APPLICATIONS OF THE ANALOG COMPUTER
TO MATHEMATICAL PROBLEMS

by

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I. INTRODUCTION

The digital computer is a highly useful aid to mathematical analysis, and is indispensable to present-day computation. It is used by mathematicians everywhere. The analog computer has not received such widespread recognition. However, it also has the potentiality of becoming highly useful to mathematicians.

The abilities of the analog computer are varied. A fully-equipped analog computer can be programmed to add, subtract, multiply, divide, and integrate continuously changing variables. In addition, it can be programmed to perform many other mathematical operations, such as taking the square root of a given quantity; and to perform certain logical operations, such as actuating a switch when a particular variable changes sign. It is also capable of generating analytic functions.

This thesis is intended to be an introductory mathematical presentation of analog computation. An attempt was made to explain in concise mathematical language, how an electronic analog computer works, why it works, and the simplicity of its use. Examples of computer solutions of several different types of mathematical problems are presented. In each case, the electrical knowledge required is meager.
II. THE BASIC THEORY OF ANALOG COMPUTATION

1. History

A computer (9, p. 2) may be defined as any device that is capable of accepting quantitative information, performing mathematical and logical operations on this information, and making the results of these operations available as output.

Computers may be separated into two classes, the analog and the digital. A digital computer performs serial operations upon discrete variables. An analog computer performs parallel operations upon continuous variables. Many different devices can be classified as analog computers. However, the only analog computers considered in this thesis are differential analyzers, that is, devices that solve differential equations. The analog computer utilizes the analog between the mathematical model of a given problem and the mathematical model of a mechanical or electrical circuit that is programmed on the computer.

In 1927, Vannevar Bush (9, p. 4) started work on the first differential analyzer. He used mechanical devices to perform the operations of addition, multiplication by a constant, and integration. The
use of such mechanical devices made the computer very large and unwieldy. The answer to this problem was to use electronic devices. However, before World War II, the components necessary to build an accurate electronic differential analyzer were not available. During the war, special-purpose electronic computers were built. By the end of the war, the necessary components were available; and in 1947, the first reasonably accurate electronic analog computer (9, p. 4) was constructed.

During recent years, computer components have been improved. Analog computers are now becoming standard tools in the field of dynamic analysis.

This paper is concerned only with electronic analog computers, and in particular with the 15-amplifier Heath Kit analog computer, model ES-400; and the 9-amplifier Heath Kit analog computer, model EC-1.

2. Building Blocks

The problem to be solved is simulated on the computer by an electrical circuit whose mathematical description is the same as that of the given problem. Each variable in the problem is represented by a voltage.
2.1 Basic Components. The three basic components used in this simulation are resistors, capacitors, and amplifiers. A resistor is an element that dissipates energy; a capacitor is an element that stores energy; and an amplifier is a device that magnifies voltages. Assuming that each of the three components is ideal, the following relationships hold:

(2.11) Resistor: \( E(s) = RI(s) \)

(2.12) Capacitor: \( E(s) = \frac{I(s)}{Cs} + \frac{e(0)}{s} \)

(2.13) Amplifier: \( E_o(s) = -\mu E_g(s) \)

where:

- \( e \) = voltage across the resistor (capacitor), volts
- \( i \) = current through the resistor (capacitor), amperes
- \( R \) = resistance, ohms
- \( C \) = capacitance, farads
- \( s \) = Laplace variable
- \( \mu \) = open loop gain of the amplifier
- \( e_o, e_g \) = output and input of the amplifier, volts.

The capital letters are used to denote the Laplace transforms of the variables. The error introduced into the solution of any problem by the assumption
that these elements are ideal will be discussed later. The symbols used for these three components are shown in Figure 1. The amplifier (9, p. 515) used is commonly called a high gain amplifier because the constant, $\mu$, is usually very large, $10^5$ to $10^8$.

These three basic components are combined in various ways to obtain devices capable of adding, integrating, and multiplying variables by constants.

2.2 Operational Amplifiers. Both a resistor and a capacitor affect the current in a circuit. The measure of this effect is called impedance. For example, the impedance of a resistor is its resistance, $R$. In Figure 2, a "feedback" impedance, $Z_f$, is connected across the input and output terminals of the amplifier; and input impedances, $Z_1$, ..., $Z_n$ are connected to the input of the amplifier. The point of connection is called the summing junction, $J$. An amplifier, together with such connections is called an operational amplifier, $\mathcal{O}$. By choosing the proper combination of resistors and capacitors, $\mathcal{O}$ acts as an integrator or as an adder. Each combination of these components will be assumed to be ideal.
THE RESISTOR

THE CAPACITOR

THE HIGH-GAIN AMPLIFIER

FIGURE 1. THE BASIC COMPONENTS
OF THE ANALOG COMPUTER

FIGURE 2. THE OPERATIONAL AMPLIFIER
Therefore, assuming that
1. The open loop gain, \( \mu \), is infinite;
2. The output impedance of the amplifier, \( Z_0 \), is zero;
3. The voltage at the summing junction is zero;

then

\[
E_0(s) = - Z_f(s) \sum_{i=1}^{n} \frac{E_i(s)}{Z_i(s)}
\]  

2.3 The Adder. If \( Z_f, Z_1, \ldots, Z_n \) are resistances \( R_f, R_1, \ldots, R_n \), then

\[
E_0(s) = - R_f \sum_{i=1}^{n} \frac{E_i(s)}{R_i}
\]

Hence, \( E \) is a combination adder and sign changer. If \( n=1 \) and \( R_f = R_1 \), then \( E \) is called an inverter.

Let \( Z_f, Z_1, \ldots, Z_n \) be capacitances, \( \frac{1}{C_f}, \frac{1}{C_1}, \ldots, \frac{1}{C_n} \), then

\[
E_0(s) = - \frac{1}{C_f} \sum_{i=1}^{n} \frac{1}{C_i} E_i(s)
\]

Therefore, \( E \) is a combination adder and sign changer for this case, too. However, capacitors are more expensive
than resistors, and not as durable. Consequently, this combination is not used.

2.4 The Integrator. If $Z_f$ is a capacitance $\frac{1}{C_f s}$, and $Z_1, \ldots, Z_n$ are resistances, $R_1, \ldots, R_n$, then

$$E_0(s) = -\frac{1}{C_f s} \sum_{i=1}^{n} \frac{E_i(s)}{R_i}.$$ 

In this case, $O$ is a combination integrator, adder, and sign changer. Here it was assumed that $e_0(0) = 0$. If $e_0(0) \neq 0$, then a voltage source in series with a relay must be connected across the input and output of the amplifier as shown in Figure 3. The voltage source is set equal to $e_0(0)$. Consequently, the capacitor will have an initial voltage across it equal to $e_0(0)$, and

$$E_0(s) = -\frac{1}{C_f s} \sum_{i=1}^{n} \frac{E_i(s)}{R_i} + \frac{e_0(0)}{s}.$$ 

Theoretically, if $e_0(0) = 0$, then the only feedback element required is the capacitor. Practically, this is not true. A relay without a voltage supply must be connected across the input and output terminals of the amplifier. When the relay is closed, the capacitor is short-circuited; the voltage across the capacitor is zero. Thus, the relay maintains the desired zero initial
FIGURE 3. THE INTEGRATOR

FIGURE 4. COMBINATION FEEDBACK
conditions. Without the relay, the capacitor could become charged by stray voltages in the computer.

In both cases, zero and non-zero initial conditions, the relay controls the integration. For \( t > 0 \), the relay is closed. Therefore, the capacitor charge is maintained constant, regardless of the input to the amplifier. At \( t = 0 \), the relay is opened. The charge on the capacitor is no longer maintained constant, and integration begins. When the desired integration is complete, the relay is closed, automatically resetting the initial condition on the integrator.

Thus, the integration of a function corresponds to the charging or discharging of a capacitor on the computer. If the value of the integral increases from \( t_1 \) to \( t_2 \), then the charge on the capacitor increases. If the value decreases, then the charge decreases.

2.5 The Differentiator. Let \( Z_1, \ldots, Z_n \) be capacitances, \( \frac{1}{C_1 s}, \ldots, \frac{1}{C_n s} \) and \( Z_f \) be a resistance \( R_f \), then

\[
E_0(s) = - R_f \sum_{i=1}^{n} C_i s E_i(s).
\]

Therefore, \( O \) is a differentiator. Differentiation is usually avoided in any computer program. Jackson
(9, pp. 145-58) calls a differentiator a noise amplifier, because it increases the noise present in the computer set-up. Any undesired signal may be classified as noise. A certain amount of noise is present in any physical system.

2.6 Special Devices. In each of the four cases considered, $Z_f$ was taken to be one element, a capacitor or a resistor. However, $Z_f$ could be any combination of resistances and capacitances. For example, let $Z_f$ be a parallel combination of a resistance, $R_f$, and a capacitance, $C_f$, as shown in Figure 4. In this case

$$Z_f(s) = \frac{R_f}{R_f C_f s + 1}$$

If $Z_1, \ldots, Z_n$ are resistances $R_1, \ldots, R_n$, then

$$E_0(s) = -R_f \sum_{i=1}^{n} \frac{E_i(s)}{R_i (R_f C_f s + 1)}$$

By using other feedback impedances, the operational amplifier, $\tilde{O}$ can be made to perform many different operations. Usually however, only the ordinary summation and integration networks are used.

2.7 General Notes. The values of resistance used are between 0.1 and 10.0 megohms. If values smaller
than these are used, the resistor will load the coefficient potentiometers. This effect will be discussed in the section on errors. The values of capacitances used are between 0.01 and 1.00 microfarads. If the value of capacitance (11, p. 21) used is too small, interactions in the computer circuit may cause the effective value of capacitance to be appreciably different from the nominal value. If the value is too large, then the physical size of the capacitor makes it inconvenient to use.

2.8 The Inductor. The third basic circuit element, the inductor, is not used as an analog computer element. For an ideal inductor,

\[(2.81) \quad E(s) = LsI(s)\]

where:

- \(e\) = voltage across the inductor, volts
- \(L\) = inductance, henrys
- \(i\) = current through the inductor, amperes
- \(s\) = Laplace transform variable.

The capital letters indicate the Laplace transforms of the variables.
Theoretically, an integrator could be obtained by using inductances for the input impedances and a resistance for the feedback impedance. The mathematical relationship obtained is

\[
E_0(s) = -\frac{R_f}{s} \sum_{i=1}^{n} \frac{E_i(s)}{L_i} + \frac{e_0(0)}{s}.
\]

However, it is impossible to have a pure inductance. Every physical inductance has an appreciable amount of resistance associated with it, so that Equation (2.82) does not hold. The actual relationship, obtained from Kirchoff's Laws, is

\[
E_0(s) = -R_f \sum_{i=1}^{n} \frac{E_i(s)}{L_i s + R_i} + \frac{e_0(0)}{s}
\]

where:

\[
R_i = \text{resistance in inductance } i, \text{ ohms.}
\]

Hence, it is physically impossible to obtain an integrator using only inductances and resistances. Even if it were possible to construct an integrator using only inductances and resistances, the physical size of the inductors required would be unreasonable: for example, if \( R = 10^5 \) ohms, then \( L = 10^5 \) henrys for
an integrator with unity gain. The feedback resistance, $R_f$, cannot be made much smaller than $10^5$ ohms because it would load the amplifier.

2.9 The Potentiometer. The last operation to consider is multiplication by a constant. It has already been shown that the adder not only adds variables, but also multiplies each one by a constant $R_f/R_1$. Multiplication by a constant is also a part of the integration process. However, the constants involved are round numbers such as 0.1, 0.5, and 10 because the values of resistance and capacitance used are round numbers. It would not be possible to use an adder to multiply a variable by constants such as 0.423 without requiring the availability of many different size resistors. For this reason, it is necessary to introduce the potentiometer.

A potentiometer is a resistor whose resistance can be varied mechanically. It consists of a fixed resistor with a movable arm, as shown in Figure 5. One end of the potentiometer is connected to ground or zero potential. The input voltage, $e_1$, is fed into the top of the potentiometer, and the output voltage, $e_0$, is measured
C = MOVABLE ARM
L = LENGTH OF $R_p$
$R_p$ = POTENTIOMETER RESISTANCE
G = GROUND
X = LENGTH OF $R_p$
BETWEEN C AND G

FIGURE 5. THE POTENTIOMETER
at the arm. For an ideal linear potentiometer the following relationship holds:

\[(2.91) \quad E_0(s) = \frac{x}{L} E_1(s)\]

where:

- \(L\) = length of the resistor
- \(x\) = length of the resistor connected between the arm and ground
- \(e_1, e_0\) = input and output voltages, volts.

If the output voltage is fed into an amplifier, "loading" of the potentiometer occurs. This effect will be discussed in the section on errors.

