Strategic Planning for the Reverse Supply Chain: Optimal End-of-Life Option, Product Design, and Pricing

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A company’s decisions on how to manage its reverse supply chain (RSC) are important for both economic and environmental reasons. From a strategic standpoint, the key decision a manufacturer makes is whether or not to collect products at their end-of-life (EOL) (i.e., when their useful lives are over), and if so, how to recover value from the recovered products. We call this decision as the EOL option of a product, and it determines how the RSC is designed and managed overall. Many EOL options exist for a product such as resale, refurbishment, remanufacturing and part salvage. However, many factors influence the optimal EOL option. These factors include the product’s: (i) characteristics, (ii) design, and (iii) pricing. A product’s characteristics are its properties that impact the various costs incurred during its production, residual part values, and customer demand. In this work, the product design is viewed as the choice of quality for each of its parts. A part’s quality-level determines, among other things, its cost, salvage value, and the likelihood of obtaining it in good condition from a disassembled used product. Finally, the manufacturer must determine how to price its new and used products. This decision depends on many considerations such as whether new and used products compete and whether competition exists from other manufacturers. The choice of appropriate EOL options for products constitutes a foundation of RSC design. In this work, we study how to optimally determine a product’s optimal EOL option and consider the impact of product design and product pricing on this decision.

We present a full description of the system that details the relationships among all entities. The system description reveals the use of a production planning type of modeling strategy. Additionally, a comprehensive and general mathematical model is presented that takes into consideration multi-period planning and product inventory. A unique aspect
of our model over previous production planning models for RSC is that we consider the product returns as being endogenous variables rather than them being exogenous. This model forms the basis of our research, and we use its special cases in our analysis.

To begin our analysis of the problem, we study the case in which the product design and price are fixed. Both non-mandated and mandated collection are considered. Our analysis focuses on a special case of the problem involving two stages: in the first stage, new products are produced, and in the second stage, the EOL products are collected for value recovery. For fixed product design and price, our analysis reveals a fundamental mapping of product characteristics onto optimal EOL options. It is germane to our understanding of the problem in general since a multi-period problem is separable into multiple two-stage problems. Necessary and sufficient optimality conditions are also presented for each possible solution of this two-stage problem. For the two-part problem, a graphical mapping of product characteristics onto optimal EOL options is also presented, which reveals how EOL options vary with product characteristics.

Additionally, we study the case of product design under mandated collection, as encountered in product leasing. We assume new production cost, part replacement cost, and part salvage value to be functions of the quality-level of a part along with the likelihood of recovering a good-part from a returned product. These are reasonable assumptions for leased products since the customer is paying for the usage of the product over a fixed contract period. In this case, the two-stage model can still be used to gain insights. For the two-part problem, a method for mapping part yields onto optimal EOL options is presented. Closed-form optimality conditions for joint determination of part yields and EOL options are not generally attainable for the two-stage case; however, computationally efficient methods for this problem are developed for some relatively non-restrictive special cases. It is found that, typically, a part may belong to one of three major categories: (i) it is of low quality and will need to be replaced to perform remanufacturing, (ii) it is of high quality and its surplus will be salvaged, or (iii) it is of moderate quality and just enough of its amount is collected to meet remanufactured product demand.

Finally, we consider the problem of determining optimal prices for new and remanufactured products under non-mandated manufacturer’s choice of collection. New and remanufactured products may or may not compete, depending on market conditions. Additionally, we assume the manufacturer to have a monopoly on the product. Again, the two-
stage problem is used and efficient solution methods are developed. Efficient solution methods and key insights are presented.
Dedication

To Amanda.
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Executive Summary

The decisions a company makes to manage its reverse supply chain (RSC) are important from both economic and environmental standpoints. A key strategic decision in this regard is whether or not to collect end-of-life (EOL) products (products whose useful lives are over), and if so, how to recover value from the recovered products. We call this decision as the EOL option of a product, and it determines how to design and manage the RSC. The typical EOL options for a product are: resale, refurbishment, remanufacturing and part salvage. The choice of an EOL option for a product is influenced by the following factors belonging to the product:

i. characteristics,

ii. design, and

iii. price.

A product’s characteristics are its properties that describe the various costs of production, residual part values, and demand for the product. In this work, a product’s design is viewed as the quality of each of its parts. A part’s quality-level determines, among other things, its cost, its salvage value, and the likelihood of obtaining a good part from a used product. Finally, the manufacturer must determine how to price the new and used products. This decision depends on whether new and remanufactured products compete and whether competition exists from other manufacturers. The choice of EOL option for a product constitutes the foundation of RSC design. This decision will drive capacity and infrastructure requirements as well as RSC network design. The product characteristics, choice of product design, and
pricing form the pillars of the EOL option decision.

Product recovery can be a lucrative business. In many products, a good deal of residual value remains in a product at its EOL. This value can be viewed as either (i) material value or (ii) functional value. The material value is recovered after having destroyed the form and function of the product. This is known as recycling, and it is, typically, one of the least desirable value recovery methods. A much more desirable option is product resale through either used-product sale or repair/refurbishment/remanufacturing. These options tend to recover much of the value added during the manufacturing of a new product.

In fact, there are numerous examples of original equipment manufacturers (OEMs) and third party companies that are engaged in recovering value from used products. A ubiquitous industry in this regard is the automotive parts industry. A search on the website of Advance Auto Parts with the search term “remanufactured” returns 32,845 results (Advance Auto Parts 2014). Volvo, a major large truck and construction equipment manufacturer, remanufactures its engines and transmissions as well as entire pieces of construction equipment. Kodak’s single use camera is remanufactured multiple times before the product is disposed-of. Other industries that leverage their RSCs include the military-industrial complex, high-tech electronics, and aviation. However, in many industries value-recovery from products tends to be an after-thought. The reverse supply chain activities have a significant impact on the economy. For example, the U.S. Trade Commission estimated that the remanufacturing industry grew 15 percent between 2009 and 2011 to at least $43 billion in the U.S.

Additionally, value recovery from products also helps to conserve resources and reduce consumption of energy. It is estimated that, on average, producing a remanufactured product requires 25% of the energy required for producing a new product (Lund 1996). For these reasons, and others, the European Union has legislated the OEMs of vehicles and electrical/electronic equipment to be responsible for handling the end-of-life (EOL) of their products through the End-of-life Vehicle (ELV) (European Union 2011a) and Waste Electrical and Electronic Equipment (WEEE) (European Union 2011b) directives.
In general, we call the various value-recovery and disposal options as EOL options for a product. These include landfilling, recycling, reselling, refurbishing, remanufacturing, and part salvage. These options can apply to either the entire product or its parts. The reverse supply chain comprises those activities that the products and parts undergo at their EOL.

A product’s EOL option will be determined by its particular characteristics and the nature of the consumer. Furthermore, there are two key system properties that must be accounted for by a model. Firstly, remanufactured products are made from a combination of recovered parts and replacement parts. As such, we must represent a product as a collection of parts, and consider the likelihood of obtaining good-parts from a returned product. Secondly, the availability of products to recover depends on previous sales of new products. For these reasons, we use a production and materials requirements planning model to address the problem. Its purpose is to aid in making the critical strategic-level decisions regarding which EOL option to choose for a product and which design to select for the product. In general, the model can be multi-period and consider factors including: (i) whether or not the product can be re-designed, (ii) whether or not the product’s price can be changed, (iii) nature of the product take-back laws in the market or if there is contractual take-back requirements (such as leasing), (iv) market conditions for the product and its parts, and (v) the cost structure of product production, collection, and value recovery options. By appropriately changing parameters to decision variables and vice-versa, we are able to determine optimal value-recovery decisions for a product under various environments. An environment is defined by: (i) whether or not the product can be re-designed, (ii) whether or not the product’s price can be changed, and (iii) whether or not the product take-back is mandated by law or contract (as in leasing).

In Chapter 1, we provide a review of literature on various aspects of RSC. It covers the production planning, product design and optimal pricing aspects related to RSC. Our review has unveiled a few significant gaps in the literature. Firstly, there is a lack of in-depth analysis of the production planning models for the RSC. Although adequate models for production planning are presented, few insights are obtained. Secondly, there is insufficient integration of product design and product pricing decisions in these production planning models. In this work, we attempt to close these gaps by: (i) providing an in-depth analysis of a simple, yet representative, production planning model that aids in
determining the optimal EOL option of a product, (ii) investigating how to jointly determine EOL option and product design, and (iii) presenting models for jointly determining EOL option and product pricing.

Next, in Chapter 2, a full description of the RSC system that we consider is provided. It details relationships among its various entities. It unveils the use of a production planning type of modeling strategy for analysis. Additionally, a comprehensive and general mathematical model is presented that takes into consideration multi-period planning and product inventory. The unique aspect of our model over previous production planning models for RSC is that our model regards product returns as endogenous variables rather than them being exogenous. This model forms the basis of our research, and we use its special cases in our analysis.

To begin our analysis of the problem, in Chapter 3, we study the case in which the product design and pricing are fixed. Both choice of collection and mandated collection are considered. Our analysis focuses on a special case of the problem involving two stages: in the first stage, new products are produced, and in the second stage, EOL products are collected for value recovery. For fixed product design and pricing, our analysis of the two-stage problem reveals a fundamental mapping of product characteristics onto optimal EOL options. It is germane to our understanding of the problem in general since a multi-period problem is separable into multiple two-stage problems. Necessary and sufficient optimality conditions are presented for each possible solution of this two-stage problem. The key decision is the choice of a part to base the remanufacturing around, designated as the “central” part. The choice of a central part determines the parts that would be in surplus and the parts that would be in deficit, and thus, require purchase of replacement parts. This decision is made by comparing: (i) the profit margin from a remanufactured product (remanufactured product price minus the cost to reassemble parts) against (ii) the sum of the opportunity cost of using a part in a remanufactured product (i.e. the part’s salvage value) and the additional cost of purchasing replacement parts for the remanufactured product for various choices of central part. The best central part is called the “key” part of the product and is given by the part for which (i) and (ii) have the smallest positive difference among all possible choices of central part. If the profit margin from a remanufactured product is less than the sum of the salvage values for all the parts, then all the parts will be salvaged and no remanufacturing occurs. When enough of a particular part has been obtained, the marginal benefit of collecting additional cores becomes negative. This part
is called the “limiting” part since it limits the number of product returns the manufacturer should optimally collect. The determination of a product’s “key” part and “limiting” part specifies the “EOL type” for the product, which is not dependent on the demands for the new or remanufactured products. The optimal EOL option is determined by using the demand information of new and remanufactured products along with the product’s EOL type. We present a graphical mapping of product characteristics onto optimal EOL type for the two-part problem that is generated using optimality conditions. This mapping constitutes a major contribution of our work by providing a graphical representation among product characteristics to determine EOL options.

In Chapter 4, we study the case of product design under mandated collection, as in the case of product leasing. We assume new production cost, part replacement cost, and part salvage value to be functions of the quality-level of the part along with the likelihood of recovering a good-part from a returned product. These are reasonable assumptions for leased products since the customer pays for the usage of the product over a fixed contract period. In this case, the two-stage model can still be used to gain insights. For the two-part problem, a method for mapping part yields onto optimal EOL options is presented. For the two-stage case, closed-form optimality conditions for joint determination of part yields and EOL options are difficult to obtain; however, computationally efficient methods for this problem are possible for some relatively non-restrictive special cases. Our analysis reveals that, in many cases, the yields of parts that are not central, should either be as small as possible or as large as possible. In the case that a part is central, then its yield may take some intermediate value. Qualitatively, we find that if a part has low part replacement cost and low disposal cost relative to the cost of adding quality, then it should be a low-yield part. On the other hand, if the salvage value and disposal cost of a part are high relative to the cost of adding quality, then the part should be made with as much quality as possible. However, if a part has a high replacement cost and low salvage value compared with its cost of quality, then the part is a key part. Essentially, the key parts are those whose replacement costs are very high, but do not have much value in the after-market. As such, we will want just enough of the key parts to be able to meet remanufactured product demand.

In Chapter 5, we consider the problem of determining optimal prices for new and remanufactured products under the manufacturer’s choice of collection. New and remanufactured products may or may not compete, depending on the
market condition. Additionally, we assume that the manufacturer has a monopoly on the product. Again, the two-stage problem provides insights, but is not generally applicable to the multi-period model. For both cases, an efficient solution method is presented.

There are the following significant contributions of this work. Firstly, we present a comprehensive and general production planning model for the RSC that considers product returns as an endogenous variable. Secondly, we study this model to develop a mapping of product characteristics onto optimal EOL options for a two-stage model. For the two-part case, this mapping can be depicted graphically. Thirdly, we study the nature of decisions for optimal product design and EOL options, when considered jointly, and present some properties of optimal solutions. Finally, we investigate the problem of determining product pricing and optimal EOL option, and provide useful insights and efficient solution methods to determine optimal EOL option and product pricing.
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Chapter 1

Introduction, Problem Statement, and Literature Review

1.1 Introduction

The decisions a company makes to manage its reverse supply chain (RSC) are important from both economic and environmental standpoints. A key strategic decision in this regard is whether or not to collect end-of-life (EOL) products (products whose useful lives are over), and if so, how to recover value from the recovered products. We call this decision as the EOL option of a product, and it determines how to design and manage the RSC. The typical EOL options for a product are: resale, refurbishment, remanufacturing and part salvage. The choice of an EOL option for a product is influenced by the following factors belonging to the product:

i. characteristics,

ii. design, and

iii. price.
A product’s characteristics are its properties that describe the various costs of production, residual part values, and demand for the product. In this work, a product’s design is viewed as the quality of each of its parts. A part’s quality-level determines, among other things, its cost, its salvage value, and the likelihood of obtaining a good part from a used product. Finally, the manufacturer must determine how to price the new and used products. This decision depends on whether new and remanufactured products compete and whether competition exists from other manufacturers. The choice of EOL option for a product constitutes the foundation of RSC design. This decision will drive capacity and infrastructure requirements as well as RSC network design. The product characteristics and choice of product design and pricing form the pillars of the EOL option decision. Figure 1.1 presents this relationship among RSC design, EOL option, and the product’s characteristics, design, and pricing.

![Figure 1.1: EOL option determines RSC design and is determined by product characteristics, design, and pricing.](image)

Product recovery can be a lucrative business. In many products, a good deal of residual value remains in a product at its EOL. This value can be viewed as either (i) material value or (ii) functional value. The material value is recovered
after having destroyed the form and function of the product. This is known as recycling, and it is, typically, one of the least desirable value recovery methods. A much more desirable option is product resale through either used-product sale or repair/refurbishment/remanufacturing. These options tend to recover much of the value added during the manufacturing of a new product.

In fact, there are numerous examples of original equipment manufacturers (OEMs) and third party companies that are engaged in recovering value from used products. A ubiquitous industry in this regard is the automotive parts industry. A search on the website of Advance Auto Parts with the search term “remanufactured” returns 32,845 results (Advance Auto Parts 2014). Volvo, a major large truck and construction equipment manufacturer, remanufactures its engines and transmissions as well as entire pieces of construction equipment. Kodak’s single use camera is remanufactured multiple times before the product is disposed-of. Other industries that leverage their RSCs include the military-industrial complex, high-tech electronics, and aviation. However, in many industries value-recovery from products tends to be an after-thought. The reverse supply chain activities have a significant impact on the economy. For example, the U.S. Trade Commission estimated that the remanufacturing industry grew 15 percent between 2009 and 2011 to at least $43 billion in the U.S.

Additionally, value recovery from products also helps to conserve resources and reduce consumption of energy. It is estimated that, on average, producing a remanufactured product requires 25% of the energy required for producing a new product (Lund 1996). For these reasons, and others, the European Union has legislated the OEMs of vehicles and electrical/electronic equipment to be responsible for handling the end-of-life (EOL) of their products through the End-of-life Vehicle (ELV) (European Union 2011a) and Waste Electrical and Electronic Equipment (WEEE) (European Union 2011b) directives.

In general, we call the various value-recovery and disposal options as EOL options for a product. These include landfilling, recycling, reselling, refurbishing, remanufacturing, and part salvage. These options can apply to either the entire product or its parts. Table 1.1 presents available EOL options. The reverse supply chain comprises those activities that the products and parts undergo at their EOL. Figure 1.2 depicts the concept of the reverse supply chain. The solid arrows represent forward supply chain activities, while the dashed arrows represent reverse supply chain
Table 1.1: Typical end-of-life (EOL) options for products.

<table>
<thead>
<tr>
<th>EOL Option</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Landfill</td>
<td>Dispose-of a product or, its part in a landfill.</td>
</tr>
<tr>
<td>Recycle</td>
<td>Recover material from the product or its part. Any value from the form of the product or its part is destroyed.</td>
</tr>
<tr>
<td>Resale</td>
<td>Sell product or its part on used market as-is.</td>
</tr>
<tr>
<td>Repair/Refurbishment</td>
<td>Fix the product or its part to some specified standard and sell them on the used market.</td>
</tr>
<tr>
<td>Remanufacturing</td>
<td>Re-make the product or its part by using a mixture of recovered and replacement parts so that it meets the &quot;like-new&quot; specification (i.e. identical warranty to that for a new product)</td>
</tr>
</tbody>
</table>

A product’s EOL option will be determined by its particular characteristics and the nature of the consumer. Table 1.2 enumerates the factors that we consider in this regard. Quantities denoted by capital letters are fixed parameters or functions of decision variables. Script capital letters represent possible decision variables.

Furthermore, there are two key system properties that must be accounted for by a model. Firstly, remanufactured products are made from a combination of recovered parts and replacement parts. As such, we must represent a product
Table 1.2: Product characteristics.

<table>
<thead>
<tr>
<th>Revenue/Cost Structure</th>
<th>Consumer Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^c$: Collection cost.</td>
<td>$D^R$: Remanufactured product demand.</td>
</tr>
<tr>
<td>$C^d$: Disassembly cost.</td>
<td>$b$: Product usage time.</td>
</tr>
<tr>
<td>$C^R$: Reassembly cost.</td>
<td>$\lambda$: Consumer willingness to return EOL product.</td>
</tr>
<tr>
<td>$S_k$: Salvage Value of part type $k$.</td>
<td>$\phi^R$: Remanufactured product price.</td>
</tr>
<tr>
<td>$C^\text{disp}_k$: disposal cost of part type $k$.</td>
<td>$\psi_k$: Proportion of good parts of type $k$ recoverable.</td>
</tr>
</tbody>
</table>

as a collection of parts, and consider the likelihood of obtaining good-parts from a returned product. Secondly, the availability of products to recover depends on previous sales of new products. In general, multiple-periods must be considered as well as other factors including: (i) whether or not the product can be re-designed, (ii) whether or not the product’s price can be changed, (iii) nature of the product take-back laws in the market or if there is contractual take-back requirements (such as leasing), (iv) market conditions for the product and its parts, and (v) the cost structure of product production, collection, and value recovery options. An environment is defined by: (i) whether or not the product can be re-designed, (ii) whether or not the product’s price can be changed, and (iii) whether or not the product take-back is mandated by law or contract (as in leasing).

1.2 Problem statement

Three related problems are addressed in this dissertation:

1. Given a product, its characteristics, a fixed product design and price, and the objective of maximizing profit, determine: (i) the optimal number of products to recover, and (ii) the optimal value recovery method.

Specifically, we focus on the remanufacturing and salvage EOL options; however, other options are considered as well. Besides just finding the optimal EOL option for a given product, we also seek to understand why that option is optimal.
2. Given a product, its characteristics, a fixed product price, and the objective of maximizing profit, determine: (i) the optimal number of products to recover, (ii) the optimal value recovery method, and (iii) the level of quality with which to build each part.

Jointly determining optimal product design and EOL option is a complex problem since product quality impacts many product characteristics. For this problem, we focus on the case in which the EOL product collection is mandated (as in leasing) and assume that the part quality impacts: (i) the cost of new parts, (ii) the cost of replacement parts, (iii) the salvage value of parts, and (iv) the proportion of good parts recoverable.

3. Given a product, its characteristics, and a fixed product design, and the objective of maximizing profit, determine: (i) the optimal number of products to recover, (ii) the optimal value recovery method, and (iii) the optimal prices of new and remanufactured products.

For the problem of jointly determining optimal product pricing and EOL option, we assume a monopoly situation under two conditions: (i) new and remanufactured products do not compete, and (ii) new and remanufactured products do compete.

1.3 Literature review

1.3.1 Motivation and current gap in literature

For a company not engaged in remanufacturing, the optimal price and market share of a product are determined by: (a) market environment, (b) cost of production, and (c) the impact that the price will have on the quantity demanded. Based on the demand function, the optimal product price specifies the quantity to produce, which constitutes an input for planning. However, if the company engages in remanufacturing as well, then it is possible for the remanufactured products to compete with new products in the marketplace. This can have significant implications on the pricing of these products. Additionally, the cost to produce remanufactured products is dependent on: (a) disassembly costs, (b) yield percentages of reusable parts from disassembled cores, (c) cost of a replacement part if a reusable part is not available from disassembled cores, and (d) availability of cores. The yield of reusable parts varies with each part
type. Therefore, as cores are dismantled, the required supply of high yield parts will be satisfied most quickly, and, as more cores are disassembled, the ratio of the usable to unusable parts acquired will increase. Disassembly will continue until it reaches a point at which it is less expensive to use a new part than to continue to disassemble. If the availability of cores is too low, then the new parts must be used to fulfill remanufactured product demand since the supply of used parts will be low. Another important consideration is that the company’s capacity to fulfill the demand of remanufactured products is, in part, dependent on the availability of cores. Since the pricing of remanufactured products is dependent on the cost incurred for their production, it is inextricably linked with the amount produced and the planning of its production. In this work, we study this aspect of the RSC and investigate how design and pricing decisions of products can be integrated with their production planning models.

The rest of this literature review is organized as follows. First, in Section 1.3.2, we present background information on RSC. In order to expose the reader to the broad context of RSC, a brief overview of the RSC literature is given in Section 1.3.3. In Sections 1.3.4 and 1.3.5, respectively, we present two distinct research streams on issues belonging to RSC, namely pricing and production planning, and also, present some models that attempt their integration. Finally, we give some concluding remarks in Section 1.3.6. Steeneck and Sarin (2013)

1.3.2 Background

The return of products by consumers to retailers and manufactures is not new, and it has been a part of the RSC for some time. The study and management of the RSC is also called reverse logistics (RL) and, it is defined by Rogers and Tibben-Lembke (1999) as:

...the process of planning, implementing and controlling the efficient, cost-effective flow of raw materials, in-process inventory, finished and related information from the point of consumption to the point of origin for the purpose of recapturing or creating value or for proper disposal.

Mainly, products enter the RSC through customer returns and EOL returns (Tibben-Lembke 2002). New-product returns occur because of customer dissatisfaction, or the product not meeting customer need. These returns can be
characterized by short return lags from purchase time and low product degradation. The EOL returns are the products returned due to failure or obsolescence. The RSC for a product directly returned by a customer to a collection center is different from the one in which a manufacturer has to purchase the used product on the open market. The EOL recoveries can be characterized by longer return lags and greater product degradation. However, depending on the condition of the return, the product may still be of substantial value.

Recapturing value from returns

A returned product, or any of its components, may undergo any of the following EOL options: landfill, recycle, reuse, repair, or remanufacture. Remanufacturing is the process of returning used products to a “like-new” condition. Reusing, repairing, remanufacturing, and recycling all recapture value from the product. However, the cost to perform these operations may be prohibitive in view of the market for the product, in which case, land-filling is the best option. Of course, the condition in which a product is returned is an important factor in determining the most cost-effective EOL option. Those components that are in good shape can be reused, whereas the worn-out components may need to be repaired or disposed of. Also, the products that are returned more quickly are in better condition, and they have stronger market demands. Thus, the timings of returns are important for value recovery.

Markets for used/remanufactured parts

Remanufactured products are sold either via a sales channel that is different from that for new products, or under a different label. Today, the customer-returns in excellent condition are sold in “B Channel” outlets so that they do not directly compete with new products (Tibben-Lembke 2002). Along the same lines, the OEMs worry about market cannibalization by remanufactured products (Atasu et al. 2008). However, the remanufactured products could be relabeled as “green” products in an effort to create a new market segment and reduce cannibalization. In fact, there are studies supporting the idea of environmentally conscious segments of the market and the benefits of capturing such markets (Ottman 1998, Roberts 1996).
Companies engaged in RSC

Remanufacturing is currently performed by numerous small firms. Lund (1996) has reported that, in 1996, there were 73,000 “remanufacturing” firms with an average sales of 2.9 million dollars and an average workforce of 24 people. However, the number of facilities that bring their products to a like-new condition was 6,000 as of a couple of years ago (Lund and Hauser 2010). A survey by Guide and Jayaraman (2000) finds that 33% of the respondents remanufactured automotive parts, 26% aerospace, 15% machinery, 10% bearings, gears, and pumps, 8% office equipment, and 8% other. Additionally, some major companies own remanufacturing facilities as well. Cisco, for example, remanufactured 410,000 units in 2008 (Carless 2008). Caterpillar has been in the diesel engine remanufacturing business since 1972 and uses a variety of programs to recover cores. These include: buy-back of unused inventory, deposits on remanufactured parts and engines, and voluntary take-back of surplus products at prices above scrap value (Stahel 1995). Xerox (Azar et al. 1995), Canon (Canon 2012), Kodak (Geyer et al. 2007), Ford (Ford 2012), and Mercedes-Benz (Mercedes-Benz 2012), also own remanufacturing facilities. In the Class 8 truck business, a Volvo subsidiary, Dex Trucks, dismantles trucks to provide spare parts to Volvo dealers (Sarin and Steeneck 2012). In the computer industry, companies such as Dell (Kumar and Craig 2007) and IBM (Fleischmann et al. 2003) leverage their RSC to obtain replacement parts for computers that need to be fixed under warranty.

1.3.3 Overview of the Literature

Several major research streams exist related to RSC. These include disassembly, reverse supply chain design, end-of-life options and product design, market segmentation for new and remanufactured products, and production planning and inventory control in the presence of product returns. In this section, we expose the reader to the breath of work reported in each of these streams.
Disassembly

Disassembly is a major process in the RSC, and it is the first stage in the material recovery process. In most cases, it is a manual process that is labor intensive (Clegg and Williams 1994) because of its low volume, and also, because cores are returned in varied conditions.

An objective of the disassembly process is to determine the level to which a core should be disassembled as there is a trade-off between the cost of disassembly and the value of the recovered parts. The design of an optimal disassembly process has been studied extensively in the literature. The work in this area can be divided into two groups: (1) those that do not consider costs based on the sequence in which the components are recovered, and (2) those that consider sequence-dependent costs. In the first category, the work by Lambert (1997) is the first to apply linear programming to determine optimal selective disassembly paths, that is, the disassembly paths, which maximize the revenue of the parts recovered minus the cost to obtain them. The problem of the second category is known to be NP-Hard (Moyer and Surendra 1997), and was first studied by Navin-Chandra (1994) who notes this problem to be a special case of the traveling salesman problem (Flood 1956). He develops heuristic algorithms that are implemented in the software ReSTAR. Later, Güngör and Gupta (2001) develop a near-optimal branch-and-bound technique. More recently, Sarin et al. (2006) have studied the problem with sequence dependent setup costs. They model the problem as an asymmetric traveling salesman problem with precedence constraints, and propose heuristic methods that produce solutions, which are within 2% of optimum. Genetic algorithms are developed by Shimizu et al. (2007) to find disassembly sequences for a wide variety of problems. For reviews of work in this area, see Guide et al. (1999) and Lambert (2003).

Although optimizing the disassembly process of a given product design is important, it is also essential to design a product for disassembly, and there has been extensive research reported on this topic. Design for disassembly (DFD) has much in common with design for assembly (DFA) (Boothroyd and Dewhurst 1987). Boothroyd and Alting (1992) have reproduced a table of best practices for DFD developed by Warnecke et al. (1992). For the reader’s convenience, we present it in Table 1.3. The table has six columns, the first of which is a list of guidelines for DFD. The second and third columns indicate the importance of each guideline for manual disassembly and automated disassembly,
respectively, where “A” means very important, “B” means important, and “C” means less important. The last three columns indicate, by an “X”, if a guideline is important for maintenance, remanufacturing, and recycling, respectively.

<table>
<thead>
<tr>
<th>Design for Disassembly Guidelines</th>
<th>PHASE I: DRAFT</th>
<th>PHASE II: DESIGN</th>
<th>PHASE III: SPECIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>PHASE I: DRAFT</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Linear and unified disassembly direction</td>
<td>B</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Sandwich structure with central joining elements</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Base part product structure</td>
<td>B</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Standardized assembly groups for variants</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Avoid non-rigid parts</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>PHASE II: DESIGN</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integration of parts</td>
<td>B</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Include nominal break points</td>
<td>B</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Operating spots for destroying separation tools</td>
<td>B</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Minimize number of joining elements</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Use joining elements that are detachable or easy to destroy</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Parts should be easy to pile or store to save room</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Non-aging material combination</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Non-corrosive material combination</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Protect assembly groups from soiling or corrosion</td>
<td>A</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Design of parts for easy transport</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Limitation to number of different materials</td>
<td>B</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Integration of poisonous substances in closed units</td>
<td>A</td>
<td>C</td>
<td>X</td>
</tr>
<tr>
<td>Avoid turning operations for disassembly</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>PHASE III: SPECIFICATION</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standardize parts for multiple use</td>
<td>B</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Standard and simple joining techniques</td>
<td>B</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Marking of central joining elements for disassembly</td>
<td>B</td>
<td>C</td>
<td>X</td>
</tr>
<tr>
<td>Open access and visibility at separation points</td>
<td>A</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Center elements on base parts</td>
<td>C</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Standard gripping spots near center of gravity</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Enable simultaneous separation and disassembly</td>
<td>B</td>
<td>B</td>
<td>X</td>
</tr>
<tr>
<td>Avoid necessity for simultaneous disassembly at different joining elements</td>
<td>B</td>
<td>A</td>
<td>X</td>
</tr>
<tr>
<td>Use of parts with narrow tolerance</td>
<td>C</td>
<td>B</td>
<td>X</td>
</tr>
</tbody>
</table>

Ishii et al. (1994) have suggested for designers to create “clumps” of components or sub-assemblies that share a physical relationship or some design intent. Bras and Hammond (1996) have developed general guidelines for designing products for remanufacturability, and also, they propose a remanufacturability metric. Ijomah et al. (2007) use industrial case studies to validate previous DFD guidelines as well as reveal the need for more in-depth investigation of DFD guidelines. For a comprehensive review on this topic, the reader is referred to Bras and McIntosh (1999).
Other aspects of disassembly that must be considered are the material explosion from disassembled parts and the variable-part-dismantling times, and they have been addressed by Guide and Srivastava (1998). Also, McGovern and Gupta (2007) study how best to balance the disassembly line using meta-heuristics since the underlying problem is NP-hard. Additional references on disassembly include Lambert (2003), Kim et al. (2007), Ilgin and Gupta (2010), and books by McGovern and Gupta (2011) and Lambert and Gupta (2005).

**Reverse supply chain design**

The reverse supply chain design requires determination of the sizes and locations of the collection centers and re-processing (remanufacturing) facilities, among others. This design is complicated by the fact that product returns are stochastic and the RSC is not independent of the forward supply chain. Extensive literature exists on the RSC design. Both deterministic and stochastic models have been proposed. In a seminal model in this area, Jayaraman et al. (1999) capture both the forward and reverse flows of materials. They use a deterministic model and have reported that having sufficient core quantities significantly affects the optimal solution, and thus, managers must be active in obtaining these cores though cooperative agreements with customers. Özeylan and Paksoy (2013) develop a mixed integer programming model for a closed-loop supply-chain (CLSC). Lehr, Thun, and Milling (2013) present a system dynamics CLSC model. Fleischmann et al. (2000) have performed a study on reverse supply configurations in different businesses. They notice that an uncertainty in the RSC increases network complexity, and they recommend inclusion of this feature in future models. Since then, some stochastic formulations have been developed, which employ robust optimization (Realff et al. 2000) and stochastic programming (Listeș 2007). Ilgin and Gupta (2010) provide a comprehensive review on the design of RSC as of 2010.

**Choice of end-of-life option and product design**

Determination of an EOL option for a product is important because it can affect its profitability. As also noted earlier, typical EOL options include landfilling, recycling, resale, refurbishment, and remanufacturing. The EOL option for a product and its design are best determined together. In case a manufacturer is required to recover products, Krikke et
al. (1998) develop a model, solved via a dynamic programming approach, to determine the optimal EOL option for the product, or for the individual parts. It is assumed that the market can absorb all the recovered products/parts, and the product design is fixed. The model allows for the optimization of EOL choices across product mixes. More recently, the design of product families has been addressed to take advantage of part commonalities. Kwak and Kim (2011) and Mangun and Thurston (2002) have considered how part commonality among products can impact effectiveness of remanufacturing operations for cell phones and PC’s, respectively. They find that part commonality improves the profitability of a company while making it more environmentally conscious. Kumar et al. (2007) propose a value-flow modeling method to determine the best EOL for a product. In this model, the recovery value of the product decreases over time. Rose and Ishii (1999) take a unique approach by developing a tool for use at the product design stage, which suggests an EOL option based only on a product’s design-independent characteristics, that is, the characteristics that a product would have regardless of the design chosen. They identify the current EOL best practices for various products and the products’ design independent characteristics. Using this data, they develop a prediction tool for determining the best EOL options for a product given its design independent characteristics. Ilgin and Gupta (2010) give a full review of the literature on EOL option choice.

A key variable in the design for remanufacture is the choice of part durability. Zhou et al. (2010) have studied product recovery and reuse for electronic products. They fix the number of life cycles for a product to go through and solve a highly nonlinear optimization problem, which determines an optimal reliability value for the parts of a product in order to minimize the cost of part recovery and replacement. The authors provide an example on the use of the model to calculate an optimal disassembly path for a PC and the best disposal option for each PC component. They have also modeled a “degradation rate” of a part, which influences its reliability and value over time. In a similar vein, Li (2013) has developed a model that also accounts for part degradation, and it considers economic, product quality, and environmental-based objectives. Atasu and Souza (2013) have addressed the issue of the level of quality to build into a product that may be returned.
New and remanufactured product market segmentation

Recently, work has been published on the problem of segmenting the market for a product into those for new products and used products. The introduction of remanufactured products into the market place has been traditionally achieved through secondary markets in order to minimize the cannibalization of the new product market by remanufactured products. However, it may be more profitable to allow remanufactured products to compete with new products since, in some cases, the value of the market created by the remanufactured products may be greater than the value of the new product market which is cannibalized. In fact, studies (Debo et al. 2005, Ferguson and Toktay 2006) have shown that it may be beneficial to produce more new products for sale at a lower price in order to increase supply of cores for the remanufacturing operation if margins are high enough for remanufactured products. We further elaborate on this topic in Section 1.3.4.

Production planning and inventory control in the presence of product returns

A remanufactured product is produced from a stream of products that are returned in varying conditions. For this reason, production planning and inventory control (PPIC) for remanufacturing is distinctly different from that for new products. Specifically, PPIC for remanufactured products may need to consider the following “complicating characteristics” of a remanufacturing environment (Guide Jr. 2000):

1. Uncertain timing and quantity of returns
2. Need to balance returns with demands
3. Disassembly of returned products
4. Uncertainty in materials recovered from returned items (i.e. the quality of the parts recovered is uncertain)
5. Requirement of a reverse logistics network
6. Material matching restrictions (e.g. controlled serial numbers for parts of a product)
7. Highly variable processing times as a consequence of varying conditions of returns

8. Stochastic routings for materials as a consequence of varying conditions of returns

These complicating characteristics of a remanufacturing operation have implications for each of the major PPIC activities, which include forecasting, production planning, and inventory control. An additional activity for remanufacturing is to plan for the reverse logistics of recovered cores. To-date, researchers have studied and modeled these PPIC activities in view of the unique environment of remanufacturing. Table 1.4, which is reproduced from the paper by Junior and Filho (2012), lists the production planning areas and characteristics that have been addressed recently. The first column lists the complicating characteristics. The headings of the subsequent columns are the PPIC activities. An “X” in a column indicates the corresponding complicating characteristic that has been addressed in the literature for the corresponding PPIC activity.

Table 1.4: Complicating characteristics of the remanufacturing environment addressed by the production and control activities (Junior and Filho 2012).

<table>
<thead>
<tr>
<th>Complicating characteristics</th>
<th>Forecasting</th>
<th>Logistics</th>
<th>Scheduling/Shop floor control</th>
<th>Inventory control and management</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Uncertain timing and quantity of returns</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(2) Need to balance returns with demands</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(3) Disassembly of returned products</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(4) Uncertainty in materials recovered from returned items</td>
<td>X</td>
<td>X</td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(5) Requirements for a reverse logistics network</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(6) Complication of material matching restrictions</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(7) Highly variable processing times</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>(8) Stochastic routings for materials for remanufacturing operations</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Work on forecasting and aggregate planning for remanufacturing remains scant in the literature. Kelle and Silver (1989) have developed methods for forecasting the returns of refillable containers such as air tanks. They develop four methods that use increasing amounts of information collected about the returns. They find it necessary to record the amount of containers issued in each period as well as the aggregate number returned in each period in order to develop an accurate forecast. It is not required to keep track of individual containers. Krupp (1992) has presented methods to forecast the number of new cores to produce based on the rate of returns of the cores and return yields if new and
remanufactured products are perfect substitutes. Maples et al. (2005) acknowledge that, due to the stochastic routings of the cores in a remanufacturing facility, traditional MCP is not valid, and instead, they study the time series data on the usage of parts in the past to forecast their demand. Aggregate planning, in case remanufacturable products are perfect substitutes for new products, is studied by Geyer et al. (2007), who develop economic models to determine the mixed production of new and remanufactured products. They model limited component durability and find that the problem is quite complicated, and that, general insights are hard to obtain. Golany et al. (2001) have studied the economic lot sizing problem for the case when remanufacturing can help satisfy new demand, and they present a polynomial-time algorithm in the case of linear cost functions. In a similar vein, Das (2012) presents a mixed-integer programming (MIP) model for a RSC where remanufactured products can be used to satisfy new product demand. Reimann and Lechner (2012) use the newsvendor framework to study production and remanufacturing policies for CLSC; however new and remanufactured products are assumed to be perfect substitutes. Wang and Huang (2013) study the problem of planning for the CLSC using robust programming in order to account for the highly variable nature of returns. In the work of Benedito and Corominas (2013), product demands and returns are assumed to be stochastic in nature, and the amount of cores available to collect is dependent on previous production.

Clegg et al. (1995), Sarin and Zaloga (1998), and Han et al. (2012) have developed deterministic production planning models that consider many EOL options including landfilling, recycling, resales and remanufacturing. An important feature that these models have in common is that the remanufactured products cannot satisfy new demand. However, the models of Clegg et al. and Han et al. do not consider core supply to be dependent on the quantity of new products produced in previous periods. The quantities returned are dependent on new production in the model by Sarin and Zaloga. Jayaraman (2006) presents a model very similar to the ones of Clegg et al. (1995) and Sarin and Zaloga (1998), except that it is solely for a remanufacturing facility. Some of these models will be discussed in more detail in Section 1.3.5.

Production planning and control also includes such activities as master production scheduling (MPS), and machine scheduling. Ferrer and Whybark (2001) are the first to address MPS for remanufacturing facilities, and they use a Material Requirements Planning (MCP) method. They also present some linear programming (LP) models for deter-
mining the number of cores to acquire and which of these to disassemble. Depuy et al. (2007) have used a modified MCP method, which also accounts for variable yield and variable processing time. Murayama et al. (2006) have used reliability models to predict the number of components, which can be recovered for reuse, and they have linked them to an MCP system to develop a production plan. Guide et al. (2005) have studied a remanufacturing scheduling problem for two products. They find that first-come-first-serve is a reasonably good dispatching rule. Teunter et al. (2006) study the dynamic lot sizing problem where demand can be satisfied by either new or remanufactured products, which can be produced on the same line (joint production), or different lines. They find a polynomial-time algorithm for the joint production case, and develop heuristics for both cases. Tang and Teunter (2006) have studied a multi-product economic lot scheduling problem with returns. They report that the optimal lot scheduling policies for new and remanufactured products on the same production line can lead to 16% reduction in total cost. Heuristic methods for the solution of this problem were first developed by Teunter et al. (2009) and later improved by Shultz (2011).

Arguably, the most studied aspects of the RSC are inventory management and control for remanufacturing. We will only discuss some early works in this area as even a brief review is outside the scope of this paper. The inventory control literature can be divided into discrete and stochastic modeling streams. The discrete stream can be further subdivided into constant and time-varying demand models. For the constant demand case, Teunter (2001) simply extends the original economic order quantity (EOQ) model to the RSC setting. However, in Kim and Goyal (2011), they consider profit maximization as the objective and the supply of remanufacturing cores to be dependent on the quantity of new products sold. In case the demand is time varying, control theory is used by Minner and Kleber (2001) where costs are assumed to be linear. Nakashima et al. (2004) study a similar problem using control theory via a dynamic programming approach, however, they consider the core supply for remanufacturing to be dependent on the number of new products sold. The stochastic modeling stream is composed of continuous-review and period-review models. Early work on inventory policies by van der Laan and Salomon (1997) uses continuous review models, and it finds that the best policy is highly sensitive to demand and return rates and, therefore, the inventory policy must be changed frequently. Periodic-review work by Inderfurth et al. (2001) indicates that the structure of the optimal policy is highly complex; however, in a certain case, a simple policy where extra cores above a certain number are disposed
of and the cores are remanufactured until a certain quantities is obtained, is near optimal.

Comprehensive reviews are also available on the aforementioned topics. For a review of the literature on the general treatment of “Green Supply Chain” management, please see Srivastava (2007). Works on quantitative models for reverse logistics are reviewed by Fleishmann (2001) and Bostel et al. (2005). Guide et al. (1999) have reviewed the literature on production planning (for contributions until 1997). Junior and Filho (2012) have provided an updated review article in the same area. For inventory management and control, Akçalı and Cetinkaya (2011) present a comprehensive review that includes a categorization of different remanufacturing environments.

### 1.3.4 Pricing of new and remanufactured products

When a remanufactured product is introduced into the market, it will compete with new products. Thus, the market gets “segmented” into new-product customers and remanufactured-product customers. The nature of this segmentation is determined by the quantity demanded for the products at their respective prices. Thus, determining prices for the new and remanufactured products is an important strategic-level decision. The literature on new product pricing is well developed and covers such topics as product competition, market segmentation, revenue management, dynamic pricing, and, more generally, pricing strategies. Dolgui and Proth (2010) provide broad overview of pricing strategies. These include:

1. high pricing - setting a high price to give the impression of exclusivity and luxury,
2. low pricing - setting a low price to sell high volumes of a product,
3. market segmentation - creating differences between products so that they appeal to different segments of the market, and pricing them accordingly,
4. price skimming - setting a high initial price and dropping it over time as the product becomes out-dated,
5. penetration pricing - setting a low price initially to gain market share, and
6. revenue management - changing the price dynamically to adjust for consumer demand.
The literature on market segmentation (or price discrimination) and revenue management are most relevant in the context of the RSC. The economic theory of market segmentation is well developed (Phlips 1983); however, the main problems in this area lie in how exactly to segment the market. Traditional market segmentation prescribes that sellers attempt to segment markets through offering different prices to different types of customers (e.g. student discounts and coupons). Moorthy (1984) argues, however, that segmenting markets in this manner is not possible in a free market, and that market segmentation must be done through “self-selection” by the consumers. That is, the seller’s problem is to determine the product line which will naturally segment the consumers into different product markets. Also, according to Moorthy and Png (1992), if not done properly, lower profit margin products could cannibalize demand for higher profit margin products, eating away profits.

Revenue management, or dynamic pricing, requires that prices change dynamically to adjust for consumer demand. Benoit and Krishna (1987) study dynamic pricing in a duopoly environment in which the production capacity of each product is chosen initially and can only be changed at some cost. A similar problem that considers strategic consumers, i.e. consumers who time their purchases based on current prices, is studied by Levin et al. (2009). More recently, Maglaras and Meissner (2006) have studied a problem in which a firm has a fixed amount of production capacity, that can be used to make various products whose demands are stochastic in nature. González-Ramírez et al. (2011) study a dynamic pricing problem in which the capacitated lot sizing problem is extended to include pricing as a decision variable. In a similar vein, Razaei and Davoodi (2012) have studied the dynamic pricing problem with lot-sizing, but they also include supplier selection. An overview of pricing models for revenue management is given by Bitran and Caldentey (2003).

In the remainder of this section, we present some seminal models on pricing and market segmentation/revenue management for new and remanufactured products. The works roughly fall into the following three categories: (1) standard one, two, and multi-period models of monopoly and duopoly, (2) models considering product-life-cycle, and (3) other important models.
Standard economic models

The basic notation used in this section is given in Table 1.5. In some of the models, the quantities given are assumed to be constants while in others they are functions of other quantities. However, the intent of their use will become clear from the context.

Table 1.5: Basic notation for pricing models.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Factor representing the acceptance of customers for choosing a remanufactured product instead of a new product, normalized such that $0 &lt; \alpha &lt; 1$.</td>
</tr>
<tr>
<td>$z$</td>
<td>Customer’s type that is determined by his valuation of the product. This valuation is assumed to be distributed according to some cumulative distribution function, $F(z)$, normalized such that $0 &lt; z &lt; 1$. We call a customer with valuation $z$ a “$z$-valuation customer”.</td>
</tr>
<tr>
<td>$F(z)$</td>
<td>Proportion of customers with valuation between 0 and $z$.</td>
</tr>
<tr>
<td>$P^i$</td>
<td>Price of product $i$, $i \in R,N$, where $R \equiv \text{Remanufactured product}$ and $N \equiv \text{New product}$ and it is normalized such that $0 \leq P^i \leq 1$.</td>
</tr>
<tr>
<td>$C^i$</td>
<td>The marginal cost of product $i$, $i \in R,N$.</td>
</tr>
<tr>
<td>$y^N$</td>
<td>Yield of a remanufacturable product returned after one use, also called the “remanufacturability” of a new product.</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Profit.</td>
</tr>
</tbody>
</table>

The first work reported on market segmentation for remanufacturing is by Ferrer (1996). The problem that they consider is to determine the conditions under which a monopolist should remanufacture. It is a one-period model, which we present next.

Let the customer’s utility functions be defined as follows:

$$U^R(z, P^R) = \alpha z - P^R$$ for the remanufactured product, and

$$U^N(z, P^N) = z - P^N$$ for the new product.

Let $z^R$ and $z^N$ be the lowest valuation customer ever to buy the remanufactured and new products, respectively. At price $P^R$, the customers with valuations between $z^R$ and $z^N$ buy the remanufactured product, whereas those with valuations greater than $z^N$ buy the new product at price $P^N$. Let $P^R(z^R)$ and $P^N(z^N, z^R)$ be the prices of remanufactured and
new products, respectively, as a functions of \( z^R \) and \( z^N \). It can be easily seen that \( P^R(z^R) = z^R \alpha \) as this is the price at which the utility of the customer with \( z^R \)-valuation is non-negative when purchasing a remanufactured product. Finally, the price of a new product needs to be found such that the utility of a \( z^N \)-valuation customer purchasing a remanufactured product is less than or equal to the utility of the same customer purchasing a new product. That is, we solve \( U^R(z^N) \leq U^N(z^N) \) for \( P^N \). By substituting \( z^R \alpha \) for \( P^R \), we find that \( P^N(z^N) \leq z^N(1 - \alpha) + z^R \alpha \), and therefore, the price which maximizes the utility for the customers, is found by using the above expression as equality.

