MATRIX ANALYSIS OF RIGID SPACE FRAMES

by

Thomas A. Grow

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Applied Mechanics

APPROVED:

Director of Graduate Studies

Dean of Engineering

APPROVED:

Head of Department

Supervisor

May, 1953

Blacksburg, Virginia

I. TABLE OF CONTENTS

List of Figures and Tables

II. DESCRIPTION OF SYMBOLS USED

a	Distance from top of column to resultant force.
a_{ij}	Element in row i and column j of a matrix.
E	Modulus of elasticity.
G	Modulus of rigidity.
I_i	Moment of inertia with respect to i-axis.
K_{ijk}	Stiffness of jk in resisting flexural moment about the i-axis.
K'_{ij}	Stiffness of member ij in torsion.
L_{ij}	Length of member ij. (Center-to-center.)
M_{ijk}	Bending moment at end j of jk about i-axis.
M_{iFjk}	Fixed-end moment at end j of jk caused by moment about i-axis.
P_i	Force or resultant force parallel to i-axis.
x_i	Element in row i of column matrix. Represents redundants in slope-deflection equations.
y_i	Element in row i of column matrix. Represents constants from loading conditions.
Δ_{ij}	Translation of joint j caused by moments about the i-axis.
θ_{ij}	Rotation of joint j caused by moments about the i-axis.
ρ_{ijk}	Rotation of member jk about the i-axis.
Σ	Algebraic summation.
ϕ	Angle of twist of a structural member in torsion.

III. INTRODUCTION

The aim of a structural designer, whether dealing with buildings, bridges, airplanes, or some other type of structure, is to design the most economical structure that will do the job required of it with safety.

In the latter part of the nineteenth century and the early part of the twentieth century, many designers went to great lengths to make their structures statically determinate and often inserted pins and hinges in the structure in order to achieve this end. With the advent of reinforced concrete and welded steel structures, however, it became evident that continuous, hence statically indeterminate structures were easier and cheaper to build and methods for designing them were devised.

One of the methods of analysis used extensively since its introduction to this country in 1915 by George A. Maney is the slope-deflection method, in which the rotations and translations of the joints are redundants and must be determined before the bending moments can be calculated. This method, as well as the other classical methods, involves the solving of simultaneous linear equations [18] - a procedure which can become very tedious in a structure of any size.

Another method, perhaps typical of newer methods of analysis, which has won wide acceptance and a great deal of popularity in its relatively short existance is Hardy Cross's moment distribution. The use of this method gives the analyst a physical picture of joint movements and at the same time saves a great deal of time in the analysis of many structures.

The usual procedure when analyzing a building frame or other three-dimensional structure is to consider it as a series of planar structures and ignore the effects of torsion in the members. The use of an appropriate factor of safety, or factor of ignorance, as many insist, makes this an acceptably safe procedure, but can also cause quite a waste of materials. The effect of torsion upon concrete has been found worthy of notice and several papers have been written on analysis of three-dimensional rigid frames, but as yet nothing has been developed that would interest very many practicing design engineers.

If rigid frames are to be analyzed in their true three-dimensional form, a convenient method of analysis must be devised. The classical methods of analysis are much too tedious, and even the most elementary type of three-dimensional frame analyzed using moment distribution is rather a tremendous undertaking, in spite of the saving of time gained by the use of this method.

IV. REVIEW OF LITERATURE

Analyses of three-dimensional structures by the classical methods have been unsuccessful from a practical point of view because of the very nature of the classical methods, most of which require the solution of a number of simultaneous equations.

In this country the three-dimensional problem has been attacked frequently by the use of moment distribution. One of the first articles, written by Paul Andersen [1], shows the importance of an exact analysis by computing the moments due to torsion in several simple concrete frames.

Phil M. Ferguson [7,8] has used moment distribution in three-dimensional analyses of beam-and-girder framing both with and without movement of columns.

The importance of exact analyses of space frames is apparently being realized and methods are being tried out, such as Jacques Heyman's [11] limit design method. The Dome of Discovery, built for the Festival of Britain in 1951, has attracted much attention and several methods have been used for analyzing it, as noted in the books [12] and articles concerning [13] it.

Matrices have been used by mathematicians for over one hundred years, but their application to the fields of engineering and physics has been rather recent, and the number of good texts concerning matrices is rather small.[8,14]

The use of matrices in statistics is increasing and Murden [15] reports on several papers by statisticians describing methods for inverting matrices, none of which work very well in the large order matrices. The problem of matrix inversion is treated very well by Guillemin [10], but most of the methods mentioned in his book are applicable only to matrices of low order. A method devised by Crout [5] is probably the best one to use when the matrix is of sixth or higher order.

The application of matrix algebra to structural problems was started in 1947 by Benscoter [2] in his analysis of continuous beams. This paper also contains a brief, but better, explanation of matrix algebra than many texts on the subject. In 1949, Chen [4] applied matrices to pin-connected structures. In 1951, Murden [15] investigated planar rigid frames by the use of matrices and also tried several methods of inverting the stiffness matrix. The conclusion of most of the discussers of the papers mentioned above and of Murden also is that matrix methods work, but from a time standpoint cannot compete with moment distribution unless many loading conditions on the structure are to be considered.

The only attempt that this writer has found in which matrices are used to solve three-dimensional structural problems is described in Brock's paper [3] on flexibility of piping systems. In this paper the matrices are large and the author feels that Crout's method of inversion is the only practical method.

V. INTRODUCTORY SUMMARY

In an effort to determine whether matrix algebra is practical or not in the analysis of three-dimensional structures, several rigid frames will be analyzed.

Equations of equilibrium for each structure will be written and then expressed in slope-deflection form. From these slope-deflection equations a matrix equation will be formed.

In solving high order matrix equations, finding the inverse of a square matrix is a long and tedious operation, and a method will be used which is readily adaptable to the type of calculating machine likely to be found in a design office.

VI. THE INVESTIGATION

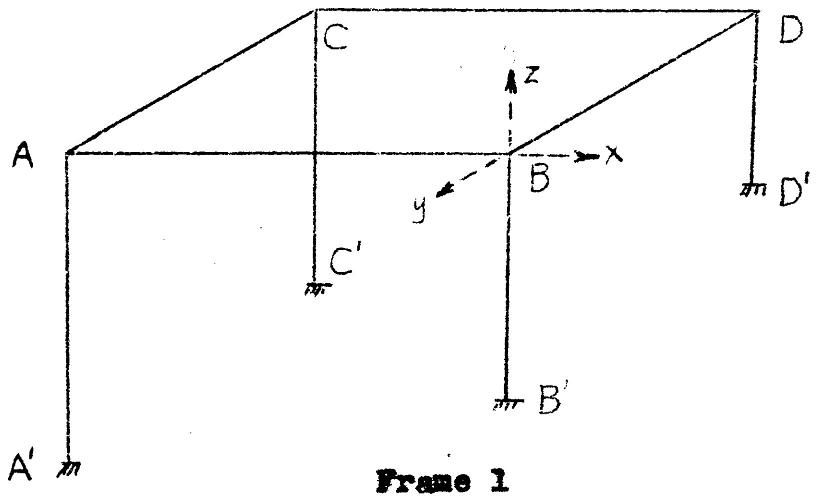
A. Investigation of Frame 1.

Frame 1, shown in Fig. 1, represents any structure of that form, having four columns and four beams or girders connecting the tops of the columns. The beams are not necessarily of the same length, nor are the columns. Any type of loading may be used.

Each joint will be assumed to have six degrees of freedom, i.e., it can rotate about any or all of the coordinate axes and can translate parallel to any or all of them.

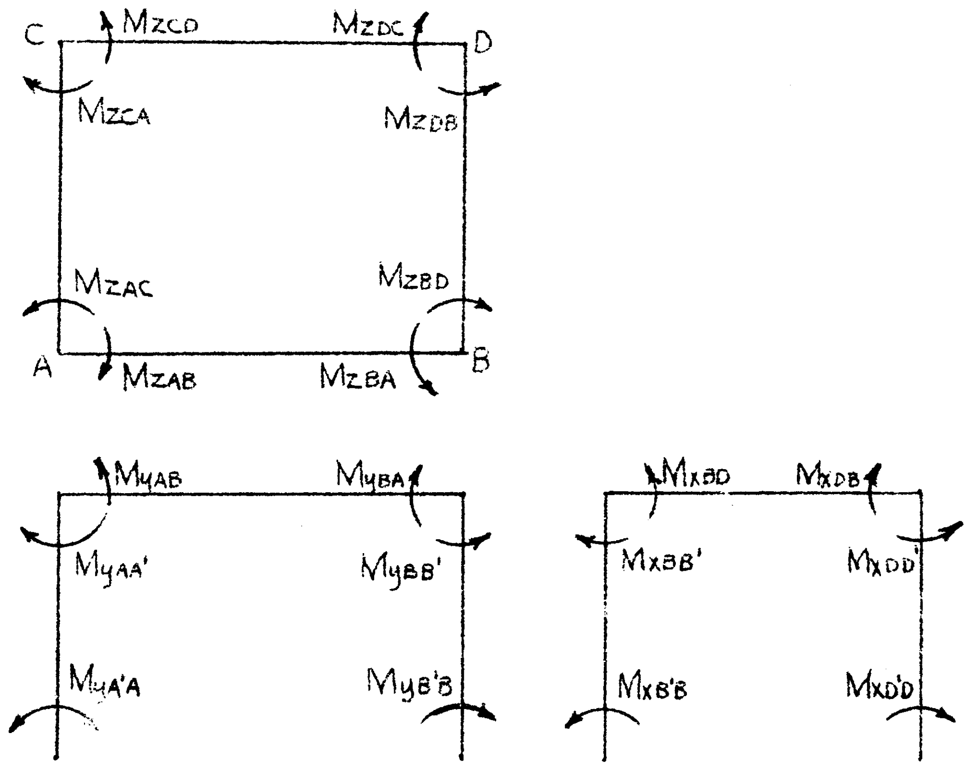
The notation used may be somewhat confusing, but is meant to conform with that used in the analysis of a planar structure. In the case of a column, for example, there would be moments about the y-axis, and symbols for all movements caused by those moments would have a "y" subscript. Hence, even though Δ_y and ρ_y are directed with respect to the x-axis, their subscripts link them with M_y and θ_y.

The sign convention used will be that of the slope-deflection method when the structure is viewed from the positive end of the axis under consideration. All bending moments, joint rotations, and joint translations will be assumed positive. In any member, a clockwise resisting moment on the member will be considered positive, and a clockwise joint rotation will be considered positive. The sign convention for fixed-end moments is the same as for the



Frame 1

Fig. 1



Frame 1, Illustrating Notation

Fig. 2

other bending moments - clockwise resisting moments will be positive.

Fig. 2 shows a plan view and two elevations of Frame 1 and illustrates some of the notation used. In this sketch, no attempt has been made to coordinate bending moments and joint movements and no particular loading has been implied.

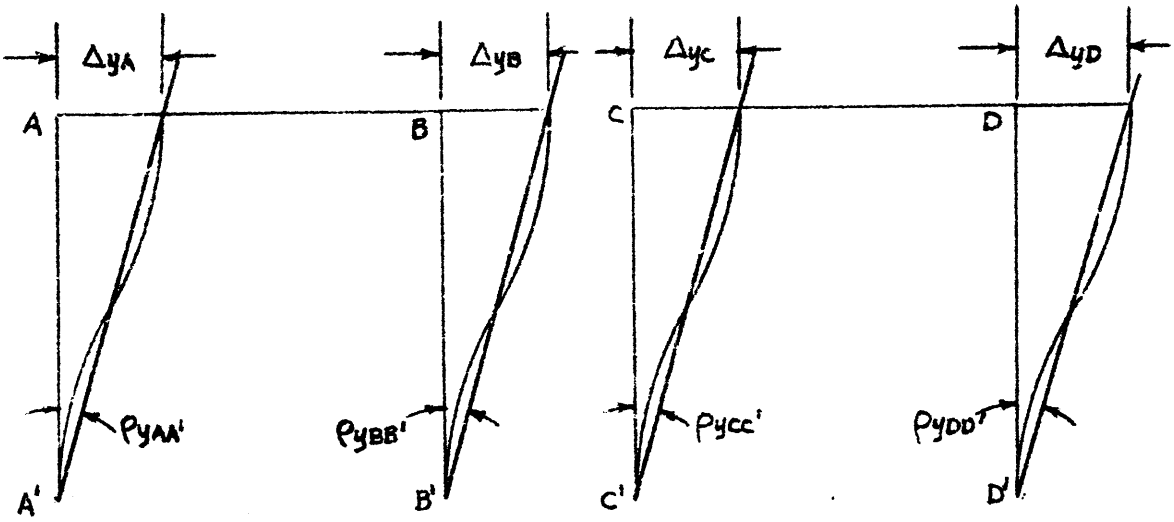
By a consideration of the geometry of deflection and by neglecting axial strains in the members, the number of redundant quantities can be reduced considerably, so that only sixteen equations of equilibrium need to be used.

The column bases are all fixed and any movements there, either translation or rotation, would be caused by foundation conditions changing, and would not be considered as unknown quantities in the analysis of this problem. Furthermore, if axial strains are neglected because of their small size as compared with other possible movements, translation of the columns' tops parallel to the z-axis will be impossible without settlement of the supports, hence these quantities also would be known quantities.

1. Equations from Geometry of Deflection

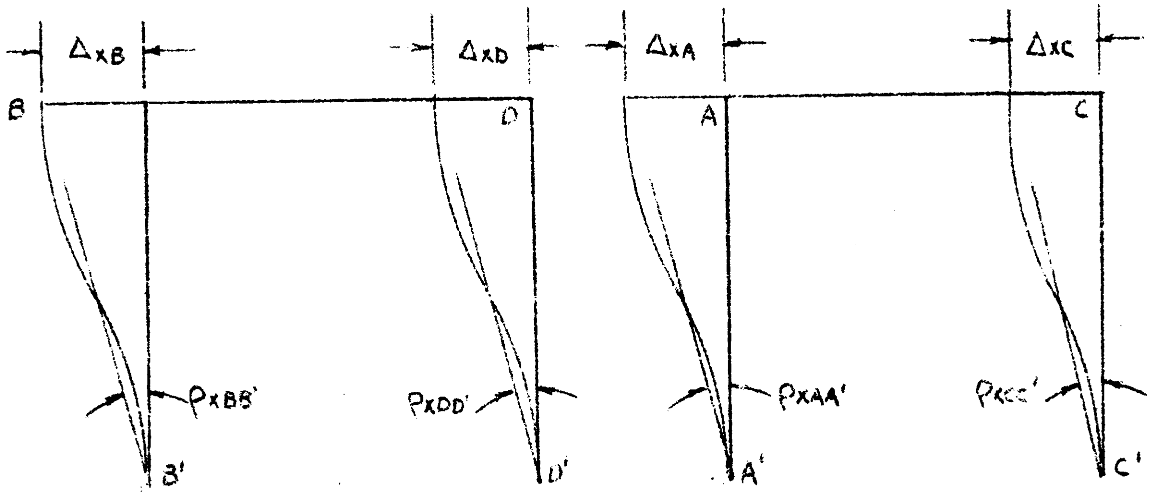
Neglecting axial strains, and referring to Fig. 3, it can be seen that the sidesway of joint A parallel to the x-axis equals that of B, and the sidesway of C equals that of D.