Clearly, the potentiometer is capable of multiplying by constants only between 0 and 1. If it is desired to multiply by some larger number, then the potentiometer must be used in conjunction with an amplifier.

2.10 Symbolic Representation. The adder, the integrator, and the potentiometer will be designated by blocks, as shown in Figure 6. Each block contains a symbol indicative of the operation performed by the block. Notice that no sign change occurs in the potentiometer.
THE POTENTIOMETER

\[ e_i(t) \quad \sum K_j \quad e_n(t) \]

\[ K_j = \frac{R_f}{R_j} \]

THE ADDER

\[ e_i(t) \quad e_n(t) \quad \sum \quad e_n(t) \]

\[ T_j = \frac{1}{R_j C_f} \]

THE INTEGRATOR

FIGURE 6. SYMBOLIC REPRESENTATIONS USED IN COMPUTER DIAGRAMS
2.11 The Independent Variable. The voltages on the computer are functions of one independent variable, computer time. Since the variables in a problem are represented on the computer by these voltages, they can be functions of at most one independent variable. This independent variable must be related to computer time in some manner, before the problem can be solved on the computer. A problem involving more than one independent variable can sometimes be solved by difference techniques. This type of situation will be discussed more thoroughly later.

2.12 Non-linear Equipment. With the three preceding operations, it is possible to solve any problem that can be described by systems of linear, ordinary, constant coefficient, differential equations. The solution of linear equations with time-varying coefficients, or of non-linear equations requires "non-linear" equipment, such as multipliers for multiplying variables; function generators for generating functions of time or of the dependent variables; comparators and relays for logical operations; and diodes for introducing discontinuities. With these additional devices and some ingenuity, many different kinds of mathematical operations can be performed.
3. Problem Solutions

Jackson (9, pp. 60-3) lists these three methods of obtaining the solution of a problem on the analog computer:

1. The direct method,
2. The indirect method,
3. The implicit method.

The method of solution is said to be direct if all of the functions that are to be operated upon are known functions of time that can be generated directly; for example, the evaluation of certain definite integrals.

The method of solution is said to be indirect if it is based upon the assumption that a certain function that is not known is known; for example, the solution of a differential equation. The method of solution is implicit if equations implied by the given equations are solved instead of the given equations; for example, the division of variable quantities. Each method is used in many different problems. In fact, it is possible that all three methods may be used in the same problem.

3.1 Procedure. Regardless of the method or methods used, there are three steps to solving a problem on the analog computer:
1. The problem is defined in terms of differential equations.

2. These equations are converted into an equivalent electrical circuit.

3. The variables in the problem are related to the computer variables by magnitude and time scaling.

Usually, the problem is given as a set of differential equations or as a set of differential and algebraic equations. A problem for which the given set of equations has to be transformed into an equivalent set of differential equations is presented in the section on characteristic values. In the second step, the operational blocks are joined together in such a way that the mathematical description of the block circuit coincides with that of the problem. The third step is necessary because the linearity range of an amplifier (the voltage range of the output for which the output is a linear function of the input), and the speed at which the computer components and the auxiliary computer devices can accurately follow a changing signal are limited.

3.2 The Linear Differential Equation. Consider the solution of the differential equation,

\[
(3.21) \quad \sum_{k=0}^{n} a_k D_x^k y(x) = f(x) \quad a_n \neq 0
\]
This equation cannot be solved on the computer, unless a complete set of initial conditions is given:

\[(3.22) \quad D_x^k y(0) = b_k \quad k = 0, 1, \ldots, n-1.\]

The computer solution of a differential equation is analogous to the Laplace transform solution of such an equation. The initial conditions are fed into the system at the beginning of the solution, and the function obtained is valid only for these particular initial conditions.

The indirect method is applicable. Step 1 is complete. Therefore, consider step 2, obtaining the computer circuit. This circuit is obtained by assuming that $D_x^n y(x)$ is known. If $D_x^n y$ is known, then $y(x)$ and the remaining $(n-1)$ derivatives can be obtained by integrating $n$ times, inserting the appropriate initial conditions on the integrators. Thus, the open loop circuit, shown in Figure 7a, with input $D_x^n y$ and output $(-1)^n y(x)$ is obtained. Equation (3.21) may be rewritten as

\[(3.23) \quad D_x^n y(x) = -\sum_{k=1}^{n-1} \left( \frac{a_k}{a_n} \right) D_x^k y(x) + \frac{f(x)}{a_n}.\]

Thus, for the given problem, $D_x^n y$ must satisfy Equation (3.23). The open loop system is closed by feeding back into the first amplifier $D_x^n y(x)$ expressed in terms of
A. THE OPEN LOOP CIRCUIT

B. THE CLOSED LOOP CIRCUIT

FIGURE 7. UNSCALED COMPUTER DIAGRAMS FOR
THE SOLUTION OF (3.21) WITH (3.22)
the lower order derivatives. The unscaled computer
diagram is shown in Figure 7b. Each coefficient
$a_k$ was assumed to be non-negative.

Why does this set-up actually generate $y(x)$?

Set $x=t$. At time $t=0$, the charge on the capacitor of
the $(n-k)$th integrator is $(-1)^{n-k} D_t^k y(0)$; hence, the
circuit represents the differential system. When the
relays are opened, the $k$th integrator begins integrating
$D_t^{k+1} y(0)$. Instantaneous changes occur everywhere in the
circuit, changes that are determined by the mathematical
equations that describe the circuit. But, the
mathematical equations of the circuit are the same as
the mathematical equations of the problem, so
$y(x)=y(t)$ is generated.

Before actually solving this problem on the computer,
the problem variables must be related to the computer
variables by magnitude and time scaling.

3.3 Magnitude Scaling. Magnitude scaling should
be done first. The linearity range of the amplifiers on
the 15-amplifier Heath Kit is ± 100 volts. In most cases,
the ranges of the dependent variables in the problem are
not ± 100 units. Therefore, the computer variables,
$e_1, ..., e_n$ and the problem variables, $x_1, ..., x_n$ must
be related linearly:
\[ x_j = K_j e_j \]

where \( K_j \) is a constant, \( 1 \leq j \leq n \).

The magnitudes of the computer variables should be kept above 10 volts. This is accomplished by choosing

\[ K_j = \frac{x_j^{\text{max}}}{e_j^{\text{max}}} \quad 1 \leq j \leq n \]

where:

\[ e_j^{\text{max}} = 100 \text{ volts, } 1 \leq j \leq n \]

\[ x_j^{\text{max}} = \text{maximum value of } x_j \text{ during the solution time.} \]

Thus,

\[ e_j = \frac{x_j}{K_j} \]

where \( e_j \) is the output of the \( j \)th amplifier in volts.

Usually, \( K_j \neq K_k \) for \( j \neq k \). Suppose that \( K_j \neq K_{j+1} \)

where

\[ e_k = \frac{x_k}{K_k} \quad k = j, j + 1 \]

\[ x_j = -D_j x_{j+1} \]

\[ x_{j+1}(0) = 0 \]

In this case, a potentiometer must be inserted between the two amplifiers; and the setting of this potentiometer and the gain of the \((j + 1)\)st amplifier adjusted so that
The ratio of these two constants is less than one, the potentiometer may be set to this value. However, if it is greater than one, then a \( \frac{K_j}{K_{j+1}} \) is set on the potentiometer, and the gain of the amplifier is changed to \( \frac{1}{a} \).

If desired, instead of making the scale adjustments directly on the computer, new variables \( y_j = \frac{x_j}{K_j} \), \( 1 \leq j \leq n \) may be defined, and substituted directly into the given set of equations. The new equations must then be programmed on the computer. It is usually easier to make the scale adjustments directly on the computer diagram. Any magnitude scaling must include the initial conditions, and any driving functions present in the circuit.

Consider the magnitude scaling for problem one, Equations (3.21) and (3.22). In this case, \( x_k = D_x^k y(x) \) for \( k = 0, 1, \ldots, n-1 \). Note that \( x_n = D_x^n y(x) \) does not have to be considered because it does not appear in Figure 8. However, if an adder and an integrator had been used in succession, instead of the summing integrator, then \( x_n \) would have appeared on the computer diagram, and it would have to be considered. Let \( D_{\max}^k \sum_{0 \leq k \leq n-1} D_x^k y \) during the desired solution time. Usually the
maximum values are not known and have to be estimated. The first estimates may not be correct, a fact that becomes apparent when the problem is implemented on the computer and voltages larger in magnitude than 100 volts or smaller in magnitude than 10 volts are obtained. If the first estimates are not correct, then new estimates must be made, and the problem rescaled. Thus, it is easy to see why it is best to make the scale adjustments on the diagram, rather than to change the variables for each estimate.

3.4 Time Scaling. Unless the computer program is time-scaled, the computer solution time is the same as the time required for the physical system to react. If the computer were programmed to generate the transients in an electrical circuit, the solution time would probably be only a fraction of a second. Whereas, if the computer were programmed to generate the transients in a chemical reactor, the solution time could be several hours or days. In either case, the length of solution time is undesirable and must be changed.

Therefore, not only must there be relationships,

\[(3.41) \quad e_j = x_j/K_j \quad 1 \leq j \leq n\]

but there must also be a relationship
\[(3.42) \quad t = bx\]

where:
- \(t\) = independent variable on the computer, sec.
- \(x\) = independent variable in the problem
- \(b\) = constant, sec/unit of \(x\).

For problem one, first assume that \(b=1\). The program with \(b=1\) is then implemented on the computer. If the time of solution is too long or too short, the value of \(b\) must be changed. For any \(b\), the computer solution time is \(b\) times as long as the actual solution time. The choice of a time scale factor depends upon several things, such as the response of the recording equipment and drift in the amplifiers. The problem of drift will be considered in the section on errors. The solution time on the 15-amplifier Heath Kit computer should be between 10 and 30 seconds.

There are two distinct methods of time scaling:

1. The transformation \(t=bx\) is substituted directly into the system equations. The transformed equations are programmed and solved.
2. The system equations are programmed on the computer, and then the time scale of the computer itself is changed.

Problem one may be time-scaled by either one of these methods. First, let $t = bx$, then

\begin{equation}
\begin{aligned}
b^n D^n_t y &= \sum_{k=0}^{n-1} \left( \frac{a_k}{a_n} \right) (b^k D^k_t y) + \frac{f(t/b)}{a_n}, \\
D^k_t y(0) &= \frac{D^k_x y(0)}{b^k} \quad k = 0, 1, \ldots, n-1.
\end{aligned}
\end{equation}

The maximum values and the initial conditions of the new variables are not the same as those of the original variables. Consequently, if the variables are transformed directly, the magnitude scaling must be redone, and the initial conditions must be changed.

Now consider the second method. Problem one is programmed directly on the computer. The time scale of the computer is then changed by changing the effective gain of each of the integrators by an appropriate constant factor. The maximum values and the initial conditions are not changed.

To prove that time scaling can be completed in the manner indicated, consider an integrator with one input, so that
\begin{equation}
 g(t) = -\frac{1}{R_1C_f} \int_0^t f(u) du + g(0)
\end{equation}

where:
\begin{align*}
 f(t) &= \text{input, volts} \\
 g(t) &= \text{output, volts}
\end{align*}

Let \( \zeta = bt \), then from equation (3.44),
\begin{equation}
 g\left( \frac{\zeta}{b} \right) = -\frac{1/b}{R_1C_f} \int_0^{\zeta/b} f\left( \frac{w}{b} \right) dw + g(0)
\end{equation}

where:
\begin{align*}
 \zeta &= \text{new computer variable, sec.} \\
 t &= \text{original computer variable, sec.} \\
 b &= \text{constant, dimensionless.}
\end{align*}

Of the three basic blocks, only the integrator is time-dependent. Therefore, the substitution \( \zeta = bt \) affects only the integrators (Equation 3.45). Consequently, changing the over-all rate of solution, the time scale of the computer itself, is equivalent to changing the rate of integration. The dependent computer variables are not changed. Hence, the maximum values and the initial conditions are not changed. Only the rate of solution is changed, by the factor \( 1/b \). This factor must be introduced before each of the integrators. This can be accomplished by changing potentiometer settings and the
values of the input resistances and the feedback capacitance of the integrator. Figure 8 illustrates this phenomenon for an integrator with a unit step function input and zero initial conditions.

Obviously, the second method of time scaling is easier than the first. It is easier to change the effective gain on each of the integrators in the system, than it is to change the entire system of equations being programmed. The factor \( \frac{1}{b} \) should be indicated on the computer diagram, and the axes of the graphical solution should be labeled in term of \( \tau \) and then converted to \( t \).

Output Devices. Since the problem variables are represented on the computer by voltages, the value of any variable at any time during the solution of the problem may be read on the voltmeter on the computer. Usually, complete records of the variables as functions of time are desired. Three devices used for this purpose are strip-chart recorders, x-y plotters, and oscilloscopes.