Let \( \psi^R \) and \( \psi^N \) be the reusable part yields from the remanufactured and new products, respectively, and it is expressed as the fraction of parts needed in the remanufactured products that are obtained from remanufactured and new product returns, respectively. Additionally, let \( u \) be the maximum number of used parts, which may be used in a remanufactured product, normalized such that the upper bound coincides with its technological limit, and \( 0 \leq u \leq 1 \). Finally, let \( s \) be the largest savings that can be obtained from employing used parts. Note that, a remanufactured product will cost \( C^R - us \). Normalization assures that \( 0 < s < C^R < 1 \). The manufacturer’s problem is to choose the used-part content of the remanufactured product and to identify the boundary customers of \( z^R \) and \( z^N \) such that the profit is maximized.

Assuming that the customers maximize their utility, we have the following problem:

\[
\max_{z^R, z^N, u} \quad \Pi = (P^R - C^N + us)(F(z^N) - F(z^R)) + (P^N - C^N)(1 - F(z^N)),
\]

subject to:

\[
\psi^R(F(z^N) - F(z^R)) + \psi^N(1 - F(z^N)) \geq u(F(z^N) - F(z^R)), \quad \text{(1.2)}
\]

\[
0 \leq u \leq 1,
\]

\[
0 \leq z^R \leq z^N \leq 1,
\]

\[
P^R(z^R) = z^R \alpha, \quad \text{(1.5)}
\]

\[
P^N(z^N) = z^N(1 - \alpha) + z^R \alpha. \quad \text{(1.6)}
\]

The objective function given by (1.1) represents the profit of the new and remanufactured sales. Constraint (1.2) ensures that enough parts are obtained from new and remanufactured product returns to satisfy the remanufacturing
demand. The proportion of used parts in a remanufactured product is ensured to be between 0 and 1 by Constraint (1.3), while the requirement that the valuation at which customers begin buying remanufactured products is less than the valuation at which they start buying new products, is enforced by (1.4). Finally, Constraints (1.5) and (1.6) ensure optimal pricing for a given $z^R$ and $z^N$.

This model determines the segmentation of customers, $z^R$ and $z^N$, and therefore, the optimal prices of the new and the remanufactured products. Additionally, the model finds the used-part content, $u$, of a remanufactured product. The results indicate that: (1) the cost savings in the production of a remanufactured product, and (2) a customer’s willingness to pay (WTP) for a remanufactured product, impact a firm’s decision to remanufacture.

Majumder and Groenevelt (2001) study a two-period scenario in which the OEM produces products in the first period, recovers some of the cores for remanufacturing, and then, competes with local remanufacturers in the second period. They use a game theoretic approach to find equilibrium conditions. Results indicate that it is beneficial for the OEM to increase remanufacturing costs for the local remanufacturer by using proprietary assembly techniques. Also, surprisingly, the remanufacturer benefits when the OEM’s remanufacturing cost goes down since the OEM has incentive to increase new production to raise the supply of cores for its own remanufacturing process. However, this model considers new and remanufactured products to be perfect substitutes. Ferrer and Swaminathan (2006) focus more on the case of duopoly than monopoly, and they extend the model of Majumder and Groenevelt (2001) to the case of multi-period. Similarly, they assume that new and remanufactured products are perfect substitutes. More recently, Ferrer and Swaminathan (2010) study the case where new and remanufactured product are differentiated, however, only for the case of a monopoly.

Debo et al. (2005) present a more complete work, which investigates both the pricing of competitive new and remanufactured products and the level of “remanufacturability” to initially build into a product. This problem is formulated such that the manufacturer is a monopolist in both the new and remanufactured markets. The proportion of customers who value a product at $z$ is given by the function $F(z) = 1 - (1 - z)^\kappa$ where $\kappa \in (0, \infty)$. Additionally, a customer of type $z$ is willing to pay $\eta(z)$ for a remanufactured product. If a customer’s WTP is linear in $z$, then $\eta(z) = \alpha z$. Let $v$ be the production quantity pair $(Q^N, Q^R)$, where $Q^N$ is the number of new products produced and $Q^R$ is the number of
remanufactured products produced. As it turns out, prices can be defined in terms of production volumes. Let \( P_N(v) \) and \( P_R(v) \) be the prices of new and remanufactured products, respectively, as a function of \( v \). Revenue can be defined as, \( \text{Rev}(v) = Q_N P_N(v) + Q_R P_R(v) \). Let \( C_N(y_N) \) be the cost of a new product as a function of \( y_N \). Thus, profit is defined as, \( \Pi(v, y_N) = \text{Rev}(v) - C_N(y_N) Q_N - C_R(y_N) Q_R \). A solution is called an implementable path, which is defined as \( \mathcal{F} = \{ v_t, t \geq 0 | v_t \in \mathbb{D}, I_0 = I, I_t = I_{t-1} + y_N Q_N, \forall t \geq 1, \text{ and } Q_R \leq I_t, \forall t \geq 0 \} \), where \( \mathbb{D} \) represents all the possible production pairs, and \( I_t \) is the inventory of used products at time \( t \). The problem is to find an optimal implementable path such that the sum of the future profits, discounted at rate \( \beta \), is maximized. Let \( V_{\beta}(I; y_N) \) be the optimal profit for this problem at initial inventory \( I \), defined as follows:

\[
V_{\beta}(I; y_N) = \max_{F \in I(I)} \sum_{t=0}^{\infty} \beta^t \Pi(v_t, y_N),
\]

where \( I(I) \) is the set of all implementable paths at initial inventory \( I \). If the problem is to determine “how remanufacturable” to make the product, we solve

\[
\max_{y_N \in [0,1]} V_{\beta}(I; y_N) - k(y_N),
\]

where \( k(y_N) \) represents the investment cost required to produce a new product with \( y_N \) remanufacturability.

Using this framework, it can be shown that as more customers become low-valuation customers, the firm will supply fewer new products, which, in turn, reduces the core supply of the remanufacturing operation, thus, decreasing remanufacturing potential. Additionally, as the cost of producing a new product increases to increase the remanufacturability of the product, the marginal profit of remanufacturing increases, but the supply for the remanufacturing process decreases. A full list of parameters and their effect on remanufacturing potential is given in Table 1.6. Note that, \( k'(0) \) is the derivative of \( k(y_N) \) with respect to \( y_N \) at \( y_N = 0 \). Likewise, \( C_N'(0) \) is the derivative of \( C_N(y_N) \) with respect to \( y_N \) at \( y_N = 0 \).

Moreover, if the discount factor is high and most consumers have low product valuations, it is beneficial to make the product more remanufacturable. In case the discount factor is low, then the fixed cost of adding remanufacturing capability is not trivial, and it can tend to dominate at higher levels of remanufacturability. However, this cost is
Table 1.6: Effect of increasing values of parameters on remanufacturing potential (Debo et al. 2005).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interpretation</th>
<th>Impact on remanufacturing potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_r(0)$</td>
<td>Remanufacturing cost</td>
<td>Negative</td>
</tr>
<tr>
<td>$k'(0)$</td>
<td>Fixed cost</td>
<td>Negative</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>Positive</td>
</tr>
<tr>
<td>$1 - \alpha$</td>
<td>Perceived depreciation</td>
<td>Negative</td>
</tr>
<tr>
<td>$C_N(0)$</td>
<td>Unit cost of new production</td>
<td>Negative</td>
</tr>
<tr>
<td>$C_N'(0)$</td>
<td>Single-use production cost</td>
<td>Depends</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Consumer profile</td>
<td>Negative</td>
</tr>
</tbody>
</table>

less at lower levels of remanufacturability, and if the customers have diverse valuations, it may still be profitable to remanufacture as the products sold to higher-valuation customers will lead to building inventory of remanufactured products for low valuation customers.

The analysis even reveals bizarre behavior for new product pricing. If there are enough low valuation customers to whom remanufactured products can be sold, it can be profitable to sell new products at a loss in order to increase core supply for remanufacturing! Thus, the new product becomes a “loss leader”. Finally, decreasing remanufacturing costs causes prices to fall for new products in order to supply the remanufacturing operation with returns to sell profitably.

A limiting assumption on the aforementioned models is that the user holds the product for exactly one period. It serves as an approximation for those products for which the usage period is much shorter than the overall life of the product.

**Model with product life-cycle considerations**

Debo et al. (2006) have considered the nature of a product’s life cycle curve and its role in determining: (1) a product’s level of remanufacturability, and (2) the joint prices of new and remanufactured products. They use a modified Bass product diffusion modeling technique (Bass 1969), which is a well-known and powerful demand forecasting technique. We present this model next.

Let $M_t$ be the proportion of the total market for the new and remanufactured products that has been penetrated in period $t$. Also let $i_{NJ}$, or the “installed base”, be the proportion of the market owning a new product. $M_t$ and $i_{NJ}$ are
normalized to be in \([0, 1]\). The size of (potential) new customers is given by

\[
m_t = M_t - M_{t-1} = (a + bi_{N,t-1})(1 - M_{t-1}),
\]

(1.8)

where \(a\) can be considered an “innovation” coefficient and \(b\) the “imitation” coefficient (e.g. word of mouth). Some of these potential new customers will buy the new product, the rest will buy the remanufactured products. The number of potential repeat customers is given by

\[
\Theta_t = \phi \sum_{\tau=1}^{\min(L,t)} \lambda_{\tau}(m_{t-\tau} + \Theta_{t-\tau}),
\]

(1.9)

where \(\phi\) is the probability that a customer will make a repeat purchase, \(\lambda_{\tau}\) is the fraction of customers who return a product after \(\tau\) periods, and \(L\) is the time horizon of the problem.

Let \(P_N^t\) and \(P_R^t\) be the prices of new and remanufactured products, respectively, in period \(t\). Also, let \(\omega_i(P_N^t, P_R^t)\) be the fraction of customers who buy product \(i, i = R, N\), as a function of \(P_N^t\) and \(P_R^t\). The sales volume in period \(t\) of each product type is given by:

\[
Q^N_t = (m_t + \Theta_t)\omega_N(P_R^t, P_R^t), \quad \text{and} \quad Q^R_t = (m_t + \Theta_t)\omega_R(P_R^t, P_R^t).
\]

(1.10)

The installed base is modeled as

\[
i^N_{t+1} = i^N_t + n_t - \sum_{\tau=1}^{\min(L,t)} \lambda_{\tau}h_{t-\tau}.
\]

(1.11)

Let \(i^N_t = \sum_{\tau=1}^{\min(L,t-\Delta)} \lambda_{\tau}h_{t-\Delta-\tau}\) and \(i^R_t = \sum_{\tau=1}^{\min(L,t-\Delta)} \lambda_{\tau}r_{t-\Delta-\tau}\) be the number of new and remanufactured returns, respectively, if it takes \(\Delta\) periods to collect the items. Also, let \(f_t\) be the volume of remanufacturable products disposed of in period \(t\). In this model, all returned remanufactured products in period \(t\), \(i^R_t\), are disposed of. The total number of products disposed of,

\[
D_t = f_t + (1 - y^N)\epsilon^N_t + \epsilon^R_t.
\]

(1.12)
The inventory is, then, given by

\[ I_{t+1} = I_t - r_t - f_t + y^N \varepsilon_{t+1}^N, \]  

(1.13)

where

\[ r_t + f_t \leq I_t. \]  

(1.14)

Let \( B^R_t \) and \( B^N_t \) be the capacities to manufacture the new and remanufactured products, respectively, in time \( t \), and \( J_N[x] \) and \( J_R[x] \) be the cost for an \( x \) magnitude of change in the capacities of the remanufactured and new product, respectively. The capacity-related costs in period \( t \) are:

\[ J(B_t, B_{t-1}) = J_N|B^R_t - B^N_{t-1}| + J_R|B^R_{t-1} - B^R_t| + C^F(B^R_t + B^R_{t-1}), \]

where \( C^F \) is the cost to operate capacity. Also,

\[ Q^N_t \leq B^N_t \quad \text{and} \quad Q^R_t \leq B^R_t. \]  

(1.15)

Then, the problem is to determine an implementable path defined as

\[ \mathcal{F}_d = \{(\mathcal{P}^R_t, \mathcal{P}^N_t, m_t, \Theta_t, i^R_t, B_t), t \geq 0, \text{s.t.} (1.8), (1.9), (1.10), (1.11), (1.12), (1.13), (1.15) \forall t \geq 1, \text{ and } (1.14) \forall t \geq 0 \}. \]  

(1.16)

Let the profit in period \( t \) be defined as

\[ \Pi_t(p, y^N) = \xi_N(\mathcal{P}^R_t, \mathcal{P}^N_t)(\mathcal{P}^N - C^N(y^N)) + \xi_R(\mathcal{P}^R_t, \mathcal{P}^R_t)(\mathcal{P}^R - C^R), \]  

(1.17)

where \( \xi_N(\mathcal{P}^R_t, \mathcal{P}^N_t) \) and \( \xi_R(\mathcal{P}^R_t, \mathcal{P}^R_t) \) are the proportion of customers who purchase the new product and the remanufactured product, respectively. We wish to maximize

\[ V_\beta(y^N) = \max_{\mathcal{F}_d \in \mathbb{I}(0)} \sum_{t=0}^{\infty} B^\gamma(\Pi(\mathcal{P}_t, y^N)(m_t + \varepsilon^R_t) - C^{disp} \mathcal{P}_t - K(C_t, C_{t-1}), \]  

(1.18)

where \( C^N(y^N) \) is the cost of manufacturing a new product with remanufacturability \( y^N \), \( C^R \) is the cost to remanufacture a returned product, and \( C^{disp} \) is the cost of disposal. The problem cannot be solved analytically in most cases. This
work has lead to the following major insights:

1. Capacity investment should be higher for fast diffusing products and high repeat sales.

2. Slow diffusing products are the best candidates for remanufacturing.

3. Reverse channel responsiveness is only important if there is demand for remanufactured products.

A pertinent idea to enhance the use of remanufactured products is to employ leasing of products and to provide an appropriate maintenance service. Robotis et al. (2012) extend the work of Debo et al. (2006) to investigate the optimal pricing and duration of leases, which include service and maintenance. They report that if a product has a long life cycle (slow diffusion), and a high remanufacturing savings, then the lease-term should be long, and a pricing policy should be employed that slowly reduces price over time for new customers (price skimming).

**Variable marginal remanufacturing cost model**

The above models do not consider the marginal cost of remanufactured products as a function of the quantity remanufactured. A model in this regard is presented by Ferguson and Toktay (2006). Through a two-period model, they study how the OEM can deter a local remanufacturer from entering the market. They model the marginal remanufacturing cost as an increasing function of remanufactured production volume because of two factors: (1) the transportation cost of cores to remanufacturing facilities, and (2) the fact that the cores are typically processed in the decreasing order of the quality of their conditions. Regarding the first point, as more cores are collected, they are first collected from high population density areas, and then, from lower population density areas; thus, each additional core becomes costlier to collect. As for the second point, since the cores are remanufactured in decreasing order of the quality of their condition, each additional core is more expensive to remanufacture. They use a monopolist model that we present next.

Let $g Q^R$ be the cost of remanufacturing $Q^R$ products, where $g$ is a constant. For the monopolist case, the optimal prices for the new and remanufactured products are developed for the first and second periods based on the customers’
utility functions and a uniformly distributed customer valuation for the products. Ferguson and Toktay (2006) show that, in the first period, $P_N^1 = 1 - Q_N^1$ and in the second period, $P_N^2 = 1 - Q_N^2 - \alpha Q_R^2$ and $P_R^2 = \alpha(1 - Q_N^2 - Q_R^2)$.

The problem is analyzed by first characterizing the optimal policies for the second period, and then, using these to solve the two-period problem. Let $\Pi_2(Q_N^2, Q_R^2 | Q_N^1)$ be the second period profit as a function of $Q_N^2, Q_R^2$ given the first period production quantity of new products, $Q_N^1$. The firm’s second period objective is to

$$
\max_{Q_N^2, Q_R^2} \Pi_2(Q_N^2, Q_R^2 | Q_N^1) = (P_N^2 - C_N)Q_N^2 + (P_R^2 - gQ_R^2)
$$

subject to

$$Q_R^2 \leq \lambda Q_N^1
$$

where, $h$, is the fraction of customers who return products that were purchased in the first period. The first term of the objective function, (1.19), represents the profit from the new products, and the second term captures profit from the remanufactured products. Constraint (1.20) ensures that the number of remanufactured products is less than the number of returned cores.

It is instructive to study the behavior of the optimal policy in the second period under different values of $g$ and $\lambda Q_N^1$. In general, it is found that when $g$ is low, remanufacturing becomes more attractive. In fact, for $g \leq \alpha(C_N - 1 + \alpha)/(1 - C_N)$, it is preferable to sell only remanufactured products in the second-period. For any other value of $g$, if there is a low core recovery rate, i.e., $\lambda Q_N^1 < \alpha C_N/[2(g + \alpha - \alpha^2)]$, all cores will be remanufactured; otherwise, $\alpha C_N/[2(g + \alpha - \alpha^2)]$ cores will be remanufactured.

Using the above second-period characterization of the optimal policy, the two-period model, with objective function:

$$(P_N^1 - C_N)Q_N^1 + \Pi_1^2(Q_N^1)$$

is solved to determine $Q_N^1$. It is found that, if $g$ is very high, then no remanufacturing occurs in the second period. Else, if $g$ is low, then more products are produced in the first period than would be produced if there were no remanufacturing so that a greater number remanufactured products can be sold at a profit in the second period. They have also studied the problem where a competitor attempts to enter the market as a remanufacturer in the second period. Major findings in this regard are that the OEM can increase cost of entry by increasing collection efforts, and that, even if both the OEM and the remanufacturer have the same cost structure,
it may be profitable for the remanufacturer to produce even if it is not so for the OEM because the OEM risks new product cannibalization.

**Other pricing models**

Ovchinnikov (2011) has introduced a new perspective in the remanufactured product pricing literature by assuming that the percentage of consumers, who would switch from purchasing a new product to purchasing a remanufactured product as a function of remanufactured product price, follows an inverted “U” shape. This is designed to reflect the fact that many consumers view low product prices as a signal of low quality. Additionally, there is a pool of low valuation consumers who will purchase a remanufactured product, but never a new product because of its high cost. This has beneficial implications for the manufacturer as it reduces cannibalization of new product sales.

Vorasayan and Ryan (2009) have extended the work of Debo et al. (2005) by using a queuing network model to account for highly variable product residence times and other sources of variability. They find that, for this scenario, either a vast majority of products are remanufactured, or none at all.

Atasu et al. (2008) include a green market segment comprising customers who have a higher valuation of remanufactured products than those of others because of the “green” nature of remanufactured products. They have studied this problem under static monopoly, two-period monopoly with life cycle considerations, static competition, and two-period model with competition considering product life cycle.

The work by Guide et al. (2003) addresses a different pricing problem that applies only to a remanufacturing firm. The questions they address are: (1) how to price remanufactured products, and (2) how much to offer to purchase a used product so that the supply and demand are balanced and the profit is maximized. Their main finding is that actively acquiring products through offering incentives is an effective way to increase profits or mitigate the effects of new environmental protection legislation.

Chen and Chang (2011) study the pricing problem under demand uncertainty using the framework of the newsvendor problem. They find that it is only beneficial to remanufacture if the remanufactured product is a strong substitute for
the new product and remanufacturing costs are low.

An unexpected result that decentralized control of remanufacturing operations may be more profitable is presented in the work of Zhou (2013). They use a multi-stage model which, in the first stage the OEM choose either centralized or decentralized remanufacturing strategy, in the second stage the supplier chooses a part price, and in the third stage the OEM chooses the production quantity of new and remanufactured products. It turns out, that in the decentralized case, the supplier is incentivized to drop the price of the part in order to sell more parts, and thus, in some cases, makes decentralized control more profitable for the OEM.

1.3.5 Production planning models for remanufacturing

In the work reviewed on the pricing models for the RSC, the variable cost of remanufacturing has either been assumed to be constant, or a function of “remanufacturability”. Only in the paper by Ferguson and Toktay (2006) has the variable cost of remanufacturing been considered as a function of the quantity remanufactured. In general, both the optimal pricing and production quantities are sensitive to the variable cost of remanufacturing. Thus, it is important to determine the nature of the variable cost function. This can best be studied via the production planning problem that, typically, focuses on production cost minimization. The traditional PPIC literature is concerned with either capacity planning and lot-sizing, or inventory control. A typical model in this area does not consider the costs associated with remanufacturing in detail. These costs are combined into a constant representing the per unit remanufacturing cost of returned cores. Our aim here is to consider the per remanufactured-unit cost in conjunction with a pricing model.

The work of Ferrer and Whybark (2001) on material planning methods for remanufacturing using MCP is one of the first papers in the PPC literature for remanufacturing. Material planning in remanufacturing is quite unique since much of the materials used come from dismantled core returns of variable quality. The other sources of materials are the suppliers, as usual. They use an LP model to determine the expected minimal number of cores required to obtain the necessary parts required to fill remanufactured product demand. Additionally, they present a model to determine which cores to disassemble so as to minimize the inventory of parts at the end of the planning horizon. These models
capture many relevant features related to remanufacturing, including, disassembly yields, part commonality among products, and the need to use new parts in remanufactured products. However, there is no consideration of costs in these models.

Jayaraman (2006) has presented a model called the Remanufacturing Aggregate Production Planning (RAPP) model, which determines an aggregate production plan that minimizes the total cost per remanufactured unit. The model, formulated as an integer program (IP), is quite extensive and includes the following parameters: (1) part commonality among cores, (2) nominal quality level of returns, (3) disassembly time, (4) remanufacturing time, (5) reassembly cost, (6) replacement-part costs, (7) core acquisition cost based on quality, (8) inventory costs, and (9) disposal costs of cores and parts. The decisions to be made are: (1) number of cores of various quality to disassemble, (2) number of cores of various qualities to dispose of, (3) number of cores of various qualities to remanufacture, (4) number of cores of various qualities to acquire, and (5) number of new parts to purchase. The objective is to minimize the sum of the inventory, disassembly, disposal, remanufacturing, core acquisition, and new part purchasing costs. The proposed model is quite detailed and arguably captures all the essential features of aggregate production planning for remanufacturing under the assumption that demand and core supply can be modeled as exogenous factors.

A model described in a research report by Sarin and Zaloga (1998) is, in someways, even more comprehensive than the RAPP model in that the supply of cores is endogenous, and it is linked to new product sales through a fixed new product residence time. Additionally, many end-of-life options are modeled that include remanufacturing, resale, recycling, and disposal. It does not, however, take into account the quality of the returns like RAPP does. It is formulated as an IP. Clegg et al. (1995) develops an LP model for production planning with remanufacturing. Their model is similar to that of Sarin and Zaloga’s, but they consider supply of cores to be exogenous.

Galbreth and Blackburn (2006) have studied the problem of determining an optimal number of cores for a remanufacturing process in order to minimize total cost. In their model, they assume that the cores are sorted by the condition they are in, and that, they are processed in the descending order of the quality of their conditions. The models that they have presented are useful if the remanufactured item contains one reusable part. Table 1.7 introduces the notation needed to present their models.
Table 1.7: Notation of Galbreth and Blackburn (2006).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^R$</td>
<td>Demand for remanufactured products</td>
</tr>
<tr>
<td>$t$</td>
<td>Core condition</td>
</tr>
<tr>
<td>$G(t)$</td>
<td>Cumulative density function (CDF) defined over the condition of a core</td>
</tr>
<tr>
<td>$k$</td>
<td>Index for the condition of ordered item within an acquired lot ($k = 1$ indicates the best condition; $k = C$ indicates the worst condition)</td>
</tr>
<tr>
<td>$C^c$</td>
<td>Unit acquisition and inspection costs</td>
</tr>
<tr>
<td>$S$</td>
<td>Unit scrap cost</td>
</tr>
<tr>
<td>$C^R$</td>
<td>Fixed portion of unit remanufacturing cost</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Variable (condition-dependent portion of unit remanufacturing cost)</td>
</tr>
<tr>
<td>$Q$</td>
<td>Quantity of used items acquired</td>
</tr>
</tbody>
</table>

Let $G(t)$ be the CDF for the condition of a core, and let it be uniformly distributed. Let $X_{(1)}, \ldots, X_{(Q)}$ denote the order statistics of the $Q$ items. Since the X’s are uniformly distributed, the $k^{th}$ order statistic has density

$$
C \binom{\frac{C-1}{k-1}}{t^{k-1}(1-t)^{C-k}}; \quad t \in [0, 1].
$$

Then, the expected cost function if the variable cost is a linear function of product condition, is given by

$$
f(Q) = C^cG + S(G - D^R) + \sum_{k=1}^{D^R} [C^R + \int_0^1 C \binom{\frac{C-1}{k-1}}{(\gamma t) t^{k-1}(1-t)^{C-k}} dt].
$$

The first term in (1.21) is the acquisition cost of the cores. The second term represents the scrap cost. The third term is the expected sum of remanufacturing cost, which includes a fixed cost, $A$, and the expected variable cost. The optimal quantity can be shown to be

$$
\bar{Q}^* = \max\{D^R, \left\lfloor \sqrt{\frac{\gamma D^R(D^R + 1)}{2(C^c + S)}} - 0.5 \right\rfloor \}. \quad (1.22)
$$

Galbreth and Blackburn (2006) have additionally studied cases where the variable cost of remanufacturing a core is a nonlinear function of core condition, given by $\gamma \phi^x$, where $x$ is greater than 0. Except for the case where $x = 2$, this problem requires a numerical approximation method. In case the products can be split into two categories of high cost...
and low cost, then the cost function becomes

\[ f(c) = lC + S(c - D) + \gamma_1 D + \bar{\delta} \sum_{n=0}^{P-1} \binom{c}{n} l^n (1 - l)^{c-n} [D - N], \]  

(1.23)

where \( l \) is the probability of a product requiring low cost to remanufacture, \( n \) is the actual number of used items in the low-cost category, and \( \gamma_1 \) is the cost to remanufacture low-cost items. Additionally, \( \bar{\delta} \) is the incremental cost for a high-cost item, i.e., the additional cost of remanufacturing a high-cost core over-and-above the low-cost items. In Equation (1.23), the first term captures the acquisition cost, the second term is the scrap cost, the third term represents the cost incurred if all the products were low-cost items, and the last term is the expected incremental cost of remanufacturing the high-cost items.

In a similar vein to that of Galbreth and Blackburn (2006), Denizel et al. (2010) consider a multi-period problem, which is solved using a stochastic programming approach. Multiple grading categories are defined with associated remanufacturing costs. Parameters in the model include exogenous demand of remanufactured products, core quantity arriving at the beginning of each period, available capacity, remanufactured product price, unit remanufacturing, salvage and holding costs for various grades of cores, and backlogging costs. Decision variables are: the number of cores to grade in each period, and the quantities of cores remanufactured and those salvaged in each period. For most problem instances, they find that ungraded cores should be disposed of at the end of each period. Additionally, they find the profit to be heavily influenced by the shape of the remanufacturing cost versus core quality curve as well as salvage values and grading costs. Backlogging and inventory costs are not found to be strong cost drivers.

1.3.6 Concluding remarks

In this literature review, we have provided background information on the RSC and a brief exposure on the breadth of RSC literature, focused of remanufacturing. Two research streams, namely, market segmentation of new and remanufactured products and production planning and control in the presence of remanufacturing, are reviewed in some depth. Additionally, a third stream that is pertinent to this work is on product design for RSC, which is also reviewed,
however the topic is not as well developed in the literature.

Our review has unveiled a few significant gaps in the literature. Firstly, in-depth analysis of the production planning models for RSC seems to be lacking. Although adequate models for production planning are presented, few insights are obtained. Secondly, there is insufficient integration of product design and product pricing decisions with these production planning models and vice-versa. Already we have argued that this integration is key for making strategic supply chain planning decisions. In this work, we attempt to close these gaps by (i) providing in-depth analysis of a simple, yet representative production planning model that aids in determining the optimal EOL option of a product, (ii) investigating how to jointly determine EOL option and product design, and (iii) presenting models for jointly determining EOL option and product pricing.

1.4 Description of subsequent chapters

Next, in Chapter 2, a full description of the RSC system that we consider is provided. It details relationships among its various entities. It unveils the use of a production planning type of modeling strategy for analysis. Additionally, a comprehensive and general mathematical model is presented that takes into consideration multi-period planning and product inventory. The unique aspect our model over previous production planning modes for RSC is that we model the product returns as endogenous variables rather than them being exogenous. This model forms the basis of our research, and we use its special cases in our analysis.

To begin our analysis of the problem, in Chapter 3, we study the case in which the product design and pricing are fixed. Both choice of collection and mandated collection are considered. Our analysis focuses on a special case of the problem involving two stages: in the first stage, new products are produced, and in the second stage, EOL products are collected for value recovery. For fixed product design and pricing, our analysis of the two-stage problem reveals a fundamental mapping of product characteristics onto optimal EOL options. Necessary and sufficient optimality conditions are presented for each possible solution of this two-stage problem. We present a graphical mapping of product characteristics onto optimal EOL type for the two-part problem that is generated using the optimality conditions.
In Chapter 4, we study the case of product design under mandated collection, as in the case of product leasing. We assume new production cost, part replacement cost and part salvage value to be functions of the quality-level of the part along with the likelihood of recovering a good-part from a returned product. Again, a two-stage model is used. For the two-part problem, a method for mapping part yields onto optimal EOL options is presented. For the two-stage case, closed-form optimality conditions for joint determination of part yields and EOL options are not generally attainable; however, computationally efficient methods for this problem are possible for some relatively non-restrictive special cases. Additionally, a categorization scheme for describing the various parts is presented.

In Chapter 5 we consider the problem of determining optimal prices for new and remanufactured products under the manufacturer’s choice of collection. New and remanufactured products may or may not compete, depending on the market condition. Additionally, we assume the manufacturer has a monopoly on the product. Again, we used a two-stage model. Efficient solution methods is are presented with some analysis.
Chapter 2

System Description and Mathematical Formulation

Our approach for the problem at hand is essentially a multi-period production and materials requirements planning model consisting of the following decisions: (i) how many new products to produce, (ii) how many cores to collect, (iii) what EOL options for recovered products and parts to select, (iv) which product design through choice of part reliability to choose, and (v) what prices of new and used products to use. A system description is given in Section 2.1 and the corresponding mathematical model is presented in 2.3. Our overall approach for analyzing this problem is discussed in Section 2.4.

2.1 System description

New products are manufactured and sold after which customers use them for a specific duration. At the end of this usage period, some proportion of customers are willing to return the product back to the manufacturer. The other customers either no longer have the product, or will never be willing to return the product. The EOL products
available to be reclaimed are called cores and the manufacturer must decide how many of them to recover. We assume that the cores can be graded, i.e. their conditions are evaluated, in the following sense. A core is labeled as a *good-core* or a *bad-core*. Some proportion of the cores are labeled as good-cores, and these can undergo any of the EOL options at some cost or be saved in inventory, while a bad-core is disposed-of. Both remanufacturing and part-salvage require the full disassembly of cores into their component parts, whereas the other EOL options do not. As such, the manufacturer decides how many cores to disassemble, in order to perform remanufacturing and part salvage. Moreover, the manufacturer must decide how many cores will undergo the various other EOL options that do not require disassembly. Through disassembly, parts can be recovered from cores. However, the recovered parts will be in varying conditions. Let a *good part* be the one which can be reused in a remanufactured product or the one for which the value can be profitably recovered. Let a *bad-part* be the one which must be disposed-of. A good part is *salvaged* if it is not used in a remanufactured product, at its value is recovered by using the most profitable EOL option available or it is saved in inventory. A bad-part is disposed-of by using the least costly EOL option available for that part. Note that, a “salvaged” good part may undergo the same EOL option as a “disposed-of” bad-part. Remanufactured products contain some mixture of recovered and replacement parts; however, we assume that as many recovered parts are used in remanufacturing as possible. In the same vein, a remanufactured product cannot contain all new parts. Products at the end of their second life-cycle are not recovered. This system is depicted in Figure 2.1 with the notation defined in Tables 2.1, 2.2, and 2.3.

Additionally, the product design plays a large role in determining the nature of a product’s characteristics and, therefore, its optimal EOL-option. In the context of this problem, we are interested in designing how much “quality” to build into each part. A given part-quality may be achieved by a variety of part designs; however, the least-cost design that meets all other design requirements is typically the one most desirable. *Thus, a choice of part quality is a choice of part design as well.* The choice of part quality, and therefore, part design impacts many aspects of this problem including new production cost, part replacement cost, part salvage value, proportion of good-cores recovered from customers, and the proportion of good parts obtained from disassembly. As a proxy for the quality of Part *k*, we use proportion of good parts of type *k* obtained from disassembling a good-core, or \( \psi_k \). Note that, \( \psi_k = 0 \) means that Part
\(k\) is made with just enough quality so that virtually all the parts of type \(k\) are only useful for one product usage period, i.e. they would be expected to fail if re-used for another usage period.

We assume that the part replacement cost increases with increment in choice of part quality. Let \(C_p^k(\psi_k)\) be the function defining the trade-off between the replacement cost and yield of Part \(k\). We assume the following properties of \(C_p^k(\psi_k)\) for Part \(k\): (a) \(C_p^k(\psi_k)\) is increasing in \(\psi_k\), (b) \(C_p^k(\psi_k) > 0 \forall \psi_k\), and (c) \(\lim_{\psi_k \to 1} C_p^k(\psi_k) = \infty\). Property (a) results from the assumption that part replacement cost increases with part quality. Property (b) states that part replacement costs will always be positive. Property (c) captures the assumption that an increment in part yields become increasingly expensive to achieve, so much so that it is not possible to obtain perfect part quality at any cost. Property (c) is also justified by the fact that product warranty periods are not typically very long, suggesting high failure rates after the warranty period and high cost to obtain a longer warranty period. Figure 2.2 depicts an example form of \(C_p^k(\psi_k)\) produced using the functional form

\[
C_p^k(\psi_k) = \frac{1}{\psi_k^{-a}} + \frac{1}{1-\psi_k^b} - 1.
\]

In Figure 2.2, we notice that for \(\psi_k\) large enough, there is a "knee-in-the-curve" - a region where the slope of the curve begins to rise quickly, prior to which the slope of the curve is roughly constant. Let \(\psi_k = 0\) be the yield corresponding...
to the quality level below which the part is unacceptable to use in the product, and let $\psi^H_k$ be the yield level at which the slope of the curve begins to rise quickly. A linear approximation of $C^p_k(\psi_k)$ is given by the line segment connecting $(0,C^p_k(0))$ and $(\psi^H_k,C^p_k(\psi^H_k))$ (see Figure 2.2). We assume that the cost of the part and the part salvage value have similarly shaped quality trade-off functions; however, they are scaled differently.

The willingness of customers to return cores, $\lambda$, and the proportion of good-cores obtained, $\phi$ will be functions of the quality of the parts. With increment in part quality, the function values will increase.

Product pricing, however, determines the quantity demanded for each type of product. In general, the quantity demanded of a product decreases with increment in product price. If products compete, as is the case with new and remanufactured products, then an increase in one product’s price may lead to an increase in the quantity demanded of the other product.
2.2 Discussion of EOL options

The EOL options can be divided into two categories: (i) EOL options that require disassembly, and (ii) EOL options that do not require disassembly. The end of life options that do require disassembly include remanufacturing and part salvage/disposal. Figure 2.3 depicts this part of the system.

Some demand is observed for remanufactured products that may or may not be filled, depending on the number of cores collected and whether it is profitable or not to remanufacture. Remanufactured products are made from a mixture of recovered parts and replacement parts; the number of cores disassembled will determine the quantity of each part that will in excess and those that will be in deficit. Thus, the unit cost of remanufacturing is dependent on the number of cores collected and disassembled.

In the case that all good parts obtained from a core are salvaged, then that core undergoes the salvage EOL option.
Other EOL options such as resale, refurbishment, and disposal may require little to no disassembly. This situation is depicted in Figure 2.4. In this case, the unit cost of each EOL option is independent of the number of cores collected.

![Figure 2.4: System description for EOL options requiring little to no disassembly.](image)

### 2.3 Mathematical formulation

We now present a mathematical formulation for the problem on-hand. The notation for model parameters that are not functions of part yield is given in Table 2.1. Notation for variables whose values are functions of part yield is given in Table 2.2. In Table 2.3, we give the notation for the decision variables. Important decision variables are given by capital script letters with the exception of good part yield.

Let \( \psi \) be the vector of good part recovery yields. Also, let \( \Pi \), \( R \), and \( C \) be the profit, revenue, and cost of the firm, respectively. Revenue is generated by the sales of new, remanufactured and other EOL products as well by the
Table 2.1: Model parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Time horizon, i.e., number of periods.</td>
</tr>
<tr>
<td>$K$</td>
<td>Number of part types.</td>
</tr>
<tr>
<td>$b$</td>
<td>Fixed number of periods for which a customer uses a product before it is available for collection.</td>
</tr>
<tr>
<td>$N$</td>
<td>New product type.</td>
</tr>
<tr>
<td>$R$</td>
<td>Remanufactured product type.</td>
</tr>
<tr>
<td>$E$</td>
<td>Recovered core product type.</td>
</tr>
<tr>
<td>$J$</td>
<td>Number of EOL options requiring little to no disassembly.</td>
</tr>
<tr>
<td>$E_j$</td>
<td>EOL option $j$, $j = \ldots J$ that requires little to no disassembly. The EOL options are indexed in order of ascending grade of core required.</td>
</tr>
<tr>
<td>$V$</td>
<td>Set of product types: $V = {N, R, E, E_1, \ldots, E_J}$.</td>
</tr>
<tr>
<td>$C^c$</td>
<td>Cost to collect a core.</td>
</tr>
<tr>
<td>$C^d$</td>
<td>Cost to disassemble a core.</td>
</tr>
<tr>
<td>$C^R$</td>
<td>Cost to reassemble parts to make a remanufactured product.</td>
</tr>
<tr>
<td>$C^E_j$</td>
<td>Cost to perform EOL option $E_j$, $j = 1, \ldots, J$.</td>
</tr>
<tr>
<td>$C^{\text{disp}}$</td>
<td>Cost to dispose-of a bad core.</td>
</tr>
<tr>
<td>$C^{\text{disp}}_k$</td>
<td>Cost to dispose-of Part $k$, $k = 1, \ldots, K$.</td>
</tr>
<tr>
<td>$D^v_t$</td>
<td>Demand for product of type $v$ in time period $t$, $v \in V$, $t = 1, \ldots, T$. $D^v_t$ is function of $\psi$ (price of product $v$ in time period $t$).</td>
</tr>
<tr>
<td>$\xi_k$</td>
<td>Number of Part $k$ contained in a product, $k = 1, \ldots, K$.</td>
</tr>
<tr>
<td>$M$</td>
<td>A very large number.</td>
</tr>
</tbody>
</table>

Table 2.2: Notation for quantities that are functions of $\psi$.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^N(\psi)$</td>
<td>Cost of producing a new product.</td>
</tr>
<tr>
<td>$C^p_k(\psi)$</td>
<td>Cost for replacement of Part $k$, $k = 1, \ldots, K$.</td>
</tr>
<tr>
<td>$C^{\text{hp}}_k(\psi)$</td>
<td>Holding cost of Part $k$, $k = 1, \ldots, K$.</td>
</tr>
<tr>
<td>$S_k(\psi)$</td>
<td>Salvage value of Part $k$, $k = 1, \ldots, K$.</td>
</tr>
<tr>
<td>$\lambda(\psi)$</td>
<td>Proportion of a product, produced $b$ periods ago, available for collection.</td>
</tr>
<tr>
<td>$\phi(\psi)$</td>
<td>Proportion of cores collected which are good-cores.</td>
</tr>
</tbody>
</table>

The salvaging of parts, and it is stated as follows:

$$
R = \sum_{t=1}^{T} \psi^N_t + \sum_{t=b+1}^{T} \psi^R_t + \sum_{t=b+1}^{T} \sum_{j=1}^{J} \psi^E_j + \sum_{t=b+1}^{T} \sum_{k=1}^{K} S_k. \psi_k. \tag{2.1}
$$

Let $C$ be the sum of the cost terms in Table 2.4, i.e.,

$$
C = C^N + C^R + \sum_{j=1}^{J} C^E_j + C^d + C^c + C^{\text{disp}} + C^{\text{disp}, P} + C^p + C^h + C^h + C^{h,p} + \sum_{j=1}^{J} C^{h,E_j}. \tag{2.2}
$$
We assume that as many reused components as possible are used to make the remanufactured products as possible. For

\[
\Pi = R - C. \tag{2.3}
\]

We assume that as many reused components as possible are used to make the remanufactured products as possible. For
example, production of one group of remanufactured products with Part 1 as the used part, while the remaining parts
are replacement parts and another group with Part 2 as the used part, with the remaining parts being replacement parts,
is not allowed. We justify this assumption by recognizing that realizing the benefit of remanufacturing usually involves
preserving as much of the original value of the product as possible, while adding as little new value as possible in order
to obtain like new status. By using very few reused parts in a product, we do still preserve the original value, but we
must create many new parts. In this sense, this is not remanufacturing, but making a new product with a few reused
parts. Along the same lines, we disallow “remanufactured” products produced from all replacement components. This
is accomplished by limiting the number of remanufactured products to be produced to the number of that part on hand
with maximum count. As such, all used parts will be used together in the remanufactured products or salvaged.

Note that, the inventory of the low quality parts will always be smaller than the inventory of the high quality parts
because there are less of them and a greater proportion of the ones available are used in remanufacturing. Additionally,
the number of high yielding parts used in remanufacturing is greater than or equal to the number of low yielding parts
since the availability of the low yielding parts is smaller. Thus, we need only limit the number of remanufactured
products made by the availability of the highest yielding part and its inventory level.

We have the following Production Planning Model for the RSC with Choice of Collection, Product Design and Product
Pricing (PPMRSC-CCPDPP):

\[
\text{max } \Pi - M \sum_{t} X_t \\
\text{subject to:}
\]

\[
I_{N}^t = I_{N}^{t-1} + \delta_{N}^t - W_{N}^t, \quad t = 1, \ldots, T, \quad (2.5)
\]

\[
I_{R}^t = I_{R}^{t-1} + \delta_{R}^t - W_{R}^t, \quad t = b + 1, \ldots, T, \quad (2.6)
\]

\[
I_{E_j}^{t} = I_{E_j}^{t-1} + \delta_{E_j}^t - W_{E_j}^t, \quad j = 1, \ldots, J, \quad t = b + 1, \ldots, T, \quad (2.7)
\]

\[
I_{k,t+1}^p = I_{k,t}^p + A_{k,t}^p - X_{k,t} - \beta_{k,t} - I_{k,t}^p, \quad k = 1, \ldots, K, \quad t = 1, \ldots, T, \quad (2.8)
\]
\[ G_t \leq \lambda W^{N}_{t-b}, \quad t = b + 1, \ldots, T, \quad (2.9) \]

\[ \phi G_t = \phi D^j + \sum_{j=1}^{J} \phi t_i^j, \quad t = b + 1, \ldots, T, \quad (2.10) \]

\[ \phi t_i^j \leq \phi t_i, \quad j = 1, \ldots, J, \quad t = 1, \ldots, T, \quad (2.11) \]

\[ \phi t_i^D \leq \phi t_i, \quad t = 1, \ldots, T, \quad (2.12) \]

\[ A^p_{k,t} = \psi_k \phi t_i^D \xi_k, \quad k = 1, \ldots, K, \quad t = b + 1, \ldots, T, \quad (2.13) \]

\[ \varphi t_i^R = \lambda t_i^{mu} + \varphi t_i + \chi_{k,\lambda}, \quad k = 1, \ldots, K, \quad t = b + 1, \ldots, T, \quad (2.14) \]

\[ X_t \geq (\chi_{k,\lambda} + \lambda t_i^{mu}) / \xi_k, \quad k = 1, \ldots, K, \quad t = b + 1, \ldots, T, \quad (2.15) \]

\[ \varphi t_i^R \leq X_t, \quad t = b + 1, \ldots, T, \quad (2.16) \]

\[ \psi t_i^v \leq D^v_i, \quad v \in V, \quad t = 1, \ldots, T, \quad (2.17) \]

\[ \psi t_i^v \leq \varphi t_i^R + I^v_{t-1}, \quad v \in V, \quad t = b + 1, \ldots, T, \quad (2.18) \]

\[ \varphi t_i^R + \varphi t_i^D \leq A^p_{k,t}, \quad k = 1, \ldots, K, \quad t = b + 1, \ldots, T, \quad (2.19) \]

\[ \psi t_i^R, \varphi t_i^R, \phi t_i, \text{ and } \phi t_i^D = 0, \quad t = 1, \ldots, b, \quad (2.20) \]

\[ \psi t_i^E_j, \varphi t_i^E_j, \text{ and } \phi t_i^j = 0, \quad j = 1, \ldots, J, \quad t = 1, \ldots, b, \quad (2.21) \]

\[ \varphi t_i^R, \varphi t_i^D, \text{ and } A^p_{k,t}, I^mu_{k,t} \text{ and } \varphi t_i^E_j = 0, \quad k = 1, \ldots, K, \quad t = 1, \ldots, b, \quad (2.22) \]

\[ I^N_0 = 0, \quad t = 0, \ldots, b, \quad (2.23) \]

\[ I^R_t = 0, \quad t = 0, \ldots, b, \quad (2.24) \]

\[ I^E_j = 0, \quad j = 1, \ldots, J, \quad t = 0, \ldots, b, \quad (2.25) \]

\[ I^D_j = 0, \quad k = 1, \ldots, K, \quad t = 0, \ldots, b, \quad (2.26) \]

\[ \psi_k \leq \psi_k^H, \quad k = 1, \ldots, K, \quad (2.27) \]

\[ \psi t_i^N, \psi t_i^R, \varphi t_i^N, \varphi t_i^R, \phi t_i^E_j, \phi t_i^D, A_t, \text{ and } X_t \geq 0, \quad t = 1, \ldots, T, \quad (2.28) \]

\[ \psi t_i^E_j, \varphi t_i^E_j, \phi t_i^j, \text{ and } I^E_j \geq 0, \quad j = 1, \ldots, J, \quad t = 1, \ldots, T, \quad (2.29) \]
\[ A^{p}_{k,j}, \forall_{k,j}, I^{p}_{k,j}, I^{pu}_{k,j}, \chi_{k,t} \geq 0 \quad k = 1, \ldots, K, t = 1, \ldots, T, \] (2.30)

\[ \psi_{k} \geq 0, \quad k = 1, \ldots, K, \] (2.31)

\[ P^{N} \text{ and } P^{R} \geq 0. \] (2.32)

Constraints (2.5)-(2.8) enforce inventory balance for new products, remanufactured products, products with EOL option \( E_{j}, j = 1, \ldots, J \), and that for parts, respectively, while Constraints (2.23) - (2.26) specify initial inventory levels. The number of cores collected in each period is limited by constraints (2.9) and link availability of cores with production in previous periods. Constraints (2.10) ensure a balance between the total number of good-cores collected and their usage. Constraints (2.11) ensure that the number of cores selected for a given EOL-option does not exceed the number of cores available. That the number of cores disassembled is less than the number of cores collected is enforced by Constraints (2.12). Constraints (2.13) specify the number of good parts of each type recovered from the disassembled cores. Constraints (2.14) ensure that an adequate supply of parts from inventory, disassembly, and purchasing are available for producing remanufactured products. Constraints (2.15) and (2.16) along with the \( -M \sum_{t} X_{t} \) term in the objective function ensures that as many recovered parts as possible will be used in a remanufactured product, and that, no remanufactured products will be made out of all new parts. Additionally, the sales of a product in the period \( t \) must be less than both the demand for the product in that period and the number of products available to be sold in that period (quantity produced + quantity in inventory), which are captured by constraints (2.17) and (2.18), respectively. Constraints (2.19) ensure that the number of parts salvaged plus the number of parts to be reused for remanufacturing do not exceed the number of parts recovered from the cores. Constraints (2.20) - (2.22) ensure that no value recovery activities occur prior to the first period that cores are available for recovery, i.e. period \( b + 1 \). Constraints (2.27) ensure that the upper bounds on part qualities are not exceeded. Lastly, the non-negativity conditions are imposed by constraints (2.28)- (2.32).

