GLR Control Charts for Process Monitoring with Sequential Sampling

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Abstract

The objective of this dissertation is to investigate GLR control charts based on a sequential sampling scheme (SS GLR charts). Phase II monitoring is considered and the goal is to quickly detect a wide range of changes in the univariate normal process mean parameter and/or the variance parameter. The performance of the SS GLR charts is evaluated and design guidelines for SS GLR charts are provided so that practitioners can easily apply the SS GLR charts in applications. More specifically, the structure of this dissertation is as follows:

We first develop a two-sided SS GLR chart for monitoring the mean μ of a normal process. The performance of the SS GLR chart is evaluated and compared with other control charts. The SS GLR chart has much better performance than that of the fixed sampling rate GLR chart. It is also shown that the overall performance of the SS GLR chart is better than that of the variable sampling interval (VSI) GLR chart and the variable sampling rate (VSR) CUSUM chart. The SS GLR chart has the additional advantage that it requires fewer parameters to be specified than other VSR charts. The optimal parameter choices are given, and regression equations are provided to find the limits for the SS GLR chart.

If detecting one-sided shifts in μ is of interest, the above SS GLR chart can be modified to be a one-sided chart. The performance of this modified SS GLR chart is investigated.

Next we develop an SS GLR chart for simultaneously monitoring the mean μ and the variance σ^2 of a normal process. The performance and properties of this chart are evaluated. The design methodology and some illustrative examples are provided so that the SS GLR chart can be easily used in applications. The optimal parameter choices are given, and the performance of the SS GLR chart remains very good as long as the parameter choices are not too far away from the optimized choices.

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- Xu, L.; **Peng, Y.** & Reynolds, M. R., Jr. (2013). An Individuals' GLR Control Chart for Monitoring Linear Profiles. *Quality and Reliability Engineering International.* DOI: 10.1002/qre.1616.

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Chapter 1. Introduction

In the industrial and business environment, quality usually refers to how well one or more desirable characteristics that a product or service should possess meet the requirements for use. Quality can be influenced by many factors, such as the condition of manufacturing machines or the training and supervision of the workforce. Statistical Process Control (SPC) is a collection of problem-solving tools useful for maintaining the quality at a desired level. Nowadays SPC is widely applied in the industrial field, as well as in many other fields including health surveillance and computer networking.

It is desirable that all processes have high quality. However the product from a process cannot be exactly identical all the time and there is always a certain amount of variability in the process. Montgomery (2012) classifies the source of variability as two types: chance causes of variation and assignable causes. Chance causes of variation are also known as common causes. The variability produced by chance causes of variation is the natural variability and inherent in the process. It can be considered as the cumulative effect of many small inevitable causes. A process is defined to be in control if chance causes of variation are the only source of variability present in the process.

The other source of variability is assignable causes, also known as special causes. The variability produced by an assignable cause is not natural in the process and can be eliminated with proper actions after the assignable cause is identified. It can be considered as some incident that occurs in the process, for example a part in the machine is broken after long-term use. Such variability may occasionally be present and generally results in changes in the process parameters associated with the distribution of the quality characteristics of interest, for example a change in the process mean or variance. A process is defined to be out of control if any assignable causes are present in the process.

The objective of SPC is to reduce the variability in the process so that quality can be maintained or improved. This can be achieved by quickly detecting any assignable causes of process changes and then taking actions to identify and eliminate the assignable causes of the changes. The control chart is a widely used SPC technique to monitor the process. Basically it is a graphical tools designed to quickly detect any assignable causes in the process. When using a control chart, samples of $n \geq 1$ observations are usually taken from the process at regular time intervals. Then a control chart statistic is computed and plotted at each sampling point. If the statistic falls outside of the predetermined control limits, a signal is given indicating that there has been a change in the process. If the signal is trigged by assignable causes, then the process is in the out-of-control state and action should be taken to find and eliminate the assignable causes. If the signal is trigged by chance causes of variation, then the process is in the in-control state and no action is needed. Such a signal is considered to be a false alarm. Note that if only chance causes are present then the process is in control by definition, but if the control statistic is inside the control limits then the process could actually be in-control or out-of-control. If the statistic is inside the control limits then we would act as if is in control, but it may not be in control.

There are two stages in process monitoring: Phase I and Phase II. In Phase I a set of historical process data is gathered and used for estimating unknown parameters and constructing trial control limits to determine whether the process has been in control. See Jones-Farmer et al. (2014) for an overview of Phase I methods and ideas. If it appears that the process has been in control then these estimates can be used as the target value in Phase II and reliable control limits can be constructed. The objective of real-time monitoring in Phase II is to detect any changes in the process. Throughout this dissertation, we focus on the Phase II monitoring.

In the application of control charts, data on quality characteristics can be classified into two categories: attributes data and variables data. Attribute data are usually discrete data taking the form of counts or the presence/absence of some quality characteristic. For example, each attributes data point represents either a defective or a non-defective item. On the other hand, variables data are usually continuous measurements, such as length or temperature. Control charts applied to attributes data and variables data are called attributes control charts and variables control charts, respectively. In this dissertation, we will study variables control charts.

Many quality characteristics can be expressed in terms of a continuous random variable which is assumed to have a normal distribution. A common problem in univariate normal process monitoring is to detect shifts in the mean and/or variance parameters. Real time monitoring in Phase II is considered, and presumably the in-control values of parameters are either known or accurately estimated from Phase I.

There has been a lot of work on monitoring the normal process. The first control charts were developed by Shewhart (1931), and control charts of this general type are referred to as Shewhart charts. These charts use the information only from the current sample data. They are generally effective for detecting large-size shifts in the parameters of interest, but are not effective for detecting shifts of small or moderate size. In order to quickly detect shifts of small and moderate size, the cumulative sum (CUSUM) chart proposed by Page (1954) and the exponential weighted moving average (EWMA) chart introduced by Roberts (1959) have been widely studied. These charts use the information from the current and the past sample data. They can be very effective for detecting a specified shift size of interest based on selecting the value of some tuning parameters. However, if the actual shift size is not close to the specified shift size, then these charts will not be very effective. For example, if the smoothing parameter in the EWMA statistic is small, then this chart is effective for detecting small-size shifts, but is not effective for detecting large-size shifts.

In practice when some parameter shift occurs in the process, the size of the shift is usually unknown. Thus it is desirable that a wide range of shift sizes can be detected effectively. One option to obtain good performance for detecting a wide range of shift sizes is to use a combination of two control charts together. In the combined control charts, usually one chart can quickly detect small shifts and the other can quickly detect large shifts. See Westgard et al. (1977), Lucas (1982) and Wu et al. (2008) for more detail. Three or more control charts can be used together in combination as a natural extension. References to such combinations include Lorden (1971), Dragalin (1997), Sparks (2000) and Han et al. (2007). One disadvantage of such combinations is that many charts parameters need to be specified. This complicates the design of the control charts and practitioners may have little knowledge about how to specify multiple parameters.

As an alternative, adaptive control charts have been proposed to detect a wide range of shifts effectively. Recent research on adaptive charts includes Sparks (2000), Capizzi and Masarotto (2003), Reynolds and Stoumbos (2006), Shu and Jiang (2006), Jiang et al. (2008) and Wu et al. (2009). The basic idea of adaptive control charts is to use data from the process to estimate the size of any possible shift, and then tune the control charts to respond quickly to this estimated size of shift. However, these adaptive charts also have the disadvantage that multiple chart parameters need to be determined and it may not be clear to practitioners how to choose the parameters and the control limits.

Another option to effectively detect a wide range of shifts is to use control charts based on likelihood ratio tests and a change-point model. These control charts are usually referred to as generalized likelihood ratio (GLR) charts. The idea of the GLR approach was first proposed by Lorden (1971). Willsky and Jones (1976) developed the GLR approach to detect jumps in a linear stochastic system. Lai (1995, 1998) studied the window-limited GLR scheme and discussed its asymptotic optimality property. Apley and Shi (1999) investigated the GLR chart for detection of mean shifts in some autocorrelated processes. Lai (2001) gave a review of the GLR approach and pointed out the potential opportunity for the application of GLR charts. Hawkins and Zamba (2005) designed a GLR chart to detect changes in the normal process mean and variance where there is no Phase I period and monitoring starts after only a few observations.

In GLR charts, the size of the possible parameter shift is estimated from the process data so practitioners don't need to have any prior knowledge about the possible shift size. Many of the existing papers are theoretically orientated, and evaluated the asymptotic properties and investigated the performance of the GLR charts through some comparison studies. However, it may not be clear to practitioners how to apply GLR charts considering the relatively heavy computational burden. GLR charts have not received as much attention in control charts applications as Shewhart, CUSUM, and EWMA charts.

Recently there has been some research on GLR charts from the practitioner's point of view. Reynolds and Lou (2010) evaluated a GLR control chart to detect changes in the mean parameter of a normal process in the Phase II period. They showed that the overall performance of the GLR chart is at least as good as other competing charts. A specific window size is recommended and a table of control limits was provided based on the desired in-control performance, so practitioners can easily apply this GLR chart in applications. Reynolds et al. (2013) further investigated a GLR chart to detect changes in the mean and/or variance parameters of a normal process. Without the need to specify multiple control chart parameters, the GLR chart is shown to have overall good performance. Xu et al. (2012) investigated a GLR chart for monitoring linear profiles. Huang et al. (2012) studied a binomial GLR chart for monitoring a proportion. Wang and Reynolds (2012) applied the GLR approach to monitor a multivariate process.

The traditional practice when applying a control chart in process monitoring is to sample from the process at a fixed sampling rate (FSR). An FSR control chart takes samples with fixed sample size (FSS) using a fixed sampling interval (FSI) between samples. The performance of the FSR control charts can be improved upon by varying the sampling rate as a function of the data from the process. Control charts that allow the sampling rate to vary as a function of the process data are usually called variable sampling rate (VSR) control charts. The basic idea of VSR control charts is that the sampling rate should be increased when there is an indication of a process change and decreased when there is no such indication. If the indication of a process change is strong enough, then a signal is given as in FSR control charts. See, for example, Tagaras (1998) and Reynolds and Stoumbos (2008).

There are several approaches to varying the sampling rate as a function of the process data. One approach is to vary the sampling interval. Control charts based on this sampling scheme are called variable sampling interval (VSI) control charts. The idea is that after the current sampling point, if there is an indication of a process change, a short sampling interval is used next. If there is no indication of a process change, a long sampling interval is used next. References on VSI control charts include Reynolds et al. (1988), Reynolds and Arnold (1989), Rendtel (1990), Reynolds et al. (1990), Runger and Pignatiello (1991), Reynolds (1996a, 1996b). It has been shown that given the same false alarm rate and the same sampling rate, VSI charts can detect most shifts in the process much faster than FSR charts.

Another approach to varying the sampling rate is to vary the sample size. Control charts based on this sampling pattern are called variable sample size (VSS) control charts. A large sample size is used next if there is an indication that a change has occurred in the process, and a small sample size is used next if there is no indication of a process change. VSS charts have been studied by a number of papers, including Prabhu et al. (1993), Park and Reynolds (1999) and Arnold and Reynolds (2001). The VSS charts take samples at fixed sampling times and provide more effective detection than the corresponding FSR charts. The VSI and VSS features can be combined together to further improve the chart's performance.

Besides VSS and VSI control charts, there is another approach to varying the sampling rate, which is sequential sampling. With sequential sampling, observations are taken in groups of one or more at each sampling point. After a group of observations is taken at a sampling point, we wait until the next sampling point to take another group if there is no indication of a process change, or take another group at the current sampling point if there is an indication of a process change. A signal is given if the indication of a process change is strong enough. Sequential sampling is appropriate for the situation in which the time required to take a group of one or more observations is so short that it is negligible. Thus it is possible to take multiple samples sequentially at a sampling point.

As a natural extension of the ideas from sequential analysis, sequential sampling methods have been widely used in the area of acceptance sampling but not so much in process monitoring. If the objective is to detect a parameter change in one direction, a sequential probability ratio test (SPRT) can be applied at each sampling point and a control chart can be constructed based on SPRT's. SPRT charts usually assume that observations are available individually and a decision about either to continue or to stop sampling can be made after each observation. See, for example, Stoumbos and Reynolds (1996, 1997), Reynolds and Stoumbos (1998) and Stoumbos and Reynolds (2001). SPRT charts are typically very effective for detecting changes in one direction in the process, with some tuning parameters specified.

The objective of this dissertation is to investigate GLR control charts based on a sequential sampling scheme. We will refer to such charts as SS GLR charts from this point. Phase II monitoring is considered and the goal is to quickly detect a wide range of changes in the univariate normal process mean parameter and/or the process variance parameter. The performance of SS GLR charts will be evaluated and the design of SS GLR charts will be provided so that practitioners can easily apply the SS GLR charts in applications. More specifically, the goals of this dissertation are as follows:

(1) Develop a two-sided SS GLR chart for monitoring the mean parameter μ of a normal process. The standard FSR GLR chart has been shown to have overall performance as good as or better than other competing charts, so we expect that the performance can be

further improved by using a sequential sampling scheme. The guidelines for designing this SS GLR chart will be provided and its performance will be evaluated through some comparison studies.

(2) If detecting one-sided shifts in μ is of interest, the above SS GLR chart can be modified to be a one-sided chart. The performance of the modified SS GLR chart will be investigated.

(3) Develop an SS GLR chart for simultaneously monitoring the mean parameter μ and the variance parameter σ^2 of a normal process. We will consider two-sided shifts for the mean and one-sided (increasing) shifts for the variance. The performance and properties of this chart will be evaluated. Design guidelines for this SS GLR chart will be given.

The structure of this dissertation is as follows. In Chapter 2 we provide the definition of the process being monitored and explain the metrics used to evaluate the performance of control charts. In addition, we review the control charts that have been developed for monitoring normal processes and discuss different sampling plans. In Chapter 3 we start with the derivation of the SS GLR chart for monitoring the process mean μ, then investigate the effect of the choice of the chart parameters and provide design guidelines to help practitioners apply this SS GLR chart. We also consider the situation in which the goal of process monitoring is to quickly detect one-sided shifts in μ. The SS GLR chart will be modified to apply to such a situation. In Chapter 4, the performance of the SS GLR chart is evaluated and compared with the standard FSR GLR chart and the VSI GLR chart, as well as with the VSSVSI CUSUM chart. It is shown that the SS GLR chart has better overall performance than other competing charts, and it has the advantage that fewer chart parameters need to be specified. We also investigate the performance of the modified one-sided SS GLR chart and compare it with the SPRT chart. In Chapter 5, we first propose a SS GLR chart for monitoring the process mean μ and/or variance σ^2, then study the choice of the chart parameters and provide design guidelines for practitioners. In Chapter 6, the performance of the SS GLR chart is evaluated and compared with the FSR GLR chart for monitoring μ and/or σ^2. The performance improvement in the GLR chart resulting from using sequential sampling is shown. The fact that the SS GLR chart is easy to design and use in applications makes it to be an attractive option for practitioners. In Chapter 7, we summarize the current results and outline some future research directions.

Chapter 2. Background

2.1 Definition of the Process

Suppose that in the process the variable of interest X has a normal distribution with mean μ and variance σ^2. When the process is in control, we assume that the in-control values of μ and σ^2 are μ_0 and σ_0^2, respectively. These in-control values are the target values for the process parameters being monitored. They are presumably either known or can be accurately estimated from the historical data in Phase I.

A sequential sampling scheme can be applied during process data collection where observations are taken one-by-one. At each sampling point, an initial observation is obtained. If there is some indication of a change in the process after this initial observation, then another observation is obtained. Additional observations are obtained as long as there is some indication of a change in the process. If the indication of a process change is strong enough at any time, then a signal is given. If there is no indication of a process change after the current observation, then we wait until the next sampling point to take another observation.

Let X_{ij} represent the j^{th} observation at sampling point i, for $i = 1,2,\dots$ and $j = 1,2,\dots,n_i$, where n_i represents the sample size at sampling point i, $n_i = 1,2,\dots$. This sample size is a random variable depending on the process data. Each observation is assumed to be taken in negligible time within a sample. Let $d > 0$ be the time interval between any two consecutive samples. Observations within and between samples are assumed to be independent. After l^{th} observation in sample k is obtained, we have the data $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k$, where $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{in_i})$ for $i = 1,2,\dots,k-1$ and $\mathbf{X}_k = (X_{k1}, X_{k2}, \dots, X_{kl})$. Notice that the l^{th} observation may not be the last observation in sample k. More explanation will be provided later in this dissertation. Also notice if the FSR scheme is applied, n_i is a constant, say n, for all i, where n represents the fixed sample size. And in this case l is equal to n. The time interval d is the fixed sampling interval.

Consider the problem in which an assignable cause occurs and changes the parameters in the process at some unknown change-point τ^*, where $\tau^* \geq 0$ and there exist two consecutive sampling points τ and $\tau + 1$ such that $\tau \leq \tau^* < \tau + 1$. Generally there are three types of parameter changes considered in process monitoring: sustained shifts, drifts, and transient shifts. For example for monitoring the process mean, a sustained shift causes μ to jump from μ_0 to some μ_1, where $\mu_1 \neq \mu_0$. The value of μ will remain at μ_1 until the change is detected and this assignable cause is removed from the process. Therefore we have in-control $X_{ij} \sim N(\mu_0, \sigma_0^2)$ for $i = 1, 2, \ldots, \tau$ and out-of-control $X_{ij} \sim N(\mu_1, \sigma_0^2)$ for $i = \tau + 1, \tau + 2, \ldots$ until the assignable cause is removed.

A drift in μ means that, instead of a sudden jump from μ_0 to μ_1, μ gradually drifts away from μ_0, and the drift continues until it is detected and the cause is removed. A transient shift means that μ only goes from μ_0 to μ_1 for a short period of time and then returns to μ_0, since the assignable cause of a process change may be present only for a short period of time, even if the change is not detected.

In this dissertation we focus on sustained shifts in μ and/or σ^2. From this point, the use of the term "shift" will refer to a sustained shift. This assignable cause occurs and changes the parameters from the in-control values (μ_0, σ_0^2) to some out of control values (μ_1, σ_1^2), where $\mu_1 \neq \mu_0$ as defined above and/or $\sigma_1^2 \neq \sigma_0^2$.

In Chapter 3 and 4 we only consider shifts in μ, where the value of σ^2 maintains at σ_0^2. If there is a change in the process due to some assignable cause, then it will be reflected by a shift in μ from its in-control value μ_0. It is assumed that the primary objective of process monitoring is to quickly detect a wide range of shifts in μ in both increasing and decreasing directions. The case of detecting a one-sided shift in μ is also investigated. In Chapter 5 and 6, the primary objective of process monitoring is assumed to be the quick detection of a wide range of two-sided shifts in μ and/or increasing shifts in σ^2, i.e., $\mu_1 \neq \mu_0$ and/or $\sigma_1^2 > \sigma_0^2$. We consider only increases in variance because first, decreases in variance mean quality improvement and might be of less concern, and second, considering two-sided GLR methods for the variance increases complication. If observations are very similar to each other, the variance estimate would be very close to zero, and then the GLR statistics would be inflated so that a signal would be triggered. In order to maintain the specified in-control performance, the control limit would need to be increased. Then the detection ability would be worse. Therefore, detecting decreases in variance is not a trivial generalization of the chart for detecting increases in variance, and we decided not to investigate the case of decreases in variance in this dissertation.

2.2 Control Chart Review

In this section, we give a review of several well-known control charts. Most of them are related to the likelihood ratio test of $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$, where θ denotes a vector of parameters of interest. We have the test statistic $l_k = \ln \frac{L(\theta_1|\mathbf{X}_k)}{L(\theta_0|\mathbf{X}_k)}$ at sampling point k, where $L(\theta_1|\mathbf{X}_k)$ and $L(\theta_0|\mathbf{X}_k)$ represent the likelihood function under θ_1 and θ_0, respectively. The review in this section is based on FSR charts, so all $\mathbf{X}_i$ have the fixed sample size n. For each control chart described below, we will first give its general form related to l_k, then show its specific form for monitoring μ, and its specific form for monitoring σ^2.

2.2.1 Shewhart Control Charts

The first control charts were developed by Shewhart (1931), and control charts of this general type are usually called Shewhart charts. These control charts are well known throughout the world and have been widely used in many applications. The original Shewhart charts were designed to detect changes in the mean μ or variance σ^2 of a normal random variable. Then this method was generalized to other process monitoring problems and different charts were developed correspondingly. For example, the Hotelling T^2 chart was designed for multivariate process monitoring, and the np-chart was designed for monitoring attributes data.

Shewhart type charts are directly related with the likelihood ratio test. The chart signals at the time $t_s = \min\{k: l_k \geq h\}$, where h is some threshold value.

