INFLATED CYLINDRICAL ENVELOPE
SUBJECTED TO AXIAL COMPRESSION LOAD

by

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Thesis submitted to the Graduate Faculty of the Virginia Polytechnic Institute in candidacy for the degree of MASTER OF SCIENCE in Engineering Mechanics

August 1960
Blacksburg, Virginia
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II. INTRODUCTION

Inflated fabric is being considered as a new structural material at the present time. It can be used in certain applications with the advantage of reducing the weight of structures, it is adaptable as an architectural element of construction; moreover, it may be developed to be one of the most economical, and simple structural materials in the future.

A number of experimental investigations of these inflated fabric structures has been studied by some research units of airship and fabric companies. However, due to the difficulties of solving such problems by analysis, there is still lack of theoretical methods; even approximate solutions.

The purpose of this thesis is to investigate a theoretical analysis for finding the relations between the applied load and the deflections, stresses, and also the end shortening of an inflated cylindrical fabric envelope subjected to axial compression, by the energy method. A cylindrical shape is selected because sphere and cylinder are considered more general in use and more easily to be treated than any other geometrical shapes. Also, for the sake of simplicity, a constant internal pressure is assumed in the analysis.

The use of large deflection theory for finding the critical buckling loading of thin shells was first advanced by Von Karman and Tsien (reference 6 and 7). Based on their conception, numerous studies concerning the buckling strength under various loadings have been investigated by others subsequently. The strain-displacement relation in their papers
is expressed in the following form including terms up to second order:

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - \frac{w}{R}
\]

In this thesis, although the same idea is applied to develop an analysis by the energy method, the strain-displacement relation is expressed in a different way which will be shown in the following sections.

Generally, in avoiding the mathematical difficulty of solving the differential equations obtained from the energy expression, most boundary-value problems in the theory of elasticity may be solved by assuming a solution in the form of a series which satisfies the boundary conditions, then minimizing the energy expression to determine the values of unknown parameters in the assumed solution. In this thesis, instead of using the variational method mentioned above, a graphical method for solving the differential equations is presented. However, owing to the fact that not all of the boundary conditions are specified at one point, the final results have to be obtained by trial and error.
III. NOMENCLATURE

E  Modulus of elasticity.
L  Half of the length of the cylindrical envelope.
R₀  Radius of the cylinder.
U  Total strain energy due to stretch of the envelope.
V  Total potential energy of the system.
W₁ Work done by applied compressive load.
W₂ Work done by internal pressure.
h  Thickness of the envelope.
p  Internal pressure in the cylindrical envelope.
q  Axial compressive load per unit area.
t  Component of displacement of a point on middle surface of
the envelope in y direction.
u  Component of displacement of a point on middle surface of
the envelope in x direction.
v₀ Initial volume of the cylinder before axial load applied.
v  Volume after deformation.
w  Component of displacement of a point on middle surface of
the envelope in z direction.
w₀  Maximum radial displacement.
x  Coordinate measured in longitudinal direction of the
cylinder.
y  Coordinate measured in circumferential direction of the
cylinder.
Coordinate measured in radial direction of the cylinder.

End-shortening of the half length of the cylinder.

Strain component in the x direction.

Strain component in the y direction.

Initial stress in the x direction before axial load applied.

Amount of stress increased in the x direction after loading.

Initial stress in the y direction.

Amount of stress increased in the y direction after loading.

Poisson's ratio.
IV. DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

DERIVED BY THE ENERGY METHOD

A. Description of the Structure

The cylindrical shell used for the analysis is assumed to be made of synthetic fabric which is capable of undergoing large deflections and strains. The thickness of the shell is very small compared with the other dimensions of the structure.

Figure 1
Both ends of the cylinder are made of rigid metal. The fabric is flexible hence no bending moment is produced at the ends.

As mentioned before, although the internal pressure varies as deformation takes place, one can use the following artifice to reduce the problem to the case of a constant pressure. Imagine a tube connecting the envelope to a very large chamber, then as deformation takes place, essentially no pressure variation results. Consider that under this condition, the load $P$ is applied, then held constant while a valve in the connecting tube is closed, and subsequently the load $P$ is slowly released. If, now keeping the valve closed, $P$ is applied again, the resulting internal pressure as well as the deformed shape of the envelope is clearly the same as that at the end of the first application of $P$. Hence, by assuming this fictitious process, the pressure may be regarded as constant. Notice, that the constant pressure that is to be assumed would be the pressure at the end of the true loading process.

