THE DEVELOPMENT OF A DYNAMICALLY COUPLED AXIAL-TORSIONAL-LATERAL POINT TRANSFER MATRIX FOR A WHIRLING ECCENTRIC MASS

by

Robert Carter Johnstone

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APPROVED:

L. D. Mitchell, Chairman

N. S. Eiss, Jr.

H. H. Mabie

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NOMENCLATURE*

1. MATHEMATICAL SYMBOLS

X,Y,Z	Absolute cartesian right-handed coordinate system
x,y,z	Rotating cartesian right-handed coordinate system
$\bar{I},\bar{J},\bar{K}$	Unit vectors in the absolute coordinate system
$\bar{i},\bar{j},\bar{k}$	Unit vectors in the rotating coordinate system
x	Vector cross product
{ }	State vector
[ ]	Transfer matrix
$\| \;\|$	Matrix determinant

2. GENERAL SYMBOLS

a,b,c and d	Lengths related to the undeflected position of the two lumped masses as seen in y-z plane, (m)
c	Damping constant, $(\frac{\text{N-sec}}{\text{m}})$
e	Eccentricity, (m)
$\bar{f}$	Force in rotating coordinate system, (N)
$\bar{F}$	Total force in absolute coordinate system, (N)
$\tilde{F}$	Field transfer matrix
g	Acceleration due to gravity, $(\frac{\text{m}}{\text{sec}^2})$
G	Center of mass of unbalanced disk
i_g	General indicator of radius of gyration, (m)

*Units used are taken from "ASME Orientation and Guide for Use of Metric Units," ASME, New York, N.Y., 1972.

I	Mass moment of inertia, $(kg \cdot m^2)$
k	Spring stiffness, $(\frac{N}{m})$
m	Mass, (kg)
$\bar{m}$	Moment in the rotating coordinate system, $(N \cdot m)$
$\bar{M}$	Total moment in the absolute coordinate system, $(N \cdot m)$
M	Bending moment state variable in the absolute coordinate system, $(N \cdot m)$
N	Normal force state variable in the absolute coordinate system, (N)
O	Point where line connecting axial centers of support bearings intersects the plane of a disk
$\tilde{P}$	Point transfer matrix
$\bar{r}$	Displacement vector from undeflected shaft center to either deflected shaft center or to deflected masses, (m)
$\bar{R}$	Whirl orbit in absolute coordinate system, (m)
S	Geometric center of unbalanced disk
t	Time, (sec)
T	Torque state variable, $(N \cdot m)$
u	Displacement state variable in X direction, (m)
v	Displacement state variable in Y direction, (m)
V	Shear force state variable in the absolute coordinate system, (N)
w	Displacement state variable in Z direction, (m)
x	Translational displacement state variable, (m)
$\bar{z}$	State vector

3. GREEK SYMBOLS

α	Phase-lag angle, (rad)
E	Eccentricity angle, (rad)
Θ	Rotation about Z axis, (rad)
ρ	Distance from deflected shaft center to center of each lumped mass, (m)
ϕ	Rotation about X axis, (rad)
ψ	Rotation about Y axis, (rad)
ω	Circular frequency, $(\frac{rad}{sec})$
Ω	Forcing circular frequency or shaft rotary circular rate, $(\frac{rad}{sec})$

4. SUPERSCRIPTS

L	Left side of imaginary cut
R	Right side of imaginary cut

5. SUBSCRIPTS

abs	Absolute coordinate system form
c	Cosine harmonic function indicator
g	Gravitational source
i	Discrete point indicator signalling the i^{th} element out of n finite elements
in	Inertial source
ℓ	Mass number indicator (1 or 2)
n	Natural circular frequency when used with ω

rot	Rotating coordinate system form
s	Sine harmonic function indicator
tot	Total vector sum of moments and forces in the rotating coordinate system
x,y,z	Rotating coordinate system state variable indicators
Y,Z	Absolute coordinate system state variable indicators

1. <u>INTRODUCTION</u>

1.1 THE PROBLEM

With the continuing demand for increased production, engineers have had to go to increased operating speeds for rotating equipment of all types. As a consequence of this increase in rotational speeds, the ability to accurately predict frequencies of vibrations becomes more important than ever before. This prediction capability is valuable to an engineer because he is expected to maintain a high degree of reliability in the machines he designs. Torsional fatigue failures, bearing fatigue failures, support fatigue failures, seal fatigue failures and shaft bending failures cause unscheduled shut-downs. These failures result in losses of thousands to tens of thousands of dollars a day and can often be traced to excessive amplitudes of vibrations. Continued advancements in high speed rotor system design technology will result in lower total lifetime costs by reducing the initial cost, the downtime cost and the operating cost of such rotating machinery.

