and 3.5	23.57 (13.09)	14.08 (7.82)
σ_D , °F (°C) see Figs. 3.4 and 3.5	1.98 (1.10)	8.75 (4.86)
σ_w , Psi (N/m ²)	10.48 (7.22 E04)	3.42 (2.36 E04)
$\sigma_{\dot{w}}$, Psi/hr (N/m ² /hr)	20.85 (1.44 E05)	6.87 (4.74 E04)

and 5.2 a,b. Similar results are obtained.

The behavior of the crossing rate as a function of the difference between strength and stress is shown in Fig. 5.11 a,b. Fig. 5.12 a,b shows the variation of the thermal stresses and strengths for thirty years at Yuma. The corresponding crossing rates and first passage probabilities are shown in Fig. 5.13 a,b with $\delta_R = 0.1$, $\delta_D = 0.2$ and Fig. 5.14 a,b with $\delta_R = 0.2$, $\delta_D = 0.2$.

The reductions in the stress and strength due to aging and cumulative damage are included. The aging factors for the strength and modulus are shown in Fig. 5.15 a,b, respectively. A comparison between damage effects at Barrow and Yuma is shown in Fig. 5.16 a,b. The effect of damage at Barrow is more severe than that at Yuma. In the absence of aging effects, Figs. 5.17 a,b and 5.18 a,b result, with

$\delta_R = 0.2$, $\delta_D = 0.2$. The reduction of the stress by aging is higher than that of the strength. Thus including aging effects yield lower first passage probabilities as can be seen by comparing Figs. 5.14 b and 5.18 b. Approach II is used in Figs. 5.19 a,b and 5.20 a,b to produce the equivalent to Figs. 5.12 a,b and 5.13 a,b.

The crossing rates in all the above figures are calculated from Eq. 4.40. A comparison between this rigorous expression and the classical one given in Eq. 4.32 indicates that there is little difference between the two expressions. The reason for this can be traced to the fact that the time rate of change of the barrier is insignificant for this particular problem. Had there been a higher rate of change of the barrier due to higher frequency, or faster degradation of the strength by aging or damage, the effect of Eq. 4.40 would have been more significant. To

illustrate the difference between Eqs. 4.32 and 4.40, a higher frequency is used in Figs. 5.21 and 5.22. In these figures, the higher curves correspond to Eqs. 4.40 and 4.58. As another example, the ratio, r , between Eq. 4.40 and 4.32 for a linear barrier, $a+bt$, can be written as,

$$r = \exp\left(-\frac{b^2}{2\sigma_x^2}\right) + \frac{\sqrt{2\pi}}{2\sigma_x} b \left(2\Phi\left(\frac{b}{\sigma_x}\right) - 1\right) \quad (5.1)$$

A plot of this equation is shown in Fig. 5.23. It indicates that r increases as $\frac{b}{\sigma_x}$ increases or decreases.

A comparison between the Poisson approach and Markov approach for a barrier with a higher frequency is shown in Figs. 5.24 and 5.25. The higher curve in Fig. 5.24 corresponds to the Markov crossing rate. Because the Markov effect starts at a late time in the life of the structure, the Poisson first passage probabilities and the Markov ones overlap as shown in Fig. 5.25.

Figs. 5.26-5.28 represent the treatment of section 4.6. First passage probabilities are shown for random barriers with Normal, Log-Normal, and Weibull distributions. Differences in the first passage probabilities calculated from these distributions exist especially at low probability levels. These differences decrease when the probabilities are high enough to cause concern. The Normal counterpart in Fig. 5.26 is less conservative than the ones obtained in Fig. 5.8.

In conclusion, the first passage problem has been introduced in the context of environmental thermal stresses. The material is viscoelastic and is afflicted with statistical variability and is subject to degradation by aging and cumulative damage. Therefore statistically and time

varying barriers have been considered. This makes the treatment presented here unique since almost all other first passage problems deal with constant deterministic barriers.

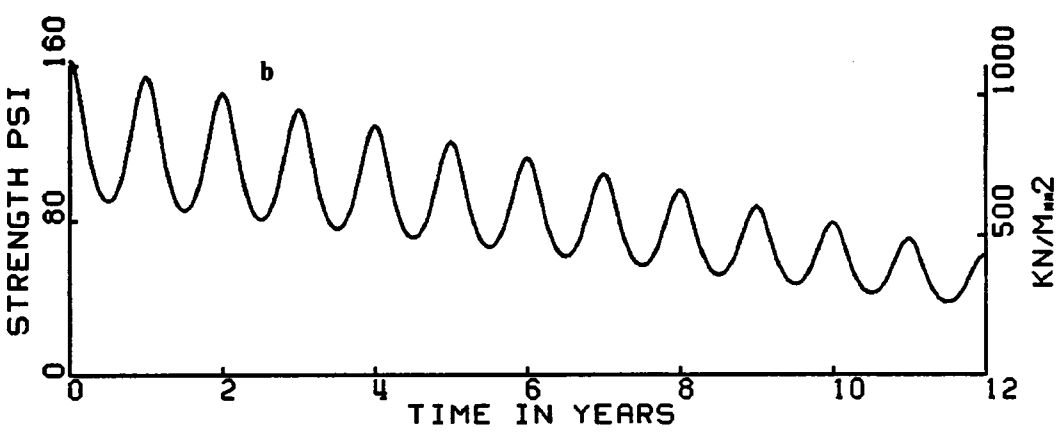
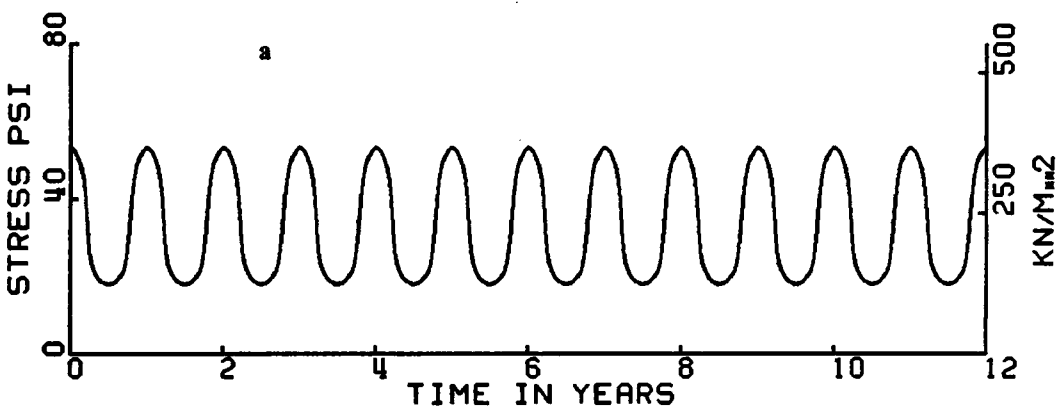


Figure 5.1 a) Mean Stress and b) Mean Strength at Barrow, Cumulative Damage is Included.

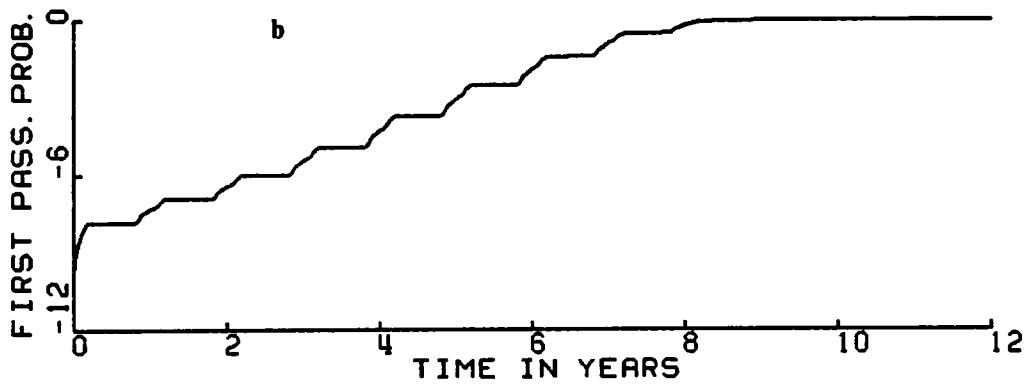
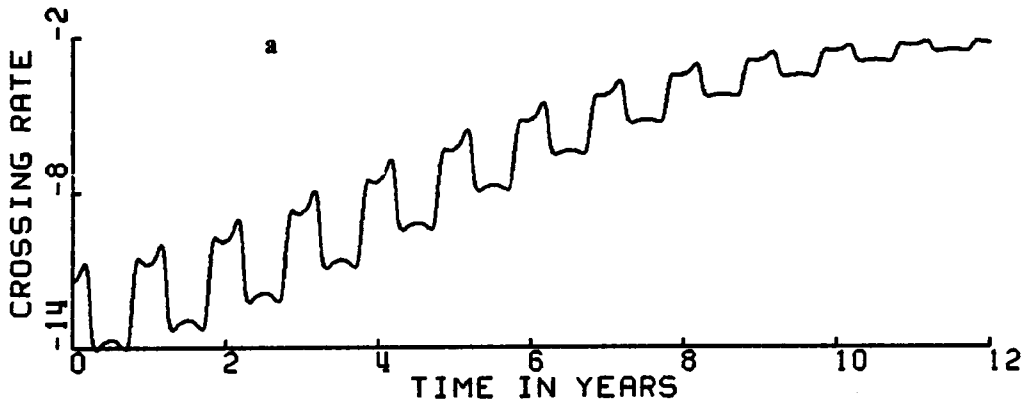


Figure 5.2 a) Crossing Rate and b) First Passage Probability at Barrow with $\delta_R = 0.1$, $\delta_D = 0.2$.

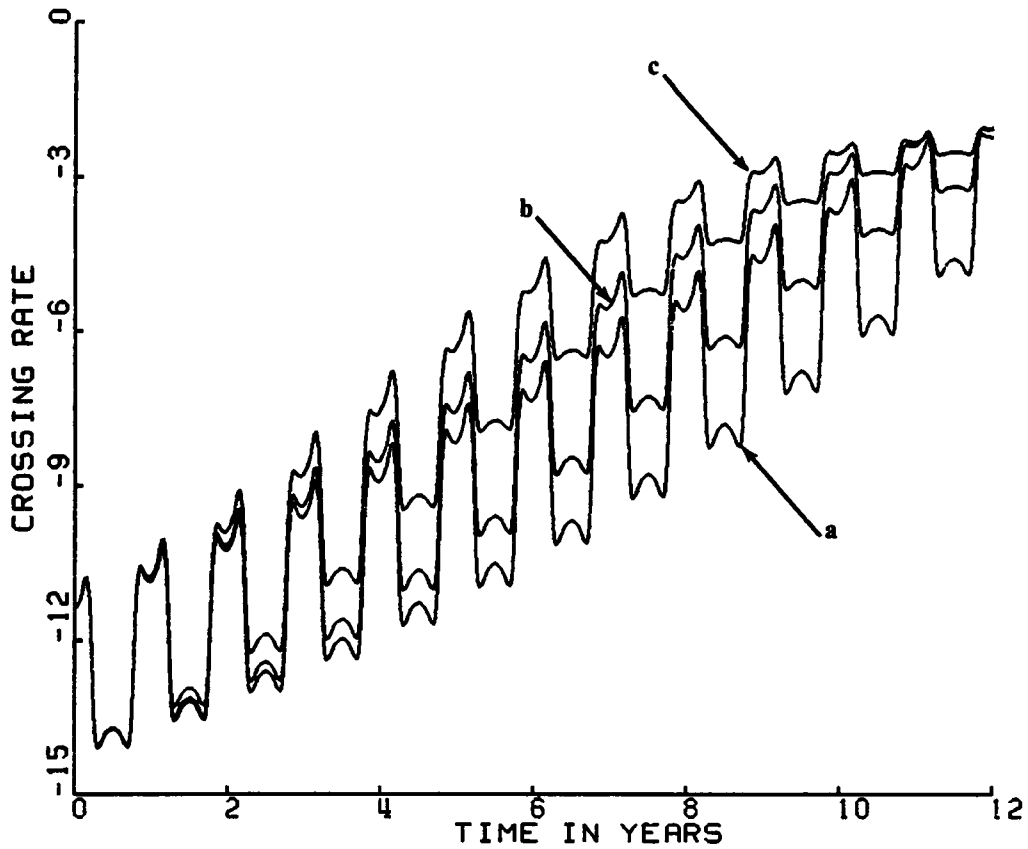


Figure 5.3 Crossing Rates at Barrow: a) $\delta_R = 0.1$, $\delta_D = 0.0$;
b) $\delta_R = 0.1$, $\delta_D = 0.1$; and c) $\delta_R = 0.1$, $\delta_D = 0.2$.