B. Assumptions

In deriving the theoretical analysis, some assumptions are necessary to make the whole problem as simple as possible:

1. The material of the envelope is perfectly elastic and obeys Hooke's law under the specified loading conditions; also, its elastic properties are the same in any direction perpendicular to the thickness.

2. The geometry of the shell is entirely defined by specifying
the form of the middle surface.

3. The thickness of the envelope is very small and may be considered to be constant during deformation.

4. The stress components normal to the middle surface are small compared with those along the surface and may be neglected. Also, it is assumed that there are no bending or compressive stresses in the envelope.

5. The weight of the envelope is neglected so the two parts of the deformed cylinder remain symmetrical to each other with respect to the middle circular plane.

C. Strain-Displacement Relation

Let \( u \), \( t \), and \( w \) be the displacement components of any point on the middle surface of the cylindrical shell in the \( x \), \( y \), and \( z \) directions after deformation. Since the cylinder is symmetrically loaded with respect to its longitudinal axis, all the points lying in the same horizontal circle have the same radial displacement \( w \), also, there is no displacement in the circumferential direction for any point in the cylinder; that means, \( t \) is equal to zero.

Consider a small area element cut out from the cylindrical shell by two radial planes and by two adjacent horizontal sections perpendicular to the longitudinal axis, with the side lengths \( dx \) and \( dy \).

The circumferential strain component along the parallel circles \( \varepsilon_y \), as seen from Figure 2, is
\[ \epsilon_y = \frac{d y' - d y}{d y} \]

\[ = \frac{w}{R_0} \]  

(1)

where the radial deflection \( w \) is considered positive outward.

**Figure 2**
Denote the end points of the infinitesimal length \( dx \) by \( A \) and \( B \); after deformation, these two points move to \( A' \) and \( B' \) respectively. Since there is no circumferential displacement \( t \) for any point in this case, the segment \( AB \) remains in the same axial plane (\( xz \)-plane in Figure 2). Therefore this can be considered as a two-dimensional problem.

If the \( x, z \) components of the displacement from \( A \) to \( A' \) are \( u \) and \( w \), the components of the displacement from \( B \) to \( B' \) can be written as \( u + \frac{du}{dx} \, dx \) and \( w + \frac{dw}{dx} \, dx \). It is shown in the Figure 2, that

\[
A'C' = dx + \frac{du}{dx} \, dx
\]

\[
B'C' = \frac{dw}{dx} \, dx
\]

\[
A'B' = \sqrt{(1 + \left(\frac{du}{dx}\right)^2 + (\frac{dw}{dx})^2) \, dx}
\]

\[
= \frac{1}{\sin \theta} \, \frac{dw}{dx} \, dx
\]

The angle \( \theta \) denotes the change of direction of \( dx \) after deformation. The longitudinal strain component in the \( x \) direction is then:

\[
\varepsilon_x = \frac{A'B' - AB}{AB}
\]

\[
= \frac{1}{\sin \theta} \, \frac{dw}{dx} - 1 \quad (2)
\]
D. Total Potential Energy of the System

1. Strain Energy due to Stretch of the Shell

When an elastic body is under the action of external forces, the body deforms and work is done by these forces. When in equilibrium, the work done in straining such a body equals the energy stored in the body, that is the strain energy.

Consider that the element of area cut out from the cylindrical shell is acted by two pairs of forces \( \sigma_x h dy \) and \( \sigma_y h dx \) in the \( x \) and \( y \) directions before loading, then during deformation, first assume that the element is acted on by the force in the \( x \) direction only; the stress component increases from \( \sigma_x \) to the value of \( \sigma_x + \sigma_x \), where \( \sigma_x \) is the stress increase after loading; while the strain increases an amount \( \varepsilon_x \). Therefore, the net increase in strain energy stored in the element is

\[
dU_1 = \frac{1}{2} \left( \sigma_x \varepsilon_x h dx dy + \sigma_x \varepsilon_x h dx dy - \mu \sigma_y \varepsilon_x h dx dy \right) = \frac{h}{2E} \left( \sigma_x^2 + 2 \sigma_x \sigma_x - 2 \mu \sigma_y \sigma_0 \sigma_x \right) dx dy
\]

where from Hooke's law \( \varepsilon_x \) is

\[
\varepsilon_x = \frac{\sigma_x}{E}
\]

Then assume that the element is acted by \( \sigma_y \) in the \( y \) direction only. The strain increases an amount \( \varepsilon_y \), and the additional energy stored
in the element
\[
d U_2 = \frac{h}{2E} \left( \sigma_y'^2 + 2 \sigma_y \sigma_y' - 2 \mu \sigma_x \sigma_y' - 2 \mu \sigma_y \sigma_y' \right) \, dx \, dy
\]

where from Hooke's law \( \varepsilon_y \) is
\[
\varepsilon_y = \frac{\sigma_y}{E}
\]

Since the cylinder is symmetrically loaded with respect to its axis, there is no shearing stress in this case.