The increase in rotating speeds creates a number of problems in rotor-bearing system dynamics. The first and second problems deal with potentially harmful phenomena. The first problem concerns the prediction of rotor-bearing system critical speeds. A critical speed is characterized by Eshleman [1]* as corresponding to resonant frequencies of the flexible rotor-bearing system. This resonant phenomenon

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*Numbers in brackets designate References at the end of the thesis.

is one in which the applied force reinforces the system's response by occurring at a natural frequency. A critical speed responds with a linearly increasing amplitude with time until limited by system damping. The form of equation (2-21) on page 35 in Pestel and Leckie [2] clearly verifies this statement. Along with the ability to accurately predict system critical speeds, engineers must also have the ability to control, or design around these system criticals.

The second problem area is that of stability. An oscillatory system is unstable if the envelope of the displacement increases monotonically with time. Also, a non-oscillatory system is unstable if the displacement amplitude increases monotonically with time. If the system is in a state of dynamic instability it is likely that serious damages will occur. It is possible that the condition may lead to the total destruction of a rotor-bearing system. Therefore, it is desirable to investigate the nature of the system's stability. The distinctions, then, between critical speeds and stability are that critical speeds occur at discrete values of operating speed and the associated amplitude increases linearly with time while instabilities occur over a range of operating speeds and their associated increase in amplitude is exponential in time.

The third problem involves the balancing of a rotor system. In the area of balancing, the shaft is normally assumed to be either rigid or flexible. A shaft is considered rigid when the bending stiffness of the shaft rotating at Ω radians per second is greater than the equivalent spring constant for the bearings. In such a case the rotor whirls in a cylindrical mode. Little shaft bending occurs, therefore it is a rigid rotor. As rotational speed increases there is an increase in the centrifugally induced whirl forces which must be balanced by equally larger resistive forces. Above the first critical two things can happen which increase the resistive force. The first possibility is that the total effect of the bearing deflection and the gyroscopic stiffening moments cause the increase in the resistive force. This causes a rigid rotor second critical to develop which has a double conical whirl mode shape. The second cause of an increase in the resistive force is that the rotor itself is sufficiently flexible that the inertia forces cause the shaft system as well as the bearings to deflect. In certain shaft bending situations the shaft whirls in a sinusoidal shape with little deflection at the bearings. In this case the bearings do not contribute significantly to the resistive force necessary to balance the inertia force. The balancing of these forces is provided by the forces caused by internal shaft deflection. This, then, is a case of flexible rotor whirl. Imbalance can have harmful effects on the system; premature shaft bending failure, excessive bearing wear, bearing babbit fatigue and possibly even bearing seizure are some of the consequences which may result from imbalance. In general, it can be stated that the more accurately balanced a rotor-bearing system is, the smaller the dynamic support reactions will be due to a decrease in vibrations. Ultimately this leads to longer and more reliable service from the machine system.

The fourth point of interest is signature analysis; that is, the use of frequency spectra to ascertain the operational characteristics of a system. A correlation between a spectral frequency component and mechanical motions or resonances can often reveal problems, either existing or impending, in the operation of a rotor system. With the aid of signature analysis and rotor tracking techniques engineers can diagnose and hopefully remedy problems which might otherwise have caused equipment failure resulting in serious outages. Diagnosis techniques are discussed in the papers by Mitchell and Lynch [3] and Kerfoot, Hauck and Palm [4]. In the paper by Kerfoot, et al. [4] additional information is provided on the application of current instrumentation for vibration monitoring and analysis.