Model PPMRSC-CCPDPP may be modified to address any situation by simply fixing certain decision variables as parameters. If we wish to consider the case of Fixed Product Design (FD), the values of \( \psi_{k}, k = 1, \ldots, K \) are now problem parameters. Likewise for Fixed Product Pricing (FP), the values of \( P^{v} \) are now problem parameters. If there
is mandated collection (MC), then \( C = \lambda Q^N \). The complexity of the model is determined by which quantities are variable. If both design and pricing are fixed, then the model is a linear program. If there is a choice in the design, the model is a nonlinear program with a non-convex constraint set (see Constraints (2.13)). However, if the pricing is a variable, then the model is a nonlinear programming problem, and its nature further depends on the demand functions.

For brevity, we designate a model only by its unique features, e.g. PPMRSC with mandated collection, fixed product design, and fixed product pricing is designated by MCFDFP

### 2.4 Approach

Using the model given in Section 2.3, we derive key insights regarding the use of various EOL options for products and the impact of their designs on these options. However, we do not need to consider the model in its entirety.

Firstly, note that, except for remanufacturing and part salvage, the other EOL options do not require disassembly. A remanufactured product is created out of a combination of recovered parts and replacement parts, and it is possible to have a surplus of certain parts after the remanufacturing demand has been met. Thus, when performing remanufacturing, the resulting remanufactured products are sold, while surplus parts are salvaged or disposed-of. As such, the marginal profit from each core obtained for the remanufacturing option is actually a function of the number of cores collected. Other product EOL options are more straightforward to analyze since the marginal profit from each core collected remains invariant of the number of cores collected. We, therefore, focus on the decision between remanufacturing and part salvage and show how to incorporate the other EOL options in a posteriori manner.

Secondly, in case product design and pricing are fixed, without loss of generality, analysis of the following two-stage problem is sufficient for our understanding of the problem: new products are produced in Stage 1, and in Stage 2 they are recovered, disassembled and the number of remanufactured products to produce is chosen. We depict this conceptualization in Figure 2.5. This two-stage problem represents the problem in general because, ignoring inventory, a multi-period problem may be decomposed into multiple two-stage problems with period \( t - b \) as the first stage and period \( t \) as the second stage for \( t = b + 1, \ldots, T \). In case product design or price change, results from this two-stage
problem help explain the underlying principles determining optimal product design; however, due to the optimal product design’s dependence on core supply and product demand, in some cases, a multi-period model must be used.

As it is germane to understanding the impact of relevant variables on the EOL option of a product its design and pricing, the crux of our approach is in developing an in-depth understanding of the two-stage problem for various scenarios.
Chapter 3

Mapping Product Characteristics onto Optimal EOL Options

A key pillar of strategic planning for the reverse supply chain is the ability to map product characteristics onto optimal EOL options. Although both product design and product pricing impact product characteristics, the insights developed in this Chapter will aid in making decisions in general. We study this problem under two situations. The first situation is that the manufacturer has the ability to decide how many cores to collect. Section 3.1 provides a detailed analysis of this situation. In this section, the manufacture must collect the product back from the customer. This may occur by government mandate, by contractual agreement, or by signing a lease agreement. Our study of this scenario in Section 3.2 reveals that the results for both choice of collection and mandated collection are similar.

3.1 Choice of collection

The manufacturer may be free to choose the number of cores to recover, but might not be able to change the design of the product or the price. In this case, we modify the PPMRSC-CCPDPP model by fixing the value of $\psi_k$, for all
The parts are indexed according to descending part yield. We designate this problem as CCFDFP (Choice of Collection, Fixed Design, Fixed Price). CCFDFP is a linear programming problem and, as such, is easy to solve. However, here we seek insights of the problem. As alluded to earlier, we consider a two-stage system in which a new product is produced at stage 1 and, at stage 2, the products are recovered at their EOL to build remanufactured products or for salvaging parts (see Figure 2.5). We assume no inventory, and thus, we sell all that we produce at stage 1. In general, a product is composed of \( K \) types of parts, and we call this as the \( K \)-part case. A mathematical formulation for the \( K \)-part case is presented in Section 3.1.1. Analysis of the formulation’s structure is given in Section 3.1.2. We formally introduce the concepts of EOL type and EOL option in Section 3.1.3. In Section 3.1.4, we give the optimality conditions for each solution based on our analysis of the problem formulation. We demonstrate how to incorporate EOL options that do not require disassembly into the analysis in Section 3.1.5. For the purposes of building intuition about the problem, a graphical mapping for the two-part case in Section 3.1.6 is presented. Finally, in Section 3.1.7, a sensitivity analysis on the part yield is given.

### 3.1.1 Mathematical formulation

For the two-stage model, the \( t \) subscript will be dropped from the notation as the stage to which the variables correspond is self evident. Additionally, without loss of generality, we assume that the manufacturer is able to perform a cost-less initial inspection and only collects good-cores, i.e. \( \phi = 1 \). For the sake of brevity, we replace some expressions with symbols. Table 3.1 gives the list of symbols and the expressions they replace. The marginal profit from a new product is given by \( \mathcal{P}^N - C^N \) and is denoted as \( \rho^N \). We call \( \mathcal{P}^R - C^R \) as the remanufacturing profit margin and denote it by \( \rho^R \). The marginal cost of collecting a core, disassembling it, and then disposing of the bad-parts, \( C^c + C^d + \sum_{k=1}^{K} C_{disp}^k (1 - \psi_k) \), is denoted by \( \theta \). We denote the availability of cores, \( \lambda Q^N \), by \( A \).

The model for the two-stage, \( K \)-part problem is a special case of the CCFPFD, and we call it as CCFPFD with two stages (2S) and \( K \)-parts (KP) model, and is as follows:
Table 3.1: Some additional notation.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho^N$</td>
<td>$\mathcal{D}^N - C^N$</td>
</tr>
<tr>
<td>$\rho^R$</td>
<td>$\mathcal{D}^R - C^R$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$C^c + C^d + \sum_{k=1}^{K} C_{disp}^k (1 - \psi_k)$</td>
</tr>
<tr>
<td>$A$</td>
<td>$\lambda \mathcal{D}^N$</td>
</tr>
</tbody>
</table>

CCFDPF-2SKP

$$\max \Pi = \rho^N \mathcal{D}^N + \rho^R \mathcal{D}^R + \sum_{k=1}^{K} S_k \mathcal{R}_k - \sum_{k=1}^{K} C_k^p \mathcal{A}_k - \theta \mathcal{C}$$

subject to:

$$\mathcal{C} \leq \lambda \mathcal{D}^N,$$  \hspace{1cm} (3.2)

$$\mathcal{D}^R \leq \psi_1 \mathcal{C},$$  \hspace{1cm} (3.3)

$$\mathcal{D}^R \leq D^R;$$  \hspace{1cm} (3.4)

$$\mathcal{A}_k^p = \psi_k \mathcal{C}, \quad k = 1 \ldots K,$$  \hspace{1cm} (3.5)

$$\mathcal{D}^R = \mathcal{Z}_k + \chi_k, \quad k = 1 \ldots K,$$  \hspace{1cm} (3.6)

$$\mathcal{R}_k + \chi_k = \mathcal{A}_k^p, \quad k = 1 \ldots K,$$  \hspace{1cm} (3.7)

$$\mathcal{D}^N, \mathcal{D}^R, \mathcal{C},$$ and $A \geq 0,$  \hspace{1cm} (3.8)

$$\mathcal{R}_k, \text{ and } \mathcal{A}_k^p \geq 0, \quad k = 1, \ldots, K.$$ \hspace{1cm} (3.9)

Since the product design is fixed in this case, let the parts be indexed in order of decreasing part recovery yields, i.e., $\psi_k \geq \psi_{k+1}$ for $k = 1, \ldots, K - 1$. Constraints (2.5) - (2.8) and (2.23) - (2.26) from CCPDFD are not required since inventory does not arise in this case. Constraint (3.2) follows from Constraints (2.9) of CCPDPP. Constraint (3.3) results from constraints (2.15) of CCPDFD, i.e., the remanufactured product must contain at least one recovered part, and $\psi_1$ corresponds to the part with maximum yield. Constraints (3.5), (3.6), and (3.7) directly follow from constraints (2.13), (2.14), and (2.19), respectively.
We can reformulate the above model such that the decision variables \( A^p_k, Z_k, S_k \) and \( \chi_k \) along with constraints (3.5), (3.6), and (3.7) can be eliminated. To begin with, \( A^p_k \) and constraint (3.5) may be eliminated by substituting \( \psi_k c \) for \( A^p_k \). Also, since we assume that \( C^p_k > S_k \) for all \( k = 1, \ldots, K \), then \( Z_k \geq 0 \) only if replacement parts of type \( k \) are needed to make \( D^R \) remanufactured products, i.e. there is a deficit of Part \( k \). Thus, \( Z_k \) may be replaced by \( \max(D^R - \psi_k c, 0) \). Likewise, note that \( S \geq 0 \) only if there are more good parts available after the cores have been disassembled than that are required for remanufacturing, i.e., there is a surplus of Part \( k \). It follows that \( S_k \) may be replaced by \( \max(C_k - D^R, 0) \). Also, if \( \rho^N > 0 \), then \( D^N = D_N \) because we can choose to manufacture new products and not to collect any cores, an option that would still be profitable. We will assume this to be the case because, otherwise, \( D^N = 0 \). Additionally, since \( \rho^N D^N \) will be a constant, we drop it from the formulation without loss of generality. The reformulated problem is as follows:

\[
\max_{D^N, D^R, c} \Pi = \rho^R D^R + \sum_{k=1}^{K} S_k \left( \max \left( \psi_k c - D^R, 0 \right) \right) - \sum_{k=1}^{K} C^p_k \left( \max \left( D^R - \psi_k c, 0 \right) \right) - \theta c
\]

subject to:

\[
c \leq \lambda D^N \quad (3.11)
\]

(3.3), (3.4), and (3.8).

Note that \( \Pi \) is a piecewise linear function of \( c \) and \( D^R \), and \((c, D^R)\) for \( D^R = \psi_k c \), for some \( k = 1, \ldots, K \), defines a boundary point for the regions of \( \Pi \) such that Part \( k \) is neither in surplus nor in deficit. It follows that the optimal values for \( c \) and \( D^R \), will be at either the boundary of a segment of \( \Pi \) or at a boundary imposed by Constraints (3.2)-(3.4) and (3.8).

Table 3.2 gives a list of all potentially optimal solutions, their interpretation as EOL options, and the expression for their profit. Note the that feasibility of each solution is dependent on the availability of good-cores to recover, \( A \), and the demand for remanufactured products, \( D^R \). When \( A \) is low and \( D^R \) is high, we might collect everything, but not
meet remanufactured product demand (see Solution (3) in Table 3.2), whereas, when \( A \) is high and \( D^R \) is low, we might collect only some of the available good-cores, but still meet remanufactured product demand (see Solution (5) in Table 3.2).

Note that the convex combinations of the solutions in Table 3.2 define the envelope of \( \Pi \) as a function of \( C \) and \( Q^R \). Let \( \tilde{\Pi} \) be the set of line segments of \( \Pi \) as a function of \( C \) at the boundary values of \( Q^R \). Since we only consider potentially optimal values of \( Q^R \) in defining \( \tilde{\Pi} \), the optimal profit for any \( C \in [0, \infty) \) must be given by some segment of \( \tilde{\Pi} \) supported on \( C \). Thus, a subset of segments of \( \tilde{\Pi} \) constitutes the concave envelope of \( \Pi \) as a function of \( C \). We call these segments of \( \tilde{\Pi} \) as the regions of \( \tilde{\Pi} \). In defining these regions, we introduce a dummy variable \( \psi_{K+1} = 0 \). The regions of \( \tilde{\Pi} \) are defined as follows:

A. Let \( \tilde{\Pi}_{0,m} \) be the region of \( \tilde{\Pi} \) such that \( D^R = \psi_m C \) and \( C \in [0, D^R \psi_m) \), for some \( m = 1, \ldots, K \).

\[
\tilde{\Pi}_{0,m}(\psi_m) = \left( \rho^R \psi_m + \sum_{k=1}^{m-1} S_k(\psi_k - \psi_m) - \sum_{k=m+1}^{K} C^p_k(\psi_m - \psi_k) - \theta \right) \psi_m
\]

for \( \psi_m \in [0, D^R \psi_m) \) and \( m = 1, \ldots, K + 1 \),

\[
\frac{\partial \tilde{\Pi}_{0,m}(\psi_m)}{\partial \psi_m} (\psi_m) = \tilde{\Pi}_{0,m}'(\psi_m) = \rho^R \psi_m + \sum_{k=1}^{m-1} S_k(\psi_k - \psi_m) - \sum_{k=m+1}^{K} C^p_k(\psi_m - \psi_k) - \theta
\]

for \( \psi_m \in [0, D^R \psi_m) \) and \( m = 1, \ldots, K + 1 \).

The region \( \tilde{\Pi}_{0,m} \) gives the profit in case: (a) \( D^R = \psi_m C \), i.e., as many remanufactured products as possible are produced using Part \( m \) and (b) \( C \in [0, D^R \psi_m) \), i.e., we collect and disassemble too few cores to obtain \( D^R \) of Part \( m \). We call Part \( m \) as the “central” part to be “remanufactured around”. In this case, we have surpluses of part types \( k < m \) and deficits of part types \( k > m \). In case \( m = K + 1 \), we say that there exists no central part, and that, \( \Pi_{0,K+1} \) represents salvage EOL option. We denote \( \Pi_{0,K+1} \) as \( \Pi_S \).
Table 3.2: Possible optimal EOL options, choice of collection.

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Description</th>
<th>$\Pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $0 \ 0 \ [0, \infty)$</td>
<td>No product recovery.</td>
<td>$0$ (3.12)</td>
</tr>
<tr>
<td>(2) $A \ 0 \ [0, \infty)$</td>
<td>Collect all available good-cores and salvage the parts.</td>
<td>$(\sum_{k=1}^{K} S_k \psi_k - \theta)A$ (3.13)</td>
</tr>
<tr>
<td>(3) $A \ \psi_m A \ [0, \frac{D^R}{\psi_m}, m = 1, \ldots, K$</td>
<td>Collect all available good-cores and make as many remanufactured products as possible out of good part type $m$. Salvage excess parts and purchase needed replacement parts. Part $m$ is the “central” part.</td>
<td>$\rho^R \psi_m A + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m)A - \sum_{k=m+1}^{K} C^p_k (\psi_m - \psi_k)A - \theta A$ (3.14)</td>
</tr>
<tr>
<td>(4) $A \ D^R \ \left[ \frac{D^R}{\psi_m}, \frac{D^R}{\psi_{m+1}}, m = 1, \ldots, K - 1 \right.$</td>
<td>Collect all available good-cores and meet remanufactured product demand. There will be an excess of part type $k \leq m$ and a deficit of part type $k \geq m + 1$. Salvage excess parts and purchase needed replacement parts.</td>
<td>$\rho^R D^R + \sum_{k=1}^{m} S_k (\psi_k - D^R) - \sum_{k=m+1}^{K} C^p_k (D^R - \psi_k)A - \theta A$ (3.15)</td>
</tr>
<tr>
<td>(5) $A \ D^R \ \left[ \frac{D^R}{\psi_m}, \infty \right.$</td>
<td>Collect all available good-cores and meet remanufactured product demand. There will be excess of all parts, which will be salvaged.</td>
<td>$\rho^R D^R + \sum_{k=1}^{K} S_k (A \psi_k - D^R) - \theta A$ (3.16)</td>
</tr>
<tr>
<td>(6) $\frac{D^R}{\psi_{m+1}} \ D^R \ \left[ \frac{D^R}{\psi_l}, \infty \right.$</td>
<td>Collect enough good-cores to obtain as many good parts of type $l$ as needed to meet remanufactured demand. Salvage excess parts and purchase product needed parts. Part $l$ is the “limiting” part. (limited core collection)</td>
<td>$\rho^R D^R + \sum_{k=1}^{l-1} S_k (\frac{D^R}{\psi_l} - \psi_k - D^R) - \sum_{k=l+1}^{K} C^p_k (D^R - \frac{D^R}{\psi_l} - \psi_k)A - \theta D^R$ (3.17)</td>
</tr>
</tbody>
</table>
B. Let $\bar{\Pi}_{m,m+1}$ be the region of $\bar{\Pi}$ such that $\mathcal{D}^R = D^R$ and $\mathcal{C} \in \left[\frac{D^R}{\psi_m}, \frac{D^R}{\psi_{m+1}}\right)$.

$$
\Pi_{m,m+1}(\mathcal{C}) = \rho^RD^R + \sum_{k=1}^{m} S_k(\psi_k \mathcal{C} - D^R)
- \sum_{k=m+1}^{K} C^p_k(D^R - \psi_k \mathcal{C}) - \theta \mathcal{C}
$$

(3.20)

for $\mathcal{C} \in \left[\frac{D^R}{\psi_m}, \frac{D^R}{\psi_{m+1}}\right)$ and $m = 1, \ldots, K$.

$$
\frac{\partial \Pi_{m,m+1}}{\partial \mathcal{C}} (\mathcal{C}) = \Pi_{m,m+1}'(\mathcal{C}) = \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C^p_k \psi_k - \theta
$$

(3.22)

for $\mathcal{C} \in \left[\frac{D^R}{\psi_m}, \frac{D^R}{\psi_{m+1}}\right)$ and $m = 1, \ldots, K$.

(3.23)

The region $\bar{\Pi}_{m,m+1}$ gives the profit in case: (a) we meet remanufactured product demand without purchasing any of Part $m$, but (b) we collect and disassemble too few cores to obtain $D^R$ of parts $m+1$. Again, we call Part $m$ as our “central” part since we plan our remanufacturing around this part.

Let $\bar{\Pi}^R(\mathcal{C})$ be the function given by the union of $\bar{\Pi}_{m,m+1}(\mathcal{C})$ for all $m = 1, \ldots, K$. Note that for $m = K$, $\mathcal{C} \in \left[\frac{D^R}{\psi_K}, \infty\right)$ and $\bar{\Pi}_{K,K+1} = \bar{\Pi}_S$. Additionally, note that $\bar{\Pi}_{0,m} = \bar{\Pi}_{m-1,m} = \bar{\Pi}_{m,m+1}$ for $\mathcal{C} = \frac{D^R}{\psi_m}, m = 1, \ldots, K$. Figure 3.1 depicts an example of $\bar{\Pi}$ versus $\mathcal{C}$ for a two-part product, where each line is labeled below by its region and above by its slope.

Figure 3.1: $\bar{\Pi}$ versus $\mathcal{C}$ for a two-part product.
Note that for the $\Pi$ depicted in Figure 3.1, $\Pi_{1,2}' > 0$ and $\Pi_{2,3}' < 0$. Thus, it would be suboptimal to collect more than $\frac{D^R}{W}$ cores. In this case, we call Part 2 as the “limiting” part since it “limits” the optimal number of cores to collect. Also, note that the slope of each region of $\Pi$ is not a function of $A$ or $D^R$. As such, the shape of $\Pi$ is independent of the market conditions for the product.

3.1.2 Properties of $\Pi$

Next, we present some properties of $\Pi$, which can be used to generate the concave envelope of $\Pi$ from the regions of $\Pi$. This will be accomplished by describing how to generate a function that returns the maximum value of $\Pi$ for a given $C$, designated as $\Pi_{max}$, and by showing that $\Pi_{max}$ is concave. Formal proofs of all the properties may be found in Appendix A.

A very helpful observation in this regard is that $\Pi_{k-1,k}$ for $k = 2, \ldots, K + 1$ decreases in slope with increment in $k$, i.e. $\Pi_{D^R}$ is a concave function of $C$. This is due to the fact that $C^p_k > S_k$, $k = 1, \ldots, K$. When in region $\Pi_{k-1,k}$ for $k = 2, \ldots, K$, we have a deficit of part type $k$, and thus, must purchase replacement parts of this type to satisfy remanufactured product demand. Collecting additional cores in this region helps in reducing our deficit of Part $k$. However, when core collection increases further ($C > \frac{D^R}{W}$), we enter region $\Pi_{k,k+1}$, in which case there is a surplus of Part $k$ and each additional core collected increases this surplus. Since $S_k < C^p_k$, we gain less value from each additional core obtained while in region $\Pi_{k-1,k}$ than in region $\Pi_{k,k+1}$. This reasoning gives us our first property:

**Property 3.1.1** $\Pi'_{k-1,k} \geq \Pi'_{k,k+1}$ for $k = 2, \ldots, K$, i.e. $\Pi_{D^R}$ is a concave function of $C$.

Another key property of $\Pi$ is that $\Pi'_{0,1}$ may be less than or greater than $\Pi'_{1,2}$. If $\Pi'_{0,1}$ were always greater than $\Pi'_{1,2}$ (for example see Figure 3.7), then very little analysis would be required as the union of $\Pi_{0,1}$ and $\Pi_{D^R}$ would be a concave function of $C$. However, cases do exist in which $\Pi'_{0,1} \leq \Pi'_{1,2}$, in which case $\Pi_5$ or $\Pi_{0,k}$ for some $k = 2, \ldots, K$ will define the limiting part of the envelope of $\Pi$ emanating from the origin. Property 3.1.2 gives the conditions determining the relationship between $\Pi'_{0,1}$ and $\Pi'_{1,2}$.
Property 3.1.2 $\bar{\Pi}_{0,1}' \geq \bar{\Pi}_{1,2}' \iff \rho^R \geq S_1 + \sum_{k=2}^{K} C_k^p$.

The proof of Property 3.1.2 follows by comparing $\bar{\Pi}_{0,1}'$ and $\bar{\Pi}_{1,2}'$.

Since $\bar{\Pi}_{DR}$ is concave in $\mathcal{C}$ by Property 3.1.1, we may determine which regions of $\bar{\Pi}$ dominate one-another for a given value of $\mathcal{C}$ by simply comparing the slopes of adjacent regions of $\bar{\Pi}$.

Property 3.1.3 (Dominance Properties)

I. If for some $m$, $m = 1, \ldots, K$, the slopes of the segments of $\Pi$ are such that $\bar{\Pi}_{0,m}' \leq \bar{\Pi}_{m-1,m}'$ then:

a. $\bar{\Pi}_{0,m} \geq \bar{\Pi}_{0,n} \text{ for } \mathcal{C} \in [0, D^R / \psi_n), n = 1, \ldots, m - 1 \text{ and}$

b. $\bar{\Pi}_{0,m} \geq \bar{\Pi}_{n-1,n} \text{ for } \mathcal{C} \in [D^R / \psi_{n-1}, D^R / \psi_n), n = 2, \ldots, m - 1$.

(see Figure 3.2)

II. If for some $m$, $m = 1, \ldots, K$, the slopes of the segments of $\Pi$ are such that $\bar{\Pi}_{0,m}' \geq \bar{\Pi}_{m,m+1}'$ then:

a. $\bar{\Pi}_{0,m} \geq \bar{\Pi}_{0,n} \text{ for } \mathcal{C} \in [0, D^R / \psi_n), n = m + 1, \ldots, K \text{ and } \bar{\Pi}_{0,m}' \geq \bar{\Pi}_{S}'$ and

b. $\bar{\Pi}_{0,n} \leq \bar{\Pi}_{n-1,n} \text{ for } \mathcal{C} \in [D^R / \psi_{n-1}, D^R / \psi_n), n = m + 1, \ldots, K \text{ and } \bar{\Pi}_{S} \leq \bar{\Pi}_{K,K+1}$ for $\mathcal{C} \in [D^R / \psi_K, \infty)$.

(see Figure 3.3)

III. If for some $m$, $m = 1, \ldots, K$, the slopes of the segments of $\Pi$ are such that $\bar{\Pi}_{m,m+1}' \leq \bar{\Pi}_{0,m}' \leq \bar{\Pi}_{m-1,m}'$ then:

a. $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{0,m} \text{ for } \mathcal{C} \in [0, D^R / \psi_n)$,

b. $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{n-1,n} \text{ for } \mathcal{C} \in [D^R / \psi_{n-1}, D^R / \psi_n), n = m + 1, \ldots, K$, and

c. $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{K,K+1} \text{ for } \mathcal{C} \in [D^R / \psi_K, \infty)$.

(see Figure 3.4)

IV. If the slopes of the segments of $\Pi$ are such that $\bar{\Pi}_{S} \geq \bar{\Pi}_{0,K}'$ then $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{S}$ for $\mathcal{C} \in [0, \infty)$. (See Figure 3.5)
Note that if $\Pi_{m,m}^l \leq \Pi_{0,m}^r \leq \Pi_{m-1,m}^r$ then

$$
\Pi_{0,m} \geq \Pi_{0,m}^l = 1, \ldots, K \psi \in [0, \frac{dR}{\psi_m}).
$$

(3.24)

If Part $m$ satisfies Equation (3.24), we say Part $m$ is the "key" part.

Property 3.1.3 specifies the conditions under which the various regions dominate each other. The conditions are used to generate $\Pi_{max}$. The nature of $\Pi_{max}$ is dependent on the conditions of Property 3.1.2. If $\Pi_{0,1}^l \geq \Pi_{1,2}^r$, then $\Pi_{max}$ is simply given by the union of $\Pi_{0,1}^l$ and $\Pi_{1,2}^r$. Otherwise, $\Pi_5$ or $\Pi_{0,m}$ for $m = 2, \ldots, K$ define the concave envelope of $\Pi$ emanating from the origin. Next, we formally generate $\Pi_{max}$, which is the concave envelope of $\Pi$. 
Property 3.1.4 (Generation of $\tilde{\Pi}_{\text{max}}$).

I. If $\bar{\Pi}_{0,1}^{'} \geq \bar{\Pi}_{1,2}^{'}$ then

$$\tilde{\Pi}_{\text{max}} = \begin{cases} 
\bar{\Pi}_{0,1} & \text{for } C \in [0, \frac{\rho}{\psi_1}), \\
\bar{\Pi}_{m-1,m} & \text{for } C \in [D^R/\psi_{m-1}, \frac{\rho}{\psi_m}), \ m = 2, \ldots, K, \\
\bar{\Pi}_{K,K+1} & \text{for } C \in [D^R/\psi_K, \infty),
\end{cases}$$

(3.25)

(see Figure 3.6).

II. Else, if $\bar{\Pi}_{K,K+1}^{'} \geq \bar{\Pi}_{0,K}^{'}$ then

$$\tilde{\Pi}_{\text{max}} = \bar{\Pi}_S \text{ for } C \in [0, \infty),$$

(3.26)

(see Figure 3.7).

III. Else, there must exist some Part $m = 2, \ldots, K$ that is key. That is, $\bar{\Pi}_{m,m+1}^{'} \geq \bar{\Pi}_{0,m}^{'} \geq \bar{\Pi}_{m-1,m}^{'}$ (Case III from Property 3.1.3 applies), and therefore

$$\tilde{\Pi}_{\text{max}} = \begin{cases} 
\bar{\Pi}_{0,m} & \text{for } C \in [0, \frac{\rho}{\psi_m}), \\
\bar{\Pi}_{n-1,n} & \text{for } C \in [D^R/\psi_{n-1}, D^R/\psi_n), \ n = m+1, \ldots, K, \\
\bar{\Pi}_{K,K+1} & \text{for } C \in [\frac{\rho}{\psi_K}, \infty),
\end{cases}$$

(3.27)

(see Figure 3.8).

Case I of Property 3.1.4 represents the situation where Part 1, the highest yielding part, is key. Additionally, salvage is not a candidate solution since, for large enough $\rho^R$, it is better to remanufacture than salvage. However, in Case II of Property 3.1.4, salvage is most profitable. In Case III of Property 3.1.4, some part other than Part 1 is key.

Property 3.1.5 $\tilde{\Pi}_{\text{max}}$ is a concave function of $C$.

Based on these observations, we recognize that $\tilde{\Pi}_{\text{max}}$ is a concave function of $C$, and hence, the concave envelope
of $\Pi$. To prove Property 3.1.5, we can show that for the case $\Pi_{0,1}^t \geq \Pi_{1,2}^t$, then $\Pi$ is a piece-wise linear function of decreasing slopes, and as such, is concave. Otherwise, $\Pi_{\text{max}}$ is defined by $\Pi_S$ or $\Pi_{0,m}$ for $m = 2, \ldots, K$ emanating from the origin, after which it is a piece-wise linear function of decreasing slopes. As the initial segment of $\Pi_{\text{max}}$ has the greatest slope, $\Pi_{\text{max}}$ is concave.

### 3.1.3 EOL type and EOL option

A fundamental result obtained from our analysis is that the structure of $\Pi_{\text{max}}$ is neither dependent on the demand for remanufactured products, nor on the availability of cores. This follows by the fact that the slopes of the segments of $\Pi$ are not functions of $A$ or $D^R$ (see Equations (3.19) and (3.22)). Note that $D^R$ is a scalar parameter of $\Pi_{\text{max}}$ while $A$ truncates $\Pi_{\text{max}}$ at $C = A$. Based on the structure of $\Pi_{\text{max}}$ for a product, we can describe a product’s EOL type, or the market-independent properties of its EOL option. The conditions that define the properties of $\Pi_{\text{max}}$ are described in terms of relationships between the slopes of the regions of $\Pi$ and the sign of the slopes. By using the definitions of the slopes of these regions given by Equations (3.19) and (3.22), we can present the conditions in terms of the problem parameters, or product characteristics. Table 3.3 provides a list of relationships between the slopes of the regions of $\Pi$, the resulting expressions in terms of product characteristics, and implication for each.
The first relationship in Table 3.3 is that $\bar{\Pi}_{0,1}^\prime \geq \bar{\Pi}_{1,2}^\prime$, which means that Part 1 is the key part. This relationship can be stated in terms of the product characteristics (see Equation (3.28)). Specifically, if the remanufacturing margin, $\rho^R$ is greater than the opportunity cost of using Part 1 in the remanufactured product, $S_1$ plus the cost of purchasing the replacement parts, $\sum_{k=2}^{K} C^p_k$, then Part 1 is the key part. In general, for a choice of central Part $m$, let the opportunity cost of using parts in a remanufactured product as opposed to salvage, $\sum_{k=1}^{m} S_k$, plus the cost of replacing parts to make a remanufactured product, $\sum_{k=m+1}^{K} C^p_k$, be called as the cost margin for central Part $m$. Relationships 2 and 3 from Table 3.3 give the conditions under which Part $m$ is the key part. Thus, if $\rho^R$ exceeds the cost margin for $m$ as the central part, but is less than the cost margin if $m+1$ were the central part, then we know $m$ is actually the key part.

Finally, relationship 4 from Table 3.3 states that if the remanufacturing margin is less than the cost margin if no parts were key, $\sum_{k=1}^{K} S_k$, then part salvage is best.

To build some intuition regarding why these relationships in Table 3.3 determine which part is key, we present the following proposition (which is known to be true already as a result of Property 3) and its proof.

**Proposition 3.1.6** If

$$\sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^p_k \leq \rho^R \leq \sum_{k=1}^{m-1} S_k + \sum_{k=m}^{K} C^p_k$$

then Part $m$ is key, i.e. $\Pi_{0,m} \geq \Pi_{0,n}$, $n = 1, \ldots, K$ for $\rho^R \in \left[0, \frac{D^R}{\psi_m}\right)$. 
Table 3.3: Slope relationships from $\Pi$ and their expressions.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Expression</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\Pi'<em>{0,1} \geq \Pi'</em>{1,2}$</td>
<td>$\rho^R \geq s_1 + \sum_{k=2}^{K} c^p_k$</td>
<td>(3.28) Part 1 key.</td>
</tr>
<tr>
<td>2. $\Pi'<em>{m,m+1} \leq \Pi'</em>{0,m}$</td>
<td>$\rho^R \geq s_k + \sum_{k=m+1}^{K} c^p_k$</td>
<td>(3.29) Parts $k &gt; m$ not key.</td>
</tr>
<tr>
<td>$\Rightarrow \Pi_{0,m} \geq \Pi_{0,n}, n &gt; m$</td>
<td>$m = 1, \ldots, K$</td>
<td></td>
</tr>
<tr>
<td>3. $\Pi'<em>{0,m} \leq \Pi'</em>{m-1,m}$</td>
<td>$\rho^R \leq s_k + \sum_{k=m}^{K} c^p_k$</td>
<td>(3.30) Parts $k &lt; m$ not key.</td>
</tr>
<tr>
<td>$\Rightarrow \Pi'<em>{0,m} \leq \Pi'</em>{0,n}, n &lt; m$</td>
<td>$m = 1, \ldots, K$</td>
<td></td>
</tr>
<tr>
<td>4. $\Pi'<em>{S} \geq \Pi'</em>{0,K}$</td>
<td>$\rho^R \leq \sum_{k=1}^{K} s_k$</td>
<td>(3.31) No parts key, salvage dominant.</td>
</tr>
<tr>
<td>$m = 1, \ldots, K$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof** Recall that

\[ \Pi_{0,m} = \rho^R \psi_m + \sum_{k=1}^{m-1} s_k (\psi_k - \psi_m) - \sum_{k=m-1}^{K} c^p_k (\psi_m - \psi_k) - \theta, \] (3.33)

\[ \Pi_{0,m-1} = \rho^R \psi_{m-1} + \sum_{k=1}^{m-2} s_k (\psi_k - \psi_{m-1}) - \sum_{k=m-1}^{K} c^p_k (\psi_{m-1} - \psi_k) - \theta \] and,

\[ \Pi_{0,m+1} = \rho^R \psi_{m+1} + \sum_{k=1}^{m} s_k (\psi_k - \psi_{m+1}) - \sum_{k=m+2}^{K} c^p_k (\psi_{m+1} - \psi_k) - \theta. \] (3.35)

Assume

\[ \rho^R \leq \sum_{k=1}^{m-1} s_k + \sum_{k=m}^{K} c^p_k, \] (3.36)

and let $\Delta \psi = (\psi_{m-1} - \psi_m)$. 


By multiplying Equation (3.36) by $\Delta \psi$ and re-arranging terms we obtain

$$\rho^R \Delta \psi - \sum_{k=1}^{m-2} s_k \Delta \psi - \sum_{k=m+1}^{K} c^p_k \Delta \psi - c^m_m \Delta \psi < s_{m-1} \Delta \psi,$$

(3.37)

or equivalently,

$$\sum_{k=1}^{m-2} s_k \Delta \psi + \sum_{k=m+1}^{K} c^p_k \Delta \psi + s_{m-1} \Delta \psi + c^m_m \Delta \psi - \rho^R \Delta \psi > 0.$$

(3.38)

By adding

$$\rho^R \psi_m + \sum_{k=1}^{m-2} s_k (\psi_k - \psi_m) - \sum_{k=m+1}^{K} c^p_k (\psi_m - \psi_k) + c^m_m \psi_m - c^m_m \psi_m$$

(3.39)

to both sides of Equation (3.37), substituting $\psi_{m-1} - \psi_m$ for $\Delta \psi$, and subtracting by $\theta$ on both sides, we obtain

$$\rho^R \psi_{m-1} + \sum_{k=1}^{m-2} s_k (\psi_k - \psi_{m-1}) + \sum_{k=m+1}^{K} c^p_k (\psi_{m-1} - \psi_k) - \theta \leq$$

$$\rho^R \psi_m + \sum_{k=1}^{m-1} s_k (\psi_k - \psi_m) + \sum_{k=m+1}^{K} c^p_k (\psi_m - \psi_k) - \theta,$$

(3.40)

which is simply $\Pi'_0, m \geq \Pi'_0, m-1$. Note that

$$\rho^R \leq \sum_{k=1}^{m-1} s_k + \sum_{k=m}^{K} c^p_k \leq \cdots \leq s_1 + \sum_{k=2}^{K} c^p_k$$

(3.41)

since $s_k \leq c^p_k$ for $k = 1, \ldots, K$ and thus, by similar logic

$$\Pi_{0,m} \geq \Pi_{0,n} \text{ for } n < m$$

(3.42)

And since $\Pi_{0,m}$ and $\Pi_{0,n}$ share the same origin

$$\Pi_{0,m} \geq \Pi_{0,n} \text{ for } n < m \text{ and } \psi \in \left(0, \frac{d_R}{\psi_n}\right)$$

(3.43)
Using similar logic, by assuming
\[
\rho^R \geq \sum_{k=1}^{m} s_k + \sum_{k=m+1}^{K} c_k^R
\] (3.44)
then
\[
\Pi_{0,m} \geq \Pi_{0,n} \text{ for } n > m; \forall \xi \in [0, \frac{d_R}{\psi_n}]. \tag{3.45}
\]

An interesting relationship is given by the left-hand-side (LHS) of Equation (3.38); it describes the difference in per collected core profit between Part \( m \) and Part \( m - 1 \) as the central part. If either Part \( m \) or Part \( m - 1 \) are central, there is a surplus of parts \( k = 1, \ldots, m - 2 \) and a deficit of parts \( k = m + 1, \ldots, K \). The first term of Equation (3.38) is the addition revenue from salvage if Part \( m \) is central as opposed to Part \( m - 1 \), while the second term represents the savings from not purchasing as many replacement parts if Part \( m \) is central. The third term is the part salvage value from Part \( m - 1 \) obtained if Part \( m \) is central, while the fourth term is the savings from not needing to replace Part \( m \) if it is central. Finally, the last term is the revenue from remanufacturing that is lost by the lower yielding part, \( m \), being central. If the LHS is positive, then the extra salvage value and replacement part cost savings outweighs the loss of remanufactured product sales, and Part \( m \) central is dominate over Part \( m - 1 \) central. In a similar manner, if the addition remain sales for Part \( m \) central are preferable to the extra salvage revenue and part cost savings, for Part \( m - 1 \) central, then Part \( m \) central is preferable to Part \( m - 1 \) central.

Note that Equation 3.30 is always greater than Equation 3.29 since \( c_k^R \geq s_k \) for \( k = 1, \ldots, K \). (i.e. the cost margin if Part \( m - 1 \) is central smaller than the cost margin if Part \( m \) were central). Also, larger \( \rho^R \) implies a the key part will be of higher yield; in turn, this means the cost margin will be greater. However, this increase in cost margin is off-set by an increase in the number of remanufactured products sold due to higher key part yield.

Using the relationships in Table 3.3 and other information about the slopes of the segments in \( \bar{\Pi} \), we can broadly categorize the EOL types for products into the following four categories.
I. Salvage

If Case II of Property 3.1.4 holds for a product, then that product has a Salvage EOL type. In this case no matter how many cores are collected, the parts are salvaged (see Figure 3.7). Specifically, the following conditions are sufficient for this outcome:

a. \( \rho_R \leq \sum_{k=1}^{K} S_k \), and

b. \( \check{\Pi}_s' \geq 0 \).

II. Remanufacture and Salvage

If Case I or Case III of Property 3.1.4 holds for a product and there is no limiting part, then that product has a combination of Remanufacture and Salvage EOL types. In other words, collecting cores is beneficial both for obtaining parts to reuse in remanufactured products, and for salvage (see Figure 3.15). Specifically, the following conditions are sufficient for this outcome:

a. \( \sum_{k=1}^{K} S_k \leq \rho_R \leq \sum_{k=1}^{K-1} S_k + C_P^0 (\check{\Pi}_{K,K+1} \leq \check{\Pi}_{0,K} \geq \check{\Pi}_{K-1,K}' \), and

b. \( \check{\Pi}_{K-1,K}' \geq 0 \).

III. Remanufacture

If Case I or Case III of Property 3.1.4 holds for a product and there does exist some limiting part, \( l \), then a product has a Remanufacture EOL type. In other words remanufacturing is profitable, but collecting more than enough cores to obtain \( D^R \) of Part \( l \) is suboptimal, which implies that part salvage is not profitable (see Figure 3.6). Specifically, the following conditions are sufficient for this outcome:

a. Either

i. \( \rho_R \geq S_1 + \sum_{k=2}^{K} C_P^0 (\check{\Pi}_{0,1} \geq \check{\Pi}_{1,2}') \) if \( m = 1 \), or

ii. \( \sum_{i=1}^{m} S_k + \sum_{k=m+1}^{K} C_P^0 \leq \rho_R \leq \sum_{i=1}^{m-1} S_k + \sum_{k=m}^{K} C_P^0 (\check{\Pi}_{m,m+1} \leq \check{\Pi}_{0,m} \leq \check{\Pi}_{m-1,m}') \) and \( \check{\Pi}_{0,m}' \geq 0 \) for some \( m > 1 \) and

b. \( \check{\Pi}_{l-1,l} \geq 0 \) and \( \check{\Pi}_{l,l+1} ' \leq 0 \) for some \( l > m \).
IV. No Collection

A product has a *No Collection* EOL type if both remanufacturing and part salvage EOL options decrease profit.

(see Figure 3.10). Specifically, the following conditions are sufficient for this outcome:

a. \( \bar{\Pi}_{0,m} \leq 0 \) for all \( m = 1, \ldots, K \), and

b. \( \bar{\Pi}_S \leq 0 \).

We may further describe \( \bar{\Pi}_{\text{max}} \) depending on which parts are key in accordance with the relationships given in Table 3.3. Specifically, Equations (52) and (53) can be used to determine key parts. Notice that for given part salvage values and part replacement costs, the key part is determined by \( \rho^R \). We may also specify the optimality conditions for each solution.

Formally, we say the EOL type is defined by the function \( \bar{\Pi}_{\text{max}}(A, d_R, \psi') \). Recall that \( A \) and \( d_R \) have no effect on the shape of \( \bar{\Pi}_{\text{max}}(\psi') \); they only affect its domain and scaling, respectively. Based on the concave nature of \( \bar{\Pi}_{\text{max}} \), we may easily find \( \psi'^* \) and the corresponding \( D^R_{\text{opt}} \) by identifying the region supported by \( \psi'^* \). For example, consider \( \bar{\Pi}_{\text{max}} \) for some \( D_R \) and various values of core availabilities: \( A_1 \in [0, \frac{D_R}{\psi_1}] \), \( A_2 \in \left[ \frac{D_R}{\psi_1}, \frac{D_R}{\psi_2} \right) \), and \( A_3 \in \left[ \frac{D_R}{\psi_2}, \infty \right) \) in Figure 3.9.

![Figure 3.9: \( \bar{\Pi} \) for a three-part product with Part 2 limiting.](image-url)
In case \( A = A_1 \), then \( \mathcal{C}^* = A \) and \( \mathcal{Q}^R = \psi_1 A \) (Solution 3, \( m = 1 \), Table 3.2). If \( A = A_2 \), then \( \mathcal{C}^* = A \) and \( \mathcal{Q}^R = D^R \) (Solution 4, \( m = 1 \), Table 3.2). However, if \( A = A_3 \), then the limiting part’s (Part 2) impact is seen with \( \mathcal{C}^* = D^R \psi \) and \( \mathcal{Q}^R = D^R \) (Solution 6, \( l = 2 \), Table 3.2). Recall that the choice of \( \mathcal{C} \) and \( \mathcal{Q}^R \) specifies the EOL option. We now give the optimality conditions for the EOL option solution forms in Table 3.2.

### 3.1.4 Optimality conditions for the various EOL options

We specify the optimality conditions for each solution in Table 3.2 based on the properties presented in Section 3.1.2 and the relationships depicted in Table 3.3.

1. **Solution 1:** The solution \( \mathcal{C}^* = 0 \) and \( \mathcal{Q}^R = 0 \) is optimal if
   
   a. \( A \in [0, \infty) \)
   
   b. \( \bar{\Pi}'_{0,m} \leq 0 \) for all \( m = 1, \ldots, K \), and
   
   c. \( \bar{\Pi}'_S \leq 0 \).

   Figure 3.10 depicts an example case in which this solution is optimal. \( \bar{\Pi}_{max} \) may be generated from any case of Property 3.1.4 for which this solution is possible. Products with this solution will have the “No Collection” EOL type.

2. **Solution 2:** The solution \( \mathcal{C}^* = A \) and \( \mathcal{Q}^R = 0 \) is optimal if
   
   a. \( A \in [0, \infty) \),
   
   b. \( \rho_R \leq \sum_{k=1}^{K} S_k (\bar{\Pi}'_S \geq \bar{\Pi}'_{0,k}) \), and
   
   c. \( \bar{\Pi}'_S \geq 0 \).

   Figure 3.11 depicts an example case in which this solution is optimal. \( \bar{\Pi}_{max} \) may be generated from only Case II of Property 3.1.4 for this solution to be possible. Products with this solution will have the “Salvage” EOL type.
3. Solution 3: The solution \( \mathcal{C}^* = A \) and \( \mathcal{Q}^{R^*} = \psi_m A \) is optimal if

a. \( A \in \left[ 0, \frac{D^R}{\psi_m} \right) \),

b. Either

\[
\begin{align*}
\rho^R & \geq S_1 + \sum_{k=2}^{K} C^p_k \left( \Pi'_{0,1} \geq \Pi'_{1,2} \right) \quad & \text{if } m = 1, \\
\sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^p_k & \leq \rho_R \leq \sum_{k=1}^{m-1} S_k \\
+ \sum_{k=m}^{K} C^p_k \left( \Pi'_{m,m+1} \leq \Pi'_{0,m} \leq \Pi'_{m-1,m} \right) & \quad & \text{if } m > 1
\end{align*}
\]  

(3.46)

and

c. \( \Pi'_{0,m} \geq 0 \)

Figure 3.12 depicts an example case in which this solution is optimal. \( \Pi_{\text{max}} \) will be generated from only Case I of Property 3.1.4 for this solution to be possible. Products with this solution will have either the “Remanufacture” or “Remanufacture and Salvage” EOL type.

4. Solution 4: The solution \( \mathcal{C}^* = A \) and \( \mathcal{Q}^{R^*} = D^R \) is optimal if

a. \( A \in \left[ D^R / \psi_{m-1}, \frac{D^R}{\psi_m} \right) \) for some \( m = 2, \ldots, K \),

b. Either

\[
\begin{align*}
\rho^R & \geq S_1 + \sum_{k=2}^{K} C^p_k \left( \Pi'_{0,1} \geq \Pi'_{1,2} \right) \quad & \text{or} \\
\sum_{k=1}^{n} S_k + \sum_{k=n+1}^{K} C^p_k & \leq \rho_R \leq \sum_{k=1}^{n-1} S_k + \sum_{k=n}^{K} C^p_k \left( \Pi'_{n,n+1} \leq \Pi'_{0,n} \leq \Pi'_{n-1,n} \right) \\
& \quad & \text{for some } 1 < n < m,
\end{align*}
\]  

(3.47)

and

c. \( \Pi'_{m-1,m} \geq 0 \).

Figure 3.13 depicts an example case in which this solution is optimal. \( \Pi_{\text{max}} \) will be generated from only Cases I and III of Property 3.1.4 for this solution to be possible. Products with this solution will have either the “Remanufacture” or “Remanufacture and Salvage” EOL type.
5. Solution 5: The solution $x^* = A$ and $z^{R*} = D^R$ is optimal if

a. $A \in \left[ \frac{b_R}{w_k}, \infty \right)$,

b. $\sum_{k=1}^{K} S_k \leq \rho R \leq \sum_{k=1}^{K-1} S_k + C^P_k (\Pi^{'}_{K,K+1} \leq \Pi^{'}_{0,K} \leq \Pi^{'}_{K-1,K})$, and

c. $\Pi^{'}_{K-1,K} \geq 0$.

Figure 3.14 depicts an example case in which this solution is optimal. $\Pi_{\text{max}}'$ will be generated from only Case III of Property 3.1.4 for this solution to be possible. Products with this solution will have the “Remanufacture and Salvage” EOL type.