For monitoring the normal process mean μ, $l_k = \frac{\mu_1 - \mu_0}{\sigma_0^2/n}\left(\bar{X}_k - \frac{\mu_1+\mu_0}{2}\right)$. This can be reduced to the Shewhart $\bar{X}$ chart that developed based on the sample mean $\bar{X}$. At sampling point k, a signal is generated if $\bar{X}_k$ falls outside of the control limits,

$$\mu_0 \pm \zeta\sigma_0^2/\sqrt{n},$$

where n is the sample size and ζ is some threshold value. The value of this threshold parameter ζ is usually taken to be 3 and the resulting control limits are called 3-sigma limits. The value of ζ can also be chosen instead to give some desired in-control performance.

For monitoring the normal process variance σ^2 (or monitoring the standard deviation σ), the Shewhart S chart was developed based on the sample standard deviation s. A sampling point k, a signal is produced if s falls outside of the control limits,

$$c_4\sigma_0 \pm \zeta\sigma_0\sqrt{1-c_4^2},$$

where c_4 is a constant that depends on the sample size n and is used to adjust for the fact that s is not an unbiased estimator of σ. For more information about the value of c_4, see Montgomery (2012). For detecting increasing shifts only, the lower control limit is usually set to be 0.

In Shewhart charts, the statistic calculated and plotted at each sampling point only depends on the data from the current sample, not from the previous samples. If the process is in-control, the calculated statistics should fall within the control limits. Only when there is a sudden large shift making the process go out-of-control, will the calculated statistic be very likely to fall outside the control limits. Therefore, Shewhart charts are effective for detecting large shifts, but not effective for detecting small and moderate shifts, as the information from past data cannot be accumulated. Runs rules can be added to the Shewhart charts so that some information from the past sample can be accumulated and the performance of detecting small and moderate shifts can be improved. However, Champ and Woodall (1987) and Woodall (2000) pointed out that one disadvantage of adding runs rules is that the in-control ARL would be reduced.

The traditional Shewhart charts take samples using fixed sampling intervals with fixed sample size. A variable sampling rate scheme can be applied to Shewhart charts. VSI Shewhart charts were introduced by Reynolds et al. (1988), and further studied by Runger and Pignatiello (1991) and Reynolds (1996a, 1996b). VSS Shewhart charts were developed by Costa (1994). Reference to VSS Shewhart charts include Prabhu et al. (1993) and Park and Reynolds (1999). The VSS and VSI schemes can be combined to develop VSSVSI Shewhart charts, which have been investigated by Prabhu et al. (1994) and Costa (1997).

2.2.2 CUSUM Control Charts

In order to accumulate the information from the past sample data, CUSUM charts were proposed by Page (1954). van Dobben de Bruyn (1968) investigated the integral equation method to evaluate the performance of the CUSUM charts. A Markov chain method was proposed by Brook and Evans (1972) to obtain the performance of one-sided CUSUM charts, and this method was generalized by Woodall (1984) for two-sided CUSUM charts. Reynolds (1995) proposed a Markov chain-integral equation method, which combined these two method and improved the accuracy of performance evaluations. Hawkins and Olwell (1998) gave a review of CUSUM charts. CUSUM charts have been studied over the years, and have been widely used in many applications.

The general expression of the CUSUM chart statistic can be represented as $C_k = \max\{0, C_{k-1}\} + l_k$. The chart signals at time $t_s = \min\{k: C_k \geq h\}$, with $C_0 = 0$.

For monitoring the normal process mean μ, the two-sided CUSUM chart is based on two one-sided CUSUM statistics, one, called the upper CUSUM statistic, is for detecting increases in μ, and the other, called the lower CUSUM statistic, is for detecting decreases in μ. Let C_k^+ and C_k^- represent the upper and lower CUSUM statistics, respectively. The original statistic C_k is equivalent to charts using these two statistics, which can be written as

$$C_k^+ = \max\ \{0, C_{k-1}^+\} + (\bar{X}_k - \mu_0 - \gamma),$$

and

$$C_k^- = \min\ \{0, C_{k-1}^-\} + (\bar{X}_k - \mu_0 + \gamma),$$

where $\gamma = |\mu_1 - \mu_0|/2$ and μ_1 is a pre-specified value representing the shift of interest. This μ_1 (or equivalently γ) can be considered as a tuning parameter in the CUSUM chart. The starting values of these two statistics are usually set to be $C_0^- = C_0^+ = 0$. The two-sided CUSUM chart signals at sampling point k if

$$C_k^+ > h_c\sigma_0/\sqrt{n} \text{ or } C_k^- < -h_c\sigma_0/\sqrt{n},$$

where $h_c > 0$ is a parameter used to determine the control limits $\left(\pm h_c\sigma_0/\sqrt{n}\right)$ for this chart, and which is chosen to give the desired in-control performance.

The CUSUM chart for monitoring the normal process mean is equivalent to applying repeated sequential probability ratio test (SPRTs) for testing the parameter μ : $H_0: \mu = \mu_0\ vs. H_1: \mu = \mu_1$, and this tuning parameter μ_1 is the value of μ to be detected by the monitoring process. Notice the value of μ_1 must be specified by practitioners whether the actual value of the shift is known or not. The CUSUM chart detects a shift quickly if the actual shift is close to the specified shift μ_1. But if the actual shift is not close to the specified shift, the performance of the CUSUM chart deteriorates.

For monitoring the normal process variance σ^2, a CUSUM chart can be constructed based on the sample variance s^2. For detecting increases in σ^2, the CUSUM statistic $C_k^{v,+}$ can be written as

$$C_k^{v,+} = \max(0, C_{k-1}^{v,+} + s_k^2 - \gamma_v),$$

where the starting value $C_0^{v,+} = 0$, the tuning parameter $\gamma_v = 2\ln\left(\frac{\sigma_0}{\sigma_1}\right)\sigma_0^2\sigma_1^2/(\sigma_0^2 - \sigma_1^2)$, and σ_1^2 here is a pre-specified value representing the shift of interest. The chart signal if $C_k^{v,+} > h_{cv}$, and h_{cv} is chosen to give the specified in-control performance.

Variable sampling schemes can be applied to the CUSUM chart to improve its performance. Reynolds et al. (1990) first developed VSI CUSUM charts for monitoring the normal process mean. References to VSI CUSUM charts also include Reynolds (1995, 1996b). The VSS CUSUM chart was proposed by Annadi et al. (1995) and the VSSVSI CUSUM chart was evaluated by Arnold and Reynolds (2001).

2.2.3 EWMA Control Charts

Besides the CUSUM chart, another chart that takes past information into account is the EWMA chart proposed by Roberts (1959). The CUSUM chart assigns equal weight to some past data in calculating the chart statistic, while the EWMA chart gives geometrically decreasing weight to all the past data so that more recent samples get larger weight. Lucas and Saccucci (1990) evaluated the performance of EWMA charts using a Markov chain method and pointed out that the CUSUM and EWMA charts have approximately equal performance. There has been a lot of work on the EWMA charts. See Sweet (1986), Domangue and Patch (1991), MacGregor and Harris (1993), Yashchin (1993) and Reynolds and Stoumbos (2005) for more references.

The general form of the one-sided EWMA chart statistic can be expressed as $E_k = (1-\lambda)\max\{0, E_{k-1}\} + \lambda l_k$, where λ is a tuning parameter satisfying $0 < \lambda \leq 1$. This λ is also called the smoothing parameter. The chart signals at time $t_s = min\{k: E_k \geq h_k\}$. A two-sided

EWMA chart can be constructed based on two one-sided EWMA statistics, similar to what was done for the two-sided CUSUM chart above.

For monitoring the normal process mean μ, it is more convenient to construct a two-sided EWMA chart based directly on the sample mean $\bar{X}_k$. The resulting chart statistic E_k^m can be written as

$$E_k^m = (1-\lambda)E_{k-1}^m + \lambda \bar{X}_k ,$$

The starting value of this chart statistic is usually set to be $E_0^m = \mu_0$. A signal is given at sampling point k if E_k^m falls outside of the control limits

$$\mu_0 \pm h_E^m \sqrt{\frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2k}]\frac{\sigma_0^2}{n}},$$

where h_E^m is a parameter used to determine the width of the control limits. It is chosen to give the desired in-control performance. In most applications λ will be chosen to be far enough above 0 that the term $(1-\lambda)^{2k}$ converges to 0 quickly, so the asymptotic control limits can be used and represented as

$$\mu_0 \pm h_E^m \sqrt{\frac{\lambda}{2-\lambda}\frac{\sigma_0^2}{n}}.$$

The performance of the EWMA chart depends on the value of the tuning parameter λ. If λ is small, the EWMA chart is effective for detecting small shifts; if λ is large, the EWMA chart is effective for detecting large shifts; if $\lambda = 1$, the EWMA chart is equivalent to the Shewhart chart as the current sample gets all the weight. It is recommended by Montgomery (2012) that the tuning parameter λ should be chosen in the range of $0.05 \le \lambda \le 0.25$ to quickly detect small to moderate shifts in μ.

For monitoring the normal process variance σ^2, an EWMA chart can be developed based on the sample variance s^2. For monitoring increases in σ^2, the chart statistic E_k^v can be written as

$$E_k^v = (1-\lambda)E_{k-1}^v + \lambda s_k^2 ,$$

where the starting value $E_0^v = 0$. Using the as asymptotic control limit, the chart signal if

$$E_k^v > \sigma_0^2 + h_E^v\left(\frac{\lambda}{2-\lambda}\frac{2}{n-1}\sigma_0^4\right),$$

and h_E^v is chosen is give the specified in-control performance.

Variable sampling schemes have been investigated in EWMA charts in many papers. Saccucci et al. (1992) proposed a VSI EWMA chart and Reynolds (1996b) evaluated the performance of a VSI EWMA chart with sampling at fixed times. The VSSVSI EWMA chart was proposed by Reynolds and Arnold (2001). More references to VSR EWMA charts include Reynolds (1995, 1996a).

2.2.4 GLR Control Charts

Shewhart charts are generally effective for detecting large shifts. CUSUM and EWMA charts can be very effective for detecting shifts of a particular size by specifying the values of the tuning parameters. In most situations, practitioners do not know the possible size of the shift that may occur, and thus may not be sure how to specify the tuning parameters. It is desirable to have some control charts that can effectively detect a wide range of shifts, yet have few parameters in the chart that need to be specified. The GLR chart has been proposed and evaluated as a solution to this problem.

The general expression of the GLR chart can be represented as $R_k = \max_{0\le\tau<k} \max_\theta \sum_{i=\tau+1}^{k} l_i(\theta)$, where τ is the assumed change point, and $l_i(\theta) = \ln\frac{L(\theta|\mathbf{X}_k)}{L(\theta_0|\mathbf{X}_k)}$. The GLR chart signals at time $t_s = min\{k: R_k \ge h\}$.

For the problem of monitoring the normal process mean, the GLR chart is related to the likelihood ratio test of the hypothesis $H_0: \mu = \mu_0$ vs. $H_1: \mu \ne \mu_0$. Instead of specifying the value of μ_1 as must be done in CUSUM charts, the GLR approach use the maximum likelihood (MLE) estimate of μ_1 and considers all the possible change points to find the most likely one. After sampling point k, the GLR chart statistic can be written as

$$R_k = \max_{0\le\tau<k} \max_{\mu_1} \ln\frac{L(\tau,\mu_1|\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_k)}{L(\infty,\mu_0|\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_k)}\,, \qquad (2.1)$$

where $L(\tau,\mu_1|\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_k)$ represents the likelihood function when there is a change in mean from μ_0 to μ_1 between samples τ and $\tau+1$, and $L(\infty,\mu_0|\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_k)$ represents the likelihood function when there is no change in the parameter μ. Note that we have this equation hold

$\sum_{i=\tau+1}^{k} l_i(\mu_1) = \ln \frac{L(\tau,\mu_1|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}{L(\infty,\mu_0|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}$. The GLR chart signals at sampling point k if $R_k > h$, where $h > 0$ is the control limit chosen to achieve the desired in-control performance.

A drawback of the GLR chart is that it requires intensive computation since at each sampling point k the maximization is accomplished over all past observations from the first sample to the k^{th} sample. The situation could get worse if there is no closed form for the MLE of the parameter of interest and an iterative numerical algorithm has to be applied. Although today's modern computational power may be able to handle this issue, in order to improve the algorithm's efficiency and ease the computational burden, a moving window can be applied to the GLR chart as proposed by Willsky and Jones (1976). The GLR chart statistic with a window, modified from equation (2.1), can be written as

$$R_{k,m_1,m_2} = \max_{\max(0,k-m_1)\leq\tau\leq k-m_2} \max_{\mu_1} \ln \frac{L(\tau,\mu_1|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}{L(\infty,\mu_0|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}. \qquad (2.2)$$

When the GLR chart with a window is applied, the values of m_1 and m_2 must be specified. The purpose of using m_1 is to ease the computational burden. The idea is if at sampling point k there has not been any signal indicating that a change has occurred in the process and is m_1 large, it is unlikely that there is a change that occurred before sampling point $(k - m_1)$. Otherwise this change would have been detected. For the value of m_2, it is usually set to be the dimension of the parameter vector of interest (see, for example, Lai (2001)). This guarantees all the parameters of interest can be estimated even though the sample size may be 1 at each sampling point. In some situations the value of m_2 is set to be larger to have more accurate estimates of the parameters of interest.

For monitoring the process mean, it is sufficient to choose m_1 such that about 400 observations can be examined in calculating the FSR GLR chart statistic. The value of m_2 is set to be 1 as the only parameter to be estimated here is μ. Then the GLR chart statistic with a window in equation (2.2) can be represented as

$$R_{k,m_1} = \max_{max(0,k-m_1)\leq\tau< k} \sum_{i=\tau+1}^{k} \frac{\hat{\mu}_{1,\tau,k} - \mu_0}{\sigma_0^2/n} \left(\bar{X}_i - \mu_0 - \frac{\hat{\mu}_{1,\tau,k} - \mu_0}{2}\right),$$

where the MLE of μ_1 is

$$\hat{\mu}_{1,\tau,k} = \frac{\sum_{i=\tau+1}^{k} \bar{X}_i}{(k-\tau)}.$$

For detecting increases in the normal process variance and/or changes in the mean, a similar argument can be used to obtain the GLR chart statistic, which can be written as

$$R'_{k,m_1} = \ln \frac{\max_{max(0,k-m_1)\le\tau<k, -\infty<\mu_1<\infty, \sigma_1^2\ge\sigma_0^2} L(\tau,\mu_1,\sigma_1^2|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}{L(\infty,\mu_0,\sigma_0^2|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}$$

$$= \max_{max(0,k-m_1)\le\tau<k} \frac{n(k-\tau)}{2}\left(\frac{S_{0,\tau,k}^2}{\sigma_0^2} - \frac{S_{\tau,k}^2}{\hat{\sigma}_{1,\tau,k}^2} - \ln\left(\frac{\hat{\sigma}_{1,\tau,k}^2}{\sigma_0^2}\right)\right),$$

where $S_{0,\tau,k}^2 = \frac{\sum_{i=\tau+1}^{k}\sum_{j=1}^{n}(x_{ij}-\mu_0)^2}{n(k-\tau)}$, $S_{\tau,k}^2 = \frac{\sum_{i=\tau+1}^{k}\sum_{j=1}^{n}(x_{ij}-\hat{\mu}_{1,\tau,k})^2}{n(k-\tau)}$, $\hat{\mu}_{1,\tau,k} = \frac{\sum_{i=\tau+1}^{k}\sum_{j=1}^{n}x_{ij}}{n(k-\tau)}$, and $\hat{\sigma}_{1,\tau,k}^2 = \max\{\sigma_0^2, S_{\tau,k}^2\}$. It is sufficient to choose the window m_1 such that about 800 observations can be examined in calculating the FSR GLR chart statistic. The value of m_2 can still be set to be 1. Since we have a lower bound on the estimate of variance component, there is no estimation problem. The GLR chart signals if the statistic is greater than the control limit.

To apply the GLR chart, the only parameter that needs to be specified by practitioners is the control limit h. After a signal is given, the estimated shifted parameter(s) and the estimated change point provide information about the magnitude of the shifted parameter(s) and the time point when a change occurred in the process. This is another advantage of the GLR chart.

We are not aware of any publications on VSR GLR charts. We will focus primarily on the SS GLR chart for monitoring the process mean and the SS GLR chart for monitoring the process mean and variance. In addition, for monitoring the process mean, the VSI GLR chart will also be investigated and compared to the SS GLR chart.

2.3 The Sampling Scheme

The fixed sampling rate scheme requires taking samples with the fixed sample size n using the fixed sampling interval d between samples. The choice of n and d affects both the cost of sampling and the ability of control charts to detect changes in the process. The ratio n/d represents the sampling rate per unit time. Given the same sampling rate, a question of interest is whether we should take large samples using a long sampling interval, or we should take small samples using a short sampling interval. The answer to this question actually depends on the type of control chart being used and the type of parameter change that occurs. Reynolds and Stoumbos (2004b) systematically investigated the sampling plan and control charts for monitoring the normal process mean and variance. For a sampling rate of $n/d = 1.0$ they concluded that $n = 1$ and $d = 1$ gives the best performance for the CUSUM or EWMA chart combinations, while larger values of n and d, such as $n = 4$ and $d = 4$, produces the best performance for the Shewhart charts. In addition, and the best CUSUM or EWMA chart combinations have overall better performance than the best Shewhart chart combination. In our study, we will investigate different values of n and d for the FSR GLR charts. All the sampling plans considered will satisfy $n/d = 1$, so fair comparisons can be made assuming that the sampling cost only depends on the value of n/d.

The variable sampling rate schemes includes VSI, VSS, the VSSVSI combination, and sequential sampling. We will focus on the sequential sampling scheme but other VSR approaches will also be investigated to make a comparison study.

It may be helpful to point out that there is a relationship between the VSS approach and the sequential sampling approach. In both VSS control charts and sequential sampling control charts, the sample size can vary depending on the data from the process. With a VSS control chart, the sample size at the current sampling point is decided by the past data from the process. This sample size is fixed and known before sampling starts at the current sampling point. Therefore, VSS control charts would be appropriate for situations where the sample size needs to be known before sampling starts and it's not feasible to change the sample size during sampling at a sampling point. On the other hand, with a sequential sampling control chart, the sample size at the current sampling point depends on the data at that sampling point as well as data from past sampling points. This sample size is random and not predictable before sampling starts. Therefore, sequential sampling control charts would be appropriate for applications where multiple observations can be taken sequentially at a sampling point whenever needed.

There is also a relationship between VSI control charts and sequential sampling control charts. With VSI control charts, a decision is made to either wait a short time interval or a long time interval to take the next sample. It has been shown that the performance of VSI control charts is better when the short time interval is shorter. The value of the short time interval is typically determined by practical considerations. For example, if it is not possible to take another sample without waiting 10 minutes, the short interval used in a VSI control chart must be at least 10 minutes. On the other hand, with a sequential sampling control chart, it is assumed that the time required to take an observation is so short that it can be neglected. So whenever there is an indication of a possible change in the process, another observation can be obtained immediately. Therefore, sequential sampling could be considered as a VSI control chart with the short time interval being zero. Samples are always taken at fixed sampling points where the time between any two consecutive sampling points is the long time interval. Note that this indicates changes would presumably occur only between samples, not within a sample. Assuming zero time between observations is a "best case" situation in the sense that if the time is some positive number, even if it is close to the zero, then the performance of the SS chart would be slightly worse than when the time is zero. A positive time between observations means that the SS chart is closer to a VSI chart. See Stoumbos and Reynolds (1996) for more investigation of this issue.

As the sample size is a random variable in the sequential sampling scheme and samples are taken at fixed time points, the ratio of the average sample size over the sampling interval d is considered as a measure of the sampling cost. We will investigate different sampling patterns for the SS GLR chart, but all these sampling patterns will satisfy the constraint that the ratio of the average sample size and the sampling interval d is 1, so fair comparisons can be conducted. Note in practice this is not a requirement for practitioners to use the control charts. The ratio of the average sample size over the sampling interval can be any number as desired.

2.4 Performance Metrics

The objective of using a VSR technique such as sequential sampling is to improve the control chart's statistical efficiency. Two general approaches to using this improved efficiency have been discussed by Baxley (1995) and Reynolds (1996a, 1996b). One approach is to increase the ability to detect process changes while maintaining the same sampling cost. The other is to reduce the sampling cost while maintaining the same ability to detect process changes. In this dissertation, the average sampling costs of different control charts are set to be equal and their abilities to detect process changes are compared.