This thesis is concerned with all four of the previously discussed problems relating to rotor-bearing system dynamics. More specifically this thesis utilizes an analytical modeling technique, called the lumped parameter or transfer-matrix technique, to analyze a rotor-bearing system.* In this method a complicated system is broken into component parts using simple elastic and dynamic properties which are then placed in matrix form. The lumped parameter model description of a rotating disk with an eccentricity is formulated within this thesis. This, together with other available lumped parameter models for bearings and rotors, allows one to fully analyze a rotor system; that is, critical speed, stability and response analysis can be performed utilizing the

*Modeling is the process, experimental or analytical, whereby the behavior of physical systems can be simulated.

techniques of Pestel and Leckie [5] and Pilkey [6]. Then balancing can be achieved using the whirl orbit technique as described by Jackson [7].

1.2 GENERAL TERMS

In order to develop a lumped parameter model for the eccentric disk it is necessary to fully describe, mathematically, the behavior of the disk. Several concepts must be introduced before proceeding with such a description.

The first behavioral item is termed precession and is defined as the rotation of the shaft center about the bearing centerline. This can be understood by visualizing the bow which occurs in a skipping rope. Precession of the shaft can be in the same or opposite direction to the rotation of the shaft. Also, the precessional rate may or may not be at the same speed as the shaft rotation. Nonsynchronous precession is the term given to the case where the shaft precession speed and the shaft rotational speeds are not equal. Synchronous precession, or whirling, is the term used to describe the case where the speeds are equal.

When describing rotor-bearing system dynamics, it is necessary to talk about three basic types of vibratory motion. These are, axial, torsional and lateral (transverse) vibrations of a bar, rod and beam, respectively. Axial vibrations correspond to linear extension and compression of a system. Torsional vibrations deal with the angle of twist and torques of a system. Lateral vibrations are associated with the plane flexural vibrations of a system, such as a beam. Lateral vibrations can occur in the two orthogonal planes not already described by axial and torsional motion. Another type of vibratory motion possible in rotor systems is called coupled vibrations. Coupled vibration means the possible mutual interaction of axial, torsional and lateral motion.

The intent of this work is to develop a coupled transfer matrix for a disk with an associated eccentricity. The eccentricity can be the result of either a geometrical problem or a material inhomogeneity problem.

2. LITERATURE REVIEW

Very little work has been reported in the literature on the subject of purely axial vibrations in rotor-bearing systems. One reason is that, unless large dynamic axial thrust loads are present designers simply ignore the possibility of problems due to axial vibration. If, however, it is felt that an axial analysis should be done, reasonably good results can be obtained using an introductory vibrations textbook such as that by Thomson [8] or Vierck [9] since axial vibrations can be analyzed as one would analyze a translational spring-mass system.

Loewy and Piarulli [10] assert that pure torsional vibrations can be considered whenever effects of lateral motion are absent, such as when a shaft is very stiff laterally or when there are enough bearing locations present to eliminate the possibility of lateral motion. For example, the usual practice in analyzing vibrations of engines (both turbine and reciprocating) and geared systems has been to consider pure torsional motion because the many bearings present eliminate the possibility of transverse vibrations. In their papers, Eshleman [11] and Pollard [12] deal with the problem of torsional vibrations in engines. Eshleman describes a technique to be used in the design, development and analysis of torsional vibrations in internal combustion engines. The object of this work was to determine the forced dynamic response of a torsional system. This was accomplished using information obtained from a natural frequency and mode shape analysis. Pollard gives a method of calculating transient torsional response of engines when accelerated through resonance. This method of analysis is also applicable to systems which inherently have no significant dynamic torque fluctuations, such as those systems driven by synchronous electric motors. Impulsive or transient torque functions applied by couplings or gears can be handled using Pollard's technique. Wang and Morse [13] present a solution technique for obtaining the static and dynamic torsional response of a general gear train system. In his follow-up paper Wang [14] extends the previous work to include the case of transient torsional analysis of gear connected systems.

