

BOUNDARY VALUE AND WIENER-HOPF PROBLEMS
FOR ABSTRACT KINETIC EQUATIONS WITH
NONREGULAR COLLISION OPERATORS

by

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

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April 1986

Blacksburg, Virginia

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(ABSTRACT)

We study the linear abstract kinetic equation $T\psi(x)' = -A\psi(x)$ in the half space $\{x \geq 0\}$ with partial range boundary conditions. The function ψ takes values in a Hilbert space H , T is a self adjoint injective operator on H and A is an accretive operator. The first step in the analysis of this boundary value problem is to show that $T^{-1}A$ generates a holomorphic bisemigroup. We prove two theorems about perturbation of bisemigroups that are interesting in their own right. The second step is to obtain a special decomposition of H which is equivalent to a Wiener-Hopf factorization. The accretivity of A is crucial in this step. When A is of the form "identity plus a compact operator", we work in the original Hilbert space. For unbounded A 's we consider weak solutions in a larger space H_T , which has a natural Krein space structure. Using the Krein space geometry considerably simplifies the analysis of the question of unique solvability.

to my parents
and V.

ACKNOWLEDGMENTS

Even though a PhD is a routine step in one's education, to succeed with this accomplishment I have relied on a lot of help from many people. First I want to express my gratitude to Prof. W. Greenberg for guidance and support, advice and encouragement, spotting my mistakes, sharing his knowledge, enduring my constant pestering him with questions, helping me with many things - math or no math. Prof. C.V.M. van der Mee I thank for a fruitful collaboration, many illuminating discussions and suggestions. I owe also to Profs. G. Hagedorn, J. Slawny, and P. Zweifel for learning a great deal from them, for their encouragement and support. I have benefitted much from discussions with Profs. J. Ball, M. Klaus, and A. Rann, and enjoyed immensely the courses taught by Profs. C. Burch-Brown, D. Farkas, D. Meyers, F. Quinn, M. Williams, R. Zia, and those mentioned before. I have also learned much from the math-physics seminar. I am also indebted to Mrs. G. Henneke for her help on countless occasions. The few Bulgarians in Blacksburg in a short while have become my very close friends and I thank them for all that one is grateful to a friend. I also want to thank my friends from the Argentinian community in Blacksburg and my friends in the math-physics graduate program. Finally I acknowledge financial support from Profs. W. Greenberg and P. Zweifel and from the Cunningham Foundation for the summers and support from Prof. W. Patty to attend a conference.

Going across the ocean, there are many more people to whom I am greatly indebted. I thank Profs. Ch. Christov and I. Todorov for constant support and interest in my progress. I owe a great deal to my friends Drs. V. Dobrev, S. Petcov, S. Todorov, Ch. Vankov, O. Yordanov, and many others. Last but not most of all, I thank my family - what I have received from them all cannot be paid back and words are simply useless.

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I. INTRODUCTION

In this thesis we shall investigate the question of unique solvability of boundary value problems for an abstract stationary linear transport equation modeling numerous kinetic processes in half space geometry. The state of the system will be described by a density function ψ in phase space. If the half space in which we consider the transport process to take place corresponds to $\{x \geq 0\}$, we assume that ψ depends only on x and does not depend on the other spatial variables. For fixed x the density function $\psi(x, \cdot)$ will be an element of an abstract Hilbert space H , which in most models is realized as an L^2 space of functions of velocity coordinates. The abstract transport equation of interest is

$$\frac{d}{dx} T\psi(x) = -A\psi(x), \quad x > 0. \quad (1.1.a)$$

Here T and A are operators on H . Viewed as a stationary transport equation, the left hand side describes free streaming, and the right hand side describes the collisions. In most cases the operator T is a self adjoint operator with absolutely continuous spectrum on an interval around zero. In particular, zero is in the continuous spectrum of T , so T^{-1} is a densely defined operator with spectrum extending to both plus and minus infinity. For many physical problems the operator A can be written as $A=I-B$, where I is the identity operator and B is a compact operator. In other important cases A is a differential or an integrodifferential operator, e.g., a Sturm-Liouville operator or a compact perturbation of a Sturm-Liouville operator. As we shall assume T to be self adjoint with no eigenvalue at zero, we can define maximal positive/negative spectral projectors Q_{\pm}

for T , i.e., $Q_{\pm}^2 = Q_{\pm}$ and $TQ_{\pm}h = Q_{\pm}Th$ for all $h \in D(T)$, and $\sigma\{TQ_{\pm}\} \subset \{\pm\lambda \geq 0\}$.

We assume that the incoming flux is given at the boundary $x=0$, which in our abstract formulation translates into the following boundary condition at $x=0$:

$$Q_+\psi(0) = \varphi_+, \quad (1.1.b)$$

where φ_+ is the given incoming flux. In addition we shall assume some polynomial boundedness of the solution at infinity:

$$\|\psi(x)\| = o(x^n), \quad x \rightarrow \infty, \quad (1.1.c)$$

for some $n=0,1,2,\dots$. Obviously we can consider a boundary value problem in the left half space $\{x \leq 0\}$, in which case condition (1.1.b) will read $Q_-\psi(0) = \varphi_-$, and the limit indicated in (1.1.c) will be taken as $x \rightarrow -\infty$.

Suppose for the moment that $B=0$. Then the solution of (1.1.a-c) is given by the semigroup generated by $T^{-1}Q_+$ on the range of Q_+ , i.e., $\psi(x) = \exp(-xT^{-1})\varphi_+$. Now assume that the operator $T^{-1}A$ has maximal positive/negative spectral projectors P_{\pm} and $\exp(-xT^{-1}A)P_{\pm}$ are right/left semigroups; that is to say, $\exp(-xT^{-1}A)P_+$, $x > 0$, is a right semigroup on $\text{Ran}P_+$ and $\exp(-xT^{-1}A)P_-$, $x < 0$, is a left semigroup on $\text{Ran}P_-$ with $P_+ + P_- = I$. The family of operators $E(x, T^{-1}A) = \pm \exp(-xT^{-1}A)P_{\pm}$, $\pm x > 0$, is called the bisemigroup generated by $T^{-1}A$ and P_{\pm} are called the separating projectors. If $T^{-1}A$ is a generator of a bisemigroup it is easy to see that every solution to the boundary value problem (1.1) has the form $\exp(-xT^{-1}A)h$, $x \geq 0$, for some $h \in P_+H$, such that $Q_+h = \varphi_+$. Correspondingly, every solution to the left half space problem will be in the form

$\exp(-xT^{-1}A)h$, $x \leq 0$, for some $h \in P_-H$ such that $Q_-h = \varphi_-$. Then the issue of unique solvability translates into the following question: "Does Q_+ map $\text{Ran}P_+$ bijectively onto $\text{Ran}Q_+$?" This and the corresponding left half space question are easily seen to be equivalent to the question of whether $H = \text{Ran}Q_+ \oplus \text{Ran}P_+$, or, again equivalently, of whether the operator $V = Q_+P_+ + Q_-P_-$ is invertible. So in investigating the unique solvability of the boundary value problem (1.1) one naturally proceeds in two steps:

- (i) show that $T^{-1}A$ is a generator of a bisemigroup, with separating projectors P_{\pm} ;
- (ii) show that $H = \text{Ran}Q_- \oplus \text{Ran}P_+$.

If B is compact one proceeds with step (ii) as follows:

- (ii.a) using the expression for P_+ , show that $P_+ - Q_+$ is compact, and then from the trivial identity $I - V = (Q_+ - Q_-)(P_+ - Q_+)$ conclude that V is Fredholm of index zero;
- (ii.b) show that V has zero null space, or equivalently that $\text{Ran}P_{\pm} \cap \text{Ran}Q_{\mp} = 0$.

The boundary value problem has an equivalent formulation in terms of a Wiener-Hopf equation. If we assume T self adjoint, then T^{-1} generates a bisemigroup $E(x, T^{-1})$ with separating projectors Q_{\pm} . Set $\mathcal{H}(x) = T^{-1}E(x, T^{-1})$, also called the propagator function. On the space of integrable functions (or bounded measurable functions or bounded continuous functions) ψ from \mathbb{R} to H we define the operator \mathcal{L} to represent convolution with $\mathcal{H}(\cdot)B$:

$$(\mathcal{L}\psi)(x) = \int_{-\infty}^{\infty} \mathcal{K}(x-y)B\psi(y)dy. \quad (1.2)$$

Let P_{\pm} be the projection of the appropriate space of H -valued functions onto the subspace consisting of functions from the right/left half line R_{\pm} to H . Set $\mathcal{L}_{\pm} = P_{\pm} \mathcal{L} P_{\pm}$:

$$(\mathcal{L}_{\pm}\psi)(x) = \pm \int_0^{\pm\infty} \mathcal{K}(x-y)B\psi(y)dy, \quad \pm x \geq 0. \quad (1.3)$$

The boundary value problem (1.1) is equivalent to the Wiener-Hopf equation

$$(I - \mathcal{L}_{+})\psi(x) = \omega(x), \quad x \geq 0, \quad (1.4)$$

where $\omega(x) = \exp(-xT^{-1})\varphi_{+}$. Of course we have to make sense of the convolution integral in (1.2) and to assure that \mathcal{L} maps the space under consideration into itself. One possible strategy is to arrange that the integral kernel $\mathcal{K}(\cdot)B$ be integrable in the Bochner sense. Because clearly $\|\mathcal{K}(x)\|$ goes like $1/x$ near zero and (for unbounded T) near infinity as well, the propagator function \mathcal{K} by itself is not Bochner integrable. This can be fixed by assuming an integrability condition of the form $\text{Ran} B \subset \text{Ran} |T|^{\alpha} \cap D(|T|^{\beta})$ for some $\alpha, \beta > 0$. This allows one to "pull out of B " the necessary behavior to make $\mathcal{K}(\cdot)B$ integrable. Another approach is to consider $\mathcal{K}(\cdot)B$ not as an element of the Bochner integrable functions (which coincide with the greatest cross norm tensor product $L^1(\mathbb{R}) \otimes_{\pi} L(H)$, with $L(H)$ the bounded operators on H), but as an element of a space of weakly integrable functions, namely the least cross norm tensor product $L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$. We may then show the boundedness of convolutions with such weakly integrable functions if, in the latter tensor product, we substitute for $L(H)$ the ideal $L(H)B$ generated by a

trace class operator B.

The question of unique solvability for (1.4) is obviously the question of invertibility of the operator $(I-L_+)$. To show this one proceeds in several steps in a fashion similar to the case of the boundary value problem:

- (i') show that the full line convolution operator $(I-L)$ is invertible;
- (ii'.a) show that the compactness of B implies that $P_{\pm}LP_{\mp}$ are compact, thus $(I-L)-(I-L_+)\oplus(I-L_-)$ is compact, and so $(I-L_+)$ and $(I-L_-)$ are Fredholm operators with $\text{ind}(I-L_+)+\text{ind}(I-L_-)=0$; show that in fact $(I-L_+)$ has index zero;
- (ii'.b) show that $(I-L_+)$ has a trivial null space.

In solving the Wiener-Hopf equation one may go through the method originally proposed by Wiener and Hopf for the scalar convolution equation. The function

$$\begin{aligned}
 W(\lambda) &= I - \int_{-\infty}^{\infty} e^{\lambda x} \chi(x) B dx = & (1.5) \\
 &= I - T^{-1}(T^{-1} - \lambda)^{-1} B = (T^{-1} - \lambda)^{-1}(T^{-1} A - \lambda), \quad \text{Re } \lambda = 0,
 \end{aligned}$$

is called the symbol of the operator $(I-L)$. If the symbol is invertible along the extended imaginary axis, one may ask if $W(\lambda)$ has a canonical Wiener-Hopf factorization. Such a factorization will be equivalent to the invertibility of the half line convolution equation.

There is a close connection between the three methods, and in fact we will go back and forth between them. The fact that $T^{-1}A$ generates a bisemigroup (problem (i)) is equivalent to the invertibility of the full line convolution equation

(problem (i')). On the other hand, the invertibility of $(I-L)$, via a powerful theorem of Bochner and Phillips, is equivalent to the invertibility of the symbol $W(\lambda)$ along the extended imaginary axis. From the explicit form of the symbol (1.5), the last condition is an easily verifiable one. Going backwards, the invertibility of the symbol implies that $T^{-1}A$ is a generator of a bisemigroup. We will state this as a theorem about perturbations of bisemigroups, i.e., if T^{-1} is a generator of a bisemigroup, we will give sufficient conditions on T and A so that $T^{-1}A$ also generates a bisemigroup. This fact is interesting in its own right outside the context of the boundary value problem (1.1). The principal difficulty is that the bisemigroup consists of an invariant decomposition of the underlying space, i.e., separating projectors with semigroups on each invariant subspace, and the perturbed bisemigroup will in general have a different invariant decomposition: the P 's are different from the Q 's.

Now let us consider step (ii). Showing that the operator $V=Q_+P_++Q_-P_-$ or that $(I-L_+)$ and $(I-L_-)$ are Fredholm of index zero is relatively straightforward. Thus we are left with the question of uniqueness. It is most convenient to do this by showing that $\text{Ran}P_{\pm} \cap \text{Ran}Q_{\mp} = 0$. Here an assumption that A is an accretive operator, i.e., $\text{Re}A = (A+A^*)/2 \geq 0$, becomes crucial.

The above discussion applies to the case when the operator A is invertible. For technical reasons we assume something more:

$$\text{Ker}A = \text{Ker}(\text{Re}A) = 0. \quad (1.7)$$

This will yield that the symbol is invertible and the analysis will go through in a straightforward fashion. However in many physical situations A has a nonzero kernel, and so we will study the conservative case

$$\text{Ker}A = \text{Ker}(\text{Re}A) \neq 0. \quad (1.8)$$

Such an assumption is satisfied, for example, in various models of radiative transfer. The manner in which we will treat this case is to isolate the zero root linear manifold $Z_0(T^{-1}A) = \{f \in H : (T^{-1}A)^n f = 0 \text{ for some } n\}$, and to show that there exists a $T^{-1}A$ -invariant decomposition of H with one summand being $Z_0(T^{-1}A)$. Then we may apply the arguments related to the case of A invertible to the complement of $Z_0(T^{-1}A)$, and separately consider $Z_0(T^{-1}A)$. In fact it turns out that the Jordan chains of $Z_0(T^{-1}A)$ have length at most two. This implies that for the boundary value condition at infinity (1.1.c) we need only consider $n=0,1$ or 2 . That is to say, if we impose a boundary condition with $n>2$ we will automatically obtain that the solutions satisfying the boundary condition with $n=2$. In general, unique solvability breaks down when $\text{Ker}A \neq 0$ and the measures of nonexistence and nonuniqueness are related to the Krein space structure of $Z_0(T^{-1}A)$ with respect to the indefinite scalar product given by $(T \cdot, \cdot)$.

The analysis as indicated may be carried out for collision operators A of the form "identity plus a compact". We would like to treat also unbounded operators of the form $A=A_1+A_2$, where A_1 is in general an unbounded, self adjoint, strictly positive operator and A_2 is a compact or a trace class operator. For this problem we will assume that T is bounded. The difficulty is that we do not know how to use a Fredholm argument in this setting, so instead we will work in a larger space H_T (in this sense we say that we will have weak solutions) on which we have a Krein space structure, and use the geometry of this Krein space in place of the Fredholm argument. More precisely, we will consider the indefinite scalar product $(T \cdot, \cdot)$ on H . Completing with respect to the norm induced by this scalar

product, we will obtain a Krein space H_T . First assume that $A_2=0$. Extending the operator $T^{-1}A_1$ from H to H_T will then lead to an H_T -positive operator, with only one "critical point", namely the one at infinity. We will assume that it is a regular critical point, which is true, for example, if A_1 is a Sturm-Liouville operator. The functional calculus for definitizable operators in a Krein space will then give us the separating projectors $P_{1\pm}$ and a bisemigroup generated by $T^{-1}A_1$ on H_T . Moreover, the ranges of $P_{1\pm}$ will be maximal positive/negative definite subspaces of the Krein space H_T . Now a simple geometrical Krein space argument will show that if $P_{1+}H_T$ is a maximal positive space, then Q_+ maps it bijectively onto Q_+H_T . But this is equivalent to the unique solvability of the boundary value problem in H_T . In the case $A_2 \neq 0$ we proceed in two steps. First we define the bisemigroup generated by $T^{-1}A_1$ as indicated. Then we use perturbation arguments for bisemigroups to carry over the study of the generator $T^{-1}A_1$ to the generator $T^{-1}A$. If P_{\pm} are the separating projectors for $T^{-1}A$, then the proof that P_+H_T is a maximal positive subspace is similar to the proof that $P_+H_T \cap Q_-H_T = 0$ and involves accretiveness of A . In the case $\text{Ker}A \neq 0$ we again separate the zero root linear manifold and investigate its Krein space structure.

The early development of convolution equation theory is closely tied to the analysis of transport type equations. In particular, we note the study of the Schwarzschild-Milne integral equation [Mi], formulated for describing the transfer of light through a stellar atmosphere, and the early theory of Wiener-Hopf equations [WH]. In the 1940's and 1950's, the pioneering work of Ambarzumian [A1], [A2], Chandrasekhar [Ch], Sobolev [S] and Busbridge [Bu] in invariant imbedding theory led to the complete solution of the scalar convolution equation of radiative transfer in terms of certain nonlinear integral equations and factorization.

All of these early results dealt only with scalar equations (with the

exception of the polarized light equations studied by Chandrasekhar). In the last two decades, these studies have been extended to more general matrix and operator equations. We mention in particular the work of Gohberg, Krein, Feldman, Semencul, Heinig and Maslennikov ([Go], [GK], [GF], [GH], [GS], [F1], [F2], [Ma]) on matrix valued convolution equations, and Gohberg and Leiterer [GL] on the theory of operator Wiener-Hopf factorization.

In the 1950's the work of van Kampen [vK] and Case [Ca] on the singular eigenfunction method was followed by a spate of activity on specific kinetic models. The Case method involves a full range expansion for the concrete realizations of the operator $T^{-1}A$ and a factorization problem for the dispersion function, leading to a half range completeness theorem. For a detailed description of the method and numerous applications we refer to the book of Case and Zweifel [CZ]; see also [Z1], [Z2] for a more modern treatment.

In trying to circumvent the lack of mathematical rigor in the Case method, Larsen and Habetler [LH] introduced the so called resolvent integration method. While resolvent integration was successful in providing a rigorous and systematic method for the treatment of a variety of kinetic models, the factorization problem remained as the central question precisely as in the Case method. For a survey of resolvent integration and its applications, we refer to the papers of Greenberg and Zweifel [GZ1], [GZ2].

About the same time as the [LH] paper, Hangelbroek [H] formulated the equation of isotropic neutron transport as an abstract equation and used semigroup theory to represent the solutions. He also introduced the projectors P_{\pm} , Q_{\pm} and the operator V . Lekkerkerker [Le] realized that one can bypass the problem of WH-factorization with this approach. This observation resulted in making the semigroup approach into a powerful tool, suitable for abstract generalization and with

far reaching implications in WH theory (see [KLH], [GMP]).