2.4.1 Measures for In-Control Processes

When the process is in control, any signal generated by a control chart is a false alarm. The false alarm rate can be measured by the reciprocal of the expected time required to generate a signal. Define the average time to signal (ATS) to be the expected length of time from the starting point of process monitoring to the time that the chart signals. Also define the average number of samples to signal (ANSS) to be the expected number of samples from the starting point to the time that the chart signals. With a FSI control chart, the ATS is the product of the fixed sampling interval length and the ANSS, that is, $\text{ATS} = d \times \text{ANSS}$. The in-control ATS provides a measure of the false alarm rate. The control charts being compared are required to have the same in-control ATS.

When sequential sampling is used, the number of observations used at each sampling point is a random variable. So we need to look at the expected number of observations in order to measure the expected sampling rate. Define the average number of observations to signal (ANOS) to be the expected number of observations from the starting point of process monitoring to the time that the signal is generated. With a FSS control chart, the ANOS is the product of the fixed sample size and the ANSS. The ratio ANOS/ATS is a measure of the average sampling rate per unit time. The ratio measuring the sampling rate will usually be of most interest when the process is in control. If different control charts are required to have the same in-control ATS and ANOS, then the in-control average sampling rates and false alarm rates for these charts will be the same.

In sequential analysis the expected sample size of a sequential procedure is usually called the average sample number (ASN). In our application of sequential sampling to process monitoring we need the expected sample size at each sampling point, so we want the ASN at each sampling point. However, this ASN may not be constant across different samples because the sample size used at a sampling point depends on past samples as well as the current sample. Thus here we define the ASN to be ASN = ANOS/ANSS, which is in effect an average sample size averaged across all samples before the signal.

2.4.2 Measures for Out-of-Control Processes

When there is a shift in process, the ATS is appropriate to measure the detection time for this shift if it occurs at the time that monitoring starts. However, in many applications, it will often be more realistic to consider that the shift occurs at some unpredictable time after monitoring starts. For these applications, the detection time should be computed as the expected time from the random point in time that the shift occurs until a signal is generated. The steady-state ATS (SSATS) is appropriate for these applications. It is the expected time from the shift to signal, which assumes that the control statistic has reached its steady-state or stationary distribution by the random time point that the shift occurs. One also assumes that the probability of the shift occurring in a sampling interval of a particular length is proportional to the product of the interval's length and the stationary probability of this interval being used, where the stationary probability is conditional on no false alarm before the shift occurs. In addition, one assumes that when the shift occurs in a sampling interval, the position of the shift is uniformly distributed over the interval.

Similarly, the steady-state ANSS (SSANSS) can be defined as the expected number of samples from the time point that the shift occurs to the time that a signal is given. It might be useful to point out the relation between SSATS and SSANSS, where $\mathrm{SSATS} = \mathrm{SSANSS} \times d - d/2$. Define the steady-state ANOS (SSANOS) as the expected number of observations from the time point that the shift occurs to the time that a signal is given. Let SSASN denote the steady-state ASN as the average sample size after the chart statistic has reached its steady state distribution, so we have $\mathrm{SSASN} = \mathrm{SSANOS}/\mathrm{SSANSS}$.

The goal of using a VSR control chart instead of a FSR control chart is to detect process changes more quickly. That is, we want to detect changes as quickly as possible by increasing the sampling rate whenever there is an indication of possible changes in process. Therefore, the SSATS is used as the primary performance measure of different control charts, rather than the SSANOS. The time to signal is our most important concern, not the number of observations to signal.

The number of observations to signal is often associated with the sampling cost. Thus, the SSANOS gives us an idea about the sampling cost. In case people are interested in this, we also present some SSANOS results so that people can have a general idea about the number of observations needed to signal.

The size of the shift in μ is expressed in terms of the standardized shift δ, where $\delta = (\mu - \mu_0)/\sigma_0$. The range of the shift size δ considered in this dissertation is from 0.25 to 7.00. The size of the shift in σ is expressed in terms of $\psi = \sigma/\sigma_0$, and the range of ψ is from 1.1 to 15.0. Some different ranges were also considered, but the results are not shown here since they lead to consistent conclusions.

2.4.3 Extra Quadratic Loss

When the performance of several control charts is compared for a wide range of shift sizes, if one chart has uniformly smaller SSATS than the other charts for all the shifts, then the conclusion is easily to draw. However, if one chart has better performance for some particular shifts, but another has better performance for some other shifts, it's not clear that which chart gives the best overall performance based on the SSATS values for each particular shift. Therefore, it is desirable to have a measure that could be used to compare different control charts over a wide range of shift sizes.

In order to have a measure for comparing different control charts, we should consider (1) the likelihood that a shift of a particular size occurs, (2) the time required to detect this shift, (3) the loss per unit time when the process is being operated at this shift. For example, small shifts are generally more likely to occur than large shifts. Large shifts can presumably be detected faster than small shifts. The loss per unit time associated with operating a process at a large shift would be higher than that at a small shift. All these factors should be considered.

Since some shifts are presumably more likely to occur than others, some prior distributions $\pi(\delta)$ and $\pi(\psi)$ can be used to represent the likelihood that shifts of size δ and ψ occur, respectively. A number of prior distributions were considered, but in this dissertation we only present the results from an exponential prior distribution with parameter 1, which can be denoted as EXP(1). This EXP(1) prior distribution will be transformed to have support in the interval [0.25, 7.00] for the mean shift, and [1.10, 15.0] for the variance shift. It seems reasonable to use an exponential distribution to account for the likelihood of shifts with different sizes occurring, because small shifts are more likely to occur than large shifts. The effect of using different prior distributions will be discussed in a later section.

The time required to detect a particular shift can be measured by the SSATS. For the loss produced from operating the process at this shift, we use the quadratic loss. The quadratic loss has been used to evaluate control charts in many papers. See, for example, Domangue and Patch (1991), Reynolds and Stoumbos (2004b), and Wu et al. (2008).

Let $X(t)$ represent the process variable X at time t after the start of the shift, and let T represent the time until a signal is obtained after the start of the shift. The standardized quadratic loss when the process is out of control with mean μ and variance σ^2 is defined to be

$$E\left(\left[\frac{(X(T)-\mu_0)}{\sigma_0}\right]^2\right) = \frac{\sigma^2}{\sigma_0^2} E\left(\left[\frac{(X(T)-\mu)}{\sigma}\right]^2\right) + \left[\frac{(\mu-\mu_0)}{\sigma_0}\right]^2 = \psi^2 + \delta^2.$$

The standardized quadratic loss when the process is in-control is 1. Thus the extra standardized quadratic loss is $\psi^2 + \delta^2 - 1$. The total extra loss resulting from the process being operated at the shift size (δ, ψ) can be represented as

$$\mathrm{EQL}(\delta, \psi) = E\left(\int_0^T (\psi^2 + \delta^2 - 1) dt\right) = (\psi^2 + \delta^2 - 1)E(T) = (\psi^2 + \delta^2 - 1)\mathrm{SSATS}.$$

Consider three types of shifts in μ and σ^2: shifts in μ alone, shifts in σ^2 alone, and shifts in both μ and σ^2. Let the prior distribution $\pi(\delta)$ measure the likelihood of a shift in μ to δ, $\pi(\psi)$ measure the likelihood of a shift in σ^2 to ψ, and $\pi(\delta, \psi) = \pi(\delta)\pi(\psi)$ measure the likelihood of a shift in both μ and σ^2 to (δ, ψ). We have

$$\mathrm{EQL}_\mu = \int \mathrm{EQL}(\delta, 1)\pi(\delta)\, d\delta, \tag{2.3}$$

$$\mathrm{EQL}_\sigma = \int \mathrm{EQL}(0, \psi)\pi(\psi) d\psi, \tag{2.4}$$

$$\mathrm{EQL}_{\mu,\sigma} = \int \int \mathrm{EQL}(\delta, \psi)\pi(\delta, \psi) d\delta\, d\psi. \tag{2.5}$$

For the mean shifts only, EQL_μ is used. For the mean shifts and/or variance shifts, we combine equations (2.3) – (2.5) to have the overall performance measurement

$$\mathrm{EQL} = w_\mu \mathrm{EQL}_\mu + w_\sigma \mathrm{EQL}_\sigma + w_{\mu,\sigma}\mathrm{EQL}_{\mu,\sigma}, \tag{2.6}$$

where the weights $w_\mu = w_\sigma = w_{\mu,\sigma}$=1/3 are used here. Note that if we only use $\text{EQL}_{\mu,\sigma}$, the probability of shifts in μ or σ alone is zero. Numerical integration was used to calculate equation (2.3) – (2.6).

The evaluation of the ATS, ANOS, SSATS, and EQL of all charts considered in the dissertation were done using simulation with 1,000,000 runs. The SSATS was obtained by simulating the control chart with 400 in-control observations, and then introducing a shift in μ and/or σ^2. Without loss of generality, we set the in-control average sampling rate to be 1. That is, we take an average of one observation per unit time. Then the in-control ATS and ANOS would be equal, and they were set to be 1481.6, which allows for comparisons with results from recent papers by Reynolds and Lou (2010) and Reynolds et al. (2013). The value of the sampling interval d was the same as the ASN, given the average sampling rate being 1. The effect of d and ASN will be investigated in more detail in a later section. Some other values of those chart parameters are also considered later in order to compare with other existing methods.

Chapter 3. The GLR Chart with Sequential Sampling for Monitoring the Process Mean

3.1 Derivation of the GLR Chart with Sequential Sampling

Suppose at sampling point k we have data $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$. Consider the alternative hypothesis that a shift in the mean from μ_0 to some unknown μ_1 has occurred at some time between samples τ and $\tau + 1$, where $\tau < k$ and $\mu_1 \neq \mu_0$. This assumes that the shift can only occur between samples, not within samples. Then under the alternative hypothesis the likelihood function at observation l in the current sample k is

$$L(\tau, \mu_1 | \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k)$$
$$= (2\pi\sigma_0^2)^{-\frac{N}{2}} \times \exp\left(-\frac{1}{2\sigma_0^2}\left(\sum_{i=1}^{\tau}\sum_{j=1}^{n_i}\left(X_{ij} - \mu_0\right)^2 + \sum_{i=\tau+1}^{k-1}\sum_{j=1}^{n_i}\left(X_{ij} - \mu_1\right)^2 + \sum_{j=1}^{l}\left(X_{kj} - \mu_1\right)^2\right)\right),$$

where $N = \sum_{i=1}^{k-1} n_i + l$ is the current total number of observations. Let $N_1 = \sum_{i=\tau+1}^{k-1} n_i + l$ be the number of observations used to estimate μ_1, so that the maximum likelihood estimator of μ_1 is

$$\hat{\mu}_{1,\tau,k,l} = \frac{\sum_{i=\tau+1}^{k-1}\sum_{j=1}^{n_i} X_{ij} + \sum_{j=1}^{l} X_{kj}}{N_1}. \tag{3.1}$$

Under the null hypothesis that there has been no shift in mean, the likelihood function at sample k can be represented as

$$L(\infty, \mu_0 | \mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k) = (2\pi\sigma_0^2)^{-\frac{N}{2}} \times \exp\left(-\frac{1}{2\sigma_0^2}\left(\sum_{i=1}^{k-1}\sum_{j=1}^{n_i}\left(X_{ij} - \mu_0\right)^2 + \sum_{j=1}^{l}\left(X_{kj} - \mu_0\right)^2\right)\right).$$

Then a log likelihood-ratio statistic for determining whether there has been a shift in μ in the past k samples is

$$R_{k,l} = \ln \frac{\max\limits_{0 \le \tau < k, -\infty < \mu_1 < \infty} L(\tau, \mu_1 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)}{L(\infty, \mu_0 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)}$$

$$= \max_{0 \le \tau < k} \frac{(\hat{\mu}_{1,\tau,k,l} - \mu_0)}{\sigma_0^2} \left(\sum_{i=\tau+1}^{k-1} \sum_{j=1}^{n_i} \left(X_{ij} - \frac{1}{2}(\hat{\mu}_{1,\tau,k,l} + \mu_0) \right) + \sum_{j=1}^{l} \left(X_{kj} - \frac{1}{2}(\hat{\mu}_{1,\tau,k,l} + \mu_0) \right) \right). \quad (3.2)$$

Using Equation (3.1), $R_{k,l}$ in Equation (3.2) can be represented as

$$R_{k,l} = \max_{0 \le \tau < k} \frac{N_1}{2\sigma_0^2} (\hat{\mu}_{1,\tau,k,l} - \mu_0)^2. \quad (3.3)$$

Based on $R_{k,l}$, we need to keep track of all past data from samples 1 to k and find the maximum statistic, which requires a lot of computation. A window with size m can be applied so the maximization of $R_{k,l}$ is taken only over the past m samples. This m is equivalent to the m_1 that described in Section 2.2.4, while m_2 is set to be 1 here. The choice of the window size m will be discussed in a later section. With a size m window, $R_{k,l}$ in Equation (3.3) reduces to

$$R_{m,k,l} = \max_{\max(0,k-m) \le \tau < k} \frac{N_1}{2\sigma_0^2} (\hat{\mu}_{1,\tau,k,l} - \mu_0)^2. \quad (3.4)$$

The decision rule for the SS GLR control chart requires the specification of two limits g and h, satisfying $0 < g < h$, where h is called control limit and g is called warning limit. The decision rule is defined as follows. If the current sampling point is k and the current number of observations at this sampling point is l:

1. Compute $R_{m,k,l}$.
2. If $R_{m,k,l} \le g$, then stop sampling at sampling point k and wait until sampling point $k+1$ to sample again. Then the sample size at sampling point k is $n_k = l$.
3. If $g < R_{m,k,l} \le h$, then
   (a) take the next observation at sampling point k, and
   (b) go to step 1 above with the current number of observations at sampling point k defined as $l = l + 1$.
4. If $R_{m,k,l} > h$, then signal that there has been a change in μ.

The control chart parameters g and h can be chosen to meet specified in-control performance requirements.

3.2 The Effect of the Window Size

First we investigate the effect of the window size m. For the GLR chart with fixed sampling rate which is denoted as the FSR GLR chart, the window size is usually chosen to be large enough such that the performance of the GLR chart with this window is essentially the same as that of the GLR chart without a window because this gives the best overall performance. See Reynolds and Lou (2010) for more details. However, for the SS GLR chart, we found that a much smaller window size is preferred.

Table 3.1: SSATS and EQL values of SS GLR charts with different window sizes

SS GLR Charts, ASN=1.5, d=1.5								
m=	1	5	8	10	15	50	100	267
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0.00	1481.37	1481.74	1481.80	1481.60	1481.69	1481.66	1481.59	1481.57
0.25	136.86	103.37	94.04	90.03	83.66	71.86	69.99	71.11
0.50	27.96	20.69	19.50	19.09	18.63	18.83	19.42	20.21
0.75	11.02	8.66	8.46	8.45	8.52	9.04	9.37	9.76
1.00	5.79	4.89	4.91	4.96	5.07	5.43	5.62	5.86
1.50	2.46	2.35	2.42	2.45	2.51	2.70	2.79	2.90
2.00	1.43	1.17	1.52	1.54	1.57	1.68	1.73	1.79
3.00	0.85	0.89	0.90	0.91	0.92	0.95	0.96	0.99
4.00	0.76	0.76	0.77	0.77	0.77	0.77	0.78	0.78
5.00	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
6.00	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
7.00	0.75	0.75	0.75	0.75	0.75	0.75	0.75	0.75
EQL	6.81	5.86	5.78	5.77	5.78	5.96	6.11	6.32
h=	6.8558	7.0251	7.0402	7.0449	7.0482	7.0483	7.0440	7.0348
g=	1.0748	1.5142	1.6232	1.6718	1.7555	2.0647	2.1466	2.1755

Consider the case in which the in-control ASN is set to be 1.5 (it will be shown later that this is a good choice). That is, when the process was in-control, sometimes only one observation will be taken at a sampling point, and sometimes more than one observation will be taken, but on average 1.5 observations will be taken at each sampling point. Table 3.1 gives the SSATS values and the EQL values for mean shifts for different values of m. The control limit h and warning limit g were chosen such that the in-control ATS (shown in the row corresponding to $\delta = 0$) and ANOS were equal or very close to 1481.6.

In Table 3.1 we can see that the EQL as a function of m is relatively flat around the minimum which appears to be around 10. Generally, large window sizes can help to detect small shifts quickly, but hurt the detection rate for median to large shift sizes. Small window sizes can help to detect median to large shift sizes, but cause slower detection for small shift sizes. For very large shifts the window essentially has no effect. Overall the window size 10 achieves a good balance, and gives an EQL value close to optimal in this case.

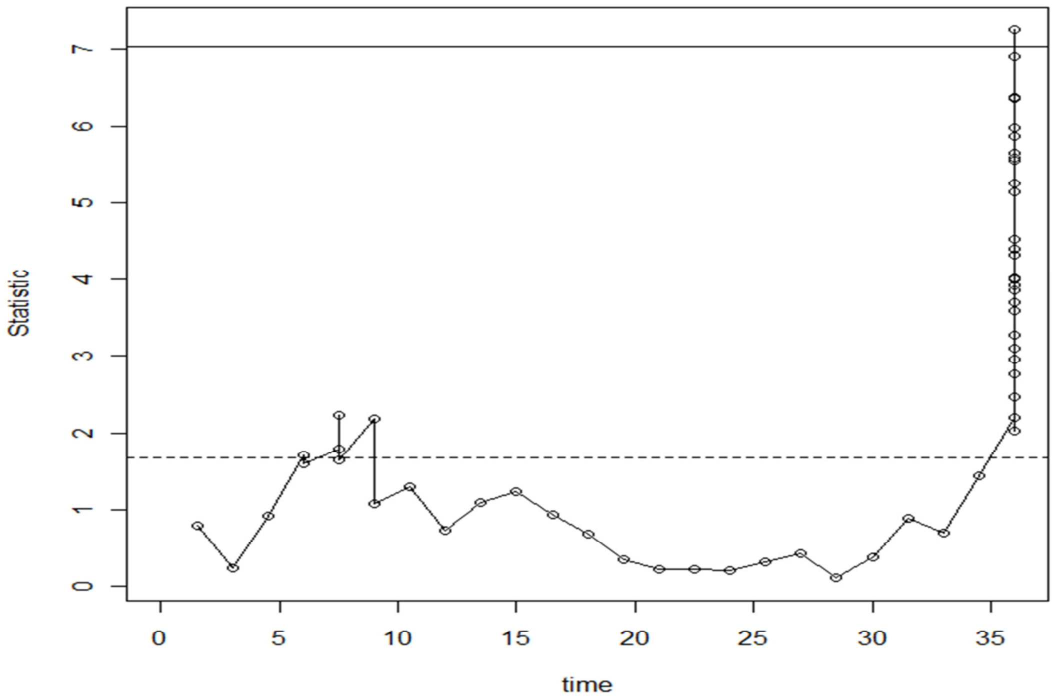

Figure 3.1: Plot of the GLR chart statistic with sequential sampling

One natural question raised here is why a small window size gives the best performance for the SS GLR chart, while a large window size is best for the FSR GLR chart. To help answer this question, let us look at a plot of the SS GLR chart.

Figure 3.1 shows a trace of the GLR chart statistic with sequential sampling based on one simulation run. Here we consider the in-control ASN of 1.5, and samples were taken every 1.5 time units. The process was in-control until time point 30, and then a mean shift to $\delta = 0.5$ was introduced. The solid horizontal line is the control limit h, and the dash horizontal line is the warning limit g. They were set to give in-control ATS and ANOS values of 1481.6. The window size here is $m = 10$.

In Figure 3.1, points connected with a vertical line indicate that they were from the same sample. For example, the first sample taken at time point 1.5 contained 1 observation, the second and third sample taken at time point 3 and 4.5 also contained 1 observation each, the fourth sample taken at time point 6 contained 2 observations, and so on. When the process was in control (before time 30), most of the samples were of size $n_i = 1$ with a few with $n_i > 1$. However, once the process went out of control (after time 30), $R_{m,k,l}$ quickly moved above g and the sampling rate increased drastically as shown in the plot. As long as there was any indication in the process data that a change might have occurred at current sampling point, more observations were taken until either there was a signal or there was evidence that there was no process change and a new sample was taken at the next sampling point. Therefore, even with a small window size, once there is any indication of a process change, more observations will be sampled as needed. The goal of using a large window size in the FSR GLR chart is to use the information from a large number of observations to detect process changes quickly. In the SS GLR chart, the information from a large number of observations would still be used even if the window size is small, as illustrated in Figure 3.1.

Comparing our Table 3.1 with Table 1 in Reynolds and Lou (2010) in the FSR case shows that the effect of m is the same in both tables for very small and very large shifts: a large m is needed to detect very small shifts, and m has essentially no effect if the shift is very large. The difference in the two tables seems to be for small to intermediate shifts such as $\delta = 0.50$. In the FSR case a relatively large m such as $m = 50$ is needed to detect this shift, while in the sequential sampling case $m = 10$ is sufficiently large. When we average over δ with a prior distribution we find that the optimal m is around $m = 10$ because it is good for intermediate shifts.