Hence the strain energy accumulated in the element is
\[
d U = d U_1 + d U_2
\]
\[
= \frac{h}{2E} \left[ \sigma_x'^2 + 2 \mu \sigma_x \sigma_y' + 2 \sigma_y' (\sigma_y' - \mu \sigma_x') \right] \, dx \, dy
\]

With this system of stresses, from Hooke's law:
\[
\varepsilon_x = \frac{1}{E} (\sigma_x' - \mu \sigma_y') ; \quad \varepsilon_y = \frac{1}{E} (\sigma_y' - \mu \sigma_x')
\]

\( dU \) may be written in the following form
\[
d U = \left[ \frac{Eh}{2(1 - \mu^2)} \left( \varepsilon_x'^2 + 2 \mu \varepsilon_x \varepsilon_y + \varepsilon_y'^2 \right) + \sigma_y' \varepsilon_x + \sigma_x' \varepsilon_y \right] \, dx \, dy
\]

The total strain energy can then be found by integrating \( dU \) over the whole surface of shell,
\[
U = \int_0^L \int_0^{R_o} \left[ \frac{Eh}{2(1 - \mu^2)} \left( \varepsilon_x'^2 + 2 \mu \varepsilon_x \varepsilon_y + \varepsilon_y'^2 \right) + \right] \, r \, dr \, d \theta
\]
\[
\sigma_{x_0} h \varepsilon_x + \sigma_{y_0} h \varepsilon_y \]
\[
= 2 \pi \int_0^L \left[ \frac{Eh}{2(1-\mu^2)} (R_0 + w) \left\{ \frac{1}{\sin^2 \theta} \left( \frac{dw}{dx} \right)^2 - 2 \frac{1}{\sin \theta} \frac{dw}{dx} + 1 + \frac{2\mu}{R_0} \frac{w}{\sin \theta} \frac{dw}{dx} \right\} - \frac{2\mu}{R_0} w + \frac{1}{R_0^2} w^2 \right] \sigma_{x_0} h (R_0 + w) \left( \frac{1}{\sin \theta} \frac{dw}{dx} - 1 \right) + \frac{\sigma_{y_0} h}{R_0} (R_0 + w) w \right] \ dx
\]

(3)

### 2. Potential Energy of the External Forces and Internal Pressure

Denote the displacement of the end point of the half length of the cylinder by

\[
\delta = \int_0^L \frac{du}{dx} \ dx
\]

(4)

the total compressive force by \( \pi R_0^2 q \), where \( q \) is the uniform distributed load per unit area. The work done by the axial compressive force is then

\[
W_1 = -\pi R_0^2 q \delta = - \int_0^L \pi R_0^2 q \frac{du}{dx} \ dx
\]

(5)

where minus sign is used because opposite directions of \( q \) and \( u \) were assumed.

An elemental volume \( dv \), with height \( dx \), cut by two adjacent horizon-
tal planes, is
\[ dv_0 = \pi R_o^2 \, dx \]

After deformation, the element of volume will change to be
\[ dv = \pi (R_o + w)^2 \left( 1 + \frac{du}{dx} \right) \, dx \]

Therefore the work done by the constant internal pressure \( p \) is
\[ W_2 = \int_0^L p \, dv = \int_0^L \pi p \left[ (R_o + w)^2 \left( 1 + \frac{du}{dx} \right) - R_o^2 \right] \, dx \quad (6) \]