Krein space theory has its roots as well in the mathematical physics of the beginning of this century. The introduction of the Minkowski metric by Minkowski [Mn] to deal with problems of special relativity accustomed the mathematical physics community to spaces with indefinite metrics. Development of the infinite dimensional theory is due largely to Krein [IKL] and Pontryagin [Po]. Spectral theory for self adjoint operators in spaces with indefinite metrics is still an area of ongoing research. The bulk of current results on spectral representations is due to Krein [KL] and Langer [L].

The theory of self adjoint operators in Pontryagin spaces was applied to the transport equation with a collision operator A having a finite dimensional negative part by Ball and Greenberg [BG] and Greenberg and van der Mee [GM]. The need to analyze the Krein space structure of the finite dimensional space $Z_0(T^{-1}A)$ was realized by Beals, Greenberg, van der Mee and Zweifel [B1], [B2], [GMZ].

In the years after 1980 a merger of the invariant imbedding techniques with the semigroup approach has led to the construction of rigorous equivalence proofs between the integrodifferential formulation and the convolution equation formulation. In the context of abstract kinetic theory the best results in this direction are due to van der Mee [M1]. Further, these combined techniques have facilitated the investigation of explicit representations of solutions and the construction of albedo operators (cf. [M4],[GMP]).

Bisemigroup theory, also referred to as the theory of dichotomous operators, in the infinite dimensional setting is in a relatively embryonic state. The development of perturbation theory of bisemigroups with applications to matrix equations is due largely to Gohberg and his school (e.g., [BGK]; important results for the infinite dimensional case are due to van der Mee [M3], [M4]).

The research in this thesis is divided into three parts. In chapter II we prove results on the perturbation of bisemigroups. In the first section we collect facts about integrals of vector valued functions and tensor products. A general reference is the monograph of Diestel and Uhl [DU]. In section two we state the Bocher-Phillips Theorem, giving a sketch of the proof. We then apply this result to obtain information on certain Banach algebras of interest in the sequel. In section three we present some new results on perturbation of bisemigroups. In particular, this will lead to an existence proof for strong solutions of the abstract kinetic equation, without the integrability condition which was necessary in previous approaches in the literature.

Chapter III is devoted to studying questions of unique solvability of the boundary value problem in the original Hilbert space for accretive collision operators of the form "identity plus a compact". We thereby extend well-posedness results previously obtained only for symmetric collision operators to a larger class, which includes significant physical applications. In section one we show unique solvability in the case of invertible A . The next two sections consider operators A with a nontrivial null space. In section two we exhibit an invariant decomposition of the Hilbert space with $Z_0(T^{-1}A)$ being one of the summands. In section three we study the Krein space structure of $Z_0(T^{-1}A)$ and investigate the question of unique solvability. In section four we give an example from radiative transfer.

Chapter IV discusses the boundary value problem for unbounded A 's in a bigger space H_T . We extensively exploit the Krein space structure of H_T to investigate the question of unique solvability. This analysis not only extends the class of collision operators that may be treated (as a possible application we mention the model of Ligou [PL], [CL], [SM], where A is of the form Sturm-Liouville plus a compact), but also provides a new way of looking at the case of symmetric A .

II. PERTURBATION OF BISEMIGROUPS

We start this chapter with a section devoted to "making sense" of the full line convolution equation - one can assume that the integral kernel $\mathcal{K}(\cdot)B$ is either Bochner integrable or Gelfand integrable. The former corresponding to the projective tensor product of the space of integrable functions and the space of bounded operators, the latter corresponds to the injective tensor product. To investigate the invertibility of the full line convolution operator we consider it as an element of an algebra of Wiener type. Finally we prove two theorems about perturbation of bisemigroups, for the cases of "strong" and "weak" integrable kernels.

1. Tensor Products and Banach Algebras

Let X and Y be Banach spaces and X^*, Y^* the dual Banach spaces. By $X \otimes Y$ we denote the algebraic tensor product consisting of all finite linear combinations $\sum x_i \otimes y_i$ with $x_i \in X, y_i \in Y$. A norm α on $X \otimes Y$ is called a reasonable cross norm if $\alpha(x \otimes y) \leq \|x\| \|y\|$ for all $x \in X, y \in Y$ and if $x^* \otimes y^*$ is a functional on the normed space $(X \otimes Y, \alpha)$ for all $x^* \in X^*, y^* \in Y^*$ with functional norm less than or equal $\|x^*\| \|y^*\|$. We will be interested in the least reasonable cross norm $\|\cdot\|_\epsilon$ called also the ϵ -, λ - or injective-tensor product norm, and the greatest reasonable cross norm $\|\cdot\|_\pi$ called also the π -, γ - or projective-tensor product norm. By definition

$$\|u\|_\epsilon = \sup\{|(x^* \otimes y^*)(u)| : \|x^*\|, \|y^*\| \leq 1\} \quad (1.1)$$

$$\|u\|_{\pi} = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}. \quad (1.2)$$

The injective tensor product $X \otimes_{\epsilon} Y$ is the completion of $X \otimes Y$ with respect to $\|\cdot\|_{\epsilon}$ and similarly the completion with respect to $\|\cdot\|_{\pi}$ is called the projective tensor product, denoted $X \otimes_{\pi} Y$. For $u \in X \otimes Y$ and any reasonable cross norm α one has $\|u\|_{\epsilon} \leq \alpha(u) \leq \|u\|_{\pi}$.

If we consider $X \otimes_{\epsilon} Y^*$ then the ϵ -norm can be written also as

$$\|u\|_{\epsilon} = \sup \{ |(x^* \otimes y)(u)| : x^* \in X^*, y \in Y \text{ and } \|x^*\| \leq 1, \|y\| \leq 1 \} \quad (1.3)$$

where $u = \sum_{i=1}^n x_i \otimes y_i^* \in X \otimes Y^*$. Indeed, using the fact that the unit ball of Y is a norming set for Y^* , we obtain

$$\begin{aligned} \|u\|_{\epsilon} &= \sup \{ |(x^* \otimes y^{**})(u)| : \|x^*\| \leq 1, \|y^{**}\| \leq 1 \} = \\ &= \sup \{ \sup \{ |\langle y^{**}, \sum_{i=1}^n \langle x^*, x_i \rangle y_i^* \rangle| : \|y^{**}\| \leq 1 \} : \|x^*\| \leq 1 \} = \\ &= \sup \{ \|\sum_{i=1}^n \langle x^*, x_i \rangle y_i^*\|_{Y^*} : \|x^*\| \leq 1 \} = \\ &= \sup \{ \sup \{ |\langle y, \sum_{i=1}^n \langle x^*, x_i \rangle y_i^* \rangle| : \|y\|_Y \leq 1 \} : \|x^*\| \leq 1 \} = \\ &= \sup \{ |(x^* \otimes y)(u)| : x^* \in X^*, y \in Y, \|x^*\| \leq 1, \|y\| \leq 1 \} \end{aligned} \quad (1.4)$$

Consider a function $f: \mathbb{R} \rightarrow X^*$ from the real line to a dual Banach space. We will say that f is weak- $*$ integrable (with respect to Lebesgue measure on \mathbb{R}) if the scalar valued function $\langle f(\cdot), x \rangle$ is in $L^1(\mathbb{R})$ for all $x \in X$.

Proposition 1.1. Let X be a Banach space and $f: \mathbb{R} \rightarrow X^*$ be weak- $*$ integrable. Then the map $X \rightarrow L^1(\mathbb{R})$ given by $x \rightarrow \langle f(\cdot), x \rangle$ is a bounded operator.

Proof. We wish to apply the closed graph theorem. Suppose x_n converge to x in X and $\langle f(\cdot), x_n \rangle$ converge to some function $\varphi(\cdot)$ in $L^1(\mathbb{R})$. Choose a subsequence $x_{n(i)}$ so that $\langle f(t), x_{n(i)} \rangle \rightarrow \varphi(t)$ pointwise a.e. Obviously, for almost every $t \in \mathbb{R}$ we have $\langle f(t), x_{n(i)} \rangle \rightarrow \langle f(t), x \rangle$. Hence $\varphi(\cdot) = \langle f(\cdot), x \rangle$. ■

From the above proposition we may conclude that there is an element x^* in X^* such that $\langle x^*, x \rangle = \int \langle f(t), x \rangle dt$ for all $x \in X$. This element x^* will be denoted $\int f(t) dt$ and is called the Gelfand integral of f . We also have that if f is weak- $*$ integrable then

$$\|f\| = \sup\{\|\langle f(\cdot), x \rangle\|_1 : x \in X, \|x\| \leq 1\} \quad (1.5)$$

is a finite number. We will call it the weak- $*$ L^1 norm or the Gelfand norm of f . Thus the space $w^*L^1(X^*)$ of all weak- $*$ integrable functions from \mathbb{R} to X^* is a normed space with the Gelfand norm.

Clearly we can identify the algebraic tensor product $L^1(\mathbb{R}) \otimes X^*$ with a subspace of $w^*L^1(X^*)$. Also the Gelfand norm (1.5) coincides with the ϵ -norm, taken in the form given by (1.4). We conclude that the closure of the weak- $*$ integrable functions is precisely $L^1(\mathbb{R}) \otimes_{\epsilon} X^*$.

Consider a Hilbert space H . The finite rank operators on H can be identified with $H \otimes H$. It is also easy to see that the ϵ -norm is the usual operator norm, so $H \otimes_{\epsilon} H = K(H)$, the compact operators on H , and the π -norm is the trace norm,

so $H \otimes_{\pi} H = K_1(H)$, the trace class operators. Moreover, viewing them as Banach spaces we have $K_1(H) = (K(H))^*$ and $(K_1(H))^* = L(H)$, the bounded operators on H .

Recall that every trace class operator T in H can be written in the form

$T = \sum_{i=1}^{\infty} \alpha_i e_i(\cdot, e'_i)$, where $\{e_i\}$ and $\{e'_i\}$ are orthogonal systems of vectors in H and α_i are positive numbers, the singular values of T , which are summable. If A is

any bounded operator, $\text{tr}(AT) = \sum_{i=1}^{\infty} \alpha_i (Ae_i, e'_i)$. We denote the trace norm $K_1(H)$

by $\|\cdot\|_{(1)}$, to differentiate it from the $\|\cdot\|_1$ norm on $L^1(\mathbb{R})$, and have

$$\|T\|_{(1)} = \text{tr}|T| = \sum_{i=1}^{\infty} \alpha_i.$$

Before proceeding, let us mention briefly the Bochner integrable functions. For more details see [Z], [Y] or the already mentioned monograph [DU]. A function $f: \mathbb{R} \rightarrow X$ from the real line with Lebesgue measure to a Banach space X is called Bochner integrable if it is weakly measurable and if the scalar valued function $\|f(\cdot)\|$ is in $L^1(\mathbb{R})$. The space of all X -valued Bochner integrable functions we will denote by $L^1(X)$. The Bochner norm is defined to be

$$\|f\|_1 = \int \|f(t)\| dt.$$

As opposed to the case of weak or weak-* integrable functions, the space $L^1(X)$ is complete. It coincides with the projective tensor product $L^1(\mathbb{R}) \otimes_{\pi} X$ and the π -norm is the Bochner norm. One can obviously define $L^p(X)$ for $1 \leq p < \infty$ with norm $\|f\|_p = \left(\int \|f(t)\|^p dt \right)^{1/p}$.

When discussing spaces of bounded or bounded continuous functions from the

real line to a Banach space X one again can consider spaces of weakly bounded or weakly bounded continuous and strongly bounded or strongly bounded continuous functions. In fact, those notions coincide. Suppose $f: \mathbb{R} \rightarrow X$ is a weakly bounded function, i.e. $\|\langle x^*, f(\cdot) \rangle\|_\infty < \infty$ for any $x^* \in X^*$. By the same reasoning as in Proposition 1.1 we conclude that $\|f\|_{w,\infty} \equiv \sup\{\|\langle x^*, f(\cdot) \rangle\|_\infty : x^* \in X^*\} < \infty$. By the uniform boundedness principle we conclude that $\|f\|_\infty \equiv \|(\|f(\cdot)\|)\|_\infty < \infty$, i.e., a weakly bounded function is strongly bounded. Moreover the two norms $\|\cdot\|_{w,\infty}$ and $\|\cdot\|_\infty$ coincide. Obviously $\|f\|_{w,\infty} \leq \|f\|_\infty$. For the converse inequality pick an arbitrary $\epsilon > 0$ and choose $t_0 \in \mathbb{R}$ such that $|\|f(t_0)\| - \|f\|_\infty| < \epsilon/2$. Next choose x_0^* such that $|\langle x_0^*, f(t_0) \rangle - \|f(t_0)\|| < \epsilon/2$. Thus we get $\|f\|_\infty - \epsilon \leq \|f\|_{w,\infty}$, but ϵ was arbitrary. So in fact $wL^\infty(X)$ coincides with $L^\infty(X)$, and we will use only the latter notation. Similarly $wC(\mathbb{R}, X)$ and $C(\mathbb{R}, X)$ coincide. In the language of tensor products we have that $L^\infty(\mathbb{R}) \otimes_\epsilon X = L^\infty(X)$ and $C(\mathbb{R}) \otimes_\epsilon X = C(\mathbb{R}, X)$.

Next we want to consider Banach algebras of weak-* integrable functions that act by convolution on $L^\infty(H)$ or $C(\mathbb{R}, H)$. Let D be a fixed trace class operator on H , and for short denote $d = \|D\|_{(1)}$. We will show that $\beta_D \equiv L^1(\mathbb{R}) \otimes_\epsilon L(H)D$ is a Banach algebra. Here $L(H)D$ is the left ideal generated by D , a general element in β_D looks like uD , where $u \in L^1(\mathbb{R}) \otimes_\epsilon L(H)$, and the norm on β_D is given by $\|uD\| \equiv d\|u\|_\epsilon$. We check the submultiplicative property of this norm. Let $uD, vD \in \beta_D$. We want to show $\|uD^*v\|_\epsilon \leq d\|u\|_\epsilon \|v\|_\epsilon$. Take a trace class operator $C = \sum \gamma_i f_i(\cdot, g_i)$ of trace class norm equal to one, i.e. $\sum \gamma_i = 1$. Assume that $D = \sum \delta_i h_i(\cdot, k_i)$ with $\sum \delta_i = d$. We have

$$\|\text{tr}(CuD^*v)\|_1 = \|\sum \gamma_i (uD^*v f_i, g_i)\|_1 \leq$$

$$\begin{aligned}
&\leq \sum \gamma_i \| (uD^*vf, g_i) \|_1 \leq \sup\{ \| (uD^*vf, g) \|_1 : \|f\|, \|g\| \leq 1 \} \sum \gamma_i = \\
&= \sup_{f, g} \| \sum \delta_i (uh_i, g)^*(vf, k_i) \|_1 \leq \sup_{f, g} \sum \delta_i \| (uh_i, g)^*(vf, k_i) \|_1 \leq \\
&\leq \sup_{f, g} \sum \delta_i \| (u h_i, g) \|_1 \| (vf, k_i) \|_1 \leq \\
&\leq \sup_{h, g} \| (uh, g) \|_1 \sup_{f, k} \| (vf, k) \|_1 \sum \delta_i \leq \|u\|_\epsilon \|v\|_\epsilon d.
\end{aligned}$$

Taking the supremum over C , we arrive at the desired submultiplicative property.

A similar argument shows that if $u \in \beta_D$ then it acts by convolution as an operator from $L^\infty(H)$ into itself with norm equal to the Banach algebra norm of u . If u is a simple function in the algebraic tensor product $L^1(\mathbb{R}) \otimes L(H)D$, and if w is a simple function in $L^1(\mathbb{R}) \otimes H$, then from the smoothing property of ordinary convolutions we conclude that $u*w \in C(\mathbb{R}) \otimes H$. Because u is a bounded operator on $L^\infty(H)$ and $C(\mathbb{R}, H)$ is a closed subspace of $L^\infty(H)$, we may extend the action of u to all of $L^\infty(H)$ and obtain that $u: L^\infty(H) \rightarrow C(\mathbb{R}, H)$. But, as noted, the operator norm of u is equal to the Banach algebra norm, so we get a unimodular representation of β_D into the operators on $L^\infty(H)$. Because the algebraic tensor product $L^\infty(\mathbb{R}) \otimes H$ is dense in $L^\infty(H)$ and using the completeness of $C(\mathbb{R}, H)$ once more, we conclude that $u: L^\infty(H) \rightarrow C(\mathbb{R}, H)$ for any u in β_D . We have obtained the following fact.

Proposition 1.2. The space β_D is a Banach algebra. It acts via convolutions as a space of bounded operators on $L^\infty(H)$ sending $L^\infty(H)$ into $C(\mathbb{R}, H)$.

2. The Wiener Property

Let Z and \mathcal{F} be Banach algebras with a unit. Assume that Z is commutative. By $Z \otimes \mathcal{F}$ we denote the algebraic tensor product consisting of finite sums $\sum z_i f_i$. Evidently this is an algebra. Because both Z and \mathcal{F} have a unit, we may assume that Z and \mathcal{F} are inside $Z \otimes \mathcal{F}$, and, moreover, Z will be in the center of $Z \otimes \mathcal{F}$. Next suppose that \mathcal{A} is a Banach algebra with unit e , such that $Z \otimes \mathcal{F}$ is dense in \mathcal{A} and the norms on Z and \mathcal{F} coincide with the norms induced on Z and \mathcal{F} as subspaces of \mathcal{A} .

Every multiplicative functional $\varphi: Z \rightarrow \mathbb{C}$ induces an algebra homomorphism $\Phi: Z \otimes \mathcal{F} \rightarrow \mathcal{F}$ via

$$\Phi(\sum z_i f_i) = \sum \varphi(z_i) f_i. \quad (2.1)$$

Following [GL] we call the algebra \mathcal{A} a $Z \otimes \mathcal{F}$ -algebra if all the induced homomorphisms ϕ are bounded operators from $Z \otimes \mathcal{F}$ with the \mathcal{A} -norm to \mathcal{F} . Therefore we can extend each induced homomorphism to a Banach algebra homomorphism $\Phi: \mathcal{A} \rightarrow \mathcal{F}$.

Wiener in [W] proved the following important lemma, which was crucial in the proof of his Tauberian theorems.

Lemma (Wiener). Let f be a function from the unit circle \mathbb{T} to the complex numbers \mathbb{C} and suppose $f(z)$ is invertible for every $z \in \mathbb{T}$, i.e., $f: \mathbb{T} \rightarrow \mathbb{C} \setminus \{0\}$. Then if f has an absolutely convergent Fourier expansion, so does $1/f$.

The above lemma was one of the first testing grounds for the maximal ideal theory for commutative Banach algebras developed by Gelfand [GRS]. Gelfand's proof was extremely short and elegant compared with the proof of Wiener.

Bochner and Phillips obtained a substantial generalization of Wiener's Lemma [BP]. Roughly speaking they changed the scalars from the field of complex numbers to a noncommutative Banach ring by tensoring the commutative Wiener algebra with the noncommutative ring. Their theorem was further generalized in [A] and [GL]. We give a short sketch of the proof following the second proof in [BP], but using the notation of [GL].

