REPRESENTATION THEORY OF THE DIAGRAM $A_n$ 
OVER THE RING $k[[x]]$

by

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Fix $R = k[[x]]$. Let $\Omega_n$ be the category whose objects are $((M_1, \ldots, M_n), (f_1, \ldots, f_{n-1}))$ where each $M_i$ is a free $R$-module and $f_i: M_i \rightarrow M_{i+1}$ for each $i = 1, \ldots, n-1$, and in which the morphisms are the obvious ones. Let $B_n$ be the full subcategory of $\Omega_n$ in which each map $f_i$ is a monomorphism whose cokernel is a torsion module. It is shown that there is a full dense functor $\Omega_n \rightarrow B_n$. If $X$ is an object of $B_n$, we say that $X$ diagonalizes if $X$ is isomorphic to a direct sum of objects $((M_1, \ldots, M_n), (f_1, \ldots, f_{n-1}))$ in which each $M_i$ is of rank one.

We establish an algorithm which diagonalizes any diagonalizable object $X$ of $B_n$, and which fails only in case $X$ is not diagonalizable.

Let $A$ be an artin algebra of finite type. We prove that for a fixed $C$ in $\text{mod}(A)$ there are only finitely many modules $A$ in $\text{mod}(A)$ (up to isomorphism) for which a short exact sequence of the form $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is indecomposable.
To my family
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Chapter I. Introduction.

Let \( R \) be a commutative ring with 1, and let \( A_n \) be the diagram \( 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \). An object in the category of representations of \( A_n \) by \( R \)-modules and \( R \)-module maps is an ordered \( n \)-tuple \( (M_1, \ldots, M_n) \) of \( R \)-modules together with an ordered \((n-1)\)-tuple \( (f_1, \ldots, f_{n-1}) \) of \( R \)-module maps such that \( f_i : M_i \rightarrow M_{i+1} \) for \( i = 1, 2, \ldots, n-1 \). Of course we think of putting the modules in at the vertices of \( A_n \) and the maps in for the arrows to obtain the picture \( \xymatrix{ M_1 \ar[r]^{f_1} & M_2 
 overhear \cdots \ar[r]^{f_{n-1}} & M_n } \).

If \( X = \xymatrix{ M_1 \ar[r]^{f_1} & M_2 
 overhear \cdots \ar[r]^{f_{n-1}} & M_n } \) and \( X' = \xymatrix{ M'_1 \ar[r]^{f'_1} & M'_2 
 overhear \cdots \ar[r]^{f'_{n-1}} & M'_n } \) are objects in this category, then a morphism \( \eta \) from \( X \) to \( X' \) is an \( n \)-tuple \( \eta = (\eta_1, \ldots, \eta_n) \) of \( R \)-module maps making the diagram

\[
\begin{array}{cccccc}
M_1 & \xrightarrow{f_2} & M_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & M_n \\
\eta_1 & \downarrow & \eta_2 & \downarrow & \cdots & \downarrow & \eta_n \\
M'_1 & \xrightarrow{f'_1} & M'_2 & \xrightarrow{f'_2} & \cdots & \xrightarrow{f'_{n-1}} & M'_n
\end{array}
\]

commute.

An object \( X \) decomposes in this category if \( X \cong X_1 \otimes X_2 \) where neither \( X_1 \) nor \( X_2 \) is zero. For now and for this entire thesis, \( R \) will denote the ring \( k[[x]] \) of formal power series in one variable over a field \( k \), and \( \Omega_n \) will denote
the category of representations of $A_n$ by finitely generated free $R$-modules and $R$-module maps; the problem of this thesis is to study certain questions about the category $Q_n$. The impetus for this study comes mainly from two directions. The first is the following result, which is just a re-statement of the invariant factor theorem.

**Proposition I.1.** Let $F_0$ and $F_1$ be finitely generated free $R$-modules, and let $X = F_1 \longrightarrow F_0$ be in $Q_2$. Then $X$ decomposes as a direct sum of representations of the types $0 \longrightarrow R$, $R \longrightarrow 0$, and $R \longrightarrow R$, where $f_1 \neq 0$.

If the cokernel of a map $g$ between free $R$-modules is a torsion module, we will call $g$ a *full* map. If the map $f$ in proposition I.1 is a full monomorphism, then $\text{rank}(F_1) = \text{rank}(F_0)$, and only maps of the type $f_1: R \longrightarrow R$ can occur. Any map from $R$ to $R$ is just multiplication by an element of $R$, so in this case we can interpret the result as saying that there exist bases for $F_1$ and $F_0$ so that the matrix of $f$ with respect to these bases is diagonal. Unfortunately even for $A_3$ not every representation $X = F_2 \overset{f}{\longrightarrow} F_1 \overset{g}{\longrightarrow} F_0$ in which $f$ and $g$ are full monomorphisms can be diagonalized, i.e. there do not always exist bases for $F_2$, $F_1$, and $F_0$ with respect to which the matrices of $f$ and $g$ are simultaneously diagonal. The main result of Chapter II is a partial remedy for this; it establishes an algorithm which determines whether a given
representation of $A_n$ by finitely generated free $R$-modules and full monomorphisms is diagonalizable, and if it is, diagonalizes it.

Our second reason to study representations of $A_n$ by free $R$-modules is that it is a natural generalization of the representations of $A_n$ by $k$-vector spaces and $k$-linear maps. Because of its intimate connection with categories of modules over artin rings, this latter class has been extensively studied and is well-understood; indeed the category of such objects is known to be of finite representation type, i.e. there are only finitely many non-isomorphic indecomposable representations (see Gabriel's paper [6721]; one consequence of this work is that if $X = k^n \xrightarrow{f_{n-1}} k^n \xrightarrow{f_1} k^n$ is a representation of $A_n$ over $k$ then $X$ diagonalizes). We see at once that the category $Q_n$ cannot be of finite type, as even $Q_2$ is not; if we let $\mu_i$ denote multiplication by $x^i$, then $(R_{\mu_i} R | i = 0, 1, \ldots)$ is an infinite family of non-isomorphic indecomposable representations of $A_2$. Yet because $R$ is so "close" to a field, it was hoped that at least the category $Q_n$ might be amenable to some "nice" description, say a description of all non-isomorphic indecomposable representations in terms of one-parameter families of these (such a category is of tame representation type; see [DR76]); this is in fact what
proposition I.1 does for $Q_2$. Unfortunately, this is not the case in general; as we shall show in Chapter III, $Q_3$ is already of wild (i.e., not tame) representation type.

We mention one more area with which our problem makes contact, although we have not studied it from this point of view. Let $K$ be the quotient field of $R$, and let $S$ be a subring of $M_n(K)$, the ring of $n \times n$ matrices with entries from $K$. An $R$-order in $S$ is a subring $A$ of $S$ such that $K \otimes A = S$ (orders can be defined much more generally; for a definition and full discussion see [CR]). A (left) $R$-lattice over the $R$-order $A$ is an $A$-module which is finitely generated and projective as an $R$-module. Orders and lattices arise naturally in the study of group representations (cf. [CR]); the orders $A$ arising there are called classical because $K \otimes A$ is a semisimple ring.

If we let $S = T_n(K)$, the ring of lower-triangular $n \times n$ matrices with entries from $K$, then $T_n(R)$ is an $R$-order in $S$. The lattices over $T_n(R)$ are most easily pictured as columns

$L_0 = \begin{pmatrix} F_{n-1} \\ F_0 \end{pmatrix}$

of finitely generated free $R$-modules together with maps $f_i : F_i \rightarrow F_{i-1}$ for $i = 1, 2, \ldots, n-1$. The action of $T_n(R)$ on

$L = \left\{ \begin{pmatrix} F_{n-1} \\ \vdots \\ F_0 \end{pmatrix}, \langle f_{n-1}, \ldots, f_1 \rangle \right\}$

is as follows. Let $z = \begin{pmatrix} z_{n-1} \\ \vdots \\ z_0 \end{pmatrix} \in L_0$ and $r = (r_{i,j}) \in T_n(R)$; then the $t^{th}$ entry of $rz$ is
The functor which sends the object \( X = \cdots \circ f_{n-1}(z_{n-1}) \circ \cdots \circ f_t(z+t) + f_t(z_t) \) sends \( X \in \Omega_n \) to \( L \) is an equivalence of categories connecting the problem of representations of \( A_n \) with questions of \( R \)-lattices over \( T_n(R) \). The order \( T_n(R) \) is the natural next step away from classical, as \( K \otimes T_n(R) \) is hereditary (i.e. every ideal is projective). Our results show that the category of \( R \)-lattices of \( T_n(R) \) is of wild representation type for \( n \geq 3 \), and provide some information about the decomposition of certain special lattices over \( T_n(R) \).

In Chapter II we confine our attention to the full subcategory \( B_n \) of \( \Omega_n \) whose objects are those representations \( f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0 \) of \( A_n \) in which each map \( f_i \) is a full monomorphism. If we choose bases for the modules \( F_i \) and write the maps as matrices, then \( B_n \) contains the representations in which each \( f_i \) is a square matrix with non-zero determinant. In this category we can discuss the problem of simultaneous diagonalization of the matrices, i.e. the problem of finding isomorphisms \( \alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_0 \) such that \( \alpha_{i-1} f_i \alpha_i^{-1} \) is a diagonal matrix for each \( i=1,2,\ldots,n-1 \). The algorithm developed solves this problem.
whenever it can be solved.

Our strategy in proving that the algorithm works as claimed is as follows. We first show that any object of $B_n$ can be written as a direct sum of an object of a special, easily diagonalized type and an object from a certain full subcategory $C_n$. We then exhibit a functor $\lambda$ from $C_n$ to a category of $(n-2)$-tuples of short exact sequences of torsion $R$-modules (satisfying some conditions), and prove that $\lambda$ is in fact a representation equivalence. This means that an object $X$ in $C_n$ decomposes iff the corresponding object $\lambda X$ decomposes. Of course any sequence of torsion $R$-modules is a sequence of $k[x]/(x^t)$-modules for some $t$, so we are able to use techniques from the theory of artin algebras to obtain information about the decomposition of $\lambda X$. We then lift this information, via $\lambda$, to information about the decomposition of $X$. Although the devices of non-commutative ring theory enter into the proof in an apparently essential way, the algorithm itself makes no reference to them; it operates only on matrices in $M_n(R)$.

From the viewpoint of representation theory, an important property of $B_n$ is that decompositions of objects in $B_n$ are unique up to isomorphism. This is easily seen from the following considerations. The category $C_n$ is, as we have said, representation equivalent to a category $T_{n-2}$ of special $(n-2)$-tuples of short exact sequences of torsion
R-modules. This latter category is itself naturally equivalent, in a way to be explained in Chapter II, to the category $U_{n-2}$ whose objects are sequences $Y = \alpha_{n-2}^{\cdot} \xrightarrow{\alpha_{n-3}^{\cdot}} M_{n-3} \xrightarrow{\alpha_{n-4}^{\cdot}} \cdots \xrightarrow{\alpha_1^{\cdot}} M_0$ of torsion $R$-modules and monomorphisms. In the same way that an object of $Q_n$ is a module over $T_n(R)$, an object of $U_{n-2}$ is a module over the ring $\mathcal{A}_{n-1,t}$ of lower-triangular $(n-1) \times (n-1)$ matrices with entries from $k[x]/(x^t)$ for some $t$. The ring $\mathcal{A}_{n-1,t}$ is artinian, so mod$(\mathcal{A}_{n-1,t})$ is a Krull-Schmidt category; so decompositions of $\mathcal{A}_{n-1,t}$-modules are unique up to isomorphism. The correspondence between decompositions in this category and decompositions in $C_n$ allows us to conclude that the latter are also unique up to isomorphism.

Chapter III is devoted to some examples of the correspondence which $\lambda$ sets up between objects in $C_3$ and short exact sequences of torsion $R$-modules. We show for a few classes of sequences what kinds of representations correspond to them, and what kind of information about the representations can be obtained from knowledge of the sequences. As an example, we show that if $X = R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ is such that $gf = x^t I$ for some $t$ (where $I$ is the nxn identity matrix), then $X$ must diagonalize.

Chapter IV is entirely concerned with the construction
of a full dense functor from $Q_n$ to $B_n$.

Chapter V deals with a slightly different topic than do the earlier sections. Its main result is a theorem about the indecomposability of short exact sequences of $A$-modules, where $A$ is an artin ring of finite representation type. This theorem says, in effect, that the exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ can be indecomposable only if $A$ is not "too big" compared to $C$, where the sizes involved are related to the lengths of the socles of $A$ and $C$. The section ends with an application of this theorem to objects in $C_3$, an interesting consequence of which is the following.