When it is felt that no significant dynamic torsional forces are present, a rotor system can be treated as a purely lateral system. There is a wealth of material in the literature which deals with pure lateral vibrations. Ekong, Eshleman and Bonthrong [15] and Huang and Huang [16] describe methods of determining critical speeds of rotor systems. Both reports treat systems consisting of multiple disks on a rotor mounted on multiple bearing supports. Gunter [17] studies the stability of a rotor system where the rotor is undergoing precession, either synchronous or nonsynchronous. In this work Gunter analyzes a system having an axially symmetric shaft on an asymmetric bearing support while maintaining a constant total rotor angular speed. Yamamota and Ota [18] studied a similar configuration with the additional possibility of rotor asymmetry. The added asymmetry was found to lead to the possibility of large amplitudes of vibrations. In this paper it is concluded that from the standpoint of stability shaft asymmetry should always be avoided. Kirk and Gunter [19] present a theory whereby a transient response simulation for a multimass rotor system may be obtained.

While dealing with purely torsional vibrations Wang and Morse [13] found that at higher frequencies (above 425 Hertz) system response is no longer dominated by torsional motion but is now a complex coupling of transverse and torsional motion. Frequencies above 425 Hertz are now of interest because (1) new high speed equipment with speeds of 40,000 rpm and up are presently being planned for some industrial operations and (2) existing geared systems have tooth passing frequencies well above 425 Hertz, thus providing chances of large coupled interactions which will result in excessive gear noise. The same conclusion is reached by Loewy and Piarulli [10]. They state that the mutual interaction of lateral and torsional motion becomes important for speeds greater than the first critical speed. It is possible, in a coupled torsional-lateral system, to get torsional instabilities occurring, due primarily to lateral instabilities. There does not presently exist, as witnessed by Badgely and Hartman [20], the analytic capability to perform a complete coupled axial-torsional-lateral response analysis for rotor systems.

Readers interested in the general notion of shaft whirl and in stability of rotor systems are referred to E. J. Gunter's work [17]. Loewy and Piarulli [10] treat the subject of lateral motion in great detail. They review the literature and add their own insight into the topics of torsional-lateral coupling and balancing. The paper by Eshleman [1] is also an excellent survey. In it Eshleman describes the current flexible rotor technology. Included are the critical speeds and response of flexible rotor systems in axial, torsional and lateral motion. A survey of modeling and computational techniques for use with such systems is also included. These last three papers, [17], [10] and [1] all have extensive bibliographies included, which should be of tremendous value to those interested in the area of rotor system dynamics.

3. METHODOLOGY

3.1 TRANSFER-MATRIX INTRODUCTION

As previously mentioned the lumped parameter or transfer-matrix modeling technique is based on the idea of breaking up a complicated system into component parts. In order to evaluate the static, stability and dynamic response of a rotor system it is necessary to have a transfer-matrix representation of each type of element involved in the system. For example, in a simple rotor-bearing system, transfer matrices are needed to describe the bearings, the rotor and the rotating disk. Then only successive matrix multiplications, according to a predetermined set of rules, are necessary to give the required system response. Computer solutions are easily obtained because of the ease of coding the rules for matrix manipulation.

In order to familiarize the reader with the methods and terminology used herein, the following introduction to the transfer-matrix technique is given. The ensuing discussion follows that given by Pestel and Leckie [21] in their Chapter 3.

3.1.1 STATE VECTOR

Consider the simple spring-mass system shown in Fig. 1(a). The state vector at a point i is the column vector containing the displacement and corresponding internal forces at the point i. In the system shown in Fig. 1(a), the displacement is the linear displacement x_i and the corresponding internal force is the force N_i in the spring.

