ON THE SOLUTION OF THE EQUATIONS OF RADIATIVE TRANSFER IN A FREE-ELECTRON ATMOSPHERE*

by

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ABSTRACT

The radiative transfer equations are solved for an electron-scattering stellar atmosphere as formulated by Chandrasekhar. The solution employs a transformation of the integrodifferential form of the transfer equations into singular integral equations for the angular intensities of the radiation field. The Milne problem is solved to illustrate the method. In addition, the relationship is found between the above method of solution and Case's normal mode expansion method. This leads to an alternate procedure for finding the normal mode expansion coefficients. As an example of the method the constant, distributed source problem is solved for a half-space medium.

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I. INTRODUCTION

During the past few years, several methods have been used to obtain exact results for neutron transport problems in slab geometry in which the theory of singular integral equations (4) plays a central role. Recently C. E. Siewert and S. K. Fraley (6) have extended one of these methods, viz., Case's (2) singular normal modes expansion method, to obtain solutions to the equations of radiative transfer for an electron-scattering stellar atmosphere as formulated by Chandrasekhar (3). Although these equations are for a free-electron atmosphere, the equations of transfer are identical in form to two-group neutron transport equations with anisotropic scattering. In the normal mode expansion method, the independent variables of the transfer equations are separated and the general solution is expressed as a normal mode expansion with arbitrary coefficients over the spectrum of the separation parameter. Application of boundary conditions leads to singular integral equations whose solutions are the expansion coefficients. By using orthogonality relations, the actual solution of these singular integral equations can be bypassed. In this thesis we extend a method developed by Rhodes (5) for the one-velocity neutron transport with anisotropic scattering to the radiative transfer equations. Our solution employs a transformation of the integrodifferential equations of radiative transfer into singular integral equations (1,5) for the angular intensity in which the spatial variable enters only as a parameter. In addition the relationship between our method and the normal mode method of Siewert and Fraley is found.

An outline of the remainder of the thesis is as follows: In Section II a description is given of the physical model and the equations of radiative transfer as formulated by Chandrasekhar (3). The procedure of Siewert and Fraley (6) is reviewed in order to compare our method with the normal mode expansion method. They find a general solution written as a normal mode expansion with two discrete eigenvectors and two singular continuum eigensolutions. Orthogonality relations for these eigensolutions as found by O. J. Smith and C. E. Siewert (7) are given and will be used to determine a relationship between the two models. In Section III our method of solution of the radiative transfer equations for an electron-scattering atmosphere are presented. To illustrate the procedure we solve the Milne problem and the law of darkening. In Section IV the relationship between our method and the normal mode formulation is given. This can be used as an alternate method for determining the normal mode expansion coefficients. As an example of the procedure, we solve the Milne problem for a semi-infinite medium containing a constant, distributed source.

II. THE RADIATIVE TRANSFER EQUATION AND ITS NORMAL MODE EXPANSION

Development of the Radiative Transfer Equation

The scattering of radiation by free electrons in semi-infinite plane-parallel atmospheres is of interest since it accounts for the principal transfer of energy in stars having high temperatures and low densities. A formulation of the transfer equations for two perpendicularly polarized intensities of the radiation field has been made by S. Chandrasekhar (3). In this work Chandrasekhar recognized that the Thompson scattering of radiation by free electrons was a special case of Rayleigh scattering. By a modification of Stokes' parameter representation for polarized light, he applied the Rayleigh scattering law to the case of electron scattering in a partially polarized beam of radiation.

Perfect scattering is defined (3) as a scattering process in which all of the light lost from an incident beam of radiation is due to scattering. Thompson scattering by free electrons is an example of perfect scattering and is assumed to be the only process involved in the flow of radiant energy through the medium. A consequence of perfect scattering in plane-parallel atmospheres is a constant net flux normal to the stratification planes of the medium. The constant net flux is provided by radiation originating in the deep interior.

Chandrasekhar's treatment of the problem (3) shows that two perpendicularly polarized intensities $\Psi_1(x,\mu)$ and $\Psi_2(x,\mu)$ are sufficient to describe the radiation field of a plane-parallel atmosphere having azimuthal symmetry and no incident radiation. The spatial variable x is the optical distance taken normal to the planes of stratification in units of the Thompson scattering coefficient, and μ is the direction cosine measured from the inward normal. This choice of direction cosine is used in order to conform with neutron transport theory and Case's method (2). The usual convention in astrophysics is to measure from the outward normal. As the Thompson scattering coefficient depends only on the charge and the mass of the electron, the intensities will be independent of the radiation frequency, and will henceforth represent intensities integrated over the frequency range.

The source-free equations of transfer (3) in matrix form are

$$\left(\mu\frac{\partial}{\partial x}+1\right)\underset{\sim}{\Psi}(x,\mu) = \int_{-1}^{1}\underset{\approx}{R}(\nu,\mu)\,\underset{\sim}{\Psi}(x,\nu)\,d\nu\,. \tag{2.1}$$

The intensity vector $\underset{\sim}{\Psi}(x,\mu)$ has two components $\Psi_1(x,\mu)$ and $\Psi_2(x,\mu)$ corresponding to two states of polarization having electric vectors vibrating in a plane perpendicular to and along the planes of stratification of the medium, respectively. Chandrasekhar's phase matrix for azimuthally symmetric, plane-parallel geometry (3) is given by the 2x2 matrix:

$$\underset{\approx}{R}(\nu,\mu) = \frac{3}{8}\begin{bmatrix} 2(1-\nu^2)(1-\mu^2)+\nu^2\mu^2 & \mu^2 \\ \nu^2 & 1 \end{bmatrix}. \tag{2.2}$$

A pencil of radiation incident from a direction described by direction cosine ν will be scattered into a direction described by direction cosine μ .

The Normal Mode Expansion Method

Following Case's method, Siewert and Fraley (6) separated the variables of Eq. (2.1) with a solution of the form

$$\underset{\sim}{\Psi}_\eta(x,\mu) = e^{-x/\eta}\ \underset{\sim}{\Phi}(\eta,\mu)\,, \tag{2.3}$$

where $\underset{\sim}{\Phi}(\eta,\mu)$ is a solution to the equation

$$(\eta-\mu)\,\underset{\sim}{\Phi}(\eta,\mu) = \eta\int_{-1}^{1}\underset{\approx}{R}(\nu,\mu)\,\underset{\sim}{\Phi}(\eta,\nu)\,d\nu. \tag{2.4}$$

Restricting η not to lie on the line (-1,1), they find a solution of Eq. (2.4) written as

$$\underset{\sim}{\Phi}(\eta,\mu) = \frac{\eta}{2}\ \frac{1}{\eta-\mu}\ \underset{\approx}{L}_\eta(\mu)\,\underset{\sim}{V_0}(\eta)\ ,\quad \eta\notin(-1,1), \tag{2.5}$$

where

$$\underset{\approx}{L}_\eta(\mu) = \begin{bmatrix} 1-P_2(\mu) & \frac{1}{4-3\eta^2}\left[1+(2-3\eta^2)P_2(\mu)\right] \\ 0 & \frac{3(1-\eta^2)}{4-3\eta^2} \end{bmatrix}, \tag{2.6}$$

and

$$\chi_0(\eta) = \int_{-1}^{1} \underset{\sim}{\Phi}(\eta,\mu)\, d\mu, \tag{2.7}$$

and $P_2(\mu)$ is the second order Legendre polynomial. Equation (2.5) is integrated over μ from -1 to 1 and a homogeneous equation for $\chi_0(\eta)$ is obtained. The determinant of the coefficient matrix for $\chi_0(\eta)$ is equated to zero in order to determine the discrete eigenvalues which are found to be the zeros of the function

$$\Omega_1(z) = -1 + 3(1-z^2)\left[1 + \frac{z}{2}\int_{-1}^{1}\frac{d\eta}{\eta - z}\right]. \tag{2.8}$$

They find a double root at infinity for $\Omega_1(z)$, and from Eq. (2.5) they determine a discrete solution

$$\underset{\sim}{\Phi}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.9}$$

Siewert and Fraley discuss a twofold degeneracy in the discrete solutions due to the double root at infinity and find a second, linearly independent, discrete eigenvector

$$\underset{\sim}{\Psi}_-(x,\mu) = (x-\mu)\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{2.10}$$

which is a solution to Eq. (2.1).

Siewert and Fraley find that the general solution of Eq. (2.4) for $\eta \in (-1,1)$ is

$$\underset{\sim}{\Phi}(\eta,\mu) = \left[\frac{\eta}{2}\,\frac{P}{\eta-\mu} + \lambda(\eta)\,\delta(\eta-\mu)\right] \underset{\approx}{L}_{\eta}(\mu)\,\underset{\sim}{V}_0(\mu), \qquad (2.11)$$

where the symbol P denotes that the Cauchy principal value is to be taken in any integration involving the following term, and $\delta(\eta-\mu)$ is the Dirac delta function. Proceeding as in the discrete case, Siewert and Fraley integrate Eq. (2.11) over μ and obtain a homogeneous equation for $\underset{\sim}{V}_0(\eta)$. After equating the determinant of the coefficient matrix for $\underset{\sim}{V}_0(\eta)$ to zero, they determine two $\lambda(\eta)$ functions which are then used in obtaining two linearly independent $\underset{\sim}{V}_0(\eta)$. They find the following continuum eigensolutions of Eq. (2.4):

$$\underset{\sim}{\Phi}_1(\eta,\mu) = \begin{bmatrix} \frac{3}{2}\,\eta(1-\mu^2)\,\frac{P}{\eta-\mu} + \lambda_1(\mu)\,\delta(\eta-\mu) \\ 0 \end{bmatrix}, \eta\in(-1,1), \qquad (2.12a)$$

and

$$\underset{\sim}{\Phi}_2(\eta,\mu) = \begin{bmatrix} -\frac{3}{2}\,\eta(\eta+\mu) \\ \frac{3}{2}\,\eta(1-\eta^2)\,\frac{P}{\eta-\mu} + \lambda_2(\mu)\,\delta(\eta-\mu) \end{bmatrix}, \eta\in(-1,1), \qquad (2.12b)$$

with the $\lambda_i(\eta)$ functions given by

$$\lambda_1(\eta) = -1 + 3(1-\eta^2)\left[1-\eta\tanh^{-1}\eta\right]$$

and

$$\lambda_2(\eta) = 1 + 3(1-\eta^2)\left[1-\eta\tanh^{-1}\eta\right]. \qquad (2.13)$$