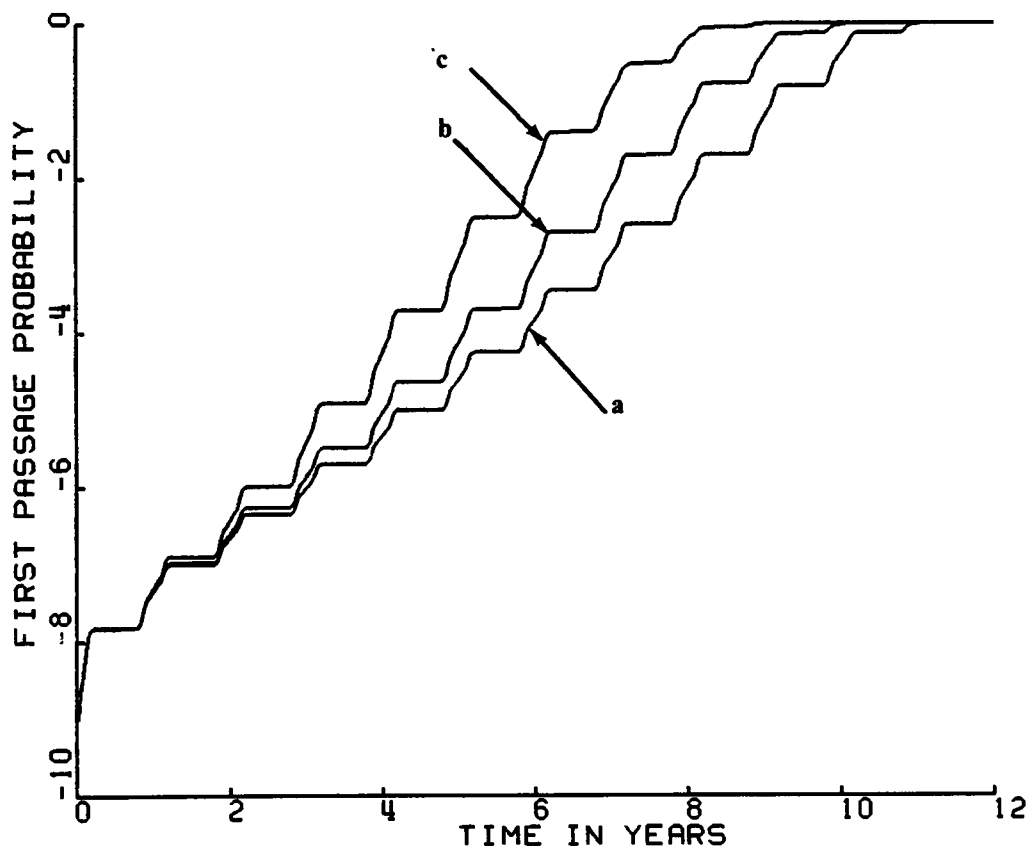


Figure 5.4 First Passage Probability at Barrow: a) $\delta_R = 0.1$, $\delta_D = 0.0$; b) $\delta_R = 0.1$, $\delta_D = 0.1$; and c) $\delta_R = 0.1$, $\delta_D = 0.2$.

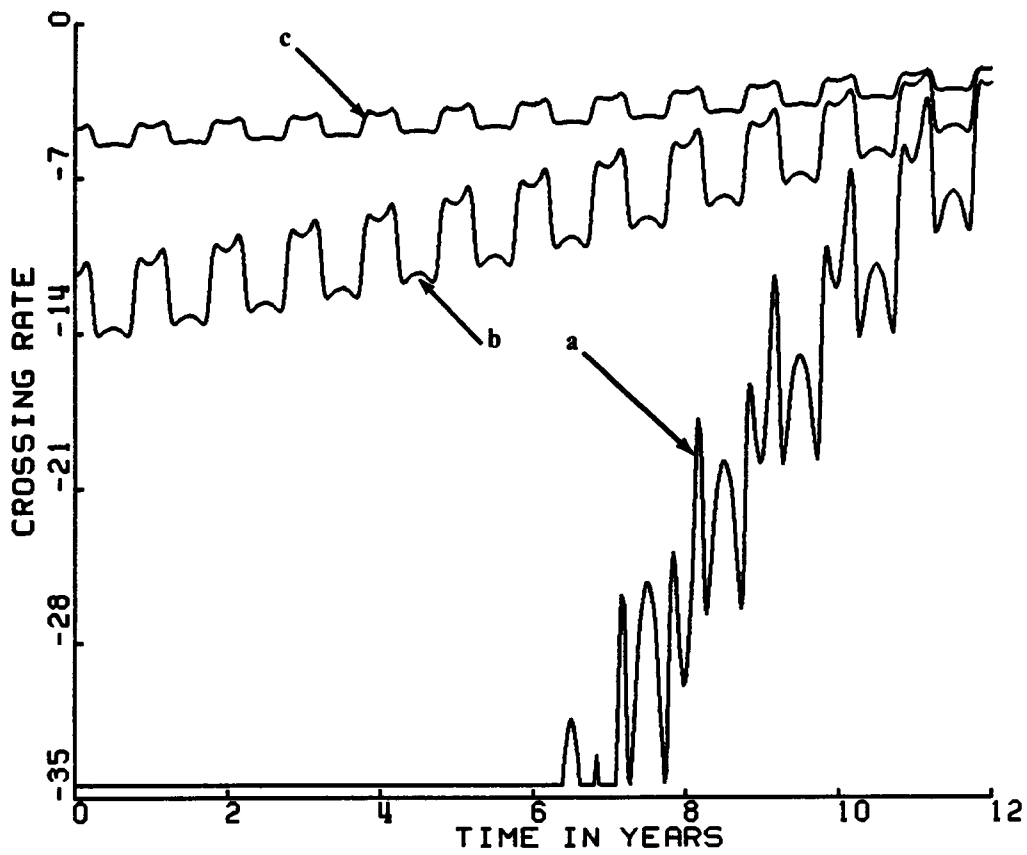


Figure 5.5 Crossing Rates at Barrow: a) $\delta_D = 0.0$, $\delta_R = 0.0$;
 b) $\delta_D = 0.0$, $\delta_R = 0.1$; and c) $\delta_D = 0.0$, $\delta_R = 0.2$.

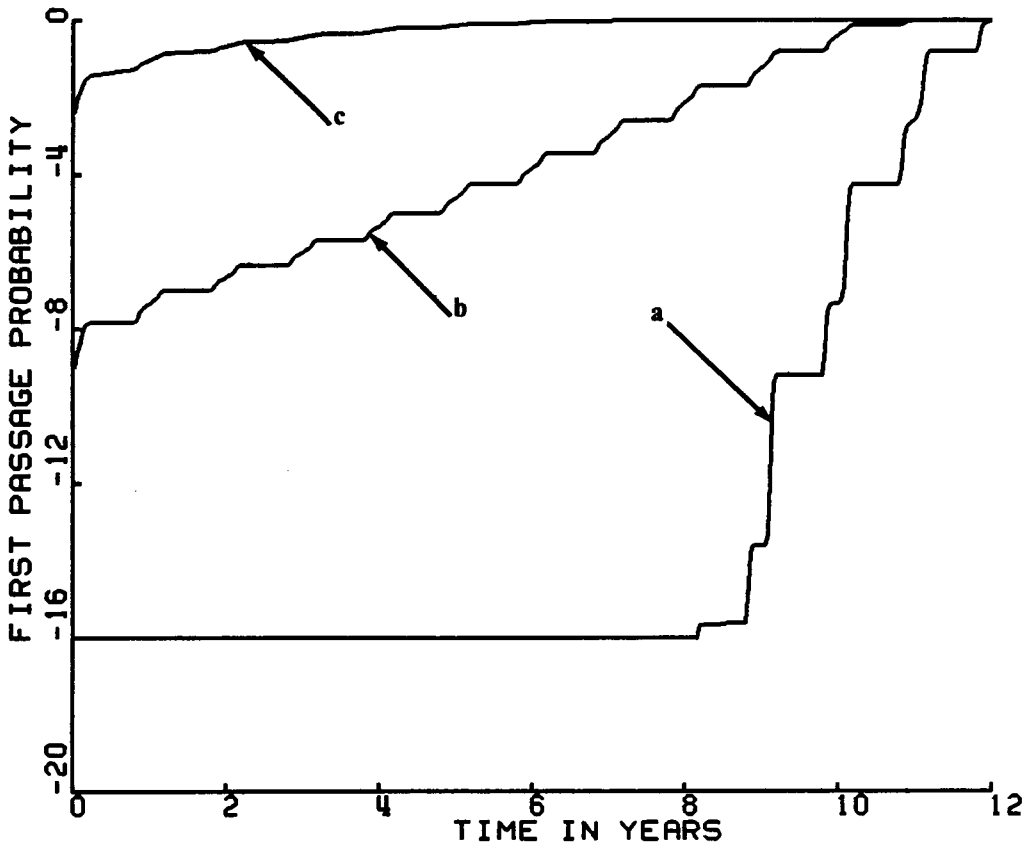


Figure 5.6 First Passage Probability at Barrow: a) $\delta_D = 0.0$, $\delta_R = 0.0$; b) $\delta_D = 0.0$, $\delta_R = 0.1$; and c) $\delta_D = 0.0$, $\delta_R = 0.2$.

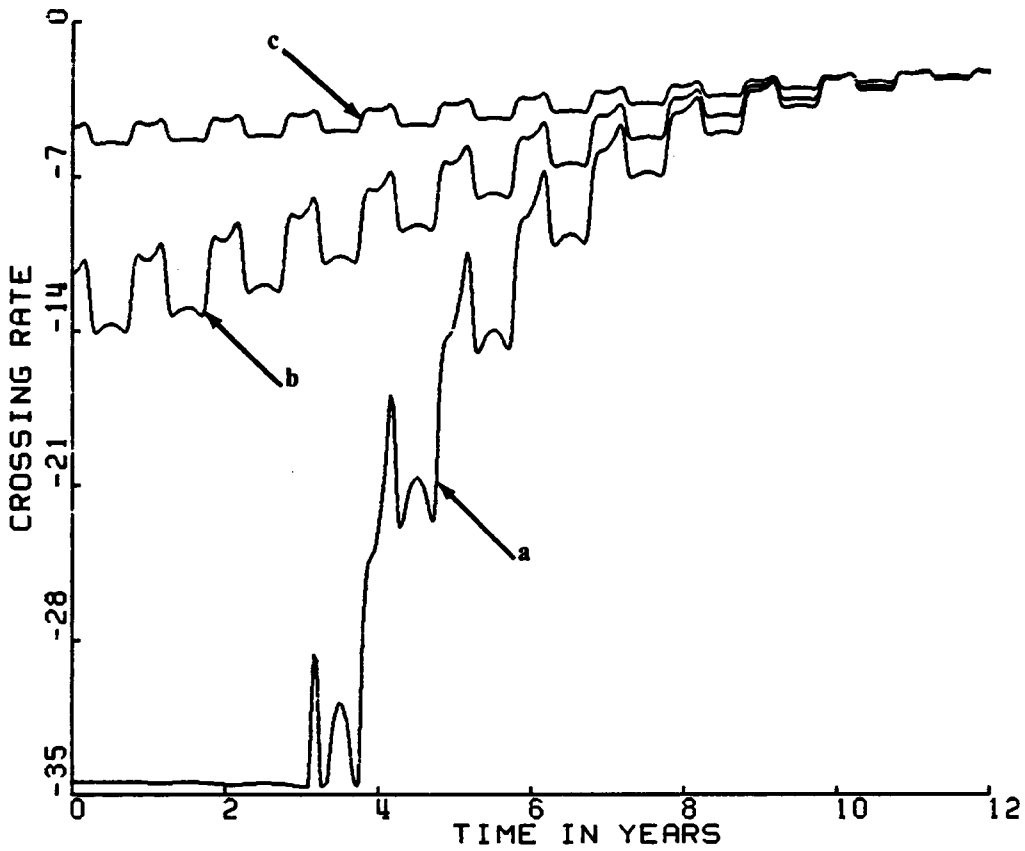


Figure 5.7 Crossing Rates at Barrow: a) $\delta_D = 0.2$, $\delta_R = 0.0$; b) $\delta_D = 0.2$, $\delta_R = 0.1$; and c) $\delta_D = 0.2$, $\delta_R = 0.2$.