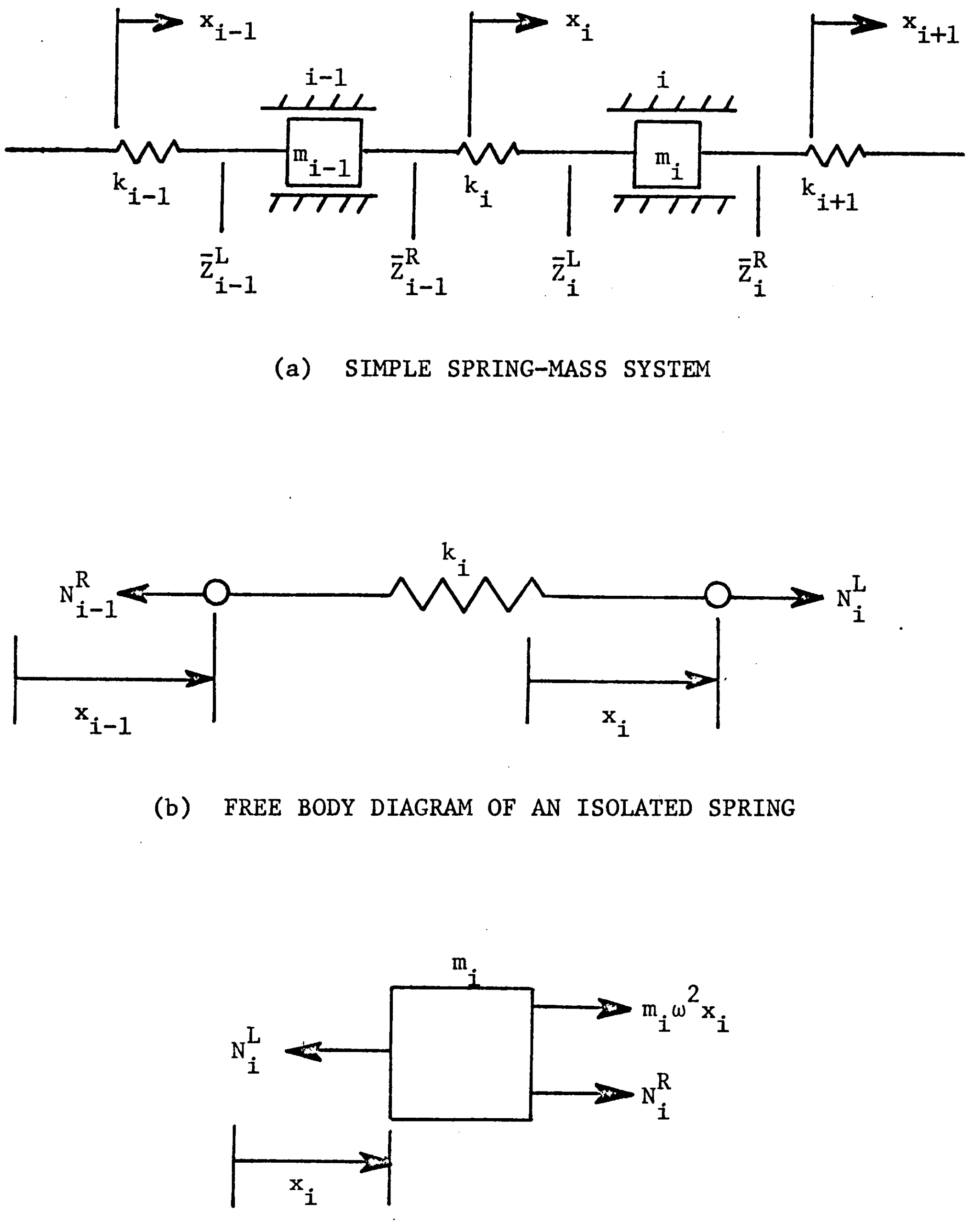


FIGURE 1. FINITE ELEMENT MODEL OF A SPRING-MASS SYSTEM

The displacement x_i and force N_i are known as state variables.

3.1.2 COORDINATE SYSTEM AND SIGN CONVENTION

Only right handed cartesian coordinate systems, with the x axis coinciding with the centroidal axis of the undeflected shaft, are used. A cut made through the elastic body will expose two forces. The force whose outward normal points in the direction of the positive x axis is termed the positive force with the other force then called the negative force. A positive displacement is coincident with the positive coordinate direction. Furthermore, a positive force can be seen on a positive (negative) face when the force vector acts in the positive (negative) coordinate direction. The right-hand rule is used in the determination of positive angular displacements and positive moments and torques.