Smith and Siewert (7) have shown that the eigensolutions $\underset{\sim}{\Phi}_+$, $\underset{\sim}{\Phi}_1(\eta,\mu)$, $\underset{\sim}{\Phi}_2(\eta,\mu)$, and $\underset{\sim}{\Psi}_-(o,\mu)$ are complete on the range $\mu \in (-1,1)$, such that an arbitrary two-component vector $\underset{\sim}{\Psi}(\mu)$ defined on $-1 \leq \mu \leq 1$ can be expanded as

$$\underset{\sim}{\Psi}(\mu) = A_+ \underset{\sim}{\Phi}_+ + A_- \underset{\sim}{\Psi}_-(o,\mu) + \int_{-1}^{1} \alpha(\eta)\, \underset{\sim}{\Phi}_1(\eta,\mu)\, d\eta + \int_{-1}^{1} \beta(\eta)\, \underset{\sim}{\Phi}_2(\eta,\mu)\, d\eta \,. \tag{2.14}$$

A consequence of this completeness relationship is that the complete solution of Eq. (2.1) can now be written as

$$\underset{\sim}{\Psi}(x,\mu) = A_+ \underset{\sim}{\Phi}_+ + A_- \underset{\sim}{\Psi}_-(x,\mu) + \int_{-1}^{1} \alpha(\eta)\, \underset{\sim}{\Phi}_1(\eta,\mu)\, e^{-x/\eta} d\eta + \int_{-1}^{1} \beta(\eta)\, \underset{\sim}{\Phi}_2(\eta,\mu)\, e^{-x/\eta} d\eta \,, \tag{2.15}$$

where A_+, A_-, $\alpha(\eta)$, and $\beta(\eta)$ are arbitrary expansion coefficients to be evaluated by boundary conditions. In addition, Siewert and Fraley have shown that $\underset{\sim}{\Phi}_+$, $\underset{\sim}{\Phi}_1(\eta,\mu)$, and $\underset{\sim}{\Phi}_2(\eta,\mu)$ are complete on the range $\mu \in (o,1)$, in the sense that an arbitrary two-component vector $\underset{\sim}{\Psi}(\mu)$ defined on the range $0 \leq \mu \leq 1$ can be expanded as

$$\underset{\sim}{\Psi}(\mu) = A_+ \underset{\sim}{\Phi}_+ + \int_{0}^{1} \alpha(\eta)\, \underset{\sim}{\Phi}_1(\eta,\mu)\, d\eta + \int_{0}^{1} \beta(\eta)\, \underset{\sim}{\Phi}_2(\eta,\mu)\, d\eta \,. \tag{2.16}$$

They use the completeness property (2.16) to solve the Milne problem.

Full-range orthogonality and normalization integrals are given by Smith and Siewert (7). They showed that

$$\int_{-1}^{1} \mu \, \tilde{\underline{\Phi}}(\eta',\mu) \, \underline{\Phi}(\eta,\mu) \, d\mu = 0, \qquad \eta \neq \eta', \tag{2.17}$$

where the tilde superscript indicates the transpose of the vector. They define the full-range scalar product,

$$\langle i | j \rangle = \int_{-1}^{1} \mu \, \tilde{\underline{\Phi}}_i(\eta',\mu) \, \underline{\Phi}_j(\eta,\mu) \, d\mu, \quad i,j = +,1,2, \tag{2.18}$$

and find

$$\langle i | j \rangle = 0, \qquad i \neq j = +, 1, 2, \tag{2.19}$$

$$\langle 1 | 1 \rangle = \eta \, \Omega_1^+(\eta) \, \Omega_1^-(\eta) \, \delta(\eta - \eta'), \tag{2.20}$$

and

$$\langle 2 | 2 \rangle = \eta \, \Omega_2^+(\eta) \, \Omega_2^-(\eta) \, \delta(\eta - \eta'), \tag{2.21}$$

where

$$\begin{aligned} \Omega_1^{\pm}(\eta) &= \lambda_1(\eta) \pm \tfrac{3}{2} \pi i \, \eta \, (1-\eta^2) \\ \Omega_2^{\pm}(\eta) &= \lambda_2(\eta) \pm \tfrac{3}{2} \pi i \, \eta \, (1-\eta^2). \end{aligned} \tag{2.22}$$

By replacing $\underline{\Phi}_j(\eta,\mu)$ in Eq. (2.18) with $\underline{\Psi}_-(0,\mu)$, an additional orthogonality relation is

$$\langle i | - \rangle = 0, \qquad i = -,1,2. \tag{2.23}$$

III. THE SINGULAR INTEGRAL EQUATION METHOD

In this section we solve the transfer equations for the intensities $\Psi_1(x,\mu)$ and $\Psi_2(x,\mu)$ by transforming the integrodifferential transfer equations into singular integral equations for the angular intensities. As an example of the method, we solve the classical Milne problem for a semi-infinite half-space with no incident radiation. The exact expression for the law of darkening, i.e., the emergent intensity at the vacuum interface, is found.

Defining the two-component vector $\underset{\sim}{\rho}(x,\mu)$ as

$$\underset{\sim}{\rho}(x,\mu) = \int_{-1}^{1} \underset{\approx}{R}(\nu,\mu)\,\underset{\sim}{\Psi}(x,\nu)\,d\nu, \tag{3.1}$$

we rewrite the equations of radiative transfer, (2.1), as

$$\left(\mu\frac{\partial}{\partial x}+1\right)\underset{\sim}{\Psi}(x,\mu) = \underset{\sim}{\rho}(x,\mu). \tag{3.2}$$

We next define the vector $\underset{\sim}{\Psi}_0(x,\mu)$ as

$$\underset{\sim}{\Psi}_0(x,\mu) = \frac{1}{2}\int_{-1}^{1}\underset{\approx}{G}(\nu,\mu)\left[\nu\,\underset{\sim}{\Psi}(x,\nu) - \mu\,\underset{\approx}{H}(\nu,\mu)\,\underset{\sim}{\Psi}(x,\mu)\right]\frac{d\nu}{\nu-\mu}, \tag{3.3}$$

where

$$\underset{\approx}{G}(\nu,\mu) = \frac{3}{2}\begin{bmatrix}1-\nu^2 & 0\\ \nu^2-\mu^2 & 1-\mu^2\end{bmatrix}, \tag{3.4}$$

and

$$\mathbf{H}(\nu,\mu) = \begin{bmatrix} 1 & 0 \\ \dfrac{-\mu(\nu-\mu)}{1-\mu^2} & 1+\dfrac{(-\mu^2+\frac{2}{3})(\nu-\mu)}{\mu(1-\mu^2)} \end{bmatrix} . \tag{3.5}$$

We operate on $\nu\boldsymbol{\Psi}(x,\nu)$ with $\left(\mu\frac{\partial}{\partial x}+1\right)$ and note the resultant identity (1,5)

$$\left(\mu\frac{\partial}{\partial x}+1\right)\nu\boldsymbol{\Psi}(x,\nu) = \mu\left(\nu\frac{\partial}{\partial x}+1\right)\boldsymbol{\Psi}(x,\nu) + (\nu-\mu)\boldsymbol{\Psi}(x,\nu)$$

$$= \mu\,\mathbf{P}(x,\nu) + (\nu-\mu)\boldsymbol{\Psi}(x,\nu). \tag{3.6}$$

We now operate on $\boldsymbol{\Psi}_0(x,\mu)$ with $\left(\mu\frac{\partial}{\partial x}+1\right)$ which, after employing Eqs. (3.2) and (3.6), yields

$$\left(\mu\frac{\partial}{\partial x}+1\right)\boldsymbol{\Psi}_0(x,\mu) = \frac{1}{2}\int_{-1}^{1}\mathbf{G}(\nu,\mu)\Big[\mu\,\mathbf{P}(x,\nu) +$$

$$+(\nu-\mu)\boldsymbol{\Psi}(x,\nu) - \mu\,\mathbf{H}(\nu,\mu)\mathbf{P}(x,\mu)\Big]\frac{d\nu}{\nu-\mu} . \tag{3.7}$$

If we define the moments

$$\rho_i^{(n)}(x) = \int_{-1}^{1}\nu^n\,\Psi_i(x,\nu)\,d\nu, \quad i=1,2, \tag{3.8}$$

then Eq. (3.1) can be written as

$$\underset{\sim}{\rho}(x,\mu) = \frac{3}{8}\begin{bmatrix} 2(1-\mu^2)\rho_1^{(0)}(x) - (2-3\mu^2)\rho_1^{(2)}(x) + \mu^2\rho_2^{(0)}(x) \\ \rho_1^{(2)}(x) + \rho_2^{(0)}(x) \end{bmatrix}. \tag{3.9}$$