6. Solution 6: The solution $x^* = \frac{b_R}{w_m}$ and $z^{R*} = D^R$ is optimal if $A \in \left[ \frac{b_R}{w_m}, \infty \right)$ and

a. Either

\[
\begin{aligned}
\rho_R &\geq S_1 + \sum_{k=2}^{K} C^P_k (\Pi^{'}_{0,1} \geq \Pi^{'}_{1,2}) \quad \text{or} \\
\sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^P_k &\leq \rho R \leq \sum_{k=1}^{n-1} S_k + \sum_{k=n}^{K} C^P_k (\Pi^{'}_{n,n+1} \leq \Pi^{'}_{0,n} \leq \Pi^{'}_{n-1,n})
\end{aligned}
\]

(3.48)

for some $1 < n < m$,

and

b. $\Pi^{'}_{m-1,m} \geq 0$ and $\Pi^{'}_{m,m+1} \leq 0$,

or

a. $\sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^P_k \leq \rho R \leq \sum_{k=1}^{m-1} S_k + \sum_{k=m}^{K} C^P_k (\Pi^{'}_{m,m+1} \leq \Pi^{'}_{0,m} \leq \Pi^{'}_{m-1,m})$, and

b. $\Pi^{'}_{0,m} \geq 0$ and $\Pi^{'}_{m,m+1} \leq 0$.

Figure 3.15 depicts an example case in which this solution is optimal. $\Pi_{\text{max}}'$ will be generated from only Cases I and III of Property 3.1.4 for this solution to be possible. Products with this solution will have the “Remanufacture” EOL type.

Using these optimality conditions, a solution procedure for determining an optimal EOL option is possible. This procedure is presented in Figure 3.16, and it is described as follows:
1. **Should cores be collected?** If $\bar{\Pi}^I \leq 0$ for all $m = 1, \ldots, K + 1$ then, no cores should be collected and Solution 1 is optimal.

2. **Does salvage dominate remanufacturing?** If $\rho^R \leq \sum_{k=1}^{K} S_k$, then salvage dominates remanufacturing and Solution 2 is optimal.

3. **Determination of key part:**
   
   a. **Part 1 Key.** Part 1 is key if $\rho^R \leq S_1 + \sum_{k=2}^{K} C^p_k$
   
   b. **Some other part, m, is key.** A part, $m$ ($m \neq 1$), is key if $\sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^p_k \leq \rho^R \leq \sum_{k=1}^{m-1} S_k + \sum_{k=m+1}^{K} C^p_k$.

4. **Is there a dearth of cores?** If $A \in [0, \frac{\rho^R}{\psi_m})$, then Solution 3 is optimal with Part $m$ key.

5. **Determination of limiting part:**
   
   a. **Key part is limiting part.** If $\sum_{k=1}^{m+1} S_k \psi_k + \sum_{k=m+2}^{K} C^p_k \psi_k - \theta \leq 0$, then the key part is also limiting.
   
   b. **Some other part is limiting.** If $l$ is such that $\bar{\Pi}^I_{l-1,l} > 0$ and $\bar{\Pi}^I_{l,l+1} < 0$ then Part $l$ is limiting
   
   c. **No limiting part.** If conditions in (a) and (b) do not hold, then there is no limiting part.

6. **Ample core supply with limiting part.** If there is a limiting part and $A \in \left[\frac{\rho^R}{\psi_l}, \infty\right)$, then Solution 6 with limiting Part $l$ is optimal.

7. **Ample core supply with no limiting part.** If there is no limiting part and $A \in \left[\frac{\rho^R}{\psi_m}, \frac{\rho^R}{\psi_l}\right)$, then Solution 5 is optimal.

8. **Neither an ample supply nor a dearth of cores.** If $A \in \left[\frac{\rho^R}{\psi_m}, \frac{\rho^R}{\psi_l}\right)$, in case there is a limiting part, or if $A \in \left[\frac{\rho^R}{\psi_m}, \psi_K\right)$, in case there is no limiting part, then Solution 4 is optimal.

### 3.1.5 Analysis of other EOL options (that do not require disassembly)

The analysis so far has focused on how to choose between (i) not collecting cores, (ii) part salvage, and (iii) product remanufacture as EOL strategies. However, there may exist other EOL options for products such as resale and refurbishment. Recall from Chapter 2 that, unlike remanufacturing, the profit margins for these options are not dependent...
on number of cores collected, and are thus relatively simple to incorporate into the analysis. Not all cores may undergo the same EOL option. For example, an OEM may decide to fill remanufacturing demand using some cores and re-sell the remainder.

Let the EOL options be indexed according to decreasing profitability. We define profit attributable to the non-
remanufacturing/salvage EOL option and its derivative with respect to $\mathcal{C}$ by

$$
\Pi_{EOL}(\mathcal{C}) = (\rho_v - C^c)\mathcal{C} \quad \text{for } v = 1, \ldots, V
$$

$$
\frac{\partial \Pi_{EOL}}{\partial \mathcal{C}} = \rho_v - C^c \quad \text{for } v = 1, \ldots, V
$$

where $\rho_v$ is the profit margin the $v^{th}$ type of EOL option.

Rather than initially considering these other EOL options directly in our analysis, we can first use the result from Section 3.1.4 to find $\Pi_{\max}$ for the remanufacturing and salvage options and then update it to include the other EOL options. The process to update $\Pi_{\max}$ to include $\Pi_{EOL}^v, v = 1, \ldots, V$ involves selecting the best EOL option from between $\Pi_{\max}$ and $\Pi_{EOL}^v, v = 1, \ldots, V$ for each additional core collected.

Beginning from $\mathcal{C} = 0$, let $\Pi_{\max}^i$ be the $i^{th}$ segment of $\Pi_{\max}$ and let $\mathcal{C}^i$ be the boundary value of $\mathcal{C}$ corresponding to the rightmost end point of $\Pi_{\max}^i$. The procedure for updating $\Pi_{\max}$ is as follows:

1. Let $\Pi_{\max}^{\text{updated}}$ be the set of segments defining the updated maximal function for $\Pi$ and let it be empty initially.

2. Select the EOL option (i.e. segment of $\Pi_{\max}$ or $\Pi_{EOL}^v, v = 1, \ldots, V$) which has the greatest slope. Add this segment to the end of the last segment in $\Pi_{\max}^{\text{updated}}$ and remove it from consideration.
iii. Repeat Step (ii) until either all segments in $\bar{\Pi}_{\text{max}}$ and $\bar{\Pi}_{\text{EOL},v}$, $v = 1, \ldots, V$ have been considered or $\bar{\Pi}_{\text{max}}^\text{updated}$ is defined for $c' > A$.

To illustrate this method, consider the example $\bar{\Pi}_{\text{max}}$ and $\bar{\Pi}_{\text{EOL},j}$ shown in Figure 3.17 and the corresponding updated $\bar{\Pi}_{\text{max}}$ in Figure 3.18.

In Figure 3.17, we see that $\bar{\Pi}_{\text{max}}^1$ has the greatest slope, and therefore it is the first segment of $\bar{\Pi}_{\text{max}}^\text{updated}$ in Figure 3.18. In this case, the first $c^1$ core collected are disassembled and their parts used for remanufacturing. However, the next best segment is $\bar{\Pi}_{\text{EOL}}^1$, in which case the next $D^1$ cores are used in EOL option 1. Core supply is not yet exhausted and the next best EOL option is $\bar{\Pi}_{\text{max}}^2$, in which case the remaining cores are collected in order to obtain additional parts for remanufacturing and salvage.

Since we are always choosing the next most profitable option when generating $\bar{\Pi}_{\text{max}}^\text{updated}$, its slopes will be non-increasing and so $\bar{\Pi}_{\text{max}}^\text{updated}$ will be concave as well.

### 3.1.6 Graphical mapping of product characteristics onto EOL type and option: two-part case

In this section, we present a graphical mapping of product characteristics onto EOL types and options. We consider a two-part problem for which such a mapping is possible using the properties of $\bar{\Pi}$ and the relationships listed in Table 3.4. Using combinations of these conditions (a-g), we can define the conditions leading to the following seven EOL strategies possible for a two-part product.

1. **Salvage**: If condition (a) that $S_1 + S_2 \geq \rho R$ (see Table 3.4) holds, then part salvage is optimal

2. **Remanufacture and Salvage, Part 1 Key**: If conditions (b) and (f) (see Table 3.4) hold, the EOL type for a product is to remanufacture and salvage. Condition (b) specifies that Part 1 is key ($S_1 + C_2^p \leq \rho R$) and condition (f) indicates that salvage is profitable ($S_1 \psi_1 + S_2 \psi_2 > \theta$).
Table 3.4: Relationships for two-part product mapping.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Implication</th>
<th>Resulting from</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (S_1 + S_2 &gt; \rho R)</td>
<td>(\Pi_3) optimal</td>
<td>Equation (3.31)</td>
</tr>
<tr>
<td>(b) (S_1 + C_2^p &lt; \rho R)</td>
<td>(\Pi_{0.1} &gt; \Pi_{0.2}) (Part 1 key)</td>
<td>Eq. (3.29)</td>
</tr>
<tr>
<td>(c) (C_2^p(\psi_1 - \psi_2) + \theta &lt; \rho R \psi_1)</td>
<td>(\Pi_{0.1} &gt; 0)</td>
<td>(\Pi_{0.1})</td>
</tr>
<tr>
<td>(d) (S_1(\psi_1 - \psi_2) + \rho R \psi_2 &gt; \theta)</td>
<td>(\Pi_{0.2} &gt; 0)</td>
<td>(\Pi_{0.2})</td>
</tr>
<tr>
<td>(e) (S_1 \psi_1 + C_2^p \psi_2 &gt; \theta)</td>
<td>(\Pi_{1.2} &gt; 0)</td>
<td>(\Pi_{1.2})</td>
</tr>
<tr>
<td>(f) (S_1 \psi_1 + S_2 \psi_2 &gt; \theta)</td>
<td>(\Pi_3 &gt; 0)</td>
<td>(\Pi_3)</td>
</tr>
<tr>
<td>(g) (\psi_1 &gt; \psi_2)</td>
<td>(\theta / \psi_1 \geq \rho R \geq \theta / \psi_2)</td>
<td>Comparison of (S_1 + C_2^p = \theta) and (S_1 \psi_1 + C_2^p \psi_2 = \theta)</td>
</tr>
</tbody>
</table>

3. **Remanufacturing and Salvage, Part 2 Key:** The conditions are the same as for Region 3 except condition (b) does not hold.

4. **Remanufacture with Part 1 Key and Limiting:** If condition (c) holds (see Table 3.4) but not condition (e), then the product is of EOL type remanufacture with Part 1 key and limiting. Condition (c) (i.e. \(C_2^p(\psi_1 - \psi_2) + \theta < \rho R \psi_1\)) indicates that Part 1 being key is profitable and Condition (e) (i.e. \(S_1 \psi_1 + C_2^p \psi_2 > \theta\)) indicates that Part 1 is limiting.

5. **Remanufacture with Part 2 Key and Limiting:** If conditions (d) and (b) but not (f) hold (see Table 3.4), then the product’s EOL type is remanufacture with Part 2 key and limiting. Condition (d) indicates that Part 2 is limiting.

6. **Remanufacture with Part 1 Key, Part 2 Limiting:** If condition (e) holds, but not (b) and (f), then the product’s EOL type is remanufacture with Part 1 key, Part 2 limiting.

7. **No Collection:** If conditions (d) and (e) do not hold, then the product's EOL type is no collection.

Based on these, conditions we generate the mapping in Figure 3.19. The x-axis depicts the replacement price of Part 2, \(C_2^p\), and the y-axis depicts the salvage value of Part 1, \(S_1\). Each region is labeled by a circled number. For each EOL type, we know the optimal EOL option based on the availability of cores. For each region in Figure 3.19, Table 3.5 gives the optimal EOL option, \(C^*\) and \(Q_{R^*}\), for each range of core availabilities. For example, if a product is in Region 5 (remanufacture with Part 2 key and limiting) and \(A \in [\frac{D^R}{\psi_2}, \infty]\), then the optimal EOL option is \(C^* = \frac{D^R}{\psi_2}\) and \(Q_{R^*} = D^R\).
Table 3.5: Optimal EOL options ($C^*$ and $Q^R*$) for various values of $A$ and EOL types.

<table>
<thead>
<tr>
<th>Region</th>
<th>EOL Type</th>
<th>Key Part</th>
<th>Limiting Part</th>
<th>$A \in (0, \frac{D_R}{\psi_1})$</th>
<th>$A \in \left[\frac{D_R}{\psi_1}, \frac{D_R}{\psi_2}\right)$</th>
<th>$A \in \left[\frac{D_R}{\psi_2}, \infty\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Salvage</td>
<td>-</td>
<td>-</td>
<td>$A$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>2</td>
<td>Remanufacturing and Salvage</td>
<td>1</td>
<td>$A$</td>
<td>$\psi_1 A$</td>
<td>$A$</td>
<td>$\frac{D_R}{\psi_1}$</td>
</tr>
<tr>
<td>3</td>
<td>Remanufacturing and Salvage</td>
<td>2</td>
<td>$A$</td>
<td>$\psi_2 A$</td>
<td>$A$</td>
<td>$\psi_2 A$</td>
</tr>
<tr>
<td>4</td>
<td>Remanufacturing</td>
<td>1</td>
<td>1</td>
<td>$A$</td>
<td>$\psi_1 A$</td>
<td>$\frac{D_R}{\psi_1}$</td>
</tr>
<tr>
<td>5</td>
<td>Remanufacturing</td>
<td>2</td>
<td>2</td>
<td>$A$</td>
<td>$\psi_2 A$</td>
<td>$A$</td>
</tr>
<tr>
<td>6</td>
<td>Remanufacturing</td>
<td>1</td>
<td>2</td>
<td>$A$</td>
<td>$\psi_1 A$</td>
<td>$A$</td>
</tr>
<tr>
<td>7</td>
<td>No Collection</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It is interesting to note that the value of $C_1^p$ does not have any bearing on the optimal EOL option. This is because Part 1 will never be replaced as it is the highest-yield part.

### 3.1.7 Value of quality

Adding quality to the parts of a product may be beneficial as it increases part yields which, in turn, increases the number of parts available to salvage and reduces the number of replacement parts that must be purchases. Additionally, adding part quality also may reduce the number of cores a company must collect to obtain sufficient parts to meet remanufactured product demand. However, this reduction in core collection may reduce the number of recovered parts which are salvaged. Therefore, a pertinent question is: how much should a company be willing to pay for an increment in part quality? In this section, we present a sensitivity analysis on part quality to help answer this question.

For this analysis, we state how much will the optimal profit change as this solution is changed in the neighborhood of this optimal solution. As such we also present the value for each part quality before the optimal solution changes. If the part quality were to be changed beyond that value, then the optimal solution must be reevaluated and the profit recalculated. By using the optimality conditions from Section 3.1.4, we can specify the range for each $\psi_k, k = 1, \ldots, K$ for which the current optimal solution remains valid. Let $\hat{\theta} = \theta + C_i^{disp}$ for $i = 1, \ldots, K$. 


1. **Salvage:** The only yield dependent condition for salvage to be optimal is that \( \sum_{k=1}^{K} S_k \psi_k \geq \theta \). Thus,

\[
\psi_i > \max(0, \frac{\hat{\psi} - \sum_{k=1, k \neq i}^{K} S_k \psi_k}{S_i + C_{i}^{disp}}) \quad \text{for} \quad i = 1, \ldots, K. \tag{3.51}
\]

2. **Remanufacture and Salvage, Key Part \( m \):** One yield dependent condition is that \( \sum_{k=1}^{K} S_k \psi_k \geq \theta \), and so we must observe the range of restrictions given by Equation (3.51). However, if \( A \in [0, \frac{D^R}{\psi_{m+1}}) \), then \( \psi_m \leq \frac{D^R}{\psi_m} \).

3. **Remanufacture, Key Part \( m \), Limiting Part \( l \):** In this case, \( \sum_{k=1}^{K} S_k \psi_k \leq \theta \) and so

\[
\psi_i < \min(\psi_i^H, \frac{\hat{\psi} - \sum_{k=1, k \neq i}^{K} S_k \psi_k}{S_i + C_{i}^{disp}}) \quad \text{for} \quad i = 1, \ldots, K. \tag{3.52}
\]

Additionally, for \( l \) to remain the limiting part, \( \Pi_{l,i+1} \) must remain negative. Thus, the range of \( \psi \) for this condition is

\[
\psi_i \leq \min(\psi_i^H, \psi_i^H + \frac{\hat{\psi} - \sum_{k=1, k \neq i}^{K} S_k \psi_k - \sum_{k=m, k \neq i}^{K} S_k \psi_k - C_{i}^{p} \psi_k}{C_{i}^{p} + C_{i}^{disp}}) \quad \text{for} \quad i = m + 1, \ldots, K \tag{3.53}
\]

If \( A \in [0, \frac{D^R}{\psi_{m+1}}) \), then \( p_{sim} \leq \frac{D^R}{\theta} \). So, the optimal \( \mathcal{D}^R \) does not change. However, if \( A \in [0, \frac{D^R}{\psi_{m+1}}] \) then \( \psi_k \geq \frac{D^R}{\psi_k} \) and \( \psi_{k+1} \leq \frac{D^R}{\psi_k} \); otherwise, different parts will be in surplus and deficit. Finally, if \( A \in [\frac{D^R}{\psi_{m+1}}, \infty) \), then \( \psi_i \geq \frac{D^R}{\psi_i} \).

Table 3.6 gives the maximum value that an additional increment in part quality provides as long as the optimal solution does not change with the final part quality.

### 3.1.8 Example

To illustrate the method described in this section, a hypothetical example from the Class 8 truck industry is presented. The major components of a class-8 truck are the cab, engine, transmission, radiator, hood, front axle assembly, and rear differential. The part replacement costs, salvage values, part yields, and disposal costs for these parts are given in Table 3.7. The remanufactured product price, cost to recover a truck, cost to disassemble a truck, demands of new
Table 3.6: Value of additional part quality for each solution.

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Value of Additional Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>A ∈ [0, ∞)</td>
</tr>
<tr>
<td>3</td>
<td>A ∈ [0, ∞)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>A ∈ [D_a / V_m^-1, D_a / V_m]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>A ∈ [D_a / V_m^-1, ∞)</td>
</tr>
<tr>
<td>6</td>
<td>A ∈ [D_a / V_m^-1, ∞)</td>
</tr>
</tbody>
</table>

and remanufactured trucks, and proportion of new trucks produced able to be recovered are given in Table 3.8. Figure 3.20 presents the plot of \( \bar{\Pi} \) for this example.

From Figure 3.20, it is clear that part salvage is the optimal EOL option. However, to verify analytically that salvage is the optimal EOL option, we must check that (i) \( \rho R \leq \sum_{k=1}^{K} s_k \) and \( \sum_{k=1}^{K} \psi_k - \theta \geq 0 \) are true. For this example, note that \( \rho R = 35,000 - 10,000 = 25,000 \) and \( \sum_{k=1}^{K} s_k = 34,102 \), and so, \( \rho R \leq \sum_{k=1}^{K} s_k \). It can also be verified that \( \sum_{k=1}^{K} \psi_k - \theta \geq 0 \) and the optimality conditions for salvage as the EOL option hold.
Table 3.7: Part replacement cost, salvage value, good-part recovery yield, and disposal cost for a class-8 truck.

<table>
<thead>
<tr>
<th>k</th>
<th>Part</th>
<th>$C^p_k$</th>
<th>$S_k$</th>
<th>$\psi_k$</th>
<th>$C^{disp}_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Cab</td>
<td>20306</td>
<td>13537</td>
<td>0.97</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>Engine</td>
<td>14480</td>
<td>9653</td>
<td>0.84</td>
<td>78.75</td>
</tr>
<tr>
<td>3</td>
<td>Transmission</td>
<td>5405</td>
<td>3604</td>
<td>0.83</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>Radiator</td>
<td>1931</td>
<td>1288</td>
<td>0.82</td>
<td>6.75</td>
</tr>
<tr>
<td>5</td>
<td>Hood</td>
<td>4049</td>
<td>2699</td>
<td>0.74</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>Front Axle Assembly</td>
<td>2257</td>
<td>1505</td>
<td>0.6</td>
<td>33.75</td>
</tr>
<tr>
<td>7</td>
<td>Rear Differential</td>
<td>2725</td>
<td>1816</td>
<td>0.59</td>
<td>112.5</td>
</tr>
</tbody>
</table>

Table 3.8: Additional product characteristics for class-8 truck.

<table>
<thead>
<tr>
<th>$\mathcal{P}^{R}$</th>
<th>$D^N$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35000</td>
<td>10000</td>
<td>4500</td>
</tr>
<tr>
<td>10000</td>
<td>$D^R$</td>
<td>4500</td>
</tr>
<tr>
<td>16000</td>
<td>$\lambda$</td>
<td>0.7</td>
</tr>
<tr>
<td>2000</td>
<td>$C^d$</td>
<td></td>
</tr>
</tbody>
</table>

### 3.2 Mandated core collection and fixed product design

In some situations, the manufacturer is required to manage the EOL of the products they produce (e.g. either with product leasing or government mandated take-back) and, therefore, there will be some quantity of cores that they must take-back. In this case product design and price are fixed and the the manufacturer can only choose how many new and remanufactured products to make. Similar to the problem of CCFDFP, this problem can also be formulated as a linear program. However, as before, we mathematically analyze the two-stage case to better understand the inherent interrelationships among different entities. The model for this problem is give below and denoted as Model MCFDFP-2SKP (Mandated Collection, Fixed Design, Fixed Price-2 Stage, K Part) and given below:

MCFDFP-2SKP
We consider two cases. In the first case, \( Q_N \) is fixed at \( D_N \), and in the second case, \( Q_N \) is considered as a decision variable. Interestingly, because \( Q_N \) and \( C \) are directly related by the equation \( C = A \) for mandated take-back, the choice of new product production quantities also dictates collection quantities. Thus, the analysis and results for this problem are identical to those presented in Section 3.1 except that, the \( \rho N Q_N \) is added back to the objective function and \( C = A \) and \( Q_N \) is the decision variable as opposed to \( C \).

In Section 3.2.1 we provide analysis for the \( K \)-part and two-part products for the case in which \( Q_N \) is fixed at \( D_N \). We present a sensitivity analysis on the quality of the parts of a product in Section 3.2.2. In Section 3.2.3 we consider \( Q_N \) as a decision variable and show how to transform Model CCFDFP-2SKP into Model MCFDFP-2SKP (Mandated Collection, Fixed Design, Fixed Pricing).

### 3.2.1 Analysis and graphical mapping for \( Q_N = D_N \)

In our analysis of this problem, we use Model CCFDFP-2SKP (see Section 3.1.1) with \( Q_N = D_N \) and designate it as Model MCFDFP-2SKP (Mandated Collection, Fixed Design, Fixed Pricing 2-Period K-Part Problem). Our only decision variable is \( Q_R \). Note that \( \Pi \) is a piece-wise linear function of \( Q_R \), where \( Q_R = 0 \) implies a pure part-salvage strategy, whereas \( Q_R = D_R \) implies a strategy in which the demand for remanufactured products is fully satisfied.

Since collection is mandated, \( C = A = \lambda D_N \). Based on the piece-wise linear nature of \( \Pi \), we know that a boundary value of \( Q_R \) will be optimal, i.e., \( Q_R * \in \{ 0, \psi_1 A, \ldots, \psi_K A, D_R \} \). Table 3.9 lists these potentially optimal solutions and
their corresponding profit expressions.

Table 3.9: Potential EOL options, mandated collection

<table>
<thead>
<tr>
<th>( \mathcal{Q} )</th>
<th>( \mathcal{D} )</th>
<th>Solutions</th>
<th>( \Pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0</td>
<td>( \mathcal{D} \in [0, \infty) )</td>
<td>( \rho^N \mathcal{D}^N + \sum_{k=1}^{K} S_k \psi_k A - \theta A ) (3.61)</td>
</tr>
</tbody>
</table>
| (2) | \( \psi_m A \) | \( \mathcal{D} \in [\psi_m A, \infty) \) | \( \rho^N \mathcal{D}^N + \rho^R \psi_m A 
+ \sum_{k=1}^{m-1} S_k (\psi_k A - \psi_m A) 
- \sum_{k=m+1}^{K} C_k^p (\psi_m A - \psi_k A) - \theta A \) (3.62) |
| (3) | \( \mathcal{D} \) | \( \mathcal{D} \in [0, \psi_m A) \) | \( \rho^N \mathcal{D}^N + \rho^R \mathcal{D} 
+ \sum_{k=1}^{m} S_k (\psi_k A - \mathcal{D}) 
- \sum_{k=m+1}^{K} C_k^p (\mathcal{D} - \psi_k A) - \theta A \) (3.63) |

We define the segments of \( \Pi \) in the following manner. Let \( \bar{\Pi}_{m+1,m} \) be the value of \( \Pi \) for \( \mathcal{Q} \in [\psi_{m+1} A, \psi_m A) \) for \( m = 1, \ldots, K - 1 \) and let \( \bar{\Pi}_{K+1,K} \) be the value of \( \Pi \) for \( \mathcal{Q} \in [0, \psi_K A) \). Specifically,

\[
\bar{\Pi}_{m+1,m}(\mathcal{Q}) = \rho^N \mathcal{Q}^N + \rho^R \mathcal{D}^R + \sum_{k=1}^{m} S_k (\psi_k A - \mathcal{Q}) 
- \sum_{k=m+1}^{K} C_k^p (\mathcal{Q} - \psi_k A) \text{ for } m = 1, \ldots, K, \quad (3.64)
\]

\[
\frac{\partial \bar{\Pi}_{m+1,m}(\mathcal{Q})}{\partial \mathcal{Q}} = \bar{\Pi}_{m+1,m}'(\mathcal{Q}) = \rho^R - \sum_{k=1}^{m} S_k - \sum_{k=m+1}^{K} C_k^p \text{ for } m = 1, \ldots, K. \quad (3.65)
\]

Since \( C_k^p > S_k \) for \( k = 1, \ldots, K \), it can be shown that \( \bar{\Pi}_{m+1,m} > \bar{\Pi}_{m,m-1} \) for \( m = 1, \ldots, K \), and thus \( \Pi \) is concave. The concave nature of \( \Pi \) gives us our optimality conditions.
A plot of profit versus the number of remanufactured products produced for a three-part product is given in Figure 3.21. Note that all parts are salvaged for a profit of $\rho^N D^N + (\sum_{k=1}^{K} \theta - \theta)$ if $\varphi^R = 0$. Each line segment is labeled above with its slope.

The optimal conditions for each solution result directly from the concave nature of $\Pi$.

1. $\varphi^R = 0$ (Salvage): Since $\Pi$ is concave, if $\tilde{\Pi}_{K+1,K} < 0$, then all other segments have negative slopes as well. By inspection of Equation (3.65), $\tilde{\Pi}_{K+1,K} < 0$ if $\rho R < \sum_{k=1}^{K}$. Note that, this is the same condition for the optimality of part salvage even if there is a choice of collection.

2. $\varphi^R = \min(D^R, \psi_mA)$, $m = 2, \ldots, K$: If $\Pi$ is initially increasing, then $\varphi^R = \psi_mA$ if the first segment of $\Pi$ with negative slope is $\tilde{\Pi}_{m+1,m}$.

3. $\varphi^R = \min(D^R, \psi_1a)$: If $\Pi$ is always increasing, then $\varphi^R = \psi_1A$.

We present a graphical mapping for the two-part problem using these results. The profit function for the two-part case is depicted in Figure 3.22. The mapping is given in Figure 3.23, and it is generated by the following rules:

a. $\rho R < S_1 + S_2$ implies that $\varphi^R < 0$.

b. $\rho R < S_1 + C_p^2$ implies that $\varphi^R = \min(\psi_1A, D^R)$.

c. otherwise $\varphi^R = \min(\psi_2A, D^R)$.

### 3.2.2 Value of quality for $\varphi^N = D^N$

Let $\Delta \psi_i$ be the changes in quality of Part $i$, $i = 1, \ldots, K$. Let $\psi'_i = \psi_i + \Delta \psi_i$. Note that as long as $\psi'_i < \psi'_m$ for $i = 1, \ldots, m-1$ and $\psi'_i > \psi'_m$ for $i = m+1, \ldots, K$, the optimal solution is unchanged. This results from the fact that the slopes of $\tilde{\Pi}_{m+1,m}$ for $m = 1, \ldots, K$ are not functions of $\psi_i$, $i = 1, \ldots, K$, but are functions of the ordering of the part qualities. As long as the optimal solution does not change with a change in part quality, the value of the change in part qualities for each solution can be found in Table 3.10.
Table 3.10: Value of additional part quality for each solution.

<table>
<thead>
<tr>
<th>$q^R$ Solutions</th>
<th>$p^R$</th>
<th>Value of Additional Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 0</td>
<td>$D^R \in [0, \infty)$</td>
<td>$(S_i + C_{isp}^i)A$ for $i = 1, \ldots, K$ (3.66)</td>
</tr>
</tbody>
</table>
| (2) $\psi_mA$  | $D^R \in [\psi_mA, \infty)$ | \[
\begin{cases}
(S_i + C_{isp}^i)A & \text{for } i = 1, \ldots, m - 1, \\
(\rho^R - \sum_{k=1}^{m-1} S_k - \sum_{k=m+1}^{K} C_p^k + C_{isp}^m)A & \text{for } i = m, \\
(C_p^m + C_{isp}^m)A & \text{for } i = m + 1.
\end{cases}
\] (3.67) |
| (3) $D^R$      | $D^R \in [0, \psi mA)$ | \[
\begin{cases}
(S_i + C_{isp}^i)A & \text{for } i = 1, \ldots, m - 1, \\
(C_p^m + C_{isp}^m)A & \text{for } i = m, \ldots, K
\end{cases}
\] (3.68) |

Note that the value of a change in part quality is not dependent of the quality of other parts. As long as the optimal solution is unchanged after changing the quality of the parts, we may calculate the value of the change in part quality by simply adding up the individual contribution of each $\Delta \psi_i$ on the profit according to the values in Table 3.10.

### 3.2.3 $Q^N$ as a decision variable

If $Q^N$ is a decision variable, then the problem is almost identical to that of CCFDFP-2SKP, except $\zeta = \lambda Q^N$ and $Q^N$ is now a decision variable. The mathematical formulation in this case is as follows.
MCFDFP-2SKP

\[
\max \mathcal{Q}^N, \mathcal{Q}^R, \mathcal{C} = \mathcal{P}^N \mathcal{Q}^N + \mathcal{P}^R \mathcal{Q}^R + \sum_{k=1}^{K} S_k \left( \max \left( \psi_k \lambda \mathcal{Q}^N - \mathcal{Q}^R, 0 \right) \right) - \sum_{k=1}^{K} C^P_k \left( \max \left( \mathcal{Q}^R - \psi_k \lambda \mathcal{Q}^N, 0 \right) \right) - \theta \lambda \mathcal{Q}^N
\]

subject to:

\[
\mathcal{Q}^N \leq D^N,
\]

\[
\mathcal{Q}^R \leq \psi_1 \lambda \mathcal{Q}^N,
\]

\[(3.4), \text{ and } (3.8).\]

The potential solutions to this problem are presented in Table 3.11. The optimal solution for this problem can be found in a similar way to those obtained for CCFDFP-2SKP. These are similar to those presented in Table 3.2, except for the inclusion of \(\mathcal{P}^N D^N\) as the first term. For this case the segments of \(\Pi\) can be analyzed as follows:

A. Let \(\Pi_{0,m}\) be the region of \(\Pi\) such that \(\mathcal{Q}^R = \psi_m \lambda \mathcal{Q}^N\) and \(D^N \in \left[0, \frac{D^R}{\lambda \psi_m}\right]\), for some \(m = 1, \ldots, K\).

\[
\Pi_{0,m}(\mathcal{Q}^N) = \mathcal{P}^N \mathcal{Q}^N + \left( \mathcal{P}^R \psi_m + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m) \right) \lambda \mathcal{Q}^N
\]

\[
- \sum_{k=m+1}^{K} C^P_k (\psi_m - \psi_k) \lambda \mathcal{Q}^N
\]

for \(\mathcal{Q}^N \in \left[0, \frac{D^R}{\lambda \psi_m}\right]\) and \(m = 1, \ldots, K + 1,\) (3.78)

\[
\frac{\partial \Pi_{0,m}(\mathcal{Q}^N)}{\partial \mathcal{Q}^N} = \Pi'_{0,m}(\mathcal{Q}^N) = \mathcal{P}^N + \mathcal{P}^R \psi_m + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m)
\]

\[
- \sum_{k=m+1}^{K} C^P_k (\psi_m - \psi_k) - \theta
\]

for \(\mathcal{Q}^N \in \left[0, \frac{D^R}{\lambda \psi_m}\right]\) and \(m = 1, \ldots, K + 1.\) (3.80)

The region \(\Pi_{0,m}\) gives the profit in case a) \(\mathcal{Q}^R = \psi_m \lambda \mathcal{Q}^N\), i.e., as many remanufactured products as possible are
Table 3.11: Possible optimal EOL options.

<table>
<thead>
<tr>
<th>Solutions</th>
<th>Description</th>
<th>( \Pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 0 0</td>
<td>No product production.</td>
<td>0 (3.72)</td>
</tr>
<tr>
<td>(2) ( D^N ) 0</td>
<td>Salvage the parts.</td>
<td>( \rho^N D^N )</td>
</tr>
<tr>
<td>(3) ( D^N \psi_m D^N ) ( m = 1, \ldots, K )</td>
<td>Meet new product demand and make remanufactured products as possible out of good part type ( m ). Salvage excess parts and purchase needed replacement parts. Part ( m ) is the central part.</td>
<td>( \rho^N D^N + (\sum_{k=1}^K S_k \psi_k - \theta) \lambda D^N ) (3.73)</td>
</tr>
<tr>
<td>(4) ( D^N D^R ) ( m = 1, \ldots, K - 1 )</td>
<td>Meet new and remanufactured product demand. There will be an excess of part type ( k \leq m ) and a deficit of part type ( k \geq m + 1 ). Salvage excess parts and purchase needed replacement parts.</td>
<td>( \rho^N D^N + \rho^R D^R + \sum_{k=1}^{m-1} S_k (\psi_k \lambda D^N - D^R) ) (3.75)</td>
</tr>
<tr>
<td>&amp;</td>
<td></td>
<td>( - \sum_{k=m+1}^K C_k^R (\psi_m - \psi_k) \lambda D^N - \theta \lambda D^N )</td>
</tr>
<tr>
<td>(5) ( D^N D^R ) ( m = 1, \ldots, K )</td>
<td>Meet new and remanufactured product demand. There will be excess of all parts, which will be salvaged.</td>
<td>( \rho^N D^N + \rho^R D^R ) (3.76)</td>
</tr>
<tr>
<td>&amp;</td>
<td></td>
<td>( + \sum_{k=1}^K S_k (\psi_k \lambda D^N - D^R) - \theta \lambda D^N )</td>
</tr>
<tr>
<td>(6) ( \frac{D^N}{\psi_m} D^R ) ( m = 1, \ldots, K )</td>
<td>Make enough new products so that enough cores are available to obtain as many good parts of type ( m ) as needed to meet remanufacturing demand. Salvage excess parts and purchase needed parts. Parts ( l ) is the “limiting” part.</td>
<td>( \rho^N D^R + \rho^R D^R + \sum_{k=1}^{m-1} S_k \frac{D^R}{\psi_l} \psi_k - D^R ) (3.77)</td>
</tr>
<tr>
<td>&amp;</td>
<td></td>
<td>( - \sum_{k=m+1}^K C_k^R \left( \frac{D^R}{\psi_l} \psi_k - \frac{D^R}{\psi_l} \right) - \theta \frac{D^R}{\psi_l} )</td>
</tr>
</tbody>
</table>
produced using Part $m$ and b) $D^N \in [0, \frac{D^R}{\lambda \psi_m})$, i.e., we make too few new products to be able to collect and disassemble enough cores to obtain $D^R$ of Part $m$.

B. Let \( \bar{\Pi}_{m,m+1} \) be the region of \( \bar{\Pi} \) such that \( \mathcal{Q} = D^R \) and \( D^N \in [\frac{D^R}{\lambda \psi_m}, \frac{D^R}{\lambda \psi_{m+1}}) \).

\[
\bar{\Pi}_{m,m+1}(\mathcal{Q}^N) = \rho^N \mathcal{Q}^N + \rho^R D^N + \sum_{k=1}^{m} S_k(\psi_k \lambda \mathcal{Q}^N - D^R) - \sum_{k=m+1}^{K} C^P_k(D^R - \psi_k \lambda \mathcal{Q}^N)
- \theta \lambda \mathcal{Q}^N \text{ for } \mathcal{Q}^N \in [\frac{D^R}{\lambda \psi_m}, \frac{D^R}{\lambda \psi_{m+1}}) \text{ and } m = 1, \ldots, K,
\]

\[
\frac{\partial \bar{\Pi}_{m,m+1}}{\partial \mathcal{Q}^N}(\mathcal{Q}^N) = \bar{\Pi}^\prime_{m,m+1}(\mathcal{Q}^N) = \rho^N + \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C^P_k \psi_k - \theta
\text{ for } \mathcal{Q}^N \in [\frac{D^R}{\lambda \psi_m}, \frac{D^R}{\lambda \psi_{m+1}}) \text{ and } m = 1, \ldots, K.
\]

The region \( \bar{\Pi}_{m,m+1} \) gives the profit in case a) we meet remanufactured product demand without purchasing any of Part $m$ and b) we make too few new products to be able to collect and disassemble enough obtain $D^R$ of $m+1$. 

Figure 3.16: Procedure for determining optimal EOL option for a product.
Figure 3.17: Example $\Pi_{\text{max}}$ and $\Pi_{\text{EOL}}$.

Figure 3.18: $\Pi_{\text{max}}^{\text{updated}}$ from example in Figure 3.17.
Figure 3.19: Mapping of product characteristic onto optimal EOL type for $A \in \left[\frac{\rho^R}{\psi_2}, \infty\right]$. 
Figure 3.20: Plot of $\Pi$ for class-8 truck example.

Figure 3.21: Profit as a function of the number products remanufactured for the MCFD-2SKP problem.

Figure 3.22: Profit as a function of the number products remanufactured for the MCFDFP-2S2P problem.
Figure 3.23: Mapping of product characteristic onto optimal EOL type for the MCFDFP-2S2P problem.
Chapter 4

Product Design for the RSC: Choice of Part Quality

Product design is a complex problem, and it involves several decisions. Among others, these include:

i. Choice of functions and features,

ii. Aesthetics (form),

iii. Level of integration/modularization of product parts,

iv. Level of standardization of parts among similar products,

v. Choice of joining techniques and part design for ease of assembly and disassembly, and

vi. Choice of part quality/reliability.

All of these product design decisions are important in their own way; however, we choose part quality as a proxy for design as it directly affects the good-part yields, $\psi$ - an important parameter in the determination of the optimal EOL option for a product.
Additionally, the choice of part quality may have a non-negligible impact on the following various parameters of the problem:

i. \textit{New production cost} ($C^N$): Greater part reliability increases part cost, which, in turn, increases new production cost.


iii. \textit{Part salvage values} ($S_r$): Greater part reliability also leads to more reliable after-market parts, and, therefore, their worthiness.

iv. \textit{Demand for new and remanufactured products}: If customers are aware that higher quality parts are used in a product, then, for a given price, a greater quantity of new and remanufactured products will be demanded.

v. \textit{Price of new and remanufactured products}: If customers are aware that higher quality parts are used in a product, then, for a given quantity of products produced, the manufacturer may charge more for the new and remanufactured products.

vi. \textit{Product lifetime}: Products produced with higher reliability parts may have longer useful lifetimes.

The above considerations are important in case the product is \textit{directly} sold to a customer who will have full control over the management of the product and its life cycle. However, if only the \textit{usage} of the product is sold to the customer (e.g. a product is leased), many of the above considerations reduce in importance. Since the customer is paying for the usage of the product, as long as the product design does not change the functionality of the product, the product design no longer impacts demand for the product (demand is now for usage of the product), product price (which is for the usage of the product), or product life-cycle length (e.g. the customer is contractually obligated to return the product after a specified length of time).

In our analysis of product design, we assume existence of some type of leasing or sale-of-service system rather than direct-sale. Thus, the manufacturer is required to handle the EOL of the product, i.e., there is mandated collection. In case products are directly sold to customers, we must consider how product design impacts customer behavior.
First, we discuss the nature of the cost/quality trade-offs in Section 4.1. Next, in Section 4.2 we present a model for the PPMRSC problem with mandated core collection and product design with fixed product pricing (MCPDPF) for the two-stage (2S), K-part (KP) case. In this model, $\psi_k, k = 1, \ldots, K$, are decision variables. The optimal EOL strategy may change with an alteration in product design, and thus, the new product design may imply a different optimal EOL option. Additionally, we find that $\Pi$ has three possible optimal modes under certain conditions and is quasi-convex between these modes. Except for in the Section 4.8, we assume this to be the case. A mapping of product design onto optimal EOL strategy is given in Section 4.3. In Section 4.4, for the special case in which $\psi_k \in [0, \psi^H_k]$, for each of the six EOL option solution-forms listed in Table 3.11, we present the principle relationships that determine optimal product design for a given EOL option. Additionally, we present methods for selecting the optimal design. We describe a method for finding both optimal product design and EOL option in Section 4.5, and then, provide an illustrative example in Section 4.6. In Section 4.7, we discuss the general case in which $\psi_k \in [0, \psi^H_k], k = 1, \ldots, K$. Finally, in Section 4.8, we discuss the most general case, in which case $\Pi$ need not have 3 optimal modes.

### 4.1 Cost/quality trade-off

Recall from Section 3.1 the nature of trade-offs between part cost and part quality and between part salvage value and part quality. Specifically, recall that they are monotone increasing with a "knee-in-the-curve" at some yield, $\psi^H_k$, which represents the greatest possible yield for that part. Yields greater than $\psi^H_k$ are either not technically feasible or are assumed to be too costly. We assume that the trade-offs for new and replacement part costs have the same functional shape as $f(\psi_k), k = 1, \ldots, K$. However, the trade-off for the salvage value of part $k$, $\psi_k = 1, \ldots, K$, may either: (i) have the same functional shape as $f(\psi_k)$ if the salvage value results from part resale, or (ii) be nonexistent with $S_k = -C^\text{disp}_k$ if salvation value is obtained through part disposal. We define $f(0) = 0$, and assume $f(\psi_k)$ to be monotone increasing in $\psi_k$. Let the new-part cost/quality trade-off for Part $k$ be

$$C_k^{N}(\psi_k) = \delta_k^{N} f(\psi_k),$$  \hspace{1cm} (4.1)
where $\delta_k^N$ is the scaling parameter. The cost/quality trade-off for the production cost of a new product is then given by

$$C^{N'} + \sum_{k=1}^{K} C_k^N(\psi_k),$$

(4.2)

where $C^{N'}$ is the cost incurred to produce a new product if all parts are made with minimal quality. The replacement part cost/quality trade-off for Part $k$ is given by

$$C_k^p(\psi_k) = C_k^{p0} + \delta_k^p f(\psi_k)$$

(4.3)

where $\delta_k^N$ is the scaling parameter and $C_k^{p0}$ is the replacement cost when $\psi_k = 0$. Let the part salvage/quality trade-off for Part $k$ be given by

$$S_k(\psi_k) = S_k^0 + \delta_k^s f(\psi_k)$$

(4.4)

where $\delta_k^s$ is the scaling parameter and $S_k^0$ is the salvage value for $\psi_k = 0$. We assume $S_k^0 = -C_k^{disp}$. Note that $\delta_k^s = 0$ if there is no after-market for part $k$. Figure 4.1 depicts examples of functions $C_k^N(\psi_k)$, $C_k^p(\psi_k)$ and $S_k(\psi_k)$.

Figure 4.1: Example $C_k^N(\psi_k)$, $C_k^p(\psi_k)$ and $S_k(\psi_k)$ functions.
4.2 Mathematical model

The model for the two-stage, K-part problem is a special case of MCPDFP, and we designate it as MCPDFP-2SKP (Mandated Collection, choice of Product Design, and Fixed Pricing) with two stages (2S) and K-parts (KP). It is as follows:

MCPDFP-2SKP

\[
\Pi = \left[ \mathcal{D}^N - (C^N + \sum_{k=1}^{K} C_i^N(\psi_k)) \right] Q_N + \rho R Q_R - \sum_{k=1}^{K} \max \left( C_p^k(\psi_k)(Q_R - \psi_k \lambda Q^N), 0 \right) + \sum_{k=1}^{K} \max \left( S_k(\psi_k)(\psi_k \lambda Q^N - Q_R), 0 \right) - \left( C^c + C^d + \sum_{k=1}^{K} C_{disp}^k(1 - \psi_k) \right) \lambda Q^N
\]

subject to:

\[
\mathcal{D}^N \leq D^N, \quad \mathcal{D}^R \leq D^R, \quad (4.6)
\]
\[
\mathcal{D}^R \leq \max(\psi_1, \ldots, \psi_K) \lambda Q^N, \quad (4.7)
\]
\[
0 \leq \psi_k \leq \psi^H_k, \quad k = 1, \ldots, K, \quad (4.8)
\]
\[
\mathcal{D}^N, \mathcal{D}^R \geq 0. \quad (4.9)
\]

Note that MCPDFP-2SKP is identical to the fixed design problem, MCFDFP-2SKP, except that: (i) \( C^N, S_k, \) and \( C_p^k \) are now functions of \( \psi_k, k = 1, \ldots, K, \) (ii) \( \psi_k, k = 1, \ldots, K, \) are decision variables whose indices no longer imply any ordering, and because of this, (iii) \( \psi_1 \) is not necessarily the highest yielding part. Thus, we must impose Constraint (4.7) to ensure that as many recovered parts as possible are used to make remanufactured products. Recall from Section 3.2.3 that our definition of \( \Pi \) is dependent on a pre-specified ordering of part yields. Specifically, it is a part's ordering with respect to the central part, \( m \), that is important since this determines if there is a surplus or a deficit of
that part. We say that Part \( k \) is a \textit{surplus part} if \( \psi_k \geq \psi_m \). Let \( S \) be the set of surplus parts. We use the convention that the central part is also in \( S \) since the central part may have a surplus, but never a deficit. We say that Part \( k \) is a \textit{deficit part} if \( \psi_k < \psi_m \). Let \( P \) be the set of deficit parts. Also, let \( K \) be the set containing all part types. Using this notation we can rewrite the regions of \( \bar{\Pi} \) as follows:

\[
\bar{\Pi}_{0,m}(Q_N) = [\mathcal{P}^N - (C^N + \sum_{k \in K} C^N_k (\psi_k))]Q_N + \rho R \psi_m \lambda \mathcal{N} \\
+ \sum_{k \in S} S_k(\psi_k)(\psi_k \lambda \mathcal{N} - \psi_m \lambda \mathcal{N}) \\
- \sum_{k \in P} C^p_k(\psi_k)(\psi_m \lambda \mathcal{N} - \psi_k \lambda \mathcal{N}) - \theta \lambda \mathcal{N}
\]

for \( Q_N \in [0, \frac{D^R}{\lambda \psi_m}] \), \( m = 1, \ldots, K + 1 \), (4.10)

\[
\frac{\partial \bar{\Pi}_{0,m}}{\partial Q_N} = \bar{\Pi}'_{0,m}(Q_N) = \mathcal{P}^N - (C^N + \sum_{k \in K} C^N_k (\psi_k)) + \rho R \psi_m \lambda \\
+ \sum_{k \in S} S_k(\psi_k)(\psi_k - \psi_m) \lambda \\
- \sum_{k \in P} C^p_k(\psi_k)(\psi_k - \psi_k^0)(\psi_m - \psi_k) \lambda - \theta \lambda
\]

for \( Q_N \in [0, \frac{D^R}{\lambda \psi_m}] \), \( m = 1, \ldots, K + 1 \), (4.11)

and

\[
\bar{\Pi}_{m,m+1}(Q_N) = [\mathcal{P}^N - (C^N + \sum_{k \in K} C^N_k (\psi_k))]Q_N + \rho R D^R \\
+ \sum_{k \in S} S_k(\psi_k)(\psi_k \lambda \mathcal{N} - D^R) \\
- \sum_{k \in P} C^p_k(\psi_k)(D^R - \psi_k \lambda \mathcal{N}) - \theta \lambda \mathcal{N}
\]

for \( Q_N \in [0, \frac{D^R}{\lambda \psi_m}] \), \( m = 1, \ldots, K \), (4.12)
A choice of product design specifies the values for $\psi_k$, $k \in K$, and thus, determines the subset to which each part belongs based on the choice of central part, $m$. However, there is no central part in case $m = K$, and thus, all parts are in $S$. For any given choice of product design, we know the optimal EOL option based on the new production cost, part salvage values, and part replacement costs associated with that product design. However, the objective of our investigation is to determine how to find the optimal part yields. While determining $\psi_k$, $k \in K$ we must consider the affect of the value of $\psi_k$ on: 1) new production cost, and either 2) replacement cost (if Part $k$ is deficit) or 3) salvage value (if Part $k$ in surplus). If Part $k$ is not central or limiting, then the contribution of $\psi_k$ to profit of region $\Pi_{0,m}$ if $\psi_k > \psi_m$ is given by the terms of Equation 4.10 that include $\psi_k > \psi_m$, i.e. $\psi_k \in S$. Thus, the contribution of $\psi_k \in S$ to $\Pi_{0,m}$ is given by

$$-C_k^N(\psi_k)\mathcal{Q}^N + S_k(\psi_k)(\psi_k\lambda\mathcal{Q}^N - \psi_m\lambda\mathcal{Q}^N) + C_k^{disp}\psi_k\lambda\mathcal{Q}^N \quad \text{for} \ \psi_k > \psi_m,$$

where the first term is the new product cost associated with Part $k$, the second term is the salvage value obtained from Part $k$, and the last term is extracted from $\theta$ and represents the disposal cost avoided due to the yield of Part $k$. By substituting the expressions for $C_k^N(\psi_k)$ (see Equation (4.1)) and $S_k(\psi_k)$ (see Equation (4.4)) into Expression (4.14), we have

$$-\delta_k^Nf(\psi_k)\mathcal{Q}^N + (S_k^0 + \delta_k^Sf(\psi_k))(\psi_k\lambda\mathcal{Q}^N - \psi_m\lambda\mathcal{Q}^N) + C_k^{disp}\psi_k\lambda\mathcal{Q}^N \quad \text{for} \ \psi_k > \psi_m.$$

Rearranging terms and substituting $-C_k^{disp}$ for $S_k^0$, we obtain:

$$f(\psi_k)\left(\lambda \mathcal{Q}^N - \psi_k\lambda\mathcal{Q}^N - \psi_m\lambda\mathcal{Q}^N\right) + C_k^{disp}\psi_k\lambda\mathcal{Q}^N \quad \text{constant} \quad \text{for} \ \psi_k > \psi_m.$$
If Expression (4.16) is a quasi-convex function of $\psi_k$, then $\psi_k^* = \psi_m$ or $\psi_k^* = \psi_k^H$. Note that the first term of Equation (4.16) is a monotone increasing function times a linear function with positive slope and negative y-intercept. Specifically, it is of the form $g(x) = f(x)(mx - b)$. Next, we investigate when such a function can be claimed to be quasi-convex. The function $g(x)$ is quasi-convex if:

$$g(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{g(x_1), g(x_2)\} \text{ for } x_1, x_2 \in \mathbb{R}. \quad (4.17)$$

**Proposition 4.2.1** If $f(x)$ is a monotone increasing function of $x$, $x \geq 0$, then there exists some $x^H > 0$ such that $g(x) = f(x)(mx - b)$ is quasi-convex for $x \in [0,x^H]$. Let $m > 0$, $b > 0$, and $x^H$ either be (i) some arbitrary value or (ii) the first value of $x > 0$ such that $g(x)$ stops increasing.