A strip-chart recorder has several channels so that two or more dependent variables may be plotted versus time. The chart paper moves by a fixed pen at a fixed speed. The pen deflects in a direction perpendicular to the direction in which the paper moves. The amount of deflection is proportional to the value of the dependent
FIGURE 8. THE EFFECT OF TIME SCALING UPON THE OUTPUT OF AN INTEGRATOR WITH A UNIT STEP INPUT
variable. The amplitude scale and the paper speed are adjustable.

An x-y plotter is used to plot one function $x(t)$ versus another $y(t)$. The paper is stationary. The recording pen moves over the paper on a movable arm. The movement of the arm in the x-direction is proportional to the value of $x(t)$; similarly, the vertical movement is proportional to $y(t)$. In general, the plotter (4, pp. 293-328) is more accurate than the strip-chart recorder. However, its dynamic response is limited. It can be used only with slowly varying functions.

The oscilloscope consists of a cathode ray tube and a fluorescent screen. It acts as an x-y plotter. The two functions to be plotted versus each other, $y(t)$ and $x(t)$, are fed into the oscilloscope. The electron beam generated by the cathode ray tube is deflected in the vertical direction by an amount proportional to the value of $y(t)$ and in the horizontal direction by an amount proportional to $x(t)$. The deflected beam is projected on the screen. Consequently, as $x(t)$ and $y(t)$ vary with $t$, the trace of the beam on the screen represents the desired curve. If permanent recordings are desired, a camera must be used in conjunction with the scope. The oscilloscope (4, pp. 327-8) can follow accurately inputs
with a maximum frequency range of 0 to 100 cps, whereas, the x-y plotter is good for only a few cycles per second. Hence, the oscilloscope should be used for problems involving functions that vary rapidly with time.

4. Error Analysis

No simple method has been devised for determining to what degree of accuracy the computer solution obtained satisfies the original problem. However, it is important to realize the possible sources of error present in an analog computer; and that the relative error differs with each type of problem, and with the particular analog computer used.

4.1 Operational Amplifiers. The components of an analog computer are not ideal, as they were assumed to be in the preceding section. The adder does not add ideally, and the integrator does not perform an ideal integration.

Jackson (9, pp. 445-67) discusses the errors inherent in operational amplifiers in detail. Such a discussion requires a thorough understanding of circuit analysis. Several sources of error listed are:

1. The output voltage of an amplifier is not identically zero for a zero input voltage;
2. The open loop gain of an amplifier is finite;
3. The voltage at the summing junction of an amplifier is not identically zero;
4. The feedback capacitor is not a pure capacitance;
5. The input and output impedances of the amplifier are not infinite and not zero, respectively.

Only the first two sources of error will be considered in this paper. The other three are usually negligible.

4.2 Drift. Each amplifier has a balancing device so that its output can be set equal to zero for a zero input. During the solution of a problem, changes within the amplifier cause the zero point of the amplifier to change, introducing a component of error into the output voltage. This phenomenon is called drift.

Amplifiers have been designed that automatically correct for drift. Such amplifiers are used in the more sophisticated computers. However, in the Heath Kit computers the amplifiers are not stabilized. The solution time of any problem on this type of computer must be kept to a minimum, since the amount of drift increases with time and is integrated with respect to time by the integrators in the problem. An excellent example of the effect of drift upon the solution of a particular problem is given in Figure 20.
4.3 Finite Gain. The most important requirement (9, p. 515) that an operational amplifier must satisfy is that it have a very large open loop gain. For accurate integration, the open loop gain of an integrator must be at least $10^5$. The gain of each amplifier on the 15-amplifier Heath Kit computer is $5 \times 10^4$. The gain of each amplifier on the 9-amplifier Heath Kit computer is only $10^3$.

A reasonable approximation (5, pp. 528-40) to the effect of finite gain upon the output of an amplifier can be obtained by assuming that there are no other sources of error. Therefore, consider an operational amplifier with $n$ inputs. Assuming that the open loop gain is finite, the following relationship, obtained from Kirchoff's Laws, holds:

$$ E_o(s) \left[ \frac{(\mu+1)}{\mu} \frac{1}{Z_f} + \frac{1}{\mu} \sum_{j=1}^{n} \frac{1}{Z_j} \right] = - \sum_{j=1}^{n} \frac{E_j(s)}{Z_j} $$

where:

- $e_o, e_j =$ output and input, respectively, volts
- $Z_j, Z_f =$ input and feedback impedances, respectively
- $\mu =$ open loop gain of the amplifier.

The capital letters indicate the Laplace transforms of the corresponding variables.
4.4 The Adder. If \( n = 1, Z_f = R_f, Z_1 = R_1 \), then

\[
\frac{E_{OA}(s)}{E_1(s)} = - \frac{R_f}{R_1} \left( \frac{1}{R_1 + R_f} \right) = - K
\]

where \( K \) is a constant independent of time. Therefore,

\[
(4.42) \quad e_{OA}(t) = - Ke_1(t).
\]

For an ideal adder,

\[
(4.43) \quad e_o(t) = - \frac{R_f}{R_1} e_1(t).
\]

Therefore, subtracting Equation (4.43) from Equation (4.42),

\[
(4.44) \quad \bar{e}(t) = \frac{R_f^2 + R_1 R_f}{R_1 R_f + (1 + \mu) R_1^2} e_1(t)
\]

where \( \bar{e}(t) \) is the error in volts.

For a particular value of \( \mu \), the maximum error in \( e_o(t) \) depends only upon the values of \( R_f \) and \( R_1 \). Several values of maximum error for \( \mu = 1000 \) for different values of \( R_f \) and \( R_1 \) are listed in Table I.

4.5 The Integrator. If \( n = 1, Z_1 = R_1, Z_f = \frac{1}{C_f s} \) and \( e_o(0) = 0 \), then

\[
(4.51) \quad \frac{E_{OA}(s)}{E_1(s)} = - \frac{1}{R_1 C_f s} \left( \frac{1}{1 + \frac{1}{\mu} \left( 1 + \frac{1}{R_1 C_f s} \right)} \right).
\]
TABLE I

The Maximum Error in the Output of an Adder
with One Input and an Open
Loop Gain - of 1000

<table>
<thead>
<tr>
<th>Input Resistance (megohms)</th>
<th>Feedback Resistance (megohms)</th>
<th>Maximum Error (volts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>1.1</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>0.3</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>negligible</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>0.2</td>
</tr>
<tr>
<td>1.0</td>
<td>10.0</td>
<td>1.1</td>
</tr>
</tbody>
</table>
Let $T = R_1C_f$, then Equation (4.51) simplifies to

$$\frac{E_{OA}(s)}{E_1(s)} = -\frac{\mu}{T(\mu+1)} \left[ \frac{1}{s+\frac{1}{T(\mu+1)}} \right]$$

In this case, the error is not only a function of the constants $R_1$ and $C_f$; but also, a function of the input to the amplifier. Equation (4.52) is the Laplace transform with zero initial conditions, of the differential equation

$$D_t e_{OA}(t) + \frac{1}{a} e_{OA}(t) = -\frac{\mu}{a} e_1(t),$$

where $a = T(1+\mu)$. The solution of Equation (4.53), with zero initial conditions is

$$e_{OA}(t) = -\frac{\mu}{a} e^{-t/A} \int_0^t e^{(t-A)} e_1(\tau) d\tau.$$

Assuming that $\mu \approx 1000$,

$$e^{-t/a} \approx (1 - t/a)$$

$$e^{t/a} \approx (1 + t/a)$$

Therefore,

$$e_{OA}(t) \approx -\frac{\mu}{a} \left[ 1 - \frac{t}{a} \right] \int_0^t \left[ 1 + \frac{\tau}{a} \right] e_1(\tau) d\tau.$$
For an ideal integrator,

\[(4.57) \quad e_0(t) = -\frac{1}{T} \int_0^t e_1(\tau) d\tau .\]

Consequently, subtracting Equation (4.57) from (4.56),

\[(4.58) \quad \bar{e}(t) = \frac{1}{T} \int_0^t e_1(\tau) d\tau - \frac{\mu}{T^2(\mu+1)^2} \int_0^t e_1(\tau) d\tau .\]

If \(e_1(t) = M\), then

\[(4.59) \quad \bar{e}(t) = \frac{Mt}{T} \left( \frac{1}{\mu+1} - \frac{\mu Mt^2}{2T^2(\mu+1)^2} \right) .\]

If \(e_1(t) = t\), then

\[(4.5.10) \quad \bar{e}(t) = \frac{t^2}{2T} \left( \frac{1}{\mu+1} - \frac{\mu t^3}{3T^2(\mu+1)^2} \right) .\]

To illustrate the effect of finite gain, the following differential system was solved on both of the Heath Kit analog computers; and the results compared with the analytical solution:

\[(4.5.11) \quad \dot{D}^2 x(t) - 2y(t) = 0\]

\[\dot{D}x(t) + (\dot{D} + 2)y(t) = 0\]

\[y(0) = x(0) = 0 \quad \dot{D}x(0) = -1 .\]
The analytical solution of this system is

\[
\begin{align*}
    x(t) &= e^{-t} \cos t - 1 \\
    y(t) &= e^{-t} \sin t
\end{align*}
\]

The computer diagram is shown in Figure 9. The resulting computer solutions are shown in Figures 10 and 11. The graph of the analytical solution is indicated by the dotted line.

4.6 Potentiometer Loading. Another possible source of error is the potentiometer. The output of a potentiometer is usually fed into an amplifier or another potentiometer. In either case,

\[
(4.61) \quad e_o \neq a_p e_i
\]

where:

- \( e_o, e_i \) = output and input, volts
- \( a_p \) = dial setting of potentiometer.

The effective setting, \( a \), of the potentiometer, is not equal to the dial setting, and the potentiometer is said to be loaded.

Consider Figure 12. The output of the potentiometer, \( e_o \), is fed into an amplifier through a resistance, \( R_i \). One end of the potentiometer, \( R_p \), is at zero potential.
FIGURE 9. A SCALED COMPUTER DIAGRAM FOR SYSTEM (4.5.11)

P = PLOTTER
TIME SCALE FACTOR, \( b = 2 \)

- \( -50D_p \times \)

- \( -50D_p \times \)

- \( -50D_p \times \)

- \( -50D_p \times \)

- \( -50D_p \times \)

- \( -50D_p \times \)

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- \( -50D_p \times \)

- \( -50D_p \times \)
FIGURE 10. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (4, 5, 11) ON THE 15-AMPLIFIER HEATH KIT COMPUTER

\[ O = \text{CALCULATED VALUES} \]
FIGURE 11. THE COMPUTER SOLUTION
OBTAINED FOR SYSTEM (4.5.11)
ON THE 9-AMPLIFIER HEATH
KIT COMPUTER
The summing junction of the amplifier is also at zero potential. Consequently, these two points may be joined together by a short circuit, a wire that has no resistance. Thus, Figures 12a and 12b are equivalent. The following relationship was obtained by applying Kirchoff's Laws to the circuit shown in Figure 12b:

\[(4.62) \quad a = \frac{a_p R_1}{R_1 + a_p R_p (1-a_p)}\]

where:

- \(a, a_p\) = effective and dial settings, respectively
- \(R_p, R_1\) = potentiometer and input resistances, respectively, ohms.

This relationship could be used to obtain for a particular potentiometer a series of curves of \(a\) versus \(a_p\) for different values of \(R_1\). The dial setting corresponding to the desired setting could then be read off of this graph. This method has disadvantages. One is the fact that resistors, each having a nominal resistance of \(k\) ohms, vary between themselves. The easiest way to correct for potentiometer loading is to implement the entire program on the computer and then to set the potentiometers after all of the connections have been made. The output resistances will then be connected
FIGURE 12. POTENTIOMETER LOADING
to the potentiometer arms, while the potentiometers are being set. Thus, the effects of loading will be compensated for automatically.

4.7 Amplifier Loading. Usually the effect of amplifier loading is negligible. However, if the amplifiers are inexpensive, as they are on the small Heath Kit computer, then amplifier loading must be considered. Figure 13a shows the output of an operational amplifier connected to the inputs of \( n \) other amplifiers. Notice that the output of this amplifier is also connected to the summing junction through the feedback resistance, \( R_f \).

As in the preceding section, Figures 13a and 13b are equivalent. The maximum voltage obtainable on the small Heath Kit computer is 60 volts. If the output of the amplifier is \( e_o \), then for the current drawn from the amplifier, \( i_o \),

\[
(4.71) \quad i_o = e_o \left[ \frac{1}{R_f} + \sum_{i=1}^{n} \frac{1}{R_i} \right].
\]

Therefore, if \( e_o = 50 \) volts, \( n = 2 \), and \( R_f = R_1 = R_2 = 0.1 \) megohms, then \( i_o = 1.5 \) milliamperes. However, the maximum current obtainable is 0.7 milliamperes. Consequently, this type of connection cannot be used on the small Heath Kit. The allowable output current of an amplifier on the large
FIGURE 13. AMPLIFIER LOADING
Heath Kit is 10 milliamperes. Hence, it is usually not necessary to worry about amplifier loading.