Using Eqs. (3.4), (3.5), (3.8) and (3.9), we find that the right side of Eq. (3.7) reduces to $\underset{\sim}{\rho}(x,\mu)$. A comparison of this result with the original equation of transfer Eq. (2.1) or (3.2), indicates that $\underset{\sim}{\Psi}_0(x,\mu)$ is a "particular solution" to Eq. (3.2). To obtain the "general solution" to Eq. (3.2), a solution to the homogeneous equation

$$\left(\mu\frac{\partial}{\partial x}+1\right)\underset{\sim}{\Psi}_1(x,\mu) = 0 \tag{3.10}$$

must be added to $\underset{\sim}{\Psi}_0(x,\mu)$. Solving Eq. (3.10) by separation of variables, we obtain a solution

$$\underset{\sim}{\Psi}_1(x,\mu) = -\frac{1}{2}\underset{\sim}{F}(\mu)\, e^{-x/\mu}, \tag{3.11}$$

where $\underset{\sim}{F}(\mu)$ is a column matrix with elements $F_1(\mu)$ and $F_2(\mu)$ which are arbitrary functions of μ and whose forms are to be determined by appropriate boundary conditions. The "general solution" is then written as

$$\underset{\sim}{\Psi}(x,\mu) = \underset{\sim}{\Psi}_0(x,\mu) + \underset{\sim}{\Psi}_1(x,\mu), \tag{3.12}$$

and by using Eqs. (3.3) and (3.11), we obtain

$$\underset{\sim}{\Psi}(x,\mu) = \frac{1}{2}\int_{-1}^{1} \underset{\approx}{G}(\nu,\mu)\Big[\nu\underset{\sim}{\Psi}(x,\nu) -$$

$$-\mu \underset{\approx}{H}(\nu,\mu)\underset{\sim}{\Psi}(x,\mu)\Big]\frac{d\nu}{\nu-\mu} - \frac{1}{2}\underset{\sim}{F}(\mu)e^{-x/\mu}. \quad (3.13)$$

Interpreting each part of the above integrals as Cauchy principal-valued-integrals, we note that the second term on the right of Eq. (3.13) can be integrated explicitly, and we obtain the singular integral equation

$$\underset{\approx}{\Lambda}(\mu)\underset{\sim}{\Psi}(x,\mu) - P\int_{-1}^{1}\underset{\approx}{G}(\nu,\mu)\underset{\sim}{\Psi}(x,\nu)\frac{\nu\, d\nu}{\nu-\mu} = -\underset{\sim}{F}(\mu)e^{-x/\mu}, \quad (3.14)$$

where we define

$$\underset{\approx}{\Lambda}(\mu) = \begin{bmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{bmatrix} \quad (3.15)$$

and the $\lambda_i(\mu)$ are given by Eq. (2.13). The transfer equations have now been transformed from the integrodifferential form of (3.2) into singular integral equations in which the spatial variable enters only as a parameter.

If the function $\underset{\sim}{F}(\mu)$ is known, the singular integral equation can be solved for $\underset{\sim}{\Psi}(x,\mu)$ by standard methods (4). However, before presenting this method for solving for $\underset{\sim}{\Psi}(x,\mu)$, we illustrate how $\underset{\sim}{F}(\mu)$ is obtained by considering Milne's problem for a semi-infinite half-space medium. We measure x positive into the medium with $x = 0$

at the boundary. The boundary conditions (6) for the problem are that there be no radiation incident on the boundary and that the solution be of exponential order at infinity, i.e.,

$$\underset{\sim}{\Psi}(0,\mu) = 0, \qquad \mu>0, \tag{3.16}$$

and

$$\lim_{x\to\infty} e^{-x}\,\underset{\sim}{\Psi}(x,\mu) = 0. \tag{3.17}$$

Applying boundary condition (3.17) to Eq. (3.14), we find that

$$\underset{\sim}{F}(\mu) = 0, \qquad \mu<0. \tag{3.18}$$

When boundary condition (3.16) is applied, we obtain the expression for $\underset{\sim}{F}(\mu)$:

$$\underset{\sim}{F}(\mu) = \int_0^1 \underset{\approx}{G}(\nu,\mu)\,\underset{\sim}{\Psi}(0,-\nu)\,\frac{\nu\,d\nu}{\nu+\mu}, \quad \mu>0, \tag{3.19}$$

where we have used the evenness of $\underset{\approx}{G}(\nu,\mu)$ in both ν and μ. We now analytically continue $\underset{\sim}{F}$ to the complex plane of z and write Eq. (3.19) as

$$\underset{\sim}{F}(z) = \int_0^1 \underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}(0,-\nu)\,\frac{\nu\,d\nu}{\nu+z}, \quad \mathrm{Re}\, z>0. \tag{3.20}$$

We likewise extend $\underset{\sim}{\Psi}$ to the complex plane and write the functional equation

$$\underset{\approx}{\Omega}(z)\,\underset{\sim}{\Psi}(x,z) - P\int_{-1}^{1}\underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}(x,\nu)\,\frac{\nu d\nu}{\nu+z} =$$

$$= \begin{cases} -\underset{\sim}{F}(z)\,e^{-x/z}, & \mathrm{Re}\ z>0, \\ 0, & \mathrm{Re}\ z<0, \end{cases} \tag{3.21}$$

where

$$\underset{\approx}{\Omega}(z) = \begin{bmatrix} \Omega_1(z) & 0 \\ 0 & \Omega_2(z) \end{bmatrix} \tag{3.22}$$

with $\Omega_1(z)$ defined by Eq. (2.8) and $\Omega_2(z) = \Omega_1(z) + 2$.

It is apparent from Eq. (3.19) that if $\underset{\sim}{\Psi}(0,-\nu)$, the emergent intensity, were known, then we could solve for $\underset{\sim}{F}(\mu)$, i.e. the intensity at any point in the system is given by its behavior at the boundary. A singular integral equation for $\underset{\sim}{\Psi}(0,-\mu)$ can be obtained from Eq. (3.14) by setting $x=0$ and restricting the direction cosine to values less than zero. After using the boundary condition (3.16) and the evenness of $\underset{\approx}{\Lambda}(\mu)$ in μ , we find a singular integral equation for $\underset{\sim}{\Psi}(0,-\mu)$:

$$\underset{\approx}{\Lambda}(\mu)\,\underset{\sim}{\Psi}(0,-\mu) - P\int_{0}^{1}\underset{\approx}{G}(\nu,\mu)\,\underset{\sim}{\Psi}(0,-\nu)\,\frac{\nu d\nu}{\nu-\mu} = 0, \quad \mu>0. \tag{3.23}$$

The last equation can be solved by standard methods as described by Muskhelishvili (4) by defining a sectionally analytic function in the complex plane, cut along (0,1) on the real axis:

$$\underset{\sim}{D}(z) = \frac{1}{2\pi i}\int_0^1 \underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}(0,-\nu)\,\frac{\nu d\nu}{\nu - z}\ , \tag{3.24}$$

Applying the Plemelj formulas (4) to $\underset{\sim}{D}(z)$, we obtain

$$\underset{\sim}{D}^+(\mu) + \underset{\sim}{D}^-(\mu) = \frac{1}{\pi i} P\int_0^1 \underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}(0,-\nu)\,\frac{\nu d\nu}{\nu - z}\ , \tag{3.25}$$

and

$$\underset{\sim}{D}^+(\mu) - \underset{\sim}{D}^-(\mu) = \mu\,\underset{\approx}{G}(\mu,\mu)\,\underset{\sim}{\Psi}(0,-\mu)$$

$$= \frac{3}{2}\mu(1-\mu^2)\,\underset{\sim}{\Psi}(0,-\mu), \tag{3.26}$$

where $\underset{\sim}{D}^+(\mu)$ and $\underset{\sim}{D}^-(\mu)$ are the limits of $\underset{\sim}{D}(z)$ as z approaches the cut $(0,1)$ from the upper and lower half planes, respectively. Using the last two equations, we can write Eq. (3.23) as

$$\underset{\approx}{\Omega}^-(\mu)\,\underset{\sim}{D}^+(\mu) - \Omega^+(\mu)\,\underset{\sim}{D}^-(\mu) = 0\ , \tag{3.27}$$

where

$$\Omega^{\pm}(\mu) = \begin{bmatrix} \Omega_1^{\pm}(\mu) & 0 \\ 0 & \Omega_2^{\pm}(\mu) \end{bmatrix} \tag{3.28}$$

and the $\Omega_1^{\pm}(\mu)$ and $\Omega_2^{\pm}(\mu)$ are defined by Eq. (2.21). Our singular integral equation (3.23) has been reduced to a homogeneous Hilbert problem: to find the nonvanishing sectionally analytic function cut along $(0,1)$ with boundary values given by Eq. (3.27). In order to solve this problem we separate Eq. (3.27) into its component equations:

$$D_i^+(\mu) - \frac{\Omega_i^+(\mu)}{\Omega_i^-(\mu)} D_i^-(\mu) = 0, \qquad i = 1,2. \tag{3.29}$$

We now look for nonvanishing sectionally analytic functions $X_i(z)$ with a cut along $(0,1)$ whose boundary values satisfy the equation

$$\frac{X_i^+(\mu)}{X_i^-(\mu)} = \frac{\Omega_i^+(\mu)}{\Omega_i^-(\mu)}, \qquad i = 1,2. \tag{3.30}$$

Siewert and Fraley (6) have found the solutions of the preceding equations to be

$$\begin{aligned} X_1(z) &= \frac{1}{1-z} \exp \frac{1}{\pi} \int_0^1 \arg \Omega_1^+(\mu) \frac{d\mu}{\mu - z} \\ X_2(z) &= \exp \frac{1}{\pi} \int_0^1 \arg \Omega_2^+(\mu) \frac{d\mu}{\mu - z}. \end{aligned} \tag{3.31}$$

With the use of Eq. (3.30), we write Eq. (3.29) as

$$\frac{D_i^+(\mu)}{X_i^+(\mu)} - \frac{D_i^-(\mu)}{X_i^-(\mu)} = 0, \qquad i = 1,2. \tag{3.32}$$

We now consider the functions

$$K_i(z) = \frac{D_i(z)}{X_i(z)}, \tag{3.33}$$

which is analytic in the finite plane except perhaps for a cut along $(0,1)$. Applying Plemelj's formulas to the last equation, we see from Eq. (3.32) that $K_i^+(\mu) - K_i^-(\mu) = 0$, so that the $K_i(z)$ are analytic everywhere in the finite plane. We note from the behavior of $D_i(z)$ and $X_i(z)$ as z approaches infinity that $K_1(z)$ behaves like a constant and that $K_2(z)$ is a polynomial of order one at most. Thus, by Liouville's theorem we can write

$$\underset{\approx}{K}(z) = \frac{1}{2\pi i}\begin{bmatrix} c & 0 \\ 0 & a+bz \end{bmatrix}, \tag{3.34}$$

and

$$\underset{\sim}{D}(z) = \underset{\approx}{K}(z)\ \underset{\sim}{X}(z), \tag{3.35}$$

where a, b, and c are constants to be determined below.