**Proof** Note that

$$g'(x) = f'(x)(mx - b) + f(x)m. \quad (4.18)$$

Initially, $g(x)$ is decreasing since $g'(0) = f'(0) \cdot -b < 0$. Also, $g(x)$ must be increasing over some regions of $x$ for $x \in (0,b/m]$ since $g'(b/m) = f(b/m)m > 0$, meaning that $g(x)$ is an increasing function at $x = b/m$. Let $x^L$ be the smallest $x$ at which $g(x)$ begins to increase (recall this must occur in the region $x \in (0,b/m]$). We have two cases.

I. $g'(x) > 0$ for all $x \in [x^L,b/m]$

In this case, $g(x)$ is increasing for $x \in [x^L,\infty)$ since $g'(x) > 0$ for $x \geq b/m$ as $mx - b > 0$. We define $x^H$ to be some arbitrary $x > 0$.

II. $g'(x) < 0$ for some $x \in [x^L,b/m]$.

In this case, $g(x)$ is decreasing in some region(s) of $x \in [x^L,b/m]$. This will be the case if $f'(x)$ becomes very large in this range. In this case, let $x^H$ be the smallest $x : g'(x) < 0$ for $x \in [x^L,b/m]$.

In both of these cases, $g(x)$ has a decreasing region in $x \in [0,x^L]$ and an increasing region for $x \in [x^L,x^H]$. Without loss
of generality, let \( x_1 < x_2 \). If both \( x_1 \) and \( x_2 \) are in \([0, x^L]\) then it is clear that

\[
g(\lambda x_1 + (1 - \lambda) x_2) \leq g(x_1)
\]  

(4.19)

is true. Similarly if \( x_1 \) and \( x_2 \) are in \([x^L, x^H]\) then it is clear that

\[
g(\lambda x_1 + (1 - \lambda) x_2) \leq g(x_2)
\]  

(4.20)

is true. If \( x_1 \in [0, x^L] \) and \( x_2 \in [0, x^L] \) then \( g(\lambda x_1 + (1 - \lambda) x_2) < g(x_1) \) for \( \lambda \) such that \( \lambda x_1 + (1 - \lambda) x_2 \leq x^L \) and

\[
g(\lambda x_1 + (1 - \lambda) x_2) < g(x_2) \text{ for } \lambda \text{ such that } \lambda x_1 + (1 - \lambda) x_2 \geq x^L.
\]

Thus

\[
g(\lambda x_1 + (1 - \lambda) x_2) \leq \max\{g(x_1), g(x_2)\}.
\]  

(4.21)

In the context of part yields, we assume \( \psi^H_t \) to be analogous to \( x^H \) in Proposition 4.2.1.

Figure 4.2 gives an example plot of Equation (4.16).

An interesting property of \( g(x) = f(x)(mx - b) \) is that \( g(x) < 0 \) for \( x : mx - b < 0 \) and \( g(x) \geq 0 \) for \( x : mx - b \geq 0 \). Additionally, we know that \( g'(x) \geq 0 \) for \( x : mx - b \geq 0 \) (see Equation 4.18) and so \( g(x) \) is increasing over \( x \in [b/m, \infty) \).

This leads us to the following proposition.

**Proposition 4.2.2** Let \( g(x) = f(x)(mx - b) \) where \( f(x) \) is a monotonically increasing function of \( x \), \( m > 0 \) and \( b > 0 \).

If \( mx - b \geq 0 \) for some \( x \in [0, x^H] \), where \( x^H \) is the upper limit of \( x \), then \( g(x^H) \geq g(x) \) for all \( x \in [0, x^H] \).

**Proof** Let \( x \in [0, x') \) be the region of \( x \) such that \( mx - b < 0 \) and \( x \in [x', \infty) \) be the region of \( x \) such that \( mx - b > 0 \).

First, note that \( g(x_1) < g(x_2) \), where \( x_1 \in [0,x') \) and \( x_2 \in [x', \infty) \) since \( g(x_1) < 0 \) and \( g(x_2) \geq 0 \). Secondly, \( g'(x_2) \geq 0 \) for all \( x_2 \in [x', \infty) \). Thus, \( g(x_2) \) is an increasing function of \( x_2 \in [x', \infty) \). If \( x' \leq x^H \), then we have that \( g(x^H) \geq g(x') \) and \( g(x') \geq g(x_1) \), \( x_1 \in [0, x') \) and thus \( g(x^H) \geq g(x) \) for \( x \in [0, x^H] \). □
In this context of part yields, we have the following result.

**Proposition 4.2.3**  If \( \psi_k > \psi_m \) and \( \psi_k^H \geq \frac{\delta N_k}{\lambda_N} + \psi_m \), then \( \psi_k^* = \psi_k^H \) if \( \prod = \bar{\prod}_{0,m} \) for some \( m \neq k, m \) and \( k \in K \).

The proof of Proposition 4.2.3 is a direct application of Proposition 4.2.2 to Equation (4.16). Figure 4.3 gives an example plot of Equation (4.16) in which \( \psi_k^H \geq \frac{\delta N_k}{\lambda_N} + \psi_m \).

However, if \( \psi_k > \psi_m \) and \( \psi_k^H \leq \frac{\delta N_k}{\lambda_N} + \psi_m \), then the contribution to profit will be negative for salvage, and the best value of \( \psi_k \geq \psi_m \) will be at either \( \psi_m \) or \( \psi_k^H \) (see Figure 4.2).

We may perform a similar analysis on the contribution to profit \( \Pi_{0,m} \) of deficit parts. The contribution to the profit \( \Pi_{0,m} \) of deficit parts is given by

\[
-C_k^N(\psi_k)\varphi^N - C_k^p(\psi_k)(\psi_m\lambda\varphi^N - \psi_k\lambda\varphi^N) + C_k^{disp}\psi_k\lambda\varphi^N. \tag{4.22}
\]

where the first term is the new product cost associated with Part \( k \), the second term is the replacement cost of Part \( k \), and the last term is extracted from \( \theta \) and represents the disposal cost avoided due to the yield of Part \( k \). We can substitute the expressions for \( C_k^N(\psi_k) \) (see Equation (4.1)) and \( S_k(\psi_k) \) (see Equation (4.3)) in to Equation (4.22) and rearrange terms to obtain

\[
\frac{f(\psi_k)}{f(x)}(\lambda\varphi^N\psi_k - \delta_k^p\psi_m\lambda\varphi^N - \delta_k^N\varphi^N) + \frac{(C_k^{p0} + C_k^{disp})\lambda\varphi^N\psi_k - C_k^{p0}\psi_m\lambda\varphi^N}{m_2x} = \frac{b}{m_1x}. \tag{4.23}
\]

Equation (4.23) has the form \( g(x) = f(x)(m_1x - b) + m_2x \), which is not generally quasi-convex in \( \psi_k \) (see Figure 4.4 for non-quasi-convex example). However, under certain circumstances, \( g(x) \) is quasi-convex. Note that

\[
g'(x) = f'(x)(m_1x - b) + f(x)m_1 + m_2. \tag{4.24}
\]

If we assume that \( f'(0)b > m_2 \), then the result of Proposition 4.2.1 holds as \( g(x) \) will be initially decreasing and only
if \( f'(x) \) is very large for \( x < b/m_1 \) will \( g(x) \) decrease. Thus, as long as 
\[
f'(0) \delta_k^\alpha \psi_m + \delta_k^N > C_k^0 + C_k^{disp},
\]
then Equation (4.23) is quasi-convex for \( \psi_k \in [0, \psi^H_k] \).

In case the profit is given by \( \Pi_{m,m+1} \), the contribution to profit of a salvaged part is:

\[
-C_k^N(\psi_k) \omega^N + S_k(\psi_k)(\psi_k \omega^N - D^R) + C_k^{disp} \psi_k \omega^N
\]

(4.25)

where the first term is the new product cost associated with Part \( k \), the second term is the salvage value obtained from Part \( k \), and the last term is extracted from \( \theta \) and represents the disposal cost avoided due to the yield of Part \( k \). We can substitute the expressions for \( C_k^N(\psi_k) \) (see Equation (4.1)) and \( S_k(\psi_k) \) (see Equation (4.4)) in to Equation (4.25) and rearrange terms to obtain

\[
f(\psi_k)(\lambda \omega^N \psi_k - \delta_k^N \omega^N - D^R + \psi_k \omega^N) + C_k^{disp} \psi_k \omega^N.
\]

(4.26)

which has the same form as Equation (4.23) and Proposition 4.2.1 applies. Similar to Proposition 4.2.3 we give the following result related to Equation (4.26).

**Proposition 4.2.4** If \( \psi_k > \psi_m \) and \( \psi_k^H \geq \frac{\delta_k^N \omega^N + \delta_k^D^R}{\omega_N \omega^N} \), then \( \psi_k^* = \psi_k^H \) if \( \Pi = \Pi_{m,m+1} \) for some \( m \neq k, m \) and \( k \in K \).

In case the profit is given by \( \Pi_{m,m+1} \), the contribution to profit of a deficit part is:

\[
-C_k^N(\psi_k) \omega^N - C_k^p(\psi_k)(D^R - \psi_k \omega^N) + C_k^{disp} \psi_k \omega^N.
\]

(4.27)

where the first term is the new product cost associated with Part \( k \), the second term is the replacement cost of Part \( k \), and the last term is extracted from \( \theta \) and represents the disposal cost avoided due to the yield of Part \( k \). We may substitute the expressions for \( C_k^N(\psi_k) \) (see Equation (4.1)) and \( S_k(\psi_k) \) (see Equation (4.3)) in to Equation (4.27) and rearrange terms to obtain

\[
f(\psi_k)(\lambda \omega^N \psi_k - \delta_k^D^R - \delta_k^N \omega^N) + (C_k^p + C_k^{disp}) \lambda \omega^N \psi_k - C_k^{disp} D^R.
\]

(4.28)
Equation (4.28) has the same form as Equation (4.23), and $\bar{\Pi}_{m,m+1}$ is a quasi-convex function of $\psi_k \in P$ only if $C_k^{p0} + C_k^{disp}$ sufficiently small.

For a given Part $k$, we may plot its contribution to the profit over its full range by concatenating the function for the Part $k$'s contribution to the profit if $k \in P$ and if $k \in S$. Figure 4.5 gives an example plot of the contribution to profit of the choice of $\psi_k$.

These results show that: (i) the profit, $\Pi$ is quasi-convex for $\psi_k < \psi_m$, if $C_k^{p0} + C_k^{disp}$ is relatively small, and (ii) for $\psi_k \geq \psi_m$, $\Pi$ is quasi-convex between $\psi_m$ and some upper limit $\psi_k^H$, which may be different for each $k \in K$. However, the location of the “knee-in-the-curve” of $f(\psi)$ is the important factor that determines the value of $\psi_k^H$, $k \in K$, and so the values of $\psi_k^H$ should be similar.

### 4.3 Mapping product design onto optimal EOL option

By the results of Section 3.1, the optimal EOL type and option for a product are known for a given product design. It is, therefore, possible to generate a mapping of product design onto EOL type and EOL option. For the two-part case, we can use the relationships given in Table 3.4 from Section 3.1.6 modified for the case of mandated collection and by
Figure 4.3: Example plot of Equation (4.16) in which $\psi_k \geq \frac{\delta^N \lambda}{\delta^N \lambda} + \psi_m$.

incorporating cost/quality trade-offs (see Table 4.1).

Table 4.1: Relationships for two-part product mapping.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Implication</th>
<th>Resulting from</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $S_1(\psi_1) + S_2(\psi_2) &gt; \rho^R$</td>
<td>$\Pi_1$ optimal</td>
<td>Equation (3.31)</td>
</tr>
<tr>
<td>b. $S_1(\psi_1) + C_2(\psi_2) &lt; \rho^R$</td>
<td>$\Pi_{0,1} &gt; \Pi_{0,2}$ (Part 1 key)</td>
<td>Eq. (3.29)</td>
</tr>
<tr>
<td>c. $\mathcal{R}^N - (C_2 + \sum_{k=1}^2 C_N(\psi_k)) + (C_2(\psi_2))(\psi_1 - \psi_2) + \theta &lt; \rho^R \psi_1$</td>
<td>$\Pi_{0,1} &gt; 0$</td>
<td>$\Pi_{0,1}$</td>
</tr>
<tr>
<td>d. $\mathcal{R}^N - (C_2 + \sum_{k=1}^2 C_N(\psi_k)) + (S_1(\psi_1))(\psi_1 - \psi_2) + \rho^R \psi_2 &gt; \theta$</td>
<td>$\Pi_{0,2} &gt; 0$</td>
<td>$\Pi_{0,2}$</td>
</tr>
<tr>
<td>e. $\mathcal{R}^N - (C_2 + \sum_{k=1}^2 C_N(\psi_k))$</td>
<td>$\Pi_{1,2} &gt; 0$</td>
<td>$\Pi_{1,2}$</td>
</tr>
<tr>
<td>f. $S_1(\psi_1) + S_2(\psi_2) &gt; \theta$</td>
<td>$S_1(\psi_1) + S_2(\psi_2) &gt; \theta$</td>
<td>$\Pi_{1,2}$</td>
</tr>
</tbody>
</table>

This mapping helps provide intuition regarding the relationship between the product’s design and its optimal EOL option. For example, consider a product with two parts with $\psi_1 \geq \psi_2$. Figure 4.6 gives an example mapping of $\psi_1$ and $\psi_2$ onto EOL type. Just as in Figure 3.19, each line bisects the mapping into regions where particular conditions hold that determine the nature of $\Pi_{max}$. 


Figure 4.4: Example plot of contribution to the profit of deficit Part $k$.

1) **Region 1: No collection**

   In this region, both $\bar{\Pi}_{0,1}$ and $\bar{\Pi}_{0,2}$ are less than 0 (Conditions c. and d. from Table 4.1 do not hold) and therefore, no collection is the best EOL option.

2) **Region 2: Salvage**

   In this region, the sum of the salvage values is greater than remanufacturing margin (Condition a. from Table 4.1), so salvage is the best EOL option.

3) **Region 3: Part 1 key, no limiting part**

   In this region, part salvage is profitable (Condition f. from Table 4.1) and Part 1 is key (Condition b. from Table 4.1).

4) **Region 4: Part 2 key, no limiting part**

   In this region, part salvage is profitable (Condition f. from Table 4.1) and Part 2 is key (Condition b. does not hold from Table 4.1).

5) **Region 5: Part 1 key, Part 2 limiting**

   In this region, Part 1 is key (Condition b. from Table 4.1) and Part 2 is limiting (Condition f. from Table 4.1).
does not hold, but Condition e. does).

(6) Region 6: Part 1 key, Part 1 limiting

In this region, Part 1 is key (Condition b. from Table 4.1) and Part 2 is limiting (Conditions f. and e. from Table 4.1 do not hold).

The lines in Figure 4.6 are labeled a-g and correspond to the conditions presented in Table 4.1. From this mapping, we can make some general observations. For very low values of \( \psi_1 \) and \( \psi_2 \), no new products are produced (Region 1) since the profit from new sales and the benefit from the few good-parts obtained (either for salvage or remanufacturing) do not outweigh collection and disassembly costs. However, for higher values of \( \psi_1 \) and \( \psi_2 \), we may be in Region 6, in which case, the option of new sales and sales of remanufactured products with Part 1 as key is best. With higher yields, we obtain more parts to make remanufactured products, and the number of replacement parts required decreases. However, the yields are not so high that making more than \( \frac{\rho_1}{\lambda \psi_1} \) new products is best since the additional profit from more new sales and the additional surplus parts of type 1 obtained along with the reduction in replacement parts to type 2 needed, does not outweigh collection and disassembly costs. However, by increasing \( \psi_1 \) and \( \psi_2 \) further,
we might be in Region 5, in which Part 1 is no longer limiting. In Region 3, surplus values and part yields are so high
that there is no limiting part. In Region 4, the surplus value of Part 1 and the yield of Part 2 are high enough to make
Part 2 as key. Finally, in Region 2, the very high yields result in so many good-parts and high salvage values that pure
part salvage is best.

Figure 4.6: Mapping of product design onto optimal EOL type for the two-stage problem.

The values of $D^R$ and $\lambda \mathcal{D}^N$ determine the specific EOL option for a product. Figure 4.7 gives an example mapping of
$\psi_1$ and $\psi_2$ onto EOL options given $D^R$ and $\lambda \mathcal{D}^N$, the regions of which are labeled by their corresponding EOL option
and solution from Table 3.2. Additionally, note that $\frac{D^R}{\lambda \mathcal{D}^N}$ represents the minimum yield of some Part $k \in \mathcal{K}$ required
to obtain enough of Part $k$ to make $D^R$ remanufactured products without needing to purchase any replacement parts of
type $k$. Thus, if both $\psi_1$ and $\psi_2$ are greater than $\frac{D^R}{\lambda \mathcal{D}^N}$, then the supply of cores is large enough to provide all the parts
needed to perform remanufacturing. By comparing $\frac{D^R}{\lambda \mathcal{D}^N}$ with $\psi_1$ and $\psi_2$ and observing the EOL type of the product
for $\psi_1$ and $\psi_2$, the optimal EOL option can be specified.

Note that this mapping does not specify which product design is optimal; we address this in the next section.
4.4 Jointly determining optimal EOL option and product design: special case of $\psi_k \in [0, \psi^H]$, $k \in K$, and $\Pi$ a quasi-convex function of $\psi_k$, $k$ not a central part

We now turn our focus on jointly determining the optimal EOL option and design of a product. In order to gain insight regarding the nature of optimal product designs, we determine some properties of the optimal product design for each EOL strategy of Table 3.11 by taking into account the product design/cost trade-offs. In this case, we assume that all parts in the product have the same greatest possible yield, i.e. $\psi_k^H = \psi^H$, $k \in K$. Additionally, we assume that $\Pi$ is a quasi-convex function of $\psi_k$, $k \in K$ and $k$ not central. We, then, integrate these results into an overall solution procedure.

1. Solution (1): $\mathcal{Q}^N = 0$ and $\mathcal{Q}^R = 0$

In Solution (1) from Table 3.11, no products are produced. In this case product design is inconsequential.
2. Solution (2): $\mathcal{O}^N = D^N$ and $\mathcal{O}^R = 0$, and $D^N \in [0, \infty)$

In Solution (2) from Table 3.11, all cores are disassembled and the good-parts are salvaged. By adding the product design trade-offs from Table 3.11 to Equation (3.73), we obtain

$$\Pi_2 = (\mathcal{O}^N - (C^N + \sum_{k \in \mathcal{K}} C_k^N (\psi_k)) D^N + \sum_{k \in \mathcal{K}} S_k(\psi_k) \lambda D^N - \theta \lambda D^N.$$  (4.29)

Note that, by assumption, $\Pi_2$ is a quasi-convex function of $\psi_k$, $k \in \mathcal{K}$, and since we are maximizing $\Pi_2$, the optimal value of $\psi_k$, $\psi_k^*$ will be on the boundary of $\psi_k$. Thus, $\psi_k^* = 0$ or $\psi_k^* = \psi^H$ for each $k \in \mathcal{K}$. Also, for each $k \in \mathcal{K}$, we say $k \in \mathcal{L}$ if $\psi_k = 0$ and $k \in \mathcal{H}$ if $\psi_k = \psi^H$. We may rewrite $\Pi_2$ in terms of $\mathcal{L}$ and $\mathcal{H}$ as follows:

$$\Pi_2 = \left[\mathcal{O}^N - (C^N + \sum_{h \in \mathcal{H}} C_h^N (\psi^H)) D^N + \sum_{h \in \mathcal{H}} (S_h(\psi^H) \lambda D^N - (C_c + C_d + \sum_{l \in \mathcal{L}} C_{disp}^l + \sum_{h \in \mathcal{H}} C_{disp}^h (1 - \psi^H)) \lambda D^N) \right.$$  (4.30)

$$+ \sum_{h \in \mathcal{H}} \left( S_h(\psi^H) \lambda D^N - C_h^N (\psi^H) D^N - C_{disp}^h (1 - \psi^H) \lambda D^N \right).$$

In Equation (4.31), the first two terms are not dependent on how the subsets, $\mathcal{L}$ and $\mathcal{H}$, of $\mathcal{K}$ are selected. The third term describes the contribution to $\Pi_2$ of the parts whose quality is 0 and the fourth term describes the contribution to $\Pi_2$ of the parts whose quality is $\psi^H$. Thus, in order to determine the quality of a given part, we simply compare its contribution to the profit if it were 0 and if it were $\psi^H$. Specifically, we see that $\psi_k^* = 0$ if

$$- C_{disp}^k \lambda D^N > S_k(\psi^H) \lambda D^N - C_k^N (\psi^H) D^N - C_{disp}^k (1 - \psi^H) \lambda D^N.$$  (4.32)
Dividing by $D^N$ and rearranging terms, we have that

$$C^N_k(\psi^H) > (S_k(\psi^H) + C^{disp}_k)\psi^H\lambda.$$  \hspace{1cm} (4.33)

This is the condition for $\psi^*_k = \psi^L$, and $\psi^*_k = \psi^H$ otherwise. This gives us the following proposition.

**Proposition 4.4.1** Let $Q^N = D^N$, $Q^R = 0$, and $D^N \in [0, \infty)$ (Solution (2)). Then, the optimal part yield for Part $k$ for this solution is such that

$$k \in \begin{cases} L, & \text{if } C^N_k(\psi^H) > (S_k(\psi^H) + C^{disp}_k)\psi^H\lambda, \\ \mathbb{H}, & \text{otherwise}. \end{cases} \hspace{1cm} (4.34)$$

3. Solution (3), $Q^N = D^N$, $Q^R = \psi_m\lambda D^N$, $D^N \in [0, \frac{D^R}{\lambda \psi_m}]$ for some $m \in \mathbb{K}$

In Solution (3) from Table 3.11, the number of remanufactured products produced is limited to the number of good-parts of type $m \in \mathbb{K}$ recovered. By adding the product design trade-offs to Equation (3.74), for a given grouping of parts into $S$ and $P$, we have that

$$\Pi_3 = [Q^N - (C^N + \sum_{k \in \mathbb{K}} C^N_k(\psi_k))]D^N + \rho^R \psi_m\lambda D^N
\sum_{k \in \mathbb{S}} S_k(\psi_k)(\psi_k\lambda D^N - \psi_m\lambda D^N)
- \sum_{k \in \mathbb{P}} C^p_k(\psi_k)(\psi_m\lambda D^N - \psi_k\lambda D^N) - \theta \lambda D^N. \hspace{1cm} (4.35)$$

Note that, by assumption, $\Pi_3$ is a quasi-convex function of $\psi_k$, $k \in \mathbb{K}, k \neq m$. Furthermore, the conditions on $D^N$ imply that $\psi_m \leq \frac{D^R}{\lambda D^N}$. However, the nature of the optimal yield for the central part, $\psi^*_m$, depends on whether $\Pi_3$ is quasi-convex in $\psi_m$ or not. The contribution of Part $m$ to the profit is given by

$$\underbrace{-C^N_m(\psi_m) + \rho^R \psi_m\lambda - \sum_{k \in \mathbb{S}} S_k(\psi_k)\psi_m\lambda D^N - \sum_{k \in \mathbb{P}} C^p_k(\psi_k)\psi_m\lambda D^N + C^{disp}_m \psi_m}_{mx} \hspace{1cm} (4.36)$$
where the first term is the cost of additional quality, the second term is the remanufacturing profit, the third term is the opportunity cost of using a part in a remanufactured product rather than salvaging that part, the fourth term is the cost of replacement parts, and the fifth term is disposal cost avoided. Equation 4.36 is of the form 

\[ g(x) = mx - f(x) \]

where \( m \) may be \( >0 \) or \( <0 \) (depending on which parts are surplus and deficit parts).

**Proposition 4.4.2** Let \( g(x) = mx - f(x) \), where \( f(x) \) is a monotonic increasing function of \( x \), \( f''(x) < 0 \) for \( x \in [0,x''] \), \( f'' > 0 \) for \( x \in [x'',\infty) \), and \( m \in [-\infty,\infty] \). \( g(x) \) is quasi-convex for the following cases:

(i) \( m < 0 \)

(ii) \( m > 0, m - f'(0) < 0 \) and \( x \in [0,\chi^H) \), where \( \chi^H \) is the smallest \( x : g(x) \) decreasing for \( x \in (0,\infty) \).

**Proof** Note that \( g'(x) = m - f'(x) \).

(i) If \( m < 0 \), then \( g'(x) < 0 \) for \( x \geq 0 \), and thus, \( g(x) \) is decreasing in \( x \geq 0 \). Therefore, \( g(x) \) is quasi-convex.

(ii) In this case, \( g(x) \) will be initially decreasing since \( g'(x) = m - f'(0) < 0 \). However, since, \( f''(x) < 0 \) for \( x \in [0,x''] \), \( g'(x) \) is increasing in \( x \in [0,x''] \). This leads to two cases:

a. \( g(x) \) never becomes increasing over \( x \in [0,x''] \). In this case, for \( x \in [x'',\infty) \), \( f'' > 0 \) and thus \( g'(x) \) is decreasing in \( x \in [x'',\infty) \), which ensures that \( g(x) \) is a decreasing function over all \( x \) and is thus quasi-convex.

b. \( g(x) \) is an increasing function over some part of the region \( x \in [0,x''] \). In this case, for \( x \in [x'',\infty) \), \( f'' > 0 \) and thus \( g'(x) \) is decreasing in \( x \in [x'',\infty) \), which means that eventually \( g(x) \) will begin decreasing again. Let \( \chi^H \) be the smallest \( x : g(x) \) is decreasing for \( x \in (x'',\infty) \). Since \( g(x) \) is decreasing for \( x \in [0,x''] \) and increasing for \( x \in [x'',\chi^H] \), \( g(x) \) is a quasi-convex function of \( x \in [0,\chi^H] \).

\[ \square \]

Based on Proposition 4.4.2, if

\[ \rho_k \psi_k \lambda - \sum_{k \in S} S_k(\psi_k) \psi_m \lambda D^N - \sum_{k \in P} C^p_k(\psi_k) \psi_m \lambda D^N + C_{disp} m \psi_m < 0, \]  

\[ (4.38) \]
then $\Pi_3$ will be a decreasing function of $\psi_m$. Thus, by rearranging terms, we find that if

$$\rho^R < \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C^p_k(\psi_k) - C^\text{disp}_m$$

(4.39)

then $\Pi_3$ will be a decreasing function.

If $\Pi_3$ is a quasi-convex function of $\psi_m$, then $\psi_m^* = 0$ or $D^R / \lambda = D^N$ if $D^R / \lambda D^N < \psi_H$. Otherwise, if $\Pi_3$ is a quasi-convex function of $\psi_m$ and $D^R / \lambda D^N \geq \psi_H$, then $\psi_m^* = \psi_H$. However, if Equation (4.38) is true, then the contribution of part is always $\leq 0$ and so $\psi_m^* = 0$ (or Solution (3) is not optimal). If $\Pi_3$ is not a quasi-convex function of $\psi_m$, then $\psi_m^* \in [0, \min\{D^R / \lambda D^N, \psi_H\}]$. We will discuss each case separately, beginning with the quasi-convex case.

(1) $\Pi_3$ is a quasi-convex function of $\psi_m$.

If $\Pi_3$ is a quasi-convex function of $\psi_m$, then the solution depends on whether $D^R / \lambda D^N < \psi_H$ or not. If $D^R / \lambda D^N < \psi_H$, then $\psi_m^* = D^R / \lambda D^N$ and if $D^R / \lambda D^N \geq 1$ then $\psi_m^* = \psi_H$.

First, consider the case, $D^R / \lambda D^N < \psi_H$. For $k \in S$, since $\psi_k \geq \psi_m$ and $\Pi_3$ is a quasi-convex function of $\psi_m$, we have $\psi_k^* = D^R / \lambda D^N$ or $\psi_H$. Similarly, for $k \in P$, since $\psi_k < \psi_m$ and $\Pi_3$ is a convex function of $\psi_k$, we have $\psi_k^* = 0$. Thus, for Solution (3), the part yields are either 0, $D^R / \lambda D^N$, or $\psi_H$. For $k \in K$, we say that $k \in L$ if $\psi_k = 0$, and $k \in G$ if $\psi_k = D^R / \lambda D^N$, and $k \in H$ if $\psi_k = \psi_H$. We may rewrite $\Pi_3$ in the following manner:

$$\Pi_3 = \{G^N - (C^N + \sum_{g \in G} C^N_g(D^R / \lambda D^N) + \sum_{h \in H} C^N_h(\psi_H))D^N + \rho^R D^R / \lambda D^N \lambda D^N$$

$$- \sum_{l \in L} (C^p_l D^R / \lambda D^N \lambda D^N + \sum_{h \in H} (S_h(\psi_H))C^p_h(\psi_H - D^R / \lambda D^N)) \lambda D^N$$

$$- (C^e + C^d + \sum_{l \in L} C^\text{disp}_l + \sum_{g \in G} C^\text{disp}_g (1 - D^R / \lambda D^N + C^\text{disp}_g (1 - \psi_H)) \lambda D^N)$$

(4.40)
or equivalently

$$\Pi_3 = (\psi^N_k - C^N_k)D^N + \rho^R D^R - (C^c + C^d)\lambda D^N$$

$$+ \sum_{l \in L} \left( -(C^p_0)D^R - C^d_{l} \lambda D^N \right)$$

$$+ \sum_{g \in G} \left( -C^N_g \left( \frac{D^R}{\lambda D^N} \right) D^N - C^d_{g} (1 - \frac{D^R}{\lambda D^N}) \lambda D^N \right)$$

$$+ \sum_{h \in H} \left( S_h (\psi^H) (\psi^H \lambda D^N - D^R) - C^N_h (\psi^H) D^N - C^d_{h} (1 - \psi^H) \lambda D^N \right).$$

(4.41)

In Equation (4.41), the first three terms are not dependent on how the subsets, \(L\), \(G\), and \(H\), of \(K\) are selected.

The fourth term depicts the contribution of the parts whose quality is 0, the fifth term captures the contribution of the parts whose quality is \(\frac{D^R}{\lambda D^N}\), and the sixth term represents the contribution of the parts whose quality is \(\psi^H\).

Thus, in order to determine the quality of a given part, we compare the contribution to the profit of each possible part yield (0, \(\frac{D^R}{\lambda D^N}\), and \(\psi^H\)). We see that a part yield of 0 is best for Part \(k\) if the following conditions hold.

i. \(\psi_k = 0\) versus \(\psi_k = \frac{D^R}{\lambda D^N}\)

$$-(C^p_0)D^R - C^d_{k} \lambda D^N > \left( C^N_k \left( \frac{D^R}{\lambda D^N} \right) D^N - C^d_{k} (1 - \frac{D^R}{\lambda D^N}) \lambda D^N \right).$$

(4.42)

By rearranging terms and dividing through by \(D^R\)

$$C^N_k \left( \frac{D^R}{\lambda D^N} \right) > (C^p_0 + C^d_{k} \frac{D^R}{\lambda D^N}).$$

(4.43)

ii. \(\psi_k = 0\) versus \(\psi_k = \psi^H\)

$$-C^p_0 D^R - C^d_{k} \lambda D^N > S_k (\psi^H) (\psi^H \lambda D^N - D^R)$$

$$-C^N_k (\psi^H) D^N - C^d_{k} (1 - \psi^H) \lambda D^N.$$

(4.44)
By rearranging terms and dividing by $D^N$, we obtain

$$C^N_k(\psi^H) > (C^0_k - \frac{D^R}{\lambda D^N} + S_k(\psi^H)(\psi^H - \frac{D^R}{\lambda D_N}) + C^{\text{disp}}_k \psi^H)\lambda. \tag{4.45}$$

On the other hand, a part yield of $\frac{D^R}{\lambda D^N}$ is best for Part $k$ if the following conditions hold.

i. $\psi_k = \frac{D^R}{\lambda D^N}$ versus $\psi_k = 0$

Relation opposite of Equation (4.43).

ii. $\psi_k = \frac{D^R}{\lambda D^N}$ versus $\psi_k = \psi^H$

$$-C^N_k(\frac{D^R}{\lambda D^N})D^N - C^{\text{disp}}_k(1 - \frac{D^R}{\lambda D^N})\lambda D^N > S_k(\psi^H)(\psi^H \lambda D^N - D^R) - C^N_k(\psi^H)D^N + C^{\text{disp}}_k(1 - \psi^H)\lambda D^N. \tag{4.46}$$

By rearranging terms and dividing though by $D^N$ we obtain

$$C^N(\psi^H) - C^N_k(\frac{D^R}{\lambda D^N}) > (S_k(\psi^H) + C^{\text{disp}}_k)(\psi^H - \frac{D^R}{\psi D^N})\lambda. \tag{4.47}$$

To check if $\psi_k^* = \psi^H$, we perform a similar comparison.

i. $\psi_k = \psi^H$ versus $\psi_k = 0$

Relation opposite of Equation (4.45).

ii. $\psi_k = \psi^H$ versus $\psi_k = \frac{D^R}{\lambda D^N}$

Relation opposite of Equation (4.47).

Based on these conditions, we have the following result.

**Proposition 4.4.3** Let $\varphi^N = D^N$ and $\varphi^R = \psi_m \lambda D^N$ (Solution (3)) for some $m = 1, \ldots, K$ and $\frac{D^R}{\lambda D^N} < \psi^H$. Also, let $\Pi_3$ be a quasi-convex function of $\psi_m$. Then, the optimal part yield for Part $k$ for this solution is such that for
each $k \in \mathbb{K}$

\[
\begin{aligned}
\mathbb{L} & \quad \text{if } C_k^N\left(\frac{p^R}{\lambda D_N}\right) > \left((C_{p^0}^k + C_{disp}^k)\frac{p^R}{\lambda D_N}\right)\lambda \quad \text{and} \\
& \quad C_k^N(\psi^H) \geq \left[\left(C_k^N - S_k(\psi^H)\right)\frac{p^R}{\lambda D_N}\right] + \left(S_k(\psi^H) + C_k^{disp}\right)\psi^H\lambda, \\
\mathbb{G} & \quad \text{if } C_k^N\left(\frac{p^R}{\lambda D_N}\right) < \left((C_{p^0}^k + C_{disp}^k)\frac{p^R}{\lambda D_N}\right)\lambda \quad \text{and} \\
& \quad C_k^N(\psi^H) - C_k^N\left(\frac{p^R}{\lambda D_N}\right) > \left[\left(S_k(\psi^H) + C_k^{disp}\right)(\psi^H - \frac{p^R}{\lambda D_N})\right]\lambda, \\
\mathbb{H} & \quad \text{otherwise}.
\end{aligned}
\]

(4.48)

Next, consider the case, $\frac{p^R}{\lambda D_N} > \psi^H$. In this case, $\psi^*_m = \psi^H$, and $\Pi_3$ is written as follows:

\[
\Pi_3 = [\mathcal{B}N - \left(C^N + \sum_{h \in H} C^N(\psi^H)\right)]D_N + p^R \psi^H \lambda D_N - \sum_{l \in L} (C^{p^0}_l) \psi^H \lambda D_N
\]

\[
-(C^c + C^d + \sum_{l \in L} C_{disp}^l + \sum_{h \in H}(1 - \psi^H))\lambda D_N,
\]

(4.49)

or equivalently,

\[
\Pi_3 = (\mathcal{B}N - C^N)D_N + p^R \psi^H \lambda D_N - (C^c + C^d)\lambda D_N
\]

\[
+ \sum_{l \in L} (-\left(C^{p^0}_l\right) \psi^H \lambda D_N - C_{disp}^l \lambda D_N)
\]

\[
+ \sum_{h \in H} -C_h^N(\psi^H)D_N - C_k^{disp}(1 - \psi^H)\lambda D_N.
\]

(4.50)

By comparing the fourth term with the fifth term, we can determine the optimal value of $\psi_k$. Specifically, $\psi^*_k = 0$

if

\[
-C_k^{p^0}\psi^H \lambda D_N - C_k^{disp} \lambda D_N > -C_k^N(\psi^H)D_N - C_k^{disp}(1 - \psi^H)\lambda D_N.
\]

(4.51)

Rearranging terms we obtain:

\[
C_k^N(\psi^H) > (C_k^{p^0} + C_k^{disp})\psi^H \lambda.
\]

(4.52)

This gives us the following proposition.
Proposition 4.4.4 Let $Q^N = D^N$ and $Q^R = \psi_m D^N$ (Solution (3)) for some $m = 1, \ldots, K$, $\Pi_3$ is a quasi-convex function of $\psi_m$, and $\frac{D^R}{D^N} \geq \psi^H$. Then, the optimal part yield of Part $k$ for this solution is such that for each $k \in K$

$\psi_k \in \begin{cases} L, & \text{if } C_k^N(\psi^H) > (C_k^0 + C_k^{disp})\psi^H \lambda, \\ H, & \text{otherwise.} \end{cases}$

(4.53)

The problem of determining the part yields for Solution (3) for non-quasi-convex $\Pi_3$ is a much more difficult than that for quasi-convex $\Pi_3$. The difficulty stems from two complicating features. The first feature is that the design of Part $m$ now affects the terms of $\Pi_3$. Therefore, we cannot decide which group to put a particular part in without first knowing the value of $\psi_m$. The second complicating feature is that the nature of $\Pi_3$ as a function of $\psi_m$ cannot be determined without knowing how the other parts are grouped. Thus, we must solve the following non-linear integer program to determine $\psi_m$ and the yields of the other parts. For $k \in K$, let $l = 1$ if $\psi_k = 0$. Otherwise, let $0_k = 1$ if $\psi_k = \psi_m$, and 0 otherwise; and let $h_k = 1$ if $\psi_k = \psi^H$, and 0 otherwise. We have:
\[
\begin{align*}
\max_{\psi_m, l_k, o_k, h_k, k \in \mathbb{K}} & \quad \phi^{\text{N}} - \phi^{\text{N}'} D^N + (C^C + C^D) \lambda D^N + \rho R \psi_m \lambda D^N \\
& + \sum_{k \in \mathbb{K}} [-C^0_k \psi_m \lambda D^N - C^{\text{disp}}_k \lambda D^N] l_k \\
& + \sum_{k \in \mathbb{K}} [-C^N_k (\psi_m) D^N - C^{\text{disp}}_k (1 - \psi_m) \lambda D^N] o_k \\
& + \sum_{k \in \mathbb{K}} [S_k (\psi^H) (\psi^H \lambda D^N - \psi_m \lambda D^N) - C^N_k (\psi^H) D^N - C^{\text{disp}}_k (1 - \psi^H) \lambda D^N] h_k
\end{align*}
\]  

(4.54)

subject to:

\[
0 \leq \psi_m \leq \min\{\frac{D^R}{\lambda D^N}, \psi^H\},
\]

(4.55)

\[
l_k + o_k + h_k = 1 \quad \forall \quad k \in \mathbb{K},
\]

(4.56)

\[
l_k, o_k, h_k \text{ Binary} \quad \forall \quad k \in \mathbb{K}.
\]

(4.57)

For a fixed value of $\psi_m$, we again may simply compare the terms of $\Pi_3$ to determine the subset to which a given part belongs. For this reason, we relax the binary constraints on $g_k, o_k,$ and $h_k$ as they will be 0 or 1 in the optimal solution.

Under certain conditions, the values of $l_k, o_k,$ and $h_k$ for $k \in \mathbb{K}$ may be known a priori. Note that in the case that $\Pi_3$ is not a quasi-convex function of $\psi_m$, we may use Equation (4.48) with $\frac{D^R}{\lambda D^N}$ substituted with $\psi_m$. Thus, if

\[
C^N_k (\psi_m) > (C^0_k + C^{\text{disp}}_k) \psi_m \lambda \quad \forall \psi_m \in [0, \min(\frac{D^R}{\lambda D^N}, \psi^H)]
\]

(4.58)

and

\[
C^N_k (\psi^H) > [(C^0_k - S_k (\psi^H)) \psi_m + (S_k (\psi^H) + C^{\text{disp}}_k) \psi^H] \lambda,
\]

(4.59)

then $\psi^*_k = 0$. If $C^0_k > S_k (\psi^H)$ then Equation 4.59 holds if

\[
C^N_k (\psi^H) > ((C^0_k - S_k (\psi^H)) \frac{D^R}{\lambda D^N} + (S_k (\psi^H) + C^{\text{disp}}_k) \psi^H) \lambda.
\]

(4.60)
However, if \( C_p^0 < S_k(\psi^H) \), then Equation (4.59) holds if

\[
C_k^N(\psi^H) > (S_k(\psi^H) + C_k^{\text{disp}})\psi^H\lambda. \tag{4.61}
\]

If Equation 4.58 does not hold for any \( \psi_m \in [0, \min(\frac{\psi}{\lambda_D}, \psi^H)] \) and

\[
C_k^N(\psi^H) - C_k^N(\psi_m) > [(S_k(\psi^H) + C_k^{\text{disp}})(\psi^H - \psi_m)]\lambda \tag{4.62}
\]

for any \( \psi_m \in [0, \min(\frac{\psi}{\lambda_D}, \psi^H)] \), then \( \psi_m^* = \psi_m \). Finally, if Equation (4.62) does not hold for all \( \psi_m \in [0, \min(\frac{\psi}{\lambda_D}, \psi^H)] \), and Equation (4.59) does not hold, then \( \psi_0^* = \psi^H \). If \( C_p^0 > S_k(\psi^H) \) then Equation (4.59) does not hold if

\[
C_k^N(\psi^H) < (S_k(\psi^H) + C_k^{\text{disp}})\psi^H\lambda. \tag{4.63}
\]

If \( C_p^0 < S_k(\psi^H) \), then Equation (4.59) does not hold if \( C_p^0 > S_k(\psi^H) \) or

\[
C_k^N(\psi^H) < ((C_p^0 - S_k(\psi^H))\frac{\psi}{\lambda_D} + (S_k(\psi^H) + C_k^{\text{disp}})\psi^H)\lambda. \tag{4.64}
\]

4. Solution (4), \( \omega^N = D^N \), \( \omega^R = D^R \), and \( D^N \in \left[ \frac{\psi}{\lambda_D}, \frac{\psi}{\lambda_D^{m+1}} \right] \) for some \( m, m = 1, \ldots, K - 1 \).

Solution (4) from Table 3.11 corresponds to the EOL option for which new and remanufacturing demands are met. In this context, \( \psi_{m+1} \) is the part yield of the next lowest yielding part after Part \( m \). By adding the product design trade-offs to Equation (3.15), for a given grouping of parts into \( S \) and \( P \), we have that

\[
\Pi_4 = \left[ \beta^{\text{NR}} - (C^N + \sum_{k \in K} C_k^N(\psi_k)) \right]D^N + \rho^RD^R + \sum_{k \in S} S_k(\psi_k)(\psi_k\lambda D^N - D^R) + \sum_{k \in P} C_k^p(\psi_k)(D^R - \psi_k\lambda D^N) - \theta \lambda D^N. \tag{4.65}
\]

Note that, by assumption, \( \Pi_4 \) is a quasi-convex function of \( \psi_k, k \in K \). Furthermore, the conditions on \( D^N \) imply that \( \psi_{m-1} \geq \frac{\psi}{\lambda_D} \) and \( \psi_m \leq \frac{\psi}{\lambda_D^m} \). Thus, \( \psi_{m-1} = \frac{\psi}{\lambda_D} \) (and is in \( S \)), and \( \psi_m = \frac{\psi}{\lambda_D^m} \) (and is in \( P \)). It
follows that \( \psi_k^* = \frac{D_p^k}{LD_N^k} \) or \( \psi^H \) for \( k \in \mathbb{S} \) and \( \psi_k^* = 0 \) or \( \frac{D_p^k}{LD_N^k} \) for \( k \in \mathbb{P} \). Thus, for Solution (4), the part yields are either 0, \( \frac{D_p^k}{LD_N^k} \), or \( \psi^H \), and we determine groupings of the parts in \( \mathbb{K} \). For \( k \in \mathbb{K} \), we say that \( k \in \mathbb{L} \) if \( \psi_k = 0 \), \( k \in \mathbb{G} \) if \( \psi_k = \frac{D_p^k}{LD_N^k} \), and \( k \in \mathbb{H} \) if \( \psi_k = \psi^H \). We can rewrite \( \Pi_4 \) in the following manner.

\[
\Pi_4 = \left[ D_N^N - \left( C_{N'} + \sum_{g \in \mathbb{G}} C_N^g \left( \frac{D_p}{LD_N} \right) + \sum_{h \in \mathbb{H}} C_N^h \right) \right] \lambda D_N^N + \rho^R D^R
\]

\[
+ \sum_{h \in \mathbb{H}} S_h(\psi^H)(\psi^H \lambda D_N^N - D^R) - \sum_{l \in \mathbb{L}} C^d_l
\]

\[
- (C^c + C^d + \sum_{l \in \mathbb{L}} C^d_{\text{disp}} + \sum_{g \in \mathbb{G}} C^\text{disp}_g (1 - \frac{D_p}{LD_N}) + \sum_{h \in \mathbb{H}} C^\text{disp}_h (1 - \psi^H)) \lambda D_N^N
\]

which is the same as \( \Pi_3 \) if \( \Pi_3 \) is a quasi-convex function of \( \psi_m \).