4.8 Checking Results. The error analysis is meaningless unless the validity of the computer solution has been established. There is no direct, general method of determining if a computer solution is correct. However, some form of checking is necessary. Fifer (4, pp. 499-557) discusses the checking of results in detail.
III. THE SOLUTION OF SEVERAL MATHEMATICAL PROBLEMS

1. Ordinary Differential Equations

The analog computer is ideally suited for the solution of systems of ordinary differential equations of the form

\[ \sum_{j=1}^{n} f_{ij}(D)y_j(x) = h_i(x) \quad 1 \leq i \leq n \]

\[ D^k y_j(0) = b_{kj} \quad 0 \leq k \leq n - 1, \ 1 \leq j \leq n \]

where:

\[ f_{ij}(D) = \sum_{k=1}^{m_{ij}} a_{kij} D^k \]

\[ D = \frac{d}{dx} \]

\[ b_{kj}, a_{kij} \text{ real numbers} \]

\[ h_i(x) = \text{analytic function of } x, \quad 1 \leq i \leq n. \]

1.1 Analytical Solutions. Systems of this type (10, pp. 45-6) can be solved analytically, using determinants. Each dependent variable, \( y_j \), is obtained from a differential equation of the form...
If the degree of the differential operator \(|(f_{ij}(D))|\), is greater than or equal to three, the solution of the auxiliary polynomial equation is cumbersome. However, on the analog computer, the method of solution is the same for any degree, only the number of components required changes.

### 1.2 Computer Solution

First, the computer diagram must be constructed. The equations given are rearranged as follows:

\[
(1.21) \quad \sum_{j=1}^{n} a_{mj} y_j^m - \sum_{j=1}^{n} g_{ij}(D)y_j + h_i(x), \quad 1 \leq i \leq n.
\]
where:

\[ m_j = \text{the order of the highest ordered derivative of } y_j \text{ appearing in the } n \text{ equations} \]

\[ a_{mjij} = \text{the coefficient of } D^{m_j}y_j \text{ in the } i \text{th equation} \]

\[ g_{ij}(D) = f_{ij}(D) - a_{mjij}D^{m_j}y_j. \]

In matrix notation,

\[(1.22) \quad A\mathbf{z} = \mathbf{Gy} + \mathbf{h}\]

where:

\[ A, \mathbf{G} = h \times n \text{ matrices, } (a_{mjij}) \text{ and } (g_{ij}(D)), \text{ respectively} \]

\[ h, \mathbf{y}, \mathbf{z} = n \times 1 \text{ matrices, } (h_i), (y_j), \text{ and } (D^{m_j}y_j), \text{ respectively}. \]

Usually, the matrix \( A \) will have only one non-zero element in each row and column. If this is true, then the computer diagram is constructed in the same manner as that of a single ordinary differential equation. The existence of \( D^{m_j}y_j \) is assumed for all \( j \). The remaining \( m_j - 1 \) derivatives of \( y_j \), and \( y_j \) can then be generated by integrating \( D^{m_j}y_j \), \( m_j \) times, setting the appropriate initial conditions on each of the integrators. The assumed values of \( D^{m_j}y_j \) are then expressed in terms of the other variables and derivatives, according to Equations (1.21).
If the matrix $A$ contains more or less than one non-zero element in each row or column, the preceding procedure cannot be applied directly. The equations corresponding to the rows of zeros must be differentiated until one or more of the $D^{m_j}y_j$ appear in each of the differentiated equations. The resulting equations are then solved simultaneously to obtain expressions for each of the $D^{m_j}y_j$, involving only the lower order derivatives. These expressions are programmed and solved on the computer.

For example, consider the following system

\[(1.23) \quad D_t^2 x + D_t^3 y = 3x + (D_t^2 + D_t)y\]
\[(D_t + 1)x + D_t y = 0\]

First the equations are rearranged as follows,

\[(1.24) \quad D_t^2 x + D_t^3 y = 3x + (D_t^2 + D_t)y\]
\[0 = -(D_t + 1)x - D_t y.\]

Therefore, the matrix

\[(1.25) \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\]
Consequently, the second equation in (1.24) must be differentiated once to obtain

\[(1.26) \quad D_t^2 x = -D_t x - D_t^2 y\]

Equations (1.26) and (1.24) are then solved simultaneously to obtain the new system

\[(1.27) \quad D_t^2 x = -D_t x - D_t^2 y\]

\[D_t^3 y = (D_t + 3)x + (2D_t^2 + D_t)y\]

which is then programmed on the computer. An unscaled computer diagram is shown in Figure 14. After the computer diagram is constructed, the problem is scaled and implemented on the computer.

In the preceding discussion, it was assumed that the value of each variable was known at \(x = 0\). These initial values are set directly on the computer as initial conditions on the integrators. If all of the boundary conditions are not initial conditions, then the introduction of these conditions is not direct. Values must be assumed for the initial conditions not given. The boundary conditions obtained for these initial conditions are examined, and if they do not agree with those given, a second assumption must be made. This
FIGURE 14. AN UNSCALED COMPUTER DIAGRAM FOR SYSTEM (1.27)
trial and error procedure is continued until the correct boundary conditions are obtained. In either case, the solution obtained is a particular solution valid only for the given boundary conditions.

1.3 Systems of Two Equations. Now consider the system

\begin{align*}
(1.31) & \quad f_{11}(D)y_1 + f_{12}(D)y_2 = 0 \\
& \quad f_{21}(D)y_1 + f_{22}(D)y_2 = 0
\end{align*}

Theoretically, this set of equations together with a set of boundary conditions can be solved analytically for any polynomial functions \(f_{11}, f_{12}, f_{21}\), and \(f_{22}\), by solving the following differential equations

\begin{align*}
(1.32) & \quad g(D)y_i = [f_{11}(D)f_{22}(D) - f_{12}(D)f_{21}(D)]y_i = 0 \\
& \quad i = 1, 2.
\end{align*}

However, if \(g(D)\) is of degree three or greater, the solution is tedious. Clearly, from equation (1.32), the solutions \(y_1\) and \(y_2\) are linear combinations of the same functions, differing only in the coefficients of these functions. Furthermore, since the roots of \(g(D)\) determine the type of solutions obtained, for example, decaying exponential, some relationship exists between the type
of solution obtained and the coefficients of the functions $f_{11}$, $f_{12}$, $f_{21}$, and $f_{22}$. Even for the case that $g(D)$ is a quadratic function, this relationship is very complicated to express analytically.

To illustrate the simplicity of solution of such a system on the analog computer and the ease of varying parameters in such systems, several families of geometric curves were generated on the computer. The parametric equations for each curve were differentiated with respect to the parameter. The derivatives obtained were combined to form the lowest order system of homogeneous differential equations whose solution is the given curve. The parameter was related to computer time, and the resulting system of equations together with the appropriate boundary conditions was solved on the computer.

1.4 The Ellipse. The first family considered was the ellipse family with centers at the origin and axes along the $x$ and $y$ axes. The parametric equations are

\[(1.41) \quad x = \cos \theta \]
\[
y = b \sin \theta .
\]

If $a = b$, then these are the parametric equations of a circle with radius $a$. 
The following system was obtained from equations (1.41) with $\theta = t$.

\begin{align*}
(1.41) & \quad b \frac{\partial}{\partial t} x + ay = 0 \\
(1.42) & \quad bx - a \frac{\partial}{\partial t} y = 0 \\
& \quad x(0) = a ; y(0) = 0 .
\end{align*}

The computer diagram for this system is shown in Figure 15, and the curves generated in Figure 16.

1.5 The Hypocycloid. The second family considered was the set of hypocycloids with centers at the origin and x-axis through a cusp. A hypocycloid is the locus of a fixed point on the circumference of a circle rolling on the inside of a fixed circle. The parametric equations (12, pp. 81-4) are

\begin{align*}
(1.51) & \quad x = (a - b) \cos t + b \cdot \cos \left(\frac{a - b}{b} t\right) \\
y & = (a - b) \sin t - b \cdot \sin \left(\frac{a - b}{b} t\right) \quad a > b
\end{align*}

where:

- $a =$ radius of the fixed circle
- $b =$ radius of the rolling circle
- $t =$ angle between the positive x-axis and the line drawn between the centers of the two circles.
FIGURE 15. A COMPUTER DIAGRAM FOR SYSTEM (1.42)

NOTE, THIS DIAGRAM IS DIRECTLY APPLICABLE ONLY FOR A=B=1.
Figure 16. The computer solution obtained for system (1.42).
When $a = 3b$, the deltoid or 3-cusped hypocycloid is generated.

The following system was obtained from Equations (1.51).

$$\begin{align*}
(1.52) \quad & D_t^2 + \left( \frac{a-b}{b} \right) x + \left( \frac{2b-a}{b} \right) D_t y = 0 \\
& \left( \frac{a-2b}{b} \right) D_t x + D_t^2 + \left( \frac{a-b}{b} \right) y = 0 \\
& D_t x(0) = D_t y(0) = y(0) = 0 ; \quad x(0) = a
\end{align*}$$

Theoretically, it is possible to generate a hypocycloid with any number of cusps. Equations (1.51) can be rewritten as

$$\begin{align*}
(1.53) \quad & x = (a-b)x_1 + bx_2 \\
& y = (a-b)x_3 - bx_4
\end{align*}$$

where $x_1$ and $x_3$ are periodic functions of period $2\pi$, and $x_2$ and $x_4$ are periodic functions of period $2\pi b/(a-b)$. Therefore, $x$ and $y$ are periodic functions of period $2\pi m$ for some integer $m$.

Suppose that $a = nb$, $n = 1, 2, \ldots, k$; then the period of $x_2$ and $x_4$ is $2\pi/(n-1)$. Therefore, as $t$ increases from 0 to $2\pi$, $x_2$ and $x_4$ complete $(n-1)$ cycles. Thus, for any integer, $x$ and $y$ are periodic functions of period $2\pi$, and the hypocycloid described by $x$ and $y$
closes after one revolution. This hypocycloid has \( n \) cusps since one cusp is obtained for each revolution of the rolling circle.

If \( a = \frac{bp}{q} \), \( p \) and \( q \) integers; then the period of \( x_2 \) and \( x_4 \) is \( 2\pi q/(p - q) \). Therefore, \( x_2 \) and \( x_4 \) complete \( (p - q)/q \) cycles as \( t \) increases from 0 to \( 2\pi \). If \( (p - q)/q = n \), then \( p/q = n + 1 \), contrary to the assumption that \( p/q \) is not an integer. But, during one period of \( x \) and \( y \); \( x_1 \), \( x_2 \), \( x_3 \), and \( x_4 \) must each complete an integral number of cycles. Therefore, the hypocycloid closes after \( q \) revolutions. Since the rolling circle completes \( p \) revolutions as \( t \) increases from 0 to \( 2q\pi \), this hypocycloid has \( p \) cusps.

Finally, let \( a = cb \), \( c \) an irrational number. In this case, the period of \( x_2 \) and \( x_4 \) is \( 2\pi/(a/b - 1) = 2\pi/d \), where \( d \) is an irrational number. Hence, \( x_2 \) and \( x_4 \) complete \( d \) cycles as \( t \) increases from 0 to \( 2\pi \). But, there do not exist integers \( m \) and \( n \) such that \( nd = m \). Therefore, the hypocycloid generated never closes and has an infinite number of cusps.

Drift in the amplifiers on the 15-amplifier Heath Kit computer made it impossible to demonstrate the effect of an irrational ratio, \( a/b \). Computer solutions were
determined only for the case \( a = nb, n \) an integer. For \( a = nb \), equations (1.52) reduce to

\[
\begin{align*}
\left[D_t^2 + (n-1)\right] x + (2-n)D_t y &= 0 \\
(n-2)D_t x + \left[D_t^2 + (n-1)\right] y &= 0 \\
D_t x(0) = D_t y(0) = y(0) = 0 & \quad x(0) = nb.
\end{align*}
\]

The scaled computer diagram for this set of equations is shown in Figure 17. The curves generated are shown in Figures 18, 19, and 20. Notice the effect of drift in the amplifiers in Figure 20. These equations were time-scaled in an attempt to minimize the effect of drift. However, as explained in section II, the time scale cannot be increased too much, otherwise, the x-y plotter will not be able to follow the output signals accurately.