By examining Eqs. (3.20) and (3.24), we find

$$\underset{\sim}{F}(z) = 2\pi i\ \underset{\sim}{D}(-z) \tag{3.36}$$

which, when substituted into Eq. (3.34), yields

$$\begin{aligned} \underset{\sim}{F}(z) &= 2\pi i\ \underset{\approx}{K}(-z)\ \underset{\sim}{X}(-z) \\ &= \begin{bmatrix} c & 0 \\ 0 & a-bz \end{bmatrix} \underset{\sim}{X}(-z). \end{aligned} \tag{3.37}$$

The above equation determines $\underset{\sim}{F}(\mu)$ by letting z tend to $\mu \in (0,1)$.

We now evaluate the constants a, b, and c. Since the net flux in the half-space is constant, we choose

$$2\int_0^1 \mu \widetilde{\begin{bmatrix}1\\1\end{bmatrix}} \underset{\sim}{\Psi}(0,-\mu)\, d\mu = M, \qquad (3.38)$$

where M is the constant flux, the tilde represents a transpose operation, and we have used boundary condition (3.16). From Eqs. (3.26) and (3.35) we have the equation

$$\mu \underset{\sim}{\Psi}(0,-\mu) = \frac{\underset{\sim}{Q}^+(\mu) - \underset{\sim}{Q}^-(\mu)}{\frac{3}{2}(1-\mu^2)}, \qquad \mu > 0$$

$$= \frac{\underset{\approx}{K}(\mu)\left[\underset{\sim}{X}^+(\mu) - \underset{\sim}{X}^-(\mu)\right]}{\frac{3}{2}(1-\mu^2)}, \quad \mu > 0. \qquad (3.39)$$

When the above result is substituted for $\mu \underset{\sim}{\Psi}(0,-\mu)$ into Eq. (3.38), we obtain one equation for the constants:

$$a\Gamma_1 + b\Gamma_2 + c\Gamma_3 = \frac{3}{4}M, \qquad (3.40)$$

where Γ_1, Γ_2, and Γ_3 are constants, independent of a, b, and c, defined by

$$\Gamma_1 = \int_0^1 \gamma_2(\mu) \frac{d\mu}{1-\mu^2}$$

$$\Gamma_2 = \int_0^1 \gamma_2(\mu) \frac{\mu\, d\mu}{1-\mu^2} \qquad (3.41)$$

$$\Gamma_3 = \int_0^1 \gamma_1(\mu)\,\frac{d\mu}{1-\mu^2}$$

with

$$\gamma_i(\mu) = \frac{1}{2\pi i}\left[X_i^+(\mu) - X_i^-(\mu)\right], \qquad i=1,2. \tag{3.42}$$

We obtain two additional equations for the constants by equating the right-hand side of Eqs. (3.24) and (3.35) for $D_2(\pm 1)$ we find

$$(a\pm b)X_2(\pm 1) = \int_0^1 \tfrac{3}{2}\mu^2\Psi_1(0,-\mu)\,d\mu \pm \int_0^1 \tfrac{3}{2}\mu\Psi_1(0,-\mu)\,d\mu\,. \tag{3.43}$$

We now note from Eqs. (3.39) and (3.34) that

$$\tfrac{3}{2}\mu\Psi_1(0,-\mu) = \frac{c}{2\pi i}\,\frac{X_1^+(\mu)-X_1^-(\mu)}{1-\mu^2}, \tag{3.44}$$

which when integrated over $0\leq\mu\leq 1$ yields the second term on the right side of Eq. (3.43) and when multiplied by μ and integrated yields the first term on the right side. The sum of these two integrations is

$$c\int_0^1 \gamma_1(\mu)\,\frac{d\mu}{1-\mu} = c\,\Gamma_4\,, \tag{3.45}$$

and the difference is

$$-c\int_0^1 \gamma_1(\mu)\,\frac{d\mu}{1+\mu} = c\,\Gamma_5\,, \tag{3.46}$$

where Γ_4 and Γ_5 are constants independent of a, b, and c. Combining the last two equations and Eq. (3.43), we obtain the equations

$$a\,X_2(+1) + b\,X_2(+1) - c\,\Gamma_4 = 0$$

and (3.47)

$$a\,X_2(-1) - b\,X_2(-1) - c\,\Gamma_5 = 0$$

which with Eq. (3.40) constitute a set of simultaneous equations that can be solved for a, b, and c. Before solving this system, we perform the integrations which appear in the definitions of the Γ's. To do this we use Cauchy's integral formula for $X_i(z)$:

$$X_i(z) = \frac{1}{2\pi i}\int_C X_i(z')\,\frac{dz'}{z'-z}\,, \qquad i = 1,2. \tag{3.48}$$

where the contour C is a positive oriented loop encircling the cut $(0,1)$.

If we shrink the contour to include only the cut $(0,1)$, we can write

$$X_i(z) = X_i(\infty) + \int_0^1 \gamma_i(\mu)\,\frac{d\mu}{\mu - z}\,, \quad i = 1,2, \tag{3.49}$$

where $\gamma_i(\mu)$ is defined by Eq. (3.42). Noting that $X_1(\infty) = 0$ and $X_2(\infty) = 1$, we find the following relations for the Γ's:

$$\Gamma_1 = \tfrac{1}{2}\left[X_2(-1) - X_2(+1)\right]$$
$$\Gamma_2 = \tfrac{1}{2}\left[2 - X_2(+1) - X_2(-1)\right]$$
$$\Gamma_3 = \tfrac{1}{2}\left[X_1(-1) - X_1(+1)\right]$$
$$\Gamma_4 = -X_1(+1) \tag{3.50}$$
$$\Gamma_5 = -X_1(-1).$$

Using the preceding equations, we find the solution to the simultaneous system of equations (3.47) and (3.40):

$$a = \tfrac{3}{4} M T$$
$$b = \tfrac{3}{4} M \tag{3.51}$$
$$c = -\tfrac{3}{4}\frac{M}{Q}$$

where we define

$$Q = X_2(-1)\, X_1(+1) - X_2(+1)\, X_1(-1)$$

and

$$T = \frac{X_2(+1)\, X_1(-1) + X_2(-1)\, X_1(+1)}{X_2(-1)\, X_1(+1) - X_2(+1)\, X_1(-1)}. \tag{3.52}$$

Substituting the values of a , b , and c above into Eq. (3.37), we find

$$\underset{\sim}{F}(\mu) = \frac{3}{4} M \begin{bmatrix} -\frac{1}{Q} X_1(-\mu) \\ (T-\mu) X_2(-\mu) \end{bmatrix} . \tag{3.53}$$

From Eqs. (3.26), (3.36), and (3.53) we obtain an expression for $\underset{\sim}{\Psi}(0,-\mu)$:

$$\underset{\sim}{\Psi}(0,-\mu) = \frac{3}{4} M \begin{bmatrix} -\frac{1}{a} & 0 \\ 0 & T+\mu \end{bmatrix} \frac{[\underset{\sim}{X}^+(\mu) - \underset{\sim}{X}^-(\mu)]}{3\pi i \mu (1-\mu^2)} . \tag{3.54}$$

Using Eq. (3.30) and the limits of the functions as z approaches infinity, Siewert and Fraley (6) have shown that

$$X_1(z)\; X_1(-z) = \frac{5}{2} \Omega_1(z) \tag{3.55a}$$

and

$$X_2(z)\; X_2(-z) = \frac{1}{2} \Omega_2(z) , \tag{3.55b}$$

If we examine the boundary values of Eq. (3.55a), we find using Eq. (2.22) that

$$\frac{5}{2} \frac{1}{X_1(-\mu)} = \frac{X_1^+(\mu) - X_1^-(\mu)}{\Omega_1^+(\mu) - \Omega_1^-(\mu)}$$

$$\frac{5}{2}\,\frac{1}{X_1(-\mu)} = \frac{X_1^+(\mu) - X_1^-(\mu)}{3\pi i\,\mu(1-\mu^2)} \;. \tag{3.56}$$

By the same fashion, Eq. (3.55b) yields

$$\frac{1}{2}\,\frac{1}{X_2(-\mu)} = \frac{X_2^+(\mu) - X_2^-(\mu)}{3\pi i\,\mu(1-\mu^2)} \;. \tag{3.57}$$

When the last two equations are substituted into Eq. (3.54), we obtain the law of darkening:

$$\underset{\sim}{\Psi}(0,-\mu) = \frac{3}{8} M \begin{bmatrix} \dfrac{-5}{Q\,X_1(-\mu)} \\[2ex] \dfrac{\mu + T}{X_2(-\mu)} \end{bmatrix} \tag{3.58}$$

which is in agreement with the result found by Siewert and Fraley (6) and by Chandrasekhar (3).