5. Solution (5), \( D_N^N = D_N^N \), \( D_N^R = D_N^R \), and \( D_N^N \in \left[ \frac{D_p}{LD_N}, \infty \right) \), where \( \psi_k \) is the lowest yielding part.

Solution (5) from Table 3.11 requires a large availability of cores. Remanufactured product demand is met; however, all cores are collected and parts are salvaged. By adding the product design trade-offs to Equation (3.16) from Table 3.11, we obtain

\[
\Pi_5 = \left[ D_N^N - \left( C_{N'} + \sum_{k \in \mathbb{K}} C_N^k \right) \right] \lambda D_N^N + \rho^R D^R + \sum_{k \in \mathbb{K}} S_k(\psi_k)(\psi_k \lambda D_N^N - D^R) - \theta \lambda D_N^N.
\]  

(4.67)

Note that \( \Pi_5 \) is a quasi-convex function of \( \psi_k, k \in \mathbb{K} \) and that all parts are in \( \mathbb{S} \). The restrictions on \( D_N^N \) imply that \( \psi_k \geq \frac{D_p}{LD_N^k} \). Since \( \psi_k \geq \psi_k \) for all \( k \in \mathbb{K} \), and we seek to maximize \( \Pi_5, \psi_k^* = \frac{D_p}{LD_N^k} \) or \( \psi^H \) for all \( k \in \mathbb{K} \). For each \( k \in \mathbb{K} \), we say \( k \in \mathbb{G} \) if \( \psi_k = \frac{D_p}{LD_N^k} \) and \( k \in \mathbb{H} \) if \( \psi_k = \psi^H \). In view of a given selection of subsets of \( \mathbb{K} \), we may rewrite \( \Pi_5 \) in the following manner.

\[
\Pi_5 = \left[ D_N^N - \left( C_{N'} + \sum_{g \in \mathbb{G}} C_N^g \left( \frac{D_p}{LD_N} \right) + \sum_{h \in \mathbb{H}} C_N^h \right) \right] \lambda D_N^N + \rho^R D^R
\]

\[
+ \sum_{h \in \mathbb{H}} S_h(\psi^H)(\psi^H \lambda D_N^N - D^R) - (C^c + C^d + \sum_{g \in \mathbb{G}} C^\text{disp}_g (1 - \frac{D_p}{LD_N}) + \sum_{h \in \mathbb{H}} C^\text{disp}_h (1 - \psi^H)) \lambda D_N^N.
\]  

(4.68)
or equivalently,

\[
\Pi_5 = [(\phi^N - C^N)D^N - (C^c + C^d)\lambda D^N] + \rho^R D^R \\
+ \sum_{g \in G} \left( -C^\text{disp}_k (1 - \frac{D^R}{\lambda D^N}) \lambda D^N - C^N (\frac{D^R}{\lambda D^N}) D^N \right) \\
+ \sum_{h \in H} (S_h(\psi^H))(\psi^H - \frac{D^R}{\lambda D^N}) \lambda D^N \\
-C^N (\psi^H) D^N - C^\text{disp}_h (1 - \psi^H) \lambda D^N).
\]  

(4.69)

In Equation (4.69), the first two terms are not dependent on how the subsets, \( G \) and \( H \), of \( K \) are selected. The third term describes the contribution to \( \Pi_5 \) of the parts whose quality is \( D^R \lambda D^N \) and the fourth term describes the contribution to \( \Pi_5 \) of the parts whose quality is \( \psi^H \). Thus, in order to determine the quality of a given part, we simply compare its contribution to the profit if it were \( D^R \lambda D^N \) or \( \psi^H \). Specifically, we see that \( \psi^*_k = \frac{D^R}{\lambda D^N} \) if

\[
-C^\text{disp}_k (1 - \frac{D^R}{\lambda D^N}) \lambda D^N - C^N (\frac{D^R}{\lambda D^N}) D^N > S_k(\psi^H)(\psi^H - \frac{D^R}{\lambda D^N}) \lambda D^N \\
-C^N (\psi^H) D^N + C^\text{disp}_k (1 - \psi^H) \lambda D^N.
\]  

(4.70)

Rearranging terms and dividing by \( D^N \) we obtain

\[
C^N (\psi^H) - C^N (\frac{D^R}{\lambda D^N}) > [(S_k(\psi^H) + C^\text{disp}_k) \psi^H - S_k(\psi^H) \frac{D^R}{\lambda D^N}] \lambda.
\]  

(4.71)

Otherwise, \( \psi^*_k = \psi^H \). This gives us the following proposition.

**Proposition 4.4.5** Let \( \phi^N = D^N \), \( \phi^R = D^R \), and \( D^N \in [\frac{D^R}{\lambda \psi^H}, \infty) \) (Solution (5)), then the optimal part yield for Part \( k \) for this solution is such that for each \( k \in K \),

\[
k \in \\
\begin{cases} 
G, & \text{if } C^N (\psi^H) - C^N (\frac{D^R}{\lambda D^N}) > [(S_k(\psi^H) + C^\text{disp}_k) \psi^H - S_k(\psi^H) \frac{D^R}{\lambda D^N}] \lambda, \\
H, & \text{otherwise.}
\end{cases}
\]  

(4.72)
6. Solution (6), \( \omega^N = \frac{D^R_k}{\lambda \psi_m}, \quad \omega^R = D^R, \) and \( D^N \in \left[ \frac{D^R_k}{\lambda \psi_m}, \infty \right) \) for some \( m = 1, \ldots, K \)

Solution (6) from Table 3.11 requires that remanufacturing demand is met and just enough new products are produced to obtain enough cores to recover enough parts of type \( m \) to perform the remanufacturing. Note that \( D^N \in \left[ \frac{D^R_k}{\lambda \psi_m}, \infty \right) \) and therefore \( \psi_m \geq \frac{D^R_k}{\lambda D^N} \). By adding the product design trade-offs to Equation (3.17), for a given grouping of parts into \( S \) and \( P \), we have that

\[
\Pi_6 = \left[ \rho^N - \frac{(C^N + \sum_{k \in P} C^N_k(\psi_k))}{\lambda \psi_m} \right] \frac{D^R_k}{\lambda \psi_m} + \rho^R D^R + \sum_{k \in S} S_k(\psi_k) \left( \frac{D^R_k}{\psi_m} \right) - D^R_k 
\]

or equivalently, we group the terms \( \Pi_6 \) to isolate those which are impacted by \( \psi_m \):

\[
\Pi_6 = \rho^R D^R - \left( \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} (C^p_k(\psi_k) + C^\text{disp}_k) ight) D^R_k 
\]

\[
+ \left( \left[ \rho^N - \frac{(C^N + \sum_{k \in S} C^N_k(\psi_k))}{\lambda \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C^p_k(\psi_m)} \right] / \lambda \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C^p_k(\psi_m) \right) \frac{D^R_k}{\psi_m} - C^\text{disp}_m - (C^c + C^d + \sum_{k \in P, k \neq m} C^\text{disp}_k(1 - \psi_k)) \frac{D^R_k}{\psi_m} 
\]

Note that, for \( k \neq m \), \( \Pi_6 \) is a convex function of \( \psi_k \). However, depending on the product characteristics for \( k = m \), \( \Pi_6 \) may be either a concave, increasing function of \( \psi_m \) or a convex, decreasing function of \( \psi_m \). Specifically, by inspecting the last term of Equation 4.74 we see that by the second order conditions for a convex function, if

\[
\left[ \rho^N - \frac{(C^N + \sum_{k \in S} C^N_k(\psi_k))}{\lambda \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C^p_k(\psi_m)} \right] / \lambda \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C^p_k(\psi_m) \frac{D^R_k}{\psi_m} - C^\text{disp}_m - (C^c + C^d + \sum_{k \in P, k \neq m} C^\text{disp}_k(1 - \psi_k)) > 0 
\]

then \( \Pi_6 \) is a convex function of \( \psi_m \) and more importantly, by first order conditions, it is a decreasing function of \( \psi_m \); otherwise, the left hand side of Equation (4.75) is negative and, thus \( \Pi_6 \) is a concave function of \( \psi_m \).
Let the left-hand-side of Equation (4.75) be denoted by \( z \). Note that, \( z \) is dependent on the selection of \( S \) and \( P \), both of which are dependent on the value of \( \psi_m \). We must, therefore, solve for \( \psi_m \) and the selection of \( S \) and \( P \) together. However, if \( z \) can be known a priori (e.g., if \( \beta^N \) is sufficiently high, then \( z > 0 \) ) then, since the range of \( D^N \) implies that \( \psi_m \geq \frac{D^r}{\lambda} \), we have:

(a) \( z < 0 \), \( \Pi_6 \) is a concave, increasing function of \( \psi_m \), and thus, \( \psi_m^* = \psi^H \).

(b) \( z > 0 \), \( \Pi_6 \) is a convex, decreasing function of \( \psi_m \), and thus, \( \psi_m^* = \frac{D^r}{\lambda} \).

Just as in Solution (3), the yield of a part may either be 0, \( \psi_H \), or \( \psi^H \). For each \( k \in K \), we say \( k \in L \) if \( \psi_k = 0 \), \( k \in G \) if \( \psi_k = \frac{D^r}{\lambda} \) and \( k \in H \) if \( \psi_k = \psi^H \). In view of a given value of \( z \) and selection of subsets of \( K \), we may rewrite \( \Pi_6 \) and derive some optimality conditions for \( \psi_k \), \( k \in K \).

\( z < 0 \) (\( \psi_m = \psi^H \))

\[
\Pi_6 = \left[ \beta^N - (C^N + \sum_{h \in H} C^N_h (\psi^H)) - \frac{D^r}{\lambda} + \rho^R D^R 
- \sum_{l \in L} (C_{l0})_l D^R 
- (C^c + C^d + \sum_{l \in L} C^d_{l} + \sum_{h \in H} C^d_{h} (1 - \psi^H)) \frac{D^r}{\psi^H} \right] D^R \tag{4.76}
\]

or equivalently,

\[
\Pi_6 = \left[ (\beta^N - C^N)/\lambda - (C^c + C^d) \frac{D^r}{\psi^H} + \rho^R D^R 
+ \sum_{l \in L} \left( -c^d_{l} \frac{D^r}{\psi^H} - (C_{l0})_l D^R \right) \right. 
\left. + \sum_{h \in H} \left( -C^N_h (\psi^H) \frac{D^r}{\lambda \psi^H} - C^d_{h} (1 - \psi^H) \frac{D^r}{\psi^H} \right) \right] D^R \tag{4.77}
\]

By comparing the third term of Equation (4.77) with its fourth term, we have that \( \psi_k^* = 0 \) if

\[
-C^d_{k} \frac{D^r}{\psi^H} - (C_{k0})_k D^R > -C^N_k (\psi^H) \frac{D^r}{\lambda \psi^H} - C^d_{k} (1 - \psi^H) \frac{D^r}{\psi^H}. \tag{4.78}
\]
Rearranging terms we obtain
\[ C_k^N(\psi^H) > (C_p^0_k + C_{k}^{disp}) \psi^H \lambda, \] (4.79)
otherwise, \( \psi'_k = \psi^H \).

Next, consider the case when \( z > 0 \) (\( \psi_m = \frac{D^R}{\lambda D^N} \)). In this case
\[
\Pi_6 = (\rho^N - C^{N'}) D^N + \rho^R D^R - (C^c + C^d) \lambda D^N
+ \sum_{l \in L} (-C_N^l \lambda D^N) + \sum_{g \in G} (-C_N^g \lambda D^N - (1 - D^R D^N) \lambda D^N)
+ \sum_{h \in H} (C_p^h \lambda D^N - D^R) - C_N^h \lambda D^N - (1 - \psi^H) \lambda D^N).
\] (4.80)
which is identical to Solution 3 for \( \frac{D^R}{\lambda D^N} < \psi^H \) and \( \Pi_3 \) quasi-convex (see Equation 4.41) in \( \psi_m \) and the same results apply. These results give us the following proposition.

**Proposition 4.4.6** Let \( D^N = \frac{D^R}{\psi_m} \), \( \varphi^R = D^R \) and \( D^N \in \left[ \frac{D^R}{\lambda D^N}, \infty \right) \). Let
\[
z = [\rho^N - (C^{N'} + \sum_{k \in K} C_N^k(\psi_k)) / \lambda + \sum_{k \in S} S_k(\psi_k) + \sum_{k \in P} C_p^k(\psi_k) + C_{m}^{disp} \]
\[ - (C^c + C^d + \sum_{k \in K, k \neq m} C_k^{disp} (1 - \psi_k)), \] (4.81)
(a) \( z < 0 \)

If \( z < 0 \), then the optimal part yield for Part \( k \in K \) for this solution is such that
\[
k \in \begin{cases} L, & \text{if } C_k^{N'}(\psi^H) > (C_p^0_k + C_k^{disp}) \psi^H \lambda, \\
H, & \text{otherwise}, \end{cases} \text{ for each } k \in K. \] (4.82)
(b) \( z > 0 \)

If \( z > 0 \), then the optimal part yield for park \( k \in K \) for this solution is such that for each \( k \in K \):

\[
\left\{ \begin{array}{ll}
\mathbb{L}, & \text{if } C_k^N \left( \frac{D^R}{\lambda} \right) > \left( (C^p_k + C^{\text{disp}}_k) \frac{D^R}{\lambda} \right) \lambda \quad \text{and} \\
& C_k^N (\psi^H) > \left( (C^p_k - S_k (\psi^H)) \frac{D^R}{\lambda} \right) \\
& \text{if } C_k^N (\psi^H) > \left( (C^p_k - S_k (\psi^H)) \frac{D^R}{\lambda} \right) \\
& + (S_k (\psi^H) + C_k^{\text{disp}}) \lambda,
\end{array} \right.
\]

The problem of determining the part yields for Solution (6) with the sign of \( z \) dependent on choice of \( \psi_k = \psi_m \) is much more difficult than for the other solutions but is similar to Solution 3 with \( \Pi_1 \) non-quasi-convex in \( \psi_m \).

Thus, we must solve the following non-linear integer program to determine \( \psi_m \) and yields of the other parts. For \( k \in K \), let \( l_k = 1 \) if \( \psi_k = 0 \), and 0, otherwise; let \( o_k = 1 \) if \( \psi_k = \psi_m \), and 0, otherwise; and let \( h_k = \psi^H \) if \( \psi_k = 1 \) and 0, otherwise.

Model Product Design (PD) - Solution 6:

\[
\max_{\psi_m, l_k, o_k, h_k, k \in K} \left[ \left( \mathcal{R}^N - C^N \right) / \lambda - (C^c + C^d) \right] \frac{D^R}{\psi_m} \\
\quad + \sum_{k \in K} \left( -C_k^{\text{disp}} \frac{D^R}{\psi_m} - (C^p_k + D^R) l_k \right) \\
\quad + \sum_{k \in K} \left( -[C_k^N (\psi_m) / \lambda + C_k^{\text{disp}} (1 - \psi_m)] \frac{D^R}{\psi_m} \right) o_k \\
\quad + \sum_{k \in K} \left( -C_k^N (\psi^H) \frac{D^R}{\lambda \psi_m} + S_k (\psi^H) (\frac{D^R}{\psi_m} - D^R) \psi^H + C_k^{\text{disp}} (1 - \psi^H) \frac{D^R}{\psi_m} \right) h_k
\]

(4.84)
subject to:

\[ \frac{D^R}{\lambda D^N} \leq \psi_m \leq \psi^H, \quad (4.85) \]

\[ l_k + o_k + h_k = 1 \quad \forall \ k \in K, \quad (4.86) \]

\[ \sum_{k \in K} o_k \geq 1, \quad (4.87) \]

\[ l_k, o_k, h_k \text{ Binary} \quad \forall \ k \in K. \quad (4.88) \]

For a fixed value of \( \psi_m \), we again may simply compare the terms of \( \Pi_6 \) to determine the subset to which a given \( k \) belongs. For this reason, we relax the binary constraint on \( g_k, o_k, \) and \( h_k \), as they will be 0 or 1 in the optimal solution.

**Proposition 4.4.7** Let \( D^N = \frac{p^R}{\psi_m}, \varphi^R = D^R \) and \( D^N \in [D^R/\lambda, \psi_m, \infty) \). The optimal product design is, then, given by the solution to:

\[ \{ \max (4.84), \text{s.t.} (4.85) - (4.87), 0 \leq g_k \leq 1, 0 \leq o_k \leq 1, 0 \leq h_k \leq 1 \quad \forall k \in K \} \quad (4.89) \]

Just as for Solution 3, we may perform some possible preprocessing of \( l_k, o_k, \) and \( h_k \). Under certain conditions, the values of \( l_k, o_k, \) and \( h_k \) for \( k \in K \) may be known a priori. Note that, in case \( \Pi_6 \) is not a quasi-convex function of \( \psi_m \), we may use Equation (4.83) with \( \frac{D^R}{\lambda D^N} \) substituted with \( \psi_m \) to possibly determine optimal \( \psi_k \). Since Equation (4.83) is the same as Equation (4.48), the conditions given in Equations (4.58), (4.59), and (4.62) apply. However, in this case, \( \psi_m \in [\frac{D^R}{\lambda D^N}, \psi^H] \). If Equation 4.58 holds for all \( \psi_m \in [\frac{D^R}{\lambda D^N}, \psi^H] \) and Equation 4.59 holds, then \( \psi_k^* = 0 \). For \( C^{p0}_k > S_k(\psi^H) \) Equation 4.59 holds if

\[ C^N_k(\psi^H) > (C^{p0}_k + C^{disp}_k) \psi^H \lambda. \quad (4.90) \]

However, if \( C^{p0}_k < S_k(\psi^H) \), then Equation 4.59 holds if

\[ C^N_k(\psi^H) > ((C^{p0}_k - S_k(\psi^H)) \frac{D^R}{\lambda D^N} + (S_k(\psi^H) + C^{disp}_k) \psi^H) \lambda. \quad (4.91) \]
If both Equation 4.58 and Equation 4.62 do not hold for any \( \psi_m \in [\frac{D^D}{\lambda}, \psi^H] \), then \( \psi^*_k = \psi_m \). Finally, if Equation 4.62 does not hold for any \( \psi_m \in [\frac{D^D}{\lambda}, \psi^H] \) and Equation 4.59 holds, then \( \psi^*_k = \psi^H \). For \( C_{p0}^k > S_k(\psi^H) \) Equation 4.59 does not hold if
\[
C_k^N(\psi^H) < [(C_{p0}^k - S_k(\psi^H)) \frac{D^R}{\lambda D^D} + (S_k(\psi^H) + C_{k}^{disp})\psi^H] \lambda. \tag{4.92}
\]
For \( C_{p0}^k < N_k(\psi^H) \), Equation 4.59 does not hold if
\[
C_k^N(\psi^H) < (C_{k}^{p0} + C_{k}^{disp}) \psi^H \lambda. \tag{4.93}
\]

**General conditions for optimality**

For each EOL option we compare the possible optimal modes of \( \Pi \) and select the best option based on a given value of \( \psi_m \). From the previous results in this section, the following relationships dictate optimal part yield in general, assuming \( \Pi \) is a quasi-convex function of \( \psi_k, k \in \mathbb{K} \):

\[
k \in \begin{cases} \mathbb{L}, & \text{if } C_k^N(\psi_m) > ((C_{k}^{p0} + C_{k}^{disp})\psi_m) \lambda \quad \text{and} \\ C_k^N(\psi^H) > [(C_{k}^{p} - S_k(\psi^H))\psi_m \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
4.5 Solution procedure for jointly determining EOL option and product design, $\psi_k \in [0, \psi^H]$, $k \in K$

From the results of Section 4.4 that describe how to determine optimal values of $\psi_k$, $k \in K$, we note that the values that $\psi_k^*$ can take depend on the nature of the problem. For the simple two-part case, we can present a plot of the potentially optimal yield values, depicted in Figure 4.8 for the case when $\frac{D_R}{\lambda_D} < \psi^H$ (ample core supply). Recall that if for Solution (3), $\Pi_3$ is not a quasi-convex function of $\psi_m$, then $\psi_m^* \in [0, \frac{D_R}{\lambda_D}]$. This case is denoted by “(3)-NQC”, otherwise, $\psi_m^* = \frac{D_R}{\lambda_D}$ for Solution (3) and is denoted by “(3)-QC”. For Solution (6), if the sign of $z$ is dependent on $f(\psi_k)$ and the selections of $S$ and $P$, then $\psi_m^* = [\frac{D_R}{\lambda_D}, \psi^H]$. We call this case as “(6)-z?” since the sign of $z$ is not known a priori. However, if $z > 0$ always, then $\psi_m^* = \frac{D_R}{\lambda_D}$, and if $z < 0$ always, then $\psi_m^* = 0$ and these cases are denoted by “(6)-z > 0” and “(6)-z < 0”, respectively. In case $\frac{D_R}{\lambda_D} > \psi^H$ (dearth of cores), only Solutions (1), (2) and (3) are possible.

Next, we describe a solution procedure for the more general case in which no restriction on the forms of trade-off function $f(\psi_k)$, $k \in K$ are placed other than that they must be monotone increasing functions of $\psi_k$. We call this as the general $f(\psi_k)$ case. Then, we discuss the case in which $f(\psi_k)$, $k \in K$ are linear functions, specifically $f(\psi_k) = \psi_k$. We call this as the linear $f(\psi_k)$ case.

4.5.1 General $f(\psi_k)$

In the most general case, the $f(\psi_k)$, $k \in K$ have no restriction on their form other than being monotone increasing.

We use the results from Section 4.4 to determine optimal conditions for various possible yields. Table 4.2 lists some relationships derived in Section 4.4 that we use to develop optimal yield conditions.

The conditions used to determine the optimal $\psi$ values are dependent on: (i) whether or not $\Pi_3$ is quasi-convex in $\psi_m$, (ii) the sign of $z$ for Solution (6), and (iii) whether or not $\frac{D_R}{\lambda_D} < \psi^H$. In some cases, we must solve an NLP to find the optimal $\psi$ for a given solution. Table 4.3 lists the conditions for the case when $\frac{D_R}{\lambda_D} < \psi^H$ for each limiting part yield.
Figure 4.8: Potentially optimal solutions for $\psi$ and their corresponding EOL option solutions.

An overall solution method is presented in Figure 4.9 for determining optimal yield if the solution is not Solution (1) or Solution (2). Essentially, the characteristics of the problem are checked to see which conditions apply in determining optimal $\psi$. First, if there is a dearth of cores ($\frac{D^R}{\lambda D^N} \geq \psi^H$), then only Solution (3) is possible and the conditions for optimal $\psi$ are based on whether $\Pi_3$ is a quasi-convex function of $\psi_m$ or not. Else, if there is ample core supply ($\frac{D^R}{\lambda D^N} < \psi^H$), then if $\Pi_3$ is not a quasi-convex function of $\psi_m$ we must solve $PD - Sol3$; otherwise, Solution (6) dominates Solution (3). Either way, Solution (6) is considered and the sign of $z$ determines which conditions are used to find optimal $\psi$. The best of either Solution (3) or Solution (6) is selected.
Table 4.2: Some key relationships to determine optimal $\psi$.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Relationship</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$C_k^N(\psi^H) &gt; (S_k(\psi^H) + C_k^{\text{disp}})\psi^H \lambda$</td>
<td>$\psi_k^* = 0$ for Solution (2).</td>
</tr>
<tr>
<td>B</td>
<td>$C_k(\frac{D^R}{\lambda D^H}) &gt; (C_k + C_k^{\text{disp}})\frac{D^R}{\lambda D^H} \lambda$</td>
<td>$\psi_k = 0$ is better than $\psi_k = \frac{D^R}{\lambda D^H}$ for Solution (3)-QC or Solution (6)-z $&gt; 0$.</td>
</tr>
<tr>
<td>C</td>
<td>$C_k^N(\psi^H) &gt; [C_k^0 \frac{D^R}{\lambda D^H} + S_k(\psi^H)(\psi^H - \frac{D^R}{\lambda D^H}) + C_k^{\text{disp}} \psi^H] \lambda$</td>
<td>$\psi_k = 0$ is better than $\psi_k = \psi^H$ for Solution (3)-QC or Solution (6)-z $&gt; 0$.</td>
</tr>
<tr>
<td>D</td>
<td>$C_k^N(\psi^H) - C_k(\frac{D^R}{\lambda D^H}) &gt; (S_k(\psi^H) + C_k^{\text{disp}})(\psi^H - \frac{D^R}{\lambda D^H})$</td>
<td>$\psi_k = \frac{D^R}{\lambda D^H}$ is better than $\psi_k = \psi^H$ for Solution (3)-QC or Solution (6)-z $&gt; 0$.</td>
</tr>
<tr>
<td>E</td>
<td>$C_k^N(\psi^H) &gt; (C_k^0 + C_k^{\text{disp}})\psi^H \lambda$</td>
<td>If this relationship holds, then $\psi_k^* = 0$ for Solution (3)-QC, $\frac{D^R}{\lambda D^H} \geq \psi^H$, or Solution (6)-z $&lt; 0$.</td>
</tr>
</tbody>
</table>

Table 4.3: Conditions from Table 4.2 required to determine optimal yield values for general $f(\psi_k)$ and $\frac{D^R}{\lambda D^H} < \psi^H$.

<table>
<thead>
<tr>
<th>$\psi_m$</th>
<th>Solution</th>
<th>0</th>
<th>$\psi_k^*$</th>
<th>1</th>
<th>$\psi_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0, \frac{D^R}{\lambda D^H}$</td>
<td>(3)-NQC</td>
<td></td>
<td>Determined by solution to PD-Sol3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{D^R}{\lambda D^H}$</td>
<td>(3)-QC or (6)-z $&gt; 0$</td>
<td>Conditions B and C</td>
<td>Condition D and C do not hold</td>
<td>o/w</td>
<td></td>
</tr>
<tr>
<td>$[\frac{D^R}{\lambda D^H}, \psi^H]$</td>
<td>(6)-z $&lt; 0$</td>
<td>Condition E</td>
<td>o/w</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: Conditions from Table 4.2 required to determine optimal yield values for general $f(\psi_k)$ and $\frac{D^R}{\lambda D^H} > \psi^H$.

<table>
<thead>
<tr>
<th>$\psi_m$</th>
<th>Solution</th>
<th>0</th>
<th>$\psi_k^*$</th>
<th>$\psi^H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(2)</td>
<td></td>
<td>Condition A</td>
<td>o/w</td>
</tr>
<tr>
<td>$\psi^H$</td>
<td>(3)</td>
<td></td>
<td>Condition E</td>
<td>o/w</td>
</tr>
</tbody>
</table>
4.5.2 Linear $f(\psi_k)$

In this case, $f(\psi_k)$ is linear; specifically, $f(\psi_k) = \psi_k$. Thus, $C_N^k(\psi_k) = \delta_N^k \psi_k$, $C_p^k(\psi_k) = C_p^0 + \delta_p^k \psi_k$, and $S_k(\psi_k) = S_k^0 + \delta_S^k \psi_k$. The linearizing of $f(\psi_k)$ has the following implications.

1. $\Pi$ is a convex function of $\psi_k$ for $k \in K$, but $k \neq m$ where $m$ is the central part. This result may be verified by inspection of $\Pi_{0,m}$ and $\Pi_{m,m+1}$ for $f(\psi_k)$ linear. It reveals that $\Pi_{0,m}$ and $\Pi_{m,m+1}$ are now convex, quadratic functions of $\psi_k$, and as such, quasi-convex in $\psi_k$.

2. $\Pi_3$ is a linear function of $\psi_m$, where $m$ is a central part. This result may be verified by inspection of Equation 4.36 for linear $f(\psi_m)$. Thus, $\psi_m^* = 0$ or $\min(\frac{D_p}{\lambda D_N}, \psi^H)$. However, $\psi_m^*$ cannot be 0, otherwise we no longer have Solution (3), and so, $\psi_m^* = \min(\frac{D_p}{\lambda D_N}, \psi^H)$.

3. However, the sign of $z$ in Solution (6) is still determined by the selection of $S$ and $P$, which are influenced by choice of $\psi_m$.

The results from Table 4.2 are still valid for linear $f(\psi_k)$, and hence, we may substitute the linear functions into the conditions of Table 4.2. For example, for linear $f(\psi_k)$ Condition A from Table 4.2 may be written as

$$\delta_N^k \psi^H > (S_k^0 + \delta_S^k \psi^H + C_k^{disp}) \psi^H \lambda.$$ (4.95)

Dividing through by $\psi^H$, Condition A for linear $f(\psi_k)$ becomes

$$\delta_N^k > (S_k^0 + \delta_S^k \psi^H + C_k^{disp}) \lambda.$$ (4.96)

A similar operation may be performed on each condition in Table 4.2. Note that Conditions A and B are identical to Conditions D and E, respectively for linear $f(\psi_k)$. The new conditions for linear $f(\psi_k)$ are given in Table 4.5. We denote $(S_k(\psi^H) + C_k^{disp}) \lambda$ by $\delta_S^k$, $(C_p^0 + C_k^{disp}) \lambda$ by $\delta_p^k$ and $\delta_N^k > [\delta_p^k \frac{D_p}{\lambda D_N}/\psi^H + \delta_S^k (1 - \frac{D_p}{\lambda D_N}/\psi^H)] \lambda$ by $\delta_G^k$. 

Table 4.5: Some key relationships to determine optimal $\psi$.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Shorthand Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A’</td>
<td>$\delta^N_k &gt; (S_k(\psi^H) + C^\text{disp}_k)\lambda$</td>
</tr>
<tr>
<td>B’</td>
<td>$\delta^N_k &gt; (C^{p_0} + C^\text{disp}_k)\lambda$</td>
</tr>
<tr>
<td>C’</td>
<td>$\delta^N_k &gt; [(C^{p_0}_k + C^\text{disp}_k)\frac{p^H}{\Delta D^N} / \psi^H$</td>
</tr>
<tr>
<td></td>
<td>$+ (S_k(\psi^H) + C^\text{disp}_k)(1 - \frac{p^H}{\Delta D^N / \psi^H})] \lambda$</td>
</tr>
</tbody>
</table>

Note, now, that for $\frac{D^R}{\Delta D^N} < \psi^H$ (ample core supply), by only finding optimal $\psi$ for EOL Solutions 1, 2 and 6, we consider every potentially optimal product design and EOL option since Solution 3 in which $\psi^*_m = \frac{p^R}{\Delta D^N}$ is considered by Solution 6. However, for $\frac{D^R}{\Delta D^N} > \psi^H$ only Solutions 1, 2, and 3, are possible.

By the value of $\frac{D^R}{\Delta D^N}$ and the value of $\psi_m$, we establish optimal EOL option and the optimal yields of all parts. Table 4.6 gives the optimal EOL options and optimal part yield conditions.

For $\frac{D^R}{\Delta D^N} > \psi^H$, Table 4.7 gives the optimal EOL options and part yield conditions.

Using the relationships in Tables 4.6 and 4.7, a solution procedure can be developed. This procedure is depicted in Figure 4.10.

From this procedure, we can generate a mapping of product characteristics onto optimal part yields. In case $\frac{D^R}{\Delta D^N} < \psi^H$ and $z > 0$, the mapping is depicted in Figure 4.11. If $\frac{D^R}{\Delta D^N} < \psi^H$, but the sign of $z$ depends on determination of $S$ and $P$, then the mapping is depicted in 4.12. Finally, if $\frac{D^R}{\Delta D^N} \geq \psi^H$ or $z < 0$, then the mapping is depicted in Figure 4.13.

### 4.6 Example

Consider a product with the part characteristics given in Table 4.8 and its other characteristics given in Table 4.9. Since $\frac{D^R}{\Delta D^N} = 0.37037$, Solution 3 and Solution 6 must be compared. Let $\psi^H = .8$.

**Solution 3**

Assume $\Pi_3$ is not a quasi-convex function of $\psi_m$. Using the preprocessing Conditions (4.58), (4.59), and (4.62), for this example we may find the optimal value of $\psi_k$, $k = 1, 2, 3$, for Solution 3.
Table 4.6: Conditions for optimal $\psi^*_k$ when $\frac{1}{\lambda D_N} < \psi^H$ (ample core supply).

<table>
<thead>
<tr>
<th>$\psi^H$</th>
<th>$\psi^*$</th>
<th>$\psi^H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_N$</td>
<td>$\lambda$</td>
<td>$\delta_k$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$D_N$</td>
<td>$\delta_k$</td>
</tr>
<tr>
<td>$\delta_k$</td>
<td>$\psi^H$</td>
<td>$\delta_k$</td>
</tr>
<tr>
<td>$\delta_k$</td>
<td>$\psi^H$</td>
<td>$\delta_k$</td>
</tr>
<tr>
<td>$\delta_k$</td>
<td>$\psi^H$</td>
<td>$\delta_k$</td>
</tr>
</tbody>
</table>

Solution to PD-Sol6

$\delta_k > \delta_k^S$
Table 4.7: Conditions for optimal $\psi_k^*$ when $\frac{\partial \pi}{\partial \psi_k} > 0$ (dearth of core supply).

<table>
<thead>
<tr>
<th>$\psi_k$</th>
<th>$\psi^H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_m$</td>
<td>Solution</td>
</tr>
<tr>
<td>0</td>
<td>&quot;(2)&quot;</td>
</tr>
<tr>
<td>$\psi^H$</td>
<td>&quot;(3)&quot;</td>
</tr>
</tbody>
</table>

**Part 1**

From the Conditions (4.58), (4.59), we may show that $\psi_1 = 0$ optimally for Solution 3. To verify that Condition (4.58) holds, consider Figure 4.14, which gives a plot of $C^N(\psi_m)$ and $(C^{p0}_1 + C^{disp}_k)\psi_m \lambda$ versus $\psi_m$. Secondly, since $C^{p0}_1 > S_1(\psi^H)$, we may use Condition 4.61 to verify Condition (4.59). Note that $C^N_1(\psi^H) = 372.358$ and $((C^{p0}_1 - S_1(\psi^H)) < 158$, and thus, Condition (4.59) holds.

**Part 2**

Similarly, for Part 2, we can verify that $\psi_2 = \psi_m$ optimally for Solution 3 by checking that Condition (4.58) does not hold and Condition (4.62) does hold. That these conditions hold can be seen in Figures 4.15 (Condition (4.58) does not hold) and 4.16 (Condition (4.62) does hold).

**Part 3**

We may verify that $\psi_3 = \psi^H$ for Solution 3 by showing that Condition (4.62) does not hold (see Figure 4.17) and that Condition (4.59) does not hold. That Condition (4.59) does not hold can be verified by checking Condition (4.63) since $C^{p0}_3 > S_3(\psi^H)$.

By setting $\psi_1 = 0$, $\psi_2 = \psi_m$ and $\psi_3 = .8$, we may maximize $\Pi_3$ over $\psi_m$. The plot of the contribution of Part 2 to the profit is given in Figure 4.18. We find that the solution is the upper boundary value of $\psi_2 = \frac{\partial \pi}{\lambda D^N} = 0.37037$. This leads to a loss of 125735.
Solution 6

Using similar preprocessing to that for Solution 3, we find that the optimal yields for Solution 6 are \( \psi_1 = 0, \psi_2 = \psi_m, \psi_3 = \psi^H \). \( \Pi_6 \) is a quasi-convex function of \( \psi_m \) and is, in fact, decreasing in \( \psi_m \) (see Figure 4.19). We have, \( \psi_m = \frac{\lambda^R}{\lambda D^N} \). Thus, \( \psi_1 = 0, \psi_2 = 0.37037, \psi_3 = 0.8 \). Therefore, the solution is the same as Solution 3.

Solution 2

To determine the optimum yields for Solution (2), we need only apply the results of Proposition 4.4.1. Checking this relation each \( k \) gives \( \psi^*_1 = 0, \psi^*_2 = 0, \) and \( \psi_1 = 0.8 \). This solution generates a loss of 644640.

Since all options generate a negative profit, the best option is to not do any leasing, which gives a profit of 0.

Table 4.8: Part Characteristics for Example.

<table>
<thead>
<tr>
<th>Part (k)</th>
<th>( C_p^0 )</th>
<th>( S_k(\psi^H) )</th>
<th>( C^{disp}_k )</th>
<th>( \delta^N_k )</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>250</td>
<td>100</td>
<td>50</td>
<td>400</td>
<td>0.5</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>275</td>
<td>25</td>
<td>200</td>
<td>300</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>250</td>
<td>200</td>
<td>100</td>
<td>200</td>
<td>0.1</td>
<td>15</td>
</tr>
</tbody>
</table>

Table 4.9: Other product characteristics for Example.

\[
\begin{array}{|c|c|}
\hline
C^c & 20 \\
C^d & 50 \\
\rho^N & 350 \\
\rho^R & 225.3 \\
D^R & 1000 \\
D^N & 3000 \\
\lambda & 0.9 \\
\hline
\end{array}
\]

4.7 Jointly determining optimal EOL option and product design: general case of \( \psi_k \in [0, \psi_k^H], k \in \mathbb{K} \)

In the special case in which \( \psi_k \in [0, \psi_k^H] \ \forall k \in \mathbb{K} \), results for how to determine optimal part yield were determined for each EOL option, but the optimal solution always satisfied Equation (4.94). However, in general, each \( \psi_k, k \in \mathbb{K} \) may
have a unique range between 0 and $\psi^H_k$, where $\psi^H_k \in [0, \psi^H]$. In this case, there may be a case in which $\psi_m > \psi^H_k$ for one or more $k \in K$, and there will be a deficit of these parts no matter how their yield is selected. Let $H$ now be defined as the set of parts $k$ such that $\psi_k = \psi^H_k$ and $\psi_k \geq \psi_m$. Parts in $H$ will be those with highest possible quality and are in surplus. Let $H^d$ be defined as the set of parts $k$ such that $\psi_k = \psi^H_k$ and $\psi_k < \psi_m$. Parts in $H^d$ will be those with highest possible quality, but are still in deficit. The general conditions for optimal part quality choice are as follows:

If $\psi^H_k \geq \psi_m$, then

$$
\begin{cases}
    L, & \text{if } C^N_k(\psi_m) > ((C^p_k + C_k^{disp})\psi_m)\lambda \\
    G, & \text{if } C^N_k(\psi_m) < (C^p_k + C_k^{disp})\psi_m\lambda \\
    H, & \text{otherwise}
\end{cases}
$$

(4.97)

If $\psi^H_k < \psi_m$ then we need only consider the conditions comparing profit for $\psi_k = 0$ and $\psi^H_k$, both of which result in part deficits.

If $\psi^H_k < \psi_m$ then

$$
\begin{cases}
    L, & \text{if } C^N_k(\psi^H_k) \geq (C^p_k \psi_m - C_k^{disp}(\psi_m - \psi^H_k) + C_k^{disp}\psi^H_k)\lambda \\
    H^d, & \text{otherwise}
\end{cases}
$$

(4.98)

To actually find the optimal part yields, we use the same process developed in Section 4.4 and Section 4.5, except we must include consideration of the new part quality category $H^d$ and allow the upper limit part yields to vary for each part. We’ll discuss the case where we assume $\Pi$ is not necessarily a quasi-convex function of $\psi_m$. For Solutions 1, 2, all results apply with $\psi^H_k$ substituted for $\psi^H$. However, for the other solutions we must consider the new part subset, $H^d$. 

Chapter 4. Product Design for RSC: Choice of Part Quality

Daniel W. Steeneck
Solution 3

In this general case, \( \Pi_3 \) is given by Equation (4.35) with \( \psi^H \) replaced by \( \psi^H_k \). Additionally, we may rewrite \( \Pi_3 \) (Equation (4.41)) with the term

\[
\sum_{k \in H_d} \left( -C_k^N(\psi^H_k)D^N - C_k^p(\psi^H_k)(\psi_m - \psi^H_k)\lambda D^N - C_k^{disp}(1 - \psi^H_k)\lambda D^N \right)
\]

(4.99)

added to reflect the new possible subset of part yields, \( \mathbb{H}_d \). Note that if \( \psi^H_k < \psi_m \), then we need only consider the conditions comparing profit of \( \Pi_3 \) for \( \psi_k = 0 \) and \( \psi^H_k \), both of which result in part deficits. For \( \psi_k = 0 \) to be better than \( \psi_k = \psi^H_k \), we have that

\[
-C_k^{p0} \psi_m \lambda D^N - C_k^{disp} \lambda D^N \geq -C_k^N(\psi^H_k)D^N - C_k^p(\psi^H_k)(\psi_m - \psi^H_k)\lambda D^N - C_k^{disp}(1 - \psi^H_k)\lambda D^N.
\]

(4.100)

Rearranging terms we obtain

\[
C_k^N(\psi^H_k) \geq (C_k^{p0} \psi_m - C_k^p(\psi_m - \psi^H_k) + C_k^{disp} \psi^H_k)\lambda D^N.
\]

(4.101)

For Solution 3, we solve the following nonlinear programming problem to determine optimal yields. Let \( h^d = 1 \) if \( k \in \mathbb{H}_d \), and 0, otherwise.
For Solution 4, each part is either in $S$ or $P$. Since $Q^R = D^R$, $\psi_k \geq \frac{D^R}{\lambda D^N}$ for $k \in S$, or $\psi_k \leq \frac{D^R}{\lambda D^N}$ for $k \in P$. Since we assume $\Pi_4$ to be a quasi-convex function of $\psi_k$, $\psi_k^H$ may only be 0, $\frac{D^R}{\lambda D^N}$, or $\psi_k^H$. We may use the conditions from

\[
\max_{\psi_m, l_k, o_k, h_k, h_k^d} \left( (\mathscr{P} - C^N) D^N + (C^C + C^D) \lambda D^N + \rho^R \psi_m \lambda D^N \right) \\
+ \sum_{k \in \mathbb{K}} \left[ -C^H \psi_m \lambda D^N - C^disp \lambda D^N \right] l_k \\
+ \sum_{k \in \mathbb{K}} \left[ -C^N (\psi^H_k) D^N - C^p (\psi^H_k) (\psi_m - \psi^H_k) \lambda D^N - C^disp (1 - \psi^H_k) \lambda D^N \right] h_k^d \\
+ \sum_{k \in \mathbb{K}} \left[ -C^N (\psi_m) D^N - C^disp (1 - \psi_m) \lambda D^N \right] o_k \\
+ \sum_{k \in \mathbb{K}} \left[ S_k (\psi^H_k) (\psi^H_k \lambda D^N - \psi_m \lambda D^N) - C^N (\psi^H_k) D^N - C^disp (1 - \psi^H_k) \lambda D^N \right] h_k
\]

subject to:

\[
0 \leq \psi_m \leq \min\left\{ \frac{D^R}{\lambda D^N}, \max\{\psi^H_1, \ldots, \psi^H_K\} \right\}, \quad (4.103)
\]
\[
\psi^H_k h^d_k \leq \psi_m, \quad \forall \; k \in \mathbb{K}, \quad (4.104)
\]
\[
l_k + o_k + h_k + h_k^d = 1 \quad \forall \; k \in \mathbb{K}, \quad (4.105)
\]
\[
l_k, o_k, h_k, h_k^d \text{ Binary } \forall \; k \in \mathbb{K}. \quad (4.106)
\]

The problem is identical to one for the special case in which all parts have the same greatest possible yield, except for the addition of the term $\sum_{k \in \mathbb{K}} \left[ -C^N (\psi^H_k) D^N - C^p (\psi^H_k) (\psi_m - \psi^H_k) \lambda D^N - C^disp (1 - \psi^H_k) \lambda D^N \right] h_k^d$ to the objective function to account for the possibility of parts in $\mathbb{P}^d$, and modification of the constraints to reflect the inclusion of $h_k^d$ as a decision variable and (i) the fact that $\psi_m$ must be less than the highest possible yield among the part (Constraint (4.103)), and (ii) that by the definition of $\mathbb{P}^d$, $\psi_m$ must be greater than $\psi^H_k$ for any Part $k \in \mathbb{P}^d$ (Constraint (4.104)).

For a fixed value of $\psi_m$, we again may simply compare the terms of $\Pi_3$ to determine the subset to which a given part belongs. For this reason, we relax the binary constrain on $g_k, o_k, h_k$ and $h_k^d$ as they will be 0 or 1 in the optimal solution.

**Solution 4**

For Solution 4, each part is either in $S$ or $P$. Since $Q^R = D^R$, $\psi_k \geq \frac{D^R}{\lambda D^N}$ for $k \in S$, or $\psi_k \leq \frac{D^R}{\lambda D^N}$ for $k \in P$. Since we assume $\Pi_4$ to be a quasi-convex function of $\psi_k$, $\psi_k^H$ may only be 0, $\frac{D^R}{\lambda D^N}$, or $\psi_k^H$. We may use the conditions from
Equations (4.97) and (4.97) with $\psi_m = \frac{D_R}{\lambda D^N}$ to determine optimal yields.

**Solution 5**

For Solution 5, each part is in $S$, and since $Q^R = D^R$, $\psi_k \geq \frac{D_R}{\lambda D^N}$ $\forall k \in K$. We may use the conditions from Equation (4.97) with $\psi_m = \frac{D_R}{\lambda D^N}$ to determine optimal yields.

**Solution 6**

A similar analysis may be performed for Solution 6, and we solve the following nonlinear programming problem that considers the new part quality category $H^d$ and allows the upper limit part yields to vary for each part to determine optimal yields.

\[
\begin{align*}
\max_{\psi_m, l_k, o_k, h_k, k \in K} & \quad \left( \left( \frac{\beta^N - C^H}{\lambda} - (C^e + C^d) \right) \frac{D_R}{\psi_m} \right) \\
& + \sum_{k \in K} \left( -C^d_{d^0} \frac{D_R}{\psi_m} - C^d_{d^0} D_R \right) l_k \\
& + \sum_{k \in K} \left( -C^N_k (\psi^H_m) \frac{D_R}{\lambda \psi_m} - C^N_k (\psi^H_m)(D^R - \frac{\psi^H_m D_R}{\psi_m}) - C^d_{d^0} (1 - \psi^H_m) \frac{D_R}{\psi_m} \right) h^d_k \\
& + \sum_{k \in K} \left( -C^N_k (\psi^H_m) \frac{D_R}{\lambda \psi_m} + S_k (\psi^H_m)(\frac{D_R}{\psi_m} - D^R) \psi^H + C^d_{d^0} (1 - \psi^H_m) \frac{D_R}{\psi_m} \right) h_k \\
\text{subject to:} & \\
\frac{D_R}{\lambda D^N} &\leq \psi_m \leq \max \{ \psi^H_1, \ldots, \psi^H_K \}, \quad (4.108) \\
\psi^H_k h^d_k &\leq \psi_m, \quad \forall \ k \in K, \quad (4.109) \\
l_k + o_k + h_k + h^d_k & = 1 \quad \forall \ k \in K, \quad (4.110) \\
\sum_{k \in K} o_k &\geq 1, \quad (4.111) \\
l_k, o_k, h_k, h^d_k &\text{ Binary } \forall \ k \in K. \quad (4.112)
\end{align*}
\]
The problem is identical to the one for the special case in which all parts have the same greatest possible yield, except that we add the term \( \sum_{k \in K} \left( -C_k^N \left( \psi_k^H \right) \frac{D^R}{\psi_m} - C_k^P \left( \psi_k^H \right) (D^R - \psi_k^H \frac{D^R}{\psi_m}) - C_k^{disp} \left( 1 - \psi_k^H \right) \frac{D^R}{\psi_m} \right) h_k^d \) to the objective value to account for the possibility of having parts in \( H^d \), and the constraints are modified to reflect the inclusion of \( h_k^d \) as a decision variable, and (i) that \( \psi_m \) must be less than the highest possible yield among the part (Constraint (4.108)), and (ii) that, by the definition of \( H^d \), \( \psi_m \) must be greater than \( \psi_k^H \) for any Part \( k \in H^d \) (Constraint (4.109)). For a fixed value of \( \psi_m \), we again may simply compare the terms of \( \Pi_3 \) to determine the subset to which a given part belongs. For this reason, we relax the binary constrain on \( g_k, o_k, h_k \) and \( h_k^d \) as they will be 0 or 1 in the optimal solution.