If $X = R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ is in $C_3$ and the degree of the determinant of $f$ is less than $n$, then $X$ must decompose.

We mention that although we have, in this thesis, restricted our attention to the category $\text{mod}(R)$, most of our techniques, and hence results, are good whenever $R$ is a local PID.
Chapter II. Diagonalization.

As was stated in the introduction, the purpose of this chapter is to develop an algorithm which generalizes the invariant factor theorem. Thus the question of this chapter is: when does a representation in $B_n$ decompose as a direct sum of representations $F_{n-1} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0$ in which each $F_i$ has rank one? If the $f_i$ are written as matrices, this is equivalent to asking when there are bases for $F_{n-1}, \ldots, F_0$ with respect to which each $f_i$ is diagonal; therefore if a representation does decompose in this way we say that it diagonalizes. We are able to supply an answer to this question in the form of an algorithm for diagonalizing an object in $B_n$ which fails only in case the object does not diagonalize.

§1. Statement of the problem. The functor $\lambda$.

The first step in the development of the algorithm is Lemma II.1.1; but first we need a few definitions.

**Definition.** Let $C_n$ be the full subcategory of $B_n$ consisting of those objects of $B_n$ for which $(f_1 \cdots f_{n-1}) (F_{n-1}) \subseteq xF_0$. (Because $R$ is local, this condition ensures that $F_0 - \text{coker}(f_1 \cdots f_{n-1})$ is a projective cover.)

**Definition.** If in the diagram
we have \( \tau_i \sigma_i = 1_{N_i} \), \( \alpha_i \sigma_i = \sigma_{i-1} \beta_i \), and \( \beta_i \tau_i = \tau_{i-1} \alpha_i \) for \( i = 2, 3, \ldots, n \), then we say that the \( \sigma_i \) are \textit{consistently split} by the \( \tau_i \) (or vice versa). If \( (\alpha_n, \ldots, \alpha_1), (\beta_n, \ldots, \beta_1) \) are morphisms in \( \Omega_n \) then we say that \( (\alpha_n, \ldots, \alpha_1) \) is consistently split by \( (\beta_n, \ldots, \beta_1) \) and that the bottom row is a summand of the top.

**Lemma II.1.1.** Let \( X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0 \) in \( \mathcal{B}_n \). If \( f_1 \circ \cdots \circ f_{n-1} (F_{n-1}) \not\subseteq \mathcal{X}F_0 \), then \( X \) has a summand of the form

\[
R \quad \cdots \quad R \quad \cdots \quad R \quad (n \text{ copies}).
\]

**Proof.** The hypothesis implies that there is an injection \( R \xrightarrow{\alpha} F_{n-1} \) such that \( f_1 \circ \cdots \circ f_{n-1} \circ \alpha(R) \supseteq R \). Because \( R \) is projective there is a splitting \( \beta \) of \( f_1 \circ \cdots \circ f_{n-1} \circ \alpha \). Set \( \sigma_{i-1} = \alpha \), \( \sigma_j = f_j \circ \cdots \circ f_{n-1} \circ \alpha \) for \( 0 \leq j < n-1 \), and set \( \tau_0 = \beta \), \( \tau_j = \beta \circ f_{j+1} \circ \cdots \circ f_{n-1} \circ \alpha \) for \( 1 \leq j < n-1 \). Then in the diagram

\[
\begin{array}{ccccccc}
F_{n-1} & \xrightarrow{f_{n-1}} & F_{n-2} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_1} & F_0 \\
\sigma_{n-1} & \uparrow & \sigma_{n-2} & \uparrow & \sigma_{n-2} & \uparrow & \sigma_0 & \uparrow & \sigma_0 \\
R & \quad \cdots \quad & R & \quad \cdots \quad & R & \quad \cdots \quad & R
\end{array}
\]
$(\sigma_{n-1},\ldots,\sigma_0)$ is a monomorphism consistently split by the epimorphism $(\tau_{n-1},\ldots,\tau_0)$, so the bottom row is a summand of the top.

Lemma II.1.1 says that an object in $B_n$ is a direct sum of an object in $C_n$ and a representation of the form $R^d \longrightarrow R^d \longrightarrow \cdots \longrightarrow R^d$, so we can concentrate on $C_n$. Essential for the analysis of objects in $C_n$ is the construction of a functor $\lambda$ from $C_n$ to a certain category of torsion $R$-modules and monomorphisms. This allows the introduction of artin algebra techniques into the problem.

Define the category $T_n$ whose objects are ordered $n$-tuples $(\epsilon_1,\ldots,\epsilon_n)$ of short exact sequences

$$\epsilon_i = 0 \rightarrow A_i \xrightarrow{a_i} B_i \xrightarrow{b_i} C_i \rightarrow 0$$

of torsion $R$-modules with the property that $B_i = A_{i-1}$ for $2 \leq i \leq n$. A morphism in $T_n$ from $(\epsilon_1,\ldots,\epsilon_n)$ to $(\epsilon'_1,\ldots,\epsilon'_n)$ is an ordered $n$-tuple of ordered triples of $R$-module maps $(\alpha_{i1}, \alpha_{i2}, \alpha_{i3}), \ldots, (\alpha_{n1}, \alpha_{n2}, \alpha_{n3})$ satisfying: (i) $\alpha_{i2} = \alpha_{i-1,1}$ for $2 \leq i \leq n$; (ii) the diagram

$$
\begin{array}{ccccccccc}
\epsilon_i &=& 0 & \rightarrow & A_i & \xrightarrow{a_i} & B_i & \xrightarrow{b_i} & C_i & \rightarrow & 0 \\
& & \downarrow{\alpha_{i1}} & & \downarrow{\alpha_{i2}} & & \downarrow{\alpha_{i3}} & & \\
\epsilon'_i &=& 0 & \rightarrow & A'_i & \xrightarrow{a'_i} & B'_i & \xrightarrow{b'_i} & C'_i & \rightarrow & 0
\end{array}
$$

commutes for $i=1,2,\ldots,n$. 
We now define the functor \( \lambda: C_n \rightarrow T_{n-2} \) (n > 2). Let \( X = F_n \overset{f_{n-1}}{\rightarrow} F_{n-2} \overset{f_{n-2}}{\rightarrow} \cdots \overset{f_1}{\rightarrow} F_0 \) be in \( C_n \). In the commutative diagram

\[
\begin{array}{ccccccccc}
F_{n-1} & \longrightarrow & F_{n-1} \\
\downarrow f_2 \cdots \cdots f_{n-1} & & \downarrow f_1 \cdots \cdots f_{n-1} \\
0 & \longrightarrow & F_1 & \overset{f_1}{\longrightarrow} & F_0 & \longrightarrow & C_1 & \longrightarrow & 0 \\
\downarrow \rho_1 & & \downarrow \rho_0 & & & & & & \\
0 & \longrightarrow & A_1 & \overset{a_1}{\longrightarrow} & B_1 & \overset{b_1}{\longrightarrow} & C_1 & \longrightarrow & 0
\end{array}
\]

\( \rho_0 \) is a projective cover. The bottom row of this diagram is the short exact sequence \( \epsilon_1 \). To construct \( \epsilon_2 \), form the diagram

\[
\begin{array}{ccccccccc}
F_{n-1} & \longrightarrow & F_{n-1} \\
\downarrow f_1 \cdots \cdots f_{n-1} & & \downarrow f_{2} \cdots \cdots f_{n-1} \\
0 & \longrightarrow & F_2 & \overset{f_2}{\longrightarrow} & F_1 & \longrightarrow & C_2 & \longrightarrow & 0 \\
\downarrow \rho_2 & & \downarrow \rho_1 & & & & & & \\
0 & \longrightarrow & A_2 & \overset{a_2}{\longrightarrow} & B_2 & \overset{b_2}{\longrightarrow} & C_2 & \longrightarrow & 0
\end{array}
\]

then \( \epsilon_2 \) is the bottom row. Clearly \( B_2 = A_1 \) as required. Continuing in this way we certainly get an object \( \lambda X \) of \( T_{n-2} \); the last step is the diagram
Suppose that \( X = \text{dom} \rightarrow \text{cod} \rightarrow \cdots \rightarrow \text{dom} \), and that \( p_n, \ldots, p_0 : X \rightarrow X' \) in \( C_n \). We get a morphism \( \lambda p : \lambda X \rightarrow \lambda X' \) in \( T_{n-2} \) in the following way. We need to fill in the diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{dom} \\
\downarrow f_n \\
\text{cod} \\
\downarrow f_{n-2} \\
\vdots \\
\downarrow f_1 \\
\text{dom} \\
\downarrow f_0 \\
0 \\
\end{array}
\end{array}
\]

with \( \alpha_{j1}, \alpha_{j2}, \alpha_{j3} \). But the left-hand "wall" induces \( \alpha_{j1} \), as its columns are exact; the middle wall induces \( \alpha_{j2} \); and
\(\alpha_j3\) can be gotten from either the middle or the bottom "floor". We will prove that the square

\[
\begin{array}{ccc}
A_j & \xrightarrow{a_j} & B_j \\
\alpha_j1 & \downarrow & \alpha_j2 \\
A_j & \xrightarrow{a_j} & B_j
\end{array}
\]

commutes; the proof that the right-hand half of the bottom floor commutes is the same. From the diagram we have that \(a_jp_1 = p_2^f_j\), so \(\alpha_j2_2a_jp_1 = \alpha_j2p_2^f_j\). Then \(\alpha_j2_2a_jp_1 = \alpha_j2p_2^f_j = p_2^f_j = \beta_jj = \beta_j2p_1;\) as \(p_1\) is an epimorphism, this gives \(\alpha_j2_2a_j = a_j\alpha_j1\) as was wanted.

Because of the way in which the maps \((\alpha_j1, \alpha_j2, \alpha_j3)\) are induced, it is clear that \((\alpha_j1, \alpha_j2, \alpha_j3, \ldots, \alpha_jn-2, \alpha_jn-2, \alpha_jn-2, \alpha_j3)\) is a morphism from \(\lambda x\) to \(\lambda x'\) in \(T_{n-2}\).

That \(\lambda\) is an additive functor is not hard to verify. We want more than this, though; we want to show that \(\lambda\) is a representation equivalence, i.e. that \(\lambda\) is full and dense, and that it reflects isomorphisms and preserves indecomposables. To prove that \(\lambda\) has these properties we need the following (non-functorial) construction, which provides a sort of inverse for \(\lambda\).

Let \(Y = (\epsilon_1, \ldots, \epsilon_{n-2})\) be in \(T_{n-2}\), and let \(F_0 \rightarrow B_1\) be the projective cover of \(B_1\) in \(\text{mod}(R)\) (so \(F_0\) is a finitely generated free \(R\)-module). Construct an object \(F_{n-1} \rightarrow F_1 \rightarrow F_0\) as follows: let \(0 \rightarrow F_1 \rightarrow F_0 \rightarrow B_1 \rightarrow 0\) be
exact (where \( 0 \rightarrow A_1 \xrightarrow{a_1} B_1 \xrightarrow{b_1} C_1 \rightarrow 0 \)). Then the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{b_1 \rho_0} & C_1 & \rightarrow & 0 \\
\downarrow{\rho_1} & & \downarrow{\rho_0} & & & & \downarrow{\rho_0} & & \\
0 & \rightarrow & A_1 & \xrightarrow{a_1} & B_1 & \xrightarrow{b_1} & C_1 & \rightarrow & 0
\end{array}
\]

commutes, where \( \rho_1 \) is just the induced map. By the Snake Lemma, \( \rho_1 \) is an epimorphism and \( \ker \rho_1 \cong \ker \rho_0 \); let \( \eta_1 : F_{n-1} \rightarrow F_1 \) be the kernel of \( \rho_1 \). Then the diagram

\[
\begin{array}{ccccccccc}
F_{n-1} & \xrightarrow{=} & F_{n-1} \\
\downarrow{\eta_1} & & \downarrow{f_1 \eta_1} \\
0 & \rightarrow & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{\rho_0} & C_1 & \rightarrow & 0 \\
\downarrow{\rho_1} & & \downarrow{\rho_0} & & & & \downarrow{\rho_0} & & \\
0 & \rightarrow & A_1 & \xrightarrow{a_1} & B_1 & \xrightarrow{b_1} & C_1 & \rightarrow & 0
\end{array}
\]

commutes. The next step is to repeat this construction with \( \rho_1 \) in place of \( \rho_0 \); we get a commutative diagram

\[
\begin{array}{ccccccccc}
F_{n-1} & \xrightarrow{=} & F_{n-1} \\
\downarrow{\eta_2} & & \downarrow{f_2 \eta_2} \\
0 & \rightarrow & F_2 & \xrightarrow{f_2} & F_0 & \xrightarrow{\rho_0} & C_2 & \rightarrow & 0 \\
\downarrow{\rho_2} & & \downarrow{\rho_0} & & & & \downarrow{\rho_0} & & \\
0 & \rightarrow & A_2 & \xrightarrow{a_2} & B_2 & \xrightarrow{b_2} & C_2 & \rightarrow & 0
\end{array}
\]
After $n-2$ steps we get an object $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in $T_n$, which we will call $\theta Y$.