We now consider the solution of the singular integral equation (3.14) for $\underset{\sim}{\Psi}(x,\mu)$. In a procedure similar to that used in solving the singular integral equation for $\underset{\sim}{\Psi}(0,-\mu)$, we consider the sectionally analytic function cut along the line $(-1,1)$:

$$\underset{\sim}{D}_0(x,z) = \frac{1}{2\pi i}\int_{-1}^{1} \underset{\sim}{G}(\nu,z)\, \underset{\sim}{\Psi}(x,\nu)\, \frac{\nu\, d\nu}{\nu - z} \;. \tag{3.59}$$

We apply the Plemelj formulas to the above equation and use the resulting boundary values to write Eq. (3.14) as

$$\frac{\Omega_i^+(\mu)}{\Omega_i^-(\mu)} D_{o_i}^-(x,\mu) - D_{o_i}^+(x,\mu) =$$

$$= \begin{cases} \dfrac{\frac{3}{2}\mu(1-\mu^2)F_i(\mu)e^{-x/\mu}}{\Omega_i^-(\mu)}, & \mu>0, \\ 0, & \mu<0, \end{cases} \tag{3.60}$$

which is an inhomogeneous Hilbert problem with a cut along $(-1,1)$. The solutions $X_{o_i}(z)$, $i=1,2$, to the homogeneous Hilbert problems with boundary values

$$\frac{X_{o_i}^+(\mu)}{X_{o_i}^-(\mu)} = \frac{\Omega_i^+(\mu)}{\Omega_i^-(\mu)}, \qquad i=1,2, \tag{3.61}$$

are

$$\begin{aligned} X_{o_1}(z) &= \frac{1}{1-z^2} \exp \frac{1}{\pi}\int_{-1}^{1} \arg \Omega_1^+(\mu) \frac{d\mu}{\mu-z}, \\ &= X_1(z)\, X_1(-z), \end{aligned} \tag{3.62}$$

and

$$\begin{aligned} X_{o_2}(z) &= \exp \frac{1}{\pi}\int_{-1}^{1} \arg \Omega_2^+(\mu) \frac{d\mu}{\mu-z}, \\ &= X_2(z)\, X_2(-z). \end{aligned} \tag{3.63}$$

With Eq. (3.61) we rewrite Eq. (3.60) as

$$\frac{D_{o_i}^-(x,\mu)}{X_{o_i}^-(\mu)} - \frac{D_{o_i}^+(x,\mu)}{X_{o_i}^-(\mu)} = \begin{cases} \dfrac{\frac{3}{2}\mu(1-\mu^2)F_i(\mu)e^{-x/\mu}}{\Omega_i^-(\mu)\, X_{o_i}^+(\mu)}, & \mu>0, \\ 0, & \mu<0. \end{cases} \tag{3.64}$$

We consider the function of x and z

$$K_{0_i}(x,z) = \frac{D_{0_i}(x,z)}{X_{0_i}(z)} + \frac{1}{2\pi i}\int_0^1 \frac{\frac{3}{2}\mu(1-\mu^2)F_i(\mu)e^{-x/\mu}}{\Omega_i^-(\mu)X_{0_i}^+(\mu)}\frac{d\mu}{\mu-z}, \quad (3.65)$$

which is analytic everywhere in the finite complex plane of z except perhaps for a cut along $(0,1)$. Using Plemelj's formulas we find $K_{0_i}^+(x,\mu) - K_{0_i}^-(x,\mu) = 0$, $\mu \in (-1,1)$. By examining the z dependence at infinity of $D_{0_i}(x,z)$ from Eq. (3.59) and $X_{0_i}(z)$ from Eq. (3.62) and (3.63), we find

$$K_{0_i}(x,z) = \frac{1}{2\pi i}\left[\sigma_i(x) + \omega_i(x)\,z\right], \; i=1,2, \quad (3.66)$$

where $\sigma_i(x)$ and $\omega_i(x)$ are coefficients dependent on x to be determined later. We now solve for $D_{0_i}(x,z)$ from Eq. (3.65) to obtain

$$D_{0_i}(x,z) = \frac{X_{0_i}(z)}{2\pi i}\Big[\sigma_i(x) + \omega_i(x)\,z - $$
$$ - \int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)F_i(\nu)e^{-x/\nu}}{\Omega_i^-(\nu)\;X_{0_i}^+(\nu)}\;\frac{d\nu}{\nu-z}\Big], i=1,2. \quad (3.67)$$

The condition of constant flux throughout the medium requires

$$-M = 2\int_{-1}^{1}\mu\left[\Psi_1(x,\mu) + \Psi_2(x,\mu)\right]d\mu, \quad (3.68)$$

where the constant M is the same as that used in Eq. (3.38). We apply the Plemelj formulas to Eq. (3.59) and obtain

$$\mu \Psi_i(x,\mu) = \frac{2}{3}\,\frac{D^+_{0i}(x,\mu) - D^-_{0i}(x,\mu)}{1-\mu^2}\,, \quad i = 1,2. \tag{3.69}$$

We determine the limits $D^+_{0i}(x,\mu)$ and $D^-_{0i}(x,\mu)$ from Eq. (3.67).

The last equation is substituted into our normalization condition (3.68) and we find an equation for the coefficients, i.e.,

$$5\,\omega_1(x) + \omega_2(x) = n_1(x) \tag{3.70}$$

where

$$n_1(x) = -\frac{3}{2}M + \int_0^1 3\nu e^{-x/\nu}\left\{\frac{F_2(\nu)}{Y_2(\nu)} - \frac{F_1(\nu)}{Y_1(\nu)}\right\}d\nu. \tag{3.71}$$

with the definition

$$\begin{aligned} Y_i(\nu) &= \Omega_i^+(\nu)\,\Omega_i^-(\nu) \\ &= \left[\lambda_i(\nu)\right]^2 + \left[\tfrac{3}{2}\pi\nu(1-\nu^2)\right]^2. \end{aligned} \tag{3.72}$$

We examine the z dependence at infinity of $D_{0_2}(x,z)$ in Eqs. (3.59) and (3.67) and find

$$\omega_2(x) = \int_{-1}^{1} \frac{3}{2}\nu\left[\Psi_1(x,\nu) + \Psi_2(x,\nu)\right]d\nu$$

which is the same integral as we had in our constant flux condition (3.68). Thus we obtain another equation for the coefficients:

$$\omega_2(x) = -\frac{3}{4}M. \tag{3.73}$$

The coefficient $\omega_1(x)$ is then completely determined by Eq. (3.70).

From Eq. (3.59) with $\ell = 1$ we find

$$D_{o2}(x,1) = -D_{o1}(x,1),$$

We now substitute for D_{o2} and D_{o1} from Eq. (3.67) with $\ell = 1$, and obtain an equation for the coefficients in Eq. (3.67):

$$5\sigma_1(x) - \sigma_2(x) = n_2(x), \tag{3.74}$$

where the right hand side is

$$n_2(x) = \int_0^1 3\nu(\nu+1)e^{-x/\nu}\left\{\frac{F_2(\nu)}{Y_2(\nu)} - \frac{F_1(\nu)}{Y_1(\nu)}\right\}d\nu, \tag{3.75}$$

and we have made use of Eqs. (3.70) and (3.73).

Using Eqs. (3.69) and (3.73) we can write the following equation for $\Psi_2(x,-\mu)$, $\mu > 0$:

$$\Psi_2(x,-\mu) = \frac{1}{2}\sigma_2(x) + \frac{3}{8}M\mu - \\ -\int_0^1 \frac{3}{2}(1-\nu^2)\frac{F_2(\nu)}{Y_2(\nu)}e^{-x/\nu}\frac{\nu d\nu}{\nu+\mu}, \quad \mu > 0. \tag{3.76}$$

We substitute Eq. (3.76) into the original integrodifferential equation (2.1) for $\Psi_2(x,-\mu)$ and obtain

$$-\frac{1}{2}\mu\left[\frac{d}{dx}\sigma_2(x)-\frac{3}{4}M\right]-\int_0^1(1-\nu^2)\frac{F_2(\nu)}{Y_2(\nu)}e^{-x/\nu}d\nu+$$

$$+\frac{1}{2}\sigma_2(x)=\frac{3}{8}\int_{-1}^1\left[\nu^2\Psi_1(x,\nu)+\Psi_2(x,\nu)\right]d\nu,\quad \mu>0. \quad (3.77)$$

In the last equation we note the right hand side is independent of μ so that the coefficient of μ must vanish. (The term independent of μ reduces to Eq. (3.74).) The vanishing of the above coefficient of μ yields the following differential equation for $\sigma_2(x)$:

$$\frac{d\sigma_2(x)}{dx}=\frac{3}{4}M$$

which has the solution

$$\sigma_2(x)=\frac{3}{4}Mx+\sigma_2(0). \quad (3.78)$$

We are now left with the evaluation of the constant term, $\sigma_2(0)$, in the expression for $\sigma_2(x)$. To find $\sigma_2(0)$ we proceed in the following manner: From the singular integral equation (3.21) with $\mathrm{Re}\, z<0$ and from the definition of $\underset{\sim}{D}_0(x,z)$ in Eq. (3.59), we see that

$$\Omega_2(z)\,\Psi_2(x,z)=2\pi i\,D_{0_2}(x,z),\quad \mathrm{Re}\, z<0. \quad (3.79)$$

We substitute for $D_{0_2}(x,z)$, $\mathrm{Re}\, z<0$, from Eq. (3.67) and note that $\Omega_2(z)$ is nonvanishing, so that we may write Eq. (3.79) as

$$\Psi_2(x,z) = \frac{X_{o_2}(z)}{\Omega_2(z)}\left[\sigma_2(x) - \frac{3}{4}Mz - \right.$$

$$\left. - \int_0^1 \frac{3\nu(1-\nu^2)\,F_2(\nu)e^{-x/\nu}}{Y_2(\nu)}\,\frac{\nu d\nu}{\nu - z}\right], \quad \mathrm{Re}\, z < 0, \tag{3.80}$$

Noting that

$$\frac{X_{o_2}(z)}{\Omega_2(z)} = \frac{1}{2}, \tag{3.81}$$

we find $\sigma_2(0)$ by setting $x = 0$ and substituting for $F_2(\nu)$ from Eq. (3.53) into Eq. (3.80) to obtain

$$\Psi_2(0,-\mu) = \frac{1}{2}\sigma_2(0) + \frac{3}{8}M\mu -$$

$$- \frac{3}{8}M\int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)(T-\nu)}{\Omega_2^-(\nu)\,X_2^+(\nu)}\,\frac{d\nu}{\nu+\mu}, \quad \mu > 0. \tag{3.82}$$