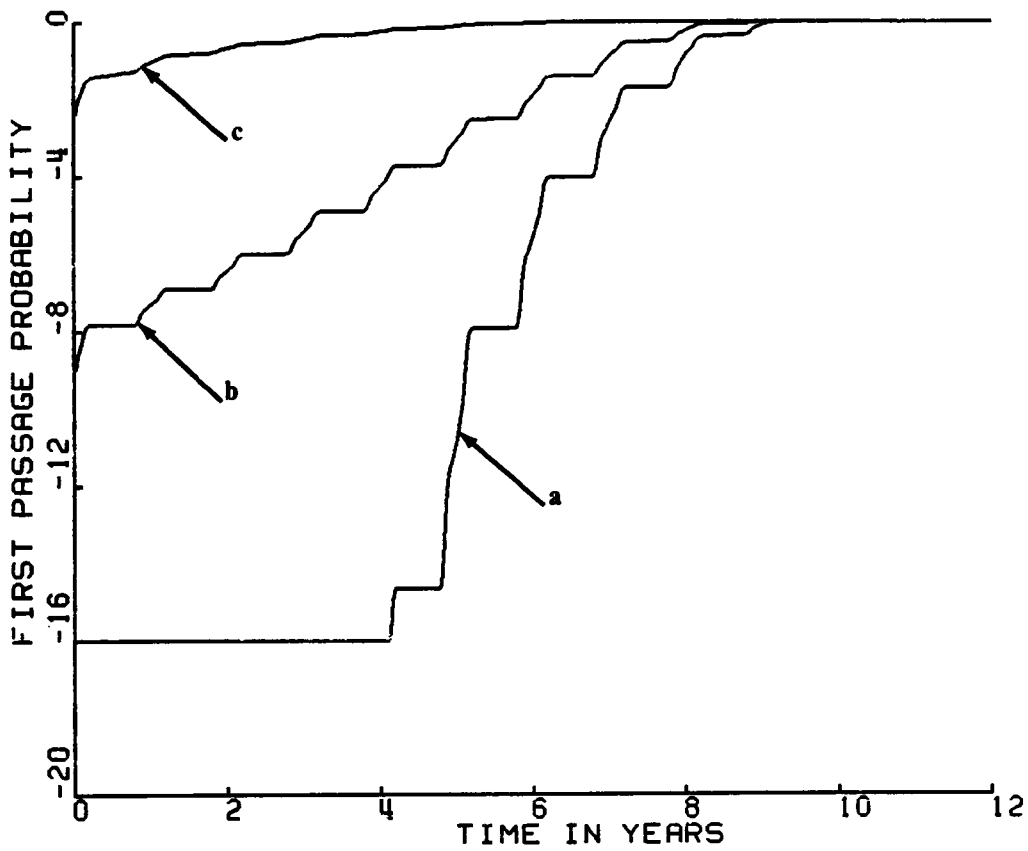


Figure 5.8 First Passage Probability at Barrow: a) $\delta_D = 0.2$, $\delta_R = 0.0$; b) $\delta_D = 0.2$, $\delta_R = 0.1$; and c) $\delta_D = 0.2$, $\delta_R = 0.2$.

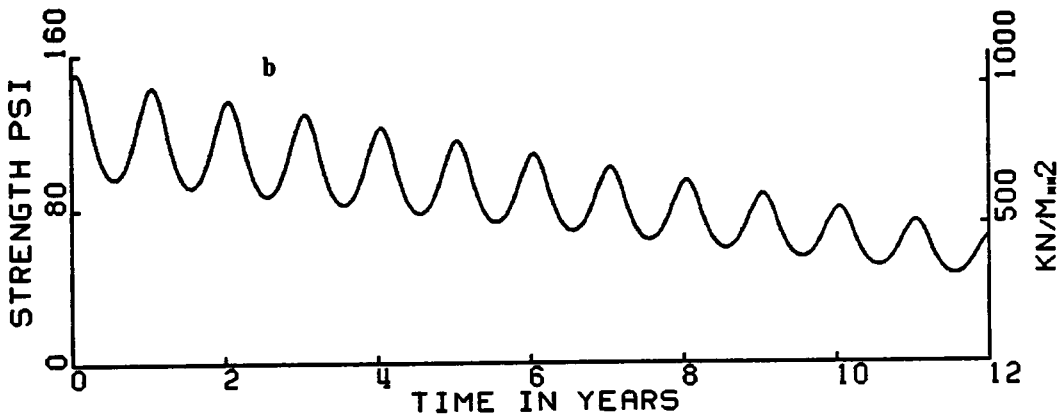
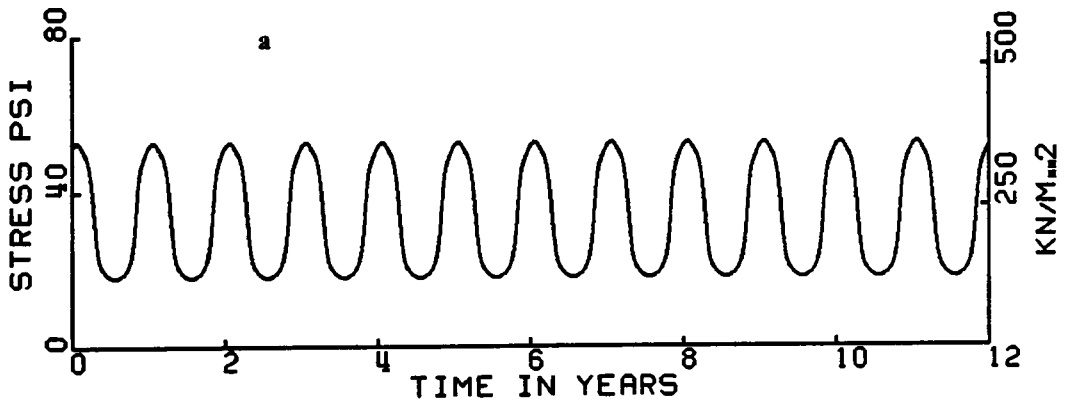


Figure 5.9 a) Mean Stress and b) Mean Strength at Barrow, Cumulative Damage is Included, Approach II.

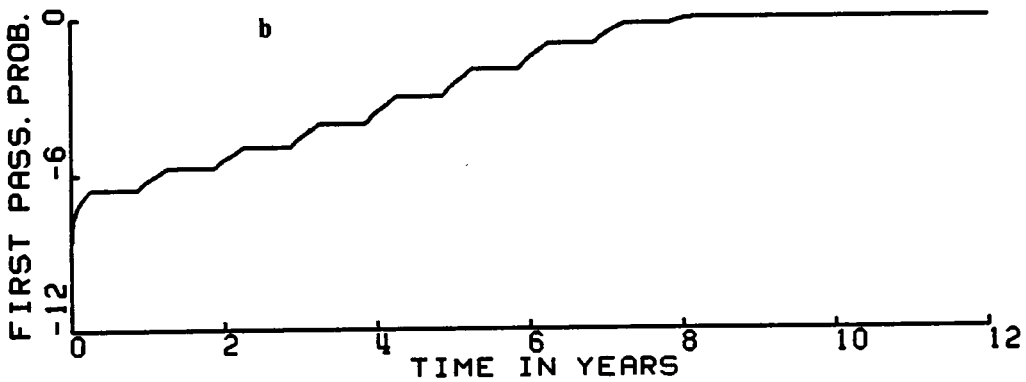
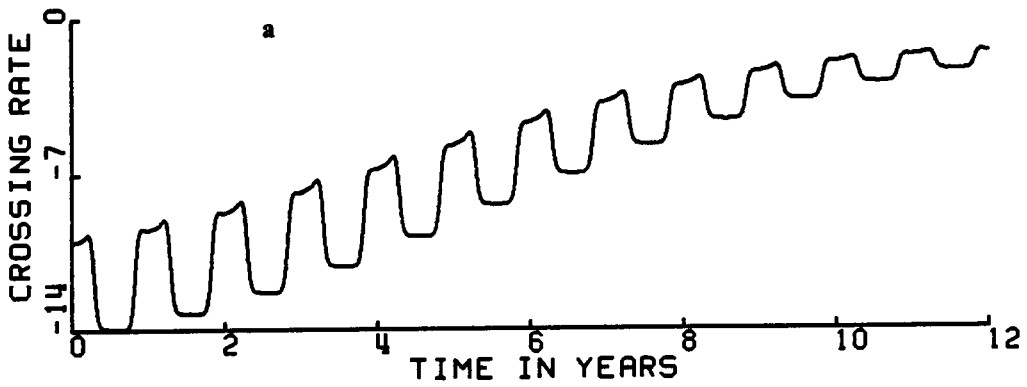


Figure 5.10 a) Crossing Rate and b) First Passage Probability at Barrow with $\delta_R = 0.1$, $\delta_D = 0.2$, Approach II.

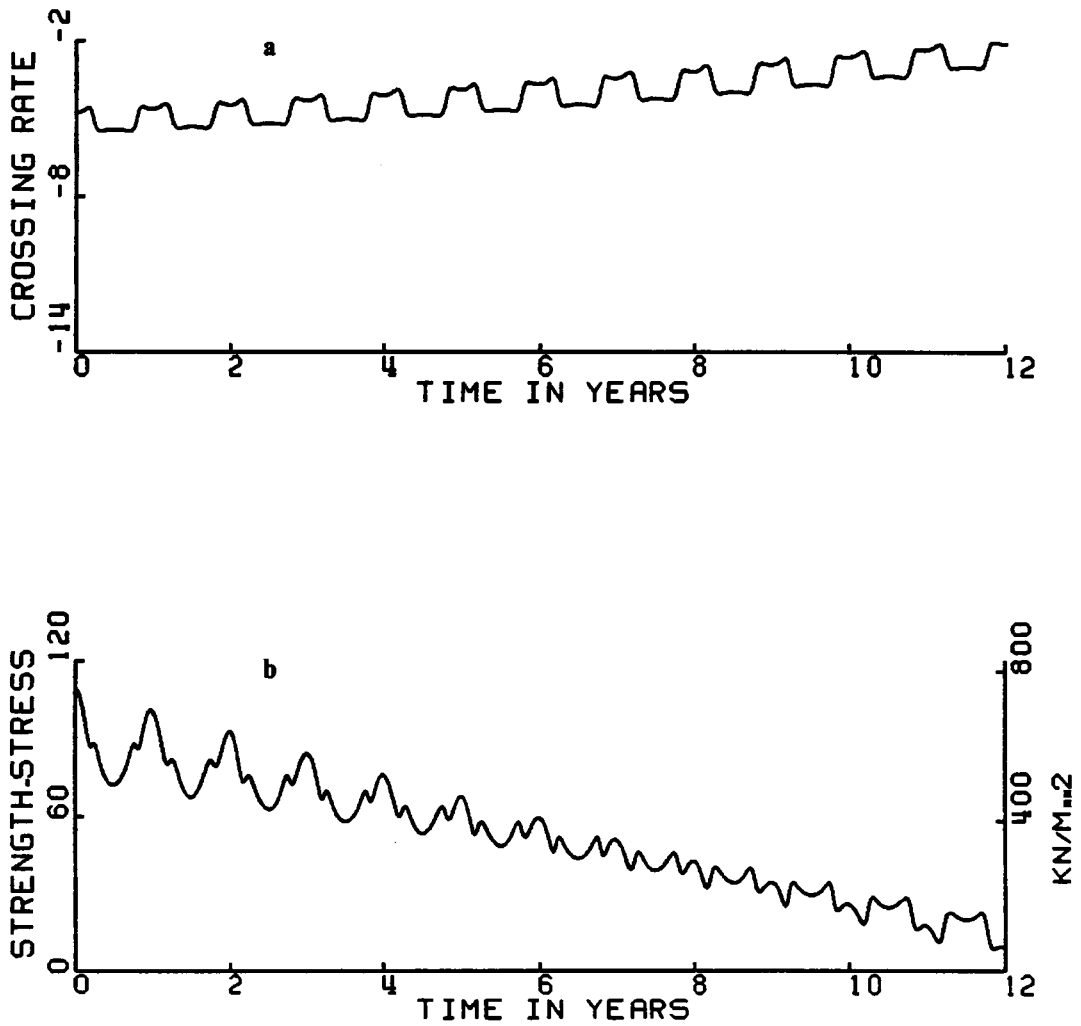


Figure 5.11 The Behavior of the Crossing Rate (a) as a Function of the Difference between Strength and Stress (b).