A similar situation exists for sidesway of these joints



Joint Translation Parallel to x-axis

Fig. 3



Joint Translation Parallel to y-axis

Fig. 4

in a direction parallel to the y-axis, as shown in Fig.4, giving rise to the following equations:

$$\rho_{yAA'} = \frac{\Delta_{yA}}{L_{AA'}} \quad \text{and} \quad \rho_{yBB'} = \frac{\Delta_{yB}}{L_{BB'}} \quad \ldots\ldots\ldots .1$$

And since $\Delta_{yA} = \Delta_{yB}$, .2

$$\rho_{yBB'} = \frac{L_{AA'}\rho_{yAA'}}{L_{BB'}} \quad \ldots\ldots\ldots .3$$

Also, $\rho_{yDD'} = \frac{L_{CC'}\rho_{yCC'}}{L_{DD'}}$ 4

For translation in the y-direction,

$$\rho_{xCC'} = \frac{L_{AA'}\rho_{xAA'}}{L_{CC'}} \quad \ldots\ldots\ldots .5$$

$$\rho_{xDD'} = \frac{L_{BB'}\rho_{xBB'}}{L_{DD'}} \quad \ldots\ldots\ldots .6$$

Sidesway will occur in a frame which is either unsymmetrically built or unsymmetrically loaded. Sway caused by moments about the y-axis will be parallel to the x-axis, while sway caused by moments about the x-axis will be parallel to the y-axis. Sway caused by a moment about the z-axis, however, will be parallel to either the x-axis or the y-axis, depending upon which beams are involved.

In Fig. 5, Δ_{zAC}, Δ_{zBD}, Δ_{zAB}, and Δ_{zCD} are really relative deflections - the difference between the sways at A and C caused by moments about the y-axis, etc. In a symmetrical frame symmetrically loaded, movements of joints A, B, C, and D would be equal, consequently the rotation of the beams about the z-axis would be zero.

Since $\rho = \frac{\Delta}{L}$

and $\Delta_{zAC} = \Delta_{yC} - \Delta_{yA'}$ 7

$$\rho_{zAC} = \frac{\rho_{yCC'} L_{CC'} - \rho_{yAA'} L_{AA'}}{L_{AC}} .$$

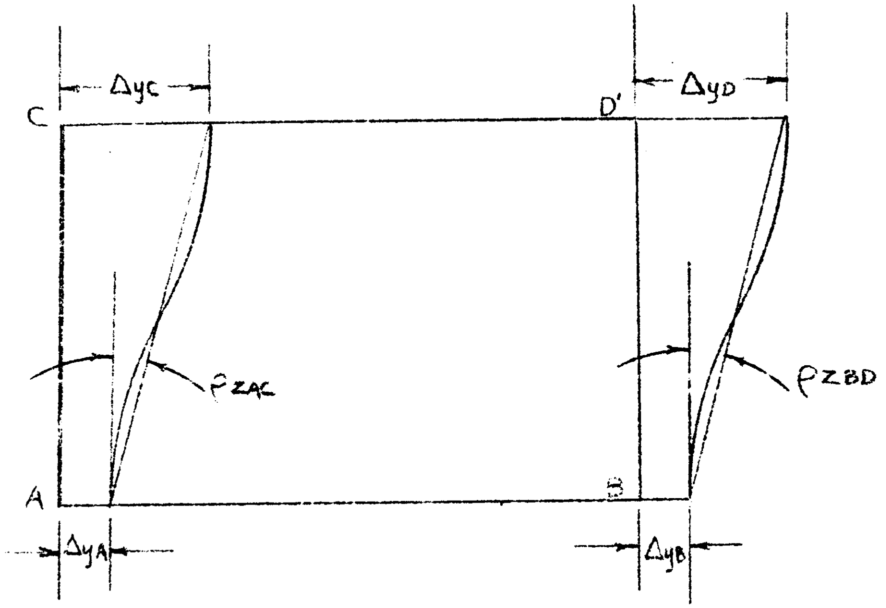
Since axial strains are neglected,

$\rho_{zBD} = \rho_{ZAC}$ 8

Also, $\Delta_{zAB} = \Delta_{xA} - \Delta_{xB}$ 9

and $\rho_{zAB} = \frac{\rho_{xAA'} L_{AA'} - \rho_{xBB'} L_{BB'}}{L_{AB}}$.

Also, $\rho_{zAB} = \rho_{zCD}$ 10



Translation Caused by Moments about z-axis

Fig. 5

2. Equations of Equilibrium

In order that member AC be in equilibrium, the twisting moment developed at A must equal that at C. Moments capable of twisting this member must be moments about the y-axis. Hence,

$$\Sigma M_{yA} + \Sigma M_{yC} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 11$$

For equilibrium in the other beams,

$$\Sigma M_{yB} + \Sigma M_{yD} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 12$$

$$\Sigma M_{xB} + \Sigma M_{xA} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 13$$

$$\Sigma M_{xD} + \Sigma M_{xC} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 14$$

The angle of twist, ϕ, of a member in torsion, is

$$\phi = \frac{ML}{K'G} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 15$$

where M is the twisting moment, L the length of the member, G is the shearing modulus of elasticity, and K' is a constant depending upon the dimensions of the cross section of the member. Values of K' are available for rolled steel sections, and Timoshenko and MacCullough [17] have tabulated data for various rectangular cross sections. This latter data will be used when numerical examples are worked.

For member AC, $\phi = \theta_{yA} - \theta_{yC}$ 16

and $$\Sigma M_{yA} + \frac{K'G}{L_{AC}}(\theta_{yA} - \theta_{yC}) = 0 \quad \ldots\ldots\ldots\ldots \quad 17$$

The other three beams can be dealt with in a similar manner, giving

$$\Sigma M_{yB} + \frac{K'G}{L_{BD}}(\theta_{yB} - \theta_{yD}) = 0 \quad \ldots\ldots\ldots\ldots \quad 18$$

$$\Sigma M_{xB} + \frac{K'G}{L_{AB}}(\theta_{xB} - \theta_{xA}) = 0 \quad \ldots\ldots\ldots\ldots \quad 19$$

$$\Sigma M_{xD} + \frac{K'G}{L_{DC}}(\theta_{xD} - \theta_{xC}) = 0 \quad \dots\dots\dots\dots\dots 20$$

From a consideration of the twist in the columns, we get the following equations:

$$\Sigma M_{zA} + \frac{K'G}{L_{AA'}}(\theta_{zA} - \theta_{zA'}) = 0 \quad \dots\dots\dots\dots\dots 21$$

$$\Sigma M_{zB} + \frac{K'G}{L_{BB'}}(\theta_{zB} - \theta_{zB'}) = 0 \quad \dots\dots\dots\dots\dots 22$$

$$\Sigma M_{zC} + \frac{K'G}{L_{CC'}}(\theta_{zC} - \theta_{zC'}) = 0 \quad \dots\dots\dots\dots\dots 23$$

$$\Sigma M_{zD} + \frac{K'G}{L_{DD'}}(\theta_{zD} - \theta_{zD'}) = 0 \quad \dots\dots\dots\dots\dots 24$$

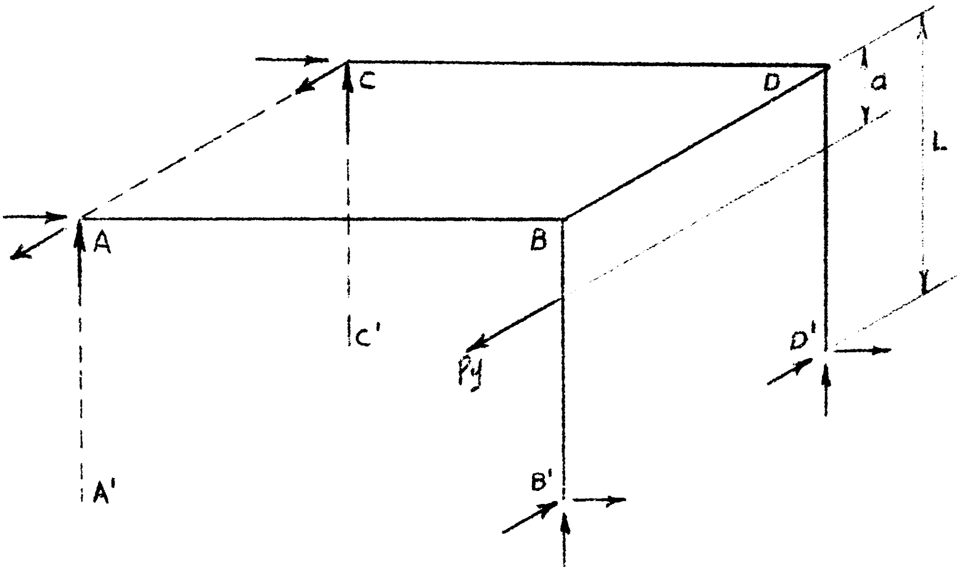
In the above equations, $\theta_{zA'}$, $\theta_{zB'}$, $\theta_{zC'}$, and $\theta_{zD'}$ will be known quantities and will be equal to zero unless some movement of the foundation occurs.

The shearing force at the base of a column without horizontal loads is equal to the change in moment from one end of the column to the other divided by the length of the member. In Fig. 6 the shearing forces in the beams and columns must equal the applied external forces. Hence,

$$\frac{M_{xBB'} + M_{xB'B}}{L_{BB'}} + \frac{M_{xDD'} + M_{xD'D}}{L_{DD'}} - \frac{M_{zBA} + M_{zAB}}{L_{AB}}$$

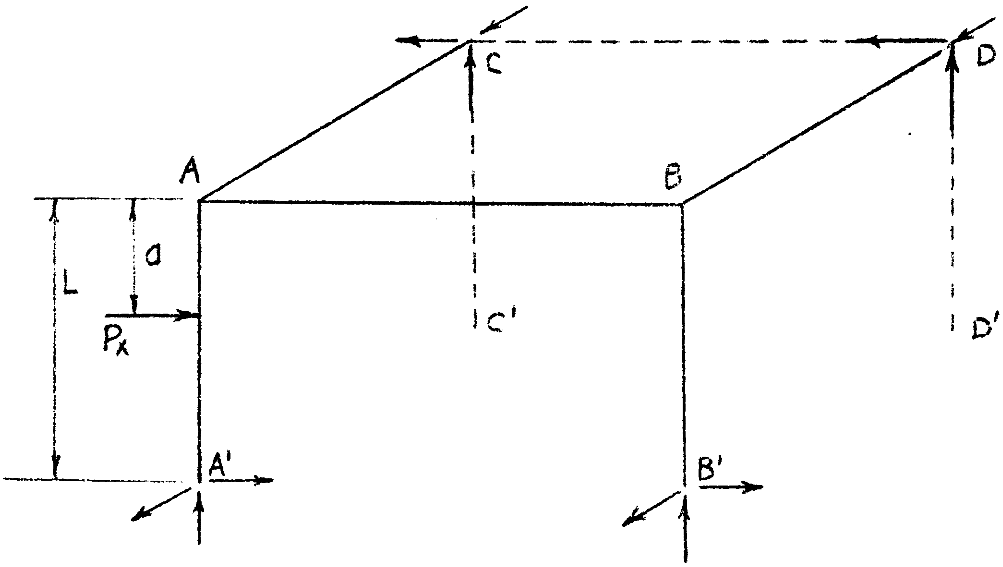
$$- \frac{M_{zDC} + M_{zCD}}{L_{CD}} = P_y - \frac{P_y a}{L} \quad \dots\dots\dots\dots\dots 25$$

where P_y is a load or resultant load parallel to the y-axis and its point of application is located by dimension a.



Right Half , Frame 1

Fig. 6



Left Half , Frame 1

Fig. 7

In Fig. 7, the sum of the shearing forces in the x-direction must equal the resultant force parallel to the x-axis. Dimension a again locates the position of the resultant external force. As before, then,

$$-\frac{M_{yAA'} + M_{yA'A}}{L_{AA'}} - \frac{M_{y3B'} + M_{yB'B}}{L_{BB'}} + \frac{M_{zAC} + M_{ZCA}}{L_{AC}}$$

$$+\frac{M_{zBD} + M_{zDB}}{L_{BD}} = P_x - \frac{P_x a}{L} \quad \ldots\ldots\ldots\ldots 26$$

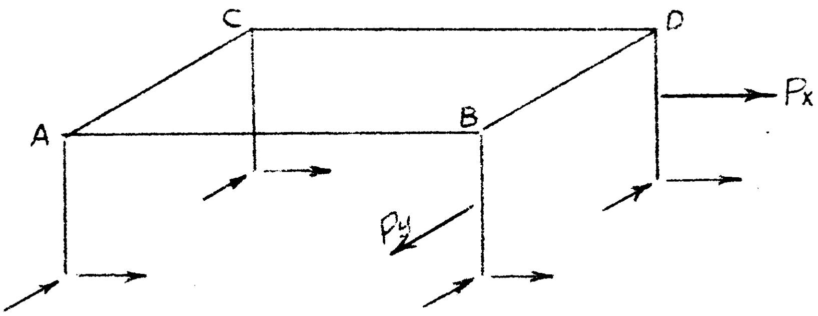
The last two equations of equilibrium equate shearing forces in the columns only to externally applied forces parallel to the horizontal axes. Referring to Fig. 8,

$$-\frac{M_{yAA'} + M_{yA'A}}{L_{AA'}} - \frac{M_{yBB'} + M_{yB'B}}{L_{BB'}} - \frac{M_{yCC'} + M_{yC'C}}{L_{CC'}}$$

$$-\frac{M_{yDD'} + M_{yD'D}}{L_{DD'}} = P_x - \frac{P_x a}{L} \quad \ldots\ldots\ldots\ldots 27$$

$$\frac{M_{xAA'} + M_{yCC'}}{L_{AA'}} + \frac{M_{yBB'} + M_{yB'B}}{L_{BB'}} + \frac{M_{yCC'} + M_{yC'C}}{L_{CC'}}$$

$$+\frac{M_{yDD'} + M_{yD'D}}{L_{DD'}} + \frac{P_y a}{L} = P_y \quad \ldots\ldots\ldots\ldots 28$$



Column Shears, Frame 1

Fig. 8

3. Equilibrium Equations in Slope-Deflection Form

The general slope-deflection equations are usually written as $M_{AB} = 2EK\ (2\theta_A + \theta_B - 3\rho) - M_{FAB}$ and $M_{BA} = 2EK\ (2\theta_B + \theta_A - 3\rho) + M_{FBA}$. M_{FAB} and M_{FBA} are fixed-end moments depending upon the loads acting upon member AB. K is the moment of inertia divided by the length of the member. This form can be shortened somewhat by using a different value for K, such that K now equals 2EK. The shorter form thus becomes: $M_{AB} = K\ (2\theta_A + \theta_B - 3\rho) - M_{FAB}$ and this form will be used throughout this thesis.