The total potential energy of the system is then
\[ V = U - W_1 - W_2 \]
\[ = 2\pi \int_0^L \left\{ \frac{Eh}{2(1 - \mu^2)} (R_o + w) \left\{ \frac{1}{\sin^2 \Theta} \left( \frac{dw}{dx} \right)^2 - 2 \frac{1}{\sin \Theta} \frac{dw}{dx} + 1 \right\} + \frac{2\mu}{R_o} \frac{w}{\sin \Theta} \frac{dw}{dx} - \frac{2\mu}{R_o} w + \frac{1}{R_o^2} w^2 \right\} \right. \]
\[ + \sigma x_o h (R_o + w) \left( \frac{1}{\sin \Theta} \frac{dw}{dx} - 1 \right) + \frac{\sigma y_o h}{R_o} w (R_o + w) \]
\[ + \frac{R_o^2 q}{2} \left( \cot \Theta \frac{dw}{dx} - 1 \right) - \frac{p}{2} (R_o + w)^2 \cot \Theta \frac{dw}{dx} + \frac{p R_o^2}{2} \right\} \, dx \quad (7) \]
E. Differential Equations of Equilibrium

By using the theory of calculus of variationsthat the equations which give a necessary condition for

\[ V = \int_{a}^{b} f \left( x, w, \theta, \frac{dw}{dx}, \frac{d\theta}{dx} \right) dx \]

to be a minimum are

\[ \frac{\partial f}{\partial \theta} - \frac{d}{dx} \left( \frac{\partial f}{\partial w} \right) = 0 \quad (8a) \]

\[ \frac{\partial f}{\partial w} - \frac{d}{dx} \left( \frac{\partial f}{\partial w'} \right) = 0 \quad (8b) \]

where \( \theta' = \frac{d\theta}{dx} \) and \( w' = \frac{dw}{dx} \).

From equation (7)

\[ \frac{\partial f}{\partial \theta} = \frac{Eh}{2(1 - \mu^2)} \left( R_o + w \right) \left\{ -2 \frac{1}{\sin \theta} \frac{dw}{dx} + \frac{2\mu}{R_o} w \right\} \frac{\cos \theta}{\sin^2 \theta} \frac{dw}{dx} \]

\[ - \sigma x_o h \left( R_o + w \right) \frac{\cos \theta}{\sin^2 \theta} \frac{dw}{dx} - \frac{1}{2} \left\{ R_o q - p \left( R_o + w \right) \right\} \frac{1}{\sin^2 \theta} \frac{dw}{dx} \]

\[ \frac{\partial f}{\partial w'} = 0 \]

equation (8a) becomes

\[ \frac{1}{\sin \theta} \frac{dw}{dx} + \frac{\mu}{R_o} w + \frac{1 - \mu}{2} \frac{2}{Eh} \left\{ \frac{R_o^2 q}{(R_o + w)} - p \left( R_o + w \right) \right\} \frac{1}{\cos \theta} + \frac{1 - \mu}{E} \sigma x_o - 1 = 0 \quad (9a) \]
Also from equation (7)

\[
\frac{\partial f}{\partial \omega} = \frac{\Phi h}{2(1 - \mu^2)} \left\{ \frac{1}{\sin^2 \theta} \left( \frac{\partial \omega}{\partial x} \right)^2 - 2 \frac{1}{\sin \theta} + 1 + \frac{2}{\sin \theta} \frac{\partial \omega}{\partial x} - \frac{2 \mu w}{\Phi h} \right\}
\]

\[
+ \frac{w^2}{\Phi h} + (R_o + w) \left( \frac{2}{\sin \theta} \frac{\partial \omega}{\partial x} - \frac{2 \mu}{\Phi h} + \frac{2w}{\Phi h^2} \right)
\]

\[
+ \frac{\sigma}{\Phi h} \left( \frac{\partial \omega}{\partial x} - 1 \right) + \frac{\sigma}{\Phi h} \left( R_o + 2w \right) - p \left( R_o + w \right) \cot \theta \frac{\partial \omega}{\partial x}
\]

\[
\frac{d}{dx} \left( \frac{\partial f}{\partial \omega} \right) = \frac{\Phi h}{2(1 - \mu^2)} \left\{ \frac{1}{\sin^2 \theta} \left( \frac{\partial \omega}{\partial x} - 1 + \frac{\mu w}{\Phi h} \right) \frac{1}{\sin \theta} \frac{\partial \omega}{\partial x}
\]

\[
+ 2 \left( R_o + w \right) \left( \frac{1}{\sin^2 \theta} \frac{\partial^2 \omega}{\partial x^2} + \frac{\mu}{\Phi h} \frac{1}{\sin \theta} \frac{\partial \omega}{\partial x} \right)
\]

\[
+ \frac{\sigma}{\Phi h} \frac{1}{\sin \theta} \frac{\partial \omega}{\partial x} - p \left( R_o + w \right) \cot \theta \frac{\partial \omega}{\partial x}
\]