The above Figure 3.1 shows how sequential sampling works and graphically illustrates that the sampling rate goes up when there is a shift so that a relatively small m is best with sequential sampling. Now we quantitatively explain why $m = 10$ is sufficiently large to detect a shift of $\delta = 0.50$ by looking at the expected number of observations in the window when the shift occurs. In the FSR GLR chart using $m = 50$ means that we are looking back at 50 observations whether the process is in control or out of control. In the SS GLR chart with ASN = 1.5, using $m = 10$ means that 15 observations on average were sampled when the process is in-control. When there is a shift of $\delta = 0.50$, the SSASN goes up significantly and an average of 41.81 observations are in the window of size 10. Many more observations are sampled on average when there is a shift in the process and this number could be even larger than 50 depending on the process data. Therefore, a small window size is preferred for the SS GLR chart. Since the best value for m actually depends on the value of the ASN, we will recommend an exact window size in a later section.

3.3 The Effect of the In-control ASN

Now we consider the effect of the in-control ASN and the choice of the sampling interval d. We are adjusting the unit of time, without loss of generality, so that the average in-control sampling rate per unit time is 1, and d is equal to the value of the in-control ASN. Note in practice the value of the in-control ASN and d could certainly be different. Here for evaluation purposes we set them to be the same so fair comparisons can be made, as all the charts have the same in-control sampling rate of 1. The question is should we sample a relatively large number of observations at each sampling point with a long time interval between samples, or should we sample a relatively small number of observations at each sampling point with a short time interval. Besides, for the same in-control ASN, different window sizes would lead to different performance as shown in Table 3.1. This means the effect of the ASN depends on the window size and therefore we need to investigate them simultaneously.

Table 3.2 shows the SSATS and EQL values for different values of the in-control ASN, each with its corresponding optimal window size. The optimization here was done by minimizing the EQL value with the EXP(1) as the prior distribution. For example, the window size 10 gives the minimum EQL value for an in-control ASN of 1.5 (with $d = 1.5$), and the window size 25 gives the minimum EQL value for an in-control ASN of 1.2 (with $d = 1.2$). We noticed that the smaller the in-control ASN, the larger the window size required to achieve the best performance. This makes sense because decreasing the in-control ASN would decrease the number of observations used to calculate the GLR statistic if m is fixed. For small shifts a large number of observations is needed to detect changes quickly, so the window size should be increased as the ASN decreases to maintain a large average number of observations in the window.

Table 3.2: SSATS and EQL values of SS GLR charts for different ASN values with the corresponding optimal window size

SS GLR charts									
ASN, d =	1.2	1.3	1.4	1.5	1.6	1.7	2	3	4
m =	25	19	16	10	9	8	6	5	4
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
0.00	1481.47	1481.44	1481.67	1481.60	1481.62	1481.67	1481.60	1481.60	1481.56
0.25	106.21	96.28	89.35	90.03	85.86	83.18	78.78	76.10	81.50
0.50	21.30	19.95	19.20	19.09	18.80	18.70	18.82	20.72	23.62
0.75	9.52	8.92	8.63	8.45	8.41	8.42	8.65	10.12	11.82
1.00	5.58	5.25	5.10	4.96	4.96	4.98	5.15	6.23	7.41
1.50	2.70	2.56	2.51	2.45	2.46	2.50	2.63	3.32	4.50
2.00	1.64	1.57	1.56	1.54	1.56	1.59	1.71	2.26	2.83
3.00	0.86	0.86	0.88	0.90	0.94	0.98	1.12	1.61	2.11
4.00	0.64	0.68	0.72	0.76	0.82	0.86	1.01	1.51	2.01
5.00	0.60	0.65	0.70	0.75	0.80	0.85	1.00	1.50	2.00
6.00	0.60	0.65	0.70	0.75	0.80	0.85	1.00	1.50	2.00
7.00	0.60	0.65	0.70	0.75	0.80	0.85	1.00	1.50	2.00
EQL	6.16	5.89	5.78	5.77	5.80	5.87	6.20	7.76	9.49
h =	7.1994	7.1462	7.0959	7.0449	6.9991	6.9561	6.8338	6.5088	6.2611
g =	2.5611	2.1780	1.9268	1.6718	1.5320	1.4141	1.1629	0.8264	0.6632

The optimal EQL value in Table 3.2 is achieved by using an in-control ASN of 1.5 with the window size 10 in column [4]. The associated sampling interval was equal to 1.5, which means samples are taken every 1.5 time units. The EQL as a function of the in-control ASN and

m are relatively flat around column [4]. As the in-control ASN increases above 3, the EQL keeps getting larger.

To illustrate how to use these results in practice, consider the situation in which a Shewhart $\overline{X}$ chart is being used to monitor a process with samples of $n = 4$ being taken every $d = 2$ hours. The sampling rate per unit time is 2 observations per hour. Note that if we wanted to adjust the time unit to give a sampling rate of 1 per time unit, we could take the time unit to be 0.5 hour, but this is not really necessary in practice. For the Shewhart chart two other sampling schemes with the same sampling rate per unit time are samples of $n = 2$ taken every $d = 1$ hour, and samples of $n = 8$ taken every $d = 4$ hours. The original sampling scheme may actually be the best choice when a Shewhart chart is being used. However, with the SS GLR chart better overall performance will be achieved by using a short sampling interval and a low in-control ASN. Two possibilities with an average in-control sampling rate of 2 observations per hour are $d = 1.0$ with an ASN of 2, and $d = 3/4$ with an ASN of 1.5. Table 3.2 shows that $d = 3/4$ with an ASN of 1.5 would be a great choice. Note that the results in Table 3.2, where $d =$ ASN, show that $d = 1.5$ and ASN$= 1.5$ is a good choice. That can be translated to the current case to conclude that $d = 0.75$ and ASN$= 1.5$ is a good choice as well. However, using a sampling interval of $d = 3/4$ may be inconvenient, so $d = 1.0$ with an ASN of 2.0 might be chosen in practice. Column [7] of Table 3.2 shows that the overall performance of $d = 1.0$ with an ASN of 2 is still very good. The performance of the Shewhart chart will be compared with the SS GLR chart in a later section.

A potential complication here is that we may need to adjust the window size for different values of the in-control ASN. So a question is whether we can use one window size for different values of the in-control ASN and still have good performance.

The performance of the SS GLR chart for different in-control ASN values with a fixed window size of $m = 10$ are displayed in Table 3.3. The in-control ATS and ANOS were set to be 1481.6. The SSATS and EQL values for this window size 10 are shown first in Table 3.3, then the optimal window size m_{opt} for those different in-control ASN values are displayed as well as the corresponding EQL_{opt}. By comparing EQL values for the window size 10 and EQL_{opt} values, we can see exactly how the performance changes from the optimal window to the fixed window.

Table 3.3: The SSATS and EQL values of SS GLR charts for different ASN values with window size 10

SS GLR charts									
ASN, d =	1.2	1.3	1.4	1.5	1.6	1.7	2	3	4
m =	10	10	10	10	10	10	10	10	10
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
0.00	1481.67	1481.71	1482.05	1481.60	1481.41	1481.30	1481.60	1481.80	1481.30
0.25	140.04	113.17	98.71	90.03	84.24	80.31	74.20	72.48	77.45
0.50	24.10	21.29	19.86	19.09	18.67	18.46	18.39	20.51	23.36
0.75	9.59	8.91	8.60	8.45	8.42	8.43	8.70	10.22	11.97
1.00	5.36	5.09	4.98	4.96	4.98	5.02	5.26	6.38	7.61
1.50	2.54	2.45	2.44	2.45	2.48	2.53	2.70	3.41	4.17
2.00	1.54	1.51	1.51	1.54	1.57	1.61	1.76	2.31	2.90
3.00	0.82	0.84	0.87	0.90	0.95	0.99	1.13	1.62	2.12
4.00	0.63	0.67	0.72	0.76	0.82	0.86	1.01	1.51	2.01
5.00	0.60	0.65	0.70	0.75	0.80	0.85	1.00	1.50	2.00
6.00	0.60	0.65	0.70	0.75	0.80	0.85	1.00	1.50	2.00
7.00	0.60	0.65	0.70	0.75	0.80	0.80	1.00	1.50	2.00
EQL	6.36	5.95	5.80	5.77	5.80	5.87	6.22	7.86	9.58
m_{opt} =	25	19	16	10	9	8	6	5	4
EQL$_{opt}$=	6.16	5.89	5.78	5.77	5.80	5.87	6.20	7.76	9.49
h =	7.1719	7.1325	7.0885	7.0449	7.0002	6.9578	6.8370	6.5069	6.2596
g =	2.3337	2.0289	1.8235	1.6718	1.5534	1.4564	1.2491	0.9092	0.7491

In Table 3.3 we can see that when the same window size is used for different values of in-control ASN, the performance of the SS GLR chart is still quite good. The EQL values are almost as good as the $\mathrm{EQL_{opt}}$ values. We also notice the consistent results as in Table 3.2 that a large in-control ASN leads to better performance for small shifts, but is worse for medium to large shifts. Based on these EQL values we recommend choosing the in-control ASN to be 1.5 or as close to 1.5 as possible, and choosing the window size to be 10. This setting gives the overall best performance of the SS GLR chart. We recommend against using large in-control ASN values such as 3 or 4, because with sequential sampling there is no need to use a large sample size when the process is in-control, and whenever there is any indication of process changes the sequential sampling scheme allows more observations to be obtained as needed.

The recommendation above is mainly based on the EQL results from the EXP(1) prior distribution. A natural question is how much these conclusions about m and the in-control ASN would change if a different prior distribution is used. Our simulation results (not displayed here) show that for different prior distributions the optimal window sizes are around 10 and the optimal in-control ASN values are between 1.1 and 1.5. The differences between the EQL value resulting from the optimal parameter setting and the EQL value resulting from the recommended parameter setting are very small, which indicates that the recommended parameter settings produce optimal or almost optimal EQL values. Thus we recommend setting the in-control ASN as 1.5 and window size as 10.

3.4 The Limits for Designing the GLR Chart with Sequential Sampling

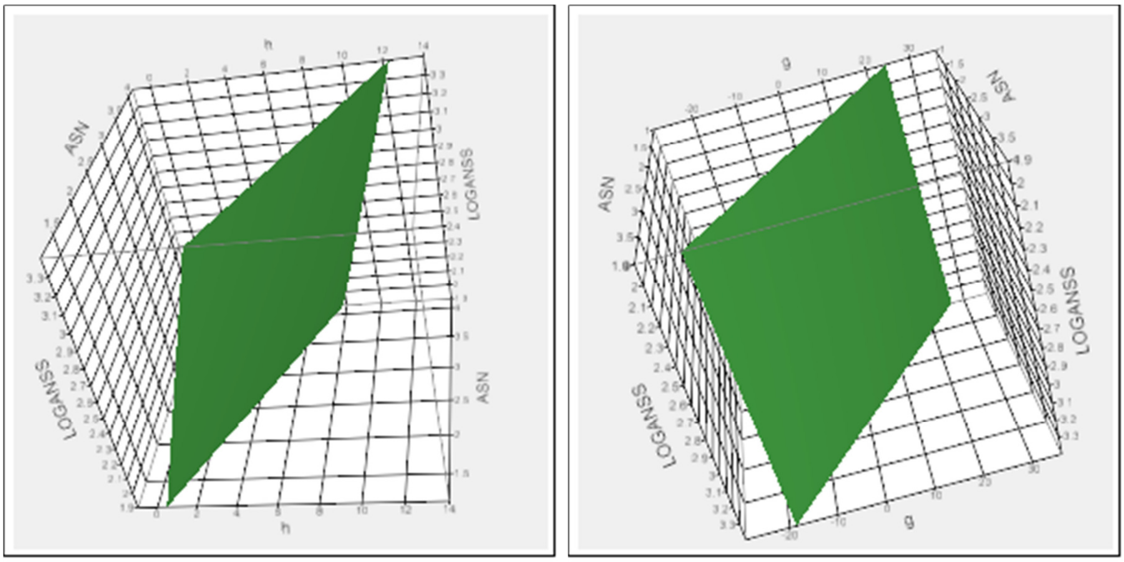

Figure 3.2: The limits h and g of the SS GLR chart as functions of the in-control ANSS and ASN.

There are two limits that need to be specified in the SS GLR chart, the control limit h and the warning limit g. These two limits determine the in-control ANSS and in-control ASN. The window size is set to be 10 as recommended.

Figure 3.2 shows plots of the limits h and g as functions of the in-control ANSS (in log scale) and the in-control ASN. It shows that both h and g are very close to linear functions of the log of the in-control ANSS and in-control ASN.

The regression equations below are provided for actually finding h and g for specified values of the in-control ASN and ANSS. Some higher order terms were included to give a more accurate fit. In these equations a represents ASN and l represents $\log_{10}$ ANSS.

$$h = -2.836582 + 2.436644a - 1.078886a^2 + 0.246254a^3 - 0.021711a^4 + 3.017296l - 0.069715l^2 - 0.180193al + 0.011538a^2l + 0.018808al^2 \quad (3.5)$$

$$g = 9.105755 - 15.116605a + 8.930549a^2 - 2.273445a^3 + 0.208830a^4 + 1.541953l - 0.184280l^2 - 0.427230al + 0.053443a^2l + 0.026885al^2 \quad (3.6)$$

Once a practitioner has some specific values for the in-control ANSS and ASN, the above equations can be used to quickly obtain the limits for the SS GLR chart. The sampling interval d can be chosen to give the desired in-control ATS or the desired in-control sampling rate. If a practitioner specifies the sampling interval d and the in-control ATS and ASN, then the in-control ANSS can be calculated using the in-control ATS$/d$, and the limits h and g can be found similarly. An illustrative example showing the use of the regression equations will be provided in next section.

The equations for h and g were obtained by fitting the regression equations using ASN values ranging from 1.2 to 4 and in-control ANSS values ranging from 80 to 2300. The coefficients of determination (model R^2) for both equations are greater than 0.999. Thus, using the above equations should give accurate values of h and g when the desired in-control ASN and ANSS are in those ranges.

3.5 An Illustrative Example

Consider a situation in which an FSR Shewhart $\overline{\mathrm{X}}$ chart is being used and testing is expensive and destructive. We consider several scenarios where performance can be improved using the SS GLR chart. In one scenario the objective is to reduce the time required to detect shifts, and we show the results for two choices of the sampling interval. Then we consider the scenario in which the objective is to reduce sampling costs. In each case we show how to set up the SS GLR chart.

Suppose the data from the process follow the distribution $N(\mu_0, \sigma^2)$. Currently the company is sampling $n = 4$ items every 4 hours and using the FSR Shewhart $\overline{\mathrm{X}}$ chart to monitor the process mean. The manager wants to reduce the time required to detect special causes so that less nonconforming items would be produced. Preliminary studies show that individual items can be sampled and tested in a fairly short time. The objective is to design a chart that will have a relatively low false alarm rate, will perform well for a wide range of shift sizes, and will use the same sampling times as the previously employed $\overline{\mathrm{X}}$ chart. The SS GLR chart is to be employed. The sampling interval is $d = 4$ hours, and the in-control ASN is set to be 4 to correspond to the Shewhart $\overline{\mathrm{X}}$ chart. This ASN value is actually larger than is recommended, but will be considered here for purposes of illustration. The window size is set to be 10, which is the recommended window size. The desired in-control ATS is set to be 1481.6, which means the in-control ANSS is set to be $\mathrm{ATS}/d = 370.4$. This corresponds to the 3-sigma limits used in the Shewhart $\overline{\mathrm{X}}$ chart. Here the in-control average sampling rate is 1 item per hour. Using the provided regression equations (3.5) and (3.6), the limits obtained are $h = 6.2596$ and $g = 0.7494$.

Table 3.4 compares the SSATS and EQL values of the SS GLR chart and the FSR Shewhart $\overline{\mathrm{X}}$ chart. The result for SS GLR chart with the above setting is given in column [2]. The h and g values obtained from the equations give an in-control ATS value quite close to the desired value. The SS GLR chart has much better performance than the FSR Shewhart $\overline{\mathrm{X}}$ chart for small shifts, but the $\bar{X}$ chart is slightly better for moderately large shifts. The EQL values indicate that overall the SS GLR chart is much better than the FSR Shewhart $\overline{\mathrm{X}}$ chart.

Given the same in-control sampling rate, the SS GLR chart will have better overall performance by using a lower in-control ASN and a shorter sampling interval. Thus instead of taking an average of 4 items every 4 hours when the process is in control, one can set the in-control ASN to be 1.5. This means that sometimes only 1 item is sampled and sometimes 2 or more items are sampled, but on average 1.5 items are taken at each sampling point. The sampling interval is then adjusted to be $d = 1.5$ to maintain the same in-control sampling rate. The in-control ANSS is equal to $\mathrm{ATS}/d = 987.73$ and the in-control ASN is 1.5. Equation (3.5) and (3.6) can be used again to obtain the limits $h = 7.0439$ and $g = 1.6766$. Column [3] in Table

3.4 shows the SSATS values of this SS GLR chart. Although the performance is a bit worse than the SS GLR chart in column [2] for the very small shifts, it is better for moderate and large shifts. Overall the SS GLR chart in column [3] has better performance than column [2], and in fact it has a smaller EQL value. The performance of the SS GLR chart in column [3] is uniformly better than that of the FSR Shewhart $\overline{\text{X}}$ chart in column [1].

Table 3.4: SSATS and EQL values for the FSR Shewhart $\overline{\text{X}}$ chart and SS GLR charts

	FSR $\overline{\text{X}}$	SS GLR		
ASN(n) =	4	4.0	1.5	1.5
d =	4.0	4.0	1.5	2.0
ASN/d =	1.0	1.0	1.0	0.75
δ	[1]	[2]	[3]	[4]
0.00	1481.60	1481.74	1479.66	1480.99
0.25	618.90	77.41	90.46	115.29
0.35	367.28	42.95	41.80	54.24
0.50	173.58	23.42	19.20	25.07
0.75	57.87	11.98	8.48	11.17
1.00	23.21	7.59	4.98	6.55
1.25	10.96	5.42	3.36	4.43
1.50	6.00	4.17	2.46	3.24
2.00	2.75	2.90	1.54	2.03
2.50	2.09	2.35	1.12	1.48
3.00	2.01	2.12	0.91	1.21
4.00	2.00	2.00	0.77	1.02
7.00	2.00	2.00	0.75	1.00
EQL	27.63	9.58	5.78	7.61
h =	0.0001	6.2596	7.0439	6.7279
g =	-	0.7494	1.6766	1.6544

In some applications the objective is to reduce the sampling cost, and it is more convenient to set the sampling interval to be an integer. Consider now the situation in which samples of items are taken every 2 hours rather than every 1.5 hours. The in-control ASN is still set to be 1.5 so the average in-control sampling rate is 0.75, which means that the sampling cost is reduced. The in-control ANSS is equal to $\text{ATS}/d = 740.8$ and the in-control ASN is 1.5. The limits $h = 6.7279$ and $g = 1.6544$ can be obtained using the regression equations (3.5) and (3.6). The performance of this SS GLR chart is shown in column [4] of Table 3.4. With a lower in-control sampling rate, the performance of the SS GLR chart in column [4] is not as good as that in column [3], but it is still uniformly better than the FSR Shewhart $\overline{\mathrm{X}}$ chart in column [1]. Therefore, the SS GLR chart can be applied in this application to provide much faster detection of process mean shifts with lower sampling cost.

3.6 The One-sided GLR Chart with Sequential Sampling

In some applications, the goal of process monitoring is to quickly detect shifts in the parameter of interest in one direction. For example, consider a chemical process where the quality characteristic of interest is the product viscosity. The target value for viscosity is 2000 centistokes at 100°**C**. There is no significant problem if viscosity decreases, but any increase in viscosity will cause serious problems and thus we should quickly detect any increase (see Montgomery (2012) for more information about this application). Suppose that viscosity follows a normal distribution with mean μ and variance σ^2, and any changes in the process will be reflected by an increase in the mean μ, while the variance σ^2 remains the same. In such cases, it is desirable to have a control chart that can effectively detect one-sided shifts in μ.

Although the SS GLR chart proposed above can be directly applied for monitoring one-sided shifts, it is in fact designed for detecting two-sided shifts. If we know the shifts will only occur in one direction, that is either increasing or decreasing, then the chart statistic in the SS GLR chart can be modified so that the one-sided shifts will be more effectively detected. In this section, we consider the goal of process monitoring to be quick detection of any increases in the mean μ. For the cases where decreases in μ are of interest, the GLR approach can be developed similarly.