The sixteen equations of equilibrium, expanded and written in slope-deflection form are as follows:

$$K_{yAA'}\ (2\theta_{yA} + \theta_{yA'} - 3\rho_{yAA'}) + K_{yAB}(2\theta_{yA} + \theta_{yB} - 3\rho_{yAB}) + K_{yCC'}(2\theta_{yC} + \theta_{yC'} - 3\rho_{yCC'}) + K_{yCD}(2\theta_{yC} + \theta_{yD} - 3\rho_{yCD}) = M_{yF} \quad \ldots\ldots\ldots\ldots\ldots\ldots 11'$$

$$K_{yBB'}(2\theta_{yB} + \theta_{yB'} - 3\rho_{yBB'}) + K_{yAB}(2\theta_{yA} + \theta_{yB} - 3\rho_{yAB}) + K_{yDD'}(2\theta_{yD} + \theta_{yD'} - 3\rho_{yDD'}) + K_{yCD}(2\theta_{yC} + \theta_{yD} - 3\rho_{yCD}) = M_{yF} \quad \ldots\ldots\ldots\ldots\ldots\ldots 12'$$

$$K_{xBB'}(2\theta_{xB} + \theta_{xB'} - 3\rho_{xBB'}) + K_{xBD}(2\theta_{xB} + \theta_{xD} - 3\rho_{xBD}) + K_{xAA'}(2\theta_{xA} + \theta_{xA'} - 3\rho_{xAA'}) + K_{xAC}(2\theta_{xA} + \theta_{xC} - 3\rho_{xAC}) = M_{xF} \quad \ldots\ldots 13'$$

$$K_{xDD'}(2\theta_{xD} + \theta_{xD'} - 3\rho_{xDD'}) + K_{xBD}(2\theta_{xD} + \theta_{xB} - 3\rho_{xBD}) + K_{xCC'}(2\theta_{xC} + \theta_{xC'} - 3\rho_{xCC'}) + K_{xAC}(2\theta_{xC} + \theta x_{A} - 3\rho_{xAC}) = M_{xF} \quad \ldots\ldots 14'$$

$$K_{yAA'}(2\theta_{yA} + \theta_{yA'} - 3\rho_{yAA'}) + K_{yAB}(2\theta_{yA} + \theta_{yB}$$
$$- 3\rho_{yAB}) + \frac{K'G}{L_{AC}}(\theta_{yA} - \theta_{yC}) = \Sigma M_{yF} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; .17'$$

$$K_{yBB'}(2\theta_{yB} + \theta_{yB'} - 3\rho_{yBB'}) + K_{yBA}(2\theta_{yB} + \theta_{yA}$$
$$- 3\rho_{yBA}) + \frac{K'G}{L_{BD}}(\theta_{yB} - \theta_{yD}) = \Sigma M_{yF} \quad . \; . \; . \; . \; . \; . \; . \; . \; .18'$$

$$K_{xBB'}(2\theta_{xB} + \theta_{xB'} - 3\rho_{xBB'}) + K_{xBD}(2\theta_{xB} + \theta_{xD}$$
$$- 3\rho_{xBD}) + \frac{K'G}{L_{AB}}(\theta_{xB} - \theta_{xA}) = \Sigma M_{xF} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; .19'$$

$$K_{xDD'}(2\theta_{xD} + \theta_{xD'} - 3\rho_{xDD'}) + K_{xBD}(2\theta_{xD} + \theta_{xB}$$
$$- 3\rho_{xBD}) + \frac{K'G}{L_{CD}}(\theta_{xD} - \theta_{xC}) = \Sigma M_{xF} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; .20'$$

$$K_{xAC}(2\theta_{zA} + \theta_{zC} - 3\rho_{zAC}) + K_{zAB}(2\theta_{zA} + \theta_{zB}$$
$$- 3\rho_{zAB}) + \frac{K'G}{L_{AA'}}(\theta_{zA} - \theta_{zA'}) = \Sigma M_{zF} \quad . \; . \; . \; . \; . \; . \; . \; . \; .21'$$

$$K_{xBD}(2\theta_{zB} + \theta_{zD} - 3\rho_{zBD}) + K_{zAB}(2\theta_{zB} + \theta_{zA}$$
$$- 3\rho_{zAB}) + \frac{K'G}{L_{BB'}}(\theta_{zB} - \theta_{zB'}) = \Sigma M_{zF} \quad . \; . \; . \; . \; . \; . \; . \; . \; .22'$$

$$K_{zCA}(2\theta_{zC} + \theta_{zA} - 3\rho_{zCA}) + K_{zCD}(2\theta_{zC} - \theta_{zD}$$
$$- 3\rho_{zCD}) + \frac{K'G}{L_{CC'}}(\theta_{zC} - \theta_{zC'}) = \Sigma M_{zF} \quad . \; . \; . \; . \; . \; . \; . \; .23'$$

$$K_{zCD}(2\theta_{zD} + \theta_{zC} - 3\rho_{zCD}) + K_{zBD}(2\theta_{zD} + \theta_{zB}$$
$$- 3\rho_{zBD}) + \frac{K'G}{L_{DD'}}(\theta_{zD} - \theta_{zD'}) = \Sigma M_{zF} \quad . \; . \; . \; . \; . \; . \; . \; .24'$$

$$\frac{K_{xBB'}}{L_{BB'}}(3\theta_{xB} + 3\theta_{xB'} - 6\rho_{xBB'}) + \frac{K_{xDD'}}{L_{DD'}}(3\theta_{xD} + 3\theta_{xD'}$$

$$- 6\rho_{xDD'}) - \frac{K_{zAB}}{L_{AB}}(3\theta_{zA} + 3\theta_{zB} - 6\rho_{zAB}) - \frac{K_{zCD}}{L_{CD}}(3\theta_{zC}$$

$$+ 3\theta_{zD} - 6\rho_{zCD}) = P_y - \frac{P_y a}{L} + \Sigma M_F \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; 25'$$

$$- \frac{K_{yAA'}}{L_{AA'}}(3\theta_{yA} + 3\theta_{yA'} - 6\rho_{yAA'}) - \frac{K_{yBB'}}{L_{BB'}}(3\theta_{yB} + 3\theta_{yB'}$$

$$- 6\rho_{yBB'}) + \frac{K_{zAC}}{L_{AC}}(3\theta_{zA} + 3\theta_{zC} - 6\rho_{zAC}) + \frac{K_{zBD}}{L_{BD}}(3\theta_{zB}$$

$$+ 3\theta_{zD} - 6\rho_{zBD}) = P_x - \frac{P_x a}{L} + \Sigma M_F \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; 26'$$

$$- \frac{K_{yAA'}}{L_{AA'}}(3\theta_{yA} + 3\theta_{yA'} - 6\rho_{yAA'}) - \frac{K_{yBB'}}{L_{BB'}}(3\theta_{yB} + 3\theta_{yB'}$$

$$- 6\rho_{yBB'}) - \frac{K_{yCC'}}{L_{CC'}}(3\theta_{yC} + 3\theta_{yC'} - 6\rho_{yCC'})$$

$$- \frac{K_{yDD'}}{L_{DD'}}(3\theta_{yD} + 3\theta_{yD'} - 6\rho_{yDD'}) = P_x - \frac{P_x a}{L} + \Sigma M_F \quad . \; . \; . \; . \; 27'$$

$$\frac{K_{xAA'}}{L_{AA'}}(3\theta_{xA} + 3\theta_{xA'} - 6\rho_{xAA'}) + \frac{K_{xBB'}}{L_{BB'}}(3\theta_{xB} + 3\theta_{xB'}$$

$$- 6\rho_{xBB'}) + \frac{K_{xCC'}}{L_{CC'}}(3\theta_{xC} + 3\theta_{xC'} - 6\rho_{xCC'})$$

$$+ \frac{K_{xDD'}}{L_{DD'}}(3\theta_{xD} + 3\theta_{xD'} - 6\rho_{xDD'}) = P_y - \frac{P_y a}{L} + \Sigma M_F \quad . \; . \; . \; . \; . \; . \; 28'$$

						$2\Sigma K_{yA}$	K_{yAB}	$2\Sigma K_{yC}$	K_{yCD}	$-3K_{yAA'}$	$-3K_{yCC'}$						$E\theta_{xA}$		$-K_{yAA'}\theta_{yA'}+3K_{yAB}\rho_{yAB}-K_{yCC'}\theta_{yC'}+3K_{yCD}\rho_{yCD}+\Sigma M_{yF}$
						K_{yAB}	$2\Sigma K_{yB}$	K_{yCD}	$2\Sigma K_{yD}$	$-3K_{yBB'}\frac{L_{AA'}}{L_{BB'}}$	$-3K_{yDD'}\frac{L_{CC'}}{L_{DD'}}$						$E\theta_{xB}$		$-K_{yBB'}\theta_{yB'}+3K_{yAB}\rho_{yAB}-K_{yDD'}\theta_{yD'}+3K_{yCD}\rho_{yCD}+\Sigma M_{yF}$
$2\Sigma K_{xA}$	$2\Sigma K_{xB}$	K_{xAC}	K_{xBD}	$-3K_{xAA'}$	$-3K_{xBB'}$												$E\theta_{xC}$		$-K_{xBB'}\theta_{xB'}+3K_{xBD}\rho_{xBD}-K_{xAA'}\theta_{xA'}+3K_{xAC}\rho_{xAC}+\Sigma M_{xF}$
K_{xAC}	K_{xBD}	$2\Sigma K_{xC}$	$2\Sigma K_{xD}$	$-3K_{xCC'}\frac{L_{AA'}}{L_{CC'}}$	$-3K_{xDD'}\frac{L_{BB'}}{L_{DD'}}$												$E\theta_{xD}$		$-K_{xDD'}\theta_{xD'}+3K_{xBD}\rho_{xBD}-K_{xCC'}\theta_{xC'}+3K_{xAC}\rho_{xAC}+\Sigma M_{xF}$
						$2\Sigma K_{yA}+\frac{K'G}{L_{AC}}$	K_{yAB}	$-\frac{K'G}{L_{AC}}$		$-3K_{yAA'}$							$E\rho_{xAA'}$		$-K_{yAA'}\theta_{yA'}+3K_{yAB}\rho_{yAB}+\Sigma M_{yF}$
						$-K_{yAB}$	$2\Sigma K_{yB}+\frac{K'G}{L_{BD}}$		$\frac{K'G}{L_{BD}}$	$-3K_{yBB'}\frac{L_{AA'}}{L_{BB'}}$							$E\rho_{xBB'}$		$-K_{yBB'}\theta_{yB'}+3K_{yAB}\rho_{yAB}+\Sigma M_{yF}$
$-\frac{K'G}{L_{AB}}$	$2\Sigma K_{xB}+\frac{K'G}{L_{AB}}$		K_{xBD}		$-3K_{xBB'}$												$E\theta_{yA}$		$-K_{xBB'}\theta_{xB'}+3K_{xBD}\rho_{xBD}+\Sigma M_{xF}$
	K_{xBD}	$-\frac{K'G}{L_{CD}}$	$2\Sigma K_{xD}+\frac{K'G}{L_{CD}}$		$-3K_{xDD'}\frac{L_{BB'}}{L_{DD'}}$												$E\theta_{yB}$		$-K_{xDD'}\theta_{xD'}+3K_{xBD}\rho_{xBD}+\Sigma M_{xF}$
				$-3K_{zAB}\frac{L_{AA'}}{L_{AB}}$	$3K_{zAB}\frac{L_{BB'}}{L_{AB}}$					$3K_{zAC}\frac{L_{AA'}}{L_{AC}}$	$-3K_{zAC}\frac{L_{CC'}}{L_{AC}}$	$2\Sigma K_{zA}+\frac{K'G}{L_{AA'}}$	K_{zAB}	K_{zAC}		X	$E\theta_{yC}$	=	$\frac{K'G}{L_{AA'}}\theta_{zA'}+\Sigma M_{zF}$
				$-3K_{zAB}\frac{L_{AA'}}{L_{AB}}$	$3K_{zAB}\frac{L_{BB'}}{L_{AB}}$					$3K_{zBD}\frac{L_{AA'}}{L_{AC}}$	$-3K_{zBD}\frac{L_{CC'}}{L_{AC}}$	K_{zBA}	$2\Sigma K_{zB}+\frac{K'G}{L_{BB'}}$		K_{zBD}		$E\theta_{yD}$		$\frac{K'G}{L_{BB'}}\theta_{zB'}+\Sigma M_{zF}$
				$-3K_{zCD}\frac{L_{AA'}}{L_{AB}}$	$3K_{zCD}\frac{L_{BB'}}{L_{AB}}$					$3K_{zAC}\frac{L_{AA'}}{L_{AC}}$	$-3K_{zAC}\frac{L_{CC'}}{L_{AC}}$	K_{zAC}		$2\Sigma K_{zC}+\frac{K'G}{L_{CC'}}$	K_{zCD}		$E\rho_{yAA'}$		$\frac{K'G}{L_{CC'}}\theta_{zC'}+\Sigma M_{zF}$
				$-3K_{zCD}\frac{L_{AA'}}{L_{AB}}$	$3K_{zCD}\frac{L_{BB'}}{L_{AB}}$					$3K_{zBD}\frac{L_{AA'}}{L_{AC}}$	$-3K_{zBD}\frac{L_{CC'}}{L_{AC}}$		K_{zBD}	K_{zCD}	$2\Sigma K_{zD}+\frac{K'G}{L_{DD'}}$		$E\rho_{yCC'}$		$\frac{K'G}{L_{DD'}}\theta_{zD'}+\Sigma M_{zF}$
	$\frac{3K_{xBB'}}{L_{BB'}}$		$\frac{3K_{xDD'}}{L_{DD'}}$	$\frac{6K_{zAB}}{L_{AB}}\frac{L_{AA'}}{L_{AB}}+\frac{6K_{zCD}}{L_{CD}}\frac{L_{AA'}}{L_{AB}}$	$-\frac{6K_{zBB'}}{L_{BB'}}-\frac{6K_{xDD'}}{L_{DD'}}\frac{L_{BB'}}{L_{DD'}}-\frac{6L_{BB'}}{L_{AB}}\left(\frac{K_{zAB}}{L_{AB}}+\frac{K_{zCD}}{L_{CD}}\right)$							$-\frac{3K_{zAB}}{L_{AB}}$	$-\frac{3K_{zAB}}{L_{AB}}$	$-\frac{3K_{zCD}}{L_{CD}}$	$-\frac{3K_{zCD}}{L_{CD}}$		$E\theta_{zA}$		$-\frac{3K_{xBB'}}{L_{BB'}}\theta_{xB'}-\frac{3K_{xDD'}}{L_{DD'}}\theta_{xD'}-\frac{P_y a}{L}+P_y+\Sigma M_F$
						$-\frac{3K_{yAA'}}{L_{AA'}}$	$-\frac{3K_{yBB}}{L_{BB'}}$			$\frac{6K_{yAA'}}{L_{AA'}}+\frac{6K_{yBB'}}{L_{BB'}}\frac{L_{AA'}}{L_{BB'}}+\frac{6K_{zAC}}{L_{AC}}\frac{L_{AA'}}{L_{AC}}+\frac{6K_{zBD}}{L_{BD}}\frac{L_{AA'}}{L_{AC}}$	$-\frac{6K_{zAC}}{L_{AC}}\frac{L_{CC'}}{L_{AC}}-\frac{6K_{zBD}}{L_{BD}}\frac{L_{CC'}}{L_{AC}}$	$\frac{3K_{zAC}}{L_{AC}}$	$\frac{3K_{zBD}}{L_{BD}}$	$\frac{3K_{zAC}}{L_{AC}}$	$\frac{3K_{zBD}}{L_{BD}}$		$E\theta_{zB}$		$\frac{3K_{yAA'}}{L_{AA'}}\theta_{yA'}+\frac{3K_{yBB'}}{L_{BB'}}\theta_{yB'}-\frac{P_x a}{L}+P_x+\Sigma M_F$
						$-\frac{3K_{yAA'}}{L_{AA'}}$	$-\frac{3K_{yBB'}}{L_{BB'}}$	$-\frac{3K_{yCC'}}{L_{CC'}}$	$-\frac{3K_{yDD'}}{L_{DD'}}$	$\frac{6K_{yAA'}}{L_{AA'}}+\frac{6K_{yBB'}}{L_{BB'}}\frac{L_{AA'}}{L_{BB'}}$	$\frac{6K_{yCC'}}{L_{CC'}}+\frac{6K_{yDD'}}{L_{DD'}}\frac{L_{CC'}}{L_{DD'}}$						$E\theta_{zC}$		$\frac{3K_{yAA'}}{L_{AA'}}\theta_{yA'}+\frac{3K_{yBB'}}{L_{BB'}}\theta_{yB'}+\frac{3K_{yCC'}}{L_{CC'}}\theta_{yC'}+\frac{3K_{yDD'}}{L_{DD'}}\theta_{yD'}-\frac{P_x a}{L}+P_x+\Sigma M_{yF}$
$\frac{3K_{xAA'}}{L_{AA'}}$	$\frac{3K_{xBB'}}{L_{BB'}}$	$\frac{3K_{xCC'}}{L_{CC'}}$	$\frac{3K_{xDD'}}{L_{DD'}}$	$-\frac{3K_{xAA'}}{L_{AA'}}-\frac{3K_{xCC'}}{L_{CC'}}\frac{L_{AA'}}{L_{CC'}}$													$E\theta_{zD}$		$-\frac{3K_{xAA'}}{L_{AA'}}\theta_{xA'}-\frac{3K_{xBB'}}{L_{BB'}}\theta_{xB'}-\frac{3K_{xCC'}}{L_{CC'}}\theta_{xC'}-\frac{3K_{xDD'}}{L_{DD'}}\theta_{xD'}-\frac{P_y a}{L}+P_y+\Sigma M_{xF}$

EQUILIBRIUM EQUATIONS IN MATRIX FORM - FRAME I
FIG. 9

4. Equilibrium Equations in Matrix Form

In constructing the matrix equation, the geometry of the structure is used to limit the number of unknowns to sixteen - the number of equilibrium equations.