We will first use $\theta$ to show that $\lambda$ is full. Suppose that $Y = (\epsilon_1, \ldots, \epsilon_{n-2})$, $Y' = (\epsilon'_1, \ldots, \epsilon'_{n-2})$ in $T_{n-2}$, $\alpha: Y \rightarrow Y'$ in $T_{n-2}$, and that

$\theta Y = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_1$ and $\theta Y' = F'_{n-1} \xrightarrow{f'_{n-1}} \cdots \xrightarrow{f'_1} F'_0$.

Then we claim that there must be maps $\beta_{n-1}$, $\beta_1$, $\beta_0$ to make

```
commute. Both $F_0 \rightarrow B_1$ and $F'_0 \rightarrow B'_1$ are projective covers, so we can lift $\alpha_{12}$ to a map $\beta_0: F_0 \rightarrow F'_0$; then $\beta_{n-1}$ is also induced. Similarly because the rows of the middle floor are exact, $\beta_1$ is defined. It is easy to check that the whole
```
diagram commutes. Note that because $F_0 \to B$ and $F'_0 \to B'$ are projective covers, $\beta_0$ is an isomorphism if $\alpha_{12}$ is; and so if the other $\alpha_{ij}$ are isomorphisms, so are $\beta_{n-1}$ and $\beta_j$.

The next step is to find $\beta_2$ to make the diagram

\[
\begin{array}{ccccccccc}
\beta_{n-1} & F_{n-1} & \cdots & F_2 & F_1 & F_0 & 0 \\
&\downarrow f_2 & & \downarrow f_2 & \downarrow A_0 & \downarrow \alpha_1 & \\
& F_2 & F_1 & F_0 & A_2 & B_2 & C_2 & 0 \\
&\downarrow f_1 & & \downarrow f_1 & \downarrow \alpha_2 & \downarrow \beta_2 & & \\
& 0 & A_2 & B_2 & C_2 & 0 & \\
\end{array}
\]

commute; but $\beta_2$ is induced in this diagram exactly as $\beta_1$ was induced in the last, and again the whole diagram commutes. Again, note that if $\alpha$ is an isomorphism then because $\beta_1$ must be an isomorphism from the first diagram, $\beta_2$ must also be an isomorphism. Continuing this process gives a morphism $\beta = (\beta_{n-1}, \ldots, \beta_0) : \Theta Y \to \Theta Y'$ in $C_n$. $\beta$ is certainly not unique, but it is clear from the construction that $\lambda \beta = \alpha$, so that $\lambda$ is full. Because $\beta$ must be an isomorphism when $\alpha$ is one, we get $\Theta \lambda X \cong X$ for $X$ in $C_n$ (take $\alpha$ to be the identity on $\lambda X$). Thus if $\lambda X \cong \lambda X'$, then $\Theta \lambda X \cong \Theta \lambda X'$, so $X \cong X'$; so $\lambda$ reflects isomorphisms. Finally if $Y$ is in $T_n$ then clearly $\lambda \Theta Y \cong Y$, so
that $\lambda$ is dense.

Because $\lambda$ is additive, $\lambda X$ decomposes if $X$ does; we want to show that the converse is also true. Suppose $\lambda X$ decomposes as $Y \otimes Y'$, where $Y = (\epsilon_1, \ldots, \epsilon_{n-2})$ and $Y' = (\epsilon'_1, \ldots, \epsilon'_{n-2})$. We will be done if we can show that $\lambda \lambda X$ decomposes, as $\lambda \lambda X \cong X$. We think of $Y \otimes Y'$ as the object $(\epsilon^n_1, \ldots, \epsilon^n_{n-2})$, where

$$
\begin{bmatrix}
  a_i & 0 \\
  0 & a_i
\end{bmatrix}
\quad
\begin{bmatrix}
  b_i & 0 \\
  0 & b_i
\end{bmatrix}
$$

$$
\epsilon^n_i \to A_i \otimes A'_i \to B_i \otimes B'_i \to C_i \otimes C'_i \to 0.
$$

The projective cover of $B_i \otimes B'_i$ is the sum

$$
\begin{bmatrix}
  p_0 & 0 \\
  0 & p_0'
\end{bmatrix}
$$

$F_0 \otimes F'_0 \to B_i \otimes B'_i$ of the projective covers. The kernel of

$$
\begin{bmatrix}
  b_1 p_0 & 0 \\
  0 & b_1 p_0'
\end{bmatrix}
$$

$F_0 \otimes F'_0 \to C_i \otimes C'_i$ is just the sum of the kernels,

namely

$$
\begin{bmatrix}
  f_1 & 0 \\
  0 & f'_1
\end{bmatrix}
$$

$F_i \otimes F'_1 \to F_0 \otimes F'_0$. In the diagram
we can take $\sigma$ to be diagonal, $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$; then its kernel is also diagonal, i.e., a map $\zeta = \begin{bmatrix} f_1 & \cdots & f_{n-1} \\ 0 & f_1' & \cdots & f_{n-1}' \end{bmatrix} : F_{n-1} \otimes F_{n-1} \to F_{1} \otimes F_{1}$. Then the diagram commutes. The only crucial part of this construction was that the map $\rho$ could be taken to be diagonal; as the map $\sigma$ takes the role of $\rho$ in the diagram for $\varepsilon_2$, this process can continue. Thus $\mathfrak{A}X$ is decomposable.
We need to mention one more property of $\lambda$, and that is that if $X = F_{n-1} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0$ with each $F_i$ of rank 1, and $\lambda X = (\epsilon_1, \ldots, \epsilon_{n-2})$, then each $\epsilon_i$ is of the form

$0 \rightarrow V_i \rightarrow V_{i+j} \rightarrow V_j \rightarrow 0$, where $V_i$ is the indecomposable $R$-module of length $i$ and the maps are the obvious inclusion and surjection. (Henceforth we will write sequences of this sort without referring to the maps; we will always mean the obvious inclusion and surjection. Also, we will let $U_0$ be the zero module.) If an object $Y$ in $T_n$ decomposes as a direct sum of objects of this type, we call $Y$ diagonalizable; by the foregoing, an object $X$ in $C_n$ is diagonalizable in $C_n$ iff $\lambda X$ is diagonalizable in $T_{n-2}$.

If $(\epsilon_1, \ldots, \epsilon_n)$ is in $T_n$, we get an object in $U_n$, the category of sequences of torsion $R$-modules and monomorphisms, by taking the sequence $A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_1 \rightarrow B_1$. By doing the obvious thing on morphisms we get a functor which is clearly an equivalence of categories. We will denote the composition of $\lambda$ with this equivalence by $\lambda'$; thus if $X \in C_n$ then $\lambda X$ is in $T_{n-2}$ and $\lambda'X$ is in $U_{n-2}$.

§2. The case of $A_3$.

Our approach to developing the promised algorithm takes
us through the cases of $A_3$ and $A_4$ before getting to the general case. At each step we will need the results obtained at the previous step. We first establish some preliminary lemmas.

Lemma 11.2.1. There is an algorithm to find a basis for the kernel of an epimorphism $\tau: R^n \rightarrow R^m$.

Proof. Let $(e_1, \ldots, e_n)$ be a basis for $R^n$. Abstractly we know that there must be a subset of $m$ 'vectors' of the set $(\tau e_1, \ldots, \tau e_n)$ which forms a basis for $R^m$, as suppose $(\tau e_{i_1}, \ldots, \tau e_{i_s})$ is a subset of minimum cardinality which still spans $R^m$. Then if $s > m$, this set is linearly dependent as a set of vectors in $K^m$, where $K$ is the quotient field of $R$; say $\sum_{i=1}^{m} \alpha_i \tau e_i = 0$. Clearing denominators, we get a dependence relation with coefficients in $R$, so $s = m$. Re-order so that $\tau e_1, \ldots, \tau e_m$ are the basis elements; then if $j > m$, $\tau e_j = \sum_{i=1}^{m} r_{ji} \tau e_i$ for some $r_{ji} \in R$, so $e_j = \sum_{i=1}^{m} r_{ji} e_i \in \ker \tau$. There are exactly $n-m$ of these, and they span a summand of $R^n$ of rank $n-m$ ($(e_1, \ldots, e_m)$ spans a free complement to it) which is inside $\ker \tau$. As $\ker \tau$ is itself a summand of $R^n$ of rank $n-m$, $(e_j - \sum_{i=1}^{m} r_{ji} e_i \mid j = m+1, \ldots, n)$ is a basis for $\ker \tau$. ■
Proposition 11.2.2. Suppose $F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ is in $B_3$. If for some fixed bases for $F_2$, $F_1$, and $F_0$ the matrix of $f$ is 
\[
\begin{pmatrix}
    f_1 & 0 \\
    0 & f_2
\end{pmatrix}
\]
and the matrix of $gf$ is 
\[
\begin{pmatrix}
    h_1 & 0 \\
    0 & h_2
\end{pmatrix},
\]
where the blocks are the same size, then the matrix of $g$ with respect to these bases is also block diagonal with blocks of the same size.

Proof. The map $f$ is a monomorphism and hence has a non-zero determinant. Thus in $M_n(K)$, $f$ has an inverse, which is $f^{-1} = \begin{pmatrix} f_1^{-1} & 0 \\ 0 & f_2^{-1} \end{pmatrix}$. Then 
\[
g = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} f_1^{-1} & 0 \\ 0 & f_2^{-1} \end{pmatrix},
\]
is block-diagonal. 

For the rest of this section we will let $X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$, and $\lambda X = 0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \xrightarrow{\delta} 0$. When we write $U_r \xrightarrow{\alpha} U_s$, we always mean that the map between these indecomposable modules is the inclusion if $r < s$, and the surjection if $r > s$.

Suppose the diagram $A \xrightarrow{\alpha} B$ commutes, where 
\[
\begin{array}{ccc}
    \tau_1 & \uparrow & \tau_1 \\
    U_r & \Downarrow & U_s \\
    \tau_2 & \uparrow & \tau_2
\end{array}
\]
$\sigma_1$, $\sigma_2$ are monomorphisms which are consistently split by the epimorphisms $\tau_1$, $\tau_2$. Then we can lift $\sigma_i$, $\tau_i$ to maps $\hat{\sigma}_i$, $\hat{\tau}_i$. 

commute; of course $\sigma_1' = \sigma_2'$ and $\tau_1' = \tau_2'$. Take bases as follows:

for $F_2$, take $\sigma_1'(1)$ together with a basis for $\ker \tau_1'$; for $F_1$, take $\sigma_1'(1)$ together with a basis for $\ker \tau_1'$; for $F_0$, take $\sigma_2(1)$ together with a basis for $\ker \tau_2$. Then $f$ clearly has the form

$$\begin{pmatrix} x^r \mu & 0 & \cdots & 0 \\ 0 & \vdots & & * \\ \vdots & & 0 & \vdots \\ 0 & & & * \end{pmatrix}$$

and $gf$ the form

$$\begin{pmatrix} x^{s-r} \mu' & 1 & 0 & \cdots & 0 \\ 0 & \vdots & & * \\ \vdots & & 0 & \vdots \\ 0 & & & * \end{pmatrix},$$

where $\mu$, $\mu'$ are units; by the proposition just proved, $g$ must have the form

$$\begin{pmatrix} x^{s-r} \mu'' & 1 & 0 & \cdots & 0 \\ 0 & \vdots & & * \\ \vdots & & 0 & \vdots \\ 0 & & & * \end{pmatrix}$$

for some unit $\mu''$. This together with Lemma II.2.1 implies that if we specify a set of monomorphisms $(\sigma_i)$ with consistent splittings $(\tau_i)$ to
make an object \( Y \) a summand of an object \( \lambda X \) in \( T_n \) (or \( \lambda'X \) in \( U_n \)), then we have in principle specified an algorithm for splitting off the corresponding summand of \( X \) in \( C_n \).