\[
+ \left\{ \frac{\Phi h}{2(1 - \mu^2)} \left( R_o + w \right) \left( -4 \frac{1}{\sin \theta} \frac{\partial \omega}{\partial x} + 2 - \frac{2 \mu w}{\Phi h} \right)
\]

\[
- \frac{\sigma}{\Phi h} \left( R_o + w \right) - \frac{R_o^2 q}{2} \frac{1}{\cos \theta} + \frac{p}{2} \left( R_o + w \right)^2 \frac{1}{\cos \theta} \right\} \cos \theta \frac{d\theta}{\sin^2 \theta} \frac{\partial \theta}{\partial x}
\]

Substituting into equation (8b)

\[
\frac{\partial f}{\partial \omega} - \frac{d}{dx} \left( \frac{\partial f}{\partial \omega} \right) = 0
\]

the second differential equation of equilibrium is thus obtained
\[
2 \left( R_0 + w \right) \frac{1}{\sin^2 \theta} \left( \frac{d^2 w}{dx^2} - \cot \theta \frac{dw}{dx} \frac{d \theta}{dx} \right) + \frac{1}{\sin^2 \theta} \left( \frac{dw}{dx} \right)^2 - \frac{3 w^2}{R_0^2}
\]

\[- (1 - 2\mu) \left( 1 + \frac{2w}{R_0} \right) + \frac{2(1 - \mu^2)}{E} \sigma \left( R_0 + w \right) - \frac{2(1 - \mu^2)}{E} \frac{\sigma y_o}{R_0} \left( R_0 + 2w \right) = 0
\]

Multiplying the above equation by \( \frac{dw}{dx} \) it is found that the resulting equation can be integrated and becomes

\[
\frac{1}{\sin^2 \theta} \left( \frac{dw}{dx} \right)^2 = \frac{w^3}{R_0^2 (R_0 + w)} - \frac{2(1 - \mu^2) \sigma x_o}{E} \frac{w}{R_0 + w} + \frac{2(1 - \mu^2) \sigma y_o}{R_0}
\]

\[+ (1 - 2\mu) \frac{w}{R_0} + \frac{C}{R_0 + w}\]  

(9b)

where \( C \) is the constant of integration.

Since

\[
\sin \theta = \frac{\frac{dw}{dx}}{\sqrt{(1 + \frac{du}{dx})^2 + \left( \frac{dw}{dx} \right)^2}}
\]

\[
\cos \theta = \frac{1 + \frac{du}{dx}}{\sqrt{(1 + \frac{du}{dx})^2 + \left( \frac{dw}{dx} \right)^2}}
\]

\[
\frac{1}{\sin \theta} \frac{dw}{dx} = \sqrt{(1 + \frac{du}{dx})^2 + \left( \frac{dw}{dx} \right)^2}
\]

Substituting the above relations into equations (9a) and (9b), and
eliminating the term $\frac{dw}{dx}$ from these two equilibrium equations, the governing differential equation involving $\frac{dw}{dx}$ in bi-quadratic form is thus obtained

$$
\left[ \frac{B^2 - C^2 - D}{2 AC^2 + 2 AD} \right]^2 \left( \frac{dw}{dx} \right)^2 + 2 \left[ \frac{A^2 - B^2 + C^2 + D}{2 A} \right] \left( \frac{B^2 - C^2 - D}{2 AC^2 + 2 AD} + \frac{1}{2} \right) \left( \frac{dw}{dx} \right)^2
$$

$$+
\left[ \frac{A^2 - B^2 + C^2 + D}{2 A} \right]^2 - C^2 - D = 0 \tag{10}
$$

where

$$
A = \frac{1 - \mu^2}{2 E} \left\{ \rho \left( R_o + w \right) - \frac{R_o^2 q}{R_o + w} \right\}
$$

$$
B = 1 - \frac{1 - \mu^2}{E} \sigma x_o - \frac{\mu}{R_o} w
$$

$$
C^2 = \frac{C}{R_o + w}
$$

$$
D = \frac{w^2}{R_o^2 (R_o + w)} + (1 - 2\mu) \frac{w}{R_o} - \frac{2(1 - \mu^2) \sigma x_o}{E} \frac{w}{R_o + w} + \frac{2(1 - \mu^2) \sigma h}{Eh} \frac{w}{R_o}
$$