The matrix equation shown in Fig. 9 consists of a square stiffness matrix of sixteenth order, a column matrix containing the redundant joint rotations and translations, and another column matrix containing the constants to be computed from loading conditions and foundation movements.

5. Numerical Example of Frame 1

The matrix equation of Fig. 9 is valid for any rectangular rigid frame consisting of four columns with beams connecting the tops of the columns. Any dimensions desired may be used and any type of loading may be used.

Shown in Fig. 10 is a simple example of this type of frame, taken from page 98 of Grinter's structural theory book [9].

All columns and beams are concrete, 8 in. x 10.6 in., and the moments of inertia of the cross section are 800 in.^{4} and 452 in.^{4} The constants shown on the sketch are 2I/L except for the constants for twisting.

From page 265 of Timoshenke and MacCullough's strength of materials book [17], $K' = \beta bc^3$, where β depends upon the ratio of the cross-sectional dimensions, and b and c are those dimensions. Thus,

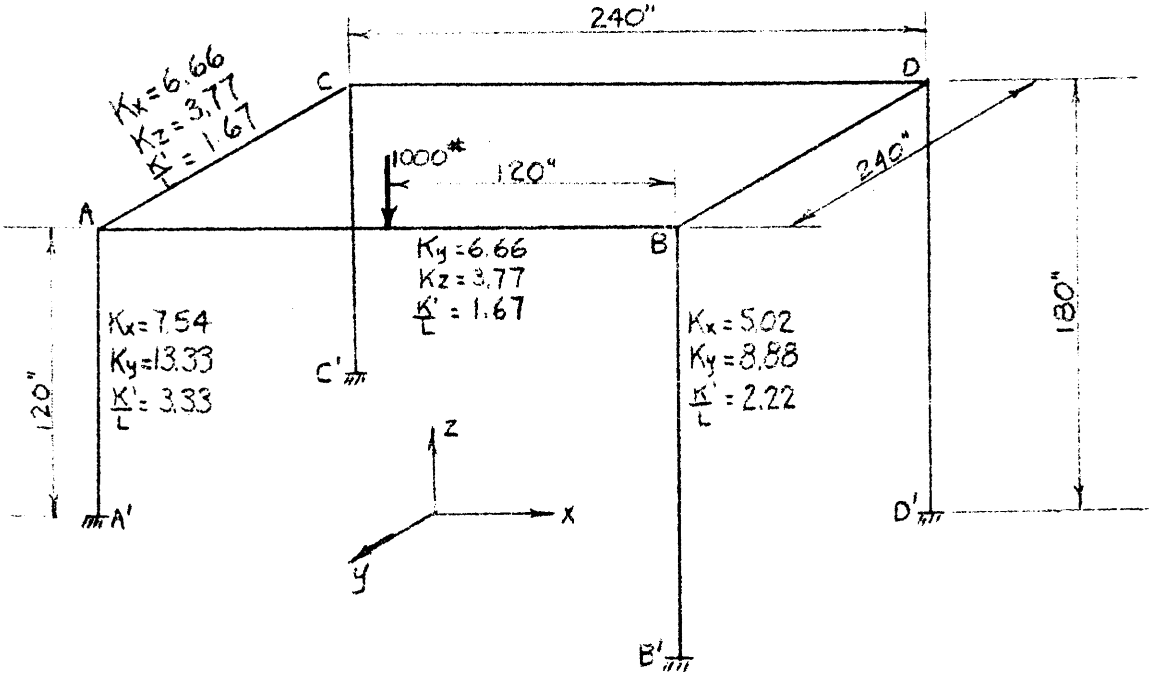
$$K' = 0.177 \times 10.6 \times 8^3$$

and $K'G = \frac{961\ E}{2.4} = 400\ E$ if Poisson's ratio for concrete is assumed to be 0.20. The value of E, the modulus of elasticity, has been removed from the stiffness constant, K, and is now included with the redundants, so that $E\theta_{yA}$, etc., must be solved for.

Fixed-end moments for member AB are:

$$M_{yFAB} = -\ PL/8 = -\ 1000 \times 240/8 = -\ 30{,}000 \text{ in. lb.}$$

$$M_{yFBA} = +\ PL/8 = 30{,}000 \text{ in. lb.}$$



Numerical Example, Frame 1

Fig. 10

E (θ_{xA}	θ_{xB}	θ_{xC}	θ_{xD}	$\rho_{xAA'}$	$\rho_{xBB'}$	θ_{yA}	θ_{yB}	θ_{yC}	θ_{yD}	ρ_{yAA}	ρ_{yCC}	θ_{zA}	θ_{zB}	θ_{zC}	θ_{zD})		EQUATION NO.
12.51	21.71	-12.59	21.79	-22.04	47.24	1117.93	479.83	86.34	-585.29	330.63	-54.30	-103.01	-109.62	-102.98	-109.60		
28.40	23.36	6.66	6.66	-22.62	-15.06												13
6.66	6.66	28.40	23.36	22.62	-15.06												14
	1.19	26.84	21.80	-17.32	-11.53												
	6.66	-1.67	25.03		-15.06												20
	6.66	-151.88	-96.97	96.93	49.46												
1.67	25.03		6.66		-15.06												19
	26.40	-595.05	-96.97	3.15	46.08												
0.19	0.08	0.19	0.08	-0.38	0.16												28
	-0.07	1.72	0.23	-0.15	-0.07												
	0.08		0.08	0.09	-0.48							0.05	-0.05	0.05	-0.05		25
	0.08	1.80	-0.24	0.11	-0.45							-0.05	-0.05	0.05	-0.05		
						-0.33	-0.15	-0.33	-0.15	0.86	0.86						27
						-0.33	-0.15	-0.33	-0.15	0.86	0.86						
						39.98	6.66	39.98	6.66	-39.99	39.99					30,000.00	11
						39.98	-11.51		-11.51	64.20	64.20					30,000.00	
						41.65	6.66	1.67		-39.99						30,000.00	17
						41.65	-12.27	-43.32	-6.66	0.12	40.11					-1980.88	
						-6.66	32.74		1.67	-17.76						-30,000.00	18
						-6.66	35.76	6.66	-32.09	164.36	188.27					62,901.37	
						6.66	31.08	6.66	31.08	-17.76	-17.76					-30,000.00	12
						6.66	28.06			156.10	156.10					43,136.40	
				-5.65	8.48					5.65	-5.65	18.41	3.77	3.77			21
				-5.65	11.11					5.65	-11.30	17.18	2.54	2.54	-1.23	-1561.31	
				-5.65	8.48					5.65	-5.65	3.77	17.30		3.77		22
				-5.65	11.11					5.65	-11.30	-14.64	13.53	-3.77	3.77		
				-5.65	8.48					5.65	-5.65		3.77	3.77	17.30		24
				-5.65	11.11					5.65	-11.30	-18.41	-17.01	4.74	12.56		
				-5.65	8.48					5.65	-5.65	3.77		18.41	3.77		23
				-5.65	11.11					5.65	-11.30	-14.64	17.30	13.59	-12.77		
						-0.33	-0.15			0.96	-0.09	0.05	0.05	0.05	0.05		26
						-0.33		0.33	0.10	0.61	-0.68	-0.98	1.00	0.12	-0.97	106.32	

WORK SHEET FOR EXAMPLE OF FRAME I

FIG. 11

Fig. 11 shows numerical values substituted into the stiffness matrix of Fig. 9 in the upper part of each square. The seventeenth column contains the constant terms, i.e., fixed-end moments only in this example. The column matrix of Fig. 9 which contained the redundants has been written as a row matrix across the top of the paper and numerical values for these quantities appear below each. All other numbers in the squares are elements in the auxiliary matrix used in solving the matrix equation. Several rows have been interchanged to prevent any element of the diagonal of the matrix from being zero. This "work sheet" thus contains both the original matrix equation and the solution to that equation, as well as all of the intermediate numbers used in solving the equation.

6. Matrix Inversion by Crout's Method

In solving a series of simultaneous equations by the use of Crout's method [5], an equivalent set of equations is formed. In this new matrix, all the elements of the principal diagonal are equal to unity, and all those below the principal diagonal are zero. Once this has been accomplished, the solutions to the equations may be found with very little difficulty.

Let us consider the following n equations involving n unknowns:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots\cdots + a_{2n}x_n = y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots\cdots + a_{nn}x_n = y_n \end{array} \quad \ldots\ldots\ldots 29$$

From these equations, an "augmented Matrix" may be formed:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} & y_2 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots\cdots & a_{nn} & y_n \end{vmatrix} \quad \ldots\ldots\ldots\ldots 30$$

or, for purposes of simplification, if y_1 is written as $a_{1,n+1}$; and y_2 as $a_{2,n+2}$; etc., the following results:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} & a_{2,n+2} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots\cdots & a_{nn} & a_{n,n+1} \end{vmatrix} \quad \ldots\ldots\ldots\ldots 31$$

If each element of the first row is divided by a_{11}, row 1 becomes:

$$1 \quad \frac{a_{12}}{a_{11}} \quad \frac{a_{13}}{a_{11}} \quad \cdots\cdots\cdots \quad \frac{a_{1,n+1}}{a_{11}}$$

Then, multiply each element of the first row by a_{21} and subtract from corresponding elements of the second row. The matrix becomes:

$$\begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots\cdots\cdots & \frac{a_{1,n+1}}{a_{11}} \\ 0 & b_{22} & b_{23} & \cdots\cdots\cdots & b_{2,n+1} \\ a_{31} & a_{32} & a_{33} & \cdots\cdots\cdots & a_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots\cdots & a_{n,n+1} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad 32$$

where $b_{22} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$

$$b_{23} = a_{23} - \frac{a_{21}a_{13}}{a_{11}}$$

$$b_{2,n+1} = a_{2,n+1} - \frac{a_{21}a_{1,n+1}}{a_{11}} \quad \ldots\ldots\ldots\ldots \quad 33$$

Next, multiply the elements of the first row by a_{31} and subtract from corresponding elements of the third row. This procedure is followed, using the general formula

$b_{sk} = a_{sk} - \frac{a_{s1}a_{1k}}{a_{11}}$ until the matrix assumes

the form:

$$\begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots\cdots\cdots & \frac{a_{1,n+1}}{a_{11}} \\ 0 & b_{22} & b_{23} & \cdots\cdots\cdots & b_{2,n+1} \\ 0 & b_{32} & b_{33} & \cdots\cdots\cdots & b_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b_{n2} & b_{n3} & \cdots\cdots\cdots & b_{n,n+1} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad 34$$

All the "b" elements are now operated on in the same manner as the "a" elements and the matrix then becomes:

$$\begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots\cdots & \frac{a_{1,n+1}}{a_{11}} \\ 0 & 1 & \frac{b_{23}}{b_{22}} & \cdots\cdots & \frac{b_{2,n+1}}{b_{22}} \\ 0 & 0 & c_{33} & \cdots\cdots & c_{3,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & c_{n3} & \cdots\cdots & c_{n,n+1} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad 35$$

where $c_{sk} = b_{sk} - \frac{b_{s2}b_{2k}}{b_{22}}$.

The process continues in the same manner for matrices of any size. The final form of a third order matrix, for example, is:

$$\begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} \\ 0 & 1 & \frac{b_{23}}{b_{22}} & \frac{b_{24}}{b_{22}} \\ 0 & 0 & 1 & \frac{c_{34}}{c_{33}} \end{vmatrix} \quad \ldots\ldots\ldots\ldots \quad 36$$

The final matrix represents a new set of equations equivalent to the first. The solutions to this equivalent set of equations are:

$$x_3 = c_{34}/c_{33}$$

$$x_2 = b_{24}/b_{22} - (b_{23}/b_{22})x_3$$

$$x_1 = a_{14}/a_{11} - (a_{13}/a_{11})x_3 - (a_{12}/a_{11})x_2 \quad \ldots\ldots \quad 37$$

The above process is not as forbidding as it may seem when applied to actual numerical problems. Although the elements below the main diagonal are not required in the final application of equations 37 in computing the values of the unknowns, they are used in computing the elements of the main diagonal and those above the main diagonal. Therefore, the following "auxiliary" matrix is very useful.

$$\left|\begin{matrix} a_{11} & \dfrac{a_{12}}{a_{11}} & \dfrac{a_{13}}{a_{11}} & \dfrac{a_{14}}{a_{11}} & \cdots \\ a_{21} & b_{22} & \dfrac{b_{23}}{b_{22}} & \dfrac{b_{24}}{b_{22}} & \cdots \\ a_{31} & b_{32} & c_{33} & \dfrac{c_{34}}{c_{33}} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}\right| \quad \cdots\cdots\cdots \quad 38$$

In this matrix, b_{sk} and c_{sk} are as given on pages 27 and 28.

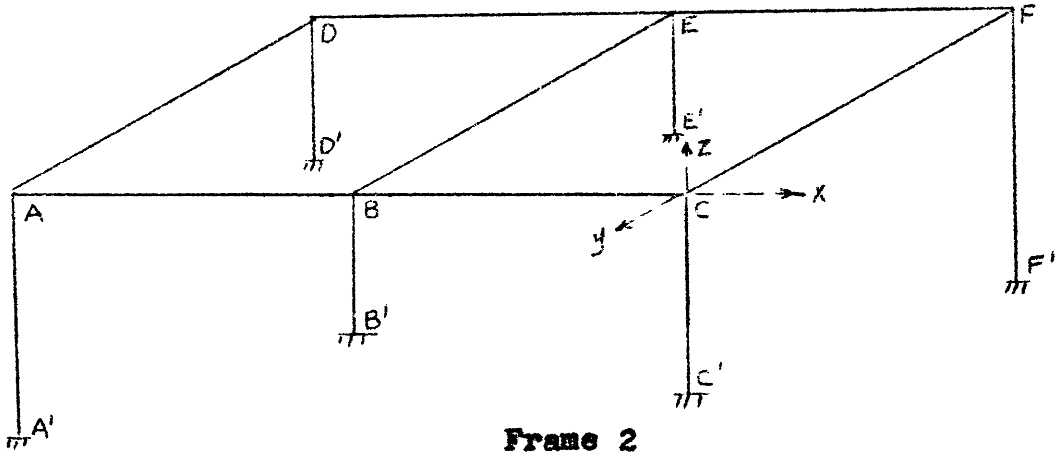
The computation of each element may be done in one continuous operation on most desk type calculators, and only those results which are to be used are recorded. The original matrix and the auxiliary matrix only are recorded.