Theorem (Bochner-Phillips). Let \mathcal{A} be a $Z \otimes \mathcal{F}$ -algebra. An element $a \in \mathcal{A}$ has a left, right or two-sided inverse in \mathcal{A} if, for each induced homomorphism Φ , the element $\Phi(a) \in \mathcal{F}$ has a left, right or two-sided inverse, respectively.

Proof. The only if part is immediate. Suppose that $a \in \mathcal{A}$ is such that $\Phi(a) \in \mathcal{F}$ has a left inverse in \mathcal{F} for every induced homomorphism Φ . We want to conclude that a has a left inverse in \mathcal{A} , or equivalently that a is not contained in any maximal left ideal of \mathcal{A} . Let I be a maximal left ideal of \mathcal{A} . Consider the left \mathcal{A} -module $\mathcal{V} = \mathcal{A}/I$ and denote by $[b]$ the coclass of $b \in \mathcal{A}$, which is an element of \mathcal{V} . The ring \mathcal{A} acts on \mathcal{V} as $c \cdot [b] = [cb]$ for $c \in \mathcal{A}$.

The module \mathcal{V} is \mathcal{A} -irreducible. Indeed, pick an arbitrary $[b] \in \mathcal{V}$, $[b] \neq 0$, and consider the submodule $\mathcal{V}_0 = \mathcal{A} \cdot [b] \subset \mathcal{V}$. Denote by S the union of all the cosets of \mathcal{V}_0 . The set S is actually a left ideal of \mathcal{A} containing I . Moreover $b \in S$, because \mathcal{A} has a unit, and $[b] \neq 0$ so $b \notin I$. By the maximality of I we are forced to have $S = \mathcal{A}$, hence $\mathcal{V}_0 = \mathcal{V}$, and \mathcal{V} is \mathcal{A} -irreducible.

From the irreducibility of \mathcal{V} and Schur's Lemma we have that the action of Z on \mathcal{V} reduces to multiplication by a complex number, i.e., for every $z \in Z$, there is $\lambda \in \mathbb{C}$ such that $z \cdot [b] = \lambda [b]$ for all $[b] \in \mathcal{V}$. One checks that this defines a multiplicative functional φ via $\varphi(z) = \lambda$. Let Φ be the induced homomorphism. Note

that for an element $\sum z_i f_i$ of $Z \otimes \mathcal{F}$ we can write

$$(\sum z_i f_i) \cdot [b] = \sum f_i \cdot (z_i \cdot [b]) = (\sum f_i \varphi(z_i)) \cdot [b] = \Phi(\sum z_i f_i) \cdot [b]$$

By the assumption that we have a $Z \otimes \mathcal{F}$ -algebra we can extend this formula to read $c \cdot [b] = \Phi(c) \cdot [b]$, $\forall b, c \in \mathcal{A}$.

By assumption, $\Phi(a)$ has a left inverse in \mathcal{F} ; say f is the left inverse, $f \cdot \Phi(a) = e$. In particular $(f \cdot \Phi(a)) \cdot [e] = [e]$. If we suppose that $a \in I$ then $\Phi(a) \cdot [e] = a \cdot [e] = [a] = [0]$, which is a contradiction because $[0] \neq [e]$. We conclude that a is not contained in any left ideal of \mathcal{A} , and therefore has a left inverse.

The proof for a right and a two-sided inverse will follow mutatis mutandis considering right and two-sided maximal ideals, respectively. ■

Suppose now, that \mathcal{F} is a Banach algebra with a unit and norm $\|\cdot\|_{\mathcal{F}}$. Let \mathcal{A} be a Banach algebra of functions from \mathbb{R} to \mathcal{F} with pointwise multiplication and norm $\|\cdot\|_{\mathcal{A}}$ satisfying

$$(i) \quad \|a\|_{\mathcal{A}} \geq \sup\{\|a(\lambda)\|_{\mathcal{F}} : \lambda \in \mathbb{R}\}, \quad \forall a \in \mathcal{A}.$$

Let Z be a central subalgebra of \mathcal{A} consisting of complex valued functions on \mathbb{R} with pointwise multiplication. Let the norm on Z be the one induced by the norm of \mathcal{A} . Also we assume that $Z \otimes \mathcal{F}$ is dense in \mathcal{A} . If \hat{Z} is the carrier space (space of maximal ideals) of Z and $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ is the one point compactification of the real line, we also assume

$$(ii) \quad \hat{Z} = \mathbb{R}^*.$$

Thus the multiplicative functionals φ on Z are precisely the evaluation functionals δ_t , i.e., $\varphi = \delta_t$ for some $t \in \mathbb{R}^*$ where $\delta_t(z) = z(t)$. If (i-ii) are satisfied, then \mathcal{A} is a $Z \otimes \mathcal{F}$ algebra. Indeed, let $\varphi = \delta_t : Z \rightarrow \mathbb{C}$ be a multiplicative functional and let Φ_t

be the induced homomorphism. Then Φ_t is given by evaluation, $\Phi_t(a) = a(t)$, $\forall a \in \mathcal{A}$. We have to check that it is a bounded map from \mathcal{A} to \mathcal{F} :

$$\sup\{\|\Phi_t(a)\|_{\mathcal{F}} : \|a\|_{\mathcal{A}} = 1\} = \sup\{\|a(t)\|_{\mathcal{F}} : \|a\|_{\mathcal{A}} = 1\} \leq 1$$

where the inequality follows from (i).

We will say that an algebra \mathcal{A} of \mathcal{F} -valued functions on the real line with the assumptions (i-ii) has the Wiener property, whence a is invertible in \mathcal{A} iff $a(t)$ is invertible in \mathcal{F} for all $t \in \mathbb{R}^*$.

Our aim will be to obtain an algebra useful for our purposes with the Wiener property. Let $u = \sum f_i A_i$ be a simple function, i.e., $u \in L^1(\mathbb{R}) \otimes L(H)$. For any $\lambda \in \mathbb{R}$ the function $t \rightarrow e^{i\lambda t} u(t)$ is obviously in $L^1(\mathbb{R}) \otimes L(H)$. Denote the Gelfand integral of this function by $\hat{u}(\lambda) = \sum \hat{f}_i(\lambda) A_i$. We obtain $\hat{u} \in C_0(\mathbb{R}) \otimes L(H)$, where $C_0(\mathbb{R})$ is the space of continuous functions on \mathbb{R} that decay at infinity (use the Riemann-Lebesgue lemma). Recall that $C_0(\mathbb{R}, L(H)) = C_0(\mathbb{R}) \otimes_{\epsilon} L(H)$. Let C be a trace class operator of trace norm one, $C = \sum \gamma_i e_i(\cdot, e'_i)$ and $\sum \gamma_i = 1$. We have

$$\begin{aligned} \sup_{\lambda} |\operatorname{tr}(C\hat{u})| &= \sup_{\lambda} \left| \sum_{i=1}^{\infty} \gamma_i (\hat{u}(\lambda) e_i, e'_i) \right| \leq \\ &\leq \sup_{\lambda} \sum_{i=1}^{\infty} \gamma_i |(\hat{u}(\lambda) e_i, e'_i)| \leq \\ &\leq \sup_{\lambda} \sup_{\|\varphi\|, \|\psi\| \leq 1} |(\hat{u}(\lambda)\varphi, \psi)| \sum \gamma_i \leq \\ &\leq \sup_{\|\varphi\|, \|\psi\| \leq 1} \|(u(\cdot)\varphi, \psi)\|_1 \leq \\ &\leq \sup\{\|\operatorname{tr}(u(\cdot)C)\|_1 : \|C\|_1 \leq 1\} = \|u\|_{\epsilon}. \end{aligned}$$

Taking the supremum over C on the left we conclude that

$$\|\hat{u}\| \leq \|u\|_\epsilon. \quad (2.2)$$

Thus the Fourier transform takes simple functions from $L^1(\mathbb{R}) \otimes_\epsilon L(H)$ into simple functions in $C_0(\mathbb{R}, L(H))$. As it is a bounded operator, it may be extended by continuity to all of $L^1(\mathbb{R}) \otimes_\epsilon L(H)$. We have shown the following fact.

Proposition 2.1. The Fourier transform acts as a bounded operator $\hat{\cdot} : L^1(\mathbb{R}) \otimes_\epsilon L(H) \rightarrow C_0(\mathbb{R}, L(H))$.

We will obtain a Banach algebra with the Wiener property by adding to our algebra $L^1(\mathbb{R}) \otimes_\epsilon L(H)D$ a sufficiently large center. Recall that D is a fixed trace class operator and the norm of an element uD is given by $\|uD\| = \|u\|_\epsilon \|D\|_{(1)}$, while multiplication is defined by convolution. Let Z be obtained from $L^1(\mathbb{R})$ by adjoining a unit and let \mathcal{F}_D be obtained by adjoining a unit to $L(H)D$. Formally we write $Z = 1 + L^1(\mathbb{R})$ and $\mathcal{F}_D = I + L(H)D$. Then $Z \otimes_\epsilon \mathcal{F}_D$ is a Banach algebra. Let \mathcal{A}_D be its image in $C(\mathbb{R}, L(H))$ under the Fourier transform. We equip \mathcal{A}_D with the norm induced from $Z \otimes_\epsilon \mathcal{F}_D$, i.e., if $u \in Z \otimes_\epsilon \mathcal{F}_D$ then the norm of the element $\hat{u} \in \mathcal{A}_D$ is $\|\hat{u}\| = \|u\|$. We immediately get that assumptions (i) and (ii) are satisfied, and thus we obtain that \mathcal{A}_D has the Wiener property. In fact, with the following lemma we have something better.

Lemma 2.2. Suppose D is a compact operator on an infinite dimensional Hilbert space H . Let $I + L(H)D$ be the algebra obtained from the ideal $L(H)D$ by formally adjoining a unit. Then $B \in I + L(H)D$ is invertible in $I + L(H)D$ iff it is invertible in $L(H)$.

Proof. In general we have $B = \alpha I + CD$ for some $\alpha \in \mathbb{C}$ and $C \in L(H)$. Suppose B is invertible in $L(H)$. Because H is infinite dimensional and D is compact, we are forced to have $\alpha \neq 0$, so, without loss of generality, assume $\alpha = 1$. Let $(I-A) \in L(H)$ be the inverse of B , i.e., $I = (I-A)(I+CD) = I - A + CD - ACD$, or $A = (I-A)CD$. Then $(I-A) \in I + L(H)D$. ■

Proposition 2.3. Let $\hat{u} \in \mathcal{A}_D \equiv (Z \otimes_{\epsilon} \mathcal{F}_D)^{\wedge}$. If $\hat{u}(\lambda)$ is invertible in $L(H)$ for every $\lambda \in \mathbb{R}^*$, then \hat{u} is invertible in \mathcal{A}_D , i.e., $\hat{u}^{-1} = \hat{v}$ for some $v \in Z \otimes_{\epsilon} \mathcal{F}_D$.

We have an immediate corollary.

Proposition 2.4. Let ψ and ω be in $L^{\infty}(H)$ or $C(\mathbb{R}, H)$. Let $u \in L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$ and let D be a trace class operator on H . Then

$$\psi - u^* D \psi = \omega \quad (2.3)$$

is a well defined equation in $L^{\infty}(H)$ or $C(\mathbb{R}, H)$. It has a solution in $L^{\infty}(H)$ or $C(\mathbb{R}, H)$ iff the symbol of this equation $W(\lambda) = I - \hat{u}(\lambda)D$, is invertible in $L(H)$ for all $\lambda \in \mathbb{R}^*$. If this is the case, then the solution is given by

$$\psi = \omega + u^x D \omega, \quad (2.4)$$

where u^x is some element of $L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$.

If instead of the injective tensor product \otimes_{ϵ} we substitute the projective tensor product \otimes_{π} , all the above facts are true and known. More precisely,

$L^1(\mathbb{R}) \otimes_{\pi} L(H)$ is the space of $L(H)$ -valued Bochner integrable functions $L^1(L(H))$, and forms a Banach algebra under convolution. The functions act via convolution as bounded operators on $L^{\infty}(H)$, sending it into $C(\mathbb{R}, H)$. Also, the algebra $(1 + L^1(\mathbb{R})) \otimes_{\pi} L(H)$ has the Wiener property. In this case, we do not need a trace class operator D .

We note that even though the Bochner integral of a compact operator valued function is a compact operator, the same is not true for $L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$. Indeed, if we have a compact operator valued function, then the Gelfand integral as defined in the previous section becomes the Dunford integral [DU], because the trace class operators are the dual of the compact operators. However, in general the Dunford integral is a map from a Banach space to its second dual, indeed, in our case from the compact operators to the bounded operators.

3. Perturbation Theory

By a strongly continuous bisemigroup $E(t)$ on a Hilbert space H (see [BGK]) we will understand a function E from $\mathbb{R} \setminus \{0\}$ into $L(H)$, the bounded operators on H , with the following properties:

- (i) $E(t)E(s) = \pm E(t+s)$ if $\text{sgn}(t) = \text{sgn}(s) = \pm$ and $E(t)E(s) = 0$ if $\text{sgn}(t) = -\text{sgn}(s)$
- (ii) $E(\cdot)$ is strongly continuous.

It is easy to check that

$$\Pi_{\pm} = s\text{-}\lim_{\pm t \downarrow 0} (\pm E(t))$$

are bounded projectors, called separating projectors, and that $\Pi_{+}\Pi_{-} = 0 = \Pi_{-}\Pi_{+}$.

In the definition of a bisemigroup we require also

$$(iii) \Pi_+ + \Pi_- = 1.$$

This is equivalent to saying that $\pm E(t)\Pi_{\pm}$, $\pm t \geq 0$, are strongly continuous right/left semigroups on $\text{Ran}\Pi_{\pm}$. An operator S is the generator of $E(t)$ if Π_{\pm} leaves $D(S)$ invariant and $S\Pi_{\pm}h = \Pi_{\pm}Sh$, $\forall h \in D(S)$, and if $E(t) = \pm \exp(-tS)\Pi_{\pm}$, $\pm t > 0$. We will also write $E(t;S)$ for the bisemigroup generated by S . If the Laplace transform of $E(t)$ exists, then it is easy to check that it is the resolvent of S on the imaginary axis, i.e.,

$$(S - \lambda)^{-1}h = \int_{-\infty}^{\infty} e^{\lambda t} E(t)h dt, \quad \text{Re } \lambda = 0, \quad h \in H.$$

The bisemigroup will be called bounded holomorphic, strongly decaying holomorphic, or exponentially decaying holomorphic if the semigroups $\pm E(t)\Pi_{\pm}$, $\pm t > 0$, have the respective properties. We have the following lemma.

Lemma 3.1. Suppose that S generates a bounded holomorphic semigroup $\exp(-tS)$ and that zero is in the spectrum of S . Then the semigroup is strongly decaying if and only if zero is in the continuous spectrum of S .

Proof. Take first $h \in D(S^{-1}) = \text{Ran}S$, which is dense by assumption. Then $h = Sk$ for some $k \in H$ and by using a property of holomorphic semigroups we have $\|\exp(-tS)h\| = \|S\exp(-tS)k\| \leq \text{const} \cdot \|k\|/t \rightarrow 0$ as $t \rightarrow \infty$. Because the semigroup is bounded and $\text{Ran}S$ is dense, we get that the strong decay holds for all $h \in H$.

If zero is an eigenvalue of S with corresponding eigenvector h , then $\exp(-tS)h = \lim_{n \rightarrow \infty} (I - tS/n)^n h = h$, whence the semigroup is not strongly decaying.

If zero is in the residual spectrum of S , we consider the adjoint semigroup,

which also is a bounded holomorphic semigroup (see [P], Corollary I.10.6) with zero an eigenvalue of its generator. As a result, the adjoint semigroup as well as the original semigroup cannot be strongly decaying. ■

In the case that S generates a decaying semigroup one observes that Π_{\pm} are the maximal positive/negative spectral projectors for the operator S , i.e. $\sigma(S\Pi_{\pm}) \subset \{\pm\text{Re}\leq 0\}$.

First we will consider the case of a Bochner integrable kernel $\mathcal{K}(\cdot)B$ and after that the case when B is trace class. Let $L_1(L(H))$ be the space of operator valued functions, Bochner integrable on the real line. As already mentioned in the previous section, the algebra obtained from $L_1(L(H))$ by formally adjoining a unit has the Wiener property.

Theorem (Bochner-Phillips [BP]). If $\mathcal{K}(\cdot) \in L_1(L(H))$ and

$$W(\lambda) = I - \int_{-\infty}^{\infty} e^{\lambda t} \mathcal{K}(t) dt, \quad \text{Re}\lambda = 0,$$

is invertible on the extended imaginary axis, then

$$W(\lambda)^{-1} = I + \int_{-\infty}^{\infty} e^{\lambda t} \mathcal{K}^X(t) dt, \quad \text{Re}\lambda = 0,$$

for some $\mathcal{K}^X(\cdot) \in L_1(L(H))$. Moreover, $\mathcal{K}^X(\cdot)$ is compact operator valued if $\mathcal{K}(\cdot)$ is.

For an angle $0 < \theta \leq \pi/2$ we denote sectors about the real axis by $\Sigma_{\theta\pm}$ with $\Sigma_{\theta\pm} = \{z \in \mathbb{C} : |\arg(\pm z)| < \theta\}$ and $\Sigma_{\theta} = \Sigma_{\theta+} \cup \Sigma_{\theta-}$. Assume that S is a spectral operator of scalar type (see [DS]) on a Hilbert space H with spectral

measure $dF(\lambda)$, i.e.,

$$S = \int_{\sigma(S)} \lambda dF(\lambda).$$

Assume also that $\sigma(S) \subset \bar{\Sigma}_{\pi/2 - \theta_1}$ for some $0 < \theta_1 < \pi/2$ and that zero is either in the resolvent set or in the continuous spectrum of S . It is immediate to check that S is a generator of a strongly decaying, holomorphic bisemigroup of angle at least θ_1 , with separating projectors given by the spectral projectors $\Pi_{\pm} = F(\sigma(S) \cap \{\pm \operatorname{Re} z \geq 0\})$. If S^{-1} is a bounded operator, the bisemigroup is exponentially decaying.

Our aim is to establish sufficient conditions so that a perturbation $S^X = SA$ will still generate a bisemigroup. Besides the assumption on S made above, suppose also that the following conditions hold:

- (i) $B = I - A$ is compact.
- (ii) B is Hölder continuous with respect to S at zero and infinity: there exist numbers $\alpha, \gamma > 0$ and bounded operators D_1, D_2 such that $B = |S|^{-\alpha} D_1$ and $B = |S|^{\gamma} D_2$, where $|S| = S(\Pi_+ - \Pi_-)$.
- (iii) The spectrum of S^X is contained in a sector around the real axis: $\sigma(S^X) \subset \bar{\Sigma}_{\pi/2 - \theta_2}$ for some $0 < \theta_2 < \pi/2$.
- (iv) $\operatorname{Ker} A = 0$.

Condition (ii) will assure the Bochner integrability of the convolution kernel. Because B is compact, (iv) implies that A^{-1} is bounded and that $S^X = SA$ is densely defined.