The boundary conditions of the system are

$$
\frac{dw}{dx} = 0 \quad \text{at} \quad x = 0
$$

$$
w = 0 \quad \text{at} \quad x = L
$$

The constant C can not be determined directly from equation (9b) with the above boundary conditions, but in equation (10) when $x = 0$

$$
\frac{dw}{dx} = 0 \text{, it is seen that}
$$
\[
\left[ \frac{A^2 - B^2 + C^2 + D}{2A} \right]^2 - C^2 - D = 0
\]

(11)

where all \(A, B\) and \(D\) are in terms of \(w_0\). Therefore from (11) the relation between \(C\) and \(w_0\) can be established as

\[
\frac{C}{R_0 + w_0} = C^2 = (B + A)^2 - D
\]

or

\[
C = (R_0 + w_0) \left[ (1 - \frac{1 - \mu^2}{E} \sigma x_0 - \frac{\mu}{R_0} w_0) + \frac{1 - \mu^2}{2EH} \left\{ p(R_0 + w_0) - \frac{R_0^2 q}{(R_0 + w_0)} \right\} \right]
\]

- \[
\left[ \frac{w_0^3}{R_0^2} + \left\{ (1 - 2\mu) + \frac{2(1 - \mu^2)}{E} \sigma y_0 \right\} (R_0 + w_0) \frac{w_0}{R_0} - \frac{2(1 - \mu^2)}{E} \sigma x_0 w_0 \right]
\]

(12)

Also, the derivative of axial displacement, \(\frac{du}{dx}\), can be found from equation (9b)

\[
\frac{du}{dx} = 1 - \sqrt{D + C^2 - \left( \frac{dw}{dx} \right)^2}
\]

\[
1 - \sqrt{\frac{w_0^3}{R_0^2 (R_0 + w)} - \frac{2(1 - \mu^2)}{E} x_0 \frac{w}{R_0 + w} + \frac{2(1 - \mu^2)}{E} \frac{y_0}{R_0} \frac{w}{(1 - 2\mu) \frac{w}{R_0}}} + \frac{C}{R_0 + w} - \left( \frac{dw}{dx} \right)^2
\]

(13)

The stress increments in the \(x\) and \(y\) directions are
\[ \sigma_x h = \frac{E_h}{1 - \mu} \left( \varepsilon_x + \mu \varepsilon_y \right) \]

\[ = \frac{E_h}{1 - \mu} \left[ \sqrt{(1 + \frac{du}{dx})^2 + \left( \frac{dv}{dx} \right)^2} - 1 + \frac{\mu w}{R_0} \right] \quad (14a) \]

\[ \sigma_y h = \frac{E_h}{1 - \mu} \left( \varepsilon_y + \mu \varepsilon_x \right) \]

\[ = \frac{E_h}{1 - \mu} \left[ \frac{W}{R_0} + \mu \left\{ \sqrt{(1 + \frac{du}{dx})^2 + \left( \frac{dv}{dx} \right)^2} - 1 \right\} \right] \quad (14b) \]

The final stresses for each point are

\[ \sigma_{xt} = \sigma_x + \sigma_x^t \quad (15a) \]

\[ \sigma_{xt} = \sigma_y + \sigma_y^t \quad (15b) \]
V. SOLUTION OF THE DIFFERENTIAL EQUATION BY GRAPHICAL METHOD

Since the exact analytical solution of equation (10) appears impossible, the graphical method of isoclines will be applied to a numerical example.

The procedure of the graphical method for finding \( \frac{dw}{dx} \) can be outlined as follows:

1. Try any value of the maximum radial deflection \( w_0 \) at \( x = 0 \).
2. The constant \( C \) can be determined from equation (12) if \( w_0 \) is known.
3. Substituting \( C \) into equation (10), find the values of \( \frac{dw}{dx} \) corresponding to different \( w_0 \)'s ranging from \( w = w_0 \) to \( w = 0 \).
4. Let \( w \) be ordinate and \( x \) be abscissa. On each line of \( w = \) constant draw a series of short parallel segments having the slope \( \frac{dw}{dx} \) found from the step (3), then the integral curve can be sketched by starting at the point where \( w = w_0 \), \( x = 0 \); and crossing through the successive lines of \( w = \) constant with indicated slopes; then intersecting the \( x \) - axis at the end point. If the value of \( w_0 \) selected is exactly the true one, this curve would intersect the \( x \) - axis at the end point where \( w = 0 \), \( x = L \).
5. Compute \( \frac{dw}{dx} \) for each point from equation (13). Find the new slope.
\[ \tan \theta = \frac{\frac{dw}{dx}}{1 + \frac{du}{dx}} \]

6. Again, starting from the point \( w = w_0, x = 0 \), plot a new curve with the slope \( \tan \theta = \frac{\frac{dw}{dx}}{1 + \frac{du}{dx}} \). This curve is obviously the true meridional deflection curve of the cylindrical envelope after loading.