The integrals appearing in this equation can be evaluated by using Cauchy's integral formula on the function $\frac{1}{X_2(z)}$ with a positive oriented contour C encircling the cut $(0,1)$, i.e.,

$$\frac{1}{X_2(z)} = \frac{1}{2\pi i}\int_C \frac{1}{X_2(z')}\,\frac{dz'}{z'-z}$$

$$= B_2(z) + \frac{1}{2\pi i}\int_0^1 \left[\frac{1}{X_2^+(\nu)} - \frac{1}{X_2^-(\nu)}\right]\frac{d\nu}{\nu - z}, \tag{3.83}$$

where $B_2(z)$ is the limit of the function $\frac{1}{X_2(z)}$ as z approaches infinity:

$$B_2(z) = 1.$$

Using Eqs. (2.22) and (3.30), we obtain from Eq. (3.83)

$$\int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)}{X_2^+(\nu)\,\Omega_2^-(\nu)}\,\frac{d\nu}{\nu - z} = 1 - \frac{1}{X_2(z)}\,, \tag{3.84}$$

and similarly

$$\int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)}{X_2^+(\nu)\,\Omega_2^-(\nu)}\,\frac{\nu\,d\nu}{\nu - z} = z - \frac{z}{X_2(z)} + \int_0^1 \gamma_2(\nu)\,d\nu\,, \tag{3.85}$$

where $\gamma_2(\nu)$ is given by Eq. (3.42). Equations (3.84) and (3.85) allow us to write Eq. (3.82) as

$$\Psi_2(0,-\mu) = \tfrac{1}{2}\sigma_2(0) + \tfrac{3}{8}M\mu + \tfrac{3}{8}M\int_0^1 \gamma_2(\nu)\,d\nu - $$
$$- \tfrac{3}{8}M(T+\mu)\left[1 - \frac{1}{X_2(-\mu)}\right],\ \mu > 0. \tag{3.86}$$

We now compare Eq. (3.86) with the law of darkening (3.58), i.e.

$$\Psi_2(0,-\mu) = \tfrac{3}{8}M\,\frac{T+\mu}{X_2(-\mu)}\,,\quad \mu > 0\,,$$

from which we find

$$\sigma_2(0) = \tfrac{3}{4}MT - \tfrac{3}{4}M\int_0^1 \gamma_2(\nu)\,d\nu,$$

so that we can write Eq. (3.78) as

$$\sigma_2(x) = \tfrac{3}{4}M\left[x + T - \int_0^1 \gamma_2(\nu)\,d\nu\right]. \tag{3.87}$$

Substitution of the last equation into Eq. (3.74) completely determines $\sigma_1(x)$. This completes the evaluation of the constants in Eq. (3.67) for $D_{0i}(x,z)$.

From the singular integral equation (3.21) and the definition of $Q_0(x,z)$ in Eq. (3.59) we can write

$$\Psi_i(x,z) = \begin{cases} \dfrac{2\pi i\, D_{0i}(x,z) - F_i(z)\, e^{-x/z}}{\Omega_i(z)}, & \mathrm{Re}\, z > 0, \\[2ex] \dfrac{2\pi i\, D_{0i}(x,z)}{\Omega_i(z)}, & \mathrm{Re}\, z < 0, \end{cases} \tag{3.88}$$

The angular intensities $\Psi_i(x,\mu)$, $i = 1, 2$, are obtained from the last equation by letting z tend to $\mu \in (-1, 1)$.

IV. THE RELATIONSHIP BETWEEN OUR APPROACH AND THE NORMAL MODE EXPANSION METHOD

In order to determine the relationship between our singular integral equation method and the normal mode expansion technique (6) we evaluate the integral

$$\int_{-1}^{1} \nu \tilde{\underline{\Phi}}_i(\mu,\nu)\, \underline{\Psi}(x,\nu)\, d\nu, \qquad i = 1,2, \tag{4.1}$$

where the tilde superscript indicates a transpose of the column vector $\underline{\Phi}_i(\mu,\nu)$. If first we substitute the explicit forms of $\underline{\Phi}_i$ from Eqs. (2.12) we obtain, after rearranging in matrix form:

$$\begin{bmatrix} \int_{-1}^{1} \nu \tilde{\underline{\Phi}}_1(\mu,\nu)\, \underline{\Psi}(x,\nu)\, d\nu \\ \int_{-1}^{1} \nu \tilde{\underline{\Phi}}_2(\mu,\nu)\, \underline{\Psi}(x,\nu)\, d\nu \end{bmatrix} = \mu \begin{bmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{bmatrix} \begin{bmatrix} \Psi_1(x,\mu) \\ \Psi_2(x,\mu) \end{bmatrix} -$$

$$-\mu\, P \int_{-1}^{1} \frac{3}{2} \begin{bmatrix} 1-\nu^2 & 0 \\ \nu^2-\mu^2 & 1-\mu^2 \end{bmatrix} \begin{bmatrix} \Psi_1(x,\nu) \\ \Psi_2(x,\nu) \end{bmatrix} \frac{\nu\, d\nu}{\nu-\mu}. \tag{4.2}$$

The integral (4.1) can also be evaluated by using the general expression for $\underline{\Psi}(x,\mu)$ given by the normal mode expansion (2.15). The two discrete terms in $\underline{\Psi}(x,\mu)$ vanish due to orthogonality relations (2.19) and (2.23) found by Smith and Siewert (7). After changing the order of integration of the two continuum terms, we perform the

ν-integration using Eqs. (2.20), (2.21), and (2.22), and obtain

$$\begin{bmatrix} \int_{-1}^{1} \nu\, \tilde{\underset{\sim}{\Phi}}_1(\mu,\nu)\underset{\sim}{\Psi}(x,\nu)\,d\nu \\ \int_{-1}^{1} \nu\, \tilde{\underset{\sim}{\Phi}}_2(\mu,\nu)\underset{\sim}{\Psi}(x,\nu)\,d\nu \end{bmatrix} = \int_{-1}^{1} \eta e^{-x/\eta} \begin{bmatrix} \alpha(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta) \\ \beta(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta) \end{bmatrix} \delta(\eta-\mu)\,d\eta,$$

$$= \mu e^{-x/\mu} \begin{bmatrix} \alpha(\mu)\Omega_1^+(\mu)\Omega_1^-(\mu) \\ \beta(\mu)\Omega_2^+(\mu)\Omega_2^-(\mu) \end{bmatrix}. \tag{4.3}$$

We equate the right hand sides of Eqs. (4.2) and (4.3) and compare the result with Eq. (3.14) which shows that the relationship between our method and the normal mode expansion method is

$$\begin{bmatrix} F_1(\mu) \\ F_2(\mu) \end{bmatrix} = -\begin{bmatrix} \alpha(\mu)\Omega_1^+(\mu)\Omega_1^-(\mu) \\ \beta(\mu)\Omega_2^+(\mu)\Omega_2^-(\mu) \end{bmatrix}. \tag{4.4}$$

Equation (4.4) indicates that solving for $\underset{\sim}{F}(\mu)$ in the singular integral equation provides an alternate means for obtaining the normal mode expansion coefficients. To illustrate, we consider the Milne problem for a half-space with a constant, distributed source $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$. The boundary conditions, similar to those in Eqs. (3.16) and (3.17), are

$$\underset{\sim}{\Psi}_S(0,\mu) = 0, \qquad \mu > 0, \tag{4.5}$$

and

$$\lim_{x \to \infty} e^{-x} \underset{\sim}{\Psi}_s(x,\mu) = 0 \;, \tag{4.6}$$

where the angular intensity vector $\underset{\sim}{\Psi}_s(x,\mu)$ represents two perpendicularly polarized intensities in the medium with sources.

The integrodifferential equation for the problem with sources (cf. Eq. (2.1)) is

$$\left(\mu \frac{\partial}{\partial x} + 1\right) \underset{\sim}{\Psi}_s(x,\mu) = \int_{-1}^{1} \underset{\sim}{R}(\nu,\mu)\, \underset{\sim}{\Psi}_s(x,\nu)\, d\nu + \underset{\sim}{S} \;, \tag{4.7}$$

and has a solution

$$\underset{\sim}{\Psi}_s(x,\mu) = \underset{\sim}{\Psi}_\infty(x,\mu) + A_{+s} \underset{\sim}{\Phi}_+ + A_{-s} \underset{\sim}{\Psi}_-(x,\mu) + $$

$$+ \int_{-1}^{1} \alpha_s(\eta)\, \underset{\sim}{\Phi}_1(\eta,\mu)\, e^{-x/\eta}\, d\eta + \int_{-1}^{1} \beta_s(\eta)\, \underset{\sim}{\Phi}_2(\eta,\mu)\, e^{-x/\eta}\, d\eta \;, \tag{4.8}$$

where $\underset{\sim}{\Psi}_\infty(x,\mu)$ is the particular solution to the constant source problem. We find the following particular solution to Eq. (4.7):

$$\underset{\sim}{\Psi}_\infty(x,\mu) = \begin{bmatrix} -\frac{3}{4}(S_1+S_2)(x^2-2x\mu) - 5S_1\mu^2 \\ -\frac{3}{4}(S_1+S_2)(x^2-2x\mu) - \frac{3}{2}(S_1+S_2)\mu^2 - \frac{3}{2}S_1 + \frac{5}{2}S_2 \end{bmatrix} . \tag{4.9}$$

We now proceed as in Eq. (4.3) and substitute $\underset{\sim}{\Psi}_s(x,\mu)$ into the integral (4.1) and obtain

$$\begin{bmatrix} \int_{-1}^{1} \nu \, \tilde{\Phi}_1(\mu,\nu) \, \Psi_s(x,\nu) \, d\nu \\ \int_{-1}^{1} \nu \, \tilde{\Phi}_2(\mu,\nu) \, \Psi_s(x,\nu) \, d\nu \end{bmatrix} = \mu e^{-x/\mu} \begin{bmatrix} \alpha_s(\mu) \, \Omega_1^+(\mu) \, \Omega_1^-(\mu) \\ \beta_s(\mu) \, \Omega_2^+(\mu) \, \Omega_2^-(\mu) \end{bmatrix} +$$

$$+ \mu \begin{bmatrix} 2S_1 \\ 4S_2 - 3\mu^2(S_1+S_2) \end{bmatrix} , \tag{4.10}$$

from which we write (cf. Eq. (3.14))

$$\Lambda(\mu) \, \Psi_s(x,\mu) - P\int_{-1}^{1} G(\nu,\mu) \, \Psi_s(x,\nu) \, \frac{\nu \, d\nu}{\nu-\mu} = -\mathcal{J}_s(\mu) e^{-x/\mu} + N(\mu) , \tag{4.11}$$

where

$$N(\mu) = \begin{bmatrix} 2S_1 \\ 4S_2 - 3\mu^2(S_1+S_2) \end{bmatrix} , \tag{4.12}$$

and

$$\mathcal{J}_s(\mu) = - \begin{bmatrix} \alpha_s(\mu) \, \Omega_1^+(\mu) \, \Omega_1^-(\mu) \\ \beta_s(\mu) \, \Omega_2^+(\mu) \, \Omega_2^-(\mu) \end{bmatrix} . \tag{4.13}$$