The writer has found it convenient, when solving problems by this method, to mark off the work sheet into squares. In the upper half of the square the original matrix is recorded, and the lower half contains the auxiliary matrix with one modification from the form of eq. 38. The elements to the right of the diagonal element in the auxiliary matrix are not divided by the diagonal element until applying eq. 37. It has also been found helpful to enclose the diagonal elements of a large matrix with a heavier square or possibly one of a different color than the other elements. The use of two straight edges at right angles to each other aids in ensuring that the proper elements are multiplied together.

The original matrix must be set up in such a fashion that none of the elements of the principal diagonal equal zero. Also, the main diagonal elements of the auxiliary matrix must not be zero, and some rearranging of rows may be necessary as the calculations proceed in order to prevent this.

B. Investigation of Frame 2.

By the addition of one bay to Frame 1 to form Frame 2, the number of redundants is increased by seven, i.e., six joint rotations and one translation parallel to the y-axis. The resulting matrix, then, will be used to solve 23 simultaneous equations.



Frame 2

Fig. 12

1. Equations from Geometry of Deflection

Any displacements of A, B, and C parallel to the x-axis will be equal if axial strains are neglected. Thus,

$$\rho_{yBB'} = \rho_{yAA'} \frac{L_{AA'}}{L_{BB'}} \quad \ldots\ldots\ldots\ldots\ldots \quad 40$$

$$\rho_{yCC'} = \rho_{yAA'} \frac{L_{AA'}}{L_{CC'}} \quad \ldots\ldots\ldots\ldots\ldots \quad 41$$

Also, deflections of joints D, E, and F parallel to the x-axis must be equal, and

$$\rho_{yEE'} = \rho_{yDD'} \frac{L_{DD'}}{L_{EE'}} \quad \ldots\ldots\ldots\ldots\ldots \quad 42$$

$$\rho_{yFF'} = \rho_{yDD'} \frac{L_{DD'}}{L_{FF'}} \quad \ldots \quad 43$$

Considering displacements parallel to the y-axis:

$$\rho_{xDD'} = \rho_{xAA'} \frac{L_{AA'}}{L_{DD'}} \quad \ldots \quad 44$$

$$\rho_{xEE'} = \rho_{xBB'} \frac{L_{BB'}}{L_{EE'}} \quad \ldots \quad 45$$

$$\rho_{xFF'} = \rho_{xCC'} \frac{L_{CC'}}{L_{FF'}} \quad \ldots \quad 46$$

As in the analysis of Frame 1, the beam translations caused by moments about the z-axis can be expressed in terms of translations parallel to the x-axis and y-axis, giving the following equations:

$$\rho_{zAB} = \rho_{xAA'} \frac{L_{AA'} - \rho_{xBB'} L_{BB'}}{L_{AB}} \quad \ldots \quad 47$$

$$\rho_{zBC} = \rho_{xBB'} \frac{L_{BB'} - \rho_{xCC'} L_{CC'}}{L_{BC}} \quad \ldots \quad 48$$

$$\rho_{zAD} = \rho_{yDD'} \frac{L_{DD'} - \rho_{yAA'} L_{AA'}}{L_{AD}} \quad \ldots \quad 49$$

$$\rho_{zBE} = \rho_{zAD} \quad \ldots \quad 50$$

$$\rho_{zCF} = \rho_{zAD} \quad \ldots \quad 51$$

$$\rho_{zDE} = \rho_{zAB} \quad \ldots \quad 52$$

$$\rho_{zEF} = \rho_{zBC} \quad \ldots \quad 53$$

2. Equations of Equilibrium

For equilibrium in AD,

$$\Sigma M_{yA} + \Sigma M_{yD} = 0 \quad \ldots \quad 54$$

For equilibrium in BE,

$$\Sigma M_{yB} + \Sigma M_{yE} = 0 \qquad \text{. . . . . 55}$$

For equilibrium in CF,

$$\Sigma M_{yC} + \Sigma M_{yF} = 0 \qquad \text{. . . . . 56}$$

Since the angle of twist in AD is the difference between the joint rotations at A and D caused by moments about the y-axis,

$$\Sigma M_{yA} = \frac{K'G}{L_{AD}} \phi_{AD} \qquad \text{. . . . . .57}$$

Considering the angle of twist in BE,

$$\Sigma M_{yB} = \frac{K'G}{L_{BE}} \phi_{BE} \qquad \text{. . . . . .58}$$

Similarly, for CF,

$$\Sigma M_{yC} = \frac{K'G}{L_{CF}} \phi_{CF} \qquad \text{. . . . . .59}$$

The sum of the shearing forces parallel to the x-axis in the columns just above the column bases must equal the sum of the external forces in the y-direction. Thus,

$$- \frac{M_{yA} + M_{yA'}}{L_{AA'}} - \frac{M_{yB} + M_{yB'}}{L_{BB'}} - \frac{M_{yC} + M_{yC'}}{L_{CC'}}$$

$$- \frac{M_{yD} + M_{yD'}}{L_{DD'}} - \frac{M_{yE} + M_{yE'}}{L_{EE'}} - \frac{M_{yF} + M_{yF'}}{L_{FF'}}$$

$$= P_x - \frac{P_x a}{L} \qquad \text{. . . . . 60}$$

For equilibrium in member ABC,

$$\Sigma M_{xA} + \Sigma M_{xB} + \Sigma M_{xC} = 0 \qquad \text{. . . . . 61}$$

For equilibrium in member DEF,

$$\Sigma M_{xD} + \Sigma M_{xE} + \Sigma M_{xF} = 0 \qquad \text{. . . . . 62}$$

The moment at C will cause a twist in BC. Therefore,

$$\Sigma M_{xC} = \frac{K'G}{L_{BC}} \phi_{BC} \quad \ldots\ldots\ldots\ldots\ldots\ldots 63$$

The moment at A will cause a twist in AB. Thus,

$$\Sigma M_{xA} = \frac{K'G}{L_{AB}} \phi_{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots 64$$

Because of the twist in EF,

$$\Sigma M_{xF} = \frac{K'G}{L_{EF}} \phi_{EF} \quad \ldots\ldots\ldots\ldots\ldots\ldots 65$$

Because of the twist in DE,

$$\Sigma M_{xD} = \frac{K'G}{L_{DE}} \phi_{DE} \quad \ldots\ldots\ldots\ldots\ldots\ldots 66$$

The sum of the shearing forces parallel to the y-axis in the columns must equal the sum of the external forces in the y-direction, giving

$$\frac{M_{xA} + M_{xA'}}{L_{AA'}} + \frac{M_{xB} + M_{xB'}}{L_{BB'}} + \frac{M_{xC} + M_{xC'}}{L_{CC'}} + \frac{M_{xD} + M_{xD'}}{L_{DD'}}$$

$$+ \frac{M_{xE} + M_{xE'}}{L_{EE'}} + \frac{M_{xF} + M_{xF'}}{L_{FF'}} = P_y - \frac{P_y a}{L} \quad \ldots\ldots 67$$

A consideration of the twist in the columns leads to the following equations containing moments about the z-axis:

$$\Sigma M_{zA} = \frac{K'G}{L_{AA'}} \phi_{AA'} \quad \ldots\ldots\ldots\ldots\ldots\ldots 68$$

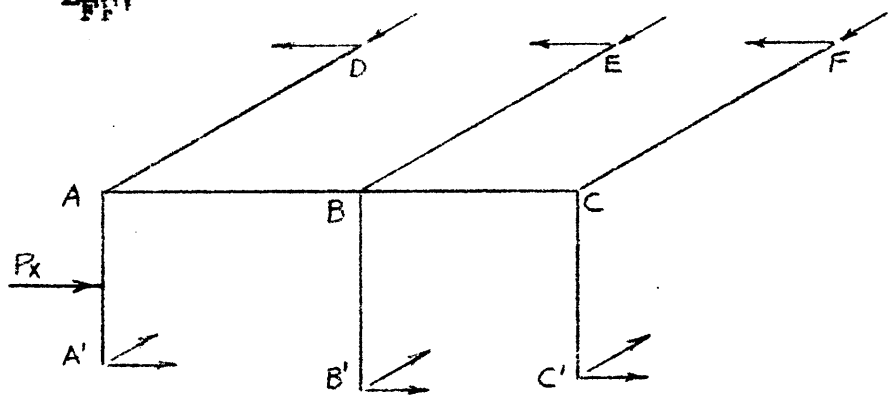
$$\Sigma M_{zB} = \frac{K'G}{L_{BB'}} \phi_{BB'} \quad \ldots\ldots\ldots\ldots\ldots\ldots 69$$

$$\Sigma M_{zC} = \frac{K'G}{L_{CC'}} \phi_{CC'} \quad \ldots\ldots\ldots\ldots\ldots\ldots 70$$

$$\Sigma M_{zD} = \frac{K'G}{L_{DD'}} \phi_{DD'} \quad \ldots\ldots\ldots\ldots\ldots\ldots 71$$

$$\Sigma M_{zE} = \frac{K'G}{L_{EE'}} \phi_{EE'} \quad \ldots\ldots\ldots\ldots\ldots\ldots 72$$

$$\sum M_{zF} = \frac{K'G}{L_{FF'}} \; \phi_{FF'} \quad \dots\dots\dots\dots\dots \; 73$$



Column and Beam Shears, Frame 2

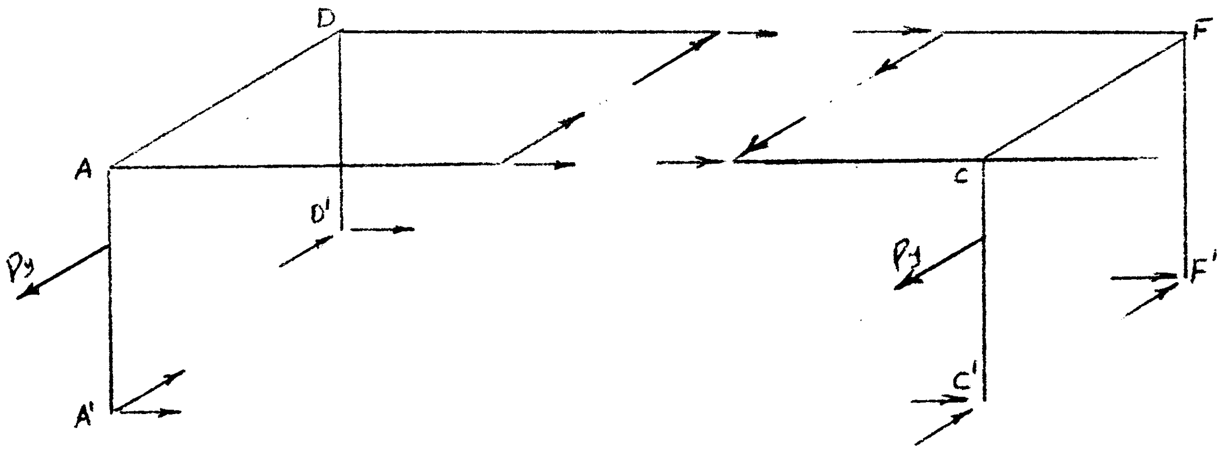
Fig. 13

In Fig. 13 the frame has been cut just above the bases of the columns and to one side of points D, E, and F. The shearing forces in the cut beams and columns parallel to the x-axis must equal the external forces in the x-direction.

$$- \frac{M_{yAA'} + M_{yA'A}}{L_{AA'}} - \frac{M_{yBB'} + M_{yB'B}}{L_{BB'}} - \frac{M_{yCC'} + M_{yC'C}}{L_{CC'}}$$
$$+ \frac{M_{zAD} + M_{zDA}}{L_{AD}} + \frac{M_{zBE} + M_{zEB}}{L_{BE}} + \frac{M_{zCF} + M_{zFC}}{L_{CF}}$$
$$= P_x - \frac{P_x a}{L} \quad \dots\dots\dots\dots\dots \; 74$$

In Fig. 14 the shearing forces parallel to the y-axis must equal the applied forces in the y-direction.

$$\frac{M_{xAA'} + M_{xA'A}}{L_{AA'}} + \frac{M_{xDD'} + M_{xD'D}}{L_{DD'}} + \frac{M_{zAB} + M_{zBA}}{L_{AB}}$$
$$+ \frac{M_{zDE} + M_{zED}}{L_{DE}} + \frac{P_y a}{L} = P_y \quad \dots\dots\dots\dots\dots \; 75$$



Left Third, Frame 2

Fig. 14

Right Third, Frame 2

Fig. 15

Fig. 15 is similar to Fig. 14 except that the right hand portion of the frame is isolated. As before, the shearing forces parallel to the y-axis will be considered.

$$\frac{M_{xCC'} + M_{xC'C}}{L_{CC'}} + \frac{M_{xFF'} + M_{xF'F}}{L_{FF'}} - \frac{M_{zBC} + M_{zCB}}{L_{BC}}$$

$$- \frac{M_{zEF} + M_{zFE}}{L_{EF}} + \frac{P_y a}{L} = P_y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad 76$$

3. Equilibrium Equations in Slope-Deflection Form

Equations 54 through 76 expressed in slope-deflection form are as follows:

$$K_{yAA'}(2\theta_{yA} + \theta_{yA'} - 3\rho_{yAA'}) + K_{yAB}(2\theta_{yA} + \theta_{yB} - 3\rho_{yAB}) + K_{yDD'}(2\theta_{yD} + \theta_{yD'} - 3\rho_{yDD'}) + K_{yDE}(2\theta_{yD} + \theta_{yE} - 3\rho_{yDE}) = \Sigma M_{yF} \quad \ldots \quad 54'$$

$$K_{yBB'}(2\theta_{yB} + \theta_{yB'} - 3\rho_{yBB'}) + K_{yBA}(2\theta_{yB} + \theta_{yB'} - 3\rho_{yBA}) + K_{yBC}(2\theta_{yB} + \theta_{yC} - 3\rho_{yBC}) + K_{yEE'}(2\theta_{yE} + \theta_{yE'} - 3\rho_{yEE'}) + K_{yED}(2\theta_{yE} + \theta_{yD} - 3\rho_{yED}) + K_{yEF}(2\theta_{yE} + \theta_{yF} - 3\rho_{yEF}) = \Sigma M_{yF} \quad \ldots \quad 55'$$

$$K_{yCC'}(2\theta_{yC} + \theta_{yC'} - 3\rho_{yCC'}) + K_{yCB}(2\theta_{yC} + \theta_{yB} - 3\rho_{yCB}) + K_{yFF'}(2\theta_{yF} + \theta_{yF'} - 3\rho_{yFF'}) + K_{yFE}(2\theta_{yF} + \theta_{yE} - 3\rho_{yFE}) = \Sigma M_{yF} \quad \ldots \quad 56'$$

$$K_{yAA'}(2\theta_{yA} + \theta_{yA'} - 3\rho_{yAA'}) + K_{yAB}(2\theta_{yA} + \theta_{yB} - 3\rho_{yAB}) + \frac{K'G}{L_{AD}}(\theta_{yA} - \theta_{yD}) = \Sigma M_{yF} \quad \ldots \quad 57'$$

$$K_{yBB'}(2\theta_{yB} + \theta_{yB'} - 3\rho_{yBB'}) + K_{yBA}(2\theta_{yB} + \theta_{yA} - 3\rho_{yBA}) + K_{yBC}(2\theta_{yB} + \theta_{yC} - 3\rho_{yBC}) + \frac{K'G}{L_{BE}}(\theta_{yB} - \theta_{yE}) = \Sigma M_{yF} \ldots 58'$$

$$K_{yCC'}(2\theta_{yC} + \theta_{yC'} - 3\rho_{yCC'}) + K_{yCB}(2\theta_{yC} + \theta_{yB} - 3\rho_{yCB}) + \frac{K'G}{L_{CF}}(\theta_{yC} - \theta_{yF}) = \Sigma M_{yF} \quad \ldots \quad 59'$$