In this section and sections dealing with the one-sided case we are talking about the SS GLR chart with a modified estimator of μ_1. For simplicity we still call it the SS GLR chart. The data setting is the same as the two-sided cased. Since we know $\mu_1 > \mu_0$, the bounded maximum likelihood estimator of μ_1 is

$$\hat{\mu}^{+}_{1,\tau,k,l} = \max\left(\mu_0\,, \frac{\sum_{i=\tau+1}^{k-1}\sum_{j=1}^{n_i} X_{ij} + \sum_{j=1}^{l} X_{kj}}{N_1}\right). \tag{3.7}$$

Then a log likelihood-ratio statistic for determining whether there has been a shift in μ in the past k samples with l observations in the last sample is

$$R^{+}_{k,l} = \ln \frac{\max_{0\le\tau<k,\mu_0<\mu_1<\infty} L(\tau,\mu_1|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}{L(\infty,\mu_0|\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_k)}$$
$$= \max_{0\le\tau<k} \frac{\left(\hat{\mu}^{+}_{1,\tau,k,l}-\mu_0\right)}{\sigma_0^2}\left(\sum_{i=\tau+1}^{k-1}\sum_{j=1}^{n_i}\left(X_{ij}-\frac{\left(\hat{\mu}^{+}_{1,\tau,k,l}+\mu_0\right)}{2}\right)+\sum_{j=1}^{l}\left(X_{kj}-\frac{\left(\hat{\mu}^{+}_{1,\tau,k,l}+\mu_0\right)}{2}\right)\right). \tag{3.8}$$

Using Equation (3.7) and a size m window, $R^{+}_{k,l}$ in Equation (3.8) can be represented as

$$R^{+}_{m,k,l} = \max_{\max(0,k-m)\le\tau<k} \frac{N_1}{2\sigma_0^2}\left(\hat{\mu}^{+}_{1,\tau,k,l}-\mu_0\right)^2. \tag{3.9}$$

The decision rule for this SS GLR chart is the same as that for the two-sided case. The choice of the window size m and the in-control ASN will be briefly discussed in the next chapter. We decided that the one-sided case may not be as important as the two-sided case, so we didn't do a complete investigation of this case. Thus, the design guidelines for this one-sided chart won't be provided in this dissertation.

Chapter 4. Performance Comparisons of Control Charts for Monitoring the Process Mean

In this chapter we consider the case of two-sided shifts in μ first and compare the performance of the SS GLR chart with other competing charts, including the FSR GLR chart, the VSI GLR chart, and the VSSVSI CUSUM chart. Then the case of one-sided shifts in μ is investigated and the SS GLR chart is compared with the SPRT chart and the VSSVSI CUSUM chart.

4.1 Comparing the SS GLR Chart with the FSR GLR Chart

The FSR GLR chart takes samples using a fixed sampling interval d and a fixed sample size n. For purposes of comparison we set up the FSR GLR chart and the SS GLR chart so that the in-control sampling rate is 1 for both charts, so this requires that $d = n$. Four different values (1, 2, 3, and 4 for the FSR CLR chart and 1.5, 2, 3, and 4 for the SS GLR chart) were investigated as the values of d and n (ASN). The window size for the FSR GLR chart is recommended to be large enough such that its performance is essentially the same as the performance of the FSR GLR chart without the window. We consider the window size $m = 400/n$. For example, when n =1 the window size is 400, and when $n = 3$ the window size is 133.

Based on the same likelihood-ratio-test argument used for the SS GLR statistics in Equations (3.1) and (3.4), the FSR GLR chart statistic $R_{m,k}^{FSR}$ after sample k can be represented as

$$R_{m,k}^{FSR} = \max_{\max(0,k-m)\leq\tau<k} \frac{(k-\tau)\times n}{2\sigma_0^2}(\hat{\mu}_{1,\tau,k}^{FSR} - \mu_0)^2, \tag{4.1}$$

where the MLE of $\hat{\mu}_{1,\tau,k}^{FSR}$ is given by

$$\hat{\mu}_{1,\tau,k}^{FSR} = \frac{\sum_{i=\tau+1}^{k}\sum_{j=1}^{n} x_{ij}}{(k-\tau)\times n}. \tag{4.2}$$

Table 4.1 gives some SSATS and EQL values for the SS GLR chart and the FSR GLR chart. The first four columns show the results for the SS GLR chart, and columns [5] to [8] show the results for the FSR GLR chart. Notice that for the FSR GLR chart, there is no warning limit g.

Table 4.1: SSATS and EQL values of the SS GLR chart and the FSR GLR chart

GLR	SS				FSR			
ASN(n) =	1.5	2	3	4	1	2	3	4
d =	1.5	2	3	4	1	2	3	4
m =	10	10	10	10	400	200	133	100
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0.00	1481.60	1481.60	1481.80	1481.30	1481.56	1481.60	1481.50	1481.59
0.25	90.03	74.20	72.48	77.45	151.71	137.50	129.95	125.01
0.50	19.09	18.39	20.50	23.36	43.76	40.60	38.92	37.80
0.75	8.45	8.70	10.22	11.97	21.02	19.74	19.07	18.57
1.00	4.96	5.26	6.38	7.61	12.49	11.82	11.43	11.13
1.50	2.45	2.70	3.41	4.17	6.00	5.68	5.47	5.29
2.00	1.54	1.76	2.31	2.90	3.55	3.34	3.18	3.07
3.00	0.90	1.13	1.62	2.12	1.67	1.53	1.62	2.02
4.00	0.76	1.01	1.51	2.01	0.95	1.04	1.50	2.00
5.00	0.75	1.00	1.50	2.00	0.62	1.00	1.50	2.00
6.00	0.75	1.00	1.50	2.00	0.51	1.00	1.50	2.00
7.00	0.75	1.00	1.50	2.00	0.50	1.00	1.50	2.00
EQL	5.77	6.22	7.86	9.58	12.41	11.85	11.85	12.15
h =	7.0449	6.8370	6.5069	6.2596	7.3288	6.5478	6.0933	5.7640
g =	1.6718	1.2491	0.9092	0.7491	-	-	-	-

The overall performance of the SS GLR chart is uniformly better than that of the FSR GLR chart in Table 4.1 when d is the same for both charts. For small shifts the SSATS values of the SS GLR charts are almost half of the SSATS values of the FSR GLR charts. For large shifts their performance is the same. When we compare across different values of d based on the EQL values, the optimal SS GLR chart in column [1] has much better performance than the optimal FSR GLR charts in column [6] and [7]. Also note that the EQL as a function of d varies more for the SS GLR charts compared to the FSR GLR charts, so the choice of d is more critical in the case of the SS GLR chart.

4.2 Comparing the SS GLR chart with the VSI GLR Chart

A VSI scheme provides an approach to varying the sampling rate when it is desirable to have a fixed sample size but the sampling interval can be varied. As an illustration, consider a situation in which a group of, say, 4 items would be tested simultaneously in a testing lab that takes 20 minutes. The sequential sampling scheme is not appropriate in this case since it does not make sense to wait 20 minutes for one item to finish the test before starting the test for another item. A technician used to test a sample of 4 items every 2 hours, and now, instead, we can apply the VSI scheme. After a sample is tested, if there is an indication of a process change, another sample of 4 items will be tested in the next 20 minutes. If there is no indication of a process change from the testing results, one will test another sample after 4 hours. The limits of the VSI scheme can be chosen so that, when the process is in control, an average of 4 items will be sampled and tested every 2 hours, so the average in-control sampling rates are the same for both FSR and VSI schemes. The VSI scheme can provide faster detection for different process shifts than the FSR scheme.

The VSI GLR chart has not been developed before, so here we develop the VSI GLR chart and compare it with the SS GLR chart. This will provide two efficient options of VSR charts for achieving improved detection. The VSI GLR chart statistic has the same form as the FSR GLR chart statistic, which means $R_{m,k}^{VSI}$ can be calculated in the same way as $R_{m,k}^{FSR}$ in equation (4.1), but addition parameters need to be specified to define the decision rule.

For the VSI chart, the time before the next sample is a function of the current value of the chart statistic. In general, this function could be in any form so the values of different sampling intervals could be anything that we specified, but previous research shows that it is sufficient to use only two possible sampling intervals to achieve good statistical properties in VSI control charts. See Stoumbos et al. (2001) for more details. Therefore in this paper we consider using only two possible sampling intervals d_1 and d_2, where $d_1 < d_0 < d_2$ and d_0 is the average sampling interval when the process is in-control. Notice there is no pre-specified chart statistic before the first sample, and so d_0 will be used as the time interval before the first sample.

The decision rule for the VSI GLR chart requires the specification of two limits g and h, satisfying $g < h$, where h is the control limit and g is the warning limit for the sampling interval. Suppose after sample k is obtained, $R_{m,k}^{VSI}$ is calculated. Then the decision rules are defined as follows. Use the long sampling interval d_2 if $R_{m,k}^{VSI} \leq g$ and the short sampling interval d_1 if $g < R_{m,k}^{VSI} \leq h$. A signal is given if $R_{m,k}^{VSI} > h$, indicating there has been a change in μ.

In the VSI GLR chart the sample size n is fixed. We consider three different values, $n = 1, 2$ or 3. In order to make fair comparisons with the SS GLR chart, we set the average in-control sampling rate to be 1. This means that the value of the average in-control sampling interval d_0 is the same as the value of n. For each d_0, it has been shown that the performance of the VSI GLR chart is better when the short interval d_1 is as short as possible, and we consider the minimum value of $d_1 = 0.1 \times d_0$. The value of the long interval d_2 is optimized so that the EQL is smallest for each d_0. The window size is set in the same way as in the FSR GLR chart.

Table 4.2: SSATS and EQL values for the SS GLR chart and the VSI GLR chart

GLR	SS			VSI		
ASN(n) =	1.5	2	3	1	2	3
$d(d_0)$ =	1.5	2	3	1	2	3
d_1 =	-	-	-	0.1	0.2	0.3
d_2 =	-	-	-	1.7	2.75	3.75
m =	10	10	10	400	200	133
Δ	[1]	[2]	[3]	[4]	[5]	[6]
0.00	1481.60	1481.60	1481.80	1482.04	1481.01	1482.13
0.25	90.03	74.20	72.48	64.32	68.55	72.71
0.50	19.09	18.39	20.50	17.76	19.26	20.54
0.75	8.45	8.70	10.22	8.53	9.39	10.04
1.00	4.96	5.26	6.38	5.14	5.73	6.14
1.50	2.45	2.70	3.41	2.64	2.96	3.22
2.00	1.54	1.76	2.31	1.72	1.98	2.25
3.00	0.90	1.13	1.62	1.06	1.41	1.85
4.00	0.76	1.01	1.51	0.88	1.34	1.84
5.00	0.75	1.00	1.50	0.83	1.34	1.84
6.00	0.75	1.00	1.50	0.82	1.34	1.84
7.00	0.75	1.00	1.50	0.82	1.34	1.84
EQL	5.77	6.22	7.86	5.91	7.01	8.07
h =	7.0449	6.8370	6.5069	7.3288	6.5478	6.0933
g=	1.6718	1.2491	0.9092	1.9018	2.1842	2.3796

Table 4.2 gives the SSATS and EQL values for some SS GLR charts and VSI GLR charts. Columns [1] – [3] give the results for the SS GLR charts, and columns [4] – [6] give the

results for the optimal VSI GLR charts for each n. The warning limit g is chosen such that the average in-control sampling interval is d_0.

In Table 4.2 column [1] shows the SSATS and EQL results for the optimal SS GLR chart. Column [2] should be compared with column [5] where $d_0 = 2$, and column [3] should be compared with column [6] where $d_0 = 3$. Based on the EQL values we can see that the SS GLR charts have better overall performance than the VSI GLR charts, no matter which d_0 we are using. The optimal SS GLR chart in column [1] has better overall performance than the optimal VSI GLR chart in column [4], although the difference is EQL values is not large.

If we focus on the SSATS values when d_0 is the same for both charts, the SS GLR charts have similar performance as the VSI GLR charts, except that the SS GLR charts are more effective for large shifts. For the VSI GLR charts we recommend that the sample size $n = 1$ should be taken with a short average time interval rather than taking a large sample with a long average time interval. We considered a small value of d_1 and optimized the value of d_2. In practice, the value of d_1 should be as small as feasible and the value of d_2 should depend on the practical interest of whether large shifts or small shifts are the most concern. A relatively large value of d_2 is effective for detecting small shifts and a relatively small value of d_2 is desired for detecting large shifts. Overall the SS GLR chart has a bit better overall performance than the VSI GLR chart, and in the SS GLR charts there is no need to specify the values of two different sampling intervals.

4.3 Comparing the SS GLR chart with the VSSVSI CUSUM Chart

In this section we compare the performance of the SS GLR chart with the VSSVSI CUSUM chart. The CUSUM chart for monitoring the process mean considers testing the parameter μ: $H_0: \mu = \mu_0$ $vs.$ $H_1: \mu = \mu_1$, where μ_1 is the value of μ to be detected by the monitoring process. Note that in a GLR chart the value of μ_1 is estimated from the process data, but in a CUSUM chart a value for μ_1 must be specified whether the actual value of the shift is known or not. Thus μ_1 is considered as a tuning parameter for a CUSUM chart. It will be convenient to specify the value of the tuning parameter as

$$\delta_1 = (\mu_1 - \mu_0)/\sigma_0,$$

which is the shift in terms of standardized units. The CUSUM chart will detect a shift quickly if the actual shift is close to the specified shift. But if the actual shift is not close to the specified shift, the performance of the CUSUM chart deteriorates.

For varying the sampling rate, we consider the combination of variable sample size and variable sampling interval. The VSSVSI CUSUM chart allows both the sample size at sample k and the time interval between samples $k-1$ and k to depend on the chart statistics at sample $k-1$. For monitoring the process mean, consider the problem of detecting an increase in μ, so that $\mu_1 > \mu_0$. Instead of using C_k^+ as the FSR CUSUM chart statistic, let Y_k^+ be the VSSVSI CUSUM chart statistic at sample k, and let $\bar{X}_k$ be the sample mean at sample k. Then we have

$$Y_k^+ = \max\ \{0, Y_{k-1}^+\} + l_k\ , \tag{4.3}$$

and

$$l_k = \frac{\mu_1 - \mu_0}{\sigma_0}\sqrt{n(Y_{k-1}^+)}[Z_k - \gamma(Y_{k-1}^+)], \tag{4.4}$$

where $n(Y_{k-1}^+)$ is the current the sample size determined by the previous statistic Y_{k-1}^+, $Z_k = \sqrt{n(Y_{k-1}^+)}\left(\frac{\bar{X}_k-\mu_0}{\sigma_0}\right)$ represents the standardized sample mean, and $\gamma(Y_{k-1}^+) = \sqrt{n(Y_{k-1}^+)}\left(\frac{\mu_1-\mu_0}{2\sigma_0}\right)$ is often called reference value.

For applications in which the goal is to detect a decrease in μ to $\mu_1 < \mu_0$, the VSSVSI CUSUM statistic Y_k^- can be developed in a similar manner and is used together with Y_k^+ from equation (4.3) to give a symmetric chart for detecting both increases and decreases in μ. Let the VSSVSI CUSUM chart denote the two-sided VSSVSI CUSUM chart from this point. The decision rules for the VSSVSI CUSUM chart can be defined in the similar way as the decision rules for the VSI GLR chart, but additional parameters need to be specified.

The first sample size and the first sampling interval must be specified to start the monitoring process since there is no pre-statistic available. Let d_0 be the average in-control sampling interval and n_0 be the average in-control sample size. We choose the first sample interval to be d_0 and the first sample size to be n_0. The ratio of d_0 and n_0 is taken to be 1 which means that the in-control sampling rate is 1, as in the previous chart comparisons considered in this paper.

For the values of the different sampling intervals, we consider using only two possible sampling intervals d_1 and d_2, where $d_1 < d_0 < d_2$, as we did for the VSI GLR chart. On the other hand, there have been no optimality results for the choice of the sample sizes in VSS control charts so far. We consider using two possible sample sizes n_1 and n_2, where $n_1 < n_0 < n_2$. We believe this choice would be preferred by practitioner because of its simplicity.

The decision rule for the VSSVSI CUSUM chart requires the specification of three limits g_n, g_d and h, satisfying $g_n < h$ and $g_d < h$, where h is the control limit, g_n is the warning limit for the sample size and g_d is the warning limit for the sampling interval. After sample k is obtained Y_k^+ and Y_k^- are calculated and the decision rules are defined as follows. Use the small sample size n_1 for next sample if $Y_k^+ \leq g_n$ and $Y_k^- \geq -g_n$. Use the large sample size n_2 if either $g_n < Y_k^+ \leq h$ or $-h \leq Y_k^- < -g_n$. Use the long sampling interval d_2 if $Y_k^+ \leq g_d$ and $Y_k^- \geq -g_d$. Use the short sampling interval d_1 if either $g_d < Y_k^+ \leq h$ or $-h \leq Y_k^- < -g_d$. A signal is given if either $Y_k^+ > h$ or $Y_k^- < -h$, indicating there has been a change in μ.

The properties of the VSSVSI CUSUM chart were evaluated using simulation with the same setting as previous charts. The control limit h and the warning limit for the sample size g_n were set to give the specified in-control ANOS and ASN. Then the warning limit for the sampling interval g_d was set to give the specified the in-control ATS. The SSATS values for a wide range of shifts and EQL values are provided in Table 4.3.

In Table 4.3, the first two columns [1] – [2] give results for the SS GLR chart, and columns [3] - [8] give results of the VSSVSI CUSUM chart. Two different values 0.5 and 1 were considered as the value of tuning parameter δ_1 in the VSSVSI CUSUM charts. Column [1] shows the results of the optimal GLR chart, which can be fairly compared with columns [3], [4], and [5] where they both have the same in-control ASN value 1.5, and the sampling interval d in the SS GLR chart is the same as the average sampling interval d_0 in the VSSVSI CUSUM chart. Similarly, column [2] can be compared with columns [6], [7], and [8], where they both have the same in-control ASN value 2, as well as the values of d and d_0.

Table 4.3: SSATS & EQL values of the SS GLR chart and the VSSVSI CUSUM chart

	SS GLR		VSSVSI CUSUM					
ASN(n) =	1.5	2.0	1.5	1.5	1.5	2	2	2
n_1 =	-	-	3	3	3	4	4	4
n_2 =	-	-	1	1	1	1	1	1
$d(d_0)$ =	1.5	2	1.5	1.5	1.5	2	2	2
d_1 =	-	-	0.150	0.150	0.150	0.200	0.200	0.200
d_2 =	-	-	2.850	1.875	2.850	3.800	2.500	3.800
m =	10	10	-	-	-	-	-	-
δ_1 =	-	-	0.5	0.5	1	0.5	0.5	1
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0.00	1481.60	1481.60	1481.59	1481.59	1481.60	1481.53	1480.99	1480.90
0.25	90.03	74.20	85.89	89.39	222.17	85.41	88.13	223.36
0.50	19.09	18.39	15.65	16.51	28.40	16.07	16.60	28.36
0.75	8.45	8.70	7.54	7.96	8.35	8.05	8.26	8.46
1.00	4.96	5.26	4.92	5.19	4.43	5.42	5.54	4.67
1.50	2.45	2.70	2.96	3.07	2.42	3.43	3.45	2.78
2.00	1.54	1.76	2.21	2.20	1.83	2.64	2.56	2.25
3.00	0.90	1.13	1.63	1.42	1.51	2.06	1.72	1.97
4.00	0.76	1.01	1.49	1.12	1.47	1.95	1.43	1.94
5.00	0.75	1.00	1.47	1.04	1.45	1.94	1.36	1.91
6.00	0.75	1.00	1.47	1.03	1.41	1.94	1.36	1.86
7.00	0.75	1.00	1.47	1.03	1.38	1.94	1.36	1.82
EQL	5.77	6.22	7.20	6.96	8.69	8.31	7.75	9.71
h =	7.0449	6.8370	10.1277	10.1277	5.608	9.8492	9.8492	5.3318
g =	1.6718	1.2491	-	-	-	-	-	-
g_n =	-	-	2.3422	2.3422	0.9812	1.7182	1.7182	0.58
g_d =	-	-	1.4471	2.5234	0.3862	1.2021	2.2773	0.2165

The comparison results from columns [1], [3], and [4] show that overall the SS GLR chart has better performance than the VSSVSI CUSUM chart. The VSSVSI CUSUM charts in columns [3] and [4] were tuned to detect the shift size $\delta_1 = 0.5$. We can see that if the actual shift size is equal (or very close) to this specified shift size, the VSSVSI CUSUM charts have better performance than the SS GLR chart. But if the actual shift size is not very close to the specified one, the VSSVSI CUSUM charts have much worse performance than the SS GLR chart. The EQL value of the GLR chart is much smaller than the EQL of the VSSVSI CUSUM charts. Similar conclusions can be reached from the comparison results of columns [1] and [5], where column [5] shows the performance of the VSSVSI CUSUM chart that is tuned to detect the shift size $\delta_1 = 1.0$. Only when the actual shift size is close to 1.0, is the VSSVSI CUSUM better. The only difference between columns [3] and [4] is the value of d_2. We can see that a relatively large value of d_2 is effective for detecting small shifts and a relatively small value of d_2 is effective for detecting large shifts. Also, it should be noted that the performance of the VSSVSI CUSUM chart in column [4] is quite different from that in column [5]. This is actually true for all the VSSVSI CUSUM charts with different values of the parameters, which indicates that the performance of VSSVSI CUSM chart is very sensitive to the parameter choice.