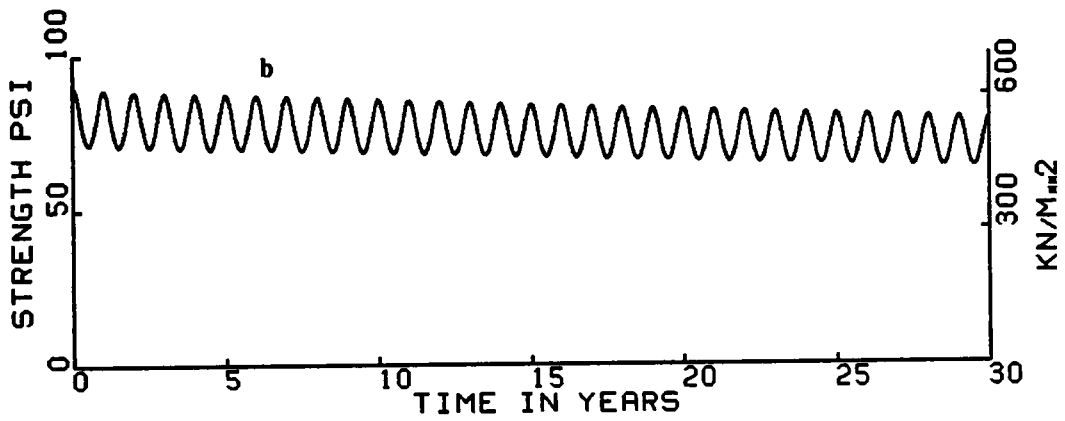
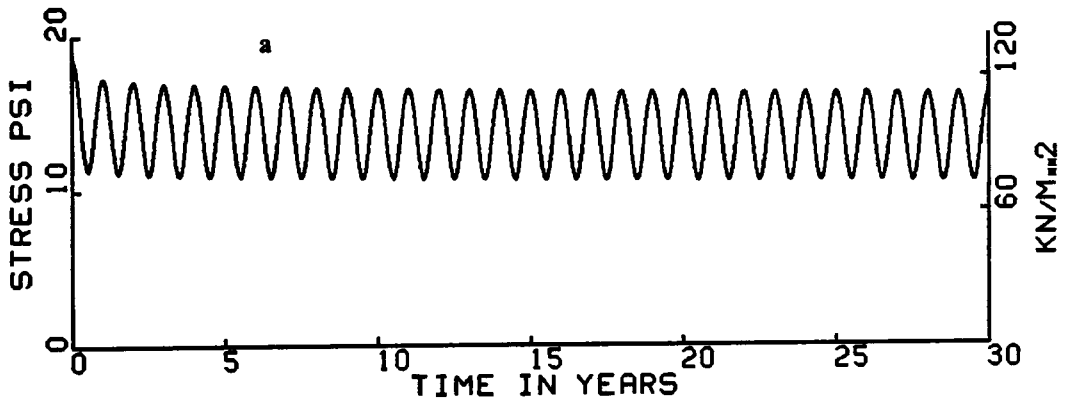


Figure 5.12 a) Mean Stress and b) Mean Strength at Yuma, Cumulative Damage and Aging are Included.

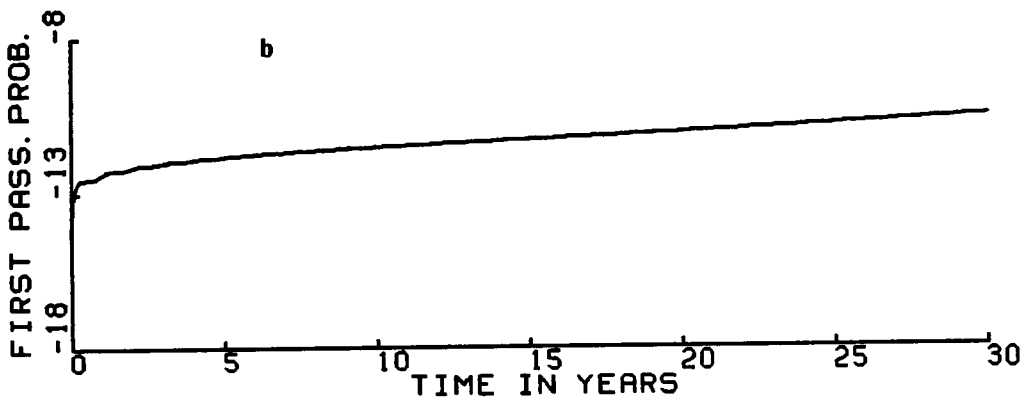
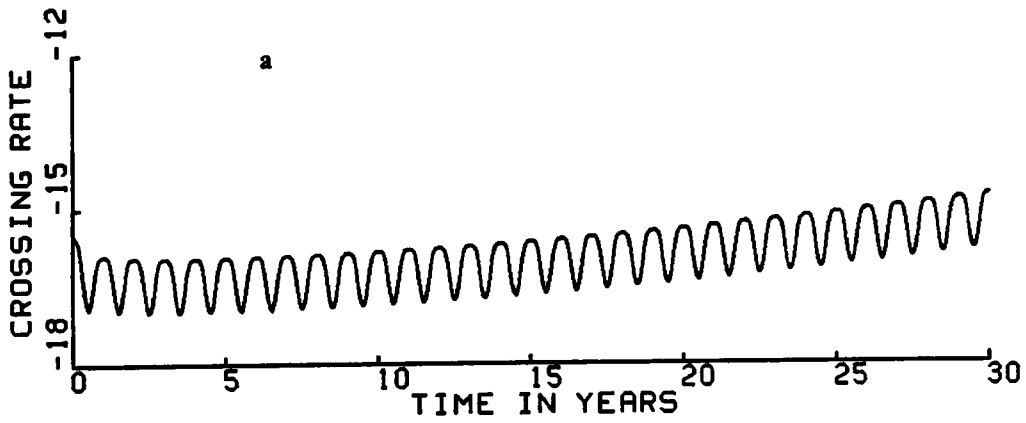


Figure 5.13 a) Crossing Rate and b) First Passage Probability at Yuma with $\delta_R = 0.1$, $\delta_D = 0.2$, Aging is Included.

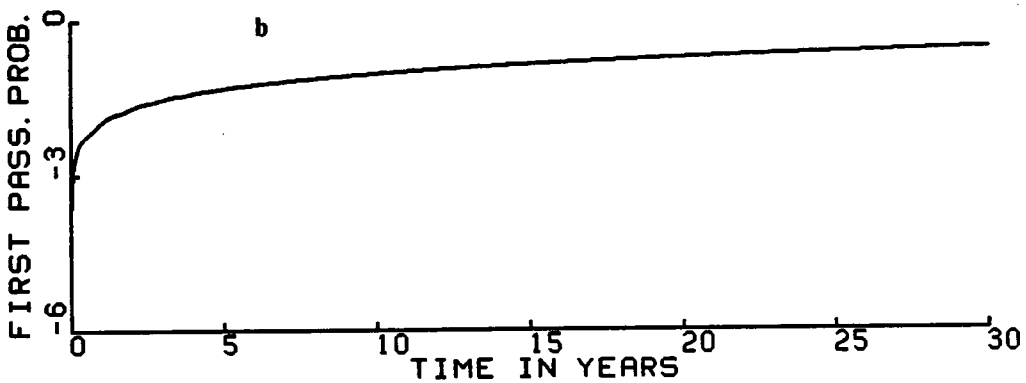
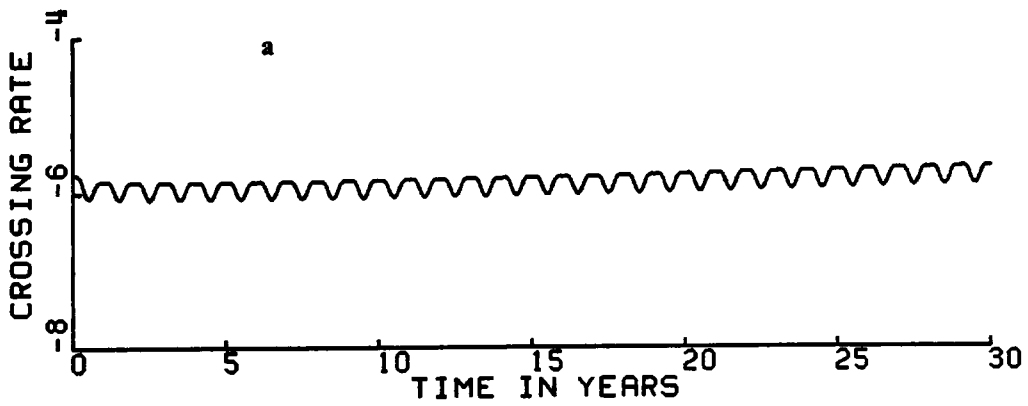


Figure 5.14 a) Crossing Rate and b) First Passage Probability at Yuma with $\delta_R = 0.2$, $\delta_D = 0.2$, Aging is Included.

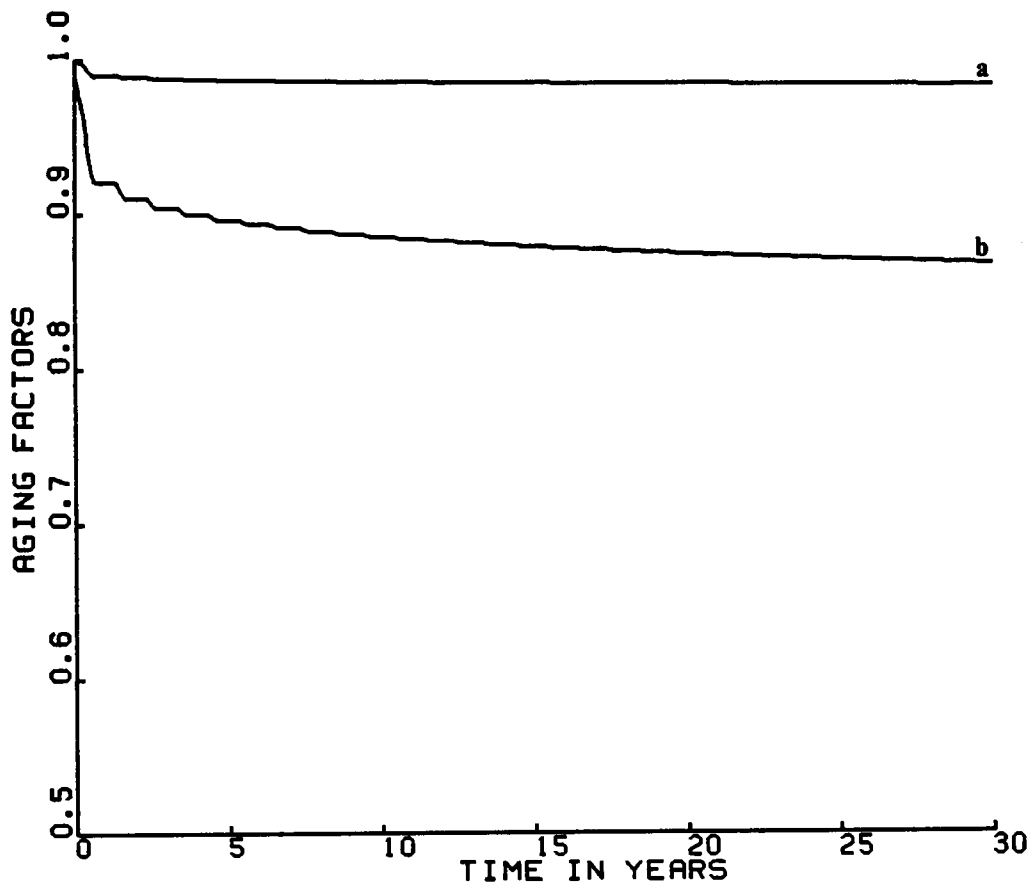


Figure 5.15 a) Strength and b) Modulus Aging Factors at Yuma.