Let us make the following remark. By assumption the spectrum of S is

contained in a sector of angle $\pi/2 - \theta_1$. Because $S - S^X$ is S -relatively compact, by a Weyl type of argument the spectrum of S^X outside of this sector will consist of at most countably many points that can accumulate only on $\sigma(S)$. To be more precise, if the parts of the resolvent set of S that contain the upper and the lower imaginary half lines are disconnected parts of the resolvent set, then we must make sure that the perturbation does not "fill a hole", i.e., there are at least two points $\lambda_{0\pm}$ such that $\lambda_{0\pm} \in \{\pm \text{Im}z > 0\} \setminus \Sigma_{\pi/2 - \theta}$ and $\lambda_{0\pm} \in \rho(S^X)$. As we will see in the next chapter, if we impose the condition $\text{Ker}A = \text{Ker}(\text{Re}A) = 0$ then S^X has no eigenvalues along the imaginary axis. So we need only assume that there is no sequence of eigenvalues which approaches zero tangentially to the imaginary axis to be able to find an angle $0 < \theta_2 < \pi/2$ such that $\sigma(S^X) \subset \bar{\Sigma}_{\pi/2 - \theta_2}$. Of course, if B is a finite rank operator there are only finitely many discrete points so there is no problem of their accumulation at zero.

Theorem 3.1. With the above assumptions on S and B , S^X generates a holomorphic bisemigroup $E^X(t)$ with separating projectors Π_{\pm}^X . For any $t \in \mathbb{R} \setminus \{0\}$ we have that $E(t) - E^X(t)$ and $\Pi_{\pm} - \Pi_{\pm}^X$ are compact, and that the bisemigroup $E^X(t)$ is strongly decaying. If $\sigma(S)$ has a gap at zero (i.e., S^{-1} is bounded), then $E^X(t)$ is exponentially decaying.

Proof. Consider the operator valued function $\mathcal{A}(t) = SE(t)B$ defined for $t \in \mathbb{R} \setminus \{0\}$. We claim that $\mathcal{A}(\cdot) \in L_1(L(H))$. Indeed, for a bounded function ϕ on $\sigma(S)$ one has, by the functional calculus, $\|\phi(S)\| = \text{ess sup} \{|\phi(z)| : z \in \sigma(S)\}$, and a simple calculation shows that the first and second Hölder conditions on B assure $\|\mathcal{A}(t)\|$ is integrable in t near infinity and near zero. Hence $\|\mathcal{A}(\cdot)\| \in L_1(\mathbb{R})$, and $\mathcal{A}(\cdot) \in L_1(L(H))$.

Next define

$$(\mathcal{L}\psi)(t) = \int_{-\infty}^{\infty} \mathcal{A}(t-s)\psi(s)ds,$$

a bounded operator on the spaces $L_p(\mathbb{H})_{-\infty}^{\infty}$ of L_p Bochner integrable vector valued functions on the real line, or on $C(\mathbb{R},\mathbb{H})$, the space of norm continuous vector valued functions on the real line. The symbol of the operator $(I-\mathcal{L})$ is given by the two-sided Laplace transform

$$W(\lambda) = I - \int_{-\infty}^{\infty} e^{\lambda t} \mathcal{A}(t) dt = (\lambda - S)^{-1}(\lambda - S^X) = I - \lambda^{-1}(S^{-1} - \lambda^{-1})^{-1}B$$

for $\operatorname{Re}\lambda=0$. From the assumptions on B it follows that $W(\lambda)$ has a bounded inverse on the extended imaginary axis. We can apply the Bochner-Phillips Theorem to obtain that the operator $I-\mathcal{L}$ on $L_p(\mathbb{H})_{-\infty}^{\infty}$ or $C(\mathbb{R},\mathbb{H})$ is invertible with inverse $I+\mathcal{L}^X$, where

$$(\mathcal{L}^X\psi)(t) = \int_{-\infty}^{\infty} \mathcal{A}^X(t-s)\psi(s)ds.$$

We claim that $E^X(t)h=(I+\mathcal{L}^X)E(t)h$, for $h\in\mathbb{H}$ and $t\in\mathbb{R}\setminus\{0\}$, is the bisemigroup generated by S^X . First let us check that $E^X(t)$ defined above is a bisemigroup. Fix some $s>0$ and $h\in\mathbb{H}$ and define $\psi(t)=E^X(t+s)h$ if $t>0$ and $\psi(t)=0$ if $t<0$. When $t>0$ we have

$$\begin{aligned} (I-\mathcal{L})\psi(t) &= E^X(t+s)h - \int_s^{\infty} \mathcal{A}(t+s-r)E^X(r)h dr = \\ &= (I-\mathcal{L})E^X(t+s)h + \int_{-\infty}^s \mathcal{A}(t+s-r)E^X(r)h dr, \end{aligned}$$

since for $r<s$ we have $t+s-r>0$ and $\mathcal{A}(t+s-r)=E(t)\mathcal{A}(s-r)$, while for $r>s$ we have

$0 = E(t) \mathcal{A}(s-r)$. We can rewrite the above as

$$(I - \mathcal{L})\psi(t) = E(t+s)h + E(t)\mathcal{L}E^X(s)h - E(t)(E(s) + \mathcal{L}E^X(s))h = E(t)E^X(s)h.$$

Therefore $\psi(t) = E^X(t)E^X(s)h$ for $t > 0$. When $t < 0$ we have

$$\begin{aligned} (I - \mathcal{L})\psi(t) &= - \int_s^\infty \mathcal{A}(t+s-r)E^X(r)h dr = E(t) \int_{-\infty}^\infty \mathcal{A}(s-r)E^X(r)h dr = \\ &= E(t)(E(s) + \mathcal{L}E^X(s))h = E(t)E^X(s)h. \end{aligned}$$

Combining the two cases and recalling the definition of $\psi(t)$, we get $E^X(t)E^X(s) = E^X(t+s)$ for $t, s > 0$ and $E^X(t)E^X(s) = 0$ for $t < 0, s > 0$.

It is easy to see that $E^X(t)$ is strongly continuous and bounded. To check that $\Pi_+^X + \Pi_-^X = I$, note that convolutions are smoothing, hence the jump of $E^X(t)$ at $t=0$ is equal to the jump of $E(t)$ at $t=0$, i.e.,

$$(\Pi_+^X + \Pi_-^X)h = E^X(+0)h - E^X(-0)h = E(+0)h - E(-0)h = (\Pi_+ + \Pi_-)h = h.$$

Therefore $E^X(t)$ is a bounded, strongly continuous bisemigroup. The above argument is adapted from [GMP].

By assumption $\sigma(S^X) \subset \bar{\Sigma}_{\pi/2 - \theta_2}$ for some $0 < \theta_2 < \pi/2$. Taking a double sided Laplace transform, we get immediately

$$(S^X - \lambda)^{-1}h = \int_{-\infty}^\infty e^{\lambda t} E^X(t)h dt, \quad \operatorname{Re} \lambda = 0, \quad \lambda \neq 0, \quad h \in H.$$

Hence S^X is the generator of $E^X(t)$ and Π_{\pm}^X are positive/negative spectral projectors for S^X .

If $\varphi \in [0, \theta)$, where $\theta = \min\{\theta_1, \theta_2\}$ then all the above applies as well for $e^{i\varphi S}$ and $e^{i\varphi S^X}$, i.e., $e^{i\varphi S^X}$ is the generator of a bounded bisemigroup, so by [K; Theorem IX.1.23], S^X generates a bounded holomorphic bisemigroup of angle at least θ .

The compactness of

$$E^X(t) - E(t) = \int_{-\infty}^{\infty} \mathcal{A}(t-s) E^X(s) ds,$$

and consequently of $\Pi_{\pm}^X - \Pi_{\pm}$, follows from the fact that the (Bochner) integral of an integrable compact operator valued function is compact.

Because $D((S^X)^{-1}) = D(S^{-1})$ is dense, zero is either in the resolvent set or in the continuous spectrum of S^X , and hence E^X is strongly decaying. If the operator S has a gap at zero, then it is immediate that $E(t)$, and hence $\mathcal{A}(t)$, is exponentially decaying. Then by the same argument as in [GRS] it can be shown that $\mathcal{A}^X(t)$ will be exponentially decaying, implying the exponential decay of $E^X(t)$. This completes the proof of the theorem. ■

Next we consider the case when instead of the integrability condition (ii) we assume:

(ii') B is a trace class operator.

The conditions on S are the same as before. Set $\mathcal{H}(x) = SE(x, S)$.

Lemma 3.2. The function \mathcal{H} is weak-* integrable, i.e., $\mathcal{H} \in L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$. In fact $\|\mathcal{H}\|_{\epsilon} = 1$.

Proof. We will assume that S is self adjoint with spectrum situated on the positive axis and of multiplicity one. The general case will only make the notation more cumbersome, but is essentially the same calculation. Thus we may take H to be the space of square integrable functions on $\sigma(S) \subset [0, \infty)$, and S the operator of multiplication by the argument. Let C be a trace class operator on H , i.e., $C = \sum \gamma_i e_i(\cdot, e'_i)$. Then we have the estimate

$$\begin{aligned}
 \int_0^\infty dt |\operatorname{tr}(C\chi(t))| &= \int_0^\infty dt \left| \sum_{i=1}^\infty \gamma_i \chi(t) e_i, e'_i \right| \leq \\
 &\leq \int_0^\infty dt \sum_{i=1}^\infty \gamma_i |\chi(t) e_i, e'_i| = \\
 &= \sum_{i=1}^\infty \gamma_i \int_0^\infty \left| \int_{\sigma(S)} \frac{1}{\mu} e^{-t/\mu} e_i(\mu) \bar{e}'_i(\mu) d\mu \right| dt \leq \\
 &\leq \sum_{i=1}^\infty \gamma_i \int_{\sigma(S)} |e_i(\mu)| |e'_i(\mu)| \left(\int_0^\infty e^{-t/\mu} \frac{dt}{\mu} \right) d\mu \leq \\
 &\leq \sum_{i=1}^\infty \gamma_i \|e_i\| \|e'_i\| = \sum_{i=1}^\infty \gamma_i = \|C\|_{(1)}. \quad \blacksquare
 \end{aligned}$$

Let ψ and ω be in $L^\infty(H)$, $C(\mathbb{R}, H)$ or $C(\mathbb{R}_+, H) \oplus C(\mathbb{R}_-, H)$, where in the last case we are allowing the functions to have a jump discontinuity at zero. Let B be a trace class operator on H with trace norm $\|B\|_{(1)}$. From the discussion in the previous sections, we have a well defined operator $\mathcal{L}: L^\infty(H) \rightarrow C(\mathbb{R}, H)$ given by convolution, i.e.,

$$(\mathcal{L}\psi)(t) = \int_{-\infty}^\infty \chi(t-s) B\psi(s) ds. \quad (3.2)$$

Moreover we have $\|\mathcal{L}\| = \|B\|_{(1)}$. Hence $(1 - \mathcal{L})$ is an operator mapping each of

$L^\infty(H)$, $C(\mathbb{R}, H)$ or $C(\mathbb{R}_+, H) \oplus C(\mathbb{R}_-, H)$ into itself. In other words the equation

$$\psi - \mathcal{L}\psi = \omega \quad (3.3)$$

is well defined. Next consider the symbol of this equation, i.e., the two sided Laplace transform ($\text{Re}\lambda=0$):

$$W(\lambda) = I - \int_{-\infty}^{\infty} e^{\lambda t} \mathcal{H}(t) B dt = I - S(S-\lambda)^{-1} B = (S-\lambda)^{-1} (S^X - \lambda), \quad (3.4)$$

where we again denote $S^X = S(1-B)$. By the Bochner-Phillips theorem equation (3.3) is uniquely solvable iff the symbol $W(\lambda)$ is invertible on the extended imaginary axis $i\mathbb{R}^*$. Moreover the solution is given by

$$\psi = \omega + \mathcal{L}^X \omega,$$

where \mathcal{L}^X is again a convolution operator,

$$(\mathcal{L}^X \omega)(t) = \int_{-\infty}^{\infty} \mathcal{H}^X(t-s) B \omega(s) ds$$

for some $\mathcal{H}^X \in L^1(\mathbb{R}) \otimes_{\epsilon} L(H)$. Moreover, because as noted in the previous section convolutions map bounded measurable functions into $C(\mathbb{R}, H)$, we obtain that, if $\omega \in C(\mathbb{R}_+, H) \oplus C(\mathbb{R}_-, H)$, then ψ has the same jump at zero as ω .

With the above remarks in mind, the proof of the following theorem will be the same as the proof of Theorem 3.1, with the exception of the compactness statement.

Theorem 3.2. If S is a spectral operator of scalar type, if its spectrum is contained in a sector of angle $\pi/2 - \theta$, $0 < \theta < \pi/2$, around the real axis, and if conditions (i), (ii'), (iii), (iv) are satisfied, then S^X generates a holomorphic bisemigroup $E^X(t)$ with separating projectors Π_{\pm}^X . For any $t \in \mathbb{R} \setminus \{0\}$ we have that $E(t) - E^X(t)$ and $\Pi_{\pm} - \Pi_{\pm}^X$ are compact. The bisemigroup $E^X(t)$ is strongly decaying. If $\sigma(S)$ has a gap at zero (i.e., S^{-1} is bounded), then $E^X(t)$ is exponentially decaying.

Proof. We will show only the compactness of $E(t) - E(t)^X$. As remarked at the end of the previous section, the Dunford integral of a compact operator valued function in general is not compact, but the Bochner integral is. Set $\Phi_n(t) = |t|^{1/n} \chi_{[-n, n]}(t)$, where $\chi_{[a, b]}$ is the characteristic function of the interval $[a, b]$. Denote $B_n = \Phi_n(S)B$. Obviously $\chi(\cdot)B_n$ is a Bochner integrable function. On the other hand, using the functional calculus for S and the dominated convergence theorem we get that $\|(\Phi_n(S) - I)h\| \rightarrow 0$ as $n \rightarrow \infty$ for every $h \in H$, hence $\|(\Phi_n(S) - I)C\| \rightarrow 0$ as $n \rightarrow \infty$ for every compact operator C . Now B can be written as $B = CD$, where C is compact and D is trace class. Indeed, let $B = \sum s_i f_i(\cdot, e_i)$ with $\{e_i\}$ and $\{f_i\}$ orthonormal families and $\{s_i\}$ the singular numbers. Denote by σ the sum of the s_i 's and by σ_n the partial sums. Choose a strictly increasing sequence of natural numbers $\{k(r)\}$, such that $\sigma - \sigma_{k(r)} \leq 3^{-r} \sigma$ for $r = 1, 2, \dots$. Set $c_i = 2^{-r}$ and $d_i = 2^r s_i$ for $k(r-1) < i \leq k(r)$. Then the desired operators are $C = \sum c_i f_i(\cdot, f_i)$ and $D = \sum d_i f_i(\cdot, e_i)$. Note that $\|D\|_{(1)} = \sum d_i \leq 3\sigma = 3\|B\|_{(1)}$. Then we will have that $\|(\Phi_n(S) - I)B\|_{(1)} \leq \|D\|_{(1)} \|(\Phi_n(S) - I)C\| \rightarrow 0$ as $n \rightarrow \infty$. From before we have

$$\left\| \int_{-\infty}^{\infty} \chi(t-s) B E^X(s) ds \right\| \leq \text{const.} \|B\|_{(1)}.$$

Set $E_n^X(t)$ to denote the bisemigroup generated by $S(I - B_n)$. We have that

$E(t) - E_n^X(t)$ is compact for any n by the argument in Theorem 3.1. By the above argument we have that $(E(t) - E_n^X(t)) \rightarrow (E(t) - E^X(t))$ in operator norm. Thus $E(t) - E^X(t)$ is compact. ■

III. STRONG SOLUTIONS

In this chapter we discuss questions of existence and uniqueness of solutions to the boundary value problem in the original Hilbert space for collision operators of the form "identity plus a compact". We consider only the case of B satisfying the "Hölder condition with respect to T ", i.e., the kernel of the convolution operator $\chi(\cdot)B$ is Bochner integrable. In fact, if B is trace class, without reference to the Hölder condition, the analysis of this chapter will go through.

1. The Strictly Accretive Case.

For clarity of the exposition we will consider first the case when the collision operator A is invertible. We will work on a Hilbert space H , assuming that $\psi(x) \in H$ for every x . Suppose that the operators T and A on H have the following properties:

- (i) T is injective and self adjoint. Denote by $D(T)$ its domain and by Q_{\pm} the maximal positive/negative spectral projectors for T .
- (ii) $B = I - A$ is compact.
- (iii) $\text{Ran} B \subset D(|T|^{\alpha}) \cap \text{Ran} |T|^{\gamma}$ for some $\alpha > 1$, $\gamma > 0$, where $|T| = T(Q_+ - Q_-)$.
- (iv) A is accretive, i.e., $\text{Re} A = (A + A^*)/2 \geq 0$.
- (v) $\text{Ker} A = \text{Ker}(\text{Re} A) = 0$.

We will use the notation $K = T^{-1}A$. Observe that K is densely defined

because $\text{Ran}A=H$ and that K is closed because $K^* \supset A^* T^{-1}$ is densely defined. From assumption (v) it follows that K has no eigenvalues on the imaginary axis. Therefore we can apply the Theorem for perturbation of bisemigroups with $S=T^{-1}$, $S^X=K$. Let us denote by P_{\pm} the spectral (separating) projectors for K .

Consider the boundary value problem

$$\frac{d}{dx} T\psi(x) = -A\psi(x), \quad x > 0, \quad (1.1a)$$

$$Q_+ \psi(0) = \varphi_+, \quad (1.1b)$$

$$\|\psi(x)\| = o(x^n) \quad (x \rightarrow \infty), \quad (1.1c)$$

for some $0 \leq n < \infty$ and $\varphi_+ \in Q_+ D(T)$. By a solution we will understand a continuous function $\psi: [0, \infty) \rightarrow D(T)$, such that $T\psi(x)$ is strongly differentiable for $x \in (0, \infty)$ and the equation and boundary conditions are satisfied. It is easy to show ([GMP], [GMW]) that every solution has the form $\psi_h(x) = \exp(-xK)P_+ h$, $x \geq 0$, for some $h \in D(T)$ such that $Q_+ P_+ h = \varphi_+$. Because K generates a strongly decaying bisemigroup, all solutions satisfy (1.1c) with $n=0$.

The unique solvability of the boundary value problem is equivalent to the bijectivity of the map Q_+ from $P_+ D(T)$ onto $Q_+ D(T)$. Introducing the operator $V = Q_+ P_+ + Q_- P_-$, we have unique solvability if V maps $D(T)$ one-to-one onto $D(T)$. Denoting by $E = V^{-1}$ the albedo operator, we will have $P_+ h = E\varphi_+$.

Lemma 1.1. The projectors P_{\pm} and the operator V leave $D(T)$ invariant. The operator $(I-V)$ is compact in H and also in $D(T)$ equipped with the T graph norm.