We now determine $\mathcal{J}_s(\mu)$ in a procedure similar to that used in Section III. We proceed to solve Eq. (4.11) subject to the boundary conditions (4.5) and (4.6). In order that our solution be of exponential order as x approaches infinity (Eq. (4.6)), we must have

$$\mathcal{J}_s(\mu) = 0 \quad , \quad \mu < 0 , \tag{4.14}$$

and from the condition of no incident radiation on the interface from the vacuum region (Eq. (4.5)), we obtain

$$\underset{\sim}{f}_s(\mu) = \int_0^1 \underset{\approx}{G}(\nu,\mu)\,\underset{\sim}{\Psi}_s(0,-\nu)\,\frac{\nu\,d\nu}{\nu+\mu} + \underset{\sim}{N}(\mu), \quad \mu>0, \qquad (4.15)$$

where we have used the fact that $\underset{\sim}{N}(\mu)$ is an even function of μ. When we extend $\underset{\sim}{f}_s$ to the complex plane, we obtain

$$\underset{\sim}{f}_s(z) = \int_0^1 \underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}_s(0,-\nu)\,\frac{\nu\,d\nu}{\nu+z} + \underset{\sim}{N}(z), \quad \mathrm{Re}\, z>0. \qquad (4.16)$$

Using Eq. (4.5) and restricting the direction cosine to values less than zero, we obtain a singular integral equation for $\underset{\sim}{\Psi}_s(0,-\mu)$:

$$\underset{\approx}{\Lambda}(\mu)\,\underset{\sim}{\Psi}_s(0,-\mu) - P\int_0^1 \underset{\approx}{G}(\nu,\mu)\,\underset{\sim}{\Psi}_s(0,-\nu)\,\frac{\nu\,d\nu}{\nu-\mu} = \underset{\sim}{N}(\mu), \quad \mu>0. \qquad (4.17)$$

We define a sectionally analytic function $\underset{\sim}{E}(z)$, cut along the line $(0,1)$, as

$$\underset{\sim}{E}(z) = \frac{1}{2\pi i}\int_0^1 \underset{\approx}{G}(\nu,z)\,\underset{\sim}{\Psi}_s(0,-\nu)\,\frac{\nu\,d\nu}{\nu-z}\,. \qquad (4.18)$$

Applying the Plemelj formulas (4), we obtain, in component form,

$$E_i^+(\mu) - \frac{\Omega_i^+(\mu)}{\Omega_i^-(\mu)}\,E_i^-(\mu) = \frac{3}{2}\,\mu(1-\mu^2)\,\frac{N_i(\mu)}{\Omega_i^-(\mu)}\,, \quad i=1,2, \qquad (4.19)$$

which is an inhomogeneous Hilbert problem. We seek a sectionally analytic function $\underset{\sim}{E}(z)$ whose boundary values satisfy Eq. (4.19).

Before solving the inhomogeneous problem, we must first consider the homogeneous Hilbert problem with boundary values given by Eq. (3.30), which has solutions $X_1(z)$ and $X_2(z)$ given by Eq. (3.31). Equation (3.30) allows us to rewrite Eq. (4.19) as

$$\frac{E_i^+(\mu)}{X_i^+(\mu)} - \frac{E_i^-(\mu)}{X_i^-(\mu)} = \frac{3}{2}\mu(1-\mu^2)\frac{N_i(\mu)}{X_i^+(\mu)\Omega_i^-(\mu)}, \quad i=1,2. \tag{4.20}$$

We now consider the functions

$$R_i(z) = \frac{E_i(z)}{X_i(z)} - \frac{1}{2\pi i}\int_0^1 \frac{\frac{3}{2}\mu(1-\mu^2)N_i(\mu)}{X_i^+(\mu)\Omega_i^-(\mu)}\frac{d\mu}{\mu-z}. \tag{4.21}$$

From Plemelj's formulas and Eq. (4.20) we see that $R_i^+(\mu) - R_i^-(\mu) = 0$, so that $R_i(z)$ is analytic everywhere in the finite plane. By examining the behavior of the right hand side of Eq. (4.21) in the limit as z approaches infinity and employing Liouville's theorem, we can write

$$\underset{\approx}{R}(z) = \frac{1}{2\pi i}\begin{bmatrix} d_1 & 0 \\ 0 & d_2+d_3 z\end{bmatrix}, \tag{4.22}$$

where the d 's are undetermined constants.

We now solve Eq. (4.21) for $E_i(z)$ to obtain

$$E_i(z) = X_i(z)\left[R_i(z) + \frac{1}{2\pi i}\int_0^1 \frac{\frac{3}{2}\mu(1-\mu^2)N_i(\mu)}{X_i^+(\mu)\Omega_i^-(\mu)}\frac{d\mu}{\mu-z}\right]. \tag{4.23}$$

The integrals in this equation are similar in form to those in Eq. (3.82). Again using Cauchy's integral formula on $\frac{1}{X_i(z)}, i=1,2,$ we find

$$\int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)}{X_1^+(\nu)\,\Omega_1^-(\nu)}\,\frac{d\nu}{\nu-z} = \mathcal{B}_1(z) - \frac{1}{X_1(z)}\,, \tag{4.24}$$

where the limit of $\frac{1}{X_1(z)}$ as z approaches infinity is

$$\begin{aligned}\mathcal{B}_1(z) &= -z + \int_0^1 \gamma_1(\nu)\;\nu\,d\nu \\ &= -z + u_1\,,\end{aligned} \tag{4.25}$$

and

$$\int_0^1 \frac{\frac{3}{2}\nu(1-\nu^2)}{X_2^+(\nu)\,\Omega_2^-(\nu)}\,\frac{\nu^2 d\nu}{\nu-z} = z^2\left[1-\frac{1}{X_2(z)}\right] + zV_0 + V_0^2 + V_1\,, \tag{4.26}$$

where the moments are

$$V_\ell = \int_0^1 \gamma_2(\nu)\;\nu^\ell d\nu. \tag{4.27}$$

Equations (4.24), (4.26), and (3.85) allow us to evaluate the integrals in Eq. (4.23) and write these equations with Eq. (4.12) and (4.22) as

$$E_1(z) = \frac{1}{2\pi i}\left\{d_1 + 2S_1(u_1 - z)\right\} X_1(z) - \frac{1}{2\pi i}\,2S_1 \tag{4.28a}$$

$$E_2(z) = \frac{1}{2\pi i}\Big\{d_2 + d_3 z + 4S_2 - $$

$$-3(S_1+S_2)\left[z^2 + zV_0 + V_0^2 + V_1\right]\Big\} X_2(z) -$$

$$-\frac{1}{2\pi i}\left\{4S_2 - 3(S_1+S_2)z^2\right\}. \tag{4.28b}$$

From Eqs. (4.16) and (4.18) we find the relationship

$$\underset{\sim}{\mathcal{J}}_s(z) = 2\pi i\, \underset{\sim}{E}(-z) + \underset{\sim}{N}(z), \tag{4.29}$$

which from Eqs. (4.28) and (4.12) can be written

$$\mathcal{J}_{s_1}(z) = \left\{d_1 + 2S_1(u_1+z)\right\} X_1(-z) \tag{4.30a}$$

$$\mathcal{J}_{s_2}(z) = \Big\{d_2 - d_3 z + 4S_2 -$$

$$-3(S_1+S_2)\left[z^2 - zV_0 + V_0^2 + V_1\right]\Big\} X_2(-z). \tag{4.30b}$$

Using Eqs. (4.30) in Eq. (4.13), we obtain the normal mode expansion coefficients for the constant source problem:

$$\alpha_s(\mu) = \frac{-\left\{d_1 + 2S_1(u_1+\mu)\right\} X_1(-\mu)}{\Omega_1^+(\mu)\,\Omega_1^-(\mu)} \tag{4.31}$$

$$\beta_s(\mu) = -\frac{1}{\Omega_2^+(\mu)\,\Omega_2^-(\mu)}\Big\{d_2 - d_3\mu + 4S_2 -$$

$$-3(S_1+S_2)\left[\mu^2 - V_0\mu + V_0^2 + V_1\right]\Big\} X_2(-\mu). \tag{4.32}$$

We have found the expansion coefficients to within the three undetermined constants d_1 , d_2 , and d_3 . The procedure for finding the constants parallels that for the source-free problem of Section III. As before we normalize the exit current at the vacuum interface:

$$K = 2\int_0^1 \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\sim} \underset{\sim}{\Psi}_S(0,-\mu)\, d\mu , \tag{4.33}$$

where we have used boundary condition (4.5). From Eqs. (4.18) and (4.28) we find an expression for $\mu \Psi_{S_i}(0,-\mu)$, $i = 1, 2,$ which we substitute into our normalization equation (4.33) to obtain

$$\begin{aligned} K = 2\int_0^1 \tfrac{2}{3} \Big\{ & \gamma_1(\mu)\left[d_1 + 2S_1(u_1 - \mu)\right] + \\ & + \gamma_2(\mu)\Big[d_2 + d_3\mu + 4S_2 - \\ & - 3(S_1+S_2)\left[\mu^2 + V_0\mu + V_0^2 + V_1\right]\Big]\Big\} \frac{d\mu}{1-\mu^2} , \end{aligned} \tag{4.34}$$

where the $\gamma_i(\mu)$ are given by Eq. (3.42). We now write this equation as

$$d_1\Gamma_3 + d_2\Gamma_1 + d_3\Gamma_2 = r_1 \tag{4.35}$$

where

$$\begin{aligned} r_1 = \tfrac{3}{4}K - 2S_1u_1\Gamma_3 + 2S_1\Gamma_5 - 4S_2\Gamma_1 + \\ + 3(S_1+S_2)\left[\Gamma_2 - \Gamma_6 + V_0\Gamma_2 + (V_0^2+V_1)\Gamma_1\right] , \end{aligned} \tag{4.36}$$

and the Γ_i , $i = 1, .., 5$, which are independent of the d 's, are given in Eqs. (3.50). The constant Γ_6 is defined as

$$\Gamma_6 = \int_0^1 \gamma_2(\mu) \frac{\mu d\mu}{1+\mu} . \tag{4.37}$$

Using the same procedure as before in evaluating the other Γ_i , we find

$$\Gamma_6 = 1 + V_0 - X_2(-1) , \tag{4.38}$$

where V_0 is defined in Eq. (4.27). Equation (4.35) represents one equation for the three undetermined constants in terms of known constants on the right hand side.