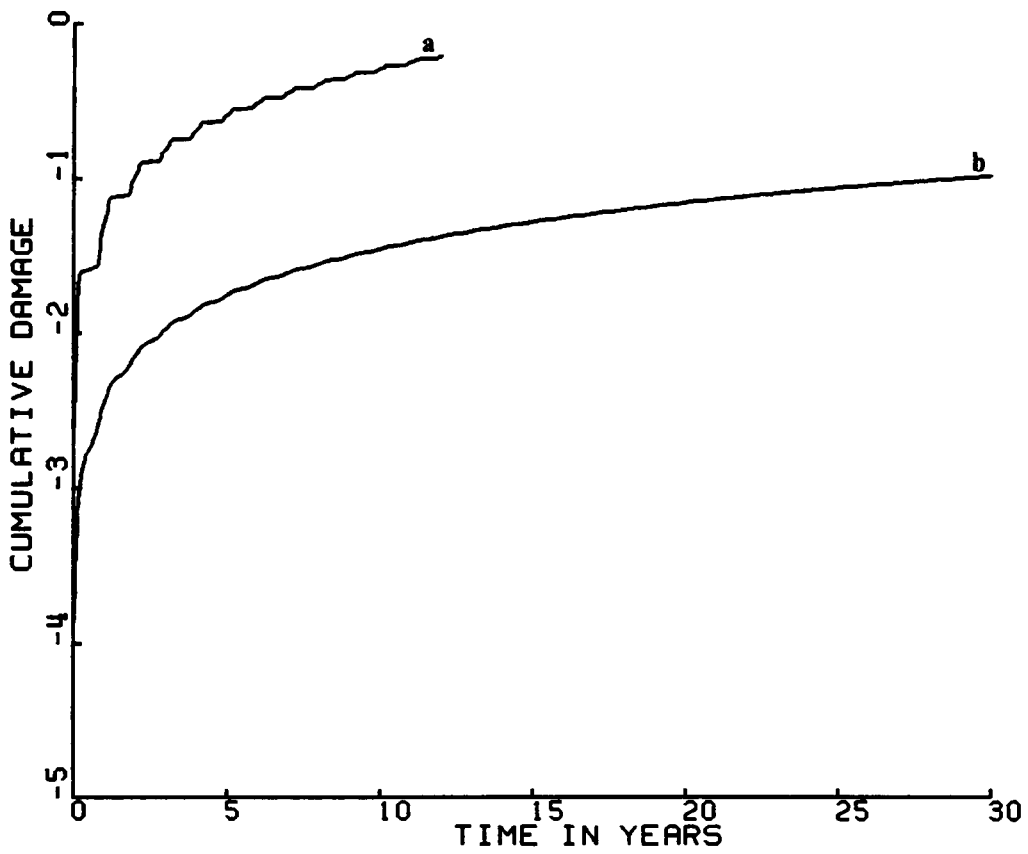


Figure 5.16 Cumulative Damage at a) Barrow and b) Yuma.

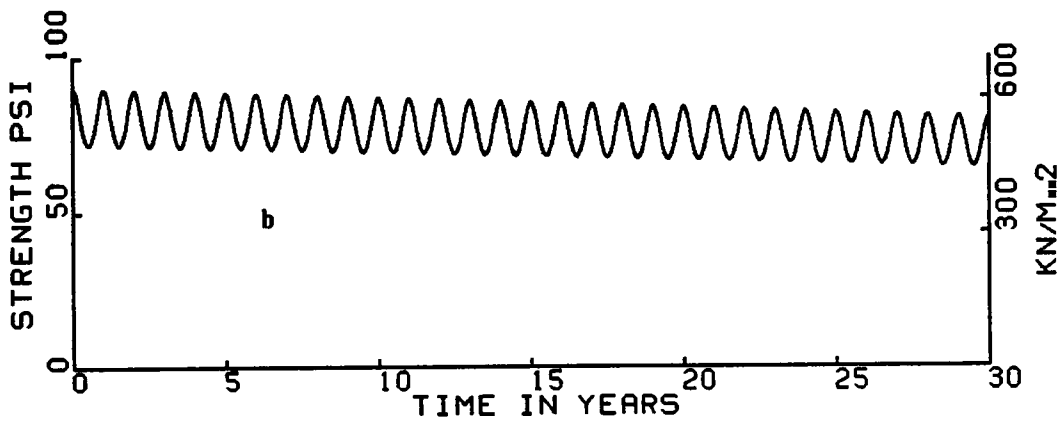
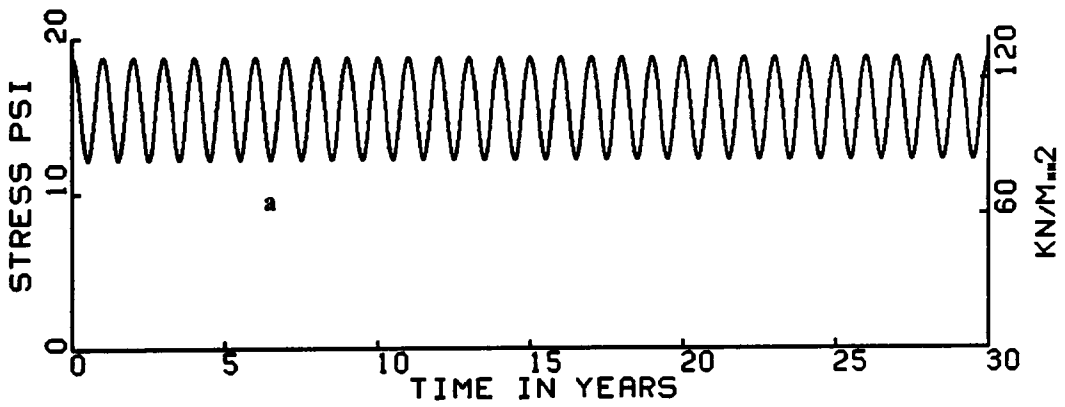


Figure 5.17 a) Mean Stress and b) Mean Strength at Yuma, Cumulative Damage is Included, Aging is not Included.

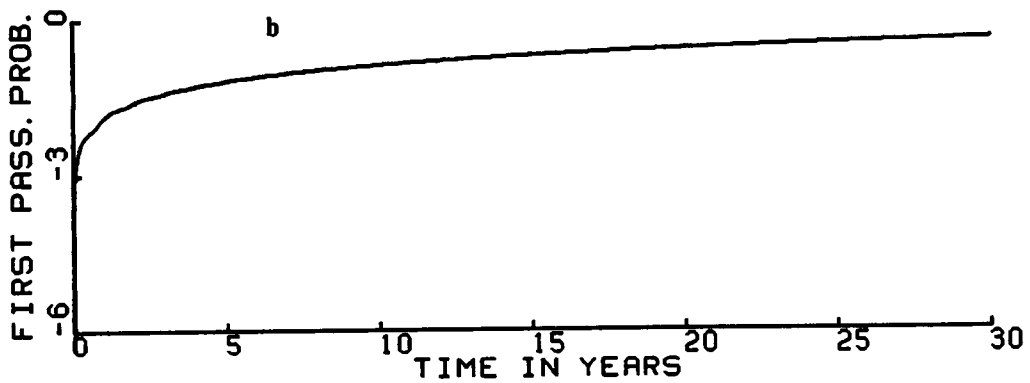
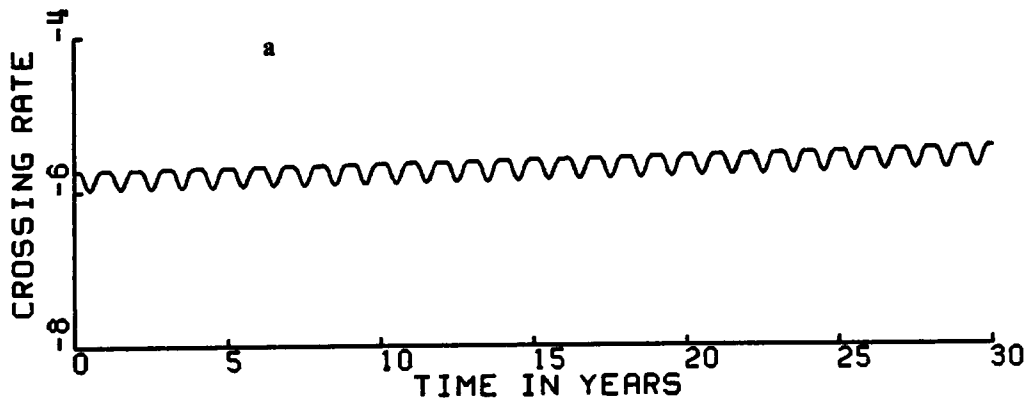


Figure 5.18 a) Crossing Rate and b) First Passage Probability at Yuma with $\delta_R = 0.2$, $\delta_D = 0.2$, Aging is not Included.

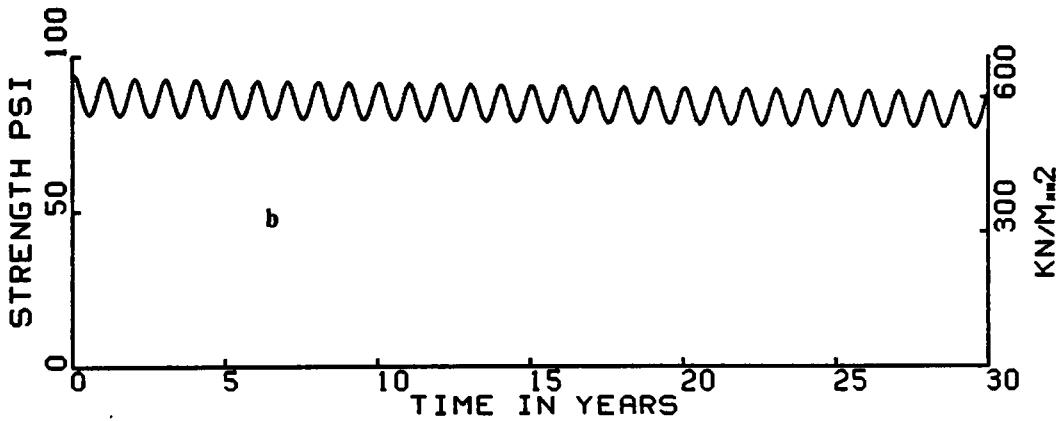
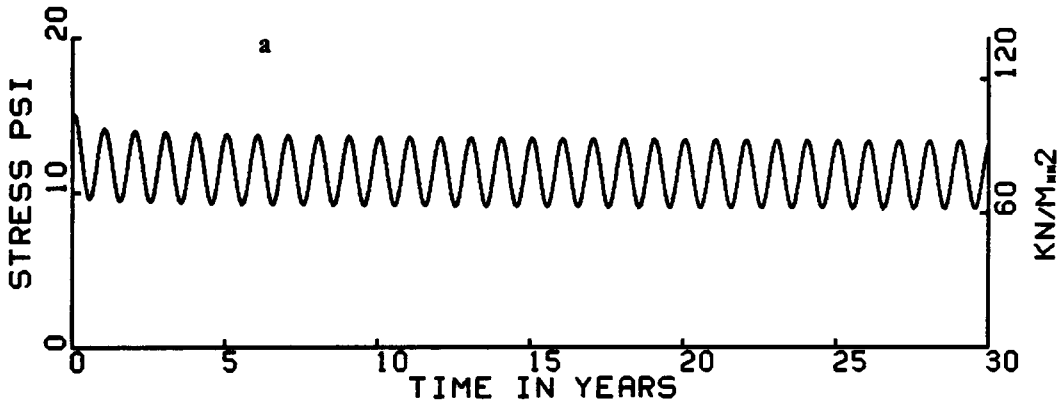


Figure 5.19 a) Mean Stress and b) Mean Strength at Yuma, Cumulative Damage and Aging are Included, Approach II.

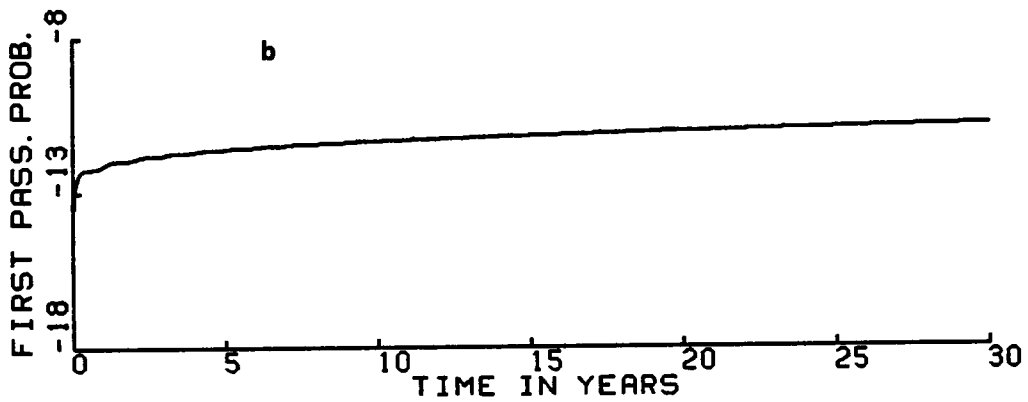
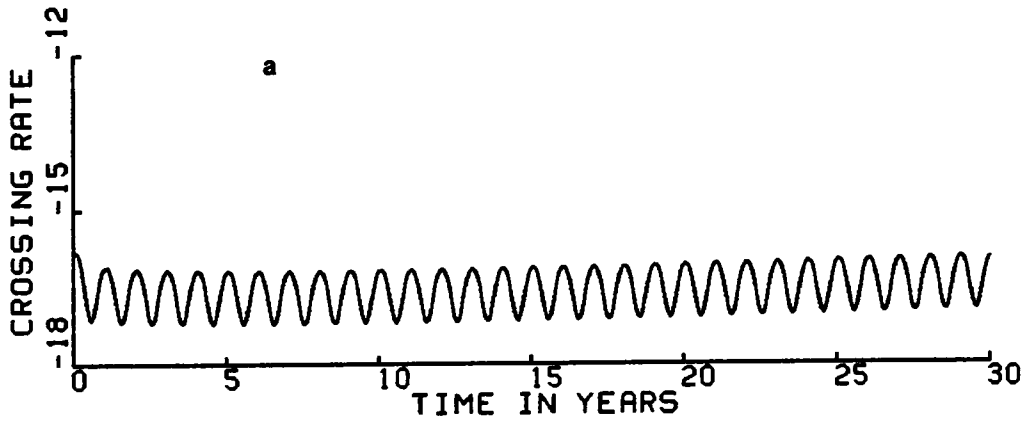


Figure 5.20 a) Crossing Rate and b) First Passage Probability at Yuma with $\delta_R = 0.1$, $\delta_D = 0.2$, Aging is Included, Approach II.