In what follows we will often write a representation
\[
F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_1} F_0 \quad \text{or an object}
\]
\[
A_{n-1} \xrightarrow{\alpha_{n-1}} A_{n-2} \xrightarrow{\alpha_{n-2}} \ldots \xrightarrow{\alpha_1} A_0 \text{ of } U_n \quad \text{as } \alpha_{n-1} \xrightarrow{\alpha_{n-2}} \ldots \xrightarrow{\alpha_1}.
\]
Maps between free modules will be written and treated as matrices without explicit mention of the bases with respect to which they are so written. In those cases in which the matrix takes a special form, it will usually be obvious that such bases exist; if there is no indication to the contrary, the standard basis for \( R^n \) will be assumed.

**Proposition II.2.3.** Fix \( i, 0 \leq i \leq n-1 \). The diagram
\[
\begin{array}{cccc}
\alpha_{n-1} & f_{n-1} & f_{n-2} & \ldots & f_1 \\
\downarrow & \downarrow & \downarrow & \ldots & \downarrow \\
\alpha_{n-2} & g_{n-1} & g_{n-2} & \ldots & g_1
\end{array}
\]
of free modules and matrices commutes iff the diagram obtained from it by replacing \( f_i, g_i \) by \( x^t f_i, x^t g_i \), respectively, commutes.

**Proof.** This is obvious. \( \square \)

The consequence of this proposition that we need is that a representation \( X = f_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_1} \) diagonalizes iff the representation obtained from \( X \) by factoring powers of \( x \)
out of each $f_i$ diagonalizes.

For an object $X$ in $C_3$, our approach to finding a diagonalization of $X$ is to detect summands of $X$ which are of the form $F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0$ where each $F_i$ has rank 1.

In $C_3$, if $X = F_2 \stackrel{f}{\longrightarrow} F_1 \stackrel{g}{\longrightarrow} F_0$, we factor powers of $x$ out of $f$ and $g$ until a unit entry appears in each; if the result is $X' = F_2 \stackrel{f'}{\longrightarrow} F_1 \stackrel{g'}{\longrightarrow} F_0$, we check to see whether $g'f'(F_2) \subseteq xF_0$.

If not, we can split off a summand of the form $R \longrightarrow R \longrightarrow R$ by Lemma 11.1.1. We would obtain an isomorphism

$$
\begin{array}{c}
F_2 \xrightarrow{f'} F_1 \xrightarrow{g'} F_0 \\
\alpha \downarrow \quad \downarrow \beta \quad \downarrow \gamma \\
F_2 \xrightarrow{f^*} F_1 \xrightarrow{g^*} F_0
\end{array}
$$

where both $f^*$ and $g^*$ are block-diagonal with a 1x1 block in the upper left-hand corner. If $f' = x^{-s}f$ and $g' = x^{-t}g$, then the diagram

$$
\begin{array}{c}
F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0 \\
\alpha \downarrow \quad \downarrow \beta \quad \downarrow \gamma \\
F_2 \xrightarrow{x^s f^*} F_1 \xrightarrow{x^t g^*} F_0
\end{array}
$$

commutes by the proposition just established, so we have split off a summand of $X$ of the form $x^s \longrightarrow x^t\mu$ for some
unit μ. What can we do if \( g'f'(\mathbb{F}_2) \subseteq \mathbb{F}_0 \)? We will want the notation \( s(M) \) for the length of the socle of a module \( M \) in \( \text{mod}(R) \).

Let \( \lambda X' = 0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0 \). Because \( g' \) has a unit entry while \( g'(\mathbb{F}_2) \subseteq \mathbb{F}_0 \), \( s(\text{coker}(g')) \lneq s(\text{coker}(g'f')) \). This means that if \( \lambda X' \) diagonalizes, it must have a summand of the form \( 0 \rightarrow V_r \rightarrow V_r \rightarrow 0 \rightarrow 0 \) for some indecomposable module \( V_r \), which corresponds to a summand of \( X \) of the form

\[
\begin{array}{c}
x' \\
\rightarrow \\
v \rightarrow \\
\end{array}
\]

for some unit \( v \). Our next few lemmas have to do with detecting such a summand.

Suppose that \( \varepsilon = 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is exact and that for some indecomposable module \( A' \) the sequence \( 0 \rightarrow A' \rightarrow A' \rightarrow 0 \rightarrow 0 \) is a summand of \( \varepsilon \). Let \( A = \bigoplus V_{a_i} \otimes A' \) be any decomposition of \( A \) in which each \( V_{a_i} \) is isomorphic to \( A' \) and \( A' \) has no summand isomorphic to \( A' \), and for each i let

\( \sigma_i : V_{a_i} \rightarrow A \) be the induced inclusion. Then the diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow \sigma_i & & \uparrow \alpha \sigma_i \\
V_{a_i} & = & V_{a_i}
\end{array}
\]

commutes.

**Lemma II.2.4.** With the notation of the last paragraph, for some i there exist epimorphisms \( \rho : A \rightarrow V_{a_i} \) and \( \tau : B \rightarrow V_{a_i} \).
which consistently split $\sigma_i$, $a\sigma_i$.

**Proof.** Let $\tau_A: A \to A'$ and $\tau_B: B \to A'$ be the consistently splittable epimorphisms we know exist. By the proof of the Krull–Schmidt theorem ([R73], p. 80), one of the compositions $\tau_A\sigma_i$ must be an isomorphism; then $\tau=(\tau_A\sigma_i)^{-1}\tau_B$ and $\rho=\tau\alpha$ consistently split $a\sigma_i$, $\sigma_i$. ■

The advantage that Lemma II.2.4 gives us is that it allows us to test conditions for the existence of a summand of $e$ for any fixed decomposition of $A_i$; if we fail to find a summand using the given decomposition, then there is no summand. Of course there is one decomposition of $A$ which is always easily found, namely the one induced by the diagonalization of $f$.

For our next step we need the special short exact sequences called the almost split or Auslander–Reiten sequences ([AR75]). One defining property of these is this:

$$0 \to A \to E \to C \to 0$$

is an almost split sequence if $A$ and $C$ are Indecomposable and, given a map $\sigma: A \to M$, then there is a map $\phi$ to make the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\sigma} & M \\
\downarrow{\alpha} & & \downarrow{\phi} \\
E & &
\end{array}
$$

commute iff $\sigma$ is not a splittable monomorphism.

**Proposition II.2.5.** In the category of torsion $R$-modules the almost split sequences are the sequences
where $\alpha = \begin{bmatrix} \text{surj} \\ \text{incl} \end{bmatrix}$ and $\beta = [-\text{incl} \; \text{surj}]$.

**Proof.** Certainly these sequences have both ends indecomposable. Suppose that $\sigma: V_r \rightarrow M$; clearly it suffices to consider the case in which $M$ is an indecomposable module $V_s$. If $\sigma$ is a splittable monomorphism in this case, then it is an isomorphism, and so if a map $\varphi: E \rightarrow V_s$ existed to make the diagram (*) commute, then $\alpha$ would be splittable. This is obviously impossible, so if $\sigma$ is a splittable monomorphism then no such map $\varphi$ exists. Suppose that $\sigma$ is not a splittable monomorphism. Then writing elements of $V_j$ as $a_0 + \cdots + a_{j-1}x^{j-1}$ we see that $\sigma$ must have the form $\sigma(a_0 + \cdots + a_{r-1}x^{r-1}) = a_0x^t + \cdots + a_{t+s-1}x^{s-1}$ for some $t > 0$. There are two cases (the case $s=r$, $t=0$ is the case of $\sigma$ a splittable monomorphism so we leave it out).

**Case 1.** If $s \neq r$, we define $\varphi$ by

$$
\varphi(a_0 + \cdots + a_{r-2}x^{r-2}, b_0 + \cdots + b_rx^r) = a_0x^t + \cdots + a_{t+s-1}x^{s-1}.
$$

**Case 2.** If $s \neq r$ then $t > 0$, and we set

$$
\varphi(a_0 + \cdots + a_{r-2}x^{r-2}, b_0 + \cdots + b_rx^r) = b_0x^{t-1} + \cdots + b_{t+s}x^{s-1}.
$$

In either case $\varphi$ makes the diagram (*) commute, and so the sequences given are almost split sequences. The work of Auslander and Reiten cited implies that almost split sequences are unique up to isomorphism, so the sequences
given are, up to isomorphism, all of the almost split sequences. \[ \] 

Henceforth we will write \( 0 \rightarrow V_r \rightarrow E_r \rightarrow V_r \rightarrow 0 \) for the unique almost split sequence corresponding to \( V_r \). The property used in defining the almost split sequence has as a consequence the fact that there is a map \( \varphi \) to make

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\uparrow \sigma & & \uparrow \varphi \\
0 & \rightarrow & V_r \\
\end{array}
\]

commute iff \( \sigma \varphi \) is not a splittable monomorphism, i.e. if the top row has no summands of the form \( 0 \rightarrow V_r \rightarrow V_r \rightarrow 0 \rightarrow 0 \).

If \( \lambda X = Y \), or if \( \lambda' X = Y \), let us say that \( X \) lies over \( Y \).

It is an easy calculation that \( (*1) \quad R \rightarrow R \rightarrow R \) lies over the almost split sequence for \( V_1 \), and that \( (*2) \quad R^2 \rightarrow R^2 \rightarrow R^2 \) lies over the almost split sequence for \( V_{a_r}, a_r \geq 1 \).

Let \( X = F_2 \rightarrow F_1 \rightarrow F_0 \). We can without loss of generality assume that \( f \) is diagonal, say \( f = \begin{pmatrix} x_1^{a_1} \mu_1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & x_n^{a_n} \mu_n \end{pmatrix} \) with each
μ_i a unit and a_1 ≤ a_2 ≤ ... ≤ a_n. We write \( \prod_{i=1}^{n} a_i \) for the induced decomposition of \( A \), and \( \sigma_r: a_r \rightarrow A \) for the specific inclusions associated with this decomposition. Then if \( \alpha \) and \( \beta \) are maps such that

\[
\begin{array}{ccc}
F_2 & \xrightarrow{f} & F_1 \\
\uparrow{\alpha} & & \uparrow{\beta} \\
R^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & x a_r \end{pmatrix}} & R^2 \\
\end{array}
\rightarrow V_r \rightarrow 0
\]

commutes, i.e. \( \alpha \) and \( \beta \) induce the inclusion \( \sigma_r \), then we want to find conditions on \( g \) which imply that there is no map \( \gamma \) such that \( (\alpha, \beta, \gamma) \) is a map from \((*2)\) to \( X \) in \( C_2 \). (We also want to do this for maps of \((*1)\) into \( X \).) If no such map \( (\alpha, \beta, \gamma) \) exists, then there is no map \( \phi \) to make the diagram II.2.6 commute, so

\[
\begin{array}{cccc}
0 & \rightarrow & V_r & = V_r \\
\downarrow{a_r} & & \downarrow{a_r} & \rightarrow 0 & \rightarrow 0
\end{array}
\]

must be a summand of \( \lambda X \), which means that \( R \xrightarrow{x a_r} R \rightarrow R \) is a summand of \( X \) for some unit \( x \). A simple condition which implies that a map \( \gamma \) does exist is that no entry in the \( r \)th column of \( g \) be a unit, as then the diagram
\[ \begin{array}{cccc}
\alpha & \downarrow f & \rightarrow & \beta \\
\downarrow (1 & 0) & (x^{a_r-1} & -1) & \rightarrow & \gamma \\
(0 & x^{a_r}) & (0 & x) & \rightarrow & \gamma \\
\end{array} \]

\( \alpha \) commutes with \( \alpha = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \), \( \beta = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \)

\( \gamma = \begin{pmatrix} 0 & x^{-1} \beta_1 + \mu_r \\ \vdots & \vdots \\ 0 & x^{-1} \beta_n + \mu_r \end{pmatrix} \) for \( a_r > 1 \); for \( a_r = 1 \) the diagram commutes with \( \alpha', \beta', \gamma' \) the second columns of the \( \alpha, \beta, \gamma \) just given.