Proof. We can write $I-V = (Q_- - Q_+)(P_+ - Q_+)$. According to the Theorem for perturbation of bisemigroups we have $P_+ - Q_+ = \int_{-\infty}^{\infty} e^{-y} E^X(y) dy$ and it is a compact operator, hence $(I-V)$ is compact in H . Assumption (iii) with $\alpha > 1$ assures

that $T\mathcal{A}(\cdot) \in L_1(L(H))_{-\infty}^{\infty}$. Therefore $(P_+ - Q_+)h \in D(T)$, $\forall h \in H$. This implies that $P_{\pm}D(T) \subset D(T)$ and $VD(T) \subset D(T)$.

Let us define $\hat{P}_{\pm}^* = AP_{\pm}A^{-1}$. Obviously $\hat{P}_+^* - Q_+ = (P_+ - Q_+) + P_+BA^{-1} - BP_+ - BP_+BA^{-1}$ is a compact operator in H . For $h \in D(T)$ we have $\hat{P}_+^*Th = AP_+A^{-1}Th = A(A^{-1}T)P_+h = TP_+h$, hence $T(P_+ - Q_+)h = (\hat{P}_+^* - Q_+)Th$. This with the compactness of $(\hat{P}_+^* - Q_+)$ implies that $P_+ - Q_+$, and hence $(I - V)$, is compact in $D(T)$ in the T -graph norm. ■

Lemma 1.2. If $h \in D(T)$ then $\psi_h(\cdot) \in L_2(H)_0^{\infty}$, $\psi_h(x) \in D(T)$ for all $x \geq 0$, and $\|T\psi_h(x)\| \rightarrow 0$ as $x \rightarrow \infty$.

Proof. Because we have a holomorphic bisemigroup, the derivative $\dot{\psi}_h(x) = -T^{-1}A\psi_h(x)$ exists and obviously belongs to $D(T)$. Since $P_+h \in D(T)$, we have that $\psi_h(x) = P_+h + \int_0^x \dot{\psi}_h(y)dy$ is in $D(T)$.

Using the shorthand notation $\omega_h(\cdot)$ for the bisemigroup generated by T^{-1} applied to a vector h , one has that $h \in D(T)$ implies $\omega_h(\cdot) \in L_2(H)_{-\infty}^{\infty}$ (the space of Bochner square integrable H -valued functions). For, using the Spectral Theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} (\omega_h(x), \omega_h(x)) dx &= \\ &= \int_{-\infty}^0 (e^{-xT^{-1}}Q_-h, e^{-xT^{-1}}Q_-h) dx + \int_0^{\infty} (e^{-xT^{-1}}Q_+h, e^{-xT^{-1}}Q_+h) dx = \\ &= \int_{-\infty}^0 dx \int_{-\infty}^0 e^{-2\frac{x}{\mu}} dE_{h,h}(\mu) + \int_0^{\infty} dx \int_0^{\infty} e^{-2\frac{x}{\mu}} dE_{h,h}(\mu) = \\ &= \int_{-\infty}^{\infty} \frac{|\mu|}{2} dE_{h,h}(\mu) = \frac{1}{2}(|T|h, h) < \infty, \end{aligned}$$

where dE is the spectral resolution of the identity for T . But $\omega_h(\cdot) \in L_2(H)_{-\infty}^{\infty}$ implies $\psi_h(\cdot) \in L_2(H)_{-\infty}^{\infty}$. Writing $T\psi_h(x) = TP_+h - A \int_0^x \psi_h(y) dy$ and taking $x \rightarrow \infty$ we get

$$\lim_{x \rightarrow \infty} T\psi_h(x) = TP_+h - A \int_0^{\infty} \psi_h(y) dy = TP_+h - AA^{-1}TP_+h = 0,$$

which completes the proof. ■

In the next lemma the accretivity assumption on A is crucial.

Lemma 1.3. $\text{Ker} V = 0$.

Proof. First note that $\text{Ker} V \subset D(T)$. Indeed, let $Vh=0$. Then $h=(I-V)h \in D(T)$, where the inclusion is assured by the first lemma. Because we have $\text{Ker} V = (\text{Ran} P_+ \cap \text{Ran} Q_-) \oplus (\text{Ran} P_- \cap \text{Ran} Q_+)$, the proof will be completed if we can show that $\text{Ran} Q_{\mp} \cap \text{Ran} P_{\pm} \cap D(T) = 0$.

Suppose that $h \in \text{Ran} P_+ \cap \text{Ran} Q_- \cap D(T)$. Following [WZM], one has

$$-(2(\text{Re} A)\psi_h(x), \psi_h(x)) = -(A\psi_h(x), \psi_h(x)) - (\psi_h(x), A\psi_h(x)) = \quad (1.2)$$

$$= \left(\frac{d}{dx} T\psi_h(x), \psi_h(x) \right) + \left(\psi_h(x), \frac{d}{dx} T\psi_h(x) \right) = \frac{d}{dx} (T\psi_h(x), \psi_h(x)).$$

But $-(Th, h) \geq 0$, because $h \in \text{Ran} Q_-$. Using the accretivity of A and Lemma 1.2 we have the estimate

$$0 \geq \lim_{\tau \rightarrow \infty} \int_0^{\tau} -(2(\text{Re} A)\psi_h(x), \psi_h(x)) dx = \lim_{\tau \rightarrow \infty} \int_0^{\tau} \frac{d}{dx} (T\psi_h(x), \psi_h(x)) dx = \quad (1.3)$$

$$= \lim_{\tau \rightarrow \infty} (T\psi_h(\tau), \psi_h(\tau)) - (T\psi_h(+0), \psi_h(+0)) = -(Th, h) \geq 0.$$

Therefore $(Th, h) = 0$. But T is injective and negative definite on $\text{Ran} Q_-$, whence we get $h = 0$. This shows that $\text{Ran} P_+ \cap \text{Ran} Q_- \cap D(T) = 0$. Analogously $\text{Ran} P_- \cap \text{Ran} Q_+ \cap D(T) = 0$ holds, and hence we have proved that V is injective. ■

Theorem 1.1. For every $\varphi_+ \in Q_+ D(T)$ the boundary value problem (1.1) has a unique solution which is given by $\psi(x) = \exp(-xK)E\varphi_+$. Regardless of n in (1.1c) it is decaying at infinity and is also square integrable in x . If T is a bounded operator, then the solutions are exponentially decaying.

Proof. Because V maps $D(T)$ into itself and $I - V$ is compact in $D(T)$ in the T graph norm, the injectivity of V implies that V maps $D(T)$ onto $D(T)$. Therefore the operator $E = V^{-1}$ is a bounded operator in $D(T)$ with the T graph norm and obviously for $\varphi_+ \in Q_+ D(T)$ we have $E\varphi_+ \in P_+ D(T)$ and $Q_+ E\varphi_+ = \varphi_+$. ■

If T is an unbounded operator, the boundary value problem (1.1) can be interpreted in another way, namely with T and the derivative interchanged in the right hand side of (1.1a). In this case we may assume that $\varphi \in Q_+ H$ rather than just $\varphi \in Q_+ D(T)$. A solution is taken to be a continuous function $\psi: (0, \infty) \rightarrow H$ which is strongly differentiable on $(0, \infty)$. We again have unique solvability for every $\varphi_+ \in Q_+ H$ and the decay of all solutions at infinity; only the square integrability of the solution may be lost. The proof follows that of Theorem 1.1 except that one needs compactness of $I - V$ only in H , and thus Lemma 1.1 is unnecessary.

2. Conservative Case. Decompositions.

In what follows the notation $K=T^{-1}A$, $\hat{K}=T^{-1}A^*$ will be used. The zero root linear manifold of K is denoted $Z_0(K) = \{f \in D(K) : K^n f = 0 \text{ for some } n \in \mathbb{N}\}$, and analogously $Z_0(\hat{K})$, $Z_0(K^*)$, and $Z_0(\hat{K}^*)$ for \hat{K} , K^* and \hat{K}^* .

We want to treat collision operators A with nontrivial kernel, so we change assumption (v) of the previous section to assumptions (v') and (vi):

$$(v') \text{ Ker}A = \text{Ker}(\text{Re}A),$$

$$(vi) Z_0(K) \subset D(|T|^{2+\gamma}) \text{ and } Z_0(\hat{K}) \subset D(T) \text{ for some } \gamma > 0.$$

First we prove a few simple facts.

Proposition 2.1.

$$(a) \quad f \in H \text{ and } (Af, f) = 0 \text{ imply } f \in \text{Ker}A, \quad (2.1a)$$

$$(b) \quad \text{Ker}A = \text{Ker}A^*, \quad (2.1b)$$

$$(c) \quad \text{Ran}A = \text{Ran}A^*, \quad (2.1c)$$

$$(d) \quad H = \text{Ker}A \oplus \text{Ran}A. \quad (2.1d)$$

Proof. (a) If $(Af, f) = 0$ then $(A^* f, f) = 0$. Adding these together gives $(\text{Re}Af, f) = 0$. But $\text{Re}A \geq 0$, so this implies $f \in \text{Ker}(\text{Re}A) = \text{Ker}A$. (b) If $f \in \text{Ker}A = \text{Ker}(\text{Re}A)$, then $0 = 2\text{Re}Af = Af + A^* f = A^* f$, and hence $f \in \text{Ker}A^*$. For the converse use (a). Finally (c) follows from (b) using the identity $(\text{Ran}A)^\perp = \text{Ker}A^*$, which is true for any bounded operator A , and (b) and (c) imply (d). ■

Proposition 2.2. The operators K and \hat{K} are densely defined and closed. Their adjoints satisfy $K^* \supset A^* T^{-1}$ and $\hat{K}^* \supset AT^{-1}$, and the intertwining properties: $TK \subset \hat{K}^* T$, $T\hat{K} \subset K^* T$.

Proof. We will show that $D(K)$ is dense. Let \tilde{A} be the inverse of $A \upharpoonright \text{Ran} A$. Hence \tilde{A} is a bounded operator from $\text{Ran} A$ onto $\text{Ran} A$. We have $D(K) = \tilde{A}(\text{Ran} T \cap \text{Ran} A) \oplus \text{Ker} A$, so it will be sufficient to show that $\text{Ran} A \cap \text{Ran} T$ is dense in $\text{Ran} A$ (We have assumed that $\text{Ker} A \subset D(T)$). Now $\text{codim Ran} A < \infty$ and $\text{Ran} T$ is dense in H , hence one can always choose a subspace $N \subset \text{Ran} T$ such that $H = N \oplus \text{Ran} A$. If $\text{Ran} A \cap \text{Ran} T$ is not dense in $\text{Ran} A$, then there is a nonzero bounded functional g on $\text{Ran} A$ such that $g(\text{Ran} A \cap \text{Ran} T) = 0$. Extend g to all of H by setting $g(N) = 0$. Then g will be a nonzero bounded functional on H with $g(\text{Ran} T) = 0$, which contradicts the fact that $\text{Ran} T$ is dense in H .

By changing A to A^* , one obtains the density of $D(\hat{K})$. The remainder of the proposition follows easily. ■

Proposition 2.3. The only possible eigenvalue of $T^{-1}A$ on the imaginary axis can be at the origin.

Proof. Let $Af = \eta Tf$ where $\eta = -\bar{\eta}$. Then $(Af, f) = \eta(Tf, f)$ and $(A^*f, f) = (f, Af) = \bar{\eta}(f, Tf) = -\eta(Tf, f)$. Adding them gives $((\text{Re} A)f, f) = 0$ and thus $f \in \text{Ker} A$. ■

Proposition 2.4. The Jordan chains of K and \hat{K} at $\lambda = 0$ have length at most two.

Proof. Suppose there are $g_0, g_1, g_2 \in H$ such that $T^{-1}Ag_0 = g_1$, $T^{-1}Ag_1 = g_2$, and $Ag_2 = 0$. Applying (2.1b) gives that $A^*g_2 = 0$. Consider $(Ag_1, g_1) = (Tg_2, g_1) = (g_2, Tg_1) = (g_2, Ag_0) = (A^*g_2, g_0) = 0$. Then $g_1 \in \text{Ker} A$ follows from (2.1a). ■

Corollary 2.1. One has the inclusion $KZ_0(K) \subset \text{Ker} K = \text{Ker} A$ and the estimate $\dim Z_0(K) = \dim(\text{Ker} A) + \dim(KZ_0(K)) \leq 2 \dim \text{Ker} A$. The analogous inclusion and estimate hold for \hat{K} .

The next theorem is an important step in the analysis of the operator K and the unique solvability of the boundary value problem.

Theorem 2.1. There is a K -invariant decomposition $H = Z_0(K) \oplus Z_1(K)$, where $Z_1(K) = (Z_0(K^*))^\perp = (TZ_0(\hat{K}))^\perp$.

It is obvious that $Z_1(K)$ defined above is K -invariant. The theorem will follow from the next two lemmas.

Lemma 2.1. If $M \subset D(T)$ is a subspace of H , then $T(TM)^\perp = M^\perp \cap \text{Ran} T$.

Proof. Let $Tx \in T(TM)^\perp$. Since T is invertible, this implies that $x \in (TM)^\perp$, and since T is self adjoint this implies $Tx \in M^\perp$. Hence $T(TM)^\perp \subset M^\perp \cap \text{Ran} T$. Conversely, let $x = Ty$ and $x \in M^\perp$, so $y \in (TM)^\perp$, and therefore $x \in T(TM)^\perp$ and $T(TM)^\perp \supset M^\perp \cap \text{Ran} T$. ■

Lemma 2.2. We have the following identities:

$$(a) \quad TZ_0(K) = Z_0(\hat{K}^*)$$

$$TZ_0(\hat{K}) = Z_0(K^*)$$

$$(b) \quad Z_0(K) \cap Z_1(K) = 0$$

$$(c) \quad KZ_0(K) = \hat{K}Z_0(\hat{K})$$

$$\hat{K}^*Z_0(\hat{K}^*) = K^*Z_0(K^*)$$

$$TKZ_0(K) = \hat{K}^*Z_0(\hat{K}^*)$$

$$T\hat{K}Z_0(\hat{K}) = K^*Z_0(K^*)$$

$$\dim Z_0(K) = \dim Z_0(K^*) = \dim Z_0(\hat{K}^*) = \dim Z_0(\hat{K})$$

Proof. (a) Let us show that $\text{Ker}(\hat{K}^*)^n = \text{Ker}(AT^{-1})^n$ for each $n \geq 1$. For $n=1$ one has $\text{Ker}\hat{K}^* = (\text{Ran}\hat{K})^\perp = (T^{-1}(\text{Ran}A^* \cap \text{Ran}T))^\perp = T(\text{Ran}A^* \cap \text{Ran}T)^\perp = T(\text{Ker}A + (\text{Ran}T)^\perp) = T\text{Ker}A = \text{Ker}(AT^{-1})$, where the second equality follows from Lemma 2.1. For the case $n=2$ assume that $f \in \text{Ker}(\hat{K}^*)^2$, hence $\hat{K}^*f \in \text{Ker}\hat{K}^* = \text{Ker}(AT^{-1}) = T\text{Ker}A$ and $\hat{K}^*f = Tg$ for some $g \in \text{Ker}A$. Moreover, $Tg \in \text{Ran}\hat{K}^* = (\text{Ker}A^*)^\perp = \text{Ran}A$, hence $Tg = Ah$ for some h . Set $k = Th$ and observe that $\hat{K}^*k = AT^{-1}k = Ah = \hat{K}^*f$ since \hat{K}^* coincides with AT^{-1} on $\text{Ran}T$. Thus $\hat{K}^*(k-f) = 0$ or $f = k + k_0$ for some $k_0 \in \text{Ker}\hat{K}^* = \text{Ker}(AT^{-1}) = T\text{Ker}A \subset \text{Ran}T$. Therefore one gets that $f \in \text{Ran}T$ and $f \in \text{Ker}(AT^{-1})^2$. Repeating this argument will prove the assertion for general n , and thus (a).

(b) Because both $Z_0(K)$ and $Z_1(K)$ are K invariant, we have $Kh \in \text{Ker}A \cap Z_1(K)$ if $h \in Z_0(K) \cap Z_1(K)$ and $Kh \neq 0$. So it is sufficient to prove that $\text{Ker}A \cap Z_1(K) = 0$. Suppose that $h \in \text{Ker}A \cap Z_1(K)$. From the self-adjointness of T , Proposition 2.1, and (a) above, it follows that $Th \in Z_0(\hat{K})^\perp \subset (\text{Ker}A^*)^\perp = \text{Ran}A = \text{Ran}A^*$, hence $Th = A^*g$ for some $g \in Z_0(\hat{K})$. We have used the fact that $h \in \text{Ker}A = \text{Ker}A^* \subset Z_0(\hat{K})$. Thus $(A^*g, g) = (Th, g) = 0$, and Proposition 2.1(a) implies $0 = A^*g = Th$. But T is injective, thus finally $h = 0$.

(c) For the first equality in (c) suppose $k \in KZ_0(K)$, i.e., $Ak = 0$ and $Tk = Ah$ for some h . Then $Tk \in \text{Ran}A = \text{Ran}A^*$ and $Tk = A^*g$ for some g . Because $k \in \text{Ker}A = \text{Ker}A^*$ it follows that $g \in Z_0(\hat{K})$ and thus $k \in \hat{K}Z_0(\hat{K})$, or $KZ_0(K) \subset \hat{K}Z_0(\hat{K})$. The converse inclusion is proved analogously. To show the third and fourth identities use (a) and Proposition 2.2. These then give the second identity immediately. Finally, the dimensional computations follow from these identities and Corollary 2.1. ■

With Lemma 2.3(b) and the dimension argument of Lemma 2.3(c), the proof of Theorem 2.1 is immediate. Let us denote by P_0 the projection H onto $Z_0(K)$ along

$Z_1(K)$, and by P_1 its complementary projection. We would like to show that KP_1 is a generator of a bisemigroup on $Z_1(K)$ with separating projectors P_{\pm} . We shall obtain this as Proposition 2.5, but instead of working with the restriction of K to $Z_1(K)$ we will work with $T^{-1}A'$ on H , where A is modified to A' , so that the latter is invertible and we can apply the Theorem for perturbation of bisemigroups. Define $A' = TP_0 + AP_1$. Using the notation $K' = T^{-1}A' = P_0 + KP_1$, the following is obvious:

$$K'P_0 = P_0K' = P_0, \quad (2.2)$$

$$K'P_1 = P_1K' = P_1K = KP_1. \quad (2.3)$$

Under the above assumptions on A and T we observe that A' has a bounded inverse, K' has no eigenvalues on the imaginary axis, and $I-A'$ satisfies the same Hölder condition as $I-A$. Indeed, suppose $A'f = 0$. Since P_0 commutes with $T^{-1}A$, we have $TP_0f = -AP_1f$ and $P_0f = -T^{-1}AP_1f$. Applying P_0 to both sides gives $P_0f = 0$ and this implies $P_1f \in \text{Ker}A$, which is impossible unless $f = 0$. Because A' is a finite rank perturbation of A , and because A has Fredholm index 0, A' is also Fredholm of index zero. Having no kernel, A' has a bounded inverse.