1.6 The Epicycloid. The third family considered was the set of epicycloids with centers at the origin and x-axis through a cusp. An epicycloid is the locus of a fixed point on the circumference of a circle rolling on the outside of a fixed circle. The parametric equations (12, pp. 81-4) are:

\[
\begin{align*}
\text{(1.61)} \quad x &= (a+b) \cos t - b \cdot \cos \frac{(a+b)}{b} t \\
y &= (a+b) \sin t - b \cdot \sin \frac{(a+b)}{b} t
\end{align*}
\]
FIGURE 17. A COMPUTER DIAGRAM FOR SYSTEM (1.54)
FIGURE 18. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (1.54) FOR N=3,5
Figure 19. The computer solution obtained for system (1.54) for $N = 4, 6$. 
FIGURE 20. THE COMPUTER SOLUTION
OBTAINED FOR SYSTEM (1.54)
FOR N = 10, 20
where:

- \(a\) = radius of the fixed circle
- \(b\) = radius of the rolling circle
- \(t\) = angle between the positive \(x\)-axis and the line drawn between the centers of the two circles, radians.

When \(a = b\), the epicycloid generated is the familiar cardioid.

The following system of equations was obtained from Equations (1.61)

\[
\begin{align*}
1.62) \quad & \left[\frac{D_t^2}{b} - \frac{(a+b)}{b}\right] x + \frac{(2b+a)}{b} D_t y = 0 \\
& - \frac{(2b+a)}{b} D_t x + \left[\frac{D_t^2}{b} - \frac{(a+b)}{b}\right] y = 0 \\
& D_t x(0) = D_t y(0) = y(0) = 0 \quad x(0) = a.
\end{align*}
\]

Equations (1.61) and (1.51) are similar. Therefore, by the same form of reasoning used for the hypocycloid, an epicycloid with any number of cusps can be generated. However, because of drift in the amplifiers only the case \(a = nb\) was considered. For this case, Equations (1.62) reduce to,
The scaled computer diagram for this set of equations is displayed in Figure 21. The curves generated are shown in Figures 22 and 23.

### 1.7 The Archimedean Spirals

The last family considered was the set of Archimedean spirals. The equation (12, p. 209) of this type of spiral in polar coordinates is

\[
(1.71) \quad \rho = a\theta
\]

where:
- \(a\) = real number
- \(\rho\) = radius vector of the point \(P(x,y)\)
- \(\theta = \arctan y/x\).

Therefore, using the relationships between polar and rectangular coordinates,

\[
(1.72) \quad x = a\theta\cos\theta \\
y = a\theta\sin\theta .
\]
FIGURE 21. A COMPUTER DIAGRAM FOR SYSTEM (1,63)
FIGURE 22. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (1.63) FOR N = 3,5
FIGURE 23. THE COMPUTER SOLUTION
OBTAINED FOR SYSTEM (1.63)
FOR N = 2, 4, 6
Set \( \theta = t \), and differentiate Equations (1.72) with respect to \( t \). Combining the derivatives, the following system is obtained:

\[
(1.73) \quad (D_t^2 - 1)x + 2D_ty = 0
\]

\[
- 2D_x + (D_t^2 - 1)y = 0
\]

\[
x(0) = y(0) = D_t y(0) = 0 \quad D_t x(0) = a .
\]

The computer diagram and the curve corresponding to \( a = 1 \) are shown in Figures 24 and 25, respectively.

1.8 Summary. The direct method of solution can be used to generate each of the above sets of curves. The sinusoidal functions in the parametric equations are solutions of particular differential equations, and hence can be generated directly. The functions \( x \) and \( y \) can then be obtained by combining these sinusoidal functions in the indicated manner. For example, consider family 4, the Archimedean spirals. In this case, the functions \( x \) and \( y \) can be generated directly. They are solutions of the following differential equation:

\[
(1.81) \quad (D^4 + 2D^2 + 1)x = 0
\]

with the appropriate boundary conditions.
FIGURE 25. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (1.73)
2. **Polynomial Equations**

The analytical solution of ordinary differential equations with constant coefficients, of characteristic value problems, and of many other mathematical problems requires the solution of a polynomial equation.

\[(2.1) \quad f(z) = a_0 z^n + \ldots + a_n = 0\]

where:

- \(a_i\) = complex number, \(0 \leq i \leq n\)
- \(n\) = positive integer
- \(z\) = complex variable.

**2.1 Analytical Solution.** The roots of polynomial equations of degree less than or equal to four may be obtained from algebraic expressions involving only the coefficients of the equation \(f(z) = 0\). However, even the solution of an equation of degree three or four is cumbersome. In Galois theory (1, pp. 69-82) in modern algebra, it is proved that polynomial equations of degree greater than four cannot be solved by using only algebraic functions of the coefficients. Transcendental functions (2, p. 236) of several variables are required for the solution of such equations. Consequently, the direct solution of these equations is too complicated.
for practical application. Instead, numerical procedures are employed.

2.2 Computer Solution. Approximate solutions of polynomial equations of any degree can be found using the analog computer. Several methods of solution have been developed. In each case, the variable $z$ in each equation is a function of two variables, $x$ and $y$ in rectangular coordinates, and $\rho$ and $\theta$ in polar coordinates. Therefore, the function $f(z)$ is actually a function of two independent variables. But, the analog computer is capable of operating with only one independent variable. Consequently, each one of these methods of solution must relate both of the variables $x$ and $y$, or $\rho$ and $\theta$ to the computer variable, time.

The following method (4, pp. 879-913) is based upon one of the fundamental theorems of complex variables, the Principle of the Argument (8, pp. 252-3): If $C$ is a simple, closed, rectifiable, oriented curve in the $z$-plane, and $f(z)$ is a function that has no singularities or zeros on $C$, and no singularities other than poles inside of $C$, then
(2.21) \[ \Delta \text{arg} \ f(z) = 2\pi \left( \sum_{p=1}^{N} n_p - \sum_{q=1}^{M} m_q \right) \]

where:

- \( n_p \): multiplicity of the zero \( z_p \) inside of \( C \)
- \( m_q \): multiplicity of the pole \( z_q \) inside of \( C \)

\( \Delta \text{arg} \ f(z) \): change in the argument of \( f(z) \) as \( z \) traces out \( C \).

In this particular section, \( w = f(z) \) is a polynomial in \( z \). Therefore, it has no poles in the finite \( z \)-plane. Furthermore, if \( C \) is a circle with center at the origin and radius \( r \), and if \( f(z) \) has no zeros on \( C \), then the preceding theorem says that the number of times that the mapping \( w = f(z) \) encircles the origin in the \( w \)-plane is equal to the number of roots, with proper multiplicities, of \( f(z) = 0 \) contained in \( C \). Thus, the solution of any given polynomial equation \( f(z) = 0 \) reduces to the generation of \( f(z) \) as a function of \( z \).

Consider any polynomial equation

(2.22) \[ f(z) = \sum_{k=0}^{m} a_{m-k}z^k = 0 \quad a_0 \neq 0 \]

For simplicity, assume that \( a_{m-k} \) is real for \( k = 0,1, \ldots, m \). Let \( z = r e^{i\theta} \), then
(2.23) \[ f(re^{i\theta}) = R(r,\theta) + iI(r,\theta) \]

where:

\[ R(r,\theta) = (a_0 r^m \cos\theta + \ldots + a_m) \]
\[ I(r,\theta) = (a_0 r^n \sin\theta + \ldots + a_{m-1} \sin\theta) \]

If \( C \) is the circle, \( |z| = r \), then for any value of \( z \) on this circle \( R(r,\theta) \) and \( I(r,\theta) \) are functions only of \( \theta \). Let \( \theta = t \), then the real and the imaginary parts of \( f(z) \) can be generated by combining the solutions of the following set of differential equations:

\[ \frac{d}{dt} y + k^2 y = 0, \quad y(0) = 0, \quad \frac{d}{dt} y(0) = k, \quad 1 \leq k \leq n. \]

For any particular value of \( r \), the functions \( I(r,\theta) \) and \( R(r,\theta) \) are plotted versus each other on the \( x-y \) plotter. If the origin is enclosed by the mapping, a smaller value of \( r \) is chosen, and the resulting curve is examined. This procedure is continued until the mapping \( w = f(z) \) passes through the origin. Clearly, the inverse image of the origin, \( z_0 \), is a root of \( f(z) = 0 \). The modulus of the inverse image of each point on this graph is equal to the radius of the circle, \( r \). Hence, \( |z| = r \). The argument of this root can be obtained by plotting \( I(r,\theta) \) versus \( \sin m\theta \) and observing the zeros of \( I(r,\theta) \).
In this manner, the entire z-plane can be covered with concentric circles and all of the roots of \( f(z) = 0 \) obtained.

If only the real roots of \( f(z) = 0 \) are desired, then the preceding process can be simplified considerably. If \( z \) is real, \( \Theta = 0 \), if \( z \) is positive, and \( \pi \), if \( z \) is negative. Therefore, for positive real values of \( z \),

\[
(2.25) \quad f(z) = f(r) = \sum_{k=0}^{m} a_{m-k} r^k.
\]

Let \( r = t \), then \( f(r) \) can be generated by solving the following differential system:

\[
(2.26) \quad D_t^m f(r) = m! a_0
\]

\[
D_t^k f(0) = k! a_{m-k}, \quad 0 \leq k \leq m-1
\]

The function \( f(r) = f(t) \) is plotted versus \( r = t \). The roots of \( f(r) = 0 \) correspond to the \( t \)-coordinates of the points at which the curve cuts the \( t \)-axis. For negative values of \( z \),

\[
(2.27) \quad f(z) = f(-r) = \sum_{k=0}^{m} (-1)^k a_{m-k} r^k.
\]
Let $r = t$, then $f(-r)$ can be generated by solving the following differential system:

$$D_t^{m} f(-r) = (-1)^m m! a_0$$

$$D_t^{k} f(0) = (-1)^k k! a_{m-k} \quad 0 \leq k \leq m - 1.$$  

The negative and positive real roots must be considered separately because computer time, $t$, is a non-negative quantity.

2.3 An Example Problem. To illustrate the preceding method consider finding the roots of the equation,

$$f(z) = z^4 - 2z^3 - z^2 + 6z - 6 = 0.$$  

A fourth degree equation was chosen because the capacity of the 15-amplifier Heath Kit computer is limited. Naturally, as the degree of the equation is increased, the amount of equipment required is increased. From Descartes' rule of signs, this equation has exactly one negative root. The real roots will be determined first.

To simplify the magnitude scaling and to allow the value of $z$ to range from negative to positive infinity, the determination of the real roots is divided into four classes. First, consider the functions
\[ f(r) = r^4 - 2r^3 - r^2 + 6r - 6 \]
and
\[ f(-r) = r^4 + 2r^3 - r^2 - 6r - 6 . \]

These two functions are used to determine if the given polynomial has a real root between -1 and +1. Next, apply the transformation \( x = 1/r \) to both of these functions. The new functions obtained are

\[ f(w) = 6w^4 - 6w^3 + w^2 + 2w - 1 \]
and
\[ f(-w) = 6w^4 + 6w^3 + w^2 - 2w - 1 . \]

The mapping \( w = 1/r \) maps the interval from 1 to \( \infty \) onto the interval 0 to 1. Hence, the last two functions can be used to determine the real roots of (2.31) between \( -\infty \) and -1 and between 1 and \( +\infty \). Notice that the ranges of \( r \) and \( w \) are the same, from 0 to 1.

Furthermore, each of the equations has the same form, differing only in coefficients. Consequently, the same computer set-up with simple modifications, such as changing amplifier gains and potentiometer settings, is applicable for all four functions. Each of the four functions was generated by integrating its fourth derivative and applying the appropriate initial
conditions to each integrator. The computer diagram is shown in Figure 26. The resulting graphs obtained are shown in Figure 27. From this graph the approximate values of the real roots are ± 1.7.

The determination of the complex roots is more complicated and involves considerable equipment. Since no trigonometric function generator was available, each of the trigonometric functions required had to be generated directly on the computer by solving the auxiliary differential Equations (2.24) for \( k = 1, 2, 3, 4 \). Consequently, Equation (2.31) was depressed to a quadratic equation before proceeding with the determination of the complex roots. The analytical solution of a quadratic is simple and does not require an analog computer or any other machine. However, this example is purely for illustration.

The depressed equation is

\[
g(z) = z^2 - 2z + 2 = 0
\]

\[
g(re^{i\theta}) = R_g(r, \theta) + i I_g(r, \theta) = 0
\]

where:

\[
R_g(r, \theta) = r^2 \cos 2\theta - 2r \cos \theta + 2
\]

\[
I_g(r, \theta) = r^2 \sin 2\theta - 2r \sin \theta
\]
FIGURE 26. A COMPUTER DIAGRAM FOR THE DETERMINATION OF THE REAL ROOTS OF EQUATION (2.31)

NOTE: THIS DIAGRAM IS DIRECTLY APPLICABLE ONLY FOR ROOTS BETWEEN 0 AND 1.
FIGURE 27. THE COMPUTER DETERMINATION OF THE REAL ROOTS OF EQUATION (2.31)
Let \( \theta = t \). The functions \( \sin kt \) and \( \cos kt \) for \( k = 1,2 \) are obtained from Equations (2.24). The circuit diagram is shown in Figure 28. The functions \( R(r,\theta) \) and \( I(r,\theta) \) are outputs of two of the summing amplifiers. The determination of the complex roots is divided into two steps.