We find two additional equations for the three constants with the same procedure as used in Section III. We set $z = \pm 1$ in Eq. (4.18) and find

$$E_2(\pm 1) = \frac{1}{2\pi i} \int_0^1 \frac{3}{2} \nu^2 \Psi_{S_1}(0,-\nu) d\nu \pm \frac{1}{2\pi i} \int_0^1 \frac{3}{2} \nu \Psi_{S_1}(0,-\nu) d\nu, \tag{4.39}$$

We note that the last equation contains integrals of $\Psi_{S_1}(0,-\nu)$ and that from Eq. (4.18) we can obtain

$$\frac{3}{2} \mu \Psi_{S_1}(0,-\mu) = \frac{\gamma_1(\mu) \left[d_1 + 2 S_1(u_1 - \mu) \right]}{1-\mu^2} , \tag{4.40}$$

which we have also used in finding Eq. (4.34). We substitute Eq. (4.40) into Eq. (4.39), perform the integrations, and obtain the following equations:

$$E_2(\pm 1) = \frac{1}{2\pi i}\left(\Gamma_3+\Gamma_5\right)\left(d_1+2S_1u_1-2S_1\right) -$$

$$-\frac{1}{2\pi i}\,2S_1\left[X_1(-1)-u_0\right] \pm$$

$$\pm\frac{1}{2\pi i}\left[\Gamma_3\left(d_1+2S_1u_1-2S_1\right)-2S_1\Gamma_5\right], \qquad (4.41)$$

where

$$u_n = \int_0^1 \gamma_1(\nu)\,\nu^n\,d\nu.$$

From Eq. (4.28b) we can obtain another equation for $E_2(\pm 1)$:

$$E_2(\pm 1) = \frac{1}{2\pi i}X_2(\pm 1)\Big\{d_2 \pm d_3 + 4S_2 -$$

$$-3(S_1+S_2)\left[1 \pm V_0 + V_0^2 + V_1\right]\Big\} -$$

$$-\frac{1}{2\pi i}\left\{S_2 - 3S_1\right\}. \qquad (4.42)$$

We now obtain the following by equating Eqs. (4.41) and (4.42) for $E_2(+1)$ and $E_2(-1)$ using Eq. (3.50) for the Γ_i :

$$\begin{cases} d_1X_1(+1) + d_2X_2(+1) + d_3X_2(+1) = r_2 \\ d_1X_1(-1) + d_2X_2(-1) - d_3X_2(-1) = r_3, \end{cases} \qquad (4.43)$$

where

$$r_2 = 2S_1(1-u_1)X_1(+1) + 2S_1u_0 - 4X_2(+1)S_2 +$$

$$+ S_2 - 3S_1 + 3(S_1+S_2)X_2(+1)\left[1+V_0+V_0^2+V_1\right] \qquad (4.44)$$

$$r_3 = 2S_1(1-u_1)\chi_1(-1) + 2S_1 u_0 - 4\chi_2(-1)S_2 -$$
$$- S_2 + 3S_1 + 3(S_1+S_2)\chi_2(-1)\left[1 - v_0 + v_0^2 + v_1\right]$$

Equations (4.35) and (4.43) constitute a system of three equations for the three unknowns, d_1 , d_2 , and d_3 having the following unique solution:

$$d_1 = \frac{1}{Q}\left\{-r_1 + \left[1-\chi_2(+1)\right]\left[r_2\chi_2(-1) + r_3\chi_2(+1)\right]\right\}$$

$$d_2 = \frac{1}{Q}\left\{r_1 W + \frac{1}{2}\left[r_3\chi_1(+1) - r_2\chi_1(-1)\right] + \frac{1}{2}W(r_2 - r_3)\right\}$$

$$d_3 = \frac{1}{2}\left\{2r_1 + r_2 + r_3\right\} \qquad (4.45)$$

where Q is defined in Eq. (3.52) and

$$W = \chi_2(+1)\chi_1(-1) + \chi_2(-1)\chi_1(+1) .$$

With the d's in Eq. (4.45) we completely determine the expansion coefficients $\alpha_s(\mu)$ and $\beta_s(\mu)$ in Eqs. (4.31) and (4.32).

The discrete expansion coefficients A_ξ and A_ζ are determined from the integral (4.1) with Φ_+ and $\Psi_-(a,\mu)$, given by Eqs. (2.9) and (2.10), replacing $\Phi_i(x,\mu)$. Substituting for the normal mode expansion $\Psi_s(x,\mu)$ from Eq. (4.8) we find, using the orthogonality relations in Section II, that

$$\int_{-1}^{1} \mu \, \tilde{\underset{\sim}{\Phi}}_{+} \underset{\sim}{\Psi}_{s}(x,\mu)\, d\mu = -\tfrac{4}{3} A_{\bar{s}} \, . \tag{4.46}$$

If instead we substitute for $\underset{\sim}{\Phi}_{+}$, we obtain our normalization condition (cf. Eq. (4.33)):

$$\int_{-1}^{1} \mu \widetilde{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \underset{\sim}{\Psi}_{s}(x,\mu)\, d\mu = -\frac{\mathcal{K}}{2} \, . \tag{4.47}$$

From the last two equations we see that

$$A_{\bar{s}} = \tfrac{3}{8} \mathcal{K} \, . \tag{4.48}$$

In the same fashion as in Eq. (4.46), we find, when we substitute the expansion (4.8),

$$\int_{-1}^{1} \mu \, \tilde{\underset{\sim}{\Psi}}_{-}(0,\mu) \underset{\sim}{\Psi}_{s}(x,\mu)\, d\mu = -\tfrac{4}{3} A_{+} + \tfrac{28}{5} S_1 - \tfrac{16}{15} S_2 \, . \tag{4.49}$$

If in Eq. (4.49) we apply the boundary condition (4.5) and use the explicit form of $\underset{\sim}{\Psi}_{-}(0,\mu)$ from Eq. (2.10), we obtain

$$\int_{0}^{1} \mu \widetilde{\begin{bmatrix} -\mu \\ -\mu \end{bmatrix}} \underset{\sim}{\Psi}_{s}(0,-\mu)\, d\mu = -\int_{0}^{1} \mu^2 \left[\Psi_{s_1}(0,-\mu) + \Psi_{s_2}(0,-\mu) \right] d\mu \, . \tag{4.50}$$

From Eqs. (4.18) and (4.28) we find expressions for $\underset{\sim}{\Psi}_{s_i}(0,-\mu)$, $i = 1, 2$, which we substitute into the last equation to obtain

$$\int_0^1 \mu \overset{\sim}{\begin{bmatrix}-\mu \\ -\mu\end{bmatrix}} \underset{\sim}{\Psi}_3(0,-\mu)\,d\mu = -\frac{2}{3}\int_0^1 \Big\{ \gamma_1(\mu)\left[d_1 + 2S_1(u_1-\mu)\right] +$$

$$+ \gamma_2(\mu) \Big[d_2 + d_3\mu + 4S_2 -$$

$$- 3(S_1+S_2)\left[\mu^2 + v_0\mu + v_0^2 + v_1\right]\Big]\Big\} \frac{\mu\, d\mu}{1-\mu^2} \,. \qquad (4.51)$$

The integrals in the last equation can be evaluated by using the Γ_ℓ in Eqs. (3.41), (3.50), and (4.37) and by using Cauchy's integral formula on $\chi_i(z)$, $i = 1, 2$, as in Eq. (3.48). After performing the integrations in Eq. (4.51), we equate the results to the right hand side of Eq. (4.49) and find

$$A_{+3} = \frac{1}{2} d_1(\Gamma_3+\Gamma_5) + \frac{1}{2} d_2\Gamma_2 + \frac{1}{2} d_3(\Gamma_2 - \Gamma_6)$$

$$+ \frac{21}{5} S_1 - \frac{4}{5} S_2 + S_1 u_1(\Gamma_3+\Gamma_5) - S_1\left(\Gamma_3 + \chi_1(-1)\right)$$

$$+ 2S_2\Gamma_2 - \frac{3}{2}(S_1+S_2)\left[\Gamma_2 - v_1 + (v_0^2+v_1)\Gamma_2 + v_0(\Gamma_2 - \Gamma_6)\right], \qquad (4.52)$$

where the d 's are given in Eq. (4.45).

V. CONCLUSIONS

In summary, we have presented a method for obtaining exact solutions for the perpendicularly polarized intensities in an electron-scattering stellar atmosphere. The solution was obtained by transforming the integrodifferential form of the radiative transfer equations for the angular intensities. As an example of this procedure, we have solved the Milne problem and the law of darkening. The connection between our singular integral equation and the normal mode expansion method was obtained using orthogonality relations. We found that this lead to an alternate procedure for obtaining the normal mode expansion coefficients. We have illustrated the procedure by solving the constant, distributed source problem for a semi-infinite medium from which we find the discrete and continuum expansion coefficients. It appears that neither method presents an appreciable advantage over the other. In the normal mode method, orthogonality relations are required to determine expansion coefficients easily; whereas, in our approach it is not always easy to find the proper number of conditions to determine the coefficients.

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