There are 8 parameters including δ_1, d_0, ANSS, n_0, d_1, d_2, n_1, and n_2 that need to be specified for the VSSVSI CUSUM chart, while there are 3 parameters including the sampling interval d, ANSS, and ASN that need to be specified for the SS GLR chart using the recommended window size 10. The optimization for the two-sided VSSVSI CUSUM chart would presumably be not easy considering a number of parameter choices, so we did not optimize it. We provide the optimal parameter choice for the SS GLR chart and we showed that even for non-optimized parameters (in-control ASN is 2 instead of 1.5), the SS GLR chart still had overall good performance. In applications, we believe that practitioners would prefer to use a chart with as few parameters that need to be specified as possible and still have good performance. The SS GLR chart has been shown here to have overall very good performance, and only a few parameters need to be specified.

4.4 Comparing the SS GLR chart with the SPRT chart in the one-sided case

For detecting one-sided shifts in μ, the SPRT chart has been investigated by Stoumbos and Reynolds (1997) and shown to be very effective. The SPRT is a sequential hypothesis test designed to test two simple hypotheses $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$. With the SS GLR chart the value of μ_1 is estimated from the process data, but in the SPRT chart the value of μ_1 is similar to the CUSUM chart in that it must be specified by the practitioner and thus it is considered as a tuning parameter. Here the SPRT chart is taken to mean the upper one-sided SPRT chart for detecting an increase in μ.

Consider the hypothesis test $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1$ where $\mu_1 > \mu_0$. The SPRT chart uses only $\mathbf{X}_k$ at sampling point k. Then a log likelihood-ratio statistic to determine whether there has been an increase in μ is

$$U_{kl} = \ln \frac{L(\mu_1 | \mathbf{X}_k)}{L(\mu_0 | \mathbf{X}_k)} = \frac{(\mu_1 - \mu_0)}{\sigma_0^2} \left(\sum_{j=1}^{l} \left(X_{kj} - \mu_0 - \frac{\mu_1 - \mu_0}{2} \right) \right). \tag{4.5}$$

The SPRT chart is a sequence of SPRT's for testing H_0 versus H_1, and each SPRT is being conducted in negligible time with a time interval d between consecutive SPRTs. The decision rule for SPRT chart is the same as the decision for the SS GLR chart except the chart statistic is calculated differently.

The control chart parameters g and h can be chosen to give specified in-control performance. For the tuning parameter μ_1, generally the SPRT chart is sensitive to small shifts if the value of μ_1 is small, and sensitive to large shifts if the value of μ_1 is large. Similar to the case of the above CUSUM chart, we specify the value of δ_1 as the tuning parameter. It is recommended by Stoumbos and Reynolds (1997) that the choice of δ_1 between 0.2 and 0.7 will give good overall performance.

Now we compare the performance of the SS GLR chart and the SPRT chart. We start comparisons with the same parameters setting for the SS GLR chart as those used for the SPRT chart in Stoumbos and Reynolds (1997). Specifically, the time interval between consecutive samples d is set to be 1.0. The in-control ASN is set to be 3.0 so the average sampling rate is 3. The tuning parameter value in the SPRT chart δ_1 is tuned to be 0.3. The in-control ATS for all charts considered in this section is equal or very close to 740.8. Any differences from 740.8 are due to simulation error.

In the SS GLR chart, a window of size m needs to be specified. We could use $m = 10$ as recommended for the two-sided case, but in order to fully understand the effect of the window size for the one-sided case, we investigate different window sizes and see how the SS GLR chart performance changes. Table 4.4 gives SSATS values and EQL values for different values of the window size m. The shift size δ considered ranges from 0.25 to 7.00, and the EQL is calculated based on shifts from 0.25 to 7.00.

Table 4.4: SSATS and EQL values for different window sizes

SS GLR, ASN=3, d=1									
m=	1	3	4	5	6	7	8	10	100
δ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
0.00	739.90	740.95	740.71	740.78	740.92	741.07	741.04	741.20	740.90
0.25	21.41	18.49	18.16	17.80	17.57	17.44	17.37	17.33	18.02
0.35	11.71	10.21	10.05	9.89	9.81	9.74	9.72	9.74	10.38
0.50	6.20	5.48	5.42	5.37	5.34	5.34	5.34	5.37	5.79
0.75	3.04	2.77	2.76	2.75	2.75	2.76	2.77	2.78	3.01
1.00	1.87	1.75	1.75	1.75	1.75	1.76	1.76	1.78	1.92
1.25	1.30	1.25	1.25	1.26	1.26	1.27	1.27	1.28	1.37
1.50	0.99	0.97	0.97	0.97	0.98	0.98	0.98	0.99	1.05
2.00	0.69	0.69	0.69	0.69	0.69	0.69	0.70	0.70	0.73
2.50	0.58	0.57	0.57	0.57	0.57	0.57	0.58	0.58	0.59
3.00	0.52	0.52	0.52	0.52	0.52	0.52	0.53	0.53	0.53
5.00	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
7.00	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
EQL	2.38	2.28	2.28	2.27	2.27	2.27	2.28	2.29	2.40
h=	6.2351	6.2973	6.2994	6.3028	6.3042	6.3042	6.3037	6.3008	6.2649
g=	0.4674	0.5639	0.5893	0.6033	0.6151	0.6254	0.6346	0.6518	0.7791

From Table 4.4 we can see that generally the window size has a small effect on the SSATS values and EQL values. With sequential sampling, it's recommended to use a relatively small window since the sampling scheme allows for a large number of observations to be examined whenever any changes occur in the process. The average in-control sampling rate considered here is 3. Compared with the two-sided case where the in-control sampling rate considered is 1, more observations are contained in the same window in the one-sided case. Thus it seems that the optimal window should be smaller than the optimal window $m = 10$ in the two-sided case. Based on the EQL values, column [5] where $m = 6$ gives the smallest EQL values if we look at more than 2 decimal places. But the differences between $m = 10$ and $m = 6$ are very small, so in order to simplify the design of the control chart we conclude that $m = 10$ can still be used for the one-sided case. When the SS GLR chart is compared to other charts with this sampling rate, both window sizes 6 and 10 will be considered.

Besides the SS GLR chart and the SPRT chart, the VSSVSI CUSUM chart is also considered as a competing chart and their performance is compared. The VSSVSI CUSUM chart also has the tuning parameter δ_1 and the value of δ_1 is tuned to be 0.3. Different values of d_1, d_2, n_1 and n_2 are used, while the average sampling interval d_0 is 1 and the average sample size n_0 is 3 when the process is in control.

Table 4.5 shows the SSATS values for the SPRT chart, the SS GLR chart and the VSSVSI CUSUM chart. Column [1] gives the results for the SPRT chart, columns [2] and [3] show the results for the SS GLR chart with window sizes 6 and 10, respectively. Columns [4] – [7] give the results for the VSSVSI CUSUM chart with different values of the parameters d_1, d_2, n_1 and n_2. These CUSUM chart results are from Stoumbos and Reynolds (1997). Note that in Table 4.5 the shift only goes up to 3.00 and there are no EQL values because Stoumbos and Reynolds (1997) did not provide them. Thus the conclusion about this table will be drawn based on the SSATS values.

Comparing the SS GLR charts with the VSSVSI CUSUM chart, the SS GLR chart has better performance for moderate and large shifts, while the VSSVSI CUSUM chart has better performance for the small shifts. This is because the tuning parameter δ_1 in the VSSVSI CUSUM chart is 0.3, which makes the CUSUM charts very sensitive to small shift sizes around 0.3. However, if the actual shift size is not close to 0.3, the SS GLR charts can detect this shift faster. Notice that as the shift size increases, the SSATS values in the SS GLR chart will converge to 0.5. But this is not true for the VSSVSI CUSUM chart, where the SSATS value will converge to some number greater than 0.5, depending on the choice of d_1 and d_2. Overall the SS GLR chart has better performance than the VSSVSI CUSUM chart.

However, the SPRT chart has the best overall performance compared with the others. The tuning parameter $\delta_1 = 0.3$ makes the SPRT chart have very good performance for small shifts. But the performance for moderate to large shifts is surprisingly good. It may be that the sequential sampling scheme protects against moderate to large shifts. With sequential sampling, whenever there is any indication of a process change, the chart statistic is likely to fall within the range of (g, h) and additional observations will be obtained until either a signal occurs (the chart statistic goes above h) or such an indication is gone (the chart statistic falls below g). When large shifts occur in the process, such an indication is relatively strong so the chart statistic is very likely to fall above g. More observations will be taken until a signal occurs and the resulting SSATS will be very small. Also considering that the in-control average sampling rate is

3, when a moderate to large shift occurs the average sampling rate will increase. With such a high average sampling rate it is not surprising that most moderate and large shifts can be detect in a few samples. Thus the SSATS values of the SPRT chart are very small and uniformly better than other charts in this case.

Table 4.5: SSATS values for SPRT, SS GLR and VSSVSI CUSUM charts

	SPRT	SS GLR		VSSVSI CUSUM			
ASN(n) =	3.0	3.0	3.0	3.0	3.0	3.0	3.0
n_1 =	-	-	-	2	2	2	2
n_2 =	-	-	-	6	6	10	10
$d(d_0)$ =	1.0	1.0	1.0	1.0	1.0	1.0	1.0
d_1 =	-	-	-	0.1	0.1	0.1	0.1
d_2 =	-	-	-	1.1	1.5	1.1	1.5
δ_1 =	0.3	-	-	0.3	0.3	0.3	0.3
m =	-	6	10	-	-	-	-
	[1]	[2]	[3]	[4]	[5]	[6]	[7]
δ	SSATS	SSATS	SSATS	SSATS	SSATS	SSATS	SSATS
0.00	740.80	740.92	741.20	740.80	740.80	740.80	740.80
0.25	12.08	17.57	17.33	15.21	13.60	16.36	14.97
0.35	6.18	9.81	9.74	9.02	8.21	10.28	9.16
0.50	3.40	5.34	5.37	5.56	5.05	6.48	5.47
0.75	1.88	2.75	2.78	3.39	2.97	3.93	3.07
1.00	1.27	1.75	1.78	2.46	2.06	2.80	2.09
1.25	0.96	1.26	1.28	1.96	1.57	2.16	1.58
1.50	0.78	0.98	0.99	1.64	1.28	1.74	1.27
2.00	0.61	0.69	0.70	1.24	0.98	1.26	0.98
2.50	0.54	0.57	0.58	0.96	0.86	0.98	0.87
3.00	0.51	0.52	0.53	0.77	0.83	0.79	0.84
h =	14.20	6.3042	6.300808	7.81	7.81	7.81	7.81
g =	0.52	0.6151	0.651754	-	-	-	-
g_n =	-	-	-	1.88	1.88	3.01	3.01
g_d =	-	-	-	3.36	1.27	3.36	1.27

It should be noted that an ASN of 3.0 is not optimal for the SS GLR chart. A question that arises is whether the conclusion would still hold if a different ASN closer to the optimal ASN was used and if a different average sampling rate was considered. Therefore we now consider the case in which the average sampling rate is set to be 1 and the value of ASN is to be optimized. The in-control ATS is set to be 1481.6 which is the same as for the two-sided case. The window size in the SS GLR chart is set to be 10. Since the SPRT chart has better overall performance than the VSSVSI CUSUM chart, we will only compare the SS GLR chart with the SPRT chart in the current case. In the SPRT chart, the tuning parameter δ_1 is optimized for each value of ASN to make fair comparisons.

Table 4.6: SSATS and EQL values for SS GLR and SPRT charts

	SS GLR			SPRT		
d, ASN =	1.1	1.5	2.0	1.1	1.5	2.0
δ_1	-	-	-	0.7	0.6	0.6
δ	[1]	[2]	[3]	[4]	[5]	[6]
0.00	1481.60	1481.60	1481.60	1481.60	1481.60	1481.60
0.25	112.12	53.08	46.96	166.24	78.57	63.47
0.35	49.95	26.96	25.33	81.93	33.58	26.03
0.50	21.11	13.46	13.43	34.48	13.63	10.58
0.75	8.53	6.39	6.70	12.64	5.69	4.60
1.00	4.80	3.87	4.19	6.45	3.33	2.87
1.25	3.17	2.67	2.95	3.91	2.27	2.07
1.50	2.28	1.99	2.24	2.62	1.69	1.64
2.00	1.39	1.30	1.53	1.42	1.12	1.23
2.50	0.96	0.99	1.21	0.92	0.89	1.08
3.00	0.74	0.84	1.08	0.70	0.80	1.02
4.00	0.58	0.76	1.01	0.57	0.75	1.00
5.00	0.55	0.75	1.00	0.55	0.75	1.00
6.00	0.55	0.75	1.00	0.55	0.75	1.00
7.00	0.55	0.75	1.00	0.55	0.75	1.00
EQL	5.63	4.59	5.16	8.00	4.62	4.50
h =	6.4993	6.3888	6.1922	5.9293	7.7856	8.1026
g =	2.2794	1.2120	0.8767	1.4804	0.8370	0.4161

Table 4.6 gives the SSATS values and EQL values for the SS GLR chart and the SPRT chart with different values of the ASN. The value of d is the same as the ASN to give the average sampling rate of 1. From columns [1] - [3] we see that ASN=1.5 in column [2] gives the best overall performance for the SS GLR chart as the EQL is minimized. From columns [4] – [6] we see that ASN=2 in column [6] gives the best overall performance for the SPRT chart. This

may be because the SPRT chart uses only the current sample to calculate the statistic. The optimal SPRT chart is overall slightly better than the optimal SS GLR chart.

The conclusion from this performance comparison study is that for applications where any shifts in μ will only occur in one direction, the SS GLR chart is quite comparable to the SPRT chart. The SPRT chart has better overall performance when the ASN is relative high, such as 2 or 3. This indicates that if we have a case where constraints require that the ASN be relatively large, the SPRT chart is a better option than the SS GLR chart. But for the cases where constraints require the ASN be small, perhaps due to the limited sampling capability or limited number of products, the SS GLR chart has better overall performance. In addition, there is no tuning parameter that needs to be specified in the SS GLR chart, but there is a tuning parameter δ_1 that needs to be specified in the SPRT chart. We believe this makes the SS GLR chart easier to design and apply than the SPRT chart. The VSSVSI CUSUM chart performs overall not as well as the other two competing charts, and multiple chart parameters must be specified such as d_1, d_2, n_1 and n_2.

Chapter 5. The GLR Chart with Sequential Sampling for Monitoring the Process Mean and Variance

5.1 Derivation of the GLR Chart with Sequential Sampling

The derivation here of the SS GLR chart for monitoring both mean and variance is similar to the previous derivation of the SS GLR chart for monitoring the mean only. But the details are given here again for completeness. Suppose that at sampling point k we have data $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k$, where $\mathbf{X}_i = (X_{i1}, X_{i2}, \dots, X_{in_i})$ for $i = 1,2, \dots, k-1$ and currently there are l observations in sample k, so that $\mathbf{X}_k = (X_{k1}, X_{k2}, \dots, X_{kl})$. Note that this l^{th} observation may not be the last observation in the k^{th} sample. Consider the alternative hypothesis that a shift in the process from the in-control $N(\mu_0, \sigma_0^2)$ distribution to an $N(\mu_1, \sigma_1^2)$ distribution has occurred at some time between samples τ and $\tau+1$, where $\tau < k$, $\mu_1 \neq \mu_0$ and/or $\sigma_1^2 \geq \sigma_0^2$, and τ, μ_1, and σ_1^2 are unknown. Note that this assumption means that the shift will only occur between samples, not within a sample. Then the likelihood function at the current l^{th} observation in the k^{th} sample is

$$L(\tau, \mu_1, \sigma_1^2 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)$$

$$= (2\pi\sigma_0^2)^{-(\sum_{i=1}^{\tau} n_i)/2} (2\pi\sigma_1^2)^{-\left(\sum_{i=\tau+1}^{k-1} n_i + l\right)/2} \times \exp\left(-\frac{1}{2\sigma_0^2} \left(\sum_{i=1}^{\tau} \sum_{j=1}^{n_i} (X_{ij} - \mu_0)^2 \right) - \frac{1}{2\sigma_1^2} \left(\sum_{i=\tau+1}^{k-1} \sum_{j=1}^{n_i} (X_{ij} - \mu_1)^2 + \sum_{j=1}^{l} (X_{kj} - \mu_1)^2 \right) \right).$$

The Maximum Likelihood Estimator (MLE) of μ_1 is

$$\hat{\mu}_{1,\tau,k,l} = \frac{\sum_{i=\tau+1}^{k-1} \sum_{j=1}^{n_i} X_{ij} + \sum_{j=1}^{l} X_{kj}}{\sum_{i=\tau+1}^{k-1} n_i + l}. \tag{5.1}$$

Let $S_{\tau,k,l}^2$ denote the MLE of σ_1^2 without the assumption $\sigma_1^2 \geq \sigma_0^2$, so that we have

$$S_{\tau,k,l}^2 = \frac{\sum_{i=\tau+1}^{k-1} \sum_{j=1}^{n_i} \left(X_{ij} - \hat{\mu}_{1,\tau,k,l}\right)^2 + \sum_{j=1}^{l} \left(X_{kj} - \hat{\mu}_{1,\tau,k,l}\right)^2}{\sum_{i=\tau+1}^{k-1} n_i + l}. \tag{5.2}$$

However, due to the restriction $\sigma_1^2 \geq \sigma_0^2$, the restricted MLE of σ_1^2 is

$$\hat{\sigma}_{1,\tau,k,l}^2 = \max\{\sigma_0^2, S_{\tau,k,l}^2\}. \tag{5.3}$$

Under the null hypothesis that there is no shift in the process, the likelihood function at sample k can be represented as

$$L(\infty, \mu_0, \sigma_0^2 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)$$
$$= (2\pi\sigma_0^2)^{-\left(\sum_{i=1}^{k-1} n_i + l\right)/2} \times \exp\left(-\frac{\sum_{i=1}^{k-1}\sum_{j=1}^{n_i}\left(X_{ij} - \mu_0\right)^2 + \sum_{j=1}^{l}\left(X_{kj} - \mu_0\right)^2}{2\sigma_0^2}\right).$$

Let $S_{0,\tau,k,l}^2$, which represents the unrestricted estimate of σ^2 under the assumption that $\mu = \mu_0$, be defined by

$$S_{0,\tau,k,l}^2 = \frac{\sum_{i=\tau+1}^{k-1}\sum_{j=1}^{n_i}\left(X_{ij} - \mu_0\right)^2 + \sum_{j=1}^{l}\left(X_{kj} - \mu_0\right)^2}{\sum_{i=\tau+1}^{k-1} n_i + l}. \tag{5.4}$$

Using equations (5.1) – (5.4), a log likelihood-ratio statistic for determining whether there has been a change in the process in the past k samples is

$$R_{k,l} = \ln \frac{\max\limits_{0 \leq \tau < k, -\infty < \mu_1 < \infty, \sigma_1^2 \geq \sigma_0^2} L(\tau, \mu_1, \sigma_1^2 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)}{L(\infty, \mu_0, \sigma_0^2 | \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_k)}$$
$$= \max_{0 \leq \tau < k} \frac{\left(\sum_{i=\tau+1}^{k-1} n_i + l\right)}{2}\left(\frac{S_{0,\tau,k,l}^2}{\sigma_0^2} - \frac{S_{\tau,k,l}^2}{\hat{\sigma}_{1,\tau,k,l}^2} - \ln\left(\frac{\hat{\sigma}_{1,\tau,k,l}^2}{\sigma_0^2}\right)\right). \tag{5.5}$$

As was done for the SS GLR chart in Chapter 3, a window of size m can be applied so that the maximization in equation (5.5) is taken over $\max(0, k - m) \leq \tau < k$. Let $R_{m,k,l}$ denote the likelihood-ratio statistic with the window. The decision rule for the SS GLR chart requires the specification of two limits h and g satisfying $0 < g < h$, where h is the control limit and g is the limit that determines whether we continue or stop sampling at the current sampling point. At the current l^{th} observation in the k^{th} sample, the decision rule is defined as follows.

1. Compute $R_{m,k,l}$.
2. If $R_{m,k,l} \leq g$, then stop sampling at sampling point k and wait until the next sampling point $k + 1$ to sample again. Then the sample size at sampling point k is $n_k = l$.