$$- K_{yAA'}(3\theta_{yA} + 3\theta_{yA'} - 6\rho_{yAA'}) - K_{yBB'}(3\theta_{yB} + 3\theta_{yB'} - 6\rho_{yBB'}) - K_{yCC'}(3\theta_{yC} + 3\theta_{yC'} - 6\rho_{yCC'}) - K_{yDD'}(3\theta_{yD} + 3\theta_{yD'} - 6\rho_{yDD'}) - K_{yEE'}(3\theta_{yE} + 3\theta_{yE'} - 6\rho_{yEE'}) - K_{yFF'}(3\theta_{yF} + 3\theta_{yF'} - 6\rho_{yFF'}) = P_x - \frac{P_x a}{L} + \Sigma M_{yF} \quad . \quad 60'$$

$$K_{yAA'}(2\theta_{xA} + \theta_{xA'} - 3\rho_{xAA'}) + K_{xAD}(2\theta_{xA} + \theta_{xD} - 3\rho_{xAD})$$
$$+ K_{xBB'}(2\theta_{xB} + \theta_{xB'} - 3\rho_{xBB'}) + K_{xBE}(2\theta_{xB} + \theta_{xE}$$
$$- 3\rho_{xBE}) + K_{xCC'}(2\theta_{xC} + \theta_{xC'} - 3\rho_{xCC'}) + K_{xCF}(2\theta_{xC}$$
$$+ \theta_{xF} - 3\rho_{xCF}) = \Sigma M_{xF} \quad \ldots\ldots\ldots\ldots\ldots \quad 61'$$

$$K_{xDD'}(2\theta_{xD} + \theta_{xD'} - 3\rho_{xDD'}) + K_{xDA}(2\theta_{xD} + \theta_{xA} - 3\rho_{xDA})$$
$$+ K_{xEE'}(2\theta_{xE} + \theta_{xE'} - 3\rho_{xEE'}) + K_{xEB}(2\theta_{xE} + \theta_{xB}$$
$$- 3\rho_{xEB}) + K_{xFF'}(2\theta_{xF} + \theta_{xF'} - 3\rho_{xFF'}) + K_{xCF}(2\theta_{xC}$$
$$+ \theta_{xF} - 3\rho_{xCF}) = \Sigma M_{xF} \quad \ldots\ldots\ldots\ldots\ldots \quad 62'$$

$$K_{xCC'}(2\theta_{xC} + \theta_{xC'} - 3\rho_{xCC'}) + K_{xCF}(2\theta_{xC} + \theta_{xF}$$
$$- 3\rho_{xCF}) + \frac{K'G}{L_{BC}}(\theta_{xC} - \theta_{xB}) = \Sigma M_{xF} \quad \ldots\ldots\ldots \quad 63'$$

$$K_{xAA'}(2\theta_{xA} + \theta_{xA'} - 3\rho_{xAA'}) + K_{xAD}(2\theta_{xA} + \theta_{xD}$$
$$- 3\rho_{xAD}) + \frac{K'G}{L_{AB}}(\theta_{xB} - \theta_{xA}) = \Sigma M_{xF} \quad \ldots\ldots\ldots \quad 64'$$

$$K_{xFF'}(2\theta_{xF} + \theta_{xF'} - 3\rho_{xFF'}) + K_{xFC}(2\theta_{xF} + \theta_{xC}$$
$$- 3\rho_{xFC}) + \frac{K'G}{L_{EF}}(\theta_{xF} - \theta_{xE}) = \Sigma M_{xF} \quad \ldots\ldots\ldots \quad 65'$$

$$K_{xDD'}(2\theta_{xD} + \theta_{xD'} - 3\rho_{xDD'}) + K_{xDA}(2\theta_{xD} + \theta_{xD'}$$
$$- 3\rho_{xDA}) + \frac{K'G}{L_{DE}}(\theta_{xE} - \theta_{xD}) = \Sigma M_{xF} \quad \ldots\ldots\ldots \quad 66'$$

$$K_{xAA'}(3\theta_{xA} + 3\theta_{xA'} - 6\rho_{xAA'}) + K_{xBB'}(3\theta_{xB} + 3\theta_{xB'}$$
$$- 6\rho_{xBB'}) + K_{xCC'}(3\theta_{xC} + 3\theta_{xC'} - 6\rho_{xCC'}) + K_{xDD'}(3\theta_{xD}$$
$$+ 3\theta_{xD'} - 6\rho_{xDD'}) + K_{xEE'}(3\theta_{xE} + 3\theta_{xE'} - 6\rho_{xEE'})$$
$$+ K_{xFF'}(3\theta_{xF} + \theta_{xF'} - 6\rho_{xFF'}) = P_y - \frac{P_y a}{L} + \Sigma M_{xF} \;.\; 67'$$

$$K_{zAB}(2\theta_{zA} + \theta_{zB} - 3\rho_{zAB}) + K_{zAD}(2\theta_{zA} + \theta_{zD}$$
$$- 3\rho_{zAD}) + \frac{K'G}{L_{AA'}}(\theta_{zA} - \theta_{zA'}) = \Sigma M_{zF} \quad \ldots\ldots\ldots \quad 68'$$

$$K_{zBA}(2\theta_{zB} + \theta_{zA} - 3\rho_{zBA}) + K_{zBE}(2\theta_{zB} + \theta_{zE} - 3\rho_{zBE})$$
$$+ K_{zBC}(2\theta_{zB} + \theta_{zC} - 3\rho_{zBC}) + \frac{K'G}{L_{BB'}}(\theta_{zB} - \theta_{zB'})$$
$$= \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 69'$$

$$K_{zCB}(2\theta_{zC} + \theta_{zB} - 3\rho_{zCB}) + K_{zCF}(2\theta_{zC} + \theta_{zF} - 3\rho_{zCF})$$
$$+ \frac{K'G}{L_{CC'}}(\theta_{zC} - \theta_{zC'}) = \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 70'$$

$$K_{zDA}(2\theta_{zD} + \theta_{zA} - 3\rho_{zDA}) + K_{zDE}(2\theta_{zD} + \theta_{zE} - 3\rho_{zDE})$$
$$+ \frac{K'G}{L_{DD'}}(\theta_{zD} - \theta_{zD'}) = \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 71'$$

$$K_{zED}(2\theta_{zE} + \theta_{zD} - 3\rho_{zED}) + K_{zEB}(2\theta_{zE} + \theta_{zB} - 3\rho_{zEB})$$
$$+ K_{zEF}(2\theta_{zE} + \theta_{zF} - 3\rho_{zEF}) + \frac{K'G}{L_{EE'}}(\theta_{zE} - \theta_{zE'})$$
$$= \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 72'$$

$$K_{zFC}(2\theta_{zF} + \theta_{zC} - 3\rho_{zFC}) + K_{zFE}(2\theta_{zF} + \theta_{zE} - 3\rho_{zFE})$$
$$+ \frac{K'G}{L_{FF'}}(\theta_{zF} - \theta_{zF'}) = \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 73'$$

$$- \frac{K_{yAA'}}{L_{AA'}}(3\theta_{yA} + 3\theta_{yA'} - 6\rho_{yAA'}) - \frac{K_{yBB'}}{L_{BB'}}(3\theta_{yB} + 3\theta_{yB'}$$
$$- 6\rho_{yBB'}) - \frac{K_{yCC'}}{L_{CC'}}(3\theta_{yC} + 3\theta_{yC'} - 6\rho_{yCC'})$$
$$+ \frac{K_{zAD}}{L_{AD}}(3\theta_{zA} + 3\theta_{zD} - 6\rho_{zAD}) + \frac{K_{zBE}}{L_{BE}}(3\theta_{zB} + 3\theta_{zE}$$
$$- 6\rho_{zBE}) + \frac{K_{zCF}}{L_{CF}}(3\theta_{zC} + 3\theta_{zF} - 6\rho_{zCF})$$
$$= P_x - \frac{P_x a}{L} + H_{yF} + \Sigma M_{zF} \quad \ldots\ldots\ldots\ldots \quad 74'$$

$$\frac{K_{xAA'}}{L_{AA'}}(3\theta_{xA} + 3\theta_{xA'} - 6\rho_{xAA'}) + \frac{K_{xDD'}}{L_{DD'}}(3\theta_{xD} + 3\theta_{xD'} - 6\rho_{xAA'}) + \frac{K_{zAB}}{L_{AB}}(3\theta_{zA} + 3\theta_{zB} - 6\rho_{zAB}) + \frac{K_{zDE}}{L_{DE}}(3\theta_{zD} + 3\theta_{zE} - 6\rho_{zDE}) = P_y - \frac{P_y a}{L} + \Sigma M_{xF} + \Sigma M_{zF} \quad . . . 75'$$

$$\frac{K_{xCC'}}{L_{CC'}}(3\theta_{xC} + 3\theta_{xC'} - 6\rho_{xCC'}) + \frac{K_{xFF'}}{L_{FF'}}(3\theta_{xF} + 3\theta_{xF'} - 6\rho_{xFF'}) - \frac{K_{zBC}}{L_{BC}}(3\theta_{zB} + 3\theta_{zC} - 6\rho_{zBC}) - \frac{K_{zEF}}{L_{EF}}(3\theta_{zE} + 3\theta_{zF} - 6\rho_{zEF}) = P_y + \Sigma M_{xF} + \Sigma M_{zF} \quad . . . . 76'$$

4. Equilibrium Equations in Matrix Form

Fig. 16 shows equations 54' through 76' in matrix form. In this form the problem is quite general and applies to any rectangular frame with six columns and six beams under any type of loading.

5. Numerical Example of Frame 2

In Fig. 17 is shown a Frame 2 type of structure. The loading, all dimensions, and flexural and twisting constants are also shown.

Fig. 18 is a work sheet used in solving the matrix equation of Fig. 16. Its form is the same as that used in the example of Frame 1, and as before, many rows have been interchanged.