### 4.8 A note about solving the problem when \( \Pi \) is not a quasi-convex function of \( \psi_k \)

If \( \Pi \) is not a quasi-convex function of \( \psi_k \), then the results in this Chapter do not directly apply. Although \( \Pi \) will always be quasi-convex for \( \psi_k \geq \psi_m \) (part salvage), the same cannot be said of \( \Pi \) for \( \psi_k < \psi_m \) (part replacement). See Figure 4.4 for any example in which \( \Pi \) is a concave function of \( \psi_k < \psi_m \). Thus, we must now solve for \( \psi_k \) and \( \psi_m \) in Solutions 3, 4, and 6. There are still only 3 or 4 possibly optimal modes of \( \psi_k \) for the special and general case, respectively. In this case, let \( l_k = 1 \) if \( \psi_k < \psi_m \), and 0, otherwise. In order to solve the problem, we must make \( \psi_k \) a decision variable and add the constraint

\[
\psi_k l_k \leq \psi_m \; \forall k \in K. \tag{4.113}
\]

Note that Constraint 4.113 defines a non-convex feasible region. There are up to \( 2^K \) possible configuration of \( l_k, k \in K \) and so this is a very difficult problem.

Note that a feasible lower bound profit may be obtained by simply using a linear approximation of \( \Pi \) for \( \psi_k < \psi_m \). In this case \( \Pi \) will be a quasi-convex function of \( \psi_k \) and the methods of this Chapter apply.
Figure 4.9: Procedure for determining optimal yield for Solution 3 and Solution 6 for general $f(\psi_k)$. 

\[
\frac{D^R}{\lambda DN} > 1 \\
\Pi_3 \text{ QC in } \psi_m \\
\text{Solution 6 dominates Solution 3} \\
z < 0 \\
z > 0 \\
\text{Solution } 3 \text{ QC in } \psi_m \\
\Pi_3 \text{ QC in } \psi_m \\
\text{Solve PD-Sol3 and retain solution} \\
\psi_k^* = 0 \\
\psi_k^* = \psi_H \\
\psi_k^* = 0 \\
\psi_k^* = \psi_H \\
\psi_k^* = \psi_H \\
\psi_k^* = \psi_H \\
\text{Select the best solution} \\
\text{Condition } E \\
\text{Condition } B \\
\text{Condition } C \\
\text{Condition } D \\
\psi_k^* = \frac{D^R}{\lambda DN} \\
\psi_k^* = \psi_H
Figure 4.10: Procedure for determining optimal yield for Solution 3 and Solution 6.
Figure 4.11: Mapping of product characteristics onto optimal product design for \( \frac{D^R}{\lambda D^N} \leq \psi^H \) and \( z > 0 \).

Figure 4.12: Mapping of product characteristics onto optimal product design for \( \frac{D^R}{\lambda D^N} \leq \psi^H \), \( z \) unknown.
Figure 4.13: Mapping of product characteristics onto optimal product design for $\frac{D^p}{\lambda D^N} > \psi^H$ or $z < 0$.

Figure 4.14: Equation 4.58 holds for all $\psi_m$ for Part 1.

Figure 4.15: Equation 4.58 does not hold for any $\psi_m$ for Part 2.
Figure 4.16: Equation 4.62 does hold for any $\psi_m$ for Part 2.

Figure 4.17: Equation 4.62 does not hold for all $\psi_m$ for Part 3.

Figure 4.18: Contribution to the profit of Part 2 for various values of $\psi_2$ for Solution 3.

Figure 4.19: Contribution to the profit of Part 2 for various values of $\psi_2$ for Solution 6.
Chapter 5

Product Pricing for the RSC

So far, we have explored two issues in strategic planning for the RSC: (i) determination of an optimal EOL option based on product characteristics, and (ii) an optimal product design in terms of choice of quality for each part. In this chapter, we turn our attention to the pricing of new and remanufactured products, which may or may not compete with each other. We study this problem in the context of consumer ownership of purchased products and choice of collection by the manufacturer, as opposed to leasing, however, the case of mandated product recovery may be studied in a similar manner.

An important function in our discussion of optimal pricing is the demand function, $D^v(\mathcal{P}^v)$, $v = N, R$, which represents the quantity demanded of a new or remanufactured product at a given price for the product. Typically, the quantity demanded for a product is a non-increasing convex function of its own price and a non-decreasing function of the price of competing products. We assume that the demand function is linear and its form depends on whether new and remanufactured products compete. In the case that new and remanufactured products do not compete, we assume the demand function for a product type to be given by

$$D^v(\mathcal{P}^v) = D^{v0} - \alpha \mathcal{P}^v, \quad v = R, N \quad (5.1)$$
where $\alpha$ and $D^0_v \geq 0$. $\alpha$ is the price elasticity of the product and $D^0_v$ is the potential market size for product type $v$. Figure 5.1 depicts a plot of a linear demand function. However, the demand function may be multivariate if the new and remanufactured products compete. We make the following assumptions about competition of new and remanufactured products:

(i) $D^R = 0$ if $P^R > P^N$,

(ii) An increment in $P^R (P^N)$ causes $D^N (D^R)$ to increase, and

(iii) An increment in $P^R (P^N)$ causes $D^R (D^N)$ to decrease.

Assumption (i) reflects the fact that customers prefer new products to remanufactured products; thus, if a remanufactured product is more expensive than a new product, there will be no quantity of remanufactured products demanded. Assumption (ii) reflects that because the products are in competition, they are substitutes, and an increase in the price of one, results in an increase in the demand of the other. Assumption (iii) just states that as the price of a product increases, smaller quantity is demanded by the market. To reflect these assumptions about the nature of customer demand, we use the following demand equation in case new and remanufactured products compete in a given stage:

\[
D^N = \begin{cases} 
D^N_0 - \alpha P^N + \gamma (P^R - P^N) & \text{for } P^R < P^N, \\
D^N_0 - \alpha P^N, & \text{otherwise,}
\end{cases}
\] (5.2)

\[
D^R = \begin{cases} 
D^R_0 - \alpha P^R + \gamma (P^N - P^R) & \text{for } P^R < P^N, \\
D^R_0 - \alpha P^R, & \text{otherwise.}
\end{cases}
\] (5.3)

$\gamma$ represents the sensitivity of the quantity demanded to the price difference between new and remanufactured products. Note that in the case that new and remanufactured products compete, as in Stage 2, the demand for new products is greatest when $P^R = P^N$, in which case the expression for $D^N$ is the same as when new and remanufactured products do not compete. In this case, new products essentially do not compete with remanufactured products.
We assume product prices to be such that $P \geq 0$ and $D_i(P) \geq 0, i = 1, 2$. Let

$$\mathcal{P}_i = \{ P : D_i(P) = 0 \}, i = 1, 2. \tag{5.4}$$

The subscript is only included if demand functions vary by stage.

Although demand is typically a non-increasing function of price, some luxury goods do not follow this pattern and their quantity demanded actually increases with increment in price beyond certain price levels. This has to do with the customers’ perception that a high-priced product is of high quality. Perhaps, the stigma associated with remanufactured products could be removed if new and remanufactured product were not priced differently.

The prices of new and remanufactured products determine the quantities demanded of each. We consider the following two situations:
i) **The new and remanufactured products do not compete.** Specifically, new products are made at Stage 1 and returned and remanufactured at Stage 2. Figure 2.5 from Chapter 2 depicts this situation. Note that this is the same situation addressed in Chapter 3.

ii) **The new and remanufactured products do compete.** Specifically, new products are made at Stage 1, and returned and remanufactured in Stage 2; however, new products are produced at Stage 2 as well. Prices are invariant between periods. Figure 5.2 depicts this situation.

![Figure 5.2: Two-stage model with new and remanufactured product competition.](image)

In either case, the optimal prices of the new and remanufactured products are dependent decisions since the quantity sold of the new product supplies the demand of the remanufactured product. However, in the second case, these decisions are even more interwoven since the two products compete. Thus, the price of one product directly impacts the quantity demanded for the other. Both situations are studied under monopoly.

Some of the results presented in Chapter 3 are price-independent and help in determining an optimal EOL option a priori. We discuss the impact of these price-independent relationships and present and discuss a mappings of pricing pairs onto optimal EOL types and options in Section 5.1. For each potential EOL option, we give closed-form expressions for optimal prices for various functional forms of the price versus demand functions. In Section 5.2, we address the case in which there is no competition between new and remanufactured products. The case of competition between new and remanufactured products is presented in Section 5.3. In Section 5.4, we discuss the broad differences between the noncompetitive and competitive cases.
5.1 Mapping of product prices onto EOL options

Recall the mapping of product characteristics onto EOL types (for the two-part case, see Figure 3.19). Note that certain decisions are price-dependent and others are not. Decisions that are dependent on price include:

a) Salvage vs. remanufacture \( (\rho_R \leq \sum_{k \in K} S_k) \),

b) No collection vs. collection \( (\rho^K \psi_m + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m) + \sum_{k=m+1}^{K} C^p_k (\psi_m - \psi_k) \leq \theta, \forall m \in K) \), and

c) Determination of key part \( (m \text{ key part if } \sum_{k=1}^{m} S_k + \sum_{k=m+1}^{K} C^p_k \leq \rho^R \leq \sum_{k=1}^{m-1} S_k + \sum_{k=m+1}^{K} C^p_k). \)

Price-independent decisions include:

a) Salvage vs. no salvage \( (\sum_{k \in K} S_k \psi_k \geq \theta) \), and

b) Limiting part \( (\sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C^p_k \psi_k \leq \theta) \).

For the two-part case, Figure 5.3 depicts a mapping of the product characteristics onto “price independent” properties of the product.

There are three such regions:

1. **Salvage profitable.** It is beneficial to collect all the cores in order to obtain parts for salvage, regardless of whether remanufacturing occurs or not.

2. **Salvage not profitable, Part 2 limiting:** Cores are collected to obtain enough of Part 2 to meet remanufactured product demand.

3. **Salvage not profitable, Part 1 limiting:** Cores are collected to obtain enough of Part 1 to meet remanufactured product demand.

A mapping of product pricing onto EOL type and option may be generated, and it depends on the region of which a product is a member. For example, let \( S_1 \psi_1 + S_2 \psi_2 \geq \theta \), i.e., the product is in Region 1. In this case, the EOL
types include: (1) salvage for \( \rho_R \leq S_1 + S_2 + C_R \), (2) remanufacturing and salvage, Part 2 key for \( S_1 + S_2 + C_R \leq \rho_R \leq S_1 + C_{P_2} + C_R \), and (3) remanufacturing and salvage, Part 1 key for \( S_1 + C_{P_2} + C_R \leq \rho_R \). Figure 5.4 depicts an example mapping of this case. Note that the mapping in Figure 5.4 does not consider those cases in which the new product price is less than the remanufactured product price. This assumption is valid for the case in which new and remanufactured products compete since there will be no demand for remanufactured products if \( \rho_N < \rho_R \), i.e., new products are preferred over remanufactured products. However, in the case that new and remanufactured products do not compete, the restriction that \( \rho_N > \rho_R \) is not necessary.

We may include demand information to determine the specific EOL option for a product. The nature of the mapping of prices onto EOL option depends on whether or not there is competition between new and remanufactured products. We discuss each case separately in Section 5.2 for the noncompetitive case, and in Section 5.3 for the competitive case.
5.2 Optimal pricing: no competition at second stage

The model for this case is almost identical to the model CCFDFP presented in Chapter 3 except that we add the revenue from profit back to the objective function, and also, include the constraint $D_N < D_N^*$. Additionally, $D_N^*$ and $D_R^*$ are now functions of $P_N$ and $P_R$, respectively. For convenience, the model is given here:

$$
\Pi = (P_N - C_N) Q_N + \rho^R Q_R + \sum_{k=1}^{K} S_k \left( \max \left( \psi_k C_k - Q_R, 0 \right) \right) \\
- \sum_{k=1}^{K} C_k^p \left( \max \left( Q_R - \psi_k C_k, 0 \right) \right) - \theta \xi
$$

subject to:

$$
0 \leq P_N \leq P_N^{N0}, \quad \text{(5.6)}
$$

$$
0 \leq P_R \leq P_R^{R0}, \quad \text{(5.7)}
$$

(3.2), (3.3), (3.4), and (3.8).

Constraints (5.6) and (5.7) restrict the range of $P_N$ and $P_R$, respectively. Since $\Pi$ is a linear function of $Q_N$, we
assume $Q = D$ since, otherwise, we do not manufacture. Additionally, note that if the demand functions are linear, the objective function is a concave, quadratic function of $P$ and $P$. The optimal prices for new and remanufactured products may be determined jointly with optimal EOL option. First, the price-independent relationships are used to isolate which EOL options are potentially optimal. Second, each potential EOL option is optimized over product prices, and the best one is selected. For brevity, we denote $D_{P} = D$, $P_{N} = P_{N}$, and $P_{R} = P_{R}$. Note that $P_{N} = D_{P} = D_{P}$. Using the mapping presented in Section 5.1 and demand information, we may map prices on optimal EOL options. By using some price independent relationships, the limiting part may be determined, defining some important properties of $\Pi$ that are true for any $(P_{N}, P_{R})$. Then, based on the nature of the product demand and certain critical values of $P_{R}$ that define boundaries between choice of key part, the optimal regions for each EOL option may be determined.

When the availability of cores matches the number of cores required to obtain $D$ of Part $m$, $\lambda_{m} = \frac{D_{P}}{V_{k}}, k \in \mathbb{K}$, and it defines a boundary between EOL options. However, the value of $\lambda_{m}$ and $D$ are functions of $P_{N}$ and $P_{R}$, respectively. Thus, $\lambda_{m} = \frac{D_{P}}{V_{k}}, k \in \mathbb{K}$ is true for a specific set of pricing pairs. Let $\mu_{k} = \{(P_{N}, P_{R}) : \lambda_{m} = \frac{D_{P}}{V_{k}}\}, k \in \mathbb{K}$. The set $\mu_{k}$ bisects the space $P_{N} \times P_{R}$ such that above $\mu_{k}$ there is an insufficient supply of cores to obtain $D$ of Part $k$ and below $\mu_{k}$ there is a surplus supply of cores.

Let

$$\rho_{m} = \sum_{k=1}^{m} S_{k} + \sum_{k=m+1}^{K} C_{k} = 1, \ldots, K. \quad (5.8)$$

Recall from Section 3.1.3 that $\rho_{m}$ defines the boundary value of $\rho_{R}$, above which some Part $1, \ldots, m$ is the key part. Additionally, if Part $m$ is the limiting part, then $\rho_{R} \leq \rho_{m+1}$ implies that $\Pi_{0,m+1} \leq 0$.

**Proposition 5.2.1** If $\Pi_{m,m+1} \leq 0$, then $\Pi_{0,m} \leq 0$ for $P_{R} - C_{R} \leq \rho_{m}$.

**Proof** Assume

$$(P_{R} - C_{R})_{m+1} + \sum_{k=1}^{m} S_{k}(\psi_{k} - \psi_{m+1}) - \sum_{k=m+2}^{K} C_{k}(\psi_{m+1} - \psi_{k}) - \theta \leq 0. \quad (5.9)$$
This may be rewritten as

\[
(P_R - C_R)\psi_{m+1} - \sum_{k=1}^{m} S_k \psi_{m+1} - \sum_{k=m+2}^{K} C_p^k \psi_{m+1} + \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+2}^{K} C_p^k \psi_k - \theta \leq 0. \tag{5.10}
\]

Adding \(C_{m+1}^p \psi_{m+1} - C_{m+1}^p \psi_{m+1}\) to the left hand side and combining terms yields:

\[
\left(\left((P_R - C_R) - \sum_{k=1}^{m} S_k - \sum_{k=m+1}^{K} C_p^k \psi_{m+1} + \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_p^k \psi_k \right) - \theta \leq 0 \right) - \left(\sum_{k=1}^{m} S_k \psi_k + \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_p^k \psi_k \right) \leq \rho^R_m - \inf \Pi'_{m,m+1}. \tag{5.11}
\]

By assumption, \(\sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_p^k \psi_k \leq 0\) and \((P_R - C_R) \leq \rho^R_m\) so Equation (5.11) is true. \(\square\)

Using the price independent relationships presented in Section 5.1, and Proposition 5.2.1 we may determine a priori certain properties of \(\Pi_{max}\) for any \((P_N, P_R)\), and generate a mapping of pricing onto EOL option:

1. If \(\sum_{k=1}^{K} S_k \psi_k - \theta > 0\), then there will be no limiting part, i.e., the product is of Salvage EOL type. Figure 5.5 depicts the mapping of pricing onto optimal EOL option, in this case, for a three-part product.

2. If \(\sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_p^k \psi_k - \theta \geq 0\) and \(\sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_p^k \psi_k \leq 0\), then Part \(m\) is limiting, \(m = 2, \ldots, K\). Remanufacturing is not optimal for \(P_R \leq \rho^R_m + C_R\). Figure 5.6 depicts the mapping of pricing onto optimal EOL option, in this case, for a three-part product.

3. If \(S_1 \psi_1 + \sum_{k=2}^{K} C_p^k \psi_k - \theta \leq 0\), then Part 1 is limiting for \(P_R \geq \rho^R_1 + C_R\). Figure 5.7 depicts the mapping of pricing onto optimal EOL option, in this case, for a three-part product.

An interesting property of these mappings is that \(\Pi_{max}\) as a function of \(P_N\) and \(P_R\) in the EOL option regions in which remanufacturing occurs, is a concave function.

**Proposition 5.2.2** In the case of no competition between new and remanufactured products, \(\Pi_{max}\) is a quasi-concave function of \(P_N\) and \(P_R\) for \(P_R \geq \sum_{k=1}^{K} S_k + C_R\).

**Proof** Note that remanufacturing is optimal for \(P_R \geq \sum_{k=1}^{K} S_k + C_R\).
Figure 5.5: Feasible regions for EOL options, $\sum_{k=1}^{K} S_k \psi_k - \theta > 0$, no competition.

Let $\bar{\Pi}_{\text{max}, 1}, \ldots, \bar{\Pi}_{\text{max}, N}$ be the segments of $\bar{\Pi}_{\text{max}}$, ordered from left to right with respect to $\mathcal{R}$. Let

$$\bar{\Pi}_{\text{max}, \text{EOL}, i} = \bar{\Pi}_{\text{max}, i} - (\mathcal{R}^N - C^N) \Theta^N, \ i = 1, \ldots, N,$$

(5.12)

i.e. $\bar{\Pi}_{\text{max}, \text{EOL}, i}$ is the contribution to the profit from the EOL stage. Recall that $\bar{\Pi}_{\text{max}, \text{EOL}, i}$ may either be $\Pi_{0,m}$ for some $m = 1, \ldots, K$, or $\Pi_{m,m+1}$ for $m = 1, \ldots, K$. Taking the partial derivative of $\Pi_{0,m}$ and $\Pi_{m,m-1}$ with respect to $\mathcal{R}^N$ we obtain:

$$\frac{\partial \Pi_{0,m}}{\partial \mathcal{R}^N} = -\alpha \lambda (\mathcal{R}^R - C^R) \psi_m + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m) - \sum_{k=m+1}^{K} C_k^R (\psi_m - \psi_k) - \theta$$

(5.13)
Figure 5.6: Feasible regions for EOL options, Part 2 limiting, no competition.

\[ \frac{\partial \Pi_m}{\partial \rho_N} = -\alpha \lambda \left( \sum_{k=1}^{m} S_k \psi_k + \sum_{k=m+1}^{K} C_k \psi_k - \theta \right), \]  
\[ (5.14) \]

which is the same as \(-\alpha \lambda \Pi_{0,m}^\prime\) and \(-\alpha \lambda \Pi_{m,m+1}^\prime\), respectively.

Note that an increment in \(\rho_N\) corresponds to a decrement in \(\lambda D_N\) (the core availability). Thus, the segments of \(\bar{\Pi}_{max,EOL}(R^N)\) are in reverse order to \(\Pi_{EOL,max}(C)\).

Since, for \(\Pi_{max,EOL}(C)\)

\[ \partial \Pi_{max,EOL,1}^\prime(C) \geq \Pi_{max,EOL,2}^\prime(C) \geq \cdots \geq \Pi_{max,EOL,N}^\prime, \]  
\[ (5.15) \]
we have

\[-\alpha \lambda \frac{\partial \Pi_{\text{max},\text{EOL},N}}{\partial \mathcal{R}^N} \geq -\alpha \lambda \frac{\partial \Pi_{\text{max},\text{EOL},N-1}(\mathcal{R}^N)}{\partial \mathcal{R}^N} \geq \cdots \geq -\alpha \lambda \frac{\partial \Pi_{\text{max},\text{EOL},1}(\mathcal{R}^N)}{\partial \mathcal{R}^N}.\] (5.16)

And, thus, $\Pi_{\text{max},\text{EOL}}(\mathcal{R}^N)$ is concave. Note that $(\mathcal{R}^N - C^N)(D^{N0} - \alpha \mathcal{R}^N)$ is a negative quadratic function and, as such, is concave. Therefore,

$$\Pi_{\text{max}}(\mathcal{R}^N) = \Pi_{\text{max},\text{EOL}} + (\mathcal{R}^N - C^N)(D^{N0} - \alpha \mathcal{R}^N)$$ (5.17)

is concave.

In case $Q^R > 0$, both salvage and no collection are not valid options. To show $\Pi_{\text{max}}(\mathcal{R}^R)$ is quasi-concave for

Figure 5.7: Feasible regions for EOL options, Part 1 limiting, no competition.
we show that $\bar{\Pi}_{\text{max}}(\mathcal{R})$ is initially an increasing function of $\mathcal{R}$, and afterwards, it is concave function of $\mathcal{R}$. For $\mathcal{R} < \sum_{k=1}^{K} S_k + C^{RA}$, part salvage is optimal, so we need not consider this case. Note that, the value of $\mathcal{R}$ determines which segment of $\bar{\Pi}$ is optimal for a given $\mathcal{R}^N$. Let $\bar{\Pi}^*$ be the optimal segment of $\bar{\Pi}$ for some $\mathcal{R}$. We have

$$\bar{\Pi}^* = \begin{cases} 
\Pi_{0,m} & \text{for } \rho^R_m \leq \mathcal{R}^R - C^R \leq \rho^R_{m-1} \text{ and } \lambda D^N \leq \frac{D^R}{\psi_m}, \\
\Pi_{m,m+1} & \text{for } \mathcal{R}^R - C^R \text{ and } \frac{D^R}{\psi_m} \leq \lambda D^N \leq \frac{D^R}{\psi_{m+1}}.
\end{cases}$$

(5.18)

Thus, as $\mathcal{R}$ is increased, then it is possible for $\bar{\Pi}^*$ to change from:

(i) $\bar{\Pi}_{0,m}$ to $\bar{\Pi}_{0,m-1}$,

(ii) $\bar{\Pi}_{0,m}$ to $\bar{\Pi}_{m-1,m}$,

(iii) $\bar{\Pi}_{0,m}$ to $\bar{\Pi}_{m,m+1}$, and

(iv) $\bar{\Pi}_{m-1,m}$ to $\bar{\Pi}_{m,m+1}$.

To show that $\bar{\Pi}^*$ is a quasi-concave function of $\mathcal{R}$, we show that $\bar{\Pi}^*$ is increasing for low values of $\mathcal{R}$, and then, is a concave function of $\mathcal{R}^R$ for larger values. First we show the nature of $\partial \bar{\Pi}^*/\partial \mathcal{R}$ as it transitions between objective functions. Consider the partial derivatives of the possible objective functions with respect to $\mathcal{R}$:

$$\frac{\partial \Pi_{0,m}}{\partial \mathcal{R}} = \psi_m \lambda D^N$$

(5.19)

and

$$\frac{\partial \Pi_{m,m+1}}{\partial \mathcal{R}} = -2\alpha \mathcal{R} + D^R + \alpha(C^R + \rho^R_m).$$

(5.20)

Consider case (i) in which increasing $\mathcal{R}$ implies that $\bar{\Pi}^*$ changes from $\bar{\Pi}_{0,m}$ to $\bar{\Pi}_{0,m-1}$. Note that, this would occur
at \( P_R = \rho_m + C^R \). Assume

\[
\frac{\partial \Pi_{0,m}}{\partial P_R} \leq \frac{\partial \Pi_{0,m-1}}{\partial P_R}.
\]  

(5.21)

Equivalently,

\[
\psi_m \lambda D^N \leq \psi_{m-1} \lambda D^N;
\]  

(5.22)

which is true, since \( \psi_m \leq \psi_{m-1} \).

Consider case (ii) in which increasing \( P_R \) implies that \( \bar{\Pi}^* \) changes from \( \bar{\Pi}_{0,m} \) to \( \bar{\Pi}_{m-1,m} \). Note that this would occur at \( P_R = \rho_m + C^R \). At this transition value of \( P_R \), assume

\[
\frac{\partial \Pi_{0,m}}{\partial P_R} \geq \frac{\partial \Pi_{m-1,m}}{\partial P_R}.
\]  

(5.23)

Equivalently,

\[
\psi_m \lambda D^N \geq -2\alpha P_R + D^{R0} + \alpha(C^R + \rho_{m-1}^R).
\]  

(5.24)

Replacing \( P_R \) with \( \rho_m + C^R \) and rearranging, terms we obtain

\[
\rho_{m-1}^R + C^R \geq \frac{D^{R0} - \psi_m \lambda D^N}{\alpha},
\]  

(5.25)

which may or may not be true, depending on the problem parameters.

Consider case (iii) in which increasing \( P_R \) implies that \( \bar{\Pi}^* \) changes from \( \bar{\Pi}_{0,m} \) to \( \bar{\Pi}_{m,m+1} \). Note that this would occur at \( P_R = \frac{D^{R0} - \psi_m \lambda D^N}{\alpha} \). At this transition value of \( P_R \), assume

\[
\frac{\partial \Pi_{0,m}}{\partial P_R} \geq \frac{\partial \Pi_{m,m+1}}{\partial P_R}.
\]  

(5.26)

Equivalently,

\[
\psi_m \lambda D^N \geq -2\alpha P_R + D^{R0} + \alpha(C^R + \rho_m^R).
\]  

(5.27)
Replacing $\mathcal{R}^R$ with $\frac{D^{R0} - \psi_0 \lambda D^N}{\alpha}$ and rearranging terms, we obtain

$$\frac{D^{R0}}{\alpha} \geq \rho^R_m + C^R,$$  

(5.28)

which is true since we consider $\mathcal{R}^R = \rho^R_m + C^R$ to be feasible (recall that $\mathcal{R}^R \leq \frac{D^{R0}}{\alpha}$).

Finally, consider case (iv) in which increasing $\mathcal{R}^R$ implies that $\bar{\Pi}^*$ changes from $\bar{\Pi}_{m-1,m}$ to $\bar{\Pi}_{m,m+1}$. Note that this would occur at $\mathcal{R}^R = \frac{D^{R0} - \psi_0 \lambda D^N}{\alpha}$. At this transition value of $\mathcal{R}^R$, assume

$$\frac{\partial \Pi_{m-1,m}}{\partial \mathcal{R}^R} \geq \frac{\partial \Pi_{m,m+1}}{\partial \mathcal{R}^R}.$$  

(5.29)

Equivalently,

$$-2\alpha \mathcal{R}^R + D^{R0} + \alpha(C^R + \rho^R_{m-1}) \geq -2\alpha \mathcal{R}^R + D^{R0} + \alpha(C^R + \rho^R_m).$$  

(5.30)

Rearranging terms we obtain

$$\rho^R_{m-1} \geq \rho^R_m,$$  

(5.31)

which we already know to be true.

Note that for $\mathcal{R}^R$ such that $\bar{\Pi}^* = \bar{\Pi}_{0,m}$, $m = 1, \ldots, K$, $\bar{\Pi}^*$ is an increasing function of $\mathcal{R}^R$. However, for $\mathcal{R}^R$ such that $\bar{\Pi}^* = \bar{\Pi}_{m-1,m}$, $m = 2, \ldots, K$, $\bar{\Pi}^*$ is a concave function of $\mathcal{R}^R$. Thus, $\bar{\Pi}^*$, in general is an increasing and then concave function of $\mathcal{R}^R$ and, thus, is a quasi-concave function of $\mathcal{R}^R$. \qed

Let the region of $P^N \times P^R$ space such that $P^R - C^R \geq \sum_{k=1}^{K} S_k$ and Constraints (5.6) and (5.7) are not violated be denoted as $X_R$, or the remanufacture region. Using the fact that $\Pi_{\text{max}}$ is concave over $X_R$, the optimal EOL option and price may be determined in a very straightforward manner. Let $X_r$ be the region of $P^N \times P^R$ over which remanufacturing EOL option $r$ is optimal. Let $g_i$ be a real-valued linear function defined over $P^N \times P^R$ for which $g_i = 0$ defines a boundary between two adjacent $X_r$'s. Let $h_i$ be a real-valued linear function defined over $P^N \times P^R$ for which $h_i = 0$ defines a boundary of $X_R$. 
Let \( \Pi_r \) be the profit function of remanufacturing EOL option \( r \). Let \( P_r' \) be the unconstrained optimal solution to \( \Pi_r \), and let \( P^R_{r*} \) be the optimal solution over \( X_R \).

Consider adjacent regions \( X_a \) and \( X_b \), and let \( g_1 = 0 \) define the boundary between them. Without loss of generality, for region \( X_a \), \( g_1 \leq 0 \), and for region \( X_b \), \( g_1 \geq 0 \). We say that boundary \( g_1 = 0 \) is violated if either \( g_1 > 0 \) for \( P_a' \) or \( g_1 < 0 \) for \( P_b' \). The boundary \( g_1 = 0 \) is doubly violated if both \( P_a' \) and \( P_b' \) violate \( g_1 = 0 \).

Proposition 5.2.3 Assume \( X_r \) such that \( g_1 \leq 0 \). \( P_r' \) violates \( g_1 = 0 \), if and only if \( P^R_{r*} \in (P^N, P^R) : g_1 > 0 \).

Proof The proof of the if part follows by contrapositive argument: If \( P^R_{r*} \in (P^N, P^R) : g_1 \leq 0 \), then \( P_r' \) does not violate \( g_1 = 0 \). If \( P^R_{r*} \in (P^N, P^R) : g_1 \leq 0 \) and since \( \Pi_{max} = \Pi_r \) is a concave function over \( X_R \), then improving directions for points on \( g_1 = 0 \) towards the region \( g_1 < 0 \). Thus, \( P_r' \) does not violate \( g_1 = 0 \).

The proof of the only if part follows by contrapositive argument: If \( P_r' \) does not violate \( g_1 = 0 \), then \( P^R_{r*} \in (P^N, P^R) : g_1 \leq 0 \).

If \( P_r' \) does not violate \( g_1 = 0 \), the improving directions of points on \( g_1 = 0 \) for \( \Pi_{max} \) are toward points in \( (P^N, P^R) \in g_1 \leq 0 \) since \( \Pi_{max} \) is concave and \( \Pi_{max} = \Pi_j \in X_r \). Thus \( P^R_{r*} \in (P^N, P^R) : g_1 \leq 0 \). \( \square \)

Proposition 5.2.4 \( g_i = 0 \) is a doubly violated boundary, if and only if \( P^R_{r*} \) lies on \( g_i = 0 \).

Proof The proof of the if part follows by contrapositive argument: If the optimal solution is not on \( g_i = 0 \), then \( g_i = 0 \) is not a doubly violated constraint. Consider two adjacent regions \( X_a \) and \( X_b \) with \( g_i = 0 \) as their boundary. By assumption, \( P^R_{r*} \) is not on \( g_i = 0 \). Let \( \mathcal{G} \) be the set of boundaries violated by \( P'_a \). There are two cases: (i) \( g_i = 0 \in \mathcal{G} \) and (ii) \( g_i = 0 \notin \mathcal{G} \).

(i) \( g_i = 0 \in \mathcal{G} \)

The points on \( g_i = 0 \) are also shared between \( X_a \) and \( X_b \) and are known to be suboptimal. For these points, we know that moving in the direction of \( X_a \) is suboptimal. Thus, improving directions must be in the direction of \( g_i \geq 0 \) and so \( P'_b \) will not violate \( g_i = 0 \).
(ii) \( g_i = 0 \in G \)

In this case, \( g_i = 0 \) is not violated by \( P'_a \) and thus, cannot be violated twice.

The proof of the only if part follows by contrapositive argument: If \( g_i = 0 \) is not doubly violated, then \( P^R^* \) is not on \( g_i = 0 \). There are two cases: (i) \( g_i = 0 \) is not violated, and (ii) \( g_i = 0 \) is violated once.

(i.) \( g_i = 0 \) is not violated

In this case the contrapositive argument of the if part of Proposition 5.2.3 applies.

(ii.) \( g_i = 0 \) is violated only once

Let \( g_i = 0 \) define the boundary between \( X_a \) and \( X_b \), and let \( P'_a \) violate \( g_i = 0 \), but \( P'_b \) does not violate \( g_i = 0 \).

By the if part of Proposition 5.2.3, we know \( P^R^* \in (P^N, P^R) : g_i \geq 0 \) since \( P'_a \) violates \( g_i = 0 \). Also, by the contrapositive argument of the only if part of Proposition 5.2.3, we know that since \( P'_b \) does not violate \( g_i = 0 \), \( P^R^* \in (P^N, P^R) : g_i \geq 0 \).

\( \square \)

**Proposition 5.2.5** \( P^R^* \) satisfies one of the following conditions:

(i) \( P^R^* = P'_r \) if \( P'_r \) does not violate any boundaries of \( X_r \),

(ii) \( P^R^* \in g_i = 0 \) if \( g_i \) is a doubly violated constraint, or

(iii) \( P^R^* \in h_i = 0 \), otherwise.

Case (i) holds clearly since \( \Pi_{\text{max}} \) is a concave function of \( (P^N, P^R) \). Case (ii) is proven in Proposition 5.2.4. Otherwise, there are no doubly violated boundaries or unconstrained optimal solutions that satisfy a region’s constraints, and so the solution must be on some boundary \( h_i \).

Using Proposition 5.2.5, we may specify the following algorithm for finding the optimal product price and EOL option. Let \( R \) be the set of EOL options, \( g'_r \) be the set of boundaries between \( X_r \)’s violated by \( P'_r \), and \( h'_r \) be the set of boundaries of \( X_R \) violated by \( P'_r \).
0. Select an arbitrary remanufacturing EOL option, $r$.

1. Find $P'_r$. Determine $g^r_v$ and $h^r_v$. Let $G^v = G^v \cup g^r_v$.

2. If $g^r_v$ and $h^r_v$ are empty, then $P^* = P'_r$.

3. If $G^v$ contains a doubly violated boundary, say $g$, then $P^*$ may be found by fixing $(P^N, P^R) : g = 0$ and solving the constrained problem for EOL option $r$.

4. If $g^r_v$ is empty, but $h^r_v$ is not, then the optimal solution lies on some $h \in h^r_v$. For each $h \in h^r_v$ fix $(P^N, P^R) : h = 0$ and solve the constrained problem for EOL option $r$. $P^*$ is given by the best of these solutions.

5. Otherwise, $R = R - r$ and $r_0 = r$. Select a new $r$ such that $X_r$ is adjacent to $X_{r_0}$ and they share some boundary in $g^r_{r_0}$. Go to step 1.

Next, the optimization problem for determining optimal $P^N$ and $P^R$ for each EOL option is presented along with their unconstrained optimal solutions and some properties of their optimal solution.

**Solution (1)**

In the case of Solution (1) from Table 3.2, we do not collect any cores. To determine optimal product prices, we solve the following optimization problem:

$$\max \Pi_1 = (P^N - C^N)D^N$$

subject to: (5.6).

Since $D^N$ is linear, we have the objective function,

$$\Pi_1 = (P^N - C^N)(- \alpha^N P^N + D^N).$$
Let \( P_{N}^{*} \) be the optimal unconstrained \( P_{N} \). \( P_{N}^{*} \) is given by

\[
P_{N}^{*} = \frac{P_{N}^{0} + C_{N}}{2}.
\] (5.34)

Note that \( P_{N}^{*} \geq 0 \).

Due to Constraint (5.6), \( P_{N}^{*} \) is given by

\[
P_{N}^{*} = \begin{cases} 
P_{N}^{*} & \text{if } P_{N}^{*} \leq P_{N}^{0}, \\ 
P_{N}^{0} & \text{otherwise.} \end{cases}
\] (5.35)

**Solution (2)**

In the case of Solution (2) from Table 3.2, all cores are collected and the parts are salvaged. To determine the optimal product prices, we solve the following optimization problem:

\[
\max_{P_{N}} \Pi_{2} = (P_{N} - C_{N})D_{N} + \sum_{k \in K} S_{k} \psi_{k} \lambda D_{N} - \theta \lambda D_{N}
\] (5.36)

subject to:

\[
0 \leq P_{N} \leq P_{N}^{0},
\] (5.37)

\[
\sum_{k=1}^{K} S_{k} \psi_{k} - \theta \geq 0.
\] (5.38)

Replacing \( D_{N} \) with its expression and rearranging terms in the objection function, we obtain

\[
\Pi_{2} = (P_{N} - C_{N} + \sum_{k \in K} S_{k} \psi_{k} \lambda - \theta)(D_{N}^{0} - \alpha_{N} P_{N}^{0}).
\] (5.39)
If $\mathcal{R}^N$ is unconstrained, then optimal $\mathcal{R}^N$ is given by

$$
\mathcal{R}^N = \frac{\mathcal{R}^N_0 + C^N - \sum_{k \in \mathbb{K}} S_k \psi_k \lambda + \theta \lambda}{2}.
$$

(5.40)

$\mathcal{R}^N$ satisfies

$$
\mathcal{R}^N = \begin{cases} 
0 & \text{if } \mathcal{R}^N < 0, \\
\mathcal{R}^N_0 & \text{if } \mathcal{R}^N > \mathcal{R}^N_0, \\
\mathcal{R}^N' & \text{o/w.}
\end{cases}
$$

(5.41)

Solution (3), central Part $m$

In this case, core availability is insufficient to acquire $D^R$ of Part $m$.

$$
\max \Pi_3 = (\mathcal{R}^N - C^N)D^N + (\mathcal{R}^R - C^R)\psi_m \lambda D^N
\]

$$
+ \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m) \lambda D^N - \sum_{k=m+1}^{K} C^p_k (\psi_m - \psi_k) \lambda D^N - \theta \lambda D^N
$$

subject to:

$$
0 \leq \mathcal{R}^N \leq \mathcal{R}^N_0, 
$$

(5.43)

$$
0 \leq \mathcal{R}^R \leq \mathcal{R}^R_0, 
$$

(5.44)

$$
\lambda D^N \leq \frac{D^R}{\psi_m}, 
$$

(5.45)

$$
\rho^R_m \leq \mathcal{R}^R - C^{ra} \leq \rho^R_{m-1}.
$$

(5.46)

Observe that $\Pi_3$ is an increasing function of $\mathcal{R}^R$. Thus, the unconstrained optimal solution is unbounded. Additionally, in regards to the constrained problem, for a given $\mathcal{R}^N$, it is optimal if to raise $\mathcal{R}^R$ until either until $D^R$ drops to $\psi_m \lambda D^N$ (Constraint (5.45) is tight, or $\mathcal{R}^R - C^{ra} = \sum_{k=1}^{m-1} S_k + \sum_{k=m}^{K} C^p_k$, above which Solution 3, for central part $m$ is no longer optimal for the problem overall. Thus, $\lambda D^N = \frac{D^R}{\psi_m}$ or $\mathcal{R}^R - C^{ra} = \sum_{k=1}^{m-1} S_k + \sum_{k=m}^{K} C^p_k$ in the optimal solution.
Solution 3, central Part m. Case 1: \( \lambda D^N = \frac{D^R}{\psi_m} \)

We may write \( P^R \) in terms of \( P^N \) by substituting the expressions for \( D^N \) and \( D^R \) into \( \lambda D^N \leq \frac{D^R}{\psi_m} \) and rearranging terms.

\[
\hat{P}_m^R = \frac{D^{R0} - \lambda \psi_m(D^{N0} - \alpha P^N)}{\alpha}. \tag{5.47}
\]

Replacing \( P^R \) by \( \hat{P}_m^R \) and rearranging terms, we obtain the following problem:

\[
\begin{align*}
\max \ \Pi_3 &= ((\mathcal{P}^N - C^N)D^N + \frac{D^{R0} - \lambda \psi_m(D^{N0} - \alpha P^N)}{\alpha} - C^R)\lambda \psi_m + \sum_{k=1}^{m-1} S_k(\psi_k - \psi_m)\lambda \\
& \quad - \sum_{k=m+1}^{K} C_k^p(\psi_m - \psi_k)\lambda - \theta \lambda)(D^{N0} - \alpha \mathcal{P}^N) \tag{5.48}
\end{align*}
\]

subject to:

\[
\begin{align*}
0 & \leq \mathcal{P}^N \leq \mathcal{P}^{N0}, \tag{5.49} \\
\mathcal{P}^N & \geq \mathcal{P}^{N0} - \frac{D^{R0}}{\psi_m \lambda}, \tag{5.50} \\
\mathcal{P}^N & \leq \mathcal{P}^{N0} + \frac{D^{R0}}{\psi_m \lambda}, \tag{5.51} \\
\hat{P}_m^R - D^{R0} & \leq \mathcal{P}^N \leq \hat{P}_m^R - D^{R0}. \tag{5.52}
\end{align*}
\]

Constraints (5.50) and (5.51) result from \( \hat{P}_m^R \geq 0 \) and \( \hat{P}_m^R \leq \mathcal{P}^{R0} \), respectively. Additionally, Constraint (5.52) is a result of Constraint (5.46).

Note that \( \Pi_3 \) remains a concave function of \( \mathcal{P}^N \). Let \( \mathcal{P}^{N'} \) be the unconstrained solution.
\[
\mathcal{P}^N = -\alpha(-CN + \frac{-\lambda\psi_mD^{N_0} + DR^0}{\alpha} - CR) + \sum_{k=1}^{m-1} S_k(\psi_k - \psi_m)\lambda
\]

\[
- \sum_{k=m+1}^{K} C_k^p(\psi_m - \psi_k)\lambda - \theta\lambda - D^{N_0}(2(1 + (\lambda\psi_m)^2))^{-1}
\]

Because $\Pi_3$ with $P^R = \hat{P}^R_m$ is a concave, univariate function of $P^N$, we can use the value of $P^N'$ and the boundary value of the constraints to determine $P^N^*$. If $P^N'$ does not violate any constraints, $P^N^* = P^N'$ and $P^R^* = \hat{P}^R_m(P^N^*)$. However, $P^N'$ may violate some constraints. Let the constraints be indexed in some arbitrary manner from $i = 1, \ldots, I$, and let $\partial P^N_i$ be the boundary value of constraint $i$. Let $V$ be the set of constraints which are violate by $P^N'$. Let

\[
\partial P^N_{max} = \{ \partial P^N_i : |P^N' - \partial P^N_i| \geq |P^N' - \partial P^N_j| \forall i \in V, j \in V \}
\]

in other words, $\partial P^N_{max}$ is the violated constraint boundary value furthest from $P^N'$.

**Proposition 5.2.6** Let $f(x)$ be a univariate, concave function of $x$ and let $x \in S$, where $S$ a polyhedral set of constraints. Let $x'$ be the unconstrained solution to max $f(x)$. If $x'$ violates some set of constraints $V$, and $\partial x_i$ is the boundary value for some $i \in V$, then the optimal solution to the constrained problem is $x^* = \partial x_{max}$ where

\[
\partial x_{max} = \{ \partial x_i : |x' - \partial x_i| \geq |x' - \partial x_j| \forall i \in V, j \in V \}
\]

**Proof** Since $f(x)$ is a univariate, concave function of $x$, any movement of $x$ away from the unconstrained optimal value of $x, x'$, decreases $f(x)$. Thus, since $\partial x_{max}$ is the furthest violated constraint boundary away from $x', x^* = \partial x_{max}$. □

Based on Proposition 5.2.6, since $\Pi_3$ for $P^R = \hat{P}^R_m$ is a univariate, concave function of $P^N$, we know that $P^N^* = \partial P^N_{max}$ if $P^N'$ violates any constraints.
Solution 3, central Part \( m \) Case 2: \( \mathcal{P}^R - C^{ra} = \sum_{k=1}^{m-1} S_k + \sum_{k=m}^{K} C^p_k \)

We may substitute \( \mathcal{P}^R \) for \( \rho_{m-1} + C^R \) and rearrange terms to obtain the following problem:

\[
\max \Pi_3 = (\mathcal{P}^N - C^N) D^N + \left( \sum_{k=1}^{m-1} S_k \psi_k + \sum_{k=m}^{K} C^p_k \psi_k - \theta \right) \lambda D^N 
\]

subject to:

\[
0 \leq \mathcal{P}^N \leq \mathcal{P}^{N0}, \quad (5.57)
\]
\[
\lambda D^N \leq \frac{\mathcal{P}^{R0} - \alpha (\rho_{m-1} + C^R)}{\alpha}, \quad (5.58)
\]

Constraint (5.58) results from Constraint (5.45).

Note that \( \Pi_3 \) remains a concave function of \( \mathcal{P}^N \) since it will remain a negative quadratic function. Let \( \mathcal{P}^{N'} \) be the unconstrained solution.

\[
\mathcal{P}^{N'} = \frac{\mathcal{P}^{N0} + C^N - \sum_{k=1}^{m-1} S_k \psi_k - \sum_{k=m}^{K} C^p_k \psi_k + \theta}{2}. \quad (5.59)
\]

Again, since the problem is a univariate in \( \mathcal{P}^N \), and \( \Pi_3 \) is a concave function \( \mathcal{P}^N \), the optimal solution is (\( \mathcal{P}^{N*}, \mathcal{P}^R^* \)) = (\( \mathcal{P}^{N'}, \sum_{k=1}^{m-1} S_k \psi_k + \sum_{k=m}^{K} C^p_k \psi_k - \theta + C^R \)) if no constraints are violated, or (\( \mathcal{P}^{N*}, \mathcal{P}^R^* \)) = (\( \partial \mathcal{P}^{N*} \max, \sum_{k=1}^{m-1} S_k \psi_k + \sum_{k=m}^{K} C^p_k \psi_k - \theta + C^R \)) if constraints are violated.

Solution (4), central Part \( m \).