First, consider values of \( r \) less than one. Then apply the transformation \( w = 1/z \) to \( g(z) \), to obtain

\[
(2.33) \quad h(w) = 2w^2 - 2w + 1 \quad \text{or} \quad h(\text{re}^{i\theta}) = R_h(r,\theta) + i I_h(r,\theta).
\]

In this case, the mapping \( w = 1/z \) maps the exterior of the unit circle onto the interior of the unit circle. The function \( h(w) \) is used to consider values of \( r \) greater than 1. Notice that the ranges of \( r \) and \( w \) are the same, 0 to 1. The modulus of \( z \) is set equal to 1 first. The resulting graph of \( I(r,\theta) \) versus \( R(r,\theta) \) is shown in Figure 29. Since the curve does not encircle the origin, the equation \( g(z) = 0 \) has no roots with modulus less than 1. Consequently, there is no reason to consider any values of \( r \) less than 1. Next, set \( w \) equal to \( \frac{3}{4} \). The resulting graph is shown in Figure 30. The graph encircles the origin twice. Consequently,
Figure 28. A computer diagram for the determination of the complex roots of equation (2.31)

\[ f(w) = R_E + i \cdot R_I \]
FIGURE 29. THE COMPUTER DETERMINATION OF THE NUMBER OF COMPLEX ROOTS OF EQUATION (2.31) FROM EQUATION (2.32) WITH $|Z| \leq 1$.
FIGURE 30. THE COMPUTER DETERMINATION OF THE NUMBER OF COMPLEX ROOTS OF EQUATION (2.31) FROM EQUATION (2.33) WITH |W| ≤ 1
there are two roots $z_1$ and $z_2$ with modulus greater than $\frac{4}{3}$. Since the equation $h(z) = 0$ has only real coefficients and only two roots, it is obvious that $z_1$ and $z_2$ are a conjugate pair, and hence, have the same modulus. Therefore, the value of $w$ was varied until the curve passed through the origin twice. The final graph is shown in Figure 31. From the graph, the modulus of $z$ is approximately 1.4. In Figure 32, the $\sin2\theta$ is plotted versus $I(r, \theta)$. Reading from this figure, the arguments of $z_1$ and $z_2$ are $\pi/4$ and $7/4\pi$ radians.

Clearly, the solution of this problem does not have to be divided into real and complex parts. Each of the real roots could have been determined in the same manner as the complex roots. However, the generation of the functions $R(r, \theta)$ and $I(r, \theta)$ would then have required the $\sin kt$ and the $\cos kt$ for $k = 1, 2, 3, 4$; and the 15-ampimeter Heath Kit is not capable of generating all of these functions and solving the problem at the same time.

2.4 The Nature of the Roots. The nature of the roots (4, pp. 882-7) of a given polynomial equation $f(z) = 0$ may be determined from the mapping $w = f(z)$ in the w-plane. For example, suppose that the equation $f(z) = 0$ has a pair of complex roots, $a \pm bi$. Then
FIGURE 31. THE COMPUTER DETERMINATION OF THE MODULI OF THE COMPLEX ROOTS OF EQUATION (2.32)
FIGURE 32. THE COMPUTER DETERMINATION OF THE ARGUMENTS OF THE COMPLEX ROOTS OF EQUATION (2.32)
In a small neighborhood of \( a + bi \), \( q(z) \) behaves approximately as a constant. Hence,

\[
(2.42) \quad f(z) = [z - (a+bi)][z - (a-bi)]A
\]

\[
= (r^2\cos^2\theta - 2\arccos\theta + a^2 + b^2)A
\]

\[
+ i(r^2\sin^2\theta - 2\arcsin\theta)A.
\]

If \( C \) is a circle with center at the origin and radius \( r \), then

\[
(2.43) \quad f(z) = 2r(rcos\theta - a). \]

For \( |a| < r \), this is the equation of a limaçon in the \( w \)-plane. Consequently, if \( f(z) = 0 \) has a pair of complex roots, the plot of \( f(z) \) in the \( w \)-plane will be a limaçon. This is illustrated in Figure 31. Similar analysis indicates that if \( f(z) = 0 \) has a single real root, a circle in the \( z \)-plane is mapped onto a circle in the \( w \)-plane; if it has a double real root, a circle is mapped into a cardioid.

If the equation \( f(z) = 0 \), has complex coefficients the same procedure is applicable. For example, let

\[
f(z) = (1 + i)z^3 + 2z^2 + z + i. \]

Then
(2.44) \( f(z) = R(r, \theta) + i I(r, \theta) \)

where:
\[
R(r, \theta) = r^3(\cos^3 \theta - \sin^3 \theta) + 2r^2 \cos 2\theta + r \cos \theta
\]
\[
I(r, \theta) = r^3(\cos^3 \theta + \sin^3 \theta) + 2r^2 \sin 2\theta + r \sin \theta + 1.
\]

The functions \( R(r, \theta) \) and \( I(r, \theta) \) can be generated in the same way as before, although the program becomes more complicated.

2.5 Other Methods of Solution. Other methods of solution have been devised. Instead of covering the z-plane with a set of concentric circles, it could be covered with a set of radial lines. In this case, the argument of \( z \) is set equal to a constant and the modulus varied. The z-plane could also be covered with a rectangular grid obtained by setting \( x \) equal to a constant and letting \( y \) vary and then reversing the roles of \( x \) and \( y \). In both cases, the principle of the argument is not applicable because the curves chosen are not even closed.

Another interesting method which has met with some success is the method of steepest descent \((5, pp. 901-7)\). A function \( F(z) \) is chosen such that its minima correspond to the zeros of the polynomial \( f(z) \). Usually, this method requires some "non-linear" equipment, such as function multipliers.
The solution of any polynomial equation can also be related to the solution of an ordinary differential equation with constant coefficients. Consider the polynomial Equation (2.22) with \( a_0 = 1 \). Assume that the coefficients are real. This equation is the auxiliary equation of the differential equation

\[
(2.51) \quad \sum_{k=0}^{m} a_k D_t^{m-k} y(t) = 0 \quad a_0 = 1.
\]

Let \( D_t^i y = x_{i+1} \) for \( 0 \leq i \leq m - 1 \). This substitution transforms Equation (2.51) into the following system of first order differential equations:

\[
(2.52) \quad D_t x_1 = x_{i+1} \quad 1 \leq i \leq m - 1
\]

\[
D_t x_m = -(a_m x_1 + a_{m-1} x_2 + \ldots + a_1 x_m).
\]

In matrix notation,

\[
(2.53) \quad \begin{pmatrix} D_t x_1 \\ \vdots \\ D_t x_m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -a_m & -a_{m-1} & \ldots & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.
\]
Theorem. The characteristic values of this matrix are the roots of the given polynomial equation.

Proof. (Mathematical induction on the degree of the polynomial.)

a. Let \( m = 1 \). Therefore, \( f(z) = z + a_1 = 0 \). The corresponding differential system is

\[
D_t x_1 = (-a_1) x_1
\]

\[
\therefore \lambda_1 = -a_1.
\]

\therefore The theorem is true for \( m = 1 \).

b. Assume that the theorem is true for \( m = k - 1 \). Then consider the equation

\[
f(z) = \sum_{j=0}^{k} a_j z^{k-j} = 0 \quad a_0 = 1.
\]

The corresponding differential system is

\[
\begin{pmatrix}
D_t x_1 \\
\vdots \\
D_t x_k
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
& & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 1 \\
-a_k & -a_{k-1} & \ldots & -a_1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_k
\end{pmatrix}
\]

or

\[
D_t x = A_k x.
\]
Therefore,

\[ \det |A_k - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ -a_k & -a_{k-1} & \ldots & -a_2 & (-a_1-\lambda) \end{vmatrix} \]

Expanding this determinant in terms of the first column,

\[ \det |A_k - \lambda I| = -\lambda \det |A_{k-1} - \lambda I| + \sum_{j=0}^{k-l} (-1)^j a_j \lambda^{(k-l)-j} \]

where:

\[ A_{k-1} = \begin{pmatrix} -\lambda & 1 & 0 & \ldots & 0 \\ 0 & -\lambda & 1 & 0 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & -\lambda & 1 \\ -a_{k-1} & -a_{k-2} & \ldots & -a_2 & (-a_1-\lambda) \end{pmatrix} \]

Therefore, by the induction hypothesis

\[ \det |A_k - \lambda I| = -\lambda \left[ (-1)^{k-l} \sum_{j=0}^{k-l} a_j \lambda^{(k-l)-j} \right] + \sum_{j=0}^{k-l} (-1)^j a_j \lambda^{(k-l)-j} \]

\[ = (-1)^k \sum_{j=0}^{k} a_j \lambda^{k-j} \]
Hence, the characteristic equation of the matrix $A$ is the given polynomial equation. Therefore, by the axiom of mathematical induction, the theorem is true for polynomial equations of any degree.

q.e.d.

Hence, if a scheme can be developed for finding the characteristic values of the matrix $A$, then the values obtained are the roots of the original polynomial equation. This relationship was presented primarily as a matter of interest. Rarely, if ever, would it be practical to convert a polynomial equation into a matrix equation before determining the roots of the equation. The determination of the characteristic values of a matrix is discussed in the next section.

2.6 Summary. The simple example illustrates that the amount of equipment required to solve a polynomial equation is considerable when no trigonometric function generators are available. For an $n$th degree polynomial with real coefficients,

\[(2.61) \quad N_A = 3(n - k) + (m + 2)\]
The preceding formula assumes that any particular amplifier is capable of accepting any number of inputs. If only the real roots are desired then

\[(2.62) \quad N_A = n + 1.\]

3. **Matrices and Characteristic Value Problems**

Frequently, the solution of a mathematical problem consists of solving a system of \(n\) linear algebraic equations in \(n\) unknowns, \(x_1, x_2, \ldots, x_n\).

\[(3.1) \quad \sum_{j=1}^{n} a_{ij}x_j = b_i \quad 1 \leq i \leq n\]

or in matrix notation,

\[(3.2) \quad Ax = b\]
where:

\[ A = n \times n \text{ matrix, } (a_{ij}) \]
\[ b = n \times 1 \text{ matrix, } (b_i) \]
\[ x = n \times 1 \text{ matrix, } (x_j) \]

Such a system can be solved on the analog computer. Apparently, the only necessary components are adders and potentiometers, since the vector \( x \) is not a function of time. Unfortunately, the direct approach leads to stable operation only if the matrix \( A \) is positive definite. The instability (4, pp. 835-42) is a direct consequence of the fact that the gain of each adder is not actually constant but is a function of the frequency of the input. In other words, if a set of algebraic equations is directly mechanized on the computer, the computer actually solves a set of differential equations. If the solutions of the system of differential equations are stable, then the steady-state values of these solutions comprise the solution of the algebraic equations. Fifer (4, pp. 839-40) proves that one of the conditions for such stability is that the matrix of coefficients \( A \) be positive definite.

3.1 Computer Solutions. Several alternative methods of solution have been proposed. For example, solving the implied system \( A'Ax = A'b \) where \( A' = (a_{ij}) \), instead of the given system \( Ax = b \). The solutions of both
systems are identical, but in the first case the matrix $A'A$ is always positive definite. This method involves lengthy preliminary calculations. In addition, special purpose computers have been designed for the express purpose of solving systems of linear equations.

Adcock (4, pp. 847-51) developed a method that is applicable to any linear system if $|A| \neq 0$. Consider the set of Equations (3.1). For simplicity assume that the coefficients $a_{ij}$ are real. Introduce the variables $E_i = E_i(x_1, x_2, \ldots, x_n, t)$ $1 \leq i \leq n$, such that

\begin{align*}
(3.11) & \quad D_tE_i = ke_i \\
& \quad e_i = \sum_{j=1}^{n} a_{ij}x_j - b_i \\
& \quad x_j = -\sum_{i=1}^{n} a_{ij}E_i \\
& \quad 1 \leq i \leq n \\
& \quad 1 \leq j \leq n.
\end{align*}

where:

- $k$ = constant of convergence
- $e_i$ = instantaneous error in equation $i$.

System (3.11) is programmed on the computer instead of system (3.1). Thus, the static problem has been transformed into a dynamic one.
Theorem. (4, pp. 847-8) System (3.11) is convergent, that is, \( e_i \to 0 \) as \( t \to +\infty \) for \( 1 \leq i \leq n \).