3.1.3 FIELD TRANSFER MATRIX AND POINT TRANSFER MATRIX

Consider the system shown in Fig. 1(a) to be a finite element model of a uniform axial rod. If it is desired to do an axial vibration analysis one assumes harmonic motion at some circular frequency, ω. The massless spring k_i connects the two rigid masses m_{i-1} and m_i. This helps clarify how the term "lumped parameter" evolved. All the stiffness connecting two points on the axial rod is placed in a spring and all the mass is lumped into the two discrete masses at the ends of the spring. We denote the state vector to the left of mass m_i as $\bar{z}_i^L$ and the state vector on the right as $\bar{z}_i^R$. Refer now to Fig.

1(b), the isolated spring k_i.

From the equilibrium of the spring it is found that

$$N_{i-1}^{R} = N_{i}^{L} \tag{3-1}$$

The stiffness property of the spring results in

$$N_{i}^{L} = k_i(x_i^L - x_{i-1}^R) \tag{3-2}$$

Equations (3-1) and (3-2) can be rewritten as

$$N_{i}^{L} = N_{i-1}^{R}$$

and

$$x_{i}^{L} = x_{i-1}^{R} + \frac{N_i^L}{k_i}$$

These two equations can then be written in matrix form as

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_i^L = \begin{bmatrix} 1 & \frac{1}{k_i} \\ 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_{i-1}^R \tag{3-3}$$

or

$$\bar{z}_i^L = \tilde{F}_i \, \bar{z}_{i-1}^R$$

The matrix $\tilde{F}_i$, known as the field transfer matrix (or field matrix), expresses the state vector $\bar{z}_i^L$ in terms of the state vector $\bar{z}_{i-1}^R$.

Consider now the mass m_i as shown in Fig. 1(c). The mass is rigid, therefore

$$x_i^L = x_i^R \tag{3-4}$$

In addition to the two forces N_i^L and N_i^R is the inertia force $m_i\omega^2 x_i^R$. From the equilibrium of forces is found

$$N_i^L = N_i^R + m_i\,\omega^2\,x_i^R \tag{3-5}$$

Equations (3-4) and (3-5) can be rewritten in matrix form as

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 \\ -m_i\omega^2 & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_i^L \tag{3-6}$$

or

$$\bar{z}_i^R = \tilde{P}_i\,\bar{z}_i^L$$

The matrix $\tilde{P}_i$ relates the two state variables $\bar{z}_i^R$ and $\bar{z}_i^L$ and is known as the point transfer matrix (point matrix).

3.1.4 EXAMPLE

As a very simple example of the use of transfer matrices, consider the spring-mass system of Fig. 2.

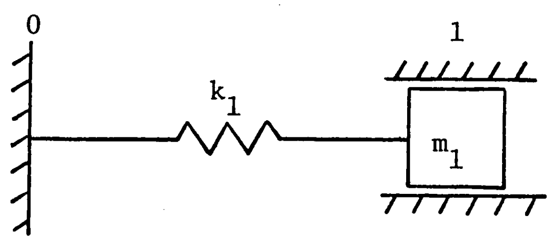


FIGURE 2. SINGLE-DEGREE-OF-FREEDOM SYSTEM

From equations (3-3) and (3-6) we obtain

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^L = \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_1 \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_0^R \quad \text{and} \quad \begin{Bmatrix} x \\ N \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_1 \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_1^L$$

Substituting the first equation into the second equation results in

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^R = \begin{bmatrix} 1 & 0 \\ -m\omega^2 & 1 \end{bmatrix}_1 \cdot \begin{bmatrix} 1 & \frac{1}{k} \\ 0 & 1 \end{bmatrix}_1 \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_0^R$$

or

$$\begin{Bmatrix} x \\ N \end{Bmatrix}_1^R = \begin{bmatrix} 1 & \frac{1}{k} \\ -m\omega^2 & 1-\frac{m\omega^2}{k} \end{bmatrix}_1 \cdot \begin{Bmatrix} x \\ N \end{Bmatrix}_0^R$$

From the boundary conditions $x_0 = 0$ and $N_1^R = 0$ the equations reduce to

$$x_1 = \frac{N_0^R}{k_1}$$

and

$$0 = (1 - \frac{m_1\omega^2}{k_1})N_0^R \quad .$$

The second equation is satisfied when $1 - \frac{m_1\omega^2}{k_1} = 0$, which results in the familiar relation for natural circular frequency $\omega_n = \sqrt{\frac{k_1}{m_1}}$.