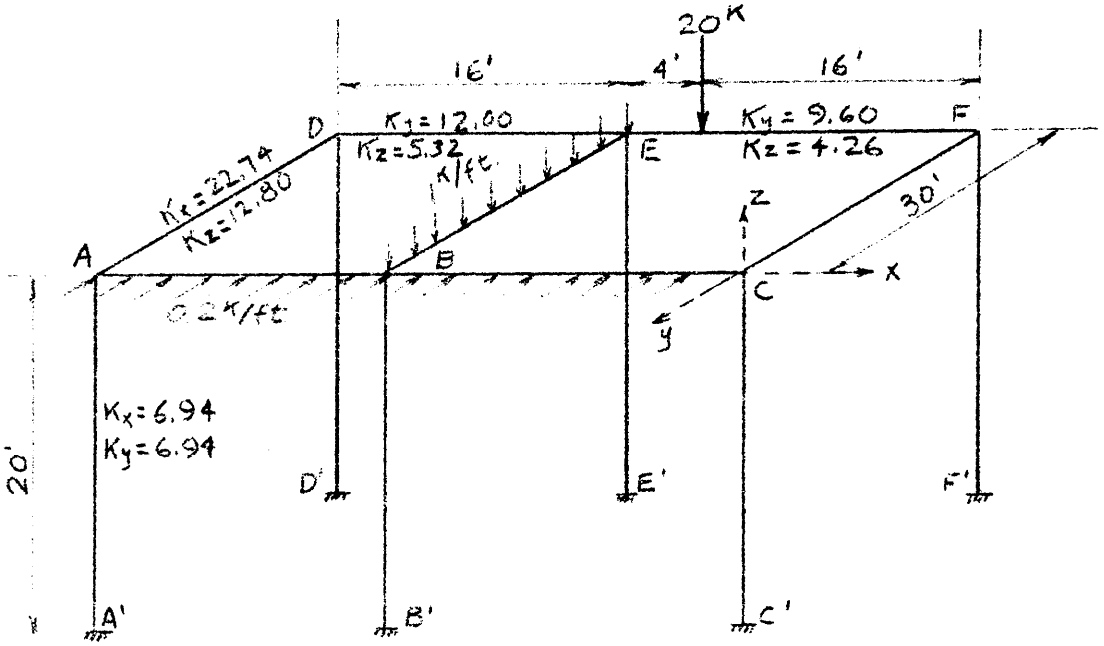
$2\Sigma K_{yA}$	K_{yAB}		$2\Sigma K_{yD}$	K_{yDE}		$-3K_{yAA'}$	$-3K_{yDD'}$															
K_{yAB}	$2\Sigma K_{yB}$	K_{yBC}	K_{yDE}	$2\Sigma K_{yE}$	K_{yEF}	$-3K_{yBB'}\frac{L_{AA'}}{L_{BB'}}$	$-3K_{yEE'}\frac{L_{DD'}}{L_{EE'}}$															
	K_{yBC}	$2\Sigma K_{yC}$		K_{yEF}	$2\Sigma K_{yF}$	$-3K_{yCC'}\frac{L_{AA'}}{L_{CC'}}$	$-3K_{yFF'}\frac{L_{DD'}}{L_{FF'}}$															
$2\Sigma K_{yA}+\frac{K'G}{L_{AD}}$	K_{yAB}		$-\frac{K'G}{L_{AD}}$			$-3K_{yAA'}$																
K_{yAB}	$2\Sigma K_{yB}+\frac{K'G}{L_{BE}}$	K_{yBC}		$-\frac{K'G}{L_{BE}}$		$-3K_{yBB'}\frac{L_{AA'}}{L_{BB'}}$																
	K_{yBC}	$2\Sigma K_{yC}+\frac{K'G}{L_{CF}}$			$-\frac{K'G}{L_{CF}}$	$-3K_{yCC'}\frac{L_{AA'}}{L_{CC'}}$																
$-\frac{3K_{yAA'}}{L_{AA'}}$	$-\frac{3K_{yBB'}}{L_{BB'}}$	$-\frac{3K_{yCC'}}{L_{CC'}}$	$-\frac{3K_{yDD'}}{L_{DD'}}$	$-\frac{3K_{yEE'}}{L_{EE'}}$	$-\frac{3K_{yFF'}}{L_{FF'}}$	$\frac{6K_{yAA'}}{L_{AA}}+\frac{6K_{yBB'}}{L_{BB'}}\frac{L_{AA'}}{L_{BB'}}+\frac{6K_{yCC'}}{L_{CC'}}\frac{L_{AA'}}{L_{CC'}}$	$\frac{6K_{yDD'}}{L_{DD'}}+\frac{6K_{yEE'}}{L_{EE'}}\frac{L_{DD'}}{L_{EE'}}+\frac{6K_{yFF'}}{L_{FF'}}\frac{L_{DD'}}{L_{FF'}}$															
								$2\Sigma K_{xA}$	$2\Sigma K_{xB}$	$2\Sigma K_{xC}$	K_{xAD}	K_{xBE}	K_{xCF}	$-3K_{xAA'}$	$-3K_{xBB'}$	$-3K_{xCC'}$						
								K_{xAD}	K_{xBE}	K_{xCF}	$2\Sigma K_{xD}$	$2\Sigma K_{xE}$	$2\Sigma K_{xF}$	$-3K_{xDD'}\frac{L_{AA'}}{L_{DD'}}$	$-3K_{xEE'}\frac{L_{BB'}}{L_{EE'}}$	$-3K_{xFF'}\frac{L_{CC'}}{L_{FF'}}$						
									$-\frac{K'G}{L_{BC}}$	$2\Sigma K_{xC}+\frac{K'G}{L_{BC}}$			K_{xCF}			$-3K_{xCC'}$						
								$2\Sigma K_{xA}-\frac{K'G}{L_{AB}}$	$\frac{K'G}{L_{AB}}$		K_{xAD}			$-3K_{xAA'}$								
										K_{xCF}		$-\frac{K'G}{L_{EF}}$	$2\Sigma K_{xF}+\frac{K'G}{L_{EF}}$			$-3K_{xFF'}\frac{L_{CC'}}{L_{FF'}}$						
								K_{xAD}			$2\Sigma K_{xD}-\frac{K'G}{L_{DE}}$	$\frac{K'G}{L_{DE}}$		$-3K_{xDD'}\frac{L_{AA'}}{L_{DD'}}$								
								$\frac{3K_{xAA'}}{L_{AA'}}$	$\frac{3K_{xBB'}}{L_{BB'}}$	$\frac{3K_{xCC'}}{L_{CC'}}$	$\frac{3K_{xDD'}}{L_{DD'}}$	$\frac{3K_{xEE'}}{L_{EE'}}$	$\frac{3K_{xFF'}}{L_{FF'}}$	$\frac{6K_{xAA'}}{L_{AA'}}-\frac{6K_{xDD'}}{L_{DD'}}\frac{L_{AA'}}{L_{DD'}}$	$-\frac{6K_{xBB'}}{L_{BB'}}-\frac{6K_{xEE'}}{L_{EE'}}\frac{L_{BB}}{L_{EE'}}$	$-\frac{6K_{xCC'}}{L_{CC'}}-\frac{6K_{xFF'}}{L_{FF'}}\frac{L_{CC'}}{L_{FF'}}$						
						$\frac{3K_{zAD}L_{AA'}}{L_{AD}}$	$-\frac{3K_{zAD}L_{DD'}}{L_{AD}}$							$-3K_{zAB}\frac{L_{AA'}}{L_{AB}}$	$3K_{zAB}\frac{L_{BB'}}{L_{AB}}$		$2\Sigma K_{zA}+\frac{K'G}{L_{AA'}}$	K_{zAB}		K_{zAD}		
						$\frac{3K_{zBE}L_{AA'}}{L_{AD}}$	$-\frac{3K_{zBE}L_{DD'}}{L_{AD}}$							$-3K_{zAB}\frac{L_{AA'}}{L_{AB}}$	$-3K_{zBC}\frac{L_{BB'}}{L_{BC}}+3K_{zAB}\frac{L_{BB'}}{L_{AB}}$	$3K_{zBC}\frac{L_{CC'}}{L_{BC}}$	K_{zAB}	$2\Sigma K_{zB}+\frac{K'G}{L_{BB'}}$	K_{zBC}		K_{zBE}	
						$\frac{3K_{zCF}L_{AA'}}{L_{AD}}$	$-\frac{3K_{zCF}L_{DD'}}{L_{AD}}$								$-3K_{zBC}\frac{L_{BB}}{L_{BC}}$	$3K_{zBC}\frac{L_{CC'}}{L_{BC}}$		K_{zBC}	$2\Sigma K_{zC}+\frac{K'G}{L_{CC'}}$			K_{zCF}
						$\frac{3K_{zAD}L_{AA'}}{L_{AB}}$	$-\frac{3K_{zAD}L_{DD'}}{L_{AD}}$							$-3K_{zDE}\frac{L_{AA'}}{L_{AB}}$	$3K_{zDE}\frac{L_{BB'}}{L_{AB}}$		K_{zAD}			$2\Sigma K_{zD}+\frac{K'G}{L_{DD'}}$	K_{zDE}	
						$\frac{3K_{zBE}L_{AA'}}{L_{AD}}$	$-\frac{3K_{zBE}L_{DD'}}{L_{AD}}$							$-3K_{zDE}\frac{L_{AA'}}{L_{AB}}$	$3K_{zDF}\frac{L_{BB'}}{L_{AB}}-3K_{zEF}\frac{L_{BB'}}{L_{BC}}$	$3K_{zEF}\frac{L_{CC'}}{L_{BC}}$		K_{zBE}		K_{zED}	$2\Sigma K_{zE}+\frac{K'G}{L_{EE'}}$	K_{zEF}
						$\frac{3K_{zCF}L_{AA'}}{L_{AD}}$	$-\frac{3K_{zCF}L_{DD'}}{L_{AD}}$								$-3K_{zEF}\frac{L_{BB'}}{L_{BC}}$	$3K_{zEF}\frac{L_{CC'}}{L_{BC}}$			K_{zCF}		K_{zEF}	$2\Sigma K_{zF}+\frac{K'G}{L_{FF'}}$
$\frac{3K_{yAA'}}{L_{AA'}}$	$\frac{3K_{yBB'}}{L_{BB'}}$	$\frac{3K_{yCC'}}{L_{CC'}}$				$6K_{yAA'}+\frac{6K_{yBB'}}{L_{BB'}}\frac{L_{AA'}}{L_{BB'}}+\frac{6K_{yCC'}}{L_{CC'}}\frac{L_{AA'}}{L_{CC'}}+\frac{6L_{AA'}}{L_{AD}}\left(\frac{K_{zAD}}{L_{AD}}+\frac{K_{zBE}}{L_{BE}}+\frac{K_{zCF}}{L_{CF}}\right)$	$\frac{6L_{DD'}}{L_{AD}}\left(\frac{K_{zAD}}{L_{AD}}+\frac{K_{zBE}}{L_{BE}}+\frac{K_{zCF}}{L_{CF}}\right)$									$\frac{3K_{zAD}}{L_{AD}}$	$\frac{3K_{zBE}}{L_{BE}}$	$\frac{3K_{zCF}}{L_{CF}}$	$\frac{3K_{zAD}}{L_{AD}}$	$\frac{3K_{zBE}}{L_{BE}}$	$\frac{3K_{zCF}}{L_{CF}}$	
								$\frac{3K_{xAA'}}{L_{AA'}}$			$\frac{3K_{xDD'}}{L_{DD'}}$			$-\frac{6K_{xAA'}}{L_{AA'}}-\frac{6K_{xDD'}}{L_{DD'}}\frac{L_{AA}}{L_{DD}}-\frac{6L_{AA'}}{L_{AB}}\left(\frac{K_{zAB}}{L_{AB}}+\frac{K_{zDE}}{L_{DE}}\right)$	$\frac{6L_{BB}}{L_{AB}}\left(\frac{K_{zAB}}{L_{AB}}+\frac{K_{zDE}}{L_{DE}}\right)$		$\frac{3K_{zAB}}{L_{AB}}$	$\frac{3K_{zAB}}{L_{AB}}$		$\frac{3K_{zDE}}{L_{DE}}$	$\frac{3K_{zDE}}{L_{DE}}$	
										$\frac{3K_{xCC'}}{L_{CC'}}$			$\frac{3K_{xFF'}}{L_{FF'}}$		$\frac{6L_{BB}}{L_{AB}}\left(\frac{K_{zBC}}{L_{BC}}+\frac{K_{zEF}}{L_{EF}}\right)$	$-\frac{6K_{zCC}}{L_{CC}}-\frac{6K_{xFF'}}{L_{FF'}}\frac{L_{CC'}}{L_{FF}}-\frac{6L_{CC'}}{L_{BC}}\left(\frac{K_{zBC}}{L_{BC}}+\frac{K_{zEF}}{L_{EF}}\right)$		$-\frac{3K_{zBC}}{L_{BC}}$	$-\frac{3K_{zBC}}{L_{BC}}$		$-\frac{3K_{zEF}}{L_{EF}}$	$-\frac{3K_{zEF}}{L_{EF}}$

X

$E\theta_{yA}$		$\Sigma M_F - K_{yAA'}\theta_{yA'} + 3K_{yAB}\rho_{yAB} - K_{yDD'}\theta_{yD'} + 3K_{yDE}\rho_{yDE}$
$E\theta_{yB}$		$\Sigma M_F - K_{yBB'}\theta_{yB'} + 3K_{yAB}\rho_{yAB} + 3K_{yBC}\rho_{yBC} - K_{yEE'}\theta_{yE'} + 3K_{yDE}\rho_{yDE} + 3K_{yEF}\rho_{yEF}$
$E\theta_{yC}$		$\Sigma M_F - K_{yCC'}\theta_{yC'} + 3K_{yBC}\rho_{yBC} - K_{yFF'}\theta_{yF'} + 3K_{yEF}\rho_{yEF}$
$E\theta_{yD}$		$\Sigma M_F - K_{yAA'}\theta_{yA'} + 3K_{yAB}\rho_{yAB}$
$E\theta_{yE}$		$\Sigma M_F - K_{yBB'}\theta_{yB'} + 3K_{yAB}\rho_{yAB} + 3K_{yBC}\rho_{yBC}$
$E\theta_{yF}$		$\Sigma M_F - K_{yCC'}\theta_{yC'} + 3K_{yBC}\rho_{yBC}$
$E\rho_{yAA'}$		$\Sigma M_F + P_x - \frac{P_x a}{L} - \frac{3K_{yAA'}}{L_{AA'}}\theta_{yA'} - \frac{3K_{yBB'}}{L_{BB'}}\theta_{yB'} - \ldots\ldots - \frac{3K_{yFF'}}{L_{FF'}}\theta_{yF'}$
$E\rho_{yDD'}$		$\Sigma M_F - K_{xAA'}\theta_{xA'} + 3K_{xAD}\rho_{xAD} - K_{xBB'}\theta_{xB'} + 3K_{xBE}\rho_{xBE} - K_{xCC'}\theta_{xC'} + 3K_{xCF}\rho_{xCF}$
$E\theta_{xA}$		$\Sigma M_F - K_{xDD'}\theta_{xD'} + 3K_{xAD}\rho_{xAD} - K_{xEE'}\theta_{xE'} + 3K_{xBE}\rho_{xBE} - K_{xFF'}\theta_{xF'} + 3K_{xCF}\rho_{xCF}$
$E\theta_{xB}$		$\Sigma M_F - K_{xCC'}\theta_{xC'} + 3K_{xCF}\rho_{xCF}$
$E\theta_{xC}$		$\Sigma M_F - K_{xAA'}\theta_{xA'} + 3K_{xAD}\rho_{xAD}$
$E\theta_{xD}$	=	$\Sigma M_F - K_{xFF'}\theta_{xF'} + 3K_{xCF}\rho_{xCF}$
$E\theta_{xE}$		$\Sigma M_F - K_{xDD'}\theta_{xD'} + 3K_{xAD}\rho_{xAD}$
$E\theta_{xF}$		$\Sigma M_F + P_y - \frac{P_y a}{L} + \frac{3K_{xAA'}}{L_{AA'}}\theta_{xA'} + \frac{3K_{xBB}}{L_{BB'}}\theta_{xB'} + \ldots\ldots + \frac{3K_{xFF'}}{L_{FF'}}\theta_{xF'}$
$E\rho_{xAA'}$		$\Sigma M_F + \frac{K'G}{L_{AA'}}\theta_{zA'}$
$E\rho_{xBB'}$		$\Sigma M_F + \frac{K'G}{L_{BB'}}\theta_{zB'}$
$E\rho_{xCC'}$		$\Sigma M_F + \frac{K'G}{L_{CC'}}\theta_{zC'}$
$E\theta_{zA}$		$\Sigma M_F + \frac{K'G}{L_{DD'}}\theta_{zD'}$
$E\theta_{zB}$		$\Sigma M_F + \frac{K'G}{L_{EE'}}\theta_{zE'}$
$E\theta_{zC}$		$\Sigma M_F + \frac{K'G}{L_{FF'}}\theta_{zF'}$
$E\theta_{zD}$		$\Sigma M_F + P_x - \frac{P_x a}{L} + \frac{3K_{yAA'}}{L_{AA'}}\theta_{yA'} + \frac{3K_{yBB'}}{L_{BB'}}\theta_{yB'} + \frac{3K_{yCC'}}{L_{CC'}}\theta_{yC'}$
$E\theta_{zE}$		$\Sigma M_F + P_y - \frac{P_y a}{L} - \frac{3K_{xAA'}}{L_{AA'}}\theta_{xA'} - \frac{3K_{xDD'}}{L_{DD'}}\theta_{xD'}$
$E\theta_{zF}$		$\Sigma M_F + P_y - \frac{P_y a}{L} - \frac{3K_{xCC'}}{L_{CC'}}\theta_{xC'} - \frac{3K_{xFF'}}{L_{FF'}}\theta_{xF'}$

EQUILIBRIUM EQUATIONS IN MATRIX FORM - FRAME 2

FIG. 16



Columns - 10"x10"	AD, BE, CF - 12"x16"	ABC, DEF - 8"x12"
$I_x = 833.33$ in.^{4}	$I_x = 4096$ in.^{4}	$I_y = 1152$ in.^{4}
$I_y = 833.33$ in.^{4}	$I_z = 2304$ in.^{4}	$I_z = 512$ in.^{4}
$K' = 0.14 \times 10 \times 1000$	$K' = 0.178 \times 16 \times 12^3$	$K' = 1204$
$= 1414$ in.^{4}	$= 4921$ in.^{4}	$\frac{K'G}{L_{AB}} = 2.61$ E
$\frac{K'G}{L} = \frac{1414\ E}{2.4 \times 240}$	$\frac{K'G}{L} = 5.69$ E	$\frac{K'G}{L_{BC}} = 2.09$ E
$= 2.44$ E		