7. The distance between the end points of the two curves is the total end shortening \( \delta \), which can be checked by summing up the area under the \( \frac{du}{dx} \) v.s. \( x \) curve by the graphical method.
VI. NUMERICAL EXAMPLE

A numerical example has been worked out for illustration. The properties and dimensions of the fabric envelope are arbitrarily assumed and given as follows:

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of Elasticity</td>
<td>$E = 1,000,000$ lb per sq ft.</td>
</tr>
<tr>
<td>Thickness of envelope</td>
<td>$h = 0.001$ ft.</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu = 0.4$</td>
</tr>
<tr>
<td>Radius of cylinder</td>
<td>$R = 10$ ft.</td>
</tr>
<tr>
<td>Half length of the cylinder</td>
<td>$L = 14.5$ ft.</td>
</tr>
<tr>
<td>Internal pressure</td>
<td>$p = 15$ lb per sq ft.</td>
</tr>
<tr>
<td>Axial load</td>
<td>$q = 10$ lb per sq ft.</td>
</tr>
</tbody>
</table>

From the membrane theory of cylindrical shells, the initial stresses $\sigma_{x_0}$ and $\sigma_{y_0}$ can be found

$$\sigma_{x_0} h = \frac{p R_0}{2} = 75 \text{ lb per ft.}$$

$$\sigma_{y_0} h = p R_0 = 150 \text{ lb per ft.}$$

The initial strains are neglected in this example. If the strains should be considered, they can be found by the same method except that $q$, $\sigma_{x_0}$, $\sigma_{y_0}$ are zero in the case.

By a number of trials, the value of the maximum radial deflection $w_0$ was found to be $0.1292$ ft.; the value of $C$ was determined from the equation (12).

$$C = 0.917936326$$

Then, following the steps listed on previous section find $\frac{dw}{dx}$ and $\frac{du}{dx}$ for each point by the graphical method.
The results of computations performed at various points of \( w \) are tabulated below:

Table 1. Values of slopes and derivatives of axial displacements

<table>
<thead>
<tr>
<th>( w ) ft.</th>
<th>( \frac{dw}{dx} )</th>
<th>( \frac{du}{dx} )</th>
<th>( \frac{du}{dx} )</th>
<th>( x ) ft.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1292</td>
<td>0.00000000</td>
<td>-0.0458183</td>
<td>0.00000000</td>
<td>0.00</td>
</tr>
<tr>
<td>0.1290</td>
<td>0.0000566</td>
<td>-0.0458124</td>
<td>0.0000593</td>
<td>0.30</td>
</tr>
<tr>
<td>0.1285</td>
<td>0.0003995</td>
<td>-0.0457977</td>
<td>0.0004187</td>
<td>0.44</td>
</tr>
<tr>
<td>0.1280</td>
<td>0.0007362</td>
<td>-0.0457831</td>
<td>0.0007715</td>
<td>0.33</td>
</tr>
<tr>
<td>0.1275</td>
<td>0.0014311</td>
<td>-0.0457690</td>
<td>0.0014997</td>
<td>0.86</td>
</tr>
<tr>
<td>0.1270</td>
<td>0.0017543</td>
<td>-0.0457548</td>
<td>0.0018384</td>
<td>0.19</td>
</tr>
<tr>
<td>0.1260</td>
<td>0.0020700</td>
<td>-0.0457258</td>
<td>0.0021692</td>
<td>0.72</td>
</tr>
<tr>
<td>0.1240</td>
<td>0.0034038</td>
<td>-0.0456703</td>
<td>0.0035667</td>
<td>0.50</td>
</tr>
<tr>
<td>0.1220</td>
<td>0.0047348</td>
<td>-0.0456166</td>
<td>0.0049611</td>
<td>10.02</td>
</tr>
<tr>
<td>0.1200</td>
<td>0.0060681</td>
<td>-0.0455648</td>
<td>0.0063578</td>
<td>10.40</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.00914700</td>
<td>-0.0451480</td>
<td>0.0203906</td>
<td>12.17</td>
</tr>
<tr>
<td>0.0800</td>
<td>0.0135402</td>
<td>-0.0450013</td>
<td>0.0330113</td>
<td>13.02</td>
</tr>
<tr>
<td>0.0600</td>
<td>0.0186654</td>
<td>-0.0448850</td>
<td>0.0488858</td>
<td>13.57</td>
</tr>
<tr>
<td>0.0400</td>
<td>0.0249312</td>
<td>-0.0453212</td>
<td>0.059187</td>
<td>13.92</td>
</tr>
<tr>
<td>0.0200</td>
<td>0.0315510</td>
<td>-0.0458048</td>
<td>0.0781297</td>
<td>14.23</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0383862</td>
<td>-0.0459958</td>
<td>0.0926483</td>
<td>14.50</td>
</tr>
</tbody>
</table>