3.1.5 GENERAL COMMENTS

Linear spring-mass systems are not the only type of mechanical systems able to be treated using the transfer-matrix approach. Two other cases often considered are the torsion system and the straight beam system. The torsion system requires the consideration of two new state variables; the displacement is ϕ_i, the angle of twist and the corresponding internal force is the torque T_i. The case of a simple straight beam able to bend in one plane requires the introduction of four new state variables. The two displacements at point i are the linear displacement w_i and the slope ψ_i. The corresponding internal forces are the shear force V_i and the moment M_i.

In order to take into account the possible presence of external forces it is possible to enlarge the transfer matrix via the addition of one row and one column. An expanded matrix of this type is called an extended transfer matrix.

One final point of great importance to the work in this paper is that in any one row of a transfer matrix there is only one harmonic function ($\sin\Omega t$, $\cos\Omega t$, $\sin 2\Omega t$, etc.) multiplying each element of the row. For ease of presentation in matrix form the harmonic function is cancelled out throughout the row. It is important to also note that in each transfer matrix only one value of the argument (Ωt, $2\Omega t$, $3\Omega t$, etc.) of the harmonic function is used throughout.

3.2 INTRODUCTION TO SYSTEM

3.2.1 ASSUMPTIONS

In the following derivation of the point transfer matrix for a whirling eccentric mass some assumptions are made. The eccentric mass can be two things; first, it can be the lumped mass associated with the model of the shaft and second, it can be a disk which has been placed on the shaft at some point.

The first and probably most important assumption made is the limitation to small displacements, both linear and angular, from the axis of rotation. The restriction to small amplitudes of motion has two very important effects to the solution of our problem. The first effect of this limitation is to take the equations of motion out of the nonlinear range and put them into the linear form. In most cases a closed analytical solution to the equations of motion is sought and often nonlinear equations are not readily solved. Linear equations are, however, readily solvable in closed form. The limitation to linear equations of motion allows one to use the principle of superposition. The importance of this principle will not be realized until the end of this work, therefore further discussion will be delayed.

In order to utilize the assumption of small displacements in the derivation that follows, any terms which are the product of two or more state variables are eliminated. This procedure is justified because a product of at least two small numbers is very small when compared to any individual state variable.

The rotational speed and the precessional speed of the rotor are both assumed to be constant and to occur in only one plane. Furthermore, these two speeds are considered to be equal. This assumption of equal speeds of rotor rotation and rotor precession limits our derivation to only the case of synchronous precession or whirl.

We assume that the shaft is supported by anisotropic bearings. Therefore, individual bearings may have different spring and damper properties in all directions normal to the axis of the shaft. The possibility also exists for the bearings to have cross-coupled stiffness and damping coefficients. For example, in the case of cross-coupled stiffness properties, a bearing when deflected in one direction can respond in a direction perpendicular to the applied force.

Another assumption made is that the materials involved in the rotor system exhibit elastic behavior, that is the material is able to regain completely its original dimensions upon removal of an external force. As previously stated, elastic properties form one of the tools needed to model a system when using the lumped parameter technique.

3.2.2 COORDINATE SYSTEMS AND STATE VARIABLES

There are two right-handed cartesian coordinate systems used in this derivation. The first system is fixed or stationary with respect to ground and the other system is rotating with the shaft at an angular speed Ω.

The X axis of the fixed coordinate system is coincident with the undeflected shaft center and with the line connecting the center of the support bearings. The Y and Z axes are then mutually perpendicular to the X axis with the Z axis directed along and in the direction of gravity acceleration. This system of fixed coordinates would normally contain rotor displacement sensors and other monitors of the rotor system's condition.

The placement of the rotating coordinate system is chosen in agreement with the assumption that the rotor whirls in only one plane, the Y-Z plane. The rotating x axis is placed coincident with the stationary X axis. The rotating y and z axes are then oriented at right angles to themselves and to the rotating x axis. By working in this coordinate system one gains insight into the rotor whirl problem.