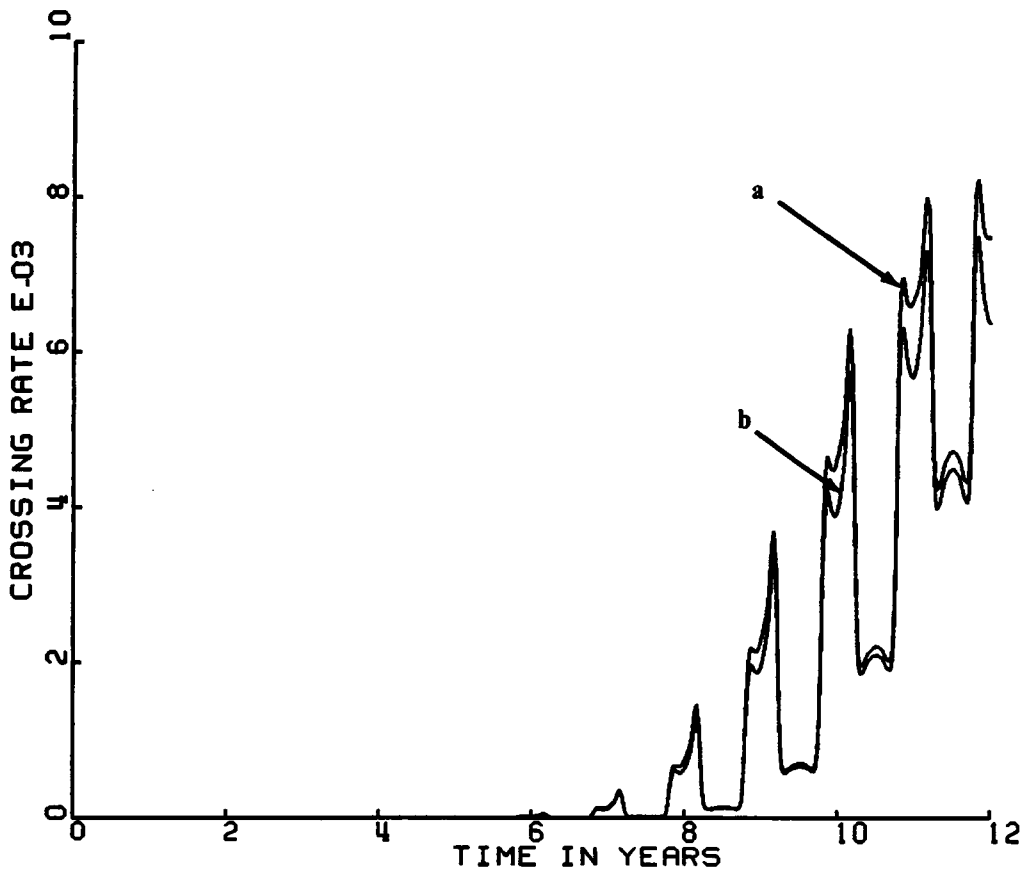


Figure 5.21 A Comparison between a) Eq. 4.32 and b) Eq. 4.40.

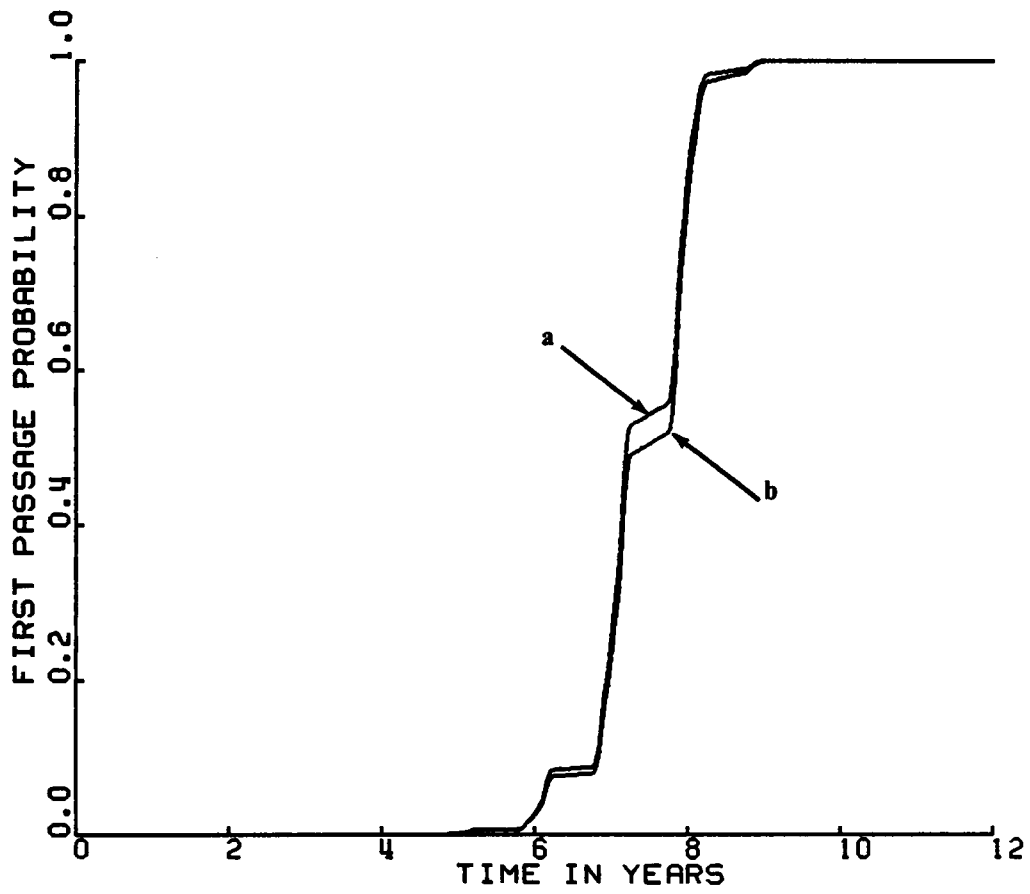


Figure 5.22 A Comparison between a) Eq. 4.59 and b) Eq. 4.58.

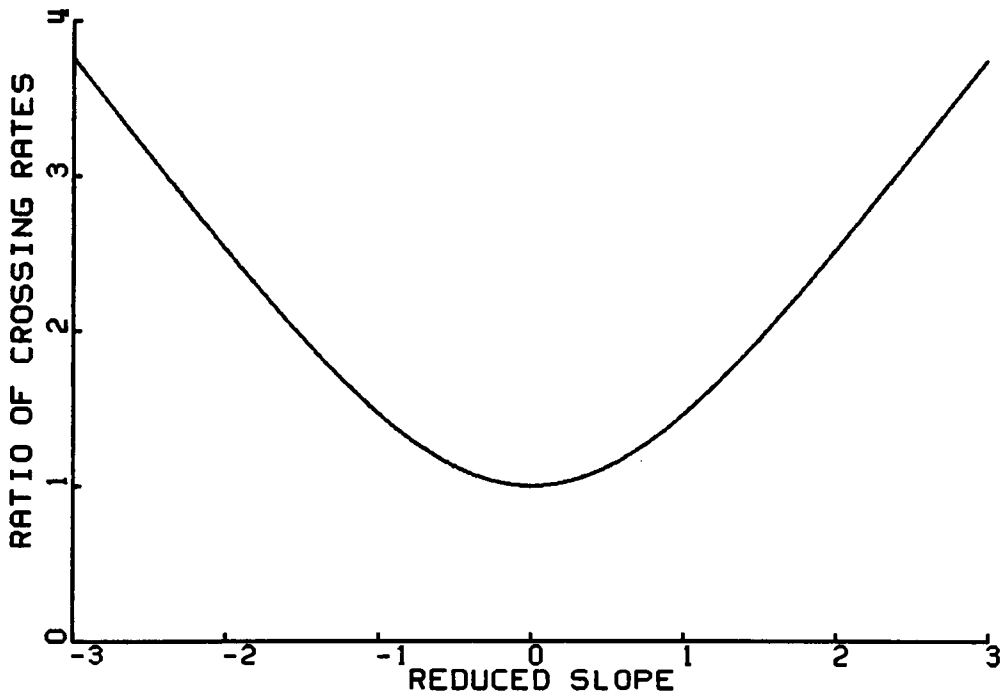


Figure 5.23 An Example of the Difference between Eqs. 4.32 and 4.40.

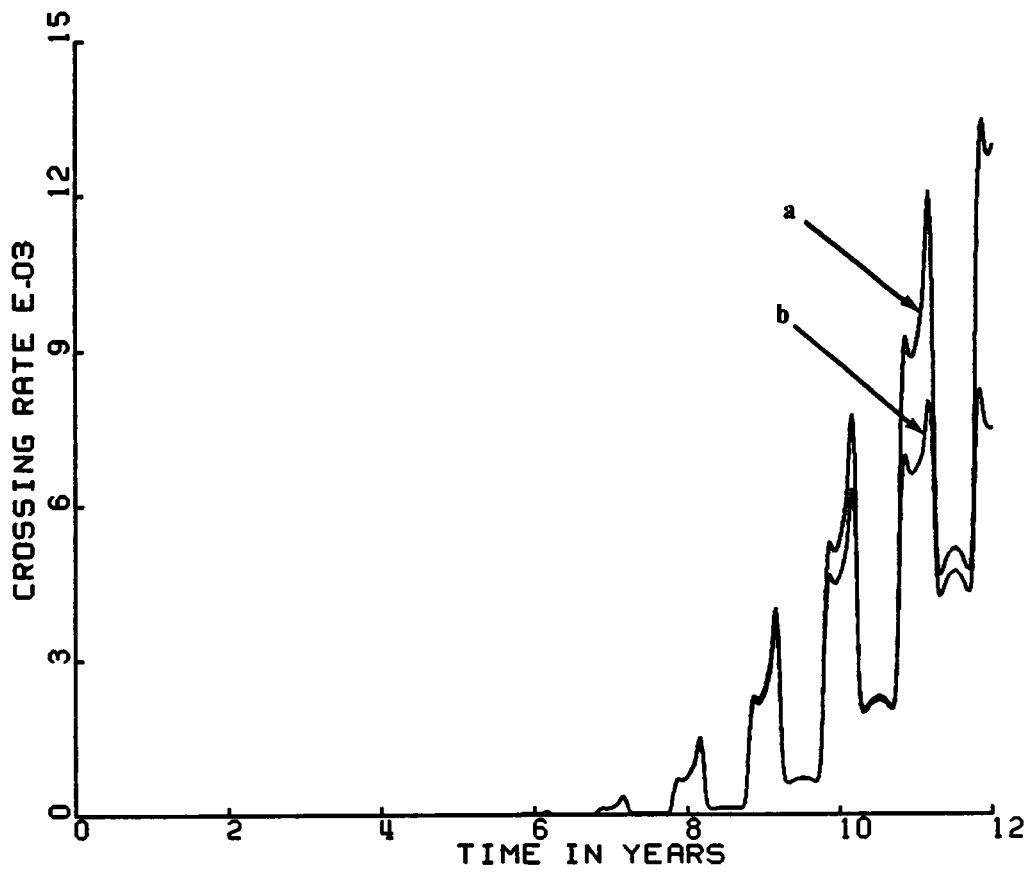


Figure 5.24 A Comparison between a) Poisson (see Eq. 4.40) and b) Markov (see Eq. 4.75) Assumptions.