Suppose next that $K'f = \eta f$ with $\eta = -\bar{\eta}$. Applying P_1 to both sides and using (2.3) gives $K(P_1f) = \eta(P_1f)$, hence by Proposition 2.3 one gets $P_1f = 0$. Applying P_0 to both sides gives $(P_0f) = \eta(P_0f)$, also impossible unless $P_0f = 0$.

The Hölder condition on $(I-A)$ implies that $(I-A) = |T|^{\alpha}D_2 = |T|^{-\gamma}D_1$ for some $\alpha > 0$, $\gamma > 1$ and bounded operators D_1 and D_2 . Now note that $Z_0(K) \subset \text{Ran}|T|^{\alpha}$. Indeed for $f \in Z_0(K)$ one has $AT^{-1}Af = 0$ by Proposition 2.4, hence $f = (1-A)f + T(1-A)T^{-1}Af = |T|^{\alpha}(D_2 + TD_1^{-1})f$. Obviously $AZ_0(K) \subset TZ_0(K) \subset \text{Ran}|T|^{\alpha}$ holds, therefore $(1-A')$ $= (1-A) + (A-T)P_0$ implies that $\text{Ran}(1-A') \subset \text{Ran}|T|^{\alpha}$. Using assumption (vi) from the beginning of

this section and a similar argument we get the other inclusion.

One can apply the Theorem for perturbation of bisemigroups to the operator K' . Let P'_\pm be the resulting positive/negative spectral projections for K' .

Proposition 2.5. The operators defined by $P_\pm = P'_\pm P_1$ are projectors, commuting with K and such that $P_+ + P_- = P_1$ and $\sigma(KP_\pm) \subset \{\lambda \in \mathbb{C} : \pm \operatorname{Re} \lambda \geq 0\}$. The families $\exp(\mp xK)P_\pm$, $x > 0$, are holomorphic, bounded semigroups. Moreover $\|\exp(\mp xK)P_\pm h\| \rightarrow 0$, $x \rightarrow \infty$, $\forall h \in H$.

Proof. Everything will go through once $(P'_\pm P_1)^2 = P'_\pm P_1$ is shown, and this obviously will follow from $P'_- P_0 = 0 = P_0 P'_-$. To check $P'_- P_0 = 0$ let $f \in Z_0(K)$. Then

$$0 \geq (K' P'_- f, P'_- f) = (P'_- K' f, P'_- f) = (P'_- f, P'_- f) \geq 0,$$

hence $P'_- f = 0$, $\forall f \in Z_0(K)$. To check $P_0 P'_- = 0$ let $f \in \operatorname{Ran} P'_-$. Because 1 is in the resolvent set of $K' P'_-$ one can write $f = (1 - K')g$ for some g . Then $P_0 f = P_0(1 - K')g = (P_0 - P_0 K')g = 0$. ■

3. Conservative Case. Boundary Value Problem.

The objective is to consider questions of existence and uniqueness of solutions to the boundary value problem (1.1), where $\varphi_+ \in Q_+ D(T)$ is the given incoming flux. Because the Jordan chains in $Z_0(K)$ have length at most two, we have three possibilities at infinity, namely

$$\lim_{x \rightarrow \infty} \|\psi(x)\| = 0, \quad (3.1a)$$

$$\|\psi(x)\| = O(1), \quad x \rightarrow \infty, \quad (3.1b)$$

$$\|\psi(x)\| = O(x), \quad x \rightarrow \infty. \quad (3.1c)$$

A solution will mean a continuous function $\psi: [0, \infty) \rightarrow D(T)$ such that $(T\psi)(x)$ is strongly differentiable on $(0, \infty)$ and the above are satisfied. For short we will denote by (A), (B), and (C) the boundary value problems (1.1a)-(1.1b)-(3.1a), (1.1a)-(1.1b)-(3.1b), and (1.1a)-(1.1b)-(3.1c), respectively.

It is clear (for a proof see [GMP], also [GMW]) that the following must be true.

Proposition 3.1. All solutions of the boundary value problem (A) are of the form $\psi(x) = \exp(-xK)h$ for some $h \in P_+D(T)$ such that $Q_+h = \varphi_+$. All solutions of (B) are of the form $\psi(x) = \exp(-xK)h_1 + h_0$ for some $h_1 \in P_+D(T)$ and $h_0 \in \text{Ker}A$ such that $Q_+(h_1+h_0) = \varphi_+$. All solutions of (C) are of the form $\psi(x) = \exp(-xK)h_1 + (1-xK)h_0$ for some $h_1 \in P_+D(T)$ and $h_0 \in Z_0(K)$ such that $Q_+(h_1+h_0) = \varphi_+$.

From the above proposition it is clear that for each of the problems (A), (B) and (C), injectivity of the map Q_+ from $P_+D(T)$, $P_+D(T) \oplus \text{Ker}A$ and $P_+D(T) \oplus Z_0(K)$, respectively, into $Q_+D(T)$ assures the uniqueness of solutions and surjectivity of the respective maps assures the existence of solutions. Below we investigate the Krein space structure of $Z_0(\hat{K})$ with respect to the indefinite scalar product $(T \cdot, \cdot)$ and show the existence of direct sum decompositions of H (Lemma 3.4) which we will use in studying the question of unique solvability.

Let us introduce the following abbreviated notation: $Z_0 = Z_0(K)$, $\hat{Z}_0 = Z_0(\hat{K})$,

$Z_1 = Z_1(K)$, $Z_{\pm} = P_{\pm}H$, $H_{\pm} = Q_{\pm}H$, $L_{\pm} = Z_{\pm} \oplus H_{\mp}$, $M_{\pm} = L_{\mp} \cap \hat{Z}_0$. Note that $Z_{\pm} \cap H_{\mp} = 0$ as in the proof of Lemma 1.3, so the definition of L_{\pm} makes sense.

Lemma 3.1. We have $\pm(Tf, f) > 0$ for $f \in M_{\pm}$, $f \neq 0$, and thus $M_+ \cap M_- = 0$.

Proof. Let $f \in \hat{Z}_0$ and $f = g + h$ with $g \in Z_+ \subset Z_1$ and $h \in H_-$. We then see that $(Th, h) = (Tf, f) + (Tg, g) - (Tf, g) - (Tg, f) = (Tf, f) + (Tg, g)$, because $f \in \hat{Z}_0$ is T -orthogonal to $g \in Z_1$ (see Lemma 2.2.a). A similar estimate as in (1.3) gives $-(Tg, g) > 0$ if $g \neq 0$ and obviously $(Th, h) > 0$ if $h \neq 0$. Because both h and g cannot be zero we get $(Tf, f) > 0$ for $f \in M_+$. An analogous argument works for M_- . ■

Lemma 3.2. $M_+ \oplus M_- = \hat{Z}_0$.

Proof. Let us define $V = Q_+P_+ + Q_-P_-$ and $V' = Q_+P_+' + Q_-P_-'$. As in the proof of Lemma 1.1 we get that $I - V'$ is a compact operator. Because $V = V'P_1$ we obtain the compactness of $P_1 - V$. Hence V is a Fredholm operator of index zero. One easily checks that $\text{Ker}V = Z_0$ and $\text{Ran}V = L_+ \cap L_-$. From $M_+ \cap M_- = 0$ follows $\text{Ran}V \cap \hat{Z}_0 = 0$. Because $\text{codimRan}V = \dim\text{Ker}V = \dim Z_0 = \dim \hat{Z}_0$ (the last equality is from Lemma 2.2.c) we get

$$(L_+ \cap L_-) \oplus \hat{Z}_0 = H \tag{3.2}$$

Combining the obvious equalities $L_+ + L_- = H$ and $L_{\pm} + \hat{Z}_0 = H$ with (5.2) gives $\dim M_{\pm} = \dim(L_{\mp} \cap \hat{Z}_0) = \dim(L_{\mp}/L_+ \cap L_-)$. We also have $(L_+/L_+ \cap L_-) \oplus (L_-/L_+ \cap L_-) \cong \hat{Z}_0$, and hence $\dim(M_+) + \dim(M_-) = \dim(\hat{Z}_0)$. This together with $M_+ \cap M_- = 0$ completes the proof of the lemma. ■

From Lemmas 3.1 and 3.2 follows that \hat{Z}_0 is a Krein space with respect to the $(T\cdot, \cdot)$ indefinite scalar product (a standard reference for Krein space theory is [B]). If M is a subspace of \hat{Z}_0 then it is called positive or positive definite if $(Tf, f) \geq 0$ or $(Tf, f) > 0$, respectively, for $0 \neq f \in M$. M is maximal positive if it is positive but is not properly contained in another positive space. Similarly negative, negative definite and maximal negative spaces are defined. The isotropic part of M is defined as $M^0 = M \cap (TM)^\perp$. Every vector in M^0 is T -orthogonal to every vector in M . If we factor M by M^0 we get a Krein space.

Obviously M_\pm are maximal positive/negative definite subspaces of \hat{Z}_0 . The dimension of a maximal positive/negative subspace of a finite dimensional Krein space is independent of the choice of subspace. Denote these invariants by $m_\pm = \dim M_\pm$.

Lemma 3.3. There are decompositions

$$\hat{Z}_0 = N_{R+} \oplus N_{R-} \quad (3.3a)$$

$$\hat{Z}_0 = N_{L+} \oplus N_{L-} \quad (3.3b)$$

such that N_{R+} and N_{L-} are subspaces of $\text{Ker} A$ with N_{R+} maximal positive and N_{L-} maximal negative in \hat{Z}_0 .

Proof. First let us show that $(\text{Ker} \hat{K})^0 = \hat{K} \hat{Z}_0$. Recall that $\text{Ker} \hat{K} = \text{Ker} A^* = \text{Ker} A$. Suppose $f \in (\text{Ker} \hat{K})^0 = (\text{Ker} A) \cap (T \text{Ker} A)^\perp$. Then $Tf \in T(\text{Ker} A)^\perp = (\text{Ker} A)^\perp \cap \text{Ran} T \subset \text{Ran} A^*$ and $Tf = A^* g$ for some g , thus $f \in \hat{K} \hat{Z}_0$. Conversely, let $f \in \hat{K} \hat{Z}_0$. Then $Tf = A^* g$ for some g and $(Tf, u) = (A^* g, u) = (g, Au) = 0$ for every $u \in \text{Ker} \hat{K}$. Therefore $(\text{Ker} \hat{K}) / (\text{Ker} \hat{K})^0$ is a Krein space and one can find a T -orthogonal, linearly independent set of vectors $\{z_1, \dots, z_n\}$ such that $(Tz_i, z_i) \neq 0$, $i=1, \dots, n$, and $(\text{Ker} \hat{K})^0 \oplus \text{span}\{z_1, \dots, z_n\} = \text{Ker} \hat{K}$. The same argument as in Proposition 3.3 shows

that $Z_0(\hat{K})$ is a Krein space, so one can choose a T-orthogonal, linearly independent set $\{y_1, \dots, y_k\}$ which is T-orthogonal to $\{z_1, \dots, z_n\}$ and such that $(\text{Ker } \hat{K}) \oplus \text{span}\{y_1, \dots, y_n\} = \hat{Z}_0$. Moreover one can assume that $y_i, i=1, \dots, n$, are negative (or that y_i are positive) because if that is not the case then one can adjust the y_i 's by the following trick [GMZ] without spoiling any of the other properties. If ζ is a real number and $x_i = \hat{K}y_i$ then

$$\begin{aligned} (T(y_i - \zeta x_i), (y_i - \zeta x_i)) &= (Ty_i, y_i) + \zeta^2 (Tx_i, x_i) - \zeta ((Tx_i, y_i) - (y_i, Tx_i)) = \\ &= (Ty_i, y_i) - 2\zeta (\text{Re}Ay_i, y_i) \end{aligned}$$

can be made negative (positive) by choosing appropriate ζ , because $(\text{Re}Ay_i, y_i) > 0$. Without loss of generality we may assume that z_1, \dots, z_m span a negative definite space in \hat{Z}_0 , and z_{m+1}, \dots, z_n span a positive definite space. Then set $N_{R+} = (\text{Ker } \hat{K})^0 \oplus \text{span}\{z_{m+1}, \dots, z_n\}$ and $N_{R-} = \text{span}\{y_1, \dots, y_k\} \oplus \text{span}\{z_1, \dots, z_m\}$. Obviously $N_{R+} \subset \text{Ker } A$, $N_{R+} \oplus N_{R-} = \hat{Z}_0$ and N_{R+} is a positive space while N_{R-} is a negative definite space.

We want to show that N_{R+} is maximal positive. Suppose not, so there is $x = x_+ + x_-$ with $(Tx, x) \geq 0$, $x_{\pm} \in N_{R\pm}$ and $x_- \neq 0$. We can modify x so that it is T-orthogonal to N_{R+} . Indeed, if x is neutral, i.e., $(Tx, x) = 0$ then it is a neutral vector in the positive space $N_{R+} \oplus \text{span}\{x\}$ and hence it will be T-orthogonal to N_{R+} . On the other hand, if $(Tx, x) > 0$, then by the same argument x will be T-orthogonal to the isotropic part N_{R+}^0 . Thus, applying orthogonalization in the definite scalar product space $(N_{R+} \oplus \text{span}\{x\}) / N_{R+}^0$ provides the desired modification.

Assuming that x is T-orthogonal to N_{R+} , one has easily $0 \geq (Tx_-, x_-) = (Tx, x) = (Tx_+, x_+) \geq 0$, and hence $(Tx_-, x_-) = 0$. But N_{R-} is negative

definite, so $x_- = 0$. This shows the maximality of N_{R+} . The proof for $N_{L\pm}$ is similar. ■

Lemma 3.4. One has the decompositions

$$H = Z_+ \oplus N_{R+} \oplus H_- \quad (3.4a)$$

$$H = Z_- \oplus N_{L-} \oplus H_+ \quad (3.4b)$$

Proof. First note that $H = L_{\pm} \oplus M_{\pm}$ holds. Indeed, as in the proof of Lemma 3.2 we have $L_{\pm} \cap M_{\pm} = 0$ and also $\text{codim}(L_{\pm}) = \dim(L_{\mp}/L_{\mp} \cap L_{\pm}) = \dim(M_{\pm})$. Next we have that

$$M_+ \oplus N_{L-} = \hat{Z}_0 \quad (3.5a)$$

$$M_- \oplus N_{R+} = \hat{Z}_0 \quad (3.5b)$$

Indeed, $M_+ \cap N_{L-} = 0$ because M_+ is positive definite and N_{L-} is negative, and also the dimensions add up correctly because both are maximal. Similarly for (3.5b). Combining these results completes the proof. ■

For the boundary value problems (A-C), as in Section 1 one has to reside in $D(T)$ rather than in H itself. Assumption (vi) in the beginning of Section 2 implies that P_0 and P_1 leave $D(T)$ invariant. Also P_{\pm} leave $D(T)$ invariant, as was shown in Section 1. Therefore all the above lemmas remain true if one intersects with $D(T)$.

Define the measure of nonexistence to be the codimension in $Q_+D(T)$ of the space of boundary values $\varphi_+ \in Q_+D(T)$ for which the problem is solvable, and the measure of nonuniqueness to be the dimension of the solution space of the

corresponding homogeneous problem. Let R , a map from $Q_+D(T)$ into $P_+D(T) \oplus N_{R+}$, be the inverse of Q_+ restricted to $P_+D(T) \oplus N_{R+}$. Its existence follows from the decomposition (3.4a). Our main results are contained in the following theorems. Of course we assume that (i-iv), (v'), and (vi) hold for the operators A and T .

Theorem 3.1. Problem (A) has at most one solution for $\varphi_+ \in Q_+D(T)$. The measure of nonexistence is given by m_+ . There is a solution if and only if $\varphi_+ \in Q_+P_+D(T)$ and it is given by $\psi(x) = \exp(-xT^{-1}A)R\varphi_+$.

Proof. Because Q_+ maps $P_+D(T) \oplus N_{R+}$ onto $Q_+D(T)$ bijectively, we have that the codimension of $Q_+P_+D(T)$ in $Q_+D(T)$ is given by $\dim(N_{R+}) = m_+$. ■

Theorem 3.2. Problem (B) has at least one solution for every $\varphi_+ \in Q_+D(T)$. A particular solution is given by $\psi_p(x) = \exp(-xT^{-1}A)P_1R\varphi_+ + P_0R\varphi_+$. The solutions of the homogeneous problem are given by $\psi_0(x) = \exp(-xT^{-1}A)h_1 + h_0$ where $h_1 \in P_+D(T)$, $h_0 \in \text{Ker}A$ and $h_1 + h_0 \in \text{Ran}Q_-$. The measure of nonuniqueness is $\dim(\text{Ker}A) - m_+$.

Proof. From Proposition 3.1 and the definition of R it is immediate that ψ_p given above is a particular solution of the boundary value problem (B). From the same proposition it is also clear that the solutions of the homogeneous problem must have the form ψ_h with $Q_+(h_1 + h_0) = 0$. Finally from (3.6.a) and $\dim(N_{R+}) = m_+$ we obtain $\dim((P_+D(T) \oplus \text{Ker}A) \cap \text{Ran}Q_-) = \dim(\text{Ker}A) - m_+$. ■

Corollary 3.1. If all Jordan chains in $Z_0(K)$ have length precisely two, i.e., $\text{Ker}A = KZ_0(K)$, then problem (B) is uniquely solvable.

Proof. From the proof of Lemma 3.3 it is clear we will have that $KZ_0(K)$ is a maximal positive (also a maximal negative) space, so the measure of nonuniqueness is zero. ■

In many physical applications there is a symmetry with respect to reflections $x \rightarrow -x$. Under such reflections the scattering remains unaltered but the velocities reverse direction. We may express this by saying that there is an unitary involution J , i.e., $J=J^*=J^{-1}$, such that $JT=-TJ$ and $JA=AJ$. It is straightforward to check that if the equation possesses such an inversion symmetry then $m_+=m_-$ (see [M1]). In this case we get a necessary and sufficient condition for solvability.

Corollary 3.2. If T and A are inversion symmetric then problem (B) is uniquely solvable if and only if all Jordan chains in $Z_0(K)$ have length precisely two.

Theorem 3.3. Problem (C) has at most one solution for every $\varphi_+ \in Q_+D(T)$. A particular solution is given by $\psi_p(x) = \exp(-xT^{-1}A)P_1R\varphi_+ + P_0R\varphi_+ - xT^{-1}AP_0R\varphi_+$. The solutions of the homogeneous problem have the form $\psi_0(x) = \exp(-xT^{-1}A)h_1 + h_0 - xT^{-1}Ah_0$ where $h_1 \in P_+D(T)$, $h_0 \in Z_0(K)$ and $h_1 + h_0 \in \text{Ran}Q_-$. The measure of nonuniqueness is given by $\dim(Z_0(K)) - m_+$.

The proof of this last theorem is analogous to the proof of Theorem 3.2, in this case, however, noting that $\dim((P_+D(T) \oplus Z_0(K)) \cap \text{Ran}Q_-) = \dim(Z_0(K)) - m_+$.

We can also consider the differential equation with the derivative and the operator T interchanged, as in the remark at the end of Section 1. Then all the above theorems hold for initial data from Q_+H instead of $Q_+D(T)$ and a solution is a strongly differentiable function from the half line to all of H .