3. If $g < R_{m,k,l} \leq h$, then take another observation at sampling point k, and go to step 1 above with $l = l + 1$.
4. If $R_{m,k,l} > h$, then signal that there has been a change in the process.

Note that $\hat{\mu}_{1,\tau,k,l}$ in equation (5.1) and $\hat{\sigma}^2_{1,\tau,k,l}$ in equation (5.3) are computed in the process of evaluating $R_{m,k,l}$, so after a signal immediate estimates of the shifted values μ_1, and σ_1^2 are available. Evaluating $R_{m,k,l}$ also requires maximization over τ, so this provides an estimate of the change point τ. Having estimates of μ_1, σ_1^2, and τ at the time that the signal occurs is a significant advantage of GLR charts.

5.2 The Effect of the Window Size and the In-control ASN

As in Chapter 3 we need to investigate the effect of the window size and the in-control ASN. The motivation and details are given here again for completeness. Although this arrangement will lead to some similar arguments as in Chapter 3, for those readers who are only interested in the case of simultaneously monitoring the process mean and variance, and/or skipped the details in Chapter 3, there is no need to go back and read the corresponding parts.

The window size m is defined in terms of the number of samples. That is, there are m samples contained in the window. The goal of using a window is to use information from a large number of observations to calculate the GLR statistic when there is a shift in the process, while reducing the computational cost at the same time.

In the FSR GLR chart, the sample size n is fixed at each sampling point. The number of observations in the window is the product of m and n, whether the process is in-control or out of control. So for a fixed sample size n, in order to let the window contains a large number of observations, we need to set m to be large. It has been shown that for the FSR GLR chart, m is recommended to be large enough such that the performance of this GLR chart is essentially the same as the performance of the FSR GLR chart without the window. Specifically, it is sufficient to choose m so that there are 800 observations in the window.

For the SS GLR chart, the sample size at each sampling point is a random variable and the in-control ASN is used as the measure of average in-control sample size. For a given in-control

ASN, it is possible that even though the window size m is small, there are still a large number of observations contained in the window when the process is out of control. If this is true, then a small window size m seems to be preferred in the SS GLR chart. Now let's look at a plot of the SS GLR chart to illustrate this point.

Figure 5.1 shows a trace of the SS GLR chart statistic based on one simulation run. The solid horizontal line h and the dash horizontal line g were set to give in-control ATS and ANOS values of 1481.6. Samples were taken every 2.0 time units with in-control ASN= 2. The window size here is $m = 5$.

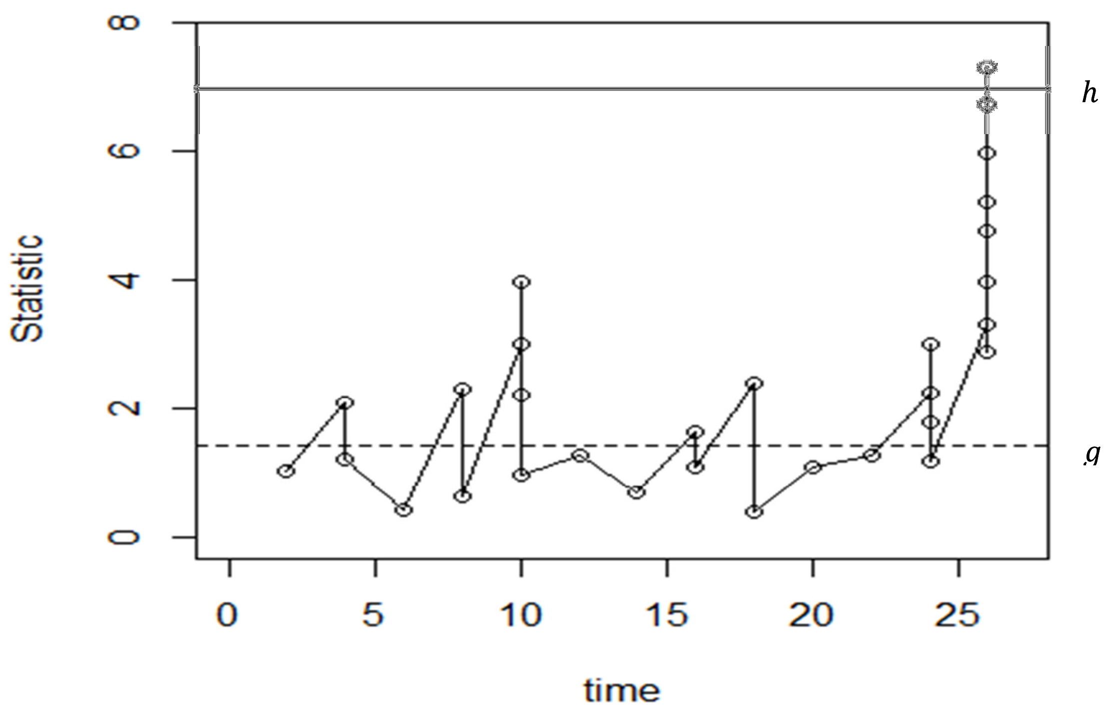

Figure 5.1: **A plot of the SS GLR chart statistic**

In Figure 5.1, the process was in-control until time point 20, and then a shift in both mean and variance of size $\delta = 0.5$ and $\psi = 1.25$ was introduced. Note that points connected with a vertical line means that they were collected at the same sampling time point. For example, the first sample taken at time 2 contained 1 observation, the second sample taken at time 4 contained

2 observations, and so on. Overall the sampling rate was low when the process was in-control. However, once the process went out of control after time point 20, the sampling rate increased drastically and more observations were taken as shown in the plot. Even with a small window, once there is any indication of a process change, sequential sampling implies more observations will be sampled so that a large number of observations will be contained in the window to help make a decision about whether there has been a change in the process.

Through our simulation (details not shown here) we noticed that as m increases while maintaining the same in-control ATS and ANOS, the limit g also increases, and the limit h slightly increases but remains almost constant. An increase in g makes it more likely that sampling will stop at a sampling point, so this has the effect of slowing the detection of any shift that occurs. Therefore, a small window is preferred in the SS GLR chart to obtain the best performance.

To determine the best choice for m, we need to consider the choice of in-control ASN as well, because both of them affect the total number of observations contained in the window, and consequently affect the chart performance. In other words, the optimal choices for m and the in-control ASN should be investigated simultaneously. Also remember that for purposes of fair comparisons, the value of the in-control ASN is set to be equal to the sampling interval d. So a related question to be considered here is whether a relatively large number of observations should be sampled at each sampling point with a long time interval between samples, or a relatively small number of observations should be sampled using a short time interval.

Table 5.1 displays the SSATS and EQL values for different values of the in-control ASN, each with its corresponding optimal window size. For example, for an in-control ASN of 1.2, the minimum EQL value is reached when $m = 12$, and for an in-control ASN of 1.4 the minimum EQL value is obtained when $m = 5$. Generally, the smaller the in-control ASN, the larger the window size that is required to achieve the best performance. This is not surprising because, if m is fixed, decreasing the in-control ASN would decrease the number of observations in the window.

The optimal EQL value in Table 5.1 is achieved by using an in-control ASN of 1.4 with the window size $m = 5$ in column [3]. The EQL as a function of the in-control ASN is relatively flat around column [3], indicating that performance is still very good for the parameter choices around column [3]. When the in-control ASN increase above 3, the EQL values get significantly larger.

Table 5.1: SSATS and EQL values and the optimal window sizes of SS GLR charts for different in-control ASN values

ASN, d =		1.2	1.3	1.4	1.5	1.6	1.7	1.8	2	3	4
m =		12	8	5	5	4	4	4	4	4	4
δ	ψ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0.00	1.00	1481.63	1481.60	1481.18	1481.30	1481.60	1481.85	1481.56	1481.71	1481.70	1481.23
0.25	1.00	144.92	134.74	133.02	120.35	117.80	111.73	107.57	102.20	97.85	103.51
0.50	1.00	25.85	25.16	25.78	24.55	24.85	24.32	24.10	23.96	26.40	29.99
0.75	1.00	10.43	10.13	10.33	10.12	10.33	10.32	10.38	10.62	12.51	14.76
1.00	1.00	5.89	5.64	5.65	5.63	5.75	5.80	5.91	6.12	7.53	9.07
1.50	1.00	2.80	2.67	2.62	2.66	2.69	2.75	2.83	2.99	3.86	4.77
2.00	1.00	1.69	1.62	1.60	1.63	1.66	1.71	1.77	1.89	2.53	3.19
3.00	1.00	0.87	0.87	0.89	0.93	0.97	1.02	1.06	1.16	1.67	2.18
4.00	1.00	0.65	0.68	0.72	0.77	0.82	0.87	0.92	1.02	1.51	2.01
5.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
6.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
7.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
0.00	1.10	272.63	246.41	228.93	212.08	201.53	192.69	185.90	176.55	164.61	169.43
0.00	1.20	73.82	66.63	62.52	58.73	56.74	55.06	54.03	52.88	54.67	59.94
0.00	1.40	19.31	17.94	17.20	16.79	16.57	16.51	16.54	16.81	19.44	22.57
0.00	1.60	9.77	9.22	8.93	8.87	8.85	8.94	9.06	9.38	11.40	13.59
0.00	1.80	6.37	6.07	5.90	5.94	5.95	6.07	6.19	6.48	8.14	9.86
0.00	2.00	4.70	4.51	4.41	4.47	4.52	4.63	4.75	5.00	6.41	7.86
0.00	2.40	3.11	3.03	3.01	3.08	3.13	3.23	3.33	3.54	4.68	5.82
0.00	3.00	2.13	2.12	2.13	2.20	2.26	2.34	2.43	2.61	3.55	4.48
0.00	5.00	1.24	1.27	1.31	1.37	1.43	1.50	1.57	1.71	2.41	3.12
0.00	7.00	1.00	1.04	1.08	1.14	1.20	1.26	1.32	1.45	2.09	2.72
0.00	10.00	0.85	0.90	0.95	1.00	1.06	1.11	1.17	1.29	1.88	2.47
0.00	15.00	0.76	0.80	0.85	0.90	0.96	1.02	1.07	1.18	1.74	2.30
0.50	1.25	15.55	14.74	14.48	14.20	14.22	14.19	14.29	14.58	17.08	19.98
1.00	1.50	4.79	4.61	4.56	4.61	4.68	4.78	4.90	5.15	6.59	8.08
1.50	1.75	2.69	2.64	2.63	2.70	2.76	2.85	2.94	3.14	4.18	5.24
2.00	2.00	1.89	1.88	1.89	1.96	2.02	2.10	2.18	2.34	3.20	4.06
3.00	2.50	1.25	1.28	1.31	1.37	1.43	1.49	1.56	1.70	2.39	3.09
4.00	3.00	1.00	1.04	1.08	1.14	1.19	1.26	1.32	1.44	2.07	2.70
0.25	1.50	11.72	11.06	10.67	10.57	10.55	10.60	10.71	11.03	13.26	15.68
0.50	2.00	4.26	4.10	4.04	4.09	4.15	4.25	4.37	4.63	5.97	7.33
0.75	2.50	2.63	2.58	2.58	2.65	2.71	2.80	2.89	3.09	4.14	5.19
1.00	3.00	1.96	1.96	1.98	2.05	2.11	2.19	2.27	2.45	3.35	4.24
1.50	4.00	1.40	1.43	1.46	1.53	1.59	1.66	1.74	1.88	2.63	3.39
2.00	5.00	1.16	1.20	1.24	1.30	1.36	1.43	1.50	1.63	2.32	3.00
EQL =		13.84	13.37	13.25	13.30	13.48	13.73	14.03	14.73	18.85	23.23
h =		7.3208	7.2699	7.2065	7.1739	7.1241	7.0893	7.0536	6.9834	6.6720	6.4256
g =		2.6770	2.2706	1.9561	1.8214	1.6635	1.5818	1.5139	1.4050	1.1083	0.9651

Similar to what was done in Chapter 3, to illustrate how to use these results in practice, consider the situation in which a Shewhart $\overline{\mathrm{X}}$&S chart is being used to monitor a process with samples of $n = 4$ being taken every $d = 4$ hours. The sampling rate per unit time is 1 observation per hour. For the Shewhart chart two other sampling schemes with the same sampling rate per unit time are samples of $n = 2$ taken every $d = 2$ hours, and samples of $n = 8$ taken every $d = 8$ hours. The original sampling scheme with the intermediate value of n may actually be the best choice when a Shewhart chart is being used. However, with the SS GLR chart better overall performance will be achieved by using a shorter sampling interval with a lower in-control ASN. Two possibilities with an average in-control sampling rate of 1 observation per hour are $d = 2.0$ with an ASN of 2.0, and $d = 1.4$ with an ASN of 1.4. Table 5.1 shows that $d = 1.4$ with an ASN of 1.4 would be a great choice. However, using a sampling interval of $d = 1.4$ hours may be inconvenient, so $d = 2.0$ with an ASN of 2.0 might be chosen in practice. Column [8] of Table 5.1 shows that the overall performance of $d = 2.0$ with an ASN of 2.0 is still very good. A comparison of the SS GLR chart with the original Shewhart chart in this example will be given later in this chapter.

One natural question raised here is that whether we can use one window size for different values of in-control ASN and still have good performance, because it adds more complication if the window size needs to be adjusted according to the value of the in-control ASN. Table 5.2 gives the performance of SS GLR charts with different in-control ASN values with a fixed window size $m = 5$. The SSATS and EQL values for this window size 5 are shown first, followed by the optimal window size m_{opt} for those different in-control ASN values, and the corresponding EQL_{opt}.

In Table 5.2 the EQL values resulting from $m = 5$ are very close to the EQL_{opt} values for different in-control ASN values. This suggests that using the fixed window size 5 still gives quite good performance for the SS GLR chart. Therefore, we recommend using the simple rule of choosing the window size to be 5, and choosing the in-control ASN to be about 1.4 with a corresponding short sampling interval. The overall best performance of the SS GLR chart is achieved with this setting, and the performance remains good as long as the in-control ASN not too far away from 1.4. We recommend against using a large in-control ASN, such as 3 or 4 with a corresponding long sampling interval, because when the process is in-control there is no need to use a large sample size, and whenever there is any indication of a process change the sequential sampling scheme allows more observations to be obtained as needed.

Table 5.2: SSATS and EQL values of SS GLR charts with window size 5 for different ASN values

ASN, $d =$		1.2	1.3	1.4	1.5	1.6	1.7	1.8	2	3	4
$m =$		5	5	5	5	5	5	5	5	5	5
δ	ψ	[1]	[2]D	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
0.00	1.00	1481.29	1481.22	1481.18	1481.30	1480.96	1481.76	1481.49	1481.54	1481.21	1480.89
0.25	1.00	189.63	152.51	133.02	120.35	112.44	106.92	103.08	98.34	95.19	100.84
0.50	1.00	32.34	27.96	25.78	24.55	23.83	23.44	23.26	23.25	25.87	29.44
0.75	1.00	11.78	10.77	10.33	10.12	10.06	10.08	10.16	10.41	12.38	14.63
1.00	1.00	6.09	5.77	5.65	5.63	5.68	5.74	5.84	6.07	7.52	9.01
1.50	1.00	2.67	2.61	2.62	2.66	2.71	2.77	2.84	3.00	3.88	4.79
2.00	1.00	1.59	1.58	1.60	1.63	1.68	1.73	1.79	1.90	2.54	3.20
3.00	1.00	0.84	0.86	0.89	0.93	0.98	1.02	1.07	1.17	1.67	2.19
4.00	1.00	0.64	0.68	0.72	0.77	0.82	0.87	0.92	1.02	1.51	2.01
5.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
6.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
7.00	1.00	0.60	0.65	0.70	0.75	0.80	0.85	0.90	1.00	1.50	2.00
0.00	1.10	288.59	252.99	228.93	212.08	199.93	191.30	184.49	175.47	163.82	168.00
0.00	1.20	79.27	68.42	62.52	58.73	56.27	54.69	53.65	52.57	54.47	59.65
0.00	1.40	19.70	17.99	17.20	16.79	16.60	16.54	16.61	16.88	19.54	22.64
0.00	1.60	9.65	9.12	8.93	8.87	8.89	9.00	9.13	9.43	11.48	13.66
0.00	1.80	6.15	5.95	5.90	5.94	6.01	6.12	6.25	6.54	8.19	9.90
0.00	2.00	4.50	4.41	4.41	4.47	4.57	4.68	4.79	5.04	6.46	7.91
0.00	2.40	2.97	2.96	3.01	3.08	3.16	3.26	3.35	3.57	4.71	5.84
0.00	3.00	2.04	2.07	2.13	2.20	2.28	2.36	2.45	2.63	3.56	4.50
0.00	5.00	1.20	1.25	1.31	1.37	1.44	1.50	1.58	1.71	2.42	3.12
0.00	7.00	0.98	1.03	1.08	1.14	1.20	1.26	1.33	1.45	2.09	2.72
0.00	10.00	0.84	0.89	0.95	1.00	1.06	1.12	1.17	1.29	1.88	2.47
0.00	15.00	0.75	0.80	0.85	0.90	0.96	1.02	1.07	1.18	1.74	2.30
0.50	1.25	16.41	15.12	14.48	14.20	14.09	14.11	14.18	14.49	17.05	19.93
1.00	1.50	4.66	4.55	4.56	4.61	4.70	4.80	4.92	5.18	6.62	8.09
1.50	1.75	2.58	2.59	2.63	2.70	2.78	2.87	2.96	3.16	4.21	5.24
2.00	2.00	1.81	1.84	1.89	1.96	2.03	2.11	2.19	2.36	3.21	4.08
3.00	2.50	1.21	1.26	1.31	1.37	1.44	1.50	1.57	1.71	2.40	3.10
4.00	3.00	0.98	1.03	1.08	1.14	1.20	1.26	1.32	1.45	2.08	2.70
0.25	1.50	11.74	11.01	10.67	10.57	10.58	10.63	10.78	11.09	13.35	15.77
0.50	2.00	4.07	4.02	4.04	4.09	4.19	4.29	4.41	4.65	6.00	7.38
0.75	2.50	2.51	2.53	2.58	2.65	2.73	2.82	2.92	3.11	4.16	5.21
1.00	3.00	1.89	1.92	1.98	2.05	2.13	2.20	2.29	2.46	3.36	4.26
1.50	4.00	1.36	1.41	1.46	1.53	1.60	1.67	1.74	1.89	2.64	3.40
2.00	5.00	1.13	1.18	1.24	1.30	1.37	1.43	1.50	1.64	2.32	3.01
EQL =		14.02	13.41	13.25	13.30	13.49	13.74	14.06	14.75	18.91	23.25
$m_{\text{opt}} =$		12	8	5	5	4	4	4	4	4	4
$\text{EQL}_{\text{opt}} =$		13.84	13.37	13.25	13.30	13.48	13.73	14.03	14.73	18.85	23.23
h=		7.2495	7.2346	7.2065	7.1739	7.1381	7.1022	7.0658	6.9943	6.6787	6.4322
g=		2.4021	2.1347	1.9561	1.8214	1.7170	1.6320	1.5611	1.4480	1.1405	0.9900

Note that the recommendations above are mainly based on the EQL values resulting from the exponential prior distribution with parameter one. A natural question is how these conclusions about m and the in-control ASN would change if a different prior distribution is used. A number of different prior distributions were investigated, including exponential prior distributions with other parameter choices, gamma prior distributions, and a uniform prior distribution. Our simulation results (not displayed here) show that for different prior distributions the optimal window sizes are around 5 and the optimal in-control ASN values are between 1.1 and 1.4. The differences between the EQL values resulting from the optimal parameter settings and the EQL values resulting from the recommended parameter settings are very small, which indicates that the recommended parameter settings produce optimal or almost optimal EQL values. Thus, we recommend the simple rule of setting the in-control ASN to be about 1.4 and window size to be 5.

It appears that the optimal m and in-control ASN here are slightly different from the optimal values in Chapter 3. The optimal values here are smaller, so on average fewer observations in the window are needed to reach the optimal performance of the SS GLR chart for monitoring μ and σ^2. This seems reasonable because changes in the two parameters (μ, σ^2) are generally easier to detect than changes in only the one parameter μ. In other words, changes in both parameters produce relatively larger effects, and therefore more easily trigger a signal, so fewer observations are needed to reach the optimal performance.

5.3 The limits of GLR Chart with Sequential Sampling

In the SS GLR chart with the recommended window $m = 5$, we need to find the values of the two limits, h and g, which determine the in-control ANSS and the in-control ASN. Figure 5.2 plots h and g as functions of the in-control ANSS (in the log scale) and the in-control ASN. Both h and g appear to be approximately linear functions of $\log_{10}$ ANSS and ASN.