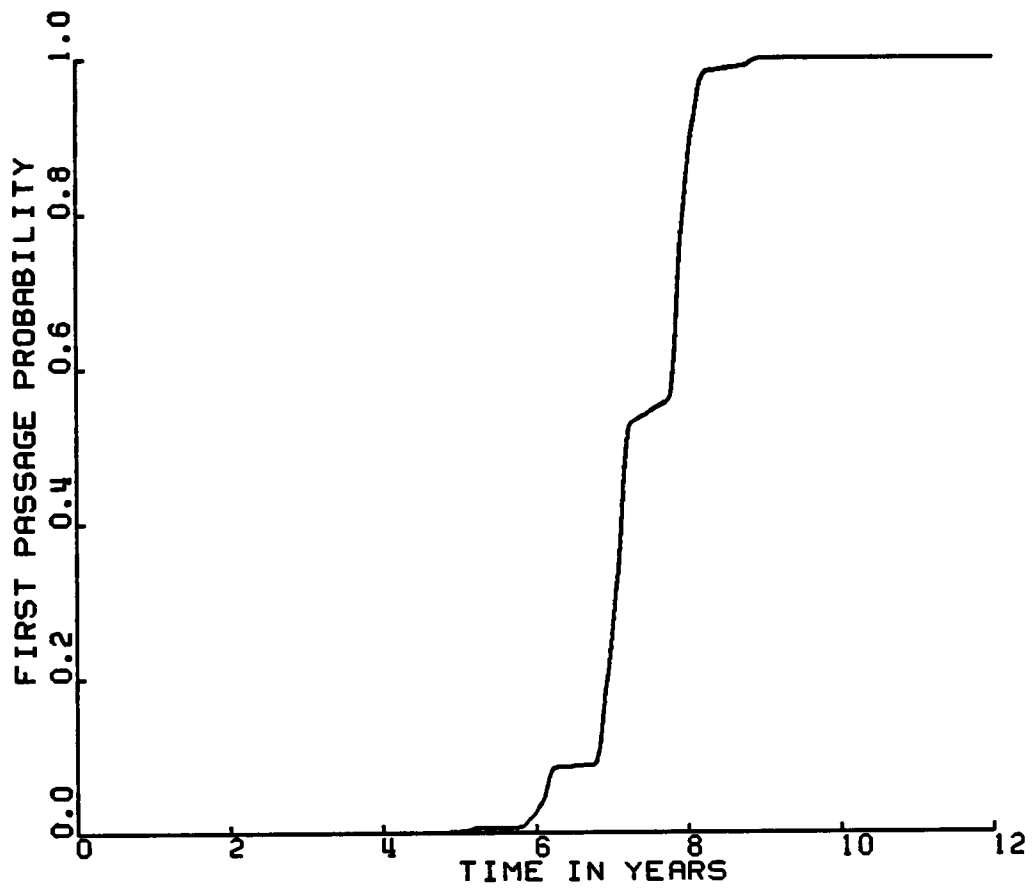


Figure 5.25 A Comparison between Poisson and Markov Assumptions, the Two Curves Overlap.

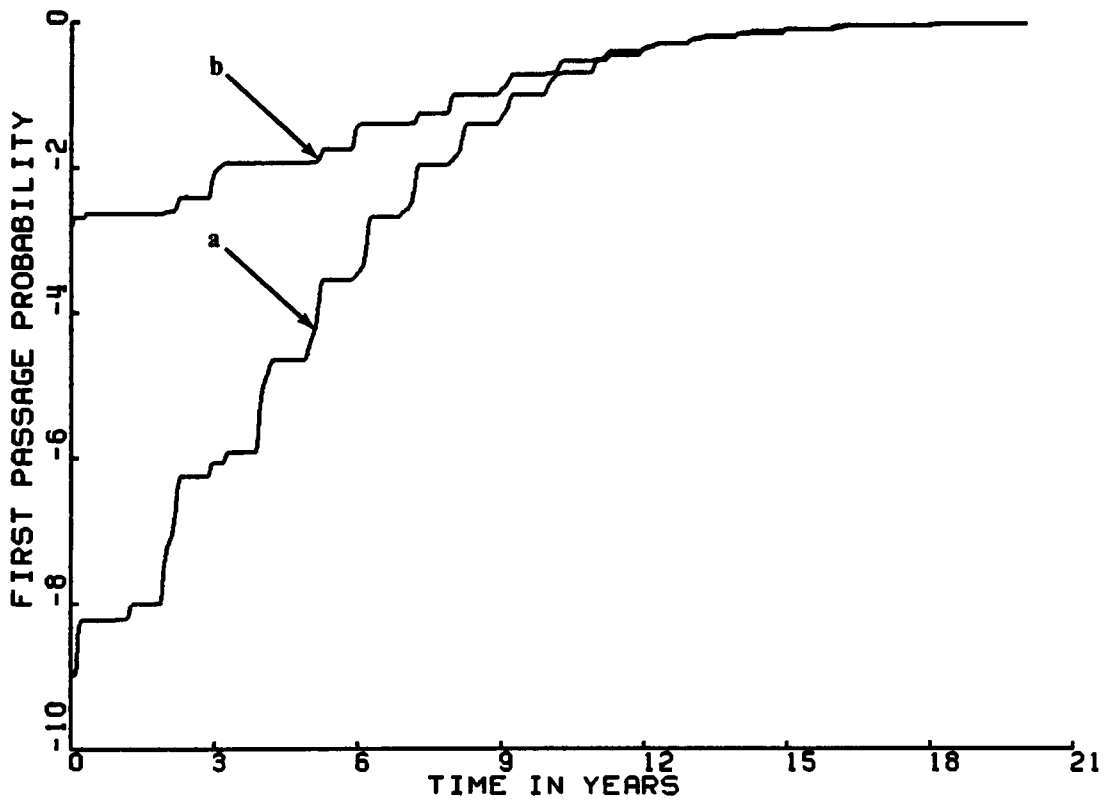


Figure 5.26 First Passage Probability at Barrow with Normal Barrier:
 a) $\delta_D = 0.2$, $\delta_R = 0.1$ and b) $\delta_D = 0.2$, $\delta_R = 0.2$.

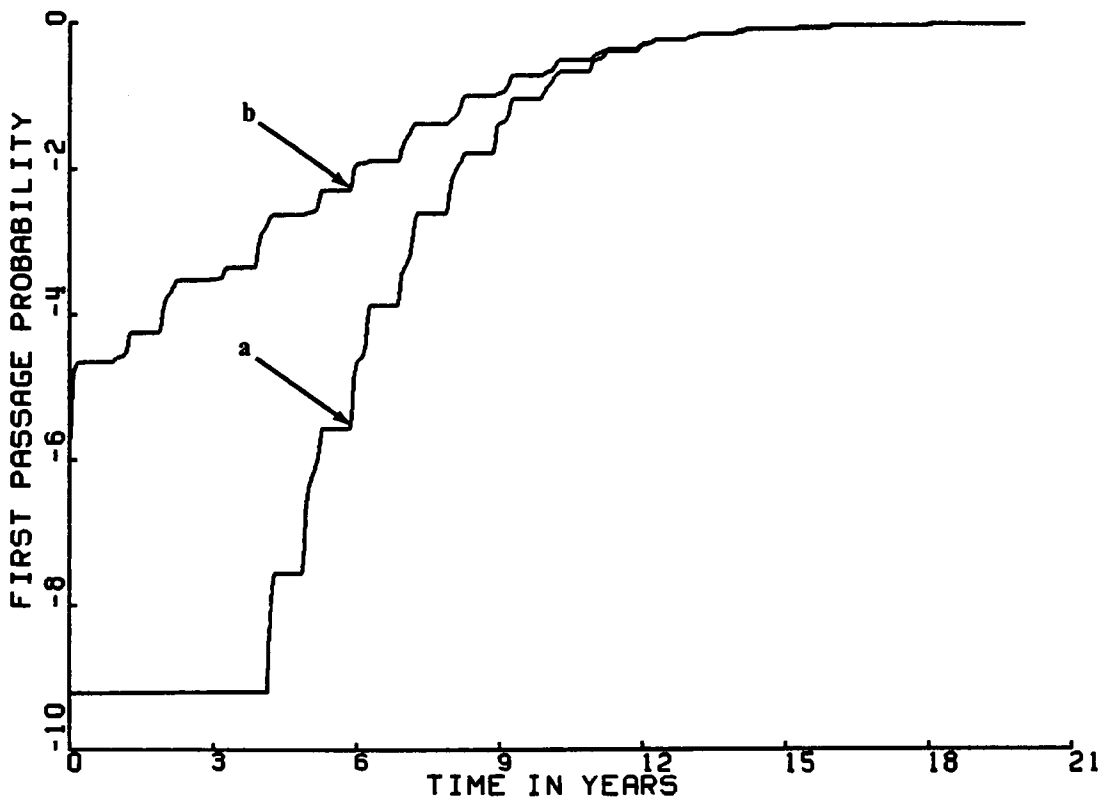


Figure 5.27 First Passage Probability at Barrow with Log-Normal Barrier: a) $\delta_D = 0.2$, $\delta_R = 0.1$ and b) $\delta_D = 0.2$, $\delta_R = 0.2$.

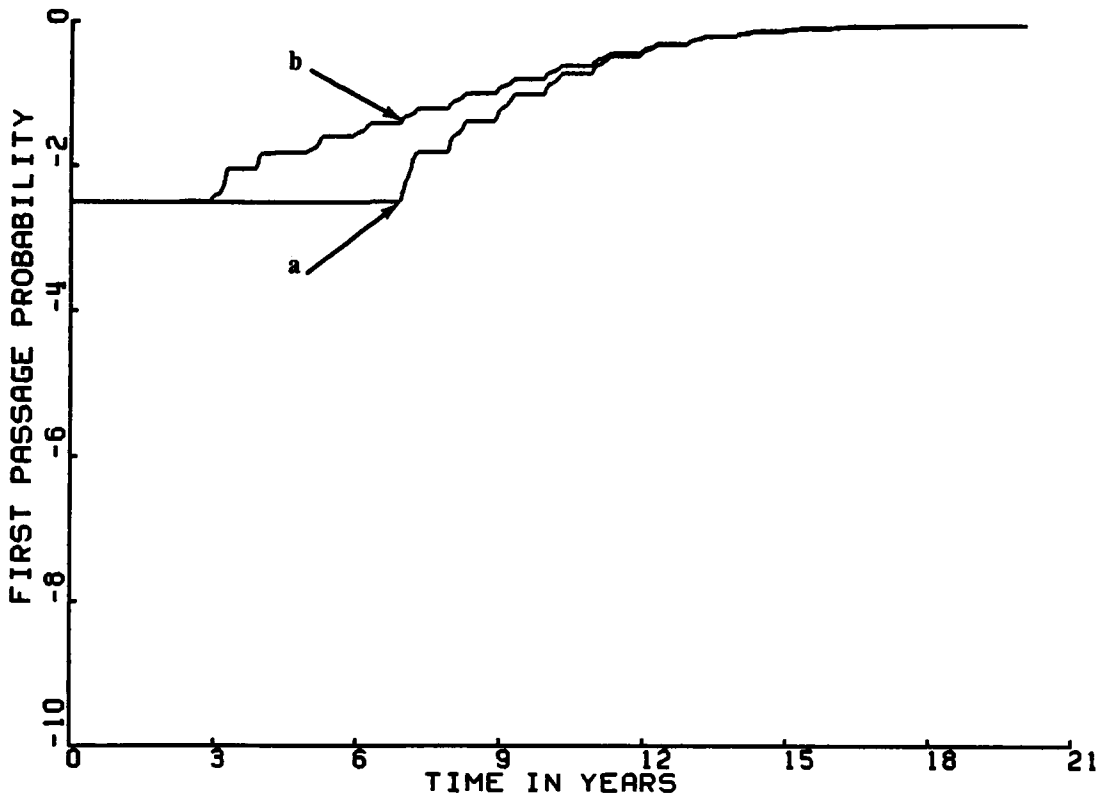


Figure 5.28 First Passage Probability at Barrow with Weibull Barrier: a) $\delta_D = 0.2$, $\delta_R = 0.1$ and b) $\delta_D = 0.2$, $\delta_R = 0.2$.