4. Transfer of Polarized Light

Consider the equation of transfer of polarized light in a plane parallel atmosphere,

$$u \frac{d}{dx} \psi(x, u, \varphi) = -\psi(x, u, \varphi) + \frac{a}{4\pi} \int_{-1}^1 d\hat{u} \int_0^{2\pi} d\hat{\varphi} Z(u, \hat{u}, \varphi - \hat{\varphi}) \psi(x, \hat{u}, \hat{\varphi}), \quad (4.1)$$

for $0 < x < \infty$, where $\psi(x, u, \varphi)$ is a four vector depending on the optical depth x , direction cosine of propagation u and azimuthal angle φ , and $0 \leq a \leq 1$ is the albedo of single scattering (see [M2]). The phase matrix Z can be represented as a product $Z(u_1, u_2, \varphi - \hat{\varphi}) = L(\pi - \sigma_2) F(\theta) L(-\sigma_1)$ of two rotation matrices

$$L(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the scattering matrix

$$F(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix},$$

where

$$\cos \theta = u_1 u_2 + (1 - u_1^2)^{1/2} (1 - u_2^2)^{1/2} \cos(\hat{\varphi} - \varphi)$$

$$\cos \sigma_1 = (u_1 - u_2 \cos \theta) / \sin \theta (1 - u_2^2)^{1/2}$$

$$\cos \sigma_2 = (u_2 - u_1 \cos \theta) / \sin \theta (1 - u_1^2)^{1/2}$$

and $\sin\sigma_1, \sin\sigma_2$ have the same sign as $\sin(\hat{\varphi}-\varphi)$. The function $a_1(\theta)$ is nonnegative and satisfies the normalization condition $\int_{-1}^1 a_1(\theta) d\cos\theta = 2$, and the matrix $F(\theta)$ leaves invariant the set of vectors $\psi = (I, Q, U, V)$ for which $I \geq (Q^2 + U^2 + V^2)^{1/2}$.

Let H be the Hilbert space of square integrable functions of u and φ , and T the operator of multiplication by u , and let $-A$ be the operator defined by the right hand side of (4.1). It is not hard to see that $1-A$ is compact, $\text{Re}A \geq 0$, and that $(I-A)$ satisfies the Hölder condition if $\int_{-1}^1 a_1(\theta)^r d\cos\theta < \infty$ for some $r > 1$ [M2]. An easy calculation gives $\text{Ker}A = \text{Ker} \text{Re}A = \text{span}\{(1,0,0,0)\}$ if $a_1(\theta) \neq a_4(\theta)$. For $a_1(\theta) \equiv a_4(\theta)$ we have $\text{Ker}A = \text{Ker} \text{Re}A = \text{span}\{(1,0,0,0), (0,0,0,1)\}$. Moreover $\text{Ker}A = \text{KZ}_0(K)$. Thus the boundary value problem (B) for the equation of transfer of polarized light has a unique solution.

5. Multigroup Neutron Transport

Consider the nonsymmetric isotropic multigroup neutron transport equation

$$\frac{d}{dx} \mu \psi(x, \mu) = -\Sigma \psi(x, \mu) + \frac{1}{2} \int_{-1}^1 C \psi(x, \mu') d\mu', \quad x \geq 0, \quad \mu \in [-1, 1], \quad (5.1)$$

where $\psi(x, \mu)$ is an n -component vector, the i -th component $\psi_i(x, \mu)$ describing the density of the i -th velocity group, Σ is diagonal with entries $\sigma_i > 0$, C has non-negative entries and in general is not symmetric. We view $\psi_i(x, \mu)$ as

$\psi_i(x, \cdot) \in L_2(-1, 1)$ and $\psi(x, \cdot) \in \bigoplus_{i=1}^n L_2(-1, 1) = V^n \otimes L_2(-1, 1)$, where V^n is an

n -dimensional vector space. Using the notation e for the constant function $2^{-1/2}$ in

$L_2(-1,1)$ and T_1 for the operator of multiplication by the argument μ in $L_2(-1,1)$, equation (5.1) can be rewritten as

$$\frac{d}{dx} T_1 \psi(x) = -(\Sigma - Ce(.,e))\psi(x),$$

where $Ce(.,e)$ should be understood as the tensor product of C acting in V^n and the rank-1 operator $e(.,e)$ acting in $L_2(-1,1)$, and T_1 and Σ should be understood as $1_{V \otimes T_1}$ and $\Sigma \otimes 1_{L_2}$, respectively. Obviously T_1 and $e(.,e)$ commute with Σ and C . To put the above equation in the form of the abstract kinetic equation we multiply both sides by Σ^{-1} and get

$$\frac{d}{dx} T\psi = -A\psi,$$

where $T = \Sigma^{-1} T_1$ and $A = 1 - \Sigma^{-1} Ce(.,e)$. Now we translate the assumptions on T and A into conditions on C and Σ . Obviously the Hölder condition is satisfied. A being injective is equivalent to $(\Sigma - C)$ being invertible and the inverse of A is given by

$$A^{-1} = 1 + (\Sigma - C)^{-1} Ce(.,e).$$

It is also easy to verify that $\operatorname{Re} A \geq 0$ is equivalent to $\Sigma - C_s \geq 0$, where $C_s = (C + C^T)/2$ is the symmetric part of C . Finally we want to see when $W(\lambda) = (T - \lambda)^{-1} (T - \lambda A)$, the transfer function, has a bounded inverse for all $\operatorname{Re} \lambda = 0$. Writing

$$W(\lambda) = (1 + (T - \lambda)^{-1} \Sigma^{-1} (\Sigma - C)^{-1} C T_1 e(.,e)) A,$$

the invertibility of W is equivalent to

$$\det(1+\Sigma^{-1}(\Sigma-C)^{-1}C((T-\lambda)^{-1}T_1 e,e)) \neq 0. \quad (5.2)$$

Note that $((T-\lambda)^{-1}T_1 e,e)$ is a diagonal $n \times n$ matrix with i -th diagonal entry

$$((T_1 \sigma_i - \lambda)^{-1} T_1 e, e) = 1 - \sigma_i \zeta \operatorname{arctg} \frac{1}{\zeta \sigma_i} = 1 - \frac{\sigma_i}{w_i(\zeta)},$$

for $\lambda = i\zeta$ and $\zeta \in \mathbb{R}$, where $w_i(\zeta)$ is defined by the last equality. Let $w(\zeta)$ be the diagonal $n \times n$ matrix with entries $w_i(\zeta)$. After factoring irrelevant factors, (5.2) becomes

$$\det(w(\zeta) - C) \neq 0, \quad \forall \zeta \in \mathbb{R}. \quad (5.3)$$

Note that $w_i: \mathbb{R} \rightarrow [\sigma_i, \infty)$. Let $\sigma = \min\{\sigma_i; i=1, \dots, n\}$. Then (5.3) will hold if $\|C\| < \sigma$. Indeed, suppose that for some ζ_0 we have $\det(w(\zeta_0) - C) = 0$. Then there will be a vector $v \in V^n$ such that $Cv = w(\zeta_0)v$. Since $w_i(\zeta_0) \geq \sigma V_i$, we get $\|Cv\| = \|w(\zeta_0)v\| \geq \sigma \|v\|$ which contradicts $\|C\| < \sigma$.

Proposition 5.1. If $\Sigma - C_s \geq 0$ and $\|C\| < \sigma$ then the half-space multigroup neutron transport problem has a unique solution.

If we assume $\operatorname{Ker} A = \operatorname{Ker}(\operatorname{Re} A) = 0$, the norm condition on C can be dropped.

Proposition 5.2. If $\Sigma - C_s \geq 0$ and $\det(\Sigma - C) \neq 0$, $\det(\Sigma - C_s) \neq 0$ then the half-space multigroup neutron transport problem has a unique solution.

IV. WEAK SOLUTIONS FOR UNBOUNDED COLLISION OPERATORS

In this chapter we consider weak solutions of the boundary value problem in a bigger space H_T , which has a natural Krein space structure. The use of the Krein space geometry makes the analysis of unique solvability simpler and more transparent. We may treat unbounded collision operators of the form "Sturm-Liouville plus a compact operator". An example of such boundary value problems is the model of Ligou [PL], [CL], [SM].

1. Krein Spaces and Positive Operators

Let H_T be a Krein space with indefinite scalar product $(\cdot, \cdot)_T$, fundamental projectors Q_{\pm} , and fundamental decomposition $H_T = Q_+ H_T \oplus Q_- H_T$. This means that $(\cdot, \cdot)_T$ is a nondegenerate sesquilinear form on H_T , $Q_+ H_T$ is a Hilbert space with scalar product $(\cdot, \cdot)_T$, $Q_- H_T$ is a Hilbert space with scalar product $-(\cdot, \cdot)_T$. The operator $Q = Q_+ - Q_-$ is an involution, i.e., $Q^2 = I$, called a fundamental symmetry. The topology on H_T is defined by the norm induced from the positive definite scalar product $(Q\cdot, \cdot)_T$. With respect to this scalar product Q is unitary. If K is an operator on H_T , its adjoint with respect to $(\cdot, \cdot)_T$, for short, called the H_T -adjoint, will be denoted $K^{\#}$. If K^* is the adjoint with respect to $(Q\cdot, \cdot)_T$, then $K^{\#} = QK^*$. A vector f in H_T is called positive, negative or neutral if $(f, f)_T > 0$, < 0 or $= 0$, respectively. A subspace M in H_T is called positive if it does not contain negative vectors, M is called positive definite if it contains only positive vectors and M is maximal positive if it is positive and is not the

proper subspace of a positive subspace. One has the obvious definitions for negative, negative definite and maximal negative subspaces. The orthogonal companion of M is $M^\perp = \{f \in H_T : (f, g)_T = 0 \text{ for all } g \in M\}$. The isotropic part of a subspace M is $M^0 = M \cap M^\perp$. If $M = M_1 \oplus M_2$ then $M^0 = M_1^0 \oplus M_2^0$. If M is a positive space then M^0 consists precisely of the neutral vectors in M . A decomposition $H_T = M_+ \oplus M_-$ with M_\pm closed positive/negative definite subspaces is a fundamental decomposition. The corresponding projectors P_\pm are called fundamental projectors.

The following simple geometric fact about Krein spaces will be crucial in the analysis of the unique solvability in the next sections.

Proposition 1.1. (see [B], Theorem 4.1) A positive subspace M_+ of H_T is maximal positive if and only if $Q_+ M_+ = Q_+ H_T$.

In Section 5 we will need the following.

Proposition 1.2. Suppose M_\pm are closed positive/negative subspaces of H_T , $M_+ \oplus M_- = H_T$ and M_- is negative definite. Then M_+ is a maximal positive subspace.

Proof. Suppose not, then there is a vector $f = f_+ + f_-$ with $f_\pm \in M_\pm$, $(f, f) \geq 0$ and $f_- \neq 0$. One can modify f so that it is orthogonal to M_+ with respect the indefinite scalar product. Indeed if f is neutral, i.e., $(f, f)_T = 0$, then it is a neutral vector in the positive space $M_+ \oplus \text{span}\{f\}$, so it will be $(\cdot, \cdot)_T$ -orthogonal to M_+ . Therefore assume $(f, f)_T > 0$. If M_+^0 is the isotropic part of M_+ , i.e., the intersection of M_+ with its $(\cdot, \cdot)_T$ -orthogonal companion, then M_+^0 will be $(\cdot, \cdot)_T$ -orthogonal to f . Thus applying orthogonalization in the Hilbert space $(M_+ \oplus \text{span}\{f\})/M_+^0$ provides the desired modification. So assume f is

$(\cdot, \cdot)_T$ -orthogonal to M_+ . Then one has $0 \geq (m_-, m_-)_T = (f, f)_T + (f_+, f_+)_T \geq 0$ hence $(m_-, m_-)_T = 0$. But by assumption $m_- \neq 0$ and M_- is negative definite, which is a contradiction. Therefore M_+ is maximal positive. ■

Next we will state the spectral theorem for positive operators in a Krein space (see [L] and [B]). A H_T -self adjoint operator K is called positive if its resolvent set is nonempty and $(Kf, f)_T \geq 0$ for all $f \in D(K)$. Let \mathcal{R} be the semiring which consists of all bounded intervals and their complements in \mathbb{R} with endpoints different from zero.

Spectral Theorem (see [L], Theorem 3.1). If K is a H_T -positive operator, then there exists a map F from \mathcal{R} into the set of bounded, H_T -self adjoint operators in H_T such that $(E, \hat{E} \in \mathcal{R})$: $F(E)F(\hat{E}) = F(E \cap \hat{E})$; $F(E \cup \hat{E}) = F(E) + F(\hat{E})$ for disjoint E and \hat{E} in \mathcal{R} ; $F(E)H_T$ is a positive/negative subspace if $E \subset \mathbb{R}_\pm$; $F(E)$ is in the double commutant of the resolvent of K ; if E is a bounded interval, then $F(E)H_T \subset D(K)$ and $K \upharpoonright F(E)H_T$ is a bounded operator; $\sigma(K \upharpoonright F(E)H_T) \subset \bar{E}$.

A point t is a critical point for the operator K if $F(E)H_T$ is an indefinite subspace for every $E \in \mathcal{R}$ with $t \in E$. The only possible critical points for a positive operator K are zero and infinity. If K has a bounded inverse it may have a critical point only at infinity. If the limits $\lim_{t \rightarrow \infty} F((t_0, t))$ and $\lim_{t \rightarrow -\infty} F((t, t_0))$ exist then we say that infinity is a regular critical point. In this case $F((0, \infty))H_T$ and $F((-\infty, 0))H_T$ are maximal positive/negative definite subspaces giving a fundamental decomposition of the Krein space.

2. Extensions and Decompositions

Let H be a Hilbert space and assume that T is a bounded, self adjoint and injective operator on H , hence T^{-1} is densely defined with domain TH . Let Q_{\pm} be the maximal positive/negative spectral projectors for T , i.e., $Q_{\pm}^2 = Q_{\pm}$, $TQ_{\pm} = Q_{\pm}T$ and $\sigma(TQ_{\pm}) \subset \{\lambda \in \mathbb{R} \mid \pm\lambda \geq 0\}$. Set $Q = Q_+ - Q_-$, obviously a unitary involution, i.e. $Q^* = Q = Q^{-1}$. By definition the absolute value of T is $|T| = TQ$, a positive operator. If (\cdot, \cdot) is the scalar product in H and $k > 0$, let the Hilbert space H_k be the space $|T|^{k/2}H$ with the scalar product $(\cdot, \cdot)_k = (|T|^{-k} \cdot, \cdot)$. For $-k > 0$ let $(\cdot, \cdot)_{-k} = (|T|^k \cdot, \cdot)$ and denote by H_{-k} the completion of H with respect to the norm $\|\cdot\|_{-k}$. In particular we have the chain of Hilbert spaces

$$H_{-2} \xrightarrow{=} H_{-1} \xrightarrow{=} H \xrightarrow{=} H_1 \xrightarrow{=} H_2 \quad (2.1)$$

where the arrows represent the unitary isomorphisms given by $|T|^{1/2}$. Because $\|\cdot\|_k$ majorizes $\|\cdot\|_h$ if $k \geq h$, we also have the chain of continuous imbeddings

$$H_{-2} \xleftarrow{\supset} H_{-1} \xleftarrow{\supset} H \xleftarrow{\supset} H_1 \xleftarrow{\supset} H_2. \quad (2.2)$$

By construction and the injectivity of T the imbeddings are dense. We remark that H_2 is the domain of T^{-1} in H and H_{-1} is denoted in other places ([B1], [GMZ]) by H_T .

Let K, \hat{K} be operators in H . Assume that

$$T D(K) \subset D(\hat{K}) \subset H_2 \subset H_1 \quad (2.3)$$

and

$$TKf = \hat{K}Tf, \quad \text{for every } f \in D(K). \quad (2.4)$$

Assume that \hat{K} is a closed operator in H . Then \hat{K} is a closed operator in H_1 . Indeed, let $f_n \in D(\hat{K})$, $\|f_n - f\|_1 \rightarrow 0$, and $\|\hat{K}f_n - g\|_1 \rightarrow 0$ for some $g \in H_1$. Because $\|\cdot\|_1$ majorizes $\|\cdot\|$ we have the same convergences in H . But we know that \hat{K} is closed in H , hence $f \in D(\hat{K})$ and $\hat{K}f = g$, hence \hat{K} is closed in H_1 . Define an extension in H_{-1} of the operator K in H by

$$T^{-1} \circ \hat{K} \circ T \quad (2.5)$$

where \hat{K} is viewed as an operator in H_1 and T and T^{-1} are viewed as isometries between H_1 and H_{-1} . Denoting this extension again by K will cause no confusion. Thus K in H_{-1} is a closed operator. We also assume that $\text{Ker}\hat{K} = T(\text{Ker}K)$. Then we will have that the kernel of K in H and H_{-1} will be the same. The zero root manifold $Z_0(K)$, which is the union of the kernels of K^n , $n=1,2,\dots$ will be the same in H and H_{-1} too. If K is densely defined in H , then its extension is densely defined in H_{-1} . Indeed if M is a set in H and $\text{clos}_{-k}M$ is the closure in H_{-k} , we have $M \subset \text{clos}_0M \subset \text{clos}_{-1}M$. Taking clos_{-1} once more we get $\text{clos}_{-1}M = \text{clos}_{-1}\text{clos}_0M$, so if $\text{clos}_0M = H$ we get also $\text{clos}_{-1}M = H_{-1}$.

If K and \hat{K} are bounded operators on H and $TK = \hat{K}T$ then K extends to a bounded operator on H_{-1} . Indeed, it is immediate that K has a bounded extension in H_{-2} . Interpolating between H_{-2} and H we get that K is bounded in H_{-1} .

Now consider an operator A in H which is Fredholm, accretive (i.e., $2\text{Re}A = A + A^* \geq 0$) and satisfies $\text{Ker}A = \text{Ker}(\text{Re}A)$. As in the previous chapter we have

the following.

Lemma 2.1.

- (a) $f \in H$ and $(Af, f) = 0$ imply $f \in \text{Ker} A$
- (b) $\text{Ker} A = \text{Ker} A^*$, $\text{Ran} A = \text{Ran} A^*$
- (c) $H = \text{Ker} A \oplus \text{Ran} A$.

Set $K = T^{-1}A$ and $\hat{K} = AT^{-1}$ (note the difference in notation from the previous chapter).

Lemma 2.2. The operators K and \hat{K} are densely defined and closed in H .

Proof. We will only show that \hat{K} is closed. The density of $D(K)$ was proved in the previous chapter, while the rest is obvious.