Proof. Let \( f(x_1, \ldots, x_n,t) = \sum_{i=1}^{n} (e_i)^2 \).

Therefore, \( D_tf = 2 \sum_{i} e_i |D_te_i| \).

From Equation 2 in (3.11),
\[
D_tf = 2 \sum_{i} e_i \left( \sum_{j} a_{ij} D_t x_j \right)
\]

From 3,
\[
D_tf = -2 \sum_{i} e_i \left( \sum_{j} a_{ij} \left( \sum_{k} a_{kj} D_tE_k \right) \right)
\]

From 1, and rearranging,
\[
D_tf = -2k \left( \sum_{j} \left( \sum_{i} a_{ij} e_i \right)^2 \right)
\]

Therefore,
\[
D_tf \leq 0 \quad \text{for all } t.
\]

Now by definition, \( f \geq 0 \) for all \( t \). Therefore, \( f \) approaches some constant value \( f_{ss} \) as \( t \) increases.

Assume \( f_{ss} \neq 0 \). Then at steady state \( D_tf = 0 \).
Hence,
\[
\sum_{i} a_{ij} e_i = 0 \quad 1 \leq j \leq n.
\]
Or in other words, this system of equations

$$A'e = 0$$

has a non-trivial solution, but $|A'| = |A| \neq 0$. Hence $f_{ss} = 0$, which implies that $e_i = 0$, $1 \leq i \leq n$.

q.e.d.

Therefore, any system can be solved in the preceding manner if $|A| \neq 0$. For a set $(4, p. 349)$ of $n$ equations in $n$ unknowns with real coefficients

$$N_A = 4n$$

$$N_p = 2n^2 - n$$

where:

$N_A, N_p$ = maximum number of amplifiers and potentiometers required, respectively.

Here it is assumed that the integrators are combination adders and integrators, and that the coefficients $a_{1i} = 1$ for all $i$. Hence, this method requires considerable computer equipment. Perhaps, it is not too practical to use a general purpose analog computer to solve such systems.

3.2 Characteristic Values. However, the preceding technique $(4, pp. 929-33)$ is applicable to another problem of importance in applied mathematics, the
determination of the characteristic values and characteristic vectors of a matrix $A$. By definition $\lambda_1$, is a characteristic value of $A$ if there exists a vector $x_1$, $x_1 \neq 0$, such that $Ax_1 = \lambda_1 x_1$; the vector $x_1$ is called a characteristic vector corresponding to $\lambda_1$. The determination of characteristic values and vectors amounts to the solution of the $n$ homogeneous equations in $(n + 1)$ unknowns:

$$ (3.21) \quad (a_{1i} - \lambda)x_i + \sum_{j \neq i}^{n} a_{ij}x_j = 0 \quad 1 \leq i \leq n. $$

For simplicity, assume that $A$ is real and that only the real characteristic vectors and values are desired. One of the variables in system (3.21) is set equal to some constant, say $x_n$. New variables $E_i = E_i(x_1, \ldots, x_n, t)$, $1 \leq i \leq n - 1$ are defined such that

$$ (3.22) \quad D_tE_i = kE_i $$

$$ x_j = - \sum_{i \neq j}^{n} (a_{ij}E_i) - (a_{jj} - \lambda)E_j \quad 1 \leq j \leq n - 1 $$

$$ e_i = \sum_{j \neq i}^{n} a_{ij}x_j - b_i + (a_{ii} - \lambda)x_i \quad 1 \leq i \leq n - 1. $$

where $k$ is the constant of convergence.
System (3.22) together with the nth equation in system (3.21) are programmed on the computer. The solution proceeds by trial and error. A value of $\lambda$ is assumed, and the resulting system of n-1 equations in n-1 unknowns is then solved on the computer, and the error in the nth equation, $e_n$, is observed. If $e_n \neq 0$, then a new value of $\lambda$ is assumed, and the solution rerun. When $e_n = 0$, the corresponding value of $\lambda$ is a characteristic value. The values of the components of the corresponding characteristic vector are outputs of amplifiers and can be read on the computer voltmeter.

3.3 An Example Problem. To examine this type of problem in more detail, consider the following example: Determine the characteristic values and vectors of the matrix

$$ A = \begin{pmatrix} 0.611 & -0.333 & 0.111 \\ -0.333 & 0.556 & -0.222 \\ 0.111 & -0.222 & 0.333 \end{pmatrix} $$

The matrix $A$ is real and symmetric. Therefore, each of the characteristic values is real.

The solution of this problem by the preceding method amounts to solving the three equations:
(3.32) 
\[(0.611 - \lambda)x_1 - 0.333x_2 + 0.111x_3 = 0 \]
\[- 0.333x_1 + (0.556 - \lambda)x_2 - 0.222x_3 = 0 \]
\[0.111x_1 - 0.222x_2 + (0.333 - \lambda)x_3 = 0 .\]

Set \(x_3 = 10\), choose \(k = 1\), and introduce the variables \(E_1\) and \(E_2\). The equations to be programmed on the computer are

\[(3.33) \quad - 0.333x_1 + (0.556 - \lambda)x_2 - 2.22 = e_1 \]
\[0.111x_1 - 0.222x_2 + (3.33 - \lambda) = e_2 \]
\[D_t E_1 = e_1 \quad i = 1, 2 \]
\[x_1 = 0.333E_1 - 0.111E_2 \]
\[x_2 = - (0.556 - \lambda)E_1 + 0.222E_2 \]
\[(0.611 - \lambda)x_1 - 0.333x_2 + 1.11 = e_3 \]

The value of \(\lambda\) is varied until \(e_3 = 0\).

The computer diagram for this problem is shown in Figure 33. Table II lists the various values assumed for \(\lambda\) and the corresponding values of \(e_3\). Only those values of \(\lambda\) sufficiently close to the final values are listed. The values of \(\lambda\) were set on a potentiometer. The corresponding values of \(x_1\) and \(x_2\) were obtained as outputs of amplifiers.
FIGURE 33. A COMPUTER DIAGRAM FOR SYSTEM (3.33)
TABLE II

The Computer Determination of the Error Present in the Solution of System (3.33) for Different Values of $\lambda$

<table>
<thead>
<tr>
<th>Assumed Value</th>
<th>Error Present</th>
<th>Components of the Characteristic Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$e_3$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>volts</td>
<td>volts</td>
<td>volts</td>
</tr>
</tbody>
</table>

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>.168</td>
<td>- 2.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.172</td>
<td>- 2.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.165</td>
<td>- 1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.160</td>
<td>- 0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.157</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*  .159</td>
<td>0.0</td>
<td>4.5</td>
<td>9.7</td>
</tr>
<tr>
<td>.332</td>
<td>0.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.340</td>
<td>1.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.330</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*  .326</td>
<td>0.0</td>
<td>-10.7</td>
<td>- 5.7</td>
</tr>
<tr>
<td>.900</td>
<td>18.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.950</td>
<td>10.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.980</td>
<td>4.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.990</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.995</td>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>*  .994</td>
<td>0.2</td>
<td>20.0</td>
<td>-20.0</td>
</tr>
</tbody>
</table>

* A characteristic value of $A$. 

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For any such problem, the maximum values of the variables \( x_i, E_i, \) and \( e_i \) are not known and vary for different values of \( \lambda \). Consequently, it is possible that the magnitude scaling factors may have to be changed several times during the solution time. The preceding method is one of trial and error. However, it is very easy to adjust the setting of a potentiometer. Hence, this method is applicable for the approximate determination of characteristic values and vectors of a matrix. The accuracy of these values can then be refined on the digital computer.

3.4 Extensions. This same procedure can be extended to the generalized characteristic value problem,

\[(3.41) \quad Ax = \lambda Bx\]

where:

- \( A, B = n \times n \) matrices, \( (a_{ij}) \) and \( (b_{ij}) \), respectively
- \( x = n \times 1 \) matrix, \( (x_i) \)
- \( \lambda = \) constant.

The equations programmed are

\[(3.42) \quad \sum_{j=1}^{n} (a_{ij} - \lambda b_{ij})x_j = e_i \quad 1 \leq i \leq n \]

\[D_i E_i = k e_i \quad 1 \leq i \leq n - 1\]

\[x_j = - \sum_{i=1}^{n} (a_{ij} - \lambda b_{ij})E_i \quad 1 \leq j \leq n - 1.\]
Values of $\lambda$ are assumed, and the resulting equations solved until $e_n = 0$.

If the matrix $A$ is not real and symmetric and the complex characteristic values and vectors are desired, this same procedure can still be used. In this case,

\[(3.43) \quad Hz = \lambda z\]

where:

- $H$ = n x n matrix, $(a_{ij} + ib_{ij})$
- $z$ = n x 1 matrix, $(x_i + iy_i)$
- $\lambda$ = constant, $\lambda_R + i\lambda_I$.

Thus, the given system can be expressed as a system of $2n$ equations in $2n$ unknowns as follows:

\[(3.44) \quad (Ax - By) - (\lambda_R x - \lambda_I y) = 0\]
\[(Bx + Ay) - (\lambda_I x + \lambda_R y) = 0\]

where:

- $A, B = (a_{ij})$ and $(b_{ij})$, respectively
- $x, y = (x_i)$ and $(y_i)$, respectively.

System (3.37) can be programmed and solved on the computer. The solution is more difficult in this case because two values must be assumed for each value of $\lambda$. 
4. Miscellaneous Problems

4.1 Ordinary Differential Equations. General analytic methods of solution have been devised for ordinary, linear differential equations with constant coefficients. Certain types of ordinary, linear differential equations with time varying coefficients, such as Bessel's equation, can be solved analytically; and graphical and numerical methods of solution may be applied to some non-linear ordinary differential equations. However, there are no general methods for the solution of these two types of equations.

The method of solution (9, pp. 148-212) on the analog computer of all three of the preceding types of equations is the same. "Non-linear" devices such as function multipliers and diodes can be used to introduce the time-varying coefficients and the non-linearities.

The computer solution of a linear equation does have some advantages over the solution of a non-linear equation. Any uniform change in the initial conditions and driving functions produces a similar change in the solution of a linear equation. This is not necessarily true for non-linear equations. In addition, it is usually easier to generate functions of time than it is to generate functions of a dependent variable. In
either case, the solution of such equations requires a fundamental understanding of the response of each of the computer components besides a thorough understanding of the problem.

4.2 Integral Equations. Another interesting problem to consider is the solution of an integral equation of the Fredholm type with a degenerate kernel:

\[(4.21) \quad u(t) + \int_{0}^{t} K(s,t) \ u(s) \ ds = f(t)\]

where:

\[K(s,t) = \sum_{i=1}^{n} g_i(s) \ h_i(t)\]

\[g_i(t), h_i(t), f(t) = \text{analytic functions of } t, \quad 1 \leq i \leq n.\]

Substituting for \(K(s,t)\) in Equation \((4.21)\), the following equation is obtained:

\[(4.22) \quad u(t) = - \sum_{i=1}^{n} h_i(t) \int_{0}^{t} u(s) \ g_i(s) \ ds + f(t).\]

The unscaled computer diagram shown in Figure 34 was obtained by assuming that \(u(t)\) was known and then proceeding in the same manner as in the solution of a differential equation. Each of the \(h_i(t)\) and \(g_i(t)\),
FIGURE 34. AN UNSCALED COMPUTER DIAGRAM FOR SYSTEM (4.21)
and $f(t)$ have to be generated by an auxiliary circuit or a function generator. In addition, numerous multipliers are required.

4.3 Definite Integrals. The analog computer can also be used to evaluate certain definite integrals that cannot be evaluated directly, such as:

$$\int_0^t e^{-at^2} dt$$

The integrand is the solution of the following linear system:

$$(4.32) \quad D_t y + 2aty = 0$$

$$y(0) = 1.$$  

This system is implemented on the computer, and the solution integrated to obtain the given integral. An unscaled diagram is shown in Figure 35.
FIGURE 35. AN UNSCALED COMPUTER DIAGRAM FOR SYSTEM (4.32)
5. Partial Differential Equations

The most general linear partial differential equation of second degree in two independent variables is of the form

\[ a \frac{\partial^2 z}{\partial x^2} + b \frac{\partial^2 z}{\partial x \partial y} + c \frac{\partial^2 z}{\partial y^2} + d \frac{\partial z}{\partial x} + e \frac{\partial z}{\partial y} + fz + g = 0 \]

where \(a, b, c, d, e, f, g\), are functions of \(x\) and \(y\).

Equations of this type (7, pp. 378-85) are classified as parabolic, elliptic, or hyperbolic in a certain region of two-dimensional space if the quantity \(b^2 - 4ac\) is zero, less than zero, or greater than zero, respectively, in the region.