We may suppose that either \( a_r < a_{r+1} \) or that \( r = n \); if not, just shift rows and columns to get it that way. Suppose \( \alpha, \beta \) are maps making \( I \rightarrow \alpha, I \rightarrow \beta \) commute and inducing the inclusion \( V_{x_r} \rightarrow \Lambda \) as before. If \( (\alpha, \beta, \gamma) \) is to be a map of \((\#2)\) into \( X \), then the following diagram must commute:
If \( e_1, e_2 \) are the usual basis elements for \( \mathbb{R}^2 \), then \( \beta \) must send \( e_1 \) into \( \text{im}(\phi) \) and \( e_2 \) to something that goes to \( I \); so if 
\[
\beta = \begin{pmatrix} c_1 & d_1 \\ \vdots & \vdots \\ c_n & d_n \end{pmatrix},
\]
then \( d_r \) must be a unit, \( x^{a_i} \) must divide \( c_i \) for all \( i \), and \( x^{a_i} \) must divide \( d_i \) for \( i \neq r \).

A further condition on \( (\alpha, \beta, \gamma) \) is that the diagram 

\[
\begin{array}{ccc}
\alpha & \xrightarrow{g} & \beta \\
\downarrow & & \downarrow \\
\mathbb{R}^2 & \xrightarrow{a_r} & \mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2 \\
\end{array}
\]

commute. Let \( \gamma = \begin{pmatrix} s_1 & t_1 \\ \vdots & \vdots \\ s_n & t_n \end{pmatrix} \); then 
\[
g \cdot \begin{pmatrix} c_1 & d_1 \\ \vdots & \vdots \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} s_1 & t_1 \\ \vdots & \vdots \\ s_n & t_n \end{pmatrix} \begin{pmatrix} x^{a_r-1} & -1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} s_1 x^{a_r-1} \\ \vdots \\ s_n x^{a_r-1} \end{pmatrix} - \begin{pmatrix} -s_1 + x t_1 \\ \vdots \\ -s_n + x t_n \end{pmatrix}.
\]

The equation for the first column is 
\[
g \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} s_1 x^{a_r-1} \\ \vdots \\ s_n x^{a_r-1} \end{pmatrix},
\]
so

\[
\begin{pmatrix}
  s_1 \\
  \vdots \\
  s_n
\end{pmatrix} = x^{1-a} \cdot g \cdot 
\begin{pmatrix}
  c_1 \\
  \vdots \\
  c_n
\end{pmatrix}; \text{ so we get } x \cdot 
\begin{pmatrix}
  t_1 \\
  \vdots \\
  t_n
\end{pmatrix} = g \cdot 
\begin{pmatrix}
  d_1 + c_1 x^{1-a} \\
  \vdots \\
  d_n + c_n x^{1-a}
\end{pmatrix}
\]

from the equation for the second column. Expanding the right-hand side of this gives

\[
x \cdot 
\begin{pmatrix}
  t_1 \\
  \vdots \\
  t_n
\end{pmatrix} = 
\begin{pmatrix}
  g_1 (d_1 + c_1 x^{1-a}) + \ldots + g_{1r} (d_r + c_r x^{1-a}) \\
  \vdots \\
  g_{n1} (d_1 + c_1 x^{1-a}) + \ldots + g_{nr} (d_r + c_r x^{1-a})
\end{pmatrix}
\]

\[
(11.2.7)
\]

\[
+ 
\begin{pmatrix}
  g_{1,r+1} (d_{r+1} + c_{r+1} x^{1-a}) + \ldots + g_{1n} (d_n + c_n x^{1-a}) \\
  \vdots \\
  g_{n,r+1} (d_{r+1} + c_{r+1} x^{1-a}) + \ldots + g_{nn} (d_n + c_n x^{1-a})
\end{pmatrix}
\]

The second term contains no units, as \(d_{r+j}\) is divisible by \(x^{a_r+j-a_r}\) and \(c_{r+j}\) is divisible by \(x^{a_r+j}\) for each \(j > r\); thus there is a solution to the mapping problem iff there are no units in the first term on the right-hand side.

For \(a_r = 1\) we get a similar condition. Suppose \(\alpha'\) and \(\beta' = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}\) make the diagram
commute; then as before we must have $x^{a_i}$ dividing $d_i$ if $i \neq r$ and $d_r$ a unit. Thus if \( \gamma \) commutes, where \( \gamma = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \),

we get (II.2.8) \( x \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = g \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \). Again there are no solutions to this iff there is a unit on the right-hand side; again these could occur only in the first $r$ terms of each entry.

Definition. If $r \in R$, $r$ can be written $r = x^\beta \mu$ for some unit $\mu$; we write $\|r\| = \beta$.

Lemma II.2.9. (i) For $a_r = 1$, there is no solution to the mapping problem above iff (a) some entry of the $r^{th}$ column of $g$ is a unit; (b) for some unit $g_{ts}$ in the $r^{th}$ column of $g$, either $g_{ts} = 0$ or $\|g_{ts}\| + a_s > 0$ (i.e., $x$ divides $g_{ts}d_s$) for each $s$ with $1 \leq s < r - 1$.

(ii) For $a_r > 1$, there is no solution to the mapping.
problem above iff (a') some entry of the r\textsuperscript{th} column of \(g\) is a unit; (b') for some unit \(g_{tr}\) in the r\textsuperscript{th} column of \(g\), either \(g_{ts}=0\) or \(\|g_{ts}\|\leq\max(1,a_r-a_s)\) for each \(s\) with \(1\leq s\leq r-1\).

Proof. We have already seen the necessity of (a) and (a'). Suppose that (a) holds but that (b) fails for some \(s\). Then \(g_{ts}\) is a unit, and the following is a solution: set \(d_i=0\) for \(i\neq s,r\); set \(d_r=1\), and set \(d_s=-g_{tr}g_{ts}^{-1}\). Now suppose that (a') holds but that (b') fails for some \(s\). Then the following is a solution: set \(c_i=0\) for \(i\neq s\) and \(d_i=0\) for \(i\neq r\); set \(d_r=1\), and \(c_s=-x^{a_r-a_s-1}g_{tr}\), where \(g_{ts}=x^p\mu\) with \(\mu\) a unit. Thus if no map \(\varphi\) exists, (a) and (b) or (a') and (b') must hold. On the other hand if (a) and (b) or (a') and (b') hold then it is easy to see that the \(t\)\textsuperscript{th} entry on the right-hand side of II.2.8, or II.2.7 respectively, is a unit while this cannot be so on the left; so there is no solution, hence no map \(\varphi\).

Lemma II.2.9 shows how to detect a summand of \(X\) of the form \(x^{a_r}y\); what we need now is a way to split off these summands once their presence has been detected, i.e. to find bases for \(F_2\), \(F_1\), and \(F_0\) so that \(f\) and \(g\) become block-diagonal with 1x1 blocks in their upper left-hand corners. The conditions just obtained for \(X\) to have a summand of this type give a way to do this; we illustrate
for a summand $x^{a_r}$. We assume that $f$ is already diagonal. We can use elementary row operations on $g$ to get $g_{tr}=1$, $g_{jr}=0$ for $j \neq t$ without affecting $f$. If $g_{t,r-1}=0$, there is no need to do anything here. If $g_{t,r-1} \neq 0$, the next step is to multiply the $r^{th}$ column of $g$ by $-g_{t,r-1}$ and add the result to the $(r-1)^{st}$ column of $g$. In order to end up with a representation isomorphic to $X$, we are then forced to multiply the $(r-1)^{st}$ row of $f$ by $g_{t,r-1}$ and add the result to the $r^{th}$ row of $f$; then the new $(r,r-1)$ entry of $f$ is $g_{t,r-1} x^{a_{r-1}} y^{a_r}$. But the conditions given in Lemma 11.2.9 imply that $||g_{t,r-1} x^{a_{r-1}} y^{a_r} || a_r$ (whether $a_r=1$ or $a_r>1$), and so we can use an elementary column operation on $f$ (which doesn't affect $g$) to get $f$ back to diagonal form. Continue this with $g_{t,r-2}$, etc.

Thus this algorithm actually says that $x^{a_r} y^{a_r}$ is a summand of $X$ iff we can find bases in which it appears as $1 \times 1$ blocks in the most naive possible way. We now have the algorithm complete. Explicitly:

Theorem 11.2.10. Let $X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ be in $B_3$. The following algorithm diagonalizes $X$ if $X$ can be diagonalized.
**Step 1.** Use Lemma II.1.1 to write $X$ as a direct sum of a diagonalized representation $Y$ and a representation $X_0 = F_2 \xrightarrow{f_0} F_1 \xrightarrow{g_0} F_0$ in $C_3$.

**Step 2.** Diagonalize $f_0$. That is, obtain a representation $X_0 = F_2 \xrightarrow{\hat{f}_0} F_1 \xrightarrow{\hat{g}_0} F_0'$, isomorphic to $X_0$, in which $\hat{f}_0$ is a diagonal matrix, say $\hat{f}_0 = \begin{pmatrix} x^{a_1} & \mu_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{a_n} \mu_n \end{pmatrix}$.

**Step 3.** Factor powers of $x$ out of $\hat{f}_0$ and $\hat{g}_0$ until a unit entry appears in each; call the result $X' = F_2 \xrightarrow{f'} F_1 \xrightarrow{g'} F_0'$. Check to see whether $g'f'(F_2) \subseteq xF_0'$. If so, proceed to step 4. If not, use Lemma II.1.1 to split off any summands of the form $R \xrightarrow{x} R \xrightarrow{x}$ that may occur. Multiply the appropriate powers of $x$ back into these to get the actual summands; the result is a sum $X'' \oplus Y''$ of representations in which $Y''$ is diagonal. If $X'' = 0$, we're done; if not, set $X_0 = X''$ and repeat step 3.

**Step 4.** Using Lemma II.2.9 check for summands of the type $x^a \xrightarrow{\nu}$, $\nu$ a unit. $X'$ must have at least one of these if it diagonalizes, so if it hasn’t, stop; $X$ does not diagonalize. (This may necessitate shifting rows and columns to get the
condition: \( a_r < a_{r+1} \) or \( r=n \), needed for Lemma II.2.9.) Again multiply the appropriate powers of \( x \) back in to obtain the actual summands; again the result will be a sum \( X^a \odot Y^b \) with 
\( Y^b \) diagonal; again if \( X^a \neq 0 \) set \( X'_0 = X^a \) and repeat step 3.

This algorithm says to diagonalize \( \begin{pmatrix} f & g \\ \end{pmatrix} \) as follows. Diagonalize \( f \), and split off any summands of the form \( R \longrightarrow R \longrightarrow R \). Then, after factoring powers of \( x \) out of \( g \), use the criteria of Lemma II.2.9 (and then elementary row and column operations) to get a \( 1 \times 1 \) block in \( g \); repeat. If at any stage this cannot be done, then \( \begin{pmatrix} f & g \\ \end{pmatrix} \) does not diagonalize. We note that, as we mentioned in the introduction, this algorithm makes no reference to the apparatus of its proof, but operates only on matrices.

§3. The general case.

Before we can settle the general case we must first manage the case of \( A_4 \). To this end set 
\[
\begin{pmatrix} f & g & h \\ \end{pmatrix} \longrightarrow \begin{pmatrix} f & g \\ \end{pmatrix} \longrightarrow \begin{pmatrix} f & g & h \\ \end{pmatrix}
\]
in \( C_4 \). As in the previous section we can factor powers of \( x \) out of each of \( f, g, \) and \( h \) until each contains a unit entry, so we will assume that this has been done. We will use the notation \( \langle\langle x\rangle\rangle \) for the module \( \text{coker}(x) \) and work in the category \( U_2 \) unless otherwise stated. In \( U_2 \) the image of \( X \) is \( \lambda' X = \langle\langle f\rangle\rangle \longrightarrow \langle\langle gf\rangle\rangle \longrightarrow \langle\langle hgf\rangle\rangle \).

Now assume that \( X \) diagonalizes. We first want to show
that $\lambda'X$ must have a summand of the form $V_{r-t} \rightarrow V_r \rightarrow V_r$.

Consider the short exact sequence

$$0 \rightarrow ((gf)) \rightarrow ((hgf)) \rightarrow ((h)) \rightarrow 0,$$

which corresponds to

$$g \rightarrow h$$

under $\lambda$. As $X$ diagonalizes, this must diagonalize; because $h$ has a unit entry, $s((h)) < s((hgf))$, so this sequence must have a summand of the form $0 \rightarrow V_r \rightarrow V_r \rightarrow 0$. This implies that $\lambda'X$ has a summand of the form $V_{r-t} \rightarrow V_r \rightarrow V_r$ as claimed.