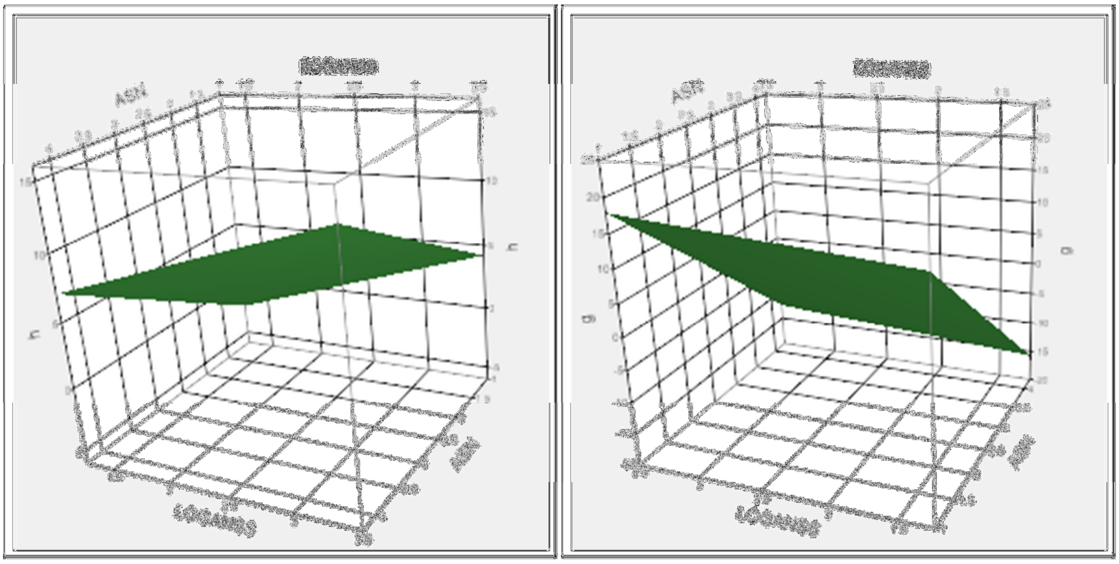

Figure 5.2: The limits h and g as functions of the in-control ANSS and ASN

In order to find the values of h and g, two regression equations were fitted with l representing the in-control $\log_{10}$ ANSS and a representing the in-control ASN. Similarly as in equations (3.5) and (3.6), some higher order terms were included to give a more accurate fit.

$$h = -3.196306 + 2.954527a - 1.297433a^2 + 0.282708a^3 - 0.023459a^4 + 2.965971l - 0.055621l^2 - 0.148813al + 0.005929a^2l + 0.019456al^2 \tag{5.6}$$

$$g = 5.942255 - 9.367659a + 5.049286a^2 - 1.182428a^3 + 0.101184a^4 + 1.459910l - 0.168979l^2 - 0.254634al + 0.028345a^2l + 0.012380al^2 \tag{5.7}$$

Once the values of the in-control ANSS and the in-control ASN are specified by practitioners, equations (5.6) and (5.7) can be easily used to calculate the values of h and g. The sampling interval d can be chosen based on practical convenience or the desired in-control sampling rate. The in-control ATS can be quickly obtained by multiplying d and the in-control ANSS. In some applications, a practitioner may specify the in-control ASN, the sampling interval d, and the in-control ATS first, and then the in-control ANSS can be calculated using the in-control ATS$/d$, and the limits h and g can be found using the equations.

In fitting the regression equations (5.7) and (5.8), the in-control ANSS ranged from 25 to 2500 and the in-control ASN ranged from 1.2 to 4. Both equations have a coefficient of determination (model R^2) larger than 0.99, indicating very accurate prediction of h and g when the in-control ANSS and in-control ASN are in those ranges.

5.4 An Illustrative Example

As in Section 3.5, we provide an illustrative example here. Consider a situation in which the FSR Shewhart $\overline{\mathrm{X}}$&S charts are being used and testing is expensive and destructive. We consider several scenarios where performance can be improved using the SS GLR chart. In one scenario the objective is to reduce the time required to detect shifts, and we show the results for two choices of the sampling interval. Then we consider the scenario in which the objective is to reduce sampling costs. In each case we show how to set up the SS GLR chart.

Currently the company is sampling $n = 4$ items every 4 hours, and using the FSR Shewhart $\overline{\mathrm{X}}\&S$ charts with 3-sigma limits to monitor the process mean and variance. Process engineers want to reduce the time to signal so that fewer nonconforming items will be produced. Preliminary studies show that individual items can be sampled and tested in a fairly short time. The objective is to design a chart that will have a relatively low false alarm rate, will perform well for a wide range of shift sizes, and will reduce the time required to detect shifts in mean and/or variance compared to the previously employed $\overline{\mathrm{X}}$&S charts.

Using 3-sigma limits with $n = 4$ and $d = 4$ for the Shewhart $\overline{\mathrm{X}}$&S charts gives an in-control ATS of 558.97. Note that using Shewhart $\overline{\mathrm{X}}$ chart alone with 3-sigma limit would give the in-control ATS of 1481.6, but using the $\overline{\mathrm{X}}$&S charts together with 3-sigma limits gives a smaller in-control ATS. Consider first employing an SS GLR chart with a sampling interval of $d = 4$ hours and an in-control ASN of 4 to correspond to the $\overline{\mathrm{X}}$&S charts. The window size is set to be 5, which is the recommended window size. The desired in-control ATS is set to be 558.97 to match the setting for the $\overline{\mathrm{X}}$&S charts, which means that the in-control ANSS will be $\mathrm{ATS}/d = 139.74$. Note that this matching would not necessarily be required in practice, but we do it here so we can compare the SS GLR chart to the existing Shewhart charts. The in-control average sampling rate is 1 item per hour. Using the equations (5.7) and (5.8), the limits are $h = 5.3424$ and $g = 0.8579$.

Table 5.3: SSATS and EQL values for SS GLR charts and the FSR Shewhart chart

		$\overline{\mathrm{X}}$&S	SS GLR		
ASN, $n =$		4	4	1.4	1.4
$d =$		4	4	1.4	2
δ	ψ	[1]	[2]	[3]	[4]
0.00	1.00	558.97	560.50	560.99	561.24
0.25	1.00	365.58	79.73	104.61	130.88
0.50	1.00	145.34	25.43	23.39	31.45
0.75	1.00	54.35	13.06	9.64	13.22
1.00	1.00	22.63	8.21	5.36	7.40
1.50	1.00	5.96	4.46	2.52	3.50
2.00	1.00	2.75	3.04	1.55	2.16
3.00	1.00	2.01	2.15	0.88	1.24
4.00	1.00	2.00	2.01	0.72	1.03
5.00	1.00	2.00	2.00	0.70	1.00
6.00	1.00	2.00	2.00	0.70	1.00
7.00	1.00	2.00	2.00	0.70	1.00
0.00	1.10	207.34	117.87	145.41	167.76
0.00	1.20	97.31	48.28	50.84	64.82
0.00	1.40	33.54	19.73	15.69	21.25
0.00	1.60	16.62	12.29	8.39	11.52
0.00	1.80	10.18	9.04	5.64	7.79
0.00	2.00	7.12	7.32	4.24	5.90
0.00	2.40	4.45	5.49	2.90	4.07
0.00	3.00	3.06	4.28	2.07	2.92
0.00	5.00	2.16	3.03	1.29	1.82
0.00	7.00	2.05	2.67	1.07	1.52
0.00	10.00	2.01	2.43	0.94	1.33
0.00	15.00	2.00	2.28	0.84	1.21
0.50	1.25	41.35	17.63	13.53	18.26
1.00	1.50	9.65	7.48	4.37	6.08
1.50	1.75	4.48	4.94	2.54	3.57
2.00	2.00	3.00	3.89	1.85	2.60
3.00	2.50	2.22	3.01	1.29	1.82
4.00	3.00	2.06	2.64	1.07	1.52
0.25	1.50	21.30	14.06	10.00	13.65
0.50	2.00	6.52	6.84	3.88	5.41
0.75	2.50	3.78	4.91	2.50	3.51
1.00	3.00	2.87	4.06	1.93	2.72
1.50	4.00	2.29	3.28	1.44	2.03
2.00	5.00	2.12	2.92	1.23	1.73
EQL=		26.80	21.59	12.42	17.10
$h =$		*	5.3424	6.0942	5.6874
$g =$		*	0.8579	1.8569	1.7960
$h_{\overline{\mathrm{X}}} =$		1.5	*	*	*
$g_{\overline{\mathrm{X}}} =$		-1.5	*	*	*
$h_{\mathrm{S}} =$		2.088	*	*	*
$g_{\mathrm{S}} =$		0	*	*	*

Table 5.3 gives SSATS and EQL values for the FSR Shewhart $\overline{\mathrm{X}}$&S charts and some SS GLR charts. Column [1] shows the results for the Shewhart $\overline{\mathrm{X}}$&S charts, and column [2] gives the results for the SS GLR chart with the above setting. The h and g values obtained from the equations give an in-control ATS value quite close to the desired value. The SS GLR chart in column [2] will detect small shifts much faster than the $\overline{\mathrm{X}}$&S charts in column [1], although the $\overline{\mathrm{X}}$&S charts are better for some moderate and large shifts. Overall the EQL value of the $\overline{\mathrm{X}}$&S charts in column [1] is larger than that of the SS GLR chart in column [2].

Given the same in-control sampling rate, the SS GLR chart will have better overall performance by using a lower in-control ASN along with a shorter sampling interval. Thus, instead of taking an average of 4 items every 4 hours when the process is in control, one can set the in-control ASN to be 1.4 and adjust the sampling interval to be 1.4 to maintain the same in-control sampling rate. The in-control ANSS is equal to $\mathrm{ATS}\,/d = 399.26$ when $d = 1.4$. Equations (5.7) and (5.8) can be used again to obtain the limits $h = 6.0942$ and $g = 1.8569$. Column [3] shows the SSATS values of this SS GLR chart. Although the performance is a bit worse than the SS GLR chart in column [2] for the very small shifts, it is much better for moderate to large shifts. Overall the SS GLR chart in column [3] has better performance than the chart in column [2], and in fact it has a much smaller EQL value. The performance of the SS GLR chart in column [3] is uniformly better than that of the FSR Shewhart $\overline{\mathrm{X}}$&S charts in column [1].

In some applications an objective is to reduce the sampling cost, and it may also be more convenient to have the sampling interval to be an integer. Consider now taking samples every 2 hours rather than every 1.4 hours. The in-control ASN is still set to be 1.4 so the average in-control sampling rate is 0.7, which means that the sampling cost is reduced. The in-control ANSS is equal to $\mathrm{ATS}/d = 279.49$ and the in-control ASN is 1.4. The limits $h = 5.6874$ and $g = 1.7960$ were obtained using the regression equations (5.7) and (5.8). The performance of this SS GLR chart is shown in column [4] of Table 5.3. With a lower in-control sampling rate, the performance of the SS GLR chart in [4] is not as good as the chart in column [3], but it is still uniformly better than the FSR Shewhart $\overline{\mathrm{X}}$&S charts in column [1]. Therefore, the SS GLR chart could be applied in this application to provide much faster detection of process mean and/or variance shifts with a lower average sampling cost.

Chapter 6. Performance Comparisons of Control Charts for Monitoring the Process Mean and Variance

For the problem of monitoring the process mean and variance, the FSR GLR chart has been shown to have very good overall performance compared to other FSR control charts, such as combinations of CUSUM charts or the likelihood ratio with EWMA estimator (ELR) chart. See Reynolds et al. (2013) for more details. A similar conclusion is expected to be obtained by comparing the SS GLR chart to other SS control charts. However, for univariate normal process monitoring, there is no other SS control chart that has been studied except for 1) the SPRT chart, which is designed for detection of a one-sided change in a single parameter (see Stoumbos and Reynolds (1997) and the results in Chapter 4 for some examples); 2) the SS GLR chart for detection of two-sided changes in the mean parameter only (see Peng and Reynolds (2014)), which we developed in the first part of this dissertation. Therefore, in this section we compare the SS GLR chart to the FSR GLR chart to see the improvement in the GLR chart resulting from using SS.

The FSR GLR chart takes samples of size n using a fixed sampling interval d. The in-control sampling rate is set to be 1 for the purpose of comparison with the SS GLR chart, so we have $d = n$. Four different values of n (1, 2, 3, and 4) are considered. The window size for the FSR GLR chart is recommended to be large enough such that the performance is essentially the same as the performance of the FSR GLR chart without the window. Here the window size used is $m = 800/n$. The FSR GLR chart statistic is the same as in the SS GLR chart except that it is calculated only after the complete sample of n observations is obtained at each sampling point.

For the SS GLR chart, the value of the average in-control ASN is set to be the same as the sampling interval d. The average in-control ASN must be larger than 1 to allow for the possibility of more than one observation at the sampling points. Four different values (1.4, 2, 3, and 4) are investigated. The window size m is taken to be 5 in all cases. Note that we already showed that a parameter setting of the in-control ASN $= 1.4$ and window $m = 5$ is a good choice in Section 5.2.

Table 6.1: SSATS and EQL values for SS and FSR GLR charts

		SS GLR charts				FSR GLR charts			
ASN(n), $d =$		1.4	2	3	4	1	2	3	4
$m =$		5	5	5	5	800	400	266	200
δ	ψ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0.00	1.00	1481.18	1481.54	1481.21	1480.89	1481.49	1481.58	1481.60	1481.56
0.25	1.00	133.02	98.34	95.19	100.84	150.74	140.20	134.64	130.88
0.50	1.00	25.78	23.25	25.87	29.44	44.07	41.77	40.53	39.74
0.75	1.00	10.33	10.41	12.38	14.63	21.24	20.34	19.87	19.50
1.00	1.00	5.65	6.07	7.52	9.01	12.64	12.18	11.90	11.67
1.50	1.00	2.62	3.00	3.88	4.79	6.08	5.85	5.69	5.54
2.00	1.00	1.60	1.90	2.54	3.20	3.60	3.44	3.30	3.19
3.00	1.00	0.89	1.17	1.67	2.19	1.69	1.56	1.65	2.02
4.00	1.00	0.72	1.02	1.51	2.01	0.96	1.04	1.50	2.00
5.00	1.00	0.70	1.00	1.50	2.00	0.62	1.00	1.50	2.00
6.00	1.00	0.70	1.00	1.50	2.00	0.52	1.00	1.50	2.00
7.00	1.00	0.70	1.00	1.50	2.00	0.50	1.00	1.50	2.00
0.00	1.10	228.93	175.47	163.82	168.00	304.31	290.11	280.95	274.72
0.00	1.20	62.52	52.57	54.47	59.65	108.75	104.26	101.66	99.71
0.00	1.40	17.20	16.88	19.54	22.64	35.66	34.81	34.30	33.93
0.00	1.60	8.93	9.43	11.48	13.66	18.60	18.43	18.30	18.17
0.00	1.80	5.90	6.54	8.19	9.90	11.89	11.92	11.88	11.83
0.00	2.00	4.41	5.04	6.46	7.91	8.49	8.58	8.59	8.60
0.00	2.40	3.01	3.57	4.71	5.84	5.26	5.38	5.42	5.48
0.00	3.00	2.13	2.63	3.56	4.50	3.31	3.44	3.52	3.64
0.00	5.00	1.31	1.71	2.42	3.12	1.59	1.75	1.98	2.28
0.00	7.00	1.08	1.45	2.09	2.72	1.15	1.37	1.68	2.08
0.00	10.00	0.95	1.29	1.88	2.47	0.90	1.18	1.57	2.02
0.00	15.00	0.85	1.18	1.74	2.30	0.74	1.08	1.52	2.00
0.50	1.25	14.48	14.49	17.05	19.93	28.86	28.06	27.67	27.33
1.00	1.50	4.56	5.18	6.62	8.09	8.94	8.94	8.91	8.86
1.50	1.75	2.63	3.16	4.21	5.24	4.66	4.72	4.74	4.77
2.00	2.00	1.89	2.36	3.21	4.08	3.02	3.09	3.16	3.27
3.00	2.50	1.31	1.71	2.40	3.10	1.74	1.86	2.04	2.31
4.00	3.00	1.08	1.45	2.08	2.70	1.25	1.42	1.71	2.09
0.25	1.50	10.67	11.09	13.35	15.77	22.20	21.90	21.69	21.52
0.50	2.00	4.04	4.65	6.00	7.38	7.58	7.67	7.70	7.69
0.75	2.50	2.58	3.11	4.16	5.21	4.29	4.41	4.47	4.54
1.00	3.00	1.98	2.46	3.36	4.26	2.98	3.10	3.20	3.33
1.50	4.00	1.46	1.89	2.64	3.40	1.90	2.05	2.23	2.48
2.00	5.00	1.24	1.64	2.32	3.01	1.46	1.63	1.88	2.21
EQL =		13.25	14.75	18.91	23.25	23.01	23.12	23.55	24.26
$h =$		7.2065	6.9943	6.6787	6.4322	7.4671	6.8345	6.4544	6.1774
$g =$		1.9561	1.4480	1.1405	0.9900	*	*	*	*

Table 6.1 shows the SSATS values for shifts in μ, for increases in σ^2, and for shifts in μ together with increases in σ^2 for the SS GLR chart and FSR GLR chart, as well as their EQL

values. The first four columns (labeled [1] – [4]) give the results for the SS GLR chart and columns [5] – [8] give the results for the FSR GLR chart.

In Table 6.1 columns [1] – [4] show that increasing the value of the in-control ASN does not improve the performance of the SS GLR chart, except for very small shifts in μ or σ^2 alone. The SS GLR chart in column [1] has the best overall performance, and the overall performance gradually deteriorates from column [1] to [4], as indicated by the EQL values. Comparing columns [5] – [8], we see that the FSR GLR chart in column [5] is better for detecting large shifts, while the FSR GLR chart in column [8] is better for detecting small shifts. The overall performance of the FSR GLR chart measured by the EQL is not sensitive to the value of n. Comparing the SS GLR chart to the FSR GLR chart, we can see that overall the SS GLR chart has much better performance than the FSR GLR chart if the in-control ASN is chosen appropriately. The EQL value for the SS GLR chart in column [1] is almost half of the EQL values for any of the FSR GLR charts, indicating the significant improvement in the performance of the GLR chart resulting from using SS instead of FSR.

It is interesting to note in Table 6.1 that the EQL for [4] is almost as large as for [8]. This seems to indicate that an SS GLR chart with a large in-control ASN, such as ASN$=4$, is not much better than a FSR GLR chart with this value of n.

Chapter 7. Conclusions and Ideas for Future Research

In this dissertation we first showed that for detecting a wide range of two-sided shifts in the normal process mean parameter μ, the SS GLR chart has very good overall performance. The one-sided GLR chart for monitoring μ has also been investigated and its performance is quite comparable with the SPRT chart. Then we developed a SS GLR chart for simultaneously monitoring the normal process mean μ and variance σ^2, and we found the SS GLR chart is very effective in detecting two-sided shifts in μ and/or increasing shifts in σ^2. In both cases, the SS GLR chart also has the advantage that it does not require practitioners to specify multiple chart parameters, and the estimates of process parameters are immediately available from the calculation of the SS GLR chart statistics. Design guidelines for the SS GLR chart are given and several examples are provided to illustrate how to use the SS GLR chart in applications. In summary, as long as the sequential sampling scheme is feasible, the SS GLR chart is recommended for monitoring μ and/or σ^2.

There are several options for future research based on the content of this dissertation.

1. Consider the situation where a gradual drift or a transient shift occurs in the process parameters. The FSR GLR chart that was designed for sustained shifts has been shown to work quite well for drifts or transient shifts. The current SS GLR chart is designed for detecting sustained shifts and it has been shown here that the SS GLR chart is effective for such shifts. If the SS GLR chart can also be shown to detect drifts or transient shifts as effectively as in the case of the FSR GLR chart, this SS GLR chart will be more attractive. If not, a possible solution is to develop an SS GLR chart for detecting drifts or transient shifts.

2. It is assumed that there is no constraint on the sample size at each sampling point in the current sequential sampling scheme. When the process is out of control, the sample size is shown to increase to detect shifts quickly. Although the average sample size will not be extremely large, it is possible that a particular sample size becomes very large. In practice it may not be possible to take such a large sample size so an upper bound could be added for the sample size at each

sampling point. It may be of interest to investigate the issue of an upper bound and see its effect. However, adding an upper bound may complicate the design of the chart.

3. Develop a GLR chart with sequential sampling for multivariate process monitoring. In the case of a fixed sampling rate, GLR charts have been developed and shown to be effective for detecting changes in the mean vector or the covariance matrix (see Wang (2012) and Wang and Reynolds (2012)). It would be interesting to see how the GLR chart with sequential sampling would perform for such process monitoring.

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