Let $f_n \in D(AT^{-1}) = T(D(A))$, so $f_n = Th_n$ for some $h_n \in D(A)$. Assume that $f_n \rightarrow f$ and $\hat{K}f_n = Ah_n \rightarrow f'$. Because $\text{Ran} A$ is closed we can write $f' = Ah$, for some $h \in D(A)$. Thus $A(h_n - h) \rightarrow 0$. By the Fredholmness of A (more precisely by Lemma 1.c) one can view A as an operator from $\text{Ran} A$ onto $\text{Ran} A$ with a bounded inverse. Whence $h_n \rightarrow h$. But T is bounded, so $Th_n \rightarrow Th$. Therefore $f = Th \in D(\hat{K})$ and $f' = Ah = \hat{K}Th = \hat{K}f$. ■

Lemma 2.3. The only possible eigenvalues of $T^{-1}A$ on the imaginary axis can be at the origin.

Lemma 2.4. The Jordan chains of $T^{-1}A$, $T^{-1}A^*$, AT^{-1} and A^*T^{-1} at $\lambda = 0$ have length at most two.

Lemma 2.5.

$$(a) \quad KZ_0(K) = (T^{-1}A^*) Z_0(T^{-1}A^*)$$

$$(b) \quad \dim Z_0(K) = \dim Z_0(T^{-1}A^*)$$

Using the remarks in the beginning of this section we extend the operator $K=T^{-1}A$ to H_{-1} . So we may consider K as a closed, densely defined operator in H_{-1} . Its zero root manifold $Z_0(K)$ will be identified with the one in H .

The space H_{-1} becomes a Krein space, denoted H_T if we introduce the indefinite scalar product $(\cdot, \cdot)_T = (Q\cdot, \cdot)_{-1} = (T\cdot, \cdot)$. Let $K^\#$ be the H_T -adjoint of K . The operator $K^\#$ is an extension of $T^{-1}A^*$ and $Z_0(K^\#) = Z_0(T^{-1}A^*)$. Let $Z_1(K)$ be the H_T -orthogonal companion of $Z_0(K^\#)$, i.e., $Z_1(K) = (Z_0(K^\#))^\perp$. It is immediate that $Z_1(K)$ is K invariant.

Lemma 2.6. $Z_0(K) \cap Z_1(K) = 0$

Proof. Let $M \subset H \subset H_T$ be a subspace then $TM^\perp \subset M^\perp$. Suppose $h \in (Z_0(K^\#))^\perp \subset (Ker A^*)^\perp = (Ker A)^\perp$. Then $Th \in (Ker A)^\perp = Ran A^*$. Hence $h = K^\# g$ for some g . Because both $Z_0(K)$ and $Z_1(K)$ are K invariant and $KZ_0(K) \subset Ker A = Ker A^*$, we need only show $Ker A^* \cap Z_1(K) = 0$. So assume also $h \in Ker(A^*) \subset Z_0(K^\#)$. This implies that $g \in Z_0(K^\#)$. Thus we have $0 = (Th, g) = (A^\# g, g)$. By Lemma 2.1 this implies $A^\# g = 0$, so $h = K^\# g = 0$. ■

Theorem 2.7. There is a K invariant decomposition of H_T : $H_T = Z_0(K) \oplus Z_1(K)$.

Proof. Using $\text{codim } M^\perp \leq \dim M$ we get $\text{codim } Z_1(K) \leq \dim Z_0(K^\#) = \dim Z_0(K)$, the equality

coming from Lemma 2.5. But above we obtained $Z_0(K) \cap Z_1(K) = 0$, hence the decomposition holds. ■

3. Strictly Positive Collision Operator

In this section we assume A positive, Fredholm and $\text{Ker} A = 0$. Even though A will be in general unbounded the assumptions we made force A^{-1} to be a bounded operator in H . By the considerations of the previous section $A^{-1}T$ has a bounded extension to H_T and $K = T^{-1}A$ has a closed, densely defined extension to H_T and a bounded inverse in H_T . The operator $A^{-1}T$ is H_T -self adjoint, H_T -positive. Indeed $(A^{-1}Tf, f)_T = (A^{-1}(Tf), (Tf)) \geq 0$ by the assumption that A is positive in H . Therefore we may view K as H_T -self adjoint, H_T -positive operator with a H_T -bounded inverse. Thus K can have at most one critical point, namely a critical point at infinity. Since T in most physical models has both positive and negative spectrum around zero we must treat the case that infinity is a critical point of K .

In order to proceed further and use the functional calculus for definitizable operators in a Krein space we have to assume that infinity is a regular critical point of K . The regularity of the critical point infinity for a positive, boundedly invertible operator in a Krein space has been investigated in detail in [C]. In particular, infinity is a regular critical point for the operator K if and only if the norms $\|\cdot\|_T$ and $\|\cdot\|_S$ are equivalent (see [B1],[B2],[C], [GMZ] or [GMP]). For example when T is multiplication by a function and A is a Sturm-Liouville operator it is shown in [B2] that the two norms are equivalent and thus infinity is a regular critical point. Also when A is a bounded operator the two norms are equivalent [GMP].

Assuming K has no singular critical points the spectral theorem for definitizable operators [L] provides us with separating projectors P_{\pm} , i.e., $P_{\pm}D(K) \subset D(K)$, $P_{\pm}Kh = KP_{\pm}h$ for $h \in D(K)$, $P_{+} + P_{-} = I$ and $\sigma(KP_{\pm}) \subset \{\lambda \in \mathbb{R} \mid \pm\lambda \geq 0\}$. From the functional calculus we obtain exponentially decaying holomorphic semigroups $\exp(-xK)P_{\pm}$, $\pm x \geq 0$. Moreover $P = P_{+} - P_{-}$ is a fundamental symmetry of the Krein space H_T and $P_{\pm}H_T$ are maximal positive/negative definite subspaces of H_T .

Now consider the boundary value problem

$$\frac{d}{dx}T\psi(x) = -A\psi(x), \quad 0 < x < \infty \quad (3.1a)$$

$$Q_{+}\psi(0) = \varphi_{+} \quad (3.1b)$$

$$\|\psi(x)\|_{-1} \rightarrow 0, \quad x \rightarrow \infty, \quad (3.1c)$$

where $\varphi_{+} \in Q_{+}H_T$ is the given incoming flux. All solutions of the boundary value problem (3.1) are in the form

$$\psi(x) = \exp(-xK)h \quad (3.2)$$

for some $h \in P_{+}H_T$ such that $Q_{+}h = \varphi_{+}$ (see [GMP],[GMW]). Thus the unique solvability is reduced to the question whether Q_{+} maps $P_{+}H_T$ bijectively onto $Q_{+}H_T$. As noted, $P_{+}H_T$ is a maximal positive subspace in H_T , so the bijectivity of Q_{+} as a map from $P_{+}H_T$ onto $Q_{+}H_T$ follows from Proposition 1.1.

4. Invertible, Accretive Collision Operator

For the collision operator A assume that it can be written as a difference

$A = A_1 - A_2$ and that

- (i) A_1 is a positive operator in H with a bounded inverse,
- (ii) the extension K_1 of the operator $T^{-1}A_1$ to the Krein space H_T has a regular critical point at infinity,
- (iii) A is an accretive operator in H , i.e. $2\operatorname{Re}A = A + A^* \geq 0$,
- (iv) $\operatorname{Ker}A = \operatorname{Ker}(\operatorname{Re}A) = 0$,
- (v) $A_1^{-1}A_2$ is a trace class operator in H_{-1} .

Following the discussion of the previous section we obtain that K_1 is similar to a self-adjoint operator. Let $P_{1\pm}$ be the separating projectors for K_1 in H_T , i.e., maximal positive/negative spectral projectors for K_1 . We also have exponentially decaying holomorphic semigroups $\exp(-x, K_1)P_{1\pm}$, $\pm x \geq 0$. Let K be the extension of $T^{-1}A$ to H_T . We want to define maximal positive/negative spectral projectors P_{\pm} for K in H_T and exponentially decaying semigroups $\exp(-xK)P_{\pm}$. To accomplish this one uses the theorem for perturbation of bisemigroups. Write $T^{-1}A = T^{-1}A_1(I - A_1^{-1}A_2)$. Condition (iv) assures that $T^{-1}A$ has no eigenvalues on the imaginary axis. This with assumption (v) gives us that K is generator of an exponentially decaying holomorphic bisemigroup if K_1 is.

Consider again the boundary value problem (3.1). Once more all solutions will be in the form $\psi(x) = \exp(-xK)h$ for some $h \in P_+H_T$ such that $Q_+h = \varphi_+$. So the question of unique solvability is equivalent to the bijectivity of Q_+ as a map from P_+H_T onto Q_+H_T . To answer this question in the affirmative using Proposition 1.1, we need to show that P_+H_T is a maximal positive subspace. Here is where the accretivity assumption (iii) becomes crucial.

Proposition 4.1. The subspaces $P_{\pm}H_T$ are positive/negative definite.

Proof. By virtue of assumptions (i) and (iv), we have that A^{-1} is a bounded operator on H . By the same reasoning as before we get that $K+K^\#$ is a strictly H_T -positive operator. Now take any $g \in P_+H_T$ and set $f(x) = \exp(-xK)g$, $x > 0$. Because KP_+ generates a holomorphic semigroup we have that $f(x) \in D(K)$ for $x > 0$. So we have

$$0 > - ((K+K^\#)f(x), f(x))_T = \frac{d}{dx}(f(x), f(x))_T, \quad x > 0.$$

Integrating both sides we obtain

$$0 > - \lim_{\tau \rightarrow \infty} \int_0^\tau ((K+K^\#)f(x), f(x))_T = \lim_{\tau \rightarrow \infty} (f(\tau), f(\tau))_T - (g, g)_T = -(g, g)_T.$$

Thus $(g, g)_T > 0$ and P_+H_T is positive definite. ■

From the fact that $P_+ + P_- = I$, P_+H_T is a positive subspace, P_-H_T is a negative definite subspace and Proposition 1.2, we conclude that P_+H_T is a maximal positive subspace. So applying once more Proposition 1.1, we obtain unique solvability.

5. Accretive Collision Operator with a Nontrivial Kernel

For the operator A assume that (i-iii) and (v) hold and change (iv) to read:

$$(iv) \quad \text{Ker}A = \text{Ker}(\text{Re}A).$$

In Section 2 we obtained the decomposition $H_T = Z_0(K) \oplus Z_1(K)$. Let P_1 be

the projection of H_T onto $Z_1(K)$ along $Z_0(K)$ and let $P_0 = I - P_1$ be the complementary projection. We proceed along the lines of the previous chapter, section 2, to define spectral projectors P_{\pm} for the restriction of K to $Z_1(K)$ and to obtain the corresponding semigroups. Let us write $A' = TP_0 + AP_1 = A + (T-A)P_0 = A_1 + (A_2 + (T-A)P_0)$. Here, P_0 and P_1 are the restrictions of the corresponding projectors to H . Note that $K' = P_0 + KP_1$. Obviously we have

$$K'P_0 = P_0K' = P_0$$

$$K'P_1 = P_1K' = P_1K = KP_1$$

Under the assumptions on A and T we observe that A' has a bounded inverse, K' has no eigenvalues on the imaginary axis, and since $A'_2 = A_2 + (T-A)P_0$ is a finite rank perturbation of A_2 , the trace class condition (v) will be satisfied. By the same arguments as in section 4 we conclude that K' is a generator of an exponentially decaying holomorphic bisemigroup with separating projectors P'_{\pm} . Now we get that the operators $P_{\pm} = P'_{\pm}P_1$ are projectors, commuting with K and such that $P_+ + P_- = P_1$, $\sigma(KP_{\pm}) \subset \{\lambda \in \mathbb{R} : \pm\lambda \geq 0\}$ and $\exp(-xK)P_{\pm}$, $\pm x \geq 0$, are exponentially decaying holomorphic semigroups on $P_{\pm}H_T$.

Consider the boundary value problem

$$\frac{d}{dx}T\psi(x) = -A\psi(x), \quad 0 < x < \infty \quad (5.1)$$

$$Q_+\psi(0) = \varphi_+ \quad (5.2)$$

where $\varphi_+ \in Q_+H_T$ is the given incoming flux. Because the Jordan chains in $Z_0(K)$

have length at most two, we have three possibilities at infinity, namely

$$\lim \|\psi(x)\| = 0, \quad x \rightarrow \infty, \quad (5.3a)$$

$$\|\psi(x)\| = O(1), \quad x \rightarrow \infty, \quad (5.3b)$$

$$\|\psi(x)\| = O(x), \quad x \rightarrow \infty. \quad (5.3c)$$

Proposition 5.1. All solutions of the boundary value problem (5.1-3.a) are of the form $\psi(x) = \exp(-xK)h$ for some $h \in P_+H_T$ such that $Q_+h = \varphi_+$. All solutions of (5.1-3.b) are of the form $\psi(x) = \exp(-xK)h_1 + h_0$ for some $h_1 \in P_+H_T$ and $h_0 \in \text{Ker}A$ such that $Q_+(h_1 + h_0) = \varphi_+$. All solutions of (5.1-3.c) are of the form $\psi(x) = \exp(-xK)h_1 + (I - xK)h_0$ for some $h_1 \in P_+H_T$ and $h_0 \in Z_0(K)$ such that $Q_+(h_1 + h_0) = \varphi_+$.

From this proposition it is clear that for each of the boundary value problems (5.1-3.a), (5.1-3.b) and (5.1-3.c), injectivity of the map Q_+ from P_+H_T , $P_+H_T \oplus \text{Ker}A$ and $P_+H_T \oplus Z_0(K)$, respectively, into Q_+H_T assures the uniqueness of solutions, and surjectivity of the respective maps assures the existence of solutions.

We want to use Proposition 1.1 once more. To this end we need to obtain a maximal positive subspace containing P_+H_T and an appropriate piece of $Z_0(K)$. A similar argument to that in Proposition 4.1 gives that $P_{\pm}H_T$ are positive/negative definite subspaces.

Proposition 5.2. One has the decomposition $H_T = Z_0(K^\#) \oplus Z_1(K)$.

Proof. Using the fact that $P_{\pm}H_T$ are definite subspaces, for the isotropic part of

$Z_1(K)$ we obtain $(Z_1(K))^0 = (P_+ H_T \oplus P_- H_T)^0 = (P_+ H_T)^0 \oplus (P_- H_T)^0 = 0$. On the other hand, from the definitions of isotropic part and of $Z_1(K)$ we have $0 = (Z_1(K))^0 = Z_1(K) \cap (Z_1(K))^\perp = Z_1(K) \cap Z_0(K^\#)$. Hence $\text{codim } Z_1(K) = \text{codim } (Z_0(K^\#))^\perp \leq \dim Z_0(K^\#)$, which completes the proof. ■

Because of the H_T -orthogonality of $Z_0(K^\#)$ and $Z_1(K)$, if we decompose $Z_0(K^\#) = M_+ \oplus M_-$ into a positive and a negative subspace we obtain that $H_T = (M_+ \oplus P_+ H_T) \oplus (M_- \oplus P_- H_T)$ is a decomposition of H_T into a positive and a negative subspace.

Proposition 5.3. There exists a decomposition $Z_0(K^\#) = M_+ \oplus M_-$ with M_+ a positive subspace contained in $\text{Ker} A$ and M_- a negative definite subspace.

Proof. First we show $(\text{Ker} A)^0 = K^\# Z_0(K^\#)$. Suppose $f \in (\text{Ker} A)^0 = (\text{Ker} A) \cap (\text{Ker} A)^\perp$. Then $f \in T(\text{Ker} A)^\perp \subset (\text{Ker} A)^\perp = \text{Ran} A^*$, so $f = K^\# g$ for some g , and hence $f \in K^\# Z_0(K^\#)$. Conversely, let $f \in K^\# Z_0(K^\#)$. Then $Tf = A^* g$ for some g , and $(f, u)_T = (Tf, u) = (A^* g, u) = (g, Au) = 0$ for every $u \in \text{Ker} A$.

Because $(\text{Ker} A)/(\text{Ker} A)^0$ is a Krein space, we can choose an H_T -orthogonal, linearly independent set of nonneutral vectors $\{z_1, \dots, z_n\} \subset \text{Ker} A$ such that $(\text{Ker} A)^0 \oplus \{z_1, \dots, z_n\} = \text{Ker} A$. The same argument as that in Proposition 5.2 shows that $Z_0(K^\#)$ is a Krein space, so we can choose an H_T -orthogonal, linearly independent set of nonneutral vectors $\{y_1, \dots, y_k\} \subset Z_0(K^\#)$ which is H_T -orthogonal to $\{z_1, \dots, z_n\}$ and such that $\text{Ker} A \oplus \text{span}\{y_1, \dots, y_k\} = Z_0(K^\#)$. One may assume that all the y_i 's are negative (or positive), because if they are not, one can adjust them to be negative (positive) without spoiling any of the other properties by the following trick [GMZ]. If ζ is a real number and $x_i = K^\# y_i$, then

$$((y-\zeta x), (y-\zeta x))_{\mathbb{T}} = (y, y)_{\mathbb{T}} + \zeta^2 (x, x)_{\mathbb{T}} - \zeta ((x, y)_{\mathbb{T}} - (y, x)_{\mathbb{T}}) = (y, y)_{\mathbb{T}} - 2\zeta ((\operatorname{Re}A)y, y)$$

can be made negative (positive) by choosing an appropriate ζ , since $(\operatorname{Re}Ay, y) > 0$.

Without loss of generality we may assume that z_1, \dots, z_m span a negative definite subspace and z_{m+1}, \dots, z_n span a positive definite subspace. Note that m is the dimension of any maximal negative subspace in the Krein space $(\operatorname{Ker}A)/(\operatorname{Ker}A)^0$, and thus is independent of the particular choice of the z_i 's. Finally, let us set $M_+ = (\operatorname{Ker}A)^0 \oplus \operatorname{span}\{z_{m+1}, \dots, z_n\}$ and $M_- = \operatorname{span}\{y_1, \dots, y_x\} \oplus \operatorname{span}\{z_1, \dots, z_m\}$. Then, by construction, M_+ is a positive subspace contained in $\operatorname{Ker}A$, M_- is negative definite, and $Z_0(K^\#) = M_+ \oplus M_-$. ■

Note that the subspace M_+ is a maximal positive subspace in both $Z_0(K^\#)$ and $\operatorname{Ker}A$. Thus its dimension is independent of the choice of the space M_+ . Denote this invariant by m_+ . Using Proposition 1.1, we conclude that $M_+ \oplus P_+ H_{\mathbb{T}}$ is a maximal positive subspace, so from Proposition 1.2 we obtain that Q_+ maps $M_+ \oplus P_+ H_{\mathbb{T}}$ bijectively onto $Q_+ H_{\mathbb{T}}$.

Define the measure of nonexistence δ to be the codimension in $Q_+ H_{\mathbb{T}}$ of the space of boundary values φ_+ for which the boundary value problem is solvable, and the measure of nonuniqueness γ to be the dimension of the solution space of the corresponding homogeneous boundary value problem. From the above considerations, we obtain the main theorem of this section.

Theorem 5.1. For the boundary value problem (5.1-3.a), we have $\delta = m_+$ and $\gamma = 0$. For the boundary value problem (5.1-3.b), we have $\delta = 0$ and $\gamma = \dim(\operatorname{Ker}A) - m_+$. For the boundary value problem (5.1-3.c), we have $\delta = 0$ and $\gamma = \dim(Z_0(K)) - m_+$.

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