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Appendix A

The state of thermal stresses for plane strain problems have been developed by several authors [26-27]. The derivation presented here is based on Ref. 28. The equilibrium equation for plane strain and axisymmetric condition in cylindrical coordinates can be written as

$$\frac{\partial S_r}{\partial r} + \frac{S_r - S_\theta}{r} = 0 \quad (\text{A.1})$$

which can be satisfied by the stress function ϕ where

$$S_r = \frac{\phi}{r} \quad (\text{A.2})$$

$$S_\theta = \frac{\partial \phi}{\partial r}$$

The stress-strain relations are written as

$$\begin{aligned} \epsilon_r &= \frac{1}{E} [S_r - \nu(S_\theta + S_z)] + \alpha T \\ \epsilon_\theta &= \frac{1}{E} [S_\theta - \nu(S_r + S_z)] + \alpha T \\ \epsilon_z &= \frac{1}{E} [S_z - \nu(S_r + S_\theta)] + \alpha T \end{aligned} \quad (\text{A.3})$$

in which E , ν and α are modulus of elasticity, Poisson's ratio, and coefficient of thermal expansion, respectively. The third equation of Eqs. A.3 yields for plane strain $\epsilon_z = 0$

$$S_z = \nu(S_r + S_\theta) - \alpha ET \quad (\text{A.4})$$

The strain-displacement relations are

$$\begin{aligned} \epsilon_r &= \frac{dU_r}{dr} \\ \epsilon_\theta &= \frac{U_r}{r} \end{aligned} \quad (\text{A.5})$$

The compatibility equation is written as

$$r \frac{\partial \epsilon_\theta}{\partial r} + \epsilon_\theta - \epsilon_r = 0 \quad (\text{A.6})$$

Substituting Eqs. A.3 into A.6, yields the governing differential equation for long hollow cylinder which can be given as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} = - \frac{\alpha E}{1-\nu} \frac{\partial T}{\partial r} \quad (\text{A.7})$$

Eq. A.7 can be rewritten as

$$\frac{\partial}{\partial r} \left[- \frac{1}{r} \frac{\partial}{\partial r} (r\phi) \right] = - \frac{\alpha E}{1-\nu} \frac{\partial T}{\partial r} \quad (\text{A.8})$$

a direct integration of Eq. A.8 gives an expression for ϕ

$$\phi = - \frac{\alpha E}{1-\nu} \frac{1}{r} \int_{r_1}^r T r \, dr + \frac{C_1 r}{2} + \frac{C_2}{r} \quad (\text{A.9})$$

From Eq. A.2 the radial stress, S_r , becomes

$$S_r = \frac{\phi}{r} = -\frac{\alpha E}{1-\nu} \frac{1}{r^2} \int_{r_1}^r Tr \, dr + \frac{C_1}{2} + \frac{C_2}{r^2} \quad (\text{A.10})$$

To obtain the stress and strains in the second layer, the following boundary conditions are applied

$$S_r = 0 \quad \text{at } r = r_1, r_3 \quad (\text{A.11})$$

$$S_r = -p' \quad \text{at } r = r_2$$

These yield

$$0 = \frac{C_1}{2} + \frac{C_2}{r_1^2} \quad (\text{A.12})$$

$$-p' = -\frac{\alpha_2 E_2}{1-\nu_2} \frac{1}{r_2^2} \int_{r_1}^{r_2} Tr \, dr + \frac{C_1}{2} + \frac{C_2}{r_2^2} \quad (\text{A.13})$$

where α_2 , E_2 , and ν_2 are the mechanical properties of the second layer. The constants, C_1 and C_2 , can be written as

$$C_2 = \frac{p' r_1^2 r_2^2}{r_2^2 - r_1^2} - \frac{\alpha_2 E_2}{1-\nu_2} \left[\frac{r_1^2}{r_2^2 - r_1^2} \right] \int_{r_1}^{r_2} Tr \, dr \quad (\text{A.14})$$

$$C_1 = -\frac{2p' r_2^2}{r_2^2 - r_1^2} + \frac{2\alpha_2 E_2}{1-\nu_2} \left[\frac{1}{r_2^2 - r_1^2} \right] \int_{r_1}^{r_2} Tr \, dr \quad (\text{A.15})$$

The thermal stresses in the second layer can now be obtained from Eq.

A.10 and A.2 as

$$S_r = -\frac{r_2^2 p'}{r_2^2 - r_1^2} \left[1 - \frac{r_1^2}{r^2}\right] + \frac{\alpha_2 E_2}{(1-\nu_2)(r_2^2 - r_1^2)} \left[1 - \frac{r_1^2}{r^2}\right] \int_{r_1}^{r_2} Tr \, dr \quad (\text{A.16})$$

$$S_\theta = -\frac{r_2^2 p'}{r_2^2 - r_1^2} \left[1 + \frac{r_1^2}{r^2}\right] + \frac{\alpha_2 E_2}{(1-\nu_2)(r_2^2 - r_1^2)} \left[1 + \frac{r_1^2}{r^2}\right] \int_{r_1}^{r_2} Tr \, dr$$

$$+ \frac{\alpha_2 E_2}{(1-\nu_2)r^2} \int_{r_1}^r Tr \, dr - \frac{\alpha_2 E_2 T}{1-\nu_2} \quad (\text{A.17})$$

using Eq. A.4

$$S_z = -\frac{2\nu_2 r_2^2 p'}{r_2^2 - r_1^2} + \frac{2\nu_2 \alpha_2 E_2}{(1-\nu_2)(r_2^2 - r_1^2)} \int_{r_1}^{r_2} Tr \, dr - \frac{\alpha_2 E_2 T}{1-\nu_2} \quad (\text{A.18})$$

substituting Eqs. A.16-A.18 into Eqs. A.3, the strains in the second layer can be written as

$$\epsilon_r = -\frac{(1+\nu_2)r_2^2 p'}{E_2(r_2^2 - r_1^2)} \left[(1-2\nu_2) - \frac{r_1^2}{r^2}\right] + \frac{(1+\nu_2)\alpha_2}{(1-\nu_2)(r_2^2 - r_1^2)}$$

$$\left[(1-2\nu_2) - \frac{r_1^2}{r^2}\right] \int_{r_1}^{r_2} Tr \, dr - \frac{\alpha_2(1+\nu_2)}{(1-\nu_2)r^2} \int_{r_1}^r Tr \, dr + \frac{1+\nu_2}{1-\nu_2} \alpha_2 T \quad (\text{A.19})$$

$$\varepsilon_{\theta} = - \frac{(1+\nu_2)r_2^2 p'}{E_2(r_2^2 - r_1^2)} \left[(1-2\nu_2) + \frac{r_1^2}{r^2} \right] + \frac{(1+\nu_2)\alpha_2}{(1-\nu_2)(r_2^2 - r_1^2)} \left[(1-2\nu_2) + \frac{r_1^2}{r^2} \right] \int_{r_1}^{r_2} \text{Tr} \, dr + \frac{\alpha_2(1+\nu_2)}{(1-\nu_2)r^2} \int_{r_1}^r \text{Tr} \, dr \quad (\text{A.20})$$

integrating Eq. A.19 to obtain the radial displacement

$$U_r = - \frac{(1+\nu_2)r_2^2 p'}{E_2(r_2^2 - r_1^2)} \left[(1-2\nu_2)r + \frac{r_1^2}{r} \right] + \frac{\alpha_2(1+\nu_2)}{(1-\nu_2)r} \int_{r_1}^r \text{Tr} \, dr + \frac{(1+\nu_2)\alpha_2}{(1-\nu_2)(r_2^2 - r_1^2)} \left[(1-2\nu_2)r + \frac{r_1^2}{r} \right] \int_{r_1}^{r_2} \text{Tr} \, dr \quad (\text{A.21})$$

To obtain the stresses and strains in the third layer, the stress function is written as

$$\phi = - \frac{\alpha_3 E_3}{1-\nu_3} \frac{1}{r} \int_{r_2}^r \text{Tr} \, dr + \frac{C_3 r}{2} + \frac{C_4}{r} \quad (\text{A.22})$$

in which α_3 , E_3 , and ν_3 are the mechanical properties of the third layer. Applying the boundary conditions, Eqs. A.11, the constants are evaluated as

$$C_4 = -p' \left[\frac{r_2^2 r_3^2}{r_3^2 - r_2^2} \right] - \frac{\alpha_3 E_3}{1-\nu_3} \frac{1}{r_3^2} \left[\frac{r_2^2 r_3^2}{r_3^2 - r_2^2} \right] \int_{r_2}^{r_3} \text{Tr} \, dr \quad (\text{A.23})$$

$$C_3 = 2 \left[\frac{r_2^2 p'}{r_3^2 - r_2^2} + \frac{\alpha_3 E_3}{(1-\nu_3)(r_3^2 - r_2^2)} \int_{r_2}^{r_3} Tr \, dr \right] \quad (\text{A.24})$$

The thermal stresses and strains in the third layer can now be written as

$$S_r = \frac{r_2^2 p'}{r_3^2 - r_2^2} \left[1 - \frac{r_3^2}{r^2} \right] + \frac{\alpha_3 E_3}{(1-\nu_3)(r_3^2 - r_2^2)} \left[1 - \frac{r_2^2}{r^2} \right] \int_{r_2}^{r_3} Tr \, dr - \frac{\alpha_3 E_3}{(1-\nu_3)r^2} \int_{r_2}^r Tr \, dr \quad (\text{A.25})$$

$$S_\theta = \frac{r_2^2 p'}{(r_3^2 - r_2^2)} \left[1 + \frac{r_3^2}{r^2} \right] + \frac{\alpha_3 E_3}{(1-\nu_3)(r_3^2 - r_2^2)} \left[1 + \frac{r_2^2}{r^2} \right] \int_{r_2}^{r_3} Tr \, dr + \frac{\alpha_3 E_3}{(1-\nu_3)r^2} \int_{r_2}^r Tr \, dr - \frac{\alpha_3 E_3 T}{1-\nu_3} \quad (\text{A.26})$$

$$\epsilon_r = \frac{(1+\nu_3)r_2^2 p'}{E_3(r_3^2 - r_2^2)} \left[(1-2\nu_3) - \frac{r_3^2}{r^2} \right] - \frac{(1+\nu_3)\alpha_3}{(1-\nu_3)r^2} \int_{r_2}^r Tr \, dr + \frac{\alpha_3(1+\nu_3)}{(1-\nu_3)(r_3^2 - r_2^2)} \left[(1-2\nu_3) - \frac{r_2^2}{r^2} \right] \int_{r_2}^r Tr \, dr + \frac{(1+\nu_3)\alpha_3 T}{1-\nu_3} \quad (\text{A.27})$$

$$\epsilon_{\theta} = \frac{(1+\nu_3)r_2^2 p'}{E_3(r_3^2 - r_2^2)} \left[(1-2\nu_3) + \frac{r_3^2}{r^2} \right] + \frac{(1+\nu_3)\alpha_3}{(1-\nu_3)(r_3^2 - r_2^2)} \left[(1-2\nu_3) + \frac{r_2^2}{r^2} \right] \int_{r_2}^{r_3} Tr \, dr + \frac{(1+\nu_3)\alpha_3}{(1-\nu_3)r^2} \int_{r_2}^r Tr \, dr \quad (\text{A.28})$$

The radial displacement can be written as

$$U_r = \frac{(1+\nu_3)r_2^2 p'}{E_3(r_3^2 - r_2^2)} \left[(1-2\nu_3)r + \frac{r_3^2}{r} \right] + \frac{(1+\nu_3)\alpha_3}{(1-\nu_3)r} \int_{r_2}^r Tr \, dr + \frac{(1+\nu_3)\alpha_3}{(1-\nu_3)(r_3^2 - r_2^2)} \left[(1-2\nu_3)r + \frac{r_2^2}{r} \right] \quad (\text{A.29})$$

To obtain an expression for p' , the following boundary condition holds

$$U_2 \Big|_{r=r_2} = U_3 \Big|_{r=r_2} \quad (\text{A.30})$$

If the third layer is very thin in comparison to the second one, then r_3 can be written as

$$r_3 = r_2 + \Delta r \quad (\text{A.31})$$

and

$$r_3^2 = r_2^2 + 2\Delta r r_2 \quad (\text{A.32})$$

Applying Eq. A.30 in conjunction with A.31 and A.32, p' can be written as,

$$p' = \frac{E_2 E_3 [2\alpha_2 (1+\nu_2) \int_{r_1}^{r_2} T r \, dr - \alpha_3 (1+\nu_3) (r_2^2 - r_1^2) T(r_2)]}{E_3 (1+\nu_2) [(1-2\nu_2) r_2^2 + r_1^2] + E_2 (1-\nu_3)^2 \frac{r_2 (r_2^2 - r_1^2)}{r_3 - r_2}} \quad (\text{A.33})$$

Eqs. 3.7-3.11 are obtained by substituting $R_2(r, \omega)$ for T in Eqs. A.16, A.17, A.19, A.20, and A.33.

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ENVIRONMENTAL THERMAL STRESSES AS A
FIRST PASSAGE PROBLEM

by

Hazim S. Zibdeh

(ABSTRACT)

Due to changes of the thermal environment, thermal stresses are produced in structures. Two approaches based on the stochastic process theory are used to describe this phenomenon.

The structure is idealized as a long hollow viscoelastic cylinder. Two sites are considered: Barrow (AK) and Yuma (AZ).

First passage concepts are applied to characterize the reliability of the system. Crossings are assumed to follow either the behavior of the Poisson process or Markov process. In both cases, the distribution of the time to first passage is taken to be the exponential distribution.

Because the material is viscoelastic, statistically and time varying barriers (strengths) with Normal, Log-Normal, or Weibull distributions are considered. Degradation of the barriers by aging and cumulative damage are incorporated in the analysis.