As will be discussed in the next section, the geometry of the rotor system will be described in the rotating coordinate reference frame.

It is necessary to define two sets of unit vectors, one in the fixed reference frame and one in the rotating reference frame, in order to use vector mechanics in the solution process. The unit vectors in the fixed coordinate system are $\bar{I}$, $\bar{J}$, $\bar{K}$ and are directed along the positive fixed X, Y and Z coordinate axes, respectively. The unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ in the rotating coordinate system are directed along the positive rotating x, y and z coordinate axes, respectively. In order to get the results out of one coordinate system and into the other the following coordinate transformation equations are necessary.

$$\begin{aligned} \bar{I} &= \bar{i} \\ \bar{J} &= \bar{j} \cos \Omega t - \bar{k} \sin \Omega t \\ \bar{K} &= \bar{j} \sin \Omega t + \bar{k} \cos \Omega t \end{aligned} \tag{3-7}$$

$$\begin{aligned} \bar{i} &= \bar{I} \\ \bar{j} &= \bar{J} \cos \Omega t + \bar{K} \sin \Omega t \\ \bar{k} &= -\bar{J} \sin \Omega t + \bar{K} \cos \Omega t \end{aligned} \tag{3-8}$$

Equation (3-7) allows transforming from the rotating reference frame to the fixed reference frame and equation (3-8) is used to transform from the fixed reference frame to the rotating reference frame.

In order to allow for general motion of the disk one expands the state variable description to include axial motion, torsional motion and lateral motion in two perpendicular directions. As described above, the X axis is placed coincident with the undeflected centroidal axis of the disk. Therefore, axial motion takes place along the X axis and torsional motion occurs in the Y-Z plane. Bending, then, can take place in both the X-Y plane and the X-Z plane. This requires subscripts to be used to distinguish between the two beam bending directions.

The state variables used to describe axial motion are u and N, the displacement and internal force, respectively. The angle of twist ϕ and the torque T are the state variables which describe torsional motion. In the X-Y bending plane the state variables v, Θ, V_Y and M_Z are used to describe the linear and angular displacements and the associated internal forces, shear and moment, respectively. The state variables w, ψ, V_Z and M_Y are the linear and angular displacements and the corresponding internal forces, shear and moment, respectively in the X-Z bending plane. In order to be consistent with Pestel and Leckie [22] a sign change must be made to two of the bending state variables above; the shear in the Y direction, V_Y, and the linear displacement w in the Z direction are both made negative state variables. This helps generate positive matrix entries.

3.2.3 SYSTEM GEOMETRY

The first consideration of the lumped parameter technique is how to model the physical system. In this thesis the disk is treated as a lumped mass which has been split into two equal halves each placed at a distance from the mass center of the disk equal to i_g, the radius of gyration of the disk. The radius of gyration, i_g, is used as a general indicator. Specific meaning will be given during the course of the derivation for each use of this symbol. The eccentricity, e, is then measured from the mass center to the geometric center of the disk. Both the distances i_g and e are shown in Fig. 3. If there are more than one disk present or if the rotor is being modeled as lumped masses at discrete points, the eccentricity is defined at each discrete point whether it be for a disk or for a rotor lumped mass.

The angle E_i is used to define the position of the eccentricity at a discrete point i along the shaft. The angle is measured in the rotating coordinate system from the positive y axis in the positive direction about the x axis. Before measuring the eccentricity angle E_i it is important to define a reference position of the shaft discontinuity (notch, hole, pin, keyway, etc.) which either already exists or can be created. To have the shaft in its reference position all that is necessary is to place the key phasor along the positive y axis. This description of the reference position of the shaft is chosen in order to conform to the experimental technique described by Jackson [7]. Now, one will be able to verify the analytical results by Jackson's experimental method.

Figure 3 shows a view of the disk with an eccentricity located by an angle E_i as defined above. The system is assumed to lie solely in the y-z plane. Note the lengths a, b, c and d shown in this figure as they will be used extensively in the derivation of the point transfer matrix.

A normal procedure adopted when using the transfer-