Examine the commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
((f)) & \xrightarrow{\alpha} & ((gf)) \\
\downarrow & & \downarrow \\
((hgf)) & \xrightarrow{\beta} & ((h)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
$$

(II.3.1)

This diagram is completely determined by $\alpha$ and $\beta$, so it is not hard to see that if $\begin{array}{c}
\alpha \\
\beta
\end{array}$ has a summand of the form $V_{r-t} \rightarrow V_r \rightarrow V_r$, this diagram is forced to have a summand of the form
Our strategy is to use the techniques developed for \( A_3 \) to find summands of the top row of II.3.1 which "extend" to summands of the type II.3.2. If none of this sort exist, then by the observations just made \( X \) cannot diagonalize. If they do exist, we must exhibit an algorithmic way to split them off, i.e. a consistently split inclusion of \( V_{r-t} \rightarrow V_r \rightarrow U_r \rightarrow 0 \).

Suppose \( V \) is an indecomposable summand of \( ((gf)) \) for which \( 0 \rightarrow V \rightarrow U \rightarrow 0 \rightarrow 0 \) is a summand of the right-hand column of II.3.1, i.e. of the sequence \( 0 \rightarrow ((gf)) \rightarrow ((hgf)) \rightarrow ((h)) \rightarrow 0 \). Then by Lemma II.2.4 there must be, in the specific decomposition of \( ((gf)) \) obtained in diagonalizing the top row of II.3.1, a summand \( V_r \rightarrow V \) for which \( 0 \rightarrow V \rightarrow U \rightarrow 0 \rightarrow 0 \) is a summand of the right-hand column of II.3.1. That is if \( \sigma_1 \uparrow \uparrow \sigma_2 \uparrow \uparrow \sigma_3 \)

\[
\begin{array}{c}
\sigma_1 \uparrow \uparrow \sigma_2 \uparrow \uparrow \sigma_3 \\
V_{r-t} \rightarrow \rightarrow V_r \rightarrow \rightarrow 0
\end{array}
\]

commutes where \( (\sigma_1, \sigma_2) \) is a monomorphism consistently split by \( (\tau_1, \tau_2) \), then there must exist a monomorphism \( \sigma_3 \) and a
consistent splitting \((\tau_1', \tau_2', \tau_3')\) of \((\sigma_1, \sigma_2, \sigma_3)\) to make the diagram commute. Then if in the diagram

\[
\begin{array}{c}
\sigma_2 \downarrow \tau_2 \quad \sigma_3 \downarrow \tau_3 \\
\uparrow v_r \quad \uparrow v_r
\end{array}
\]

we define \(\tau_2 = \tau_3 \beta\), we get a consistent splitting \((\tau_1, \tau_2, \tau_3)\) of the monomorphism \((\sigma_1, \sigma_2, \sigma_3)\). Thus if a summand of the right sort can be found, it can be algorithmically split off.

We have proved the following theorem, which really states that the algorithm of theorem II.2.10 can continue to the case of \(A_4\).

**Theorem II.3.3.** Let \(X=F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0\) be in \(B_4\). If \(X\) can be diagonalized, the following algorithm diagonalizes \(X\).

**Step 1.** Use Lemma II.1.1 to write \(X\) as a direct sum of a diagonalized representation \(Y\) and a representation \(X_0=F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0\) in \(C_4\).

**Step 2.** Factor powers of \(x\) out of \(f_0, g_0,\) and \(h_0\) until a unit entry appears in each; call the result \(X'=F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0\). Check to see whether \(h_1 g_1 f_1 (F_3) \subseteq x F'_0\). If so, proceed to step 3. If not, split off such summands
of the form $R \to R \to R \to R$ as may occur. Multiply the appropriate powers of $x$ back into these to get the actual summands; the result is a sum $X'' \oplus Y''$ with $Y''$ diagonal. If $X'' = 0$, stop; if not set $X_0 = X''$ and repeat step 2.

Step 3. Diagonalize $f' \to g'$. If this is not possible, stop; $X$ does not diagonalize. If it is possible, use the techniques developed for $A_3$ to determine whether there is any summand $V_r$ of $\langle g' f' \rangle$ for which $0 \to V_r \to V_r \to 0 \to 0$ is a summand of the right-hand column of II.3.1. If there are none, then by the observations above, $X$ cannot diagonalize. If there is one, we have found a summand $V_{r-t} \to V_r \to V_r$ of $\lambda' X$.

Step 4. Split off the summand found in Step 3 in the manner indicated above. Multiply the appropriate powers of $x$ back in as before; the result will be, as before, a sum $X'' \oplus Y''$ with $Y''$ diagonal. If $X'' = 0$, we're done; if not, set $X_0 = X''$ and go back to Step 2. □

For the general case we prove, by induction on $n$, that this algorithm can continue. Let $X = F_{n-1} \to F_{n-2} \to \cdots \to F_0$ and assume that each $f_i$ has a unit entry but that $f_1 \circ \cdots \circ f_{n-1}(F_{n-1}) \subseteq xF_0$; as before we can easily reduce to this case. For each $i$ let $C_i$ denote the module $\text{coker}(f_i)$, and let $C_{i,j}$ denote the module
\[ \text{coker}(f_1 \circ \cdots \circ f_j). \]

Then \( \lambda'X = \)

\[ C_{n-1} \xrightarrow{\alpha_{n-1}} C_{n-2,n-1} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} C_{1,n-1}. \]

Arguing in the same way as for \( A_4 \) we can show that \( \lambda'X \) must have a summand of the form \( \nu_{r_{n-1}} \xrightarrow{} \nu_{r_{n-2}} \xrightarrow{} \cdots \xrightarrow{} \nu_{r_2} \xrightarrow{} \nu_{r_2} \) if \( X \) is to diagonalize. Just as in the previous case, this forces the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & C_{n-1} \xrightarrow{\alpha_{n-1}} C_{2,n-1} \rightarrow C_{2,n-2} \rightarrow 0.
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
C_{1,n-1} & \rightarrow & C_{1,n-1} \\
\downarrow & & \downarrow \\
0 & \rightarrow & C_{2,n-2} \rightarrow C_{1,n-2} \rightarrow C_1 \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

(II.3.4)

to have a summand of the form

\[
\begin{array}{ccc}
\nu_{r_{n-1}} & \rightarrow & \nu_{r_2} \rightarrow \nu_t \\
\downarrow & & \downarrow \\
\nu_{r_2} & \rightarrow & \nu_t \\
\downarrow & & \downarrow \\
\nu_t & \rightarrow & 0.
\end{array}
\]
Now suppose $X$ diagonalizes. Our induction hypothesis is that we can diagonalize, algorithmically, any diagonalizable object in $C_{n-1}$, so by this hypothesis we can diagonalize $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{2}} F_1$. This gives a diagonalization of the top row of II.3.4; just as in the case of $A_4$, there must be an indecomposable summand $V$ of $C_{2,n-1}$ such that $0 \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow 0$ is a summand of the right-hand column of II.3.4. Again by Lemma II.2.4, there must then be, in the specific decomposition of $C_{2,n-1}$ obtained in diagonalizing $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{2}} F_1$, a summand $V_{r_2}$ for which $0 \longrightarrow V_{r_2} \longrightarrow V_{r_2} \longrightarrow 0 \longrightarrow 0$ is a summand of the right-hand column of II.3.4. Thus if $(\sigma_{n-1}, \ldots, \sigma_2)$ is the monomorphism with consistent splitting $(\tau_{n-1}, \ldots, \tau_3, \tau_2)$, found by diagonalizing $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{2}} F_1$, which makes the diagram

\[
\begin{array}{cccccccc}
C_{n-1} & \xrightarrow{\alpha_{n-1}} & C_{n-2,n-1} & \xrightarrow{\alpha_{n-2}} & \ldots & \xrightarrow{\alpha_3} & C_{2,n-1} \\
\sigma_{n-1} \downarrow \tau_{n-1} & \sigma_{n-2} \downarrow \tau_{n-2} & & & & \sigma_2 \downarrow \tau_2 \\
V_{r_{n-1}} & \longrightarrow & V_{r_{n-2}} & \longrightarrow & \ldots & \longrightarrow & V_{r_2}
\end{array}
\]

commute, then there must exist a monomorphism $\sigma_1$ and a
consistent splitting \((\tau_2', \tau_1')\) of \((\sigma_2, \sigma_1)\) to make the diagram

\[
\begin{array}{c}
\sigma_2 \rightarrow \tau_2' \rightarrow \sigma_1 \rightarrow \tau_1' \\
V_{r_2} \quad V_{r_2}
\end{array}
\]

commute. Then if in the diagram

\[
\begin{array}{c}
\alpha_{n-1} \rightarrow \alpha_{n-2} \rightarrow \ldots \rightarrow \alpha_3 \rightarrow \alpha_2 \rightarrow \alpha_1 \\
V_{r_{n-1}} \quad V_{r_{n-2}} \quad \ldots \quad V_{r_2} \quad V_{r_2}
\end{array}
\]

we define \(\tau_2 = \tau_1 \alpha_2\), we get a consistent splitting \((\tau_{n-1}, \ldots, \tau_1)\) of the monomorphism \((\sigma_{n-1}, \ldots, \sigma_1)\).

We have proved the following theorem.

**Theorem 11.3.5.** If the object \(X = F_{n-1} \rightarrow F_{n-2} \rightarrow \ldots \rightarrow F_0\) in \(B_n\) diagonalizes, then the following algorithm diagonalizes \(X\).

**Step 1.** Use Lemma 11.1.1 to write \(X\) as a direct summand of a diagonal representation \(Y\) and a representation \(X_0 = F_{n-1} \rightarrow F_{n-2} \rightarrow \ldots \rightarrow F_0\) in \(C_n\).

**Step 2.** Factor powers of \(x\) out of each \(f_i, 0\) until each contains a unit entry; call the result \(X' = F_{n-1} \rightarrow F_{n-2} \rightarrow \ldots \rightarrow F_0\). Check to see whether \(f_1' \cdots f_{n-1}' (F_{n-1}) \subseteq xF_0\). If so, proceed to step 3. If not, split off any summands of the form \(R \rightarrow R \rightarrow \ldots \rightarrow R\).
that may occur. Multiply the appropriate powers of $x$ back into these to get the actual summands; the result is a sum $X^n \odot Y^n$ with $Y^n$ diagonal. If $X^n = 0$, stop; if not, set $X_0 = X^n$ and repeat step 2.

**Step 2.** Diagonalize $F_{n-1} \xleftarrow{f_{n-1}} F_{n-2} \xleftarrow{f_{n-2}} \ldots \xleftarrow{f_2} F_1$. If this not possible, then $X$ does not diagonalize. If it is possible, use the techniques described above to determine whether there is any summand $V_{r_2}$ of $C_{2,n-1} = \langle f'_{2} \ldots f'_{n-1} \rangle$ for which $0 \rightarrow V_{r_2} \rightarrow 0$ is a summand of the right-hand column of II.3.4. If there are none, then $X$ does not diagonalize. If there is one, we have found a summand $V_{r_{n-1}} \rightarrow V_{r_{n-2}} \rightarrow \ldots \rightarrow V_{r_2} \rightarrow V_{r_2}$ of $\lambda'X$.

**Step 4.** Split off the summand found in step 3 in the manner indicated above. Multiply the appropriate powers of $x$ back in as before; the result will again be a sum $X^n \odot Y^n$ with $Y^n$ diagonal. If $X^n = 0$, we're done; if not, set $X_0 = X^n$ and go back to step 2.

Theorem II.3.5 says that the method of diagonalization established in theorems II.2.10 and II.3.3 for the cases of $A_3$ and $A_4$, respectively, can continue indefinitely. We remarked after II.2.10 that the method established there operated only on matrices; as the general algorithm is just a repetition of that method, it too operates only on
matrices.
Chapter III. Examples in $C_3$.

In this chapter we exhibit some examples of the correspondence between objects in $C_3$ and short exact sequences, and of how considerations of the latter can be made to yield information about the former. We end the chapter by sketching a proof that $C_3$ (and hence $Q_3$) is of wild representation type. For this chapter we will let $\epsilon$ be the short exact sequence

$$\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0.
\end{array}$$

**Example 1.** One term zero and split sequences.

(a) $A = 0$. Then $\epsilon$ is the sequence $0 \rightarrow 0 \rightarrow B \sim C \rightarrow 0$, and it is easy to see that if $f \epsilon$ is $\begin{array}{c}
0 \\
0 \\
0 \\
B
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
0 \\
C
\end{array}$, then $f$ must be an isomorphism. Conversely if $X = \begin{array}{c}
0 \\
0 \\
0 \\
B
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
0 \\
C
\end{array}$ in $C_3$ with $f$ an isomorphism then $\lambda X$ plainly has the form $0 \rightarrow 0 \rightarrow B \sim C \rightarrow 0$. Clearly such an object diagonalizes.