The end shortening \( \delta \) is found to be 0.6635 ft.
Table 2. Values of final stress

<table>
<thead>
<tr>
<th>w (ft.)</th>
<th>$\sigma_{x_0} h + \sigma_{X_0} h$ (lb. per ft.)</th>
<th>$\sigma_{y_0} h + \sigma_{Y_0} h$ (lb. per ft.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1292</td>
<td>26.607</td>
<td>146.024</td>
</tr>
<tr>
<td>0.1240</td>
<td>26.543</td>
<td>145.379</td>
</tr>
<tr>
<td>0.1200</td>
<td>26.482</td>
<td>144.883</td>
</tr>
<tr>
<td>0.1000</td>
<td>25.250</td>
<td>142.405</td>
</tr>
<tr>
<td>0.0600</td>
<td>25.779</td>
<td>137.454</td>
</tr>
<tr>
<td>0.0000</td>
<td>25.107</td>
<td>130.043</td>
</tr>
</tbody>
</table>
Radial Deflection (ft.), w

Longitudinal Axis (ft.), x

$w_o = 0.1292$

Figure 3. Radial Deflection Curve

$p = 15 \text{ lb. per sq. ft.}$
$q = 10 \text{ lb. per sq. ft.}$
$R_o = 10 \text{ ft.}$
$L = 14.5 \text{ ft.}$
VII. CONCLUSIONS

Due to the non-linearity of the differential equation and indeterminate constant of integration, an exact solution of the deflection equation is difficult to obtain analytically; however, numerical examples may be worked out by using a graphical method with a trial and error feature to find the final solution. The analysis presented here is only an approximate method for solving the particular problem, it is hard to see whether it may give a fairly good result unless an experimental investigation can be made for comparison.

The result of computation indicates that the deflections and slopes are unexpectedly small compared with the dimensions of the cylinder. This may be due to the fact that internal pressure is limited to be constant during loading. The accuracy of the solution depends on the determination of the length of the cylinder from the graph. A better result may be obtained by taking smaller intervals of \( w \) in the region near maximum radial deflection, \( w_0 \); also by using different scales in different regions in plotting the curve. In the problem solved, the final stresses are found less than the original values of the unloaded cylinder. The maximum stresses occur at the mid cross-section.

The following suggestions are worth considering for further improvement of the analysis.

1. If the initial pressure \( p \) is large, the structure will no longer remain in the cylindrical shape, and therefore, the initial
strains and deflections should be considered in the analysis. They can be found by the same procedure as shown in the previous sections except that $q$, $\sigma_{x_0}$ and $\sigma_{y_0}$ are zero in this case.

Assuming that the internal pressure and the external force are applied at the same time, the final strains, deflections, and stresses can be determined by the same procedure except that the terms involving $\sigma_{x_0}$ and $\sigma_{y_0}$ are omitted from the basic equations. The net increments of these quantities are then the differences between the initial and the final values.

2. The material has been assumed to be isotropic in this analysis. However, since most fabrics do not exhibit the same properties in all directions in the plane of the fabric, and are susceptible to creep effects, some deviation from the true behavior is to be expected. If the rate of application of loading is high, then no substantial error arises from neglecting the creep effects.

3. Extensions of the present study to the cases involving shear, bending and also wrinkling are recommended.
VIII. ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. R. Chicurel, major professor of this thesis, for his enthusiastic help and advice, without whose guidance and encouragement this investigation would not have been brought to a conclusion.

Appreciation is extended to Professor D. H. Pletta, who originally suggested this topic to the author, and also to Dr. T. S. Chang, for their helpful criticisms and suggestions.
IX. BIBLIOGRAPHY

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