At present, this is the only type of partial differential equation that has been studied on the analog computer. In particular, the solutions of the following three equations will be considered in the following discussion:

\[ \begin{align*}
1. & \quad \frac{\partial^2 u}{\partial x^2} = k \frac{\partial u}{\partial t} \\
2. & \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\
3. & \quad \frac{\partial^2 u}{\partial x^2} = k \frac{\partial^2 u}{\partial t^2} .
\end{align*} \]
In mathematical physics, these equations (7, pp. 415-73) are classified as the one-dimensional equation of heat conduction, Laplace's equation in two dimensions and the one-dimensional wave equation. Equation 1 arises in the study of the temperature distribution in an insulated rod; 2 arises in the study of the steady-state temperature distribution in a thin rectangular plate; and 3 arises in the analysis of a vibrating string.

5.1 Computer Solution. The voltages on the analog computer are all functions of a single independent variable, computer time. Hence, the analog computer can handle only problems with one independent variable. But, each of the three preceding differential equations contains two independent variables. Two men who attempted to surmount this difficulty were Howe and Haneman (6), who used difference techniques to solve equation 1 in (5.2).

Basically, there are two finite difference methods of solution, the "serial" and the "parallel" methods. In each method, one of the two partial derivatives is replaced by a finite difference approximation. For example, consider the following system:
The solution \( u(x,t) \) is desired over some region in the \( x-t \) plane, as shown in Figure 36. Either the \( t \)-interval or the \( x \)-interval may be divided into sub-intervals.
Consider both cases. First divide the \( x \)-interval into \( n \) sub-intervals, \([x_0, x_1], \ldots, (x_{n-1}, x_n]\) of length \( h_x \).

Then

\[
(5.12) \quad \frac{\partial^2 u}{\partial x^2} \bigg|_{x_{i-1/2}} = \frac{u(x_i,t) - u(x_{i-1},t)}{h_x} \quad 1 \leq i \leq n.
\]

Therefore,

\[
(5.13) \quad \frac{\partial^2 u}{\partial x^2} \bigg|_{x_i} = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h_x^2}
\]
or

\[
(5.14) \quad \frac{\partial^2 u}{\partial x^2} \bigg|_{x_i} = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h_x^2} \quad 1 \leq i \leq n
\]
FIGURE 36, THE REGION OF INTEREST IN THE X-T PLANE FOR SYSTEM (5.11)
where \( u(x_{-1}, t) = u(x_{n+1}, t) = 0 \). Consequently, the original partial differential equation is replaced by the \( n \) first order, ordinary differential equations,

\[
(5.15) \quad D_t u(x_i,t) = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h_x^2} \quad 1 \leq i \leq n
\]

and the solution \( u(x,t) \) is replaced by the \( n \) solutions \( u(x_i,t) \), \( 1 \leq i \leq n \). Since \( u(x,0) \) is a known function of \( x \), \( u(x_i,0) \) is known for all \( i \) and can be set directly as initial conditions on the integrators.

Next, divide the \( t \)-interval, \([0,t_0]\) where \( t_0 > 0 \) into \( n \) sub-intervals, \([0,t_1), \ldots, (t_{n-1}, t_n]\) of length \( h_t \). Then

\[
(5.16) \quad \left. \begin{array}{c}
\frac{\partial u}{\partial t} \\
\end{array} \right|_{t_1} = \frac{u(x,t_1) - u(x,t_{i-1})}{h_t}
\]

Consequently, the original partial differential equation is replaced by the \( n \) second order, ordinary differential equations,

\[
(5.17) \quad D_x^2 u(x,t) = \frac{u(x,t_i) - u(x,t_{i-1})}{h_t} \quad 1 \leq i \leq n.
\]

Since \( u(0,t) \) is a known function of \( t \), \( u(0,t_i) \) is known for \( 1 \leq i \leq n \) and can be set directly as initial conditions on the integrators generation \( u(x,t_i) \). However, the
values of $D_x u(0,t_1)$ are not known. Therefore, the initial conditions for $n$ integrators are not known. Consequently, the solution is by trial and error. Values are assumed for these initial conditions and the boundary conditions obtained under these assumptions are compared with the desired boundary conditions. The correct values are those that yield the desired boundary conditions.

The serial method of solution (5) consists of solving each of the $n$ ordinary differential equations obtained successively, using at each stage one or more of the solutions obtained from previous stages. This method requires curve-tracing equipment, which introduces errors into the solution. These errors make this method impractical. The parallel method of solution (5) consists of solving all of the $n$ equations simultaneously on the computer.

Fisher (5) applied both methods to the solutions of Equations (5.2). He concluded that the serial method of solution was inherently unstable, the instability increasing as the length of the sub-intervals was decreased. He also concluded that the parallel method is stable when applied to Equations 1 and 3 in (5.2), but not when applied to Equation 2.
5.2 An Example Problem. As an example, consider the computer solution of the following problem:

Determine the temperature \( u(x,t) \) at any point in an insulated isotropic metal rod, 10 units in length, given that \( u(x,t) \) satisfies the system,

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}
\]

\( u(x,\infty) = u(0,t) = u(10,t) = 0 \quad u(x,0) = 100 \).

First difference the \( x \)-interval \([0,10]\), \( h_x = 2 \) units. The given equation transforms into the set

\[
\begin{align*}
D_t u(x_i,t) &= \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{4} \\
1 \leq i &\leq 5.
\end{align*}
\]

The computer diagram is shown in Figure 37, the results in Figure 38.

Next, difference the \( t \)-interval \([0,20]\), \( h_t = 10 \) seconds. The given equation transforms into the set

\[
\begin{align*}
D_x^2 u(x,t_i) &= \frac{u(x,t_i) - u(x,t_{i-1})}{10} \\
1 \leq i &\leq 2.
\end{align*}
\]
Figure 37. A computer diagram for system (5.21) with the x-interval differenced.

Note: The first and last integrators have only two inputs since \( u(0,T) = u(10,T) = 0 \).
FIGURE 38. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (5.21) WITH THE X-INTERVAL DIFFERENCED
The computer diagram is shown in Figure 39, the results in Figure 40. In this case the solution was not direct, the value of $D_x u(0,10)$ was varied until $u(10,10)$ was zero. Then using this value, the value of $D_x u(0,20)$ was varied until $u(10,20)$ was zero.

Theoretically, the values of $h_x$ and $h_t$ are arbitrary, and can be chosen to yield the desired degree of accuracy in the problem. This is true for $h_x$, but not for $h_t$. Drift in the amplifiers, and inaccuracies in the meter and computer components make the trial and error solution tedious and almost impossible. Decreasing the value of $h_t$, increases the number of initial conditions that must be set by trial and error, and hence, increases the length of time required to adjust all of the initial conditions. Thus, the effect of drift increases, increasing the error in the final solution since any error in $u(x,t_i)$ becomes a part of $u(x,t_{i+1})$ and thus a part of the final solution.

A second set of solutions was obtained by changing the boundary conditions to

$$(5.24) \quad u(x,0) = 100 - 10x$$

$$u(0,t) = u(10,t) = 0.$$ 

The computer diagrams are the same as before, except for the addition of one integrator to generate $u(x,0)$. The results are shown in Figures 41 and 42.
FIGURE 39. A COMPUTER DIAGRAM FOR SYSTEM (5.21)
WITH THE T-INTERVAL DIFFERENCED
Figure 40: The computer solution obtained for system (5.21) with the T-interval differenced.
FIGURE 41. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (5.24) WITH THE X-INTERVAL DIFFERENCED
Figure 42. The computer solution obtained for system (5.24) with the t-interval differenced.
5.3 Analytical Solution. The preceding problems may be solved analytically by the method of separation of variables (7, pp. 420-3). Assume that $u(x,t) = X(x)T(t)$ then,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{T} \frac{\partial T}{\partial t} = -k^2$$

Solving Equations (5.31) and substituting in the boundary conditions,

$$u(x,t) = \sum_{n \text{ odd}} \frac{400}{n\pi} \sin \frac{n\pi x}{10} e^{-\frac{n^2\pi^2 t}{100}}$$

Case 2. $u(x,t) = \sum_{n=1}^{200} \frac{200}{n\pi} \sin \frac{n\pi x}{10} e^{-\frac{n^2\pi^2 t}{100}}$

Values of $x$ and $t$ were chosen and the corresponding values of $u(x,t)$ were approximated from (5.26) for Case 1 and for Case 2. These calculated values were plotted on Figures 38 and 41, respectively.

5.4 Summary. As Jackson (9, p. 314) suggests perhaps the next step is to subdivide both intervals and solve the problem both ways simultaneously, devising some circuit to automatically correct for errors in the solution. As in the case of the ordinary differential equation, the solution obtained is a particular solution
valid only for the given boundary conditions. The extension of the method of solution to non-linear partial differential equations would not be too easy, but the extension is a natural one. First, however, more work needs to be done on the linear equations.

6. Geometric Applications

6.1 History. The Greek geometers (2, pp. 77-8) could not prove or disprove that

1. A cube can be constructed whose volume is twice that of a given cube;
2. Any angle can be trisected;

using only a finite number of straight lines and circles.

These two problems are equivalent to proving or disproving that the real root(s) of the equations

\[ x^3 - 2 = 0 \] and \[ 4x^3 - 3x - b = 0 \]

can be constructed in the preceding manner. The first equivalence is trivial. The second one follows from the trigonometric identity \( \cos \alpha = 4 \cos^3 \alpha/3 - 3 \cos \alpha/3 \), where \( \alpha \) is any angle.

In 1837, Wantzel (2, p. 77) obtained conditions, necessary and sufficient, for the real roots of a polynomial equation with rational coefficients to be constructible as specified. Neither of the two
preceding cubics satisfied these conditions. Thus, the solution of these problems by the above method was proved impossible. However, these problems have been solved by other methods, for example, Diocles (250 to 100 B. C.), (12, pp. 26-7) used the cissoid $y^2 = x^3/(2a - x)$, to solve the cube root problem, $x^3 - c = 0$.

6.2 Computer Solution. Problem 1 contains no parameters and amounts to constructing $3\sqrt{2}$. The procedure is simple; the computer diagram is given in Figure 43. The function $f(t) = t^3 - 2$ is generated and plotted versus $t$. The distance from the origin to the point of intersection of $f(t)$ with the $t$-axis corresponds to $3\sqrt{2}$.

Problem 2 is more complicated. The problem is to determine the real roots of the equation:

(6.21) \[ f(t) = 4t^3 - 3t - b = 0. \]

Replace Equation (6.21) by the following system:

(6.22) \[
\begin{align*}
y_1 &= 4t^3 \\
y_2 &= 3t + b.
\end{align*}
\]

The circuit diagram is shown in Figure 44, and the resulting graph in Figure 45. The functions $y_1$ and $y_2$
FIGURE 43. A COMPUTER DIAGRAM

FOR GENERATING THE $\frac{3}{2}$
FIGURE 44. A COMPUTER DIAGRAM FOR SYSTEM (6.22)
FIGURE 45. THE COMPUTER SOLUTION OBTAINED FOR SYSTEM (6.22)
were generated and plotted, \( y_2 \) for \( 0 \leq b \leq 1 \). Only angles between 0 and \( \pi/2 \) radians can be trisected on this graph. However, this graph could easily be extended by letting \( b \) take on any value between -1 and +1. For any particular value of \( b \), the \( t \)-coordinate of the point of intersection of \( y_1 \) and \( y_2 \) is \( \cos \theta/3 \). Drop a perpendicular to the \( t \)-axis from this point. Connect the point of intersection of this perpendicular and the unit circle to the origin. Thus, \( \theta/3 \) is constructed.

6.3 Other Problems. The analog computer can be used for other geometrical constructions, for example, the construction of the transcendental number, \( \pi \). Equation (2.24) \( k=1 \), is programmed on the computer, and the function generated, \( \sin t \), is plotted versus \( t \). The first point of intersection of \( \sin t \) with the \( t \)-axis corresponds to the number \( \pi \).
IV. BIBLIOGRAPHY


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Abstract

This thesis is intended to be an introductory mathematical presentation of analog computation. An attempt was made to explain in concise mathematical language, how an electronic analog computer works, why it works, and the simplicity of its use.

The components of the computer are considered as operational blocks, each block performing an indicated operation. Hence, the electrical knowledge presented is meager.

The methods of solution and the corresponding computer solutions obtained for several types of mathematical problems are presented; such as, the determination of the characteristic vectors and characteristic values of a given matrix. In each case, a 15-amplifier Heath Kit analog computer model number ES-400 was used. Since this type of computer contains no devices for multiplying variable quantities, the only types of problems that could be considered were those that can be represented by a system of linear, ordinary differential equations with constant coefficients. However, similar techniques are applicable to the analogous non-linear systems and systems with variable coefficients, on a fully-equipped analog computer.