(b) $C = 0$. This time $\epsilon$ takes the form $0 \rightarrow A \sim B \rightarrow 0 \rightarrow 0$, and so if $g \epsilon = \begin{array}{c}
0 \\
0 \\
0 \\
B
\end{array} \rightarrow \begin{array}{c}
0 \\
0 \\
0 \\
C
\end{array}$ then $g$ must be an isomorphism. Once again the converse holds, and once again every such object diagonalizes.

(c) $\epsilon$ split. Then $\epsilon$ is isomorphic to a direct sum of sequences of the types found in (a) and (b), and so

$$\begin{array}{c}
\begin{pmatrix}
f_1 & 0 \\
0 & f_2
\end{pmatrix} \begin{pmatrix}
g_1 & 0 \\
0 & g_2
\end{pmatrix}
\end{array}$$

corresponds to an object of the form $\begin{array}{c}
\begin{array}{c}
f_1 \\
0
\end{array} \rightarrow \begin{array}{c}
g_1 \\
0
\end{array} \\
\begin{array}{c}
f_2 \\
g_2
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
0 \\
0
\end{array} \rightarrow \begin{array}{c}
0 \\
0
\end{array}
\end{array}$ in $C_3$. 

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where either (i) \( f_1 \) and \( g_2 \) are isomorphisms or (ii) \( f_2 \) and \( g_1 \) are isomorphisms. Again it is clear that any object of this sort diagonalizes.

**Example 2.** One term projective.

By ‘one term projective’ we mean that there is an integer \( t \) such that \( \epsilon \) is a short exact sequence of \( k[x]/(x^t) \)-modules and that one of \( A, B, \) or \( C \) is projective in \( \text{mod } (k[x]/(x^t)) \). If \( A \) is projective in this category then it is also injective, so \( \epsilon \) splits. If \( C \) is projective then of course \( \epsilon \) also splits, so the only interesting case is when \( B \) alone is projective. Suppose \( B \cong \mathbb{W}_p^t \). Then if \( \epsilon = f \xrightarrow{g} \) then it must be that \( gf \) is isomorphic to \( x^t I \), where \( I \) is the text identity matrix; conversely if \( X = f \xrightarrow{g} \) with \( gf \cong x^t I \), then \( XX \) has its middle term projective. The surprise is that these always decompose.

**Proposition III.1.** If \( \epsilon = \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} C \xrightarrow{0} \) is exact with \( B \cong \mathbb{W}_p^t \), then \( \epsilon \) diagonalizes.

**Proof.** Suppose \( s(C) = m \), and let \( \mathbb{W}_p^t \xrightarrow{m} C \) be the projective cover. By projectivity of \( B \) we get the map \( \rho \) in the diagram

\[
\begin{array}{c}
A \\
\xrightarrow{a} \\
B \\
\xrightarrow{b} \\
C \\
\xrightarrow{c} \\
0
\end{array}
\]
in which $\alpha$ is just the induced map. $\beta$ is an epimorphism
($\downarrow\beta=b$, and $\downarrow$ is essential), so there is a map $\tau: \mathbb{P} \rightarrow B$ to
split it. Because $\beta$ is an epimorphism the Snake Lemma tells
us that $\alpha$ is also an epimorphism; a diagram chase says that
we can define $c=\alpha^{-1}\tau\alpha$ to get a splitting of $\alpha$ for which
$\alpha\circ\tau\circ\alpha$. Thus the bottom row is a summand of the top, and so
$\varepsilon$ decomposes into the direct sum of the (already diagonal)
sequence $0 \rightarrow K \rightarrow \mathbb{P} \rightarrow C \rightarrow 0$ and a sequence of the form
$0 \rightarrow A' \rightarrow B' \rightarrow 0 \rightarrow 0$; so $\varepsilon$ must diagonalize.

**Example 3.** Both ends indecomposable.

Let $\delta_t = 0 \rightarrow V_i \xrightarrow{a_t} V_{i+j-t} \oplus V_t \xrightarrow{b_t} V_j \rightarrow 0$,
where $0 \leq t \leq \min(i,j)$,

$$a_t(a_0 + \cdots + a_{i-1} x^{i-1}) = (a_0 x^{j-1} + \cdots + a_{i-1} x^{i+j-t-1} + a_0 + \cdots + a_{i-1} x^{i-1}),$$
and $b_t(a_0 + \cdots + a_j + x^{j-t-1} + b_0 + \cdots + b_{i-1} x^{i-1}) = a_0 + \cdots + a_{j-t-1} x^{j-t} + (a_{j-t} - b_0) x^{i-t} + \cdots + (a_{j-t} - b_{i-1}) x^{j-1}$;

then $\delta_t = \begin{bmatrix} x^{i-t} & 0 \\ x^t & 1 \end{bmatrix} \begin{bmatrix} x^{j-t} \\ 0 \end{bmatrix}$. (We used the special case
of the almost-split sequence ($i=j, t=i-1$) in Chapter II.) We
want to sketch a proof that these are, up to isomorphism,
all the short exact sequences with left-hand end $V_i$ and right-hand end $V_j$.

First, if $0 \rightarrow V_i \xrightarrow{a} B \xrightarrow{b} V_j \rightarrow 0$ is exact, then $B \cong V_i + V_j$, as $s(B) \leq 2$. Note that $t \leq \min(i,j)$, as otherwise both indecomposable summands of $B$ have length less than $j$. If $t=i$ or $t=j$ then $\epsilon$ splits, as set $r=\max(i,j)$; then over $K[x]/(x^r)$, either $V_i$ is injective or $V_j$ is projective. Suppose then that $t < \min(i,j)$. Then $\epsilon$ must split monomorphically into $V_i + V_j$, so the isomorphism $B \cong V_i + V_j$ extends to an isomorphism of sequences, i.e. we may work with the sequence $0 \rightarrow V_i \xrightarrow{a} V_i + V_j \xrightarrow{b} V_j \rightarrow 0$. Then up to multiplication by units, $a(a_0 + \cdots + a_{i-1} x^{i-1}) = (a_0 x^{j-t} + \cdots + a_{i-1} x^{i+j-1}, a_0 x^p + \cdots + a_t + a_{t-p-1} x^{t-1})$. The cokernel map is forced to be $b(a_0 + \cdots + a_{i+j-t} x^{i+j-t-1}, b_0 + \cdots + b_{t-1} x^{t-1}) = a_0 + \cdots + (a_{j-t} - b_0) x^{j-t} + \cdots + (a_{j-1} - b_t + p-1) x^{j-1}$. Unless $p=0$, this is not an $R$-map, so $p=0$ and hence $\epsilon \cong \delta_t$.

A result of M. Loupias [L75] on representations of partially ordered sets implies that $C_3$ is of infinite representation type; to conclude this chapter we sketch a proof that $C_3$ is in fact of wild type. Let $S_t = K[x]/(x^t)$, and let $A_{2,t}$ be the ring of lower-triangular $2 \times 2$ matrices
with entries from $S_t$. We will exhibit a functor $\sigma$ which embeds the category $I_n,m$ of representations of the diagram

$$1 \to \cdots \to n+1 \to \cdots \to n+m+1$$

into $\text{Mod}(A_2, n+m+1)$. This will prove our claim, as it is known that for $n,m>3$ the category $I_n,m$ is of wild type (\cite{DR76}).

Suppose $Z = \langle q, p_1, \ldots, p_{n+m+1} \rangle : Z \to Z'$ is a morphism in $I_n,m$, i.e., if the diagram (\#)
commutes, we set \( p_M = (q, p_{n+2}, \ldots, p_{n+m+1}) : M \rightarrow M' \) and 
\( p_N = (p_1, \ldots, p_{n+m+1}) : N \rightarrow N' \). Then the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{p_M} & & \downarrow{p_N} \\
M' & \xrightarrow{f'} & N'
\end{array}
\]

commutes, so \( \sigma(q, p_1, \ldots, p_{n+m+1}) = (p_M, p_N) : (M, N, f) \rightarrow (M', N', f') \) in \( \text{mod} (A_2, n+m+1) \). It is easy to see that \( \sigma \) is a functor.

If \( Z \) and \( Z' \) are in \( I_{n,m} \) and \( \sigma Z = (M, N, f) \), 
\( \sigma Z' = (M', N', f') \), then if \( (\rho, \tau) \circ \sigma Z \rightarrow \sigma Z' \) then as \( M, M' \) and \( N, N' \)
are sums of the vector spaces \( W, W' \), \( M, M' \) in \( Z \) and \( Z' \), the
maps \( \rho \) and \( \tau \) are sums of vector space maps between these.
Because of the commutativity condition \( f' \rho = \tau f \), these maps
make a diagram like \((*)\) commute, and so give a morphism
\( \delta : Z \rightarrow Z' \) for which \( \sigma \delta = (\rho, \tau) \). Thus \( \sigma \) is full. If \( (\rho, \tau) \) is an
isomorphism, then each of the vector space maps will be an
isomorphism, so \( \sigma \) reflects isomorphisms. \( \sigma \) is clearly
faithful, so we're done: \( \sigma \) embeds \( I_{n,m} \) into \( \text{Mod}(A_2, n+m+1) \).
Chapter IV. Reduction.

In this chapter we show that there is a full dense functor from the category $\mathcal{Q}_n$ of all representations of $A_n$ by free $R$-modules to the category $\mathcal{B}_n$ in which the results of chapter II were obtained. Perhaps the best way to explain our strategy is to sketch it for an object of $\mathcal{Q}_3$.

Let $X=F_2 \rightarrow_{f} F_1 \rightarrow_{g} F_0$ be in $\mathcal{Q}_3$. Our first step will be to prove that we can split off the kernel $F'$ of $f$ and the free part $F ''$ of the cokernel of $g$ to write $X$ as the direct sum of representations $F' \rightarrow 0 \rightarrow F''$ and $X'=F_2 \rightarrow_{f'} F_1 \rightarrow_{g'} F_0$, where now $f'$ is a monomorphism and $g'$ is full.

We next want to go to a representation in which the maps are both monomorphisms. Let $H_1 = \ker g'$ and $H_2 = \ker (g'f')$; $H_1$ and $H_2$ are summands of $F_1$, $F_2$ respectively, because $\text{im}(g')$ and $\text{im}(g'f')$ are free. The first part of the reduction is to set $\mu X' = F_2 / H_2 \rightarrow_{f'} F_1 / H_1 \rightarrow_{g'} F_0$, where $f'$ is the map given by $z \in H_2 \rightarrow f'(z) + H_1$, and $g'(z + H_1) = g'(z)$. Clearly $\mu X'$ is a representation in which both $f'$ and $g'$ are monomorphisms.

Now let $Y=F_2 \rightarrow_{f} F_1 \rightarrow_{g} F_0$ be such that $f$ and $g$ are both monomorphisms and $g$ is full; this is the type of object that is in the image of $\mu$. We need to go next to an object with
the first map full also; this will be an object in $B_3$. If we let $F_1^*$ be the unique smallest summand of $F_1$ such that $f(F_2) \subseteq F_1^*$, and $F_0^*$ be the unique smallest summand of $F_0$ with $g(F_1^*) \subseteq F_0^*$, then $vY = F_2 \xrightarrow{f} F_1^* \xrightarrow{\hat{g}} F_0^*$, where $\hat{g}$ is $g$ restricted to $F_1^*$, is the desired object in $B_3$.

In order to do this in general we need to define two categories intermediate between $Q_n$ and $B_n$.

Definitions. Let $Q_n'$ be the full subcategory of $Q_n$ consisting of those objects in $\mathbf{X} = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in which $f_{n-1}$ is a monomorphism and $f_1$ is full. Let $Q_n$ be the full subcategory of $Q_n'$ comprising those objects for which each $f_i$ is also a monomorphism.

Recall that $B_n$ is the full subcategory of $Q_n$ in which each $f_i$ is also full.

We first want to show that every object in $Q_n$ can be written as the direct sum of an object in $Q_n'$ and one of the form $F' \to 0 \to \cdots \to 0 \to F^*$ (where either $F'$ or $F^*$ or both may be zero). Let $\mathbf{X} = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ be in $Q_n$. If $\ker(f_{n-1}) = F'_X$ is not zero, it is a summand, as $\im(f_{n-1})$ is free; choose a free complement $\hat{F}'_X$ to $F'_X$. Then the diagram
Now suppose that $f_1$ is not full; then $\text{coker}(f_1) \cong F \oplus T$, where $T$ is the torsion submodule of $\text{coker}(f_1)$. If
\[ F \xrightarrow{f_1} F_0 \xrightarrow{\pi} F \oplus T \rightarrow 0 \]
is exact, let $T^*_x = \pi^{-1}(T)$; then $T^*_x$ is the unique smallest summand of $F_0$ that contains $\text{im}(f_1)$.