For Solution (4) with central Part \( m \) from Table 3.2, the availability of cores is sufficient to obtain \( D^R \) Part \( m \), but not of Part \( m - 1 \). We write the profit of Solution (4) as follows.
\[ \text{max } \Pi_4 = (P^N - C^N)D^N + (P^R - C^R)D^R \]
\[ + \sum_{k=1}^{m} S_k (D^R - \psi_m \lambda D^N) \]
\[ - \sum_{k=m+1}^{K} C^R_p (\psi_m \lambda D^N - D^R) \]
\[ - \theta \lambda D^N \]
\[ (5.60) \]

subject to:
\[ 0 \leq P^N' \leq P^N_0, \]
\[ 0 \leq P^R' \leq P^R_0, \]
\[ \frac{D^R}{\psi_m} \leq \lambda D^N \leq \frac{D^R}{\psi_{m+1}}, \]
\[ P^R - C^R \geq \rho^R_m. \]
\[ (5.61) \]
\[ (5.62) \]
\[ (5.63) \]
\[ (5.64) \]

Note that \( \Pi_4 \) is a concave function of \( P^N \) and \( P^R \) as it is a negative quadratic function of these variables. Constraint 5.63 requires that the supply of cores be sufficient to obtain \( D^R \) of Part \( m \), but in sufficient to obtain \( D^R \) of Part \( m+1 \). Constraint 5.64 enforces the optimality condition that Part \( k, k \leq m \) is key. The unconstrained optimal solution is given by

\[ P^N' = \frac{P^N_0 + C^N + \left( \sum_{k=1}^{m} S_k \psi_m + \sum_{k=m+1}^{K} C^R_p \psi_m + \theta \right) \lambda}{2}, \]
\[ (5.65) \]

and

\[ P^R' = \frac{P^R_0 + C^N - \rho^R_m}{2}. \]
\[ (5.66) \]

The constrained problem has a concave objective function over a convex feasible region and is easy to solve.
Solution (5)

This is the same as Solution (4) with central Part \( m = K \) (Recall that, by definition, \( \psi_{m+1} = 0 \) and so \( \lambda D^R \leq \infty \) for \( m = K \)).

Solution (6)

In this case, core supply must exceed that needed to meet remanufactured product demand using Part \( m \). The optimization problem is given as follows:

\[
\max \Pi_6 = (P^N - C^N)D^N + (P^R - C^R)D^R \\
+ \sum_{k=1}^{m-1} S_k (\psi_k \frac{D^R}{\psi_m} - D^R) - \sum_{k=m+1}^{K} C^p_k (D^R - \psi_k \frac{D^R}{\psi_m}) - \theta \frac{D^R}{\psi_m} 
\]

subject to:

\[
0 \leq P^N \leq P^{N0}, \quad (5.68)
\]

\[
0 \leq P^R \leq P^{R0}, \quad (5.69)
\]

\[
\lambda D^N \geq \frac{D^R}{\psi_m}, \quad (5.70)
\]

\[
P^R - C^{ra} \geq \rho^R_m. \quad (5.71)
\]

Note that \( \Pi_6 \) is a concave function of \( P^N \) and \( P^R \). Constraint 5.70 requires that the supply of cores be sufficient to obtain \( D^R \) of Part \( m \) Constraint 5.71 enforces the optimality condition that Part \( k, k \leq m \) is key. The unconstrained optimal solution is given by

\[
P^{N\prime} = \frac{P^{N0} + C^N}{2}, \quad (5.72)
\]

and

\[
P^{R\prime} = \frac{P^{R0} + C^R - \sum_{k=1}^{m-1} S_k (\psi_k \psi_m - 1) + \sum_{k=m+1}^{K} C^p_k (1 - \psi_k \psi_m) + \theta}{2}. \quad (5.73)
\]
The constrained problem has a concave objective function over a convex feasible region and is easy to solve.

5.3 Optimal pricing: competition at second stage

In some cases new and remanufactured products compete for sales. This is especially true for products with relatively short usage times compared to length of product demand. That is, used products will be returned while new and remanufactured product demand is still strong.

We study the following two-stage problem: at Stage 1, new products are produced, and at Stage 2, the new products produced at Stage 1 become available for collection. Additionally, at Stage 2, new and remanufactured products are produced, which compete for sales. We assume that new product price is constant between stages. There is no competition at Stage 1; however, new and remanufactured products compete at Stage 2. To reflect these assumptions about the nature of customer demand, we use the following demand equations:

\[ D_1^N = D_0^N - \alpha P_N, \]  
\[ D_2^N = \begin{cases} 
  D_0^N - \alpha P_N + \gamma (P_R - P_N) & \text{for } P_R < P_N, \\
  D_0^N - \alpha P_N & \text{otherwise,} 
\end{cases} \]  
\[ D_R = \begin{cases} 
  D_0^R - \alpha P_R + \gamma (P_N - P_R) & \text{for } P_R < P_N, \\
  0 & \text{otherwise.} 
\end{cases} \]  

The model formulation for the two-stage problem with new and remanufactured product competition is as follows:
Π = (\( P^N - C^N \)) Q_1^N + (\( P^N - C^N \)) Q_2^N + (\( P^R - C^R \)) Q^R + \sum_{k=1}^{K} S_k \left( \max \left( \psi_k \epsilon - Q^R, 0 \right) \right)

- \sum_{k=1}^{K} C_k^p \left( \max \left( Q^R - \psi_k \epsilon, 0 \right) \right) \theta \epsilon

subject to:

\( \epsilon \leq \lambda Q^N \),

(5.78)

0 \leq Q^N \leq P^{N0}

(5.79)

0 \leq Q^R \leq P^{R0}

(5.80)

P^R \leq P^N

(5.81)

Q_1^N, Q_2^N, and \( \epsilon \geq 0 \)

(5.82)

(3.3) and (3.4).

As before, we assume \( Q^N = D^N \); otherwise, we do not make any new products. Constraint (5.78) ensures that the quantity of cores collected is less than the number of cores available. Additionally, note that for Solutions 3, 4, 5, and 6, in which new and remanufactured products compete, \( P^{N0} = \frac{D^R}{\alpha} \), \( P^{N0} = \frac{D^{N0} + \gamma P^R}{\alpha + \gamma} \), and \( P^{R0} = \frac{D^{R0} + \gamma P^N}{\alpha + \gamma} \). For Solutions 1 and 2, new and remanufactured products do not compete since there are no remanufactured products made in the second stage, and therefore, \( P^{N0} = P^{N0} \).

**Proposition 5.3.1**

\( P^{N0} \leq \frac{D^{R0} + (\alpha + \gamma) D^{N0}}{\alpha^2 + 2\alpha \gamma} \), and

(5.83)

\( P^{R0} \leq \frac{D^{N0} + (\alpha + \gamma) D^{R0}}{\alpha^2 + 2\alpha \gamma} \).

(5.84)

The proof follows from finding the intersection of constraints \( P^N \leq P^{N0} \) and \( P^R \leq P^{R0} \) where \( P^{N0} = \frac{D^{N0} + \gamma P^R}{\alpha + \gamma} \) and \( P^{R0} = \frac{D^{R0}}{\alpha} \).
Proposition 5.3.2 \( \mathcal{P}_{2}^{N \emptyset} \leq \mathcal{P}_{1}^{R \emptyset} \).

Proof From Proposition 5.3.1, we know that the maximum value that \( \mathcal{P}_{2}^{N \emptyset} \) can take is \( \frac{D_{R \emptyset}^{0} \gamma + (\alpha + \gamma) D_{N \emptyset}^{0}}{\alpha^2 + 2\alpha \gamma} \). Thus, we only need to show that

\[
\frac{D_{R \emptyset}^{0} \gamma + (\alpha + \gamma) D_{N \emptyset}^{0}}{\alpha^2 + 2\alpha \gamma} \leq \frac{D_{N \emptyset}^{0}}{\alpha}.
\]

(5.85)

By rearranging terms, we obtain \( D_{R \emptyset}^{0} \leq D_{N \emptyset}^{0} \), which is true by assumption.

Thus, \( \mathcal{P}_{2}^{N \emptyset} \leq \mathcal{P}_{1}^{N \emptyset} \) in the feasible region and so the requirement \( \mathcal{P}_{N} \leq \mathcal{P}_{1}^{N \emptyset} \) is redundant to Constraint 5.79. Thus, Constraint 5.79 restricts the range of \( \mathcal{P}_{N} \). Constraint 5.80 restricts the range of \( \mathcal{P}_{R} \). The main differences in determining optimal prices in case there is competition between new and remanufactured products and the case in which there is no competition arise from the fact that: (i) \( \mathcal{P}_{N} \geq \mathcal{P}_{R} \), and (ii) the ranges of the values that \( \mathcal{P}_{N} \) and \( \mathcal{P}_{R} \) can take are dependent on each other. We shall see that \( \mathcal{P}_{N} = \mathcal{P}_{R} \) is a potentially optimal solution and must be considered.

Again based on the mapping presented in Section 5.1 and Proposition 5.2.1, we may determine a priori certain properties of \( \Pi_{\max} \) for any \( (\mathcal{P}_{N}, \mathcal{P}_{R}) \). However, in order to describe the solution space, we now must incorporate the Constraints (5.80) and (5.81). Figure 5.8 depicts the new form of the solution space with these constraints incorporated for a three-part product example with \( \sum_{k=1}^{3} S_k \psi_k - \theta \geq 0 \).

Proposition 5.3.3 In the case of competition between new and remanufactured products, \( \Pi_{\max} \) is a concave function of \( \mathcal{P}_{N} \) and \( \mathcal{P}_{R} \) for \( \mathcal{P}_{R} \geq \sum_{k=1}^{K} S_k + C_{R} \).

The proof to Proposition 5.3.3 is similar to that for Proposition 5.2.2. Proposition 5.3.3 allows the use of the same algorithm presented in Section 5.2.

Next, the optimization problems associated with each EOL option are presented.

Solution 1

In case of Solution (1) from Table 3.2, we do not collect any cores. To determine optimal product prices, we solve the
Figure 5.8: Example of solution-space regions for a three-part product with \( \sum_{k=1}^{3} S_k - \theta \geq 0 \) in the presence of competition.

following optimization problem:

\[
\max \Pi_1 = (\mathcal{R}^N - C^N)D_1^N + (\mathcal{R}^N - C^N)D_2^N
\]

subject to:

\[
0 \leq \mathcal{R}^N \leq \mathcal{R}_1^{N_0}.
\]

Since \( D_1^N = D_2^N \) as we produce no remanufactured product, we may substitute in the expressions for \( D_1^N \) (Equation (5.75)) and \( D_2^N \) (Equation (5.76)) into the objective function to obtain

\[
\Pi_1 = 2(\mathcal{R}^N - C^N)(D_1^{N_0} - \alpha \mathcal{R}^N),
\]
which is a concave function of $\mathcal{P}^N$ since it is a negative quadratic function of $\mathcal{P}^N$. Thus,

$$\mathcal{P}^N_1 = \frac{\mathcal{P}^N_0 + C^N}{2}. \tag{5.89}$$

Due to Constraint (5.87), $\mathcal{P}^N_*$ is given by Equation (5.35).

**Solution 2**

In the case of Solution (2) from Table 3.2, all cores are collected and the parts are salvaged. To determine the optimal product prices, we solve the following optimization problem:

$$\max \Pi_2 = (\mathcal{P}^N - C^N)D_1^N + (\mathcal{P}^N - C^N)D_2^N + \sum_{k \in K} S_k \psi_k \lambda D_1^N - \theta \lambda D_1^N \tag{5.90}$$

subject to:

$$0 \leq \mathcal{P}^N \leq \mathcal{P}^N_0. \tag{5.91}$$

Since $D_1^N = D_2^N$, we may substitute the expressions for $D_1^N$ (Equation (5.75)) and $D_2^N$ (Equation (5.76)) into the objective function to obtain

$$\Pi_2 = 2(\mathcal{P}^N - C^N)(D^{N0} - \alpha \mathcal{P}^N) + (\sum_{k \in K} S_k \psi_k - \theta) \lambda (D^{N0} - \alpha \mathcal{P}^N), \tag{5.92}$$

which is a concave function of $\mathcal{P}^N$ since it is a negative quadratic function of $\mathcal{P}^N$. Thus

$$\mathcal{P}^N_* = \frac{2 \mathcal{P}^{N0} + C^N - (\sum_{k \in K} S_k \psi_k - \theta) \lambda}{4}. \tag{5.93}$$

$\mathcal{P}^N_*$ satisfies Equation (5.41).
Solution 3 central Part m

In this case, core availability is insufficient to acquire \(D^R\) of Part \(m\) and, thus, \(\lambda D^N \leq \frac{D^R}{\psi_m}\). The optimization problem is given by

\[
\begin{align*}
\max \Pi_3 &= (\mathcal{NP}^N - C^N)D^N + (\mathcal{NP}^N - C^N)D^N_2 + (\mathcal{NP}^R - C^R)\psi_m \lambda D^N_1 + \sum_{k=1}^{m-1} S_k (\psi_k - \psi_m) \lambda D^N_1 \\
&- \sum_{k=m+1}^K C_k (\psi_m - \psi_k) \lambda D^N_1 - \theta \lambda D^N_1
\end{align*}
\]

subject to:

\[
\begin{align*}
0 &\leq \mathcal{NP}^N \leq \mathcal{NP}^{N0}, \\
\mathcal{NP}^R &\leq \mathcal{NP}^{R0}, \\
\lambda D^N_1 &\leq \frac{D^R}{\psi_m}, \\
0 &\leq \mathcal{NP}^R \leq \mathcal{NP}^N, \\
\rho^R_m &\leq \mathcal{NP}^R - C^R \leq \rho_{m-1}.
\end{align*}
\]  

(5.94)  

(5.95)  

(5.96)  

(5.97)  

(5.98)  

(5.99)  

(5.100)

Firstly, note that, only terms \((\mathcal{NP}^R - C^R)\psi_m \lambda D^N_1\) and \((\mathcal{NP}^N - C^N)D^N_2\) are functions of \(\mathcal{NP}^R\). Also, as long as \(\mathcal{NP}^R - C^R > 0\), which we assume (otherwise, Solution 3 is not optimal), \((\mathcal{NP}^R - C^R)\psi_m \lambda D^N_1\) is increasing in \(\mathcal{NP}^R\). Additionally, in term \((\mathcal{NP}^N - C^N)D^N_2\), \(D^N_2\) is an increasing function of \(\mathcal{NP}^R\). Thus, we will increase \(\mathcal{NP}^R\) as much as possible. The optimal \(\mathcal{NP}^{R*}\) may take the following four possible values based on Constraints (5.97)-(5.100).

(i) \(\mathcal{NP}^{R*} = \mathcal{NP}^{R0}\) (Boundary of Constraint (5.97))

(ii) \(\mathcal{NP}^{R*} \in \{ \mathcal{NP}^R : \lambda D^N \leq \frac{D^R}{\psi_m} \}\) (Boundary of Constraint (5.98))

(iii) \(\mathcal{NP}^{R*} = \mathcal{NP}^N\) (Boundary of Constraint (5.99))

(iv) \(\mathcal{NP}^{R*} = \sum_{k=1}^{m-1} S_k + \sum_{k=m}^K \) (Constraint (5.100) is tight)
The unconstrained problem is unbounded in \( P_R \).

In a similar manner to the case without competition, we can find the optimal solution for cases (ii) - (iv) above. First, we rewrite the problem considering the tight constraint, and then, find the unconstrained solution to the problem. If constraints are violated, we can use \( \partial P_{\text{max}} \), to specify the optimal solution.

**Solution 4**

In the case of Solution 4, the core supply is at least enough to meet remanufactured product demand for Part \( m \), but there will be a deficit of Part \( m + 1 \). The problem is given by

\[
\max \Pi_4 = (P^N - C^N)D^N_1 + (P^N - C^N)D^N_2 + (P^R - C^R)D^R + \sum_{k=1}^{m-1} S_k (\psi_k \lambda D^N_1 - D^R) \\
- \sum_{k=m}^{K} C^P_k (\psi_m - \psi_k \lambda D^N_1) - \theta \lambda D^N_1
\]

subject to:

\[
0 \leq P^N \leq P^{N0}, \quad (5.103)
\]

\[
P^R \leq P^{R0}, \quad (5.104)
\]

\[
\frac{D^R}{\psi_m} \leq \lambda D^N_1 \leq \frac{D^R}{\psi_{m+1}}, \quad (5.105)
\]

\[
P^R - C^R \geq R^R_m, \quad (5.106)
\]

\[
0 \leq P^R \leq P^N. \quad (5.107)
\]

Again, note that \( \Pi_4 \) is a concave function of \( P^N \) and \( P^R \). The constrained problem has a concave objective function over a convex feasible region and can be solved easily.

**Solution (5)**

This is the same as Solution (4) with central Part \( m = K \) (Recall that, by definition, \( \psi_{m+1} = 0 \) and so \( \lambda D^R \leq \infty \) for \( m = K \)).
Solution 6

In the case of Solution 6, core supply meets or exceeds the number needed to meet remanufactured product using Part $m$.

\[
max \quad \Pi_6 = (P_N - C_N)D_1^N + (P_N - C_N)D_2^N + (P_R - C_R)\psi_m \lambda D_1^N + \sum_{k=1}^{m-1} S_k(\psi_k \frac{D_R^R}{\psi_m} - D^R) \quad (5.108)
\]

\[- \sum_{k=m+1}^{K} C_k^P(D^R + \psi_k \frac{D_R^R}{\psi_m}) - \theta \frac{D_R^R}{\psi_m} \quad (5.109)
\]

subject to:

\[0 \leq P_N \leq P_N^N, \quad (5.110)\]

\[P_R \leq P_R^R, \quad (5.111)\]

\[\lambda D_1^N \leq \frac{D^R}{\psi_m}, \quad (5.112)\]

\[P_R - C_R \geq \rho_m^R, \quad (5.113)\]

\[0 \leq P_R \leq P_N. \quad (5.114)\]

Again, note that $\Pi_6$ is a concave function of $P_N$ and $P_R$. The constrained problem has a concave objective function over a convex feasible region and can easily be solved.

### 5.4 Comparison of noncompetitive and competitive cases

To build some intuition about the general differences between the noncompetitive and competitive cases, we can look at the unconstrained optimal solution to both problems in the case that remanufacturing just requires some fixed per unit cost, i.e., we are not taking into account the salvage values and part replacement costs.
In the case of no competition, we find \((P^N_1, P^R_1) = (P^N, P^R)\) that maximizes

\[
(P^N - C^N)D^N + (P^R - C^R)D^R_{NC},
\]

where \(D^R_{NC} = D^R_0 - \alpha P^R\). It can be shown that

\[
P^N_1^* = \frac{P^N_0 + C^N}{2}, \quad \text{and} \quad P^R_1^* = \frac{P^R_0 + C^R}{2}.
\]

However, in the case the products compete, we find \((P^N_2^*, P^R_2^*) = (P^N, P^R)\) that maximizes

\[
(P^N - C^N)D^N_1 + (P^N - C^N)D^N_2 (P^R - C^R)D^R,
\]

which can easily be shown to be

\[
P^N_2^* = \frac{2\alpha^2 C^N + 3\alpha\gamma C^N + 2\alpha D^N_0 + 2\gamma D^N_0 + \gamma D^R_0}{4\alpha^2 + 6\alpha\gamma}, \quad \text{and} \quad P^R_2^* = \frac{2\alpha^2 C^R + 3\alpha\gamma C^R + 2\gamma D^N_0 + 2\alpha D^R_0 + \gamma D^R_0}{4\alpha^2 + 6\alpha\gamma}.
\]

It can be shown that \(P^N_1^* \geq P^N_2^*\) and \(P^R_1^* \geq P^R_2^*\), thus the prices are higher in the noncompetitive case, as expected.

The extent of disparity between prices depends on how sensitive the demand functions are to product competition, \(\gamma\). Higher \(\gamma\) results in larger price and demand differences.
Chapter 6

Conclusions and Future Work

In this dissertation, we have addressed the problem of determining the optimal EOL option for a product. This is an important problem because it dictates the design of a reverse supply chain (RSC). The EOL for a product is influenced by its characteristics, design, and pricing (see Figure 1.1). We have considered these aspects in determining an optimal EOL option for a product.

Specifically, we consider a problem in which the manufacturer must decide whether or not to collect products back from customers, and if so, to decide how to recover value from these products. The number of cores collected and the value recovery strategy specify the EOL option for the product. The EOL options fall into two broad categories: (i) those requiring disassembly of a product, and (ii) those that do not require disassembly of product. The EOL options that do require disassembly are remanufacturing and part salvage, and we focus on these in our analysis. A product planning framework is used to more precisely capture the nature of the cost structure of remanufacturing. For example, the unit cost of a remanufactured product is dependent on: (i) the number of cores collected, (ii) the central part around which the remanufacturing is planned, and (iii) the demand for remanufactured products.

The literature review from Chapter 1 identifies several major gaps in the literature resulting from not integrating production planning into strategic planning level analysis for RSC. Chapter 2 presents a model that addresses these
gaps and forms the foundation of our research. The model is a multi-period, production planning model that considers product returns as an endogenous variable linked to a previous period’s new production.

The problem of mapping product characteristics onto optimal EOL options is addressed in Chapter 3 under fixed product design and product pricing. A two-stage model in which new products are manufactured and sold in the first stage, and then, are recovered and subjected to an EOL option in the section stage, is used to study the problem. This two-stage model is germane to gaining insight about the problem as, disregarding inventory, the general multi-period model can be decomposed into multiple two-stage models. The centerpiece of our work is the mapping generated from this two-stage model. This mapping reveals several interesting EOL properties of products. Firstly, the optimal \textit{EOL type} for a product does not depend on product demand. Specifically, our analysis reveals: (i) the notion of the “key” part (the best “central” part, or part to around which to base remanufacturing), and “limiting” part - the lowest yielding part for which no deficit is desired, (ii) whether part salvage is profitable, and (iii) whether product recovery will even be profitable. By including the demand information, the particular optimal EOL option for a product can be determined. These results form the foundation for studying optimal product design and product pricing.

In Chapter 4, the problem of choosing initial part quality levels under mandated collection, e.g. product leasing, is addressed. We find that a part should be designed such that either: (i) it is of minimal quality and be replaced during remanufacturing, designated as a “Replacement Part”, (ii) it is of high quality and the excess salvaged, designated as a “Salvage Part”, or (iii) it is of moderate quality such that just enough parts are obtained to meet remanufactured product demand, designated as a “Key Part” or “Limiting Part”. Adding quality to Replacement parts is very expensive, however, their replacement costs are low as are their salvage values. On the other hand, the salvage value and replacement cost for Salvage parts is quite high compared with the cost of initial quality. Key and limiting parts are such that replacement cost is very high, but salvage value is very low, so it is optimal to obtain just enough of these parts to perform remanufacturing. The difficulty of determining the category to which a part belongs is in finding the optimal yield of the key or limiting part. We have developed methods to optimally determine this yield and to categorize the parts.

Chapter 5 investigates the problem of determining optimal prices for new and remanufactured products under the man-
manufacturer’s choice of collection. Again, a two-stage model is used to study this problem, and two cases are considered. In the first case, new products are made at the first stage and returned at the second stage. Only remanufactured products are produced at the second stage. In the second case, both new and remanufactured products are produced at the second stage. Thus, there is a competition between new and remanufactured products. We find that there are certain price independent relationships that determine certain properties of the EOL option. Using these properties, only a limited number of EOL options must be considered. Both the competitive and noncompetitive cases behave similarly; however, we find that the prices of new and remanufactured products will be lower and demand will be higher in the competitive case as opposed to the non-competitive case.

This dissertation make several significant contributions. Firstly, a comprehensive and general production planning model is developed for the RSC that considers product returns as an endogenous variable. Secondly, a mapping of product characteristics onto optimal EOL options is developed using a two-stage model that is a special case of the comprehensive model. This mapping can be depicted graphically for the two-part case, and it provides significant insight into the problem. Thirdly, we characterize the nature of the optimal choice of part quality and find that the parts may only take certain quality levels. Finally, the problem of product pricing is addressed and efficient methods for determining optimal EOL option and product pricing are developed.

Future Research

Reverse supply chain research is a rich field for making further contributions. This dissertation work provides a solid foundation for future research in this area. One natural extension of this work is using the mapping from Chapter 3 to map various products onto EOL options and compare them with current industry best-practices. This would serve two purposes: (i) validate the assumptions of the work in this dissertation, and (ii) reveal products whose RSC could be improved. Another avenue of research would be to use the framework presented here to study the system wide impact of the EOL option selected. This would help guide public policy debate for creating legislation such as the EOL directives in Europe. Of course, uncertainty plays a critical role in decision making. Understanding the robustness of the optimal EOL options determined by the methods presented in this dissertation can be useful. Additionally, determination of robust EOL options is vital from a practical standpoint. Some questions are still open related to
Chapter 5. In this dissertation work, we develop methods for optimally finding the price and EOL option for a product when the new and remanufactured products do not compete and when they do compete. It would very interesting to do a full comparison of both of these situations to see how competition affects pricing and EOL option.

Other areas of future work may be more than simple extensions of this dissertation work, but still share the same basic framework. Many aspects of the problem are not deterministic, such as customers usage-time of a product. The length of time for which a customer uses a product determines when it might be returned to the manufacturer. Using a stochastic modeling approach, the nature of the customer usage of the product could be more accurately captured. In addition, variable product return times impact its value and that of its parts, and this consideration can further enhance the applicability of the model. Additionally, most products are a part of a larger product family. As such, they may share common parts. The benefit of common parts in RSC is that a recovered part from one product might be reused in an other product. Finally, it can be argued that in order to fully take advantage of the potential of a RSC, customers must be willing to pay for the usage of a product rather than having to own the product. In this way, the manufacturer can take control of the entire life-cycle of the product. Research regarding the advantages of selling the usage of a product rather than the product itself needs to be investigated.
Bibliography


Appendix A

Proofs for Chapter 3

Proof of Proposition 3.1.1

Proof $\Pi'_{k-1,k} \geq \Pi'_{k,k+1}$ is expressed as

$$\sum_{k=1}^{m-1} S_k \psi_k - \sum_{k=m}^{K} C^p_k \psi_k - \theta \geq \sum_{k=1}^{m} S_k \psi_k - \sum_{k=m+1}^{K} C^p_k \psi_k - \theta. \quad (A.1)$$

By rearranging and then canceling terms we obtain

$$C^p_k \geq S_k, \quad (A.2)$$

which is true by assumption for all $k = 1, \ldots, K$. \qed

Proof of Proposition 3.1.2

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Proof  By comparison of $\bar{\Pi}_0', \bar{\Pi}_1'$ and $\bar{\Pi}_1', \bar{\Pi}_2'$: Assume $\bar{\Pi}_0', \bar{\Pi}_1' \geq \bar{\Pi}_1', \bar{\Pi}_2'$:

$$\rho \geq \sum_{k=2}^{K} C_k \psi_k - \frac{\theta}{\sum_{k=2}^{K} C_k \psi_k} \geq S_1 \psi_1 + \sum_{k=2}^{K} C_k \psi_k$$  \hspace{1cm} (A.3)

$$\Rightarrow \rho R \geq S_1 + \sum_{k=2}^{K} C_K \psi_k.$$  \hspace{1cm} (A.4)

$\Box$

Proof of Proposition 3.1.3

Proof

I. a. To show: $\bar{\Pi}_{0,m} \leq \bar{\Pi}_{0,m-1,m}$ for some $m, m = 2, \ldots, K$, then $\bar{\Pi}_{0,m} \geq \bar{\Pi}_{0,n}$ for $\forall \epsilon \in [0, D^R / \psi_n), n = 2, \ldots, m - 1$.

This is shown in three steps:

(1) First it is shown that

$$\bar{\Pi}_{0,k} \leq \bar{\Pi}_{k-1,k} \Rightarrow \bar{\Pi}_{0,k-1} \leq \bar{\Pi}_{k-2,k-1}$$  \hspace{1cm} (A.5)

for $k = 2, \ldots, K$. By reapplying the relationship in Equation (A.5) for each $k = 2, \ldots, m$ it follows that

$$\bar{\Pi}_{0,m} \leq \bar{\Pi}_{m-1,m} \Rightarrow \bar{\Pi}_{0,n} \leq \bar{\Pi}_{n-1,n}$$  \hspace{1cm} (A.6)

for $n = 2, \ldots, m - 1$.

(2) Secondly it is shown that

$$\bar{\Pi}_{0,k} \leq \bar{\Pi}_{k-1,k} \Rightarrow \bar{\Pi}_{0,k} \geq \bar{\Pi}_{0,k-1}$$  \hspace{1cm} (A.7)

for $k = 2, \ldots, m$ implies that

$$\bar{\Pi}_{0,k} \leq \bar{\Pi}_{0,k-1}.$$  \hspace{1cm} (A.8)
(3) If $\Pi'_{0,m} \leq \Pi'_{m-1,m}$ then (1) and (2) hold and thus

$$\Pi_{0,m} \geq \Pi_{0,n} \text{ for } n = 2, \ldots, m - 1.$$  \hspace{1cm} (A.9)

Let us now show (1), that $\Pi'_{0,k} \leq \Pi'_{k-1,k}$ by way of showing $\Pi'_{0,k} \leq \Pi'_{k-1,k}$ \hspace{1cm} (A.9)

First, we show (2), that $\Pi'_{0,k} \leq \Pi'_{k-1,k}$ \hspace{1cm} (A.10)

$$\Pi'_{0,k} \geq \Pi'_{0,k-1} \Rightarrow \Pi'_{0,k} \geq \Pi'_{k-2,k-1}.$$  \hspace{1cm} (A.11)

Thus,

$$\Pi'_{0,k} \geq \Pi'_{0,k-1}$$  \hspace{1cm} (A.12)

$$\Rightarrow \frac{\Pi'_{0,k} - \Pi'_{0,k-1}}{\psi_{k-1}} < \frac{\Pi'_{k-2,k-1} - \Pi'_{0,k}}{\psi_{k-1}}.$$  \hspace{1cm} (A.13)

which is true since $D^R / \psi_{k-1} < \frac{D^R}{\psi_k}$. Next we show that $\Pi'_{0,k-1} \leq \Pi'_{k-2,k-1}$. By Property 1, we know that $\Pi'_{k-2,k-1} \geq \Pi'_{k-1,k}$, thus we need only show that $\Pi'_{k-1,k} \geq \Pi'_{0,k-1}$. Note that we can compute $\Pi'_{k-1,k}$ using $\Pi'_{0,k}$ and $\Pi'_{0,k-1}$:

$$\Pi'_{k-1,k} = \frac{\Pi'_{0,k} - \Pi'_{0,k-1}}{D^R / \psi_{k-1}}.$$  \hspace{1cm} (A.14)
\[ \Pi'_{k,1,k} \geq \Pi'_{0,k-1} \]  
(A.15)

\[ \frac{\Pi'_{0,k,k} \left( DR \psi_k \right) - \Pi'_{0,k-1,1} \left( DR / \psi_{k-1} \right)}{DR / \psi_k - DR / \psi_{k-1}} \geq \Pi'_{0,k-1} \]  
(A.16)

\[ \Rightarrow \Pi'_{0,k} > \Pi'_{0,k-1} \]  
(A.17)

which we have already shown to be true.

a. To show: \( \bar{\Pi}'_{0,m} \leq \bar{\Pi}'_{m-1,m} \) for some \( m, m = 2, \ldots, K \), then \( \bar{\Pi}_{0,m} \geq \bar{\Pi}_{n-1,n} \) for \( \forall \theta \in [DR / \psi_{n-1}, DR / \psi_n) \), \( n = 2, \ldots, m - 1 \). This is shown by proving that

\[ \bar{\Pi}_{0,k} \geq \bar{\Pi}_{k-1,k} \]  
for \( \forall \theta \in [DR / \psi_{k-1}, DR / \psi_k) \)  
(A.18)

and then applying the transitive property to prove the result.

First, note that

\[ \bar{\Pi}_{k-1,k} = \bar{\Pi}_{0,k-1} DR / \psi_{k-1} + \bar{\Pi}'_{k-1,k} (\theta - DR / \psi_{k-1}) \]  
(A.19)

Thus \( \bar{\Pi}_{0,k} > \bar{\Pi}_{k-1,k} \) if

\[ \bar{\Pi}'_{0,k} > \bar{\Pi}'_{0,k-1,1} DR / \psi_{k-1} + \bar{\Pi}'_{k-1,k} (\theta - DR / \psi_{k-1}) \]  
(A.20)

Rearranging terms, we obtain

\[ (\bar{\Pi}'_{0,k} - \bar{\Pi}'_{0,k-1,1}) DR / \psi_{k-1} > (\bar{\Pi}'_{0,k-1,1} - \bar{\Pi}'_{0,k}) (\theta - DR / \psi_{k-1}) \]  
(A.21)

which is true is since \( \bar{\Pi}'_{0,k} - \bar{\Pi}'_{0,k-1,1} > 0 \) and \( \bar{\Pi}'_{0,k-1,1} - \bar{\Pi}'_{0,k} < 0 \). This proves the result.

II. a. Similar logic to (I.a.) applies and we find that \( \bar{\Pi}'_{0,m} \geq \bar{\Pi}'_{0,n+1} \) for \( n = m + 1, \ldots, K \). That \( \bar{\Pi}'_{0,m} \geq \bar{\Pi}'_{s} \) results from the fact that \( \bar{\Pi}'_{s} = \bar{\Pi}'_{k,k+1} \) and so \( \bar{\Pi}'_{0,m} \geq \bar{\Pi}'_{m,m+1} \geq \bar{\Pi}'_{k,k+1} = \bar{\Pi}'_{s} \).

b. Similar logic to (I.b.) applies.

III. This is a direct result of (I) and (II).
IV. (i) Note that $\bar{\Pi}_{S} = \bar{\Pi}_{K,K+1}$ for $m = 1, \ldots, K$. (ii) Then by (I) $\bar{\Pi}_{0,K} \geq \bar{\Pi}_{0,n}$, for $n = 1, \ldots, K - 1$ and $\bar{\Pi}_{0,K} \geq \bar{\Pi}_{n-1,n}$, for $n = 1, \ldots, K$. (iii) Since $\bar{\Pi}_{S}$ and $\bar{\Pi}_{0,K}$ share the same origin and $\bar{\Pi}_{S} \geq \bar{\Pi}_{0,K}$, $\bar{\Pi}_{S} \geq \bar{\Pi}_{0,n}$, for $n = 1, \ldots, K - 1$. (iv) It can be shown that $\bar{\Pi}_{S} \geq \bar{\Pi}_{K,K+1}$. First note:

\[
\bar{\Pi}_{K,K+1} = \bar{\Pi}_{0,K}'(\frac{D}{\Psi_{K}}) + \bar{\Pi}_{K,K+1}'(\mathcal{C} - \frac{D}{\Psi_{K}}),
\]

(A.22)

and

\[
\bar{\Pi}_{S} = \bar{\Pi}_{S}'\mathcal{C}.
\]

(A.24)

Then

\[
\bar{\Pi}_{S} \geq \bar{\Pi}_{K,K+1},
\]

(A.25)

\[
\bar{\Pi}_{S}'\mathcal{C} \geq \bar{\Pi}_{0,K}'(\frac{D}{\Psi_{K}}) + \bar{\Pi}_{S}'(\mathcal{C} - \frac{D}{\Psi_{K}}),
\]

(A.26)

\[
\implies \bar{\Pi}_{S} \geq \bar{\Pi}_{0,K}',
\]

(A.27)

which is true by (iii) and so $\bar{\Pi}_{S} \geq \bar{\Pi}_{K,K+1}$. Thus $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{S}$ for $\mathcal{C} \in [0, \frac{D}{\Psi_{K}})$ since by (ii) and (iii) $\bar{\Pi}_{S} \geq \bar{\Pi}_{0,K} \geq \bar{\Pi}_{0,n}$, $n = 1, \ldots, K - 1$ and $\bar{\Pi}_{S} \geq \bar{\Pi}_{0,K} \geq \bar{\Pi}_{n-1,n}$, $n = 1, \ldots, K$. Finally, $\bar{\Pi}_{\text{max}} = \bar{\Pi}_{S}$ for $\mathcal{C} \in [\frac{D}{\Psi_{K}}, \infty)$ by (iv). 

\[\square\]

Proof of Property 3.1.4

I. By Property 3, case II, we know that since $\bar{\Pi}_{0,1}' \geq \bar{\Pi}_{1,2}'$: (i) $\bar{\Pi}_{0,1}' \geq \bar{\Pi}_{0,n}'$, $n = 2, \ldots, K$ and $\bar{\Pi}_{0,1}' \geq \bar{\Pi}_{S}'$, and (ii) $\bar{\Pi}_{0,n}' \leq \bar{\Pi}_{n-1,n}'$, $n = 2, \ldots, K$. Additionally, we can show that based on (i) and (ii), (iii) $\bar{\Pi}_{0,k} \geq \bar{\Pi}_{0,k-1}'$ for all $k \in \{2, \ldots, K\}$ and (iv) $\bar{\Pi}_{k-1,k}' \geq \bar{\Pi}_{S}$ for all $k \in \{2, \ldots, K\}$. To show (iii) we show that $\bar{\Pi}_{0,1}' \geq \bar{\Pi}_{1,2}' \geq \ldots \bar{\Pi}_{m-1,m}' \geq \ldots \bar{\Pi}_{0,1}'$. 

Proof
\[ \tilde{\Pi}'_{0,m+1} \implies \tilde{\Pi}'_{0,m} \geq \tilde{\Pi}'_{0,m+1} \]. First, recognize that

\[ \tilde{\Pi}'_{0,m} = \frac{\tilde{\Pi}'_{0,1}}{\mathcal{Y}_1} + \sum_{k=2}^{m} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \]  

(A.28)

and

\[ \tilde{\Pi}'_{0,m+1} = \frac{\tilde{\Pi}'_{0,1}}{\mathcal{Y}_1} + \sum_{k=2}^{m+1} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \]  

(A.29)

Thus if

\[ \tilde{\Pi}'_{0,m} \geq \tilde{\Pi}'_{0,m+1} \]  

(A.30)

then

\[ \frac{\tilde{\Pi}'_{0,1}}{\mathcal{Y}_1} + \sum_{k=2}^{m} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \geq \frac{\tilde{\Pi}'_{0,1}}{\mathcal{Y}_1} + \sum_{k=2}^{m+1} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \]  

(A.31)

which implies

\[ \tilde{\Pi}'_{0,1} \frac{\mathcal{D}^R}{\mathcal{Y}_1} + \sum_{k=2}^{m} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \geq \tilde{\Pi}'_{m,m+1} \frac{\mathcal{D}^R}{\mathcal{Y}_m} \]  

(A.32)

Since \( \tilde{\Pi}'_{0,1} \geq \tilde{\Pi}'_{1,2} \geq \ldots \geq \tilde{\Pi}'_{m-1,m} \) and \( \frac{\mathcal{D}^R}{\mathcal{Y}_m} = \frac{\mathcal{D}^R}{\mathcal{Y}_1} + \sum_{k=2}^{m} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) \), then we have the result. To show (iv) we must show that \( \tilde{\Pi}'_{0,1} \geq \tilde{\Pi}'_{1,2} \geq \ldots \geq \tilde{\Pi}'_{m-1,m} \geq \tilde{\Pi}'_{m,m+1} \implies \tilde{\Pi}'_{k-1,k} \geq \tilde{\Pi}'_{S} \). Note that

\[ \tilde{\Pi}'_{m-1,m} = \tilde{\Pi}'_{0,1} \frac{\mathcal{D}^R}{\mathcal{Y}_1} + \sum_{k=2}^{m-1} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) + \tilde{\Pi}'_{m-1,m} (\mathcal{C}' / \mathcal{D}^R / \psi_{k-1}) \]  

(A.33)

Thus

\[ \tilde{\Pi}'_{S} \leq \tilde{\Pi}'_{m-1,m} \]  

(A.34)

which may be written as

\[ \tilde{\Pi}'_{S} \mathcal{C}' \leq \tilde{\Pi}'_{0,1} \frac{\mathcal{D}^R}{\mathcal{Y}_1} + \sum_{k=2}^{m-1} \tilde{\Pi}'_{k-1,k} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) + \tilde{\Pi}'_{m-1,m} (\mathcal{C}' / \mathcal{D}^R / \psi_{k-1}) \]  

(A.35)

which is true since \( \tilde{\Pi}'_{0,1} \geq \tilde{\Pi}'_{1,2} \geq \ldots \geq \tilde{\Pi}'_{m-1,m} \geq \tilde{\Pi}'_{S} \) and \( \mathcal{C}' = \frac{\mathcal{D}^R}{\mathcal{Y}_1} + \sum_{k=2}^{m-1} \left( \frac{\mathcal{D}_k}{\mathcal{Y}_k} - \frac{\mathcal{D}^R}{\mathcal{Y}_k} / \psi_{k-1} \right) + (\mathcal{C}' - \ldots \)
$D^R/\psi_{k-1})$. Now we show that a particular region is best for each range of $\mathcal{C}$.

a. $\mathcal{C} \in [0, \frac{D^R}{\psi_{K}})$: By (i) since $\bar{\Pi}_{0,m}$ at $\mathcal{C} = 0$ for all $m = 1, \ldots, K$, then $\bar{\Pi}_{0,1} \geq \bar{\Pi}_{0,n}$ for $n \in \{2, \ldots, K\}$ for $\mathcal{C} \in [0, \frac{D^R}{\psi_{K}})$. Also, since $\bar{\Pi}_{0,1} = 0$ and $\bar{\Pi}_S = 0$, $\bar{\Pi}_{0,1} \geq \bar{\Pi}_S$ for $\mathcal{C} \in [0, \infty)$.

b. $\mathcal{C} \in [D^R/\psi_{k-1}, \frac{D^R}{\psi_{K}})$: By (b) and (c) $\Pi_{k-1,k} \geq \Pi_{0,k}$ and by (d), $\Pi_{k-1,k} \geq \Pi_S$ for $\mathcal{C} \in [D^R/\psi_{k-1}, \frac{D^R}{\psi_{K}})$, and $k \in \{2, \ldots, K\}$.

c. $\mathcal{C} \in [\frac{D^R}{\psi_{K}}, \infty)$: Note that $\bar{\Pi}_S = \bar{\Pi}_{K,K+1} \leq \bar{\Pi}_{0,K}$. Since $\bar{\Pi}_S$ and $\bar{\Pi}_{0,K}$ share the same origin (Property 2) and $\bar{\Pi}_S \leq \bar{\Pi}_{0,K}$, Next we show that $\bar{\Pi} \leq \bar{\Pi}_{K,K+1}$. Recall that

$$\bar{\Pi}_{k,K+1} = \bar{\Pi}_{0,K} \left( \frac{D^R}{\psi_K} \right) + \bar{\Pi}_{k,K+1}(\mathcal{C} - \frac{D^R}{\psi_K}),$$

(A.36)

and

$$\bar{\Pi}_S = \bar{\Pi}_S \mathcal{C},$$

(A.38)

$$\bar{\Pi}_S \leq \bar{\Pi}_{K,K+1},$$

(A.39)

which can be written as

$$\bar{\Pi}_S \mathcal{C} \leq \bar{\Pi}_{0,K} \left( \frac{D^R}{\psi_K} \right) + \bar{\Pi}_S(\mathcal{C} - \frac{D^R}{\psi_K}).$$

(A.40)

Rearranging terms, we obtain

$$\bar{\Pi}_S \leq \bar{\Pi}_{0,K},$$

(A.41)

which we have shown to be true and thus $\bar{\Pi}_S \leq \bar{\Pi}_{K,K+1}$.

II. Case IV of Property 4 holds and the result follows.

III. Note that (i) since $\bar{\Pi}_{1,2} \geq \bar{\Pi}_{0,2}$ there exists some largest $k = 2, \ldots, K$, call it $k'$ such that $\bar{\Pi}_{k'-1,k} \geq \bar{\Pi}_{0,k}$. It can be shown that (ii) $\bar{\Pi}_{k,k+1} \leq \bar{\Pi}_{0,k}$ and $\bar{\Pi}_{k-1,k} \leq \bar{\Pi}_{0,k}$ implies that $\bar{\Pi}_{k-1,k} \leq \bar{\Pi}_{0,k-1}$. Note that, due to (i), we have that (iii) for all $k > k'$, $\bar{\Pi}_{k-1,k} \leq \bar{\Pi}_{0,k}$. Since $\bar{\Pi}_{K,K+1} \leq \bar{\Pi}_{0,K}$ and (iii), we have by (ii) that $\bar{\Pi}_{k',k'+1} \leq \bar{\Pi}_{0,k'}$. However,
at \( k' \), \( \Pi_{k'-1,k'} \geq \Pi_{0,k'} \) and so Case III from Property 3 holds.

\[ \square \]

**Proof of Property 3.1.5**

**Proof**

I. Eq. (3.25)

In case Eq. (3.25), then \( \Pi'_{0,1} \geq \Pi'_{1,2} \). By Property 1, \( \Pi'_{m-1,m} \geq \Pi'_{m,m+1} \) for all \( m \in \{2, \ldots, K\} \) and so \( \Pi'_{0,1} \geq \Pi'_{1,2} \geq \cdots \geq \Pi'_{k-1,k'} \geq \cdots \geq \Pi'_{K,K+1} \). Thus Eq. (3.25) is concave.

II. Eq. (3.26) In case Eq. (3.26), then Eq. (3.26) linear, which is concave.

III. Eq. (3.27) In case Eq. (3.27), then \( \Pi'_{0,k} \geq \Pi'_{k,k+1} \) for some \( k \) since Property 4, case (III) applies. By Property 1, \( \Pi'_{m-1,m} \geq \Pi'_{m,m+1} \) for all \( m \in \{2, \ldots, K\} \) and so \( \Pi'_{0,k} \geq \Pi'_{k,k+1} \geq \cdots \geq \Pi'_{m-1,m} \geq \cdots \geq \Pi'_{K,K+1} \). Thus Eq. (3.27) is concave.

\[ \square \]

**Proof**

To show: If \( \sum_{k=1}^{K} (S_k' + \delta_k' (\psi_k - \psi_k^0)) \psi_k > \theta \forall \psi_k \in [\psi_k^L, \psi_k^H], k = 1, \ldots, K \) and either (i) \( A \approx \infty \) or (ii) \( \rho R < \sum_{k \in K} (S_k' + \delta_k'^+ (\psi_k - \psi_k^0)) \forall \psi_k \in [\psi_k^L, \psi_k^H], \)

\[
\psi_k^* = \begin{cases} 
\psi_k^L & \text{if } (S_k' + \delta_k'^+ (\psi_k^L - \psi_k^0) + C_{disp}^k) \lambda < \delta_k \\
\psi_k^H & \text{otherwise}. 
\end{cases} 
\]  

(A.42)

The proof of this results from a comparison of the terms \( \Pi \) which are function of \( \psi_k \) for \( \psi_k = \psi_k^L \) and \( \psi_k = \psi_k^H \).

\[ -\delta_k (\psi_k^L - \psi_k^0) \omega^N (S_k' + \delta_k^L (\psi_k^L - \psi_k^0)) \psi_k^L A - C_{disp}^k (1 - \psi_k^L) A \]  

(A.43)

\[ > -\delta_k (\psi_k^H - \psi_k^0) \omega^N (S_k' + \delta_k^H (\psi_k^H - \psi_k^0)) \psi_k^H A - C_{disp}^k (1 - \psi_k^H) A \]  

(A.44)

(A.45)
By rearranging terms

\[-\delta_k (\psi^L_k - \psi^H_k) \mathcal{Q}^N + (S'_k - \delta_k \psi^0_k)(\psi^L_k - \psi^H_k)A + \delta_k (\psi^L_k - \psi^H_k)(\psi^L_k + \psi^H_k) + C^{\text{disp}}_k (\psi^L_k - \psi^H_k)A > 0 \quad (A.46)\]

Dividing by \((\psi^L_k - \psi^H_k)\) (which is < 0) and rearranging terms:

\[(S'_k + \delta_k (\psi^L_k + \psi^H_k - \psi^0_k) + C^{\text{disp}}_k)\lambda < \delta_k \quad (A.47)\]

which gives the result. \(\Box\)