Numerical Example, Frame 2

Fig. 17

E (θ_{yA}	θ_{yB}	θ_{yC}	θ_{yD}	θ_{yE}	θ_{yF}	$\rho_{yAA'}$	$\rho_{yCC'}$	θ_{xA}	θ_{xB}	θ_{xC}	θ_{xD}	θ_{xE}	θ_{xF}	$\rho_{xAA'}$	$\rho_{xBB'}$	$\rho_{xCC'}$	θ_{zA}	θ_{zB}	θ_{zC}	θ_{zD}	θ_{zE}	θ_{zF})		EQUATION NO.
0.82	-1.40	1.38	3.18	-11.69	6.89	0.04	-0.30	3.93	-23.79	1.18	0.08	26.51	3.88	7.83	5.31	10.11	2.19	0.45	-3.99	0.42	-0.55	-0.45		
37.88	12.00	37.88	37.88	12.00		-20.82	-20.82																	54
12.00	57.08	9.60	12.00	57.08	9.60	-20.82	-20.82																-614.40	55
	53.28	9.60		53.28	9.60	-14.23	-14.23																-614.40	
	9.60	33.08		9.60	33.08	-20.82	-20.82																153.60	56
	9.60	31.36			31.36	-18.26	-18.26																264.30	
43.57	12.00		-5.69			-20.82																		57
	-1.80	0.32	-49.26	-12.00		2.82	23.64																-23.44	
12.00	63.77	9.60		-5.69		-20.82																		58
	59.97	-1.20	-12.00	-66.54	-9.60	0.41	16.16																707.36	
	9.60	38.77			-5.69	-20.82																		59
	9.60	37.05		9.60	-43.08	3.26	21.80																-303.60	
-0.08	-0.08	-0.08	-0.08	-0.08	-0.08	0.52	0.52																	60
	-0.06	-0.07				0.43	0.42																-0.11	
						25.60	-25.60							-19.95	19.95		38.68	5.32		12.80			51.20	68
						25.60	-51.20							-19.95	19.95		38.68	5.32		12.80			57.74	
								59.36	59.36	59.36	22.74	22.74	22.74	-20.82	-20.82	-20.82							-900.00	61
								59.36	59.36	59.36	22.74	22.74	22.74	-20.82	-20.82	-20.82							-900.00	
								56.75	2.61		22.74			-20.82										64
								56.75	-54.14	-56.75	1.00	-21.74	-21.74	-0.92	19.90	19.90							860.42	
									-2.09	61.45			22.74			-20.82								63
									-2.09	63.64	-0.03	0.83	23.57	0.03	-0.76	-21.58							-33.21	
								22.74	22.74	22.74	59.36	59.36	59.36	-20.82	-20.82	-20.82							900.00	62
								22.74			50.65	50.65	50.65	-12.85	-12.85	-12.85							1244.77	
								22.74			56.75	2.61		-20.82										66
								22.74	-22.74	1.09	47.62	-44.60	-47.60	-0.39	11.71	12.06							-1186.36	
										22.74		-2.09	61.45			-20.82								65
										22.74		-2.38	55.57	0.01	-0.35	-13.75							75.16	
								0.08	0.08	0.08	0.08	0.08	0.08	-0.34	-0.34	-0.34							-7.20	67
								0.08			0.05			-0.31	-0.31	-0.31							-7.21	
						25.60	-25.60							-19.95	7.17	12.78	5.32	47.20	4.26		12.80		28.80	69
						25.60	-51.20								-12.78	12.78	-33.36	41.88	4.26	-12.80	12.80		-22.40	
						25.60	-25.60								-12.78	12.78		4.26	36.59			12.80	-80.00	70
						25.60	-51.20							19.95	-52.68	-59.85	98.83	-173.69	19.00	39.96	-52.76	12.80	-502.86	
						25.60	-25.60							-19.95	19.95		12.80			38.68	5.32			71
						25.60	-51.20										-25.88	-5.32		25.88	5.32		-51.20	
						25.60	-25.60							-19.95	7.17	12.78		12.80		5.32	47.20	4.26		72
						25.60	-51.20								-12.78		-5.32	-33.31	-4.26		33.31	4.26	-18.28	
						25.60	-25.60								-12.78	12.78			12.80		4.26	36.56		73
						25.60	-51.20							19.95	-52.68	-59.85		-4.26	-23.22			23.22	82.33	
0.08	0.08	0.08				0.94	-0.42										0.10	0.10	0.10	0.10	0.10	0.10		74
	0.06	0.07	-0.08	-0.07	-0.07	1.03	-1.43							0.55	-1.10	-1.65	-0.83	1.31	-0.94	-1.08	1.59	-1.03	-0.87	
								0.08			0.08			-0.75	0.41		0.08	0.08		0.08	0.08		-3.20	75
								0.08	-0.08		0.05	-0.05		-0.72	1.13	1.85	0.19	-1.61	1.15	0.37	-1.46	0.99	0.36	
										0.08			0.08		0.21	-0.56		-0.05	-0.05		-0.05	-0.05	-4.00	76
										0.08			0.06		0.21	-0.32	-1.06	1.76	-0.30	-1.48	-0.18	1.10	-0.50	

WORK SHEET FOR EXAMPLE OF FRAME 2

FIG 18

VII. Results

A. General

In order to determine how many decimal places should be used in solving the matrix equation for Frame 1, a similar problem was worked using first two, then five decimal places. Table I shows a comparison of the results of these solutions.

Table II shows how well these solutions satisfied the original equations. In each case the right side of the equation is zero.

In view of the magnitudes of the roots of the equations, the results using two decimals are quite satisfactory. If the solutions using five decimal places are presumed to be correct roots, there seems to be little use in carrying out the computations to more than two decimal places, since the values shown in Table I agree, for the most part, within one or two percent. Since the data in most structural problems is accurate to no more than two decimal places, two places in the solutions appear to be adequate.

Tables III and IV show similar results for a set of 23 simultaneous equations. The results shown here agree even more closely than do those of the 16th order problem.

As an example of what one error in the inversion of the matrix can do, Table V is included. Two sets of solutions to the same matrix equation are shown, one correct and the other incorrect. In arriving at the incorrect set of roots, one and only one erroe was made. About three quarters of the way through the auxiliary matrix, one number was recorded with its sign reversed. The effect of such an error can be seen by the complete lack of agreement between the correct and the incorrect solutions.

B. Frame 1

Table VI shows the moments in all the members of Frame 1.

In Table VII are shown the values of the left sides of the equations when the roots are substituted into the original equations. As before, the right side of each equation is equal to zero.

C. Frame 2

In Table VIII are shown the moments in the example of Frame 2.

Table IX shows the results when the roots of the matrix equation are substituted into the original equations.

Table I

Matrix Equation Solutions - 16th Order

Unknown	Value (2 Decimals)	Value (5 Decimals)
X_1	81.10	79.27992
X_2	-94.64	-93.04309
X_3	81.11	79.28037
X_4	-94.57	-93.04363
X_5	112.67	110.15851
X_6	-169.01	-166.36025
X_7	1380.24	1377.30578
X_8	-1105.07	-1106.56140
X_9	-175.96	-173.14930
X_{10}	39.95	41.12131
X_{11}	380.55	377.73749
X_{12}	-104.22	-101.51002
X_{13}	-35.35	-35.57252
X_{14}	-32.28	-32.50211
X_{15}	-35.30	-35.57251
X_{16}	-32.29	-32.50209

Table II

Accuracy of Equation Solutions - 16th Order

Equation Number	Value Left Side (2 Decimals)	Value Left Side (5 Decimals)
1	0.49	0.00030
2	0.88	0.00047
3	0.78	0.00027
4	1.85	0.00763
5	0.08	0.00307
6	0.07	0.00021
7	0.00	0.00000
8	2.97	0.00854
9	5.18	0.00395
10	11.48	0.00610
11	8.95	0.00429
12	0.97	0.00049
13	0.96	0.00049
14	0.86	0.00026
15	0.61	0.00034
16	0.68	0.00272

Table III

Matrix Equation Solutions - 23rd Order

Unknown	Value (2 Decimals)	Value (6 Decimals)
x_1	-4.64	-4.175650
x_2	-3.46	-3.457231
x_3	-4.93	-4.338010
x_4	0.64	0.087192
x_5	16.56	16.523149
x_6	-3.34	-4.038936
x_7	-9.01	-8.424693
x_8	9.27	8.517076
x_9	-1.38	-1.263731
x_{10}	19.80	19.978047
x_{11}	0.21	0.235504
x_{12}	-4.61	-4.538354
x_{13}	-29.08	-28.856564
x_{14}	3.16	2.720262
x_{15}	-11.68	-11.222782
x_{16}	-17.84	-17.151958
x_{17}	6.04	4.328659
x_{18}	10.68	9.846682
x_{19}	2.85	2.776267
x_{20}	6.43	6.413440
x_{21}	12.81	11.969316
x_{22}	4.10	4.021510
x_{23}	1.92	1.894197

Table IV

Accuracy of Equation Solutions - 23rd Order

Equation Number	Value Left Side (2 Decimals)	Value Left Side (6 Decimals)
1	0.26	0.000033
2	0.00	0.000027
3	0.00	0.000025
4	0.34	0.000022
5	0.99	0.000049
6	0.33	0.413051
7	0.06	0.000009
8	0.47	0.000009
9	1.14	0.000038
10	0.48	0.002882
11	0.24	0.000010
12	2.01	1.005087
13	0.78	0.001145
14	0.03	0.000006
15	0.43	0.001512
16	2.37	1.860635
17	0.63	0.001504
18	0.64	0.001501
19	2.18	0.044824
20	0.15	0.000614
21	0.20	0.009719
22	0.01	0.006763
23	0.46	0.001511

Table V

Result of One Error

Unknown	Correct Value	Incorrect Value
x_1	-7.45	-4.64
x_2	-3.69	-3.46
x_3	-4.93	-8.39
x_4	0.64	3.44
x_5	16.56	16.99
x_6	-3.34	0.11
x_7	-9.01	-12.73
x_8	9.27	12.98
x_9	-1.38	0.85
x_{10}	19.80	19.87
x_{11}	0.21	-0.34
x_{12}	-4.61	-4.13
x_{13}	-29.08	-28.73
x_{14}	3.16	2.59
x_{15}	-11.68	-9.56
x_{16}	-17.84	-17.59
x_{17}	6.04	3.90
x_{18}	10.68	15.76
x_{19}	2.85	10.02
x_{20}	6.43	-15.11
x_{21}	12.81	14.23
x_{22}	4.10	5.66
x_{23}	1.92	17.50

Table VI

Bending and Torsional Moments - Frame 1

Member	Moment about x-axis (in.lb.)	Moment about y-axis (in.lb.)	Moment about z-axis (in.lb.)
A'A	404.21	1680.11	-343.02
AA'	309.89	16582.12	-343.02
B'B	-602.45	-17472.82	-243.35
BB'	-493.46	-21733.71	-243.35
C'C	403.61	3322.36	-342.92
CC'	688.40	4473.28	-342.92
D'D	-602.40	-3036.42	-243.31
DD'	-602.40	-8233.80	-243.31
BD	434.29	176.11	936.99
DB	434.83	176.11	937.07
AC	-250.48	1722.75	1011.79
CA	-251.01	1722.75	1011.90
AB	57.08	-18304.85	-798.18
BA	57.08	31054.07	-823.10
CD	57.41	-2747.98	-797.88
DC	57.41	-7221.03	-822.84

Table VII

Accuracy of Equation Solution - Frame 1

Equation Number	Value Left Side of Equation
13	0.24
14	0.16
20	0.42
19	2.02
28	0.48
25	0.08
27	0.00
11	2.57
17	0.03
18	4.48
12	8.88
21	2.05
22	2.01
24	2.39
23	2.57
26	4.08

Table VIII

Bending and Torsional Moments - Frame 2

Member	Moment about x-axis (in. kips)	Moment about y-axis (in. kips)	Moment about z-axis (in. kips)
A'A	-135.74	5.69	5.34
AA'	-108.47	11.38	5.34
B'B	-275.65	-9.71	1.09
BB'	-440.75	-19.43	1.09
C'C	-202.30	9.57	-9.73
CC'	-194.11	19.15	-9.73
D'D	-162.46	22.06	1.02
DD'	-161.91	44.13	1.02
E'E	73.42	-81.12	-1.34
EE'	257.40	-162.25	-1.34
F'F	-183.56	47.81	-1.09
FF'	-156.63	95.63	-1.09
AB	-72.34	2.88	26.63
BA	-72.34	-23.76	-85.03
BC	52.18	-13.63	128.18
CB	52.18	13.05	-50.74
DE	68.98	-63.96	-48.74
ED	68.98	-242.40	-53.89
EF	-47.29	-772.70	54.74
FE	-47.29	173.66	55.16
AD	180.55	-13.42	69.88

Table VIII (Cont'd)

Member	Moment about x-axis	Moment about y-axis	Moment about z-axis
DA	93.00	-13.42	47.23
BE	-1379.13	58.55	12.92
EB	1564.69	58.55	0.12
CF	141.89	-31.35	-99.45
FC	203.29	-31.35	-54.14

Table IX

Accuracy of Equation Solution - Frame 2

Equation Number	Value Left Side of Equation
54	0.15
55	0.02
56	0.38
57	0.00
58	0.51
59	0.02
60	0.07
68	0.30
61	0.03
64	0.27
63	0.03
62	0.15
66	0.07
65	0.56
67	0.23
69	0.18
70	0.33
71	0.22
72	0.11
73	0.18
74	0.05
75	0.05
76	0.08

VIII. Conclusions

The matrix method of analyzing rigid space frames is a practical method when an exact analysis is desired and has certain advantages over other methods, particularly when many such analyses are to be made.

Starting with the general matrix of Fig.9, any structure of four columns and four girders can be analyzed simply by substituting numerical values for the stiffness of each member and constants from the loading conditions. Any structure similar to Frame 2 can be analyzed from the matrix of Fig. 16. Computation of the stiffness factors for the members and fixed-end moments and other constant terms are the only parts of the entire analysis requiring any engineering knowledge. Substituting into Fig. 9 or Fig. 16 and solving the resulting numerical matrix is a time-consuming task, but one not requiring a high degree of skill. Once the auxiliary matrix is found, any changes made in the loading conditions will change only the last column of the auxiliary matrix and the analysis can be made without going through the entire matrix again. Any movements of the structure caused by foundation settlement, rotation, or movement horizontally can be handled as part of the routine problem with very little additional labor. The matrix inversion itself is tedious, but routine, and can be performed by anyone familiar with a desk calculator.

There are two outstanding disadvantages of this method. Any error in the matrix inversion process may well make the solutions be completely wrong. Thus, all work must be checked completely, since the errors likely to be made are not of the compensating type and will not simply increase the time required for convergence as frequently happens in moment distribution.

To many people, one advantage of moment distribution is that the analyst can always have before him a mental picture of the joint distortions as the analysis proceeds. This is lacking in the matrix method and the analyst cannot tell until he is finished whether or not his results are reasonable.

IX. Acknowledgments

The writer wishes to express his thanks to Professor Dan Pletta for his suggestions in choosing a topic to investigate and to Professor Victor Szebehely for his able introduction to the study of matrices.

The writer is deeply indebted to Professors Pletta and Daniel Frederick for their time and patience in suggesting corrections and improvements of the manuscript.

X. Bibliography

1. Andersen, Paul, "Design of Reinforced Concrete in Torsion," Transactions, American Society of Civil Engineers, vol. 103 (1938), pp. 1503-1526.
2. Bensooter, S. U., "Matrix Analysis of Continuous Beams" Transactions, American Society of Civil Engineers, vol. 112 (1947), pp. 1109-1172.
3. Brock, J. E., "A Matrix Method for Flexibility Analysis of Piping Systems," Journal of Applied Mechanics, vol. 19, no. 4 (Dec., 1952), pp. 501-516.
4. Chen, Pei-Ping, "Matrix Analysis of Pin-Connected Structures," Transactions, American Society of Civil Engineers, vol. 114 (1949), pp. 181-192.
5. Crout, P. D., "A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients," Transactions, American Institute of Electrical Engineers, vol. 60 (1941), pp. 1235-1240.
6. Ferguson, Phil M., "Analysis of Three-Dimensional Beam-and Girder Framing," Proceedings, American Concrete Institute, vol. 22, no. 1 (Sept., 1950), pp. 61-72.
7. Ferguson, Phil M., "Analysis Of Beam-and-Girder Framing with Known Column Settlements," Journal of the American Concrete Institute, vol. 24, no. 2 (Oct., 1952) pp. 77-84.

8. Frazer, R. A., Duncan, W. J., and Collar, A. R., Elementary Matrices, Cambridge, The University Press, 1950.

9. Grinter, L. E., Theory of Modern Steel Structures, vol. II, New York, The Macmillan Co., 1949.

10. Guillemin, E. A., The Mathematics of Circuit Analysis, New York, John Wiley & Sons, 1950.

11. Heyman, Jacques, "The Limit Design of Space Frames," Journal of Applied Mechanics, vol. 18, no. 2 (June, 1951), pp. 157-162.

12. Lazarides, T. O., The Structural Analysis of the Dome of Discovery, London, Crosby Lockwood, 1952.

13. Makowski, Z. S., and Pippard, A. J. S., "Experimental Analysis of Space Structures, with Particular Reference to Braced Domes," Journal of the Institution of Civil Engineers, paper no. 5887, (1953), pp. 420-441.

14. Michal, A. D., Matrix and Tensor Calculus, New York, John Wiley & Sons, 1948.

15. Murden, Wm. P., "Matrix Analysis of Rigid Frames," MS thesis, VPI, 1951.

16. Tedaldi, R. J., "Rigid Space Frame Analysis Using Successive Corrections," MS thesis, VPI, 1951.

17. Timoshenko, S., and MacCullough, G. H., Elements of Strength of Materials, New York, D. Van Nostrand Company, 1949.

18. Wilbur, John B., "Successive Elimination of Unknowns in the Slope-Deflection Method," *Transactions, American Society of Civil Engineers*, vol. 102 (1937), pp. 346-371.

The vita has been removed from the scanned document