Choose a free complement $F^*_x$ for $T^*_x$ in $F_0$; then the diagram
\[ F_n-1 \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_1 \xrightarrow{f_1} F_0 \]
commutes. We can now write $X$ as the direct sum of
\[ \hat{F}_x \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_1 \rightarrow T^*_x \quad \text{and} \quad F'_x \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow F'_x. \]

Given any summand of a free $R$-module $F$, it is not hard to show that we can pick a free complement to it in $F$ which
has as a basis some subset of the set \( \{ e_1, \ldots, e_n \} \) of standard basis vectors for \( \mathbb{R}^n \). If \( \hat{F}_x \) and \( F_x \) are always chosen this way (where if either one of \( e_i, e_j \) is a possible choice, we choose \( e_{\min(i, j)} \)), passage from \( Q_n \) to \( Q'_n \) becomes functorial. Call this functor \( \sigma \); \( \sigma \) is clearly full and dense.

We next describe a functor \( \mu \) from \( Q'_n \) to \( Q_n \).

**Definition.** Let \( X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0 \) and \( X' = F'_{n-1} \xrightarrow{f'_{n-1}} F'_{n-2} \xrightarrow{f'_{n-2}} \cdots \xrightarrow{f'_1} F'_0 \) be in \( Q'_n \), and for each \( i = 1, 2, \ldots, n-1 \) set \( H_i = \ker(f_1 \cdots f_i) \). The object \( \mu X \) in \( Q_n \) is the representation

\[
\tilde{F}_{n-1} \xrightarrow{\tilde{f}_{n-1}} \tilde{F}_{n-2} \xrightarrow{\tilde{f}_{n-2}} \cdots \xrightarrow{\tilde{f}_2} \tilde{F}_1 \xrightarrow{\tilde{f}_1} \tilde{F}_0,
\]

where the \( \tilde{f}_i \) are the induced maps \( z + H_i \xrightarrow{f_i(z) + H_{i-1}} \). \( H_i \) is a summand for each \( i \), so \( \mu X \) is a representation of \( A_n \) by free \( R \)-modules; as \( \tilde{f}_1 \circ \cdots \circ \tilde{f}_i \) is a monomorphism for each \( i \), each \( \tilde{f}_i \) is a monomorphism, so this representation is in \( Q_n \). If \( \alpha = (\alpha_{n-1}, \ldots, \alpha_0) : X \rightarrow X' \) in \( Q'_n \), then \( \mu \alpha = (\alpha_{n-1}, \ldots, \alpha_0) : \mu X \rightarrow \mu X' \) is given by \( \alpha_i(z + H_i) = \alpha_i(z) + H'_i \) for \( 1 \leq i \leq n-1 \), and \( \alpha_0 = \alpha_0 \). It is tedious but routine to check that this actually does give a morphism from \( \mu X \) to \( \mu X' \). Another tedious but routine chore is to check that \( \mu \) preserves compositions; as \( \mu 1 = 1 \) is
obvious, $\mu$ is a functor.

$q_n$ is contained in $Q'$, and $\mu X = X$ for $X$ in $q_n$, so $\mu$ is dense. $\mu$ is clearly not faithful, but it is full as the following shows. Suppose $\beta = (\beta_{n-1}, \ldots, \beta_0) : \mu X \rightarrow \mu X'$ in $q_n$, and write $F_i = F_i / H_i \otimes H_i$, etc., for each $i$ (we take $H_0 = 0$). Then if we choose $\alpha_i = \begin{bmatrix} \beta_i & 0 \\ 0 & 0 \end{bmatrix}$ for each $i$, then $\mu \alpha = \beta$, so $\mu$ is full.

We now want to describe a functor $v$ from $q_n$ to $B_n$. If $f : F \rightarrow F'$ is a map of free $R$-modules, let $f(F)^*$ be the unique smallest summand of $F'$ that contains $f(F)$. If $X = F_{n-1} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0$ and $X' = F'_n \rightarrow F'_{n-1} \rightarrow \cdots \rightarrow F'_0$ are in $q_n$, let $f_n^{-1}(F_{n-1})^* = G_{n-2}$, $f_{n-2}(f_n^{-1}(F_{n-1})^*)^* = G_{n-3}$, etc. Then $vX$ is the object $F_{n-1} \rightarrow G_{n-2} \rightarrow G_{n-3} \rightarrow \cdots \rightarrow \hat{G}_1 \rightarrow G_0$, where $\hat{f}_i$ is the restriction of $f_i$ to $G_i$. $vX$ is certainly in $B_n$. If $\alpha = (\alpha_{n-1}, \ldots, \alpha_0) : X \rightarrow X'$ in $q_n$, then define $\alpha_i^v$ to be the restriction of $\alpha_i$ to $G_i$; then it is clear that $v\alpha = (\alpha_{n-1}, \alpha_{n-2}^v, \ldots, \alpha_0^v)$ is a morphism from $vX$ to $vX'$ in $B_n$. Again it is routine to check that $v$ is a functor.

As before, $B_n$ is contained in $q_n$ and $vX = X$ for $X$ in $B_n$.
so $v$ is dense. We will show that $v$ is full, i.e. that given $\beta = (\beta_{n-1}, \ldots, \beta_0): vX \to vX'$ in $B_n$ we can find a map $\alpha = (\alpha_{n-1}, \ldots, \alpha_0): X \to X'$ in $A_n$ for which $v\alpha = \beta$. We must have $\alpha_{n-1} = \beta_{n-1}$. For $0 \leq i \leq n-2$, let $Q_i$ be a free complement for $G_i$, and define $\alpha_i: G_i \otimes Q_i \to G_i' \otimes Q_i'$ by $\alpha_i = \begin{bmatrix} \beta_i & 0 \\ 0 & \rho_i \end{bmatrix}$, where $(\rho_{n-1}, \ldots, \rho_0)$ is any morphism from $Q_{n-1} \to Q_{n-2} \to Q_{n-3} \to \ldots \to Q_1 \to Q_0$ to $Q_{n-1}' \to Q_{n-2}' \to Q_{n-3}' \to \ldots \to Q_1' \to Q_0'$ in $A_n$. It is then clear that $v\alpha = \beta$. Also clear from this is that $v$ cannot be faithful.

The composition $v\mu \sigma$ is the promised full dense functor from $Q_n$ to $B_n$. 
Chapter V. A result on indecomposability of short exact sequences.

For this chapter we will let $A$ be an artin algebra of finite representation type, and $\text{mod}(A)$ the category of finitely generated (left) $A$-modules. Our result says that if $C$ is in $\text{mod}(A)$, then there are only finitely many modules $A$ in $\text{mod}(A)$ for which there is a short exact sequence $0\to A\to B\to C\to 0$ which is indecomposable. To prove it we need the following version of a result of M. Auslander, in which $\text{Tr}$ and $D$ are the usual artin algebra transpose and dual (see Auslander [A76]), and in which we write $(X,C)$ for $\text{Hom}_A(X,C)$. The submodule $P(X,C)$ of $(X,C)$ consists of those maps $f: X \to C$ for which there exists a factorization

$X \xrightarrow{\alpha} P \xrightarrow{\beta} C$

with $P$ projective.

**Theorem A.** Let $C$ be $\text{mod}(A)$. Let $A_1, \ldots, A_m$ be a complete list of all non-injective indecomposable modules in $\text{mod}(A)$ and let $X_i = \text{TrDA}_i$. For each $i$, $(X_i,C)/P(X_i,C)$ is an $\langle\text{End} X_i\rangle^{\text{op}}$-module of finite length. Let $S_{i_1}, \ldots, S_{i_d}$ be a complete set of non-isomorphic simple $\langle\text{End} X_i\rangle^{\text{op}}$-modules, and for each $\langle\text{End} X_i\rangle^{\text{op}}$-submodule $H$ of $(X_i,C)$ containing $P(X_i,C)$ let $n_1(A_i,H), \ldots, n_{d_i}(A_i,H)$ be the uniquely
determined non-negative integers so that the (End $X_i\rangle^{op}$-socle of $(X_i,C)/H$ is isomorphic to to $\prod_{j=1}^{d_i} n_{i,j}(A_{ij},H)$. Finally let $n(A_i) = \max(n_{i,j}(A_{ij},H))$ as $j$ runs through $1,2,...,d_i$ and as $H$ runs through all $(End X_i)^{op}$-submodules of $(X_i,C)$ containing $P(X_i,C)$. Then

(a) $n(A_i)$ is finite; and

(b) if $k > n(A_i)$ and $0 \to A_i^k \to B \to C \to 0$ is exact, then $A_i^k$ contains a submodule $A'$ (isomorphic to $A_i^{k-n(A_i)}$) such that $g(A')$ is a summand of $B$.

Using the notation of theorem A we have

**Theorem V.1.** Fix $C$ in $\text{mod}(A)$. Then there are only a finite number of modules $A$ in $\text{mod}(A)$ such that $0 \to A \to B \to C \to 0$ is indecomposable as a short exact sequence. In fact, if $A = \bigoplus_{i=1}^{m} A_i^{p_i}$, then if $p_i > n(A_i)$ for some $i$, the sequence decomposes.

**Proof.** We first mention that if $A$ has an injective summand then the sequence decomposes, so we have lost nothing by not including these in the list of $A_i$'s. Suppose $p > n(A_i)$, and form the pushout diagram
Because $p_j > n(A_j)$, $A_i^{p_j}$ has a submodule $A'$ for which $h'(A')$ is a summand of $D$ (so $A'$ is actually a summand of $A_i^{p_j}$) by theorem A. Let $A''$, $A'''$ be such that $A' \otimes A'' = A$, $A' \otimes A''' = A_i^{p_j}$. Then we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A' \otimes A'' & \overset{g}{\longrightarrow} & B & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix} & & \downarrow \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} & & \downarrow & \\
0 & \longrightarrow & A' \otimes A''' & \overset{\alpha}{\longrightarrow} & \alpha(A') \otimes B' & \longrightarrow & C & \longrightarrow & 0 \\
\end{array}
\]

in which $\alpha = h'\big|_{A'}$ is an isomorphism and $h = h'\big|_{A''}$. Then $A' \longrightarrow A'' \overset{g(A')}{\longrightarrow} B$ is a monomorphism which is split by $B \overset{\mu_1}{\longrightarrow} \alpha(A') \overset{\alpha^{-1}}{\longrightarrow} A'$, so $g(A')$ is a summand of $B$. Thus $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ decomposes as a short exact sequence. □

Remark. In mod$(R)$ it is easy to calculate that $n(V_j, C) = s(C)$ for each $V_j$. Let $s(C) = t$, and let $p > 0$ be an integer. Then theorem V.1 implies that there are no more than $1 + p + p^2 + \ldots + p^t = \frac{1 + p^{t+1}}{1 - p}$ non-isomorphic indecomposable short exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of torsion $R$-modules in
which the length of the longest indecomposable summand of $A$ is at most $p$. In chapter III we exhibited $\min(r,p)+1$ non-isomorphic indecomposable short exact sequences.

If $r=p$, then $\min(r,p)+1=1+p=\frac{1+p^2}{1-p}$, so this bound is the best possible.

**Corollary 5.2.** Let $X=R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ be in $C_3$ and let $\lambda X=0 \xrightarrow{A} B \xrightarrow{C} 0$ (so $f,g \in M_n(R)$). Suppose a diagonal form of $f$ is

$$
\begin{pmatrix}
x_1\mu_1 & 0 \\
. & \ddots \\
0 & \cdots & x_n\mu_n
\end{pmatrix}
$$

if any exponent $a_i$ appears more than $s(C)$ times, $X$ must decompose. In particular if we set $d(f)=\max(a_i_{i=1,..,n})$ then if $d(f) \cdot s(C) \cdot n$ then $X$ must decompose.

**Proof.** If $a_i$ appears more than $s(C)$ times than the module $A$ has more than $s(C)$ summands $V_{a_i}$. 

Certainly $\deg(det(f)) = d(f)$, so this corollary implies that if $\deg(det(f)) \cdot s(C) < n$ then $X$ decomposes. If $g$ is an isomorphism then $\xrightarrow{fg}$ certainly decomposes. If $g$ is not an isomorphism then $s(C) > 1$, so if $\deg(det(f)) \cdot n$ then $\xrightarrow{fg}$ decomposes.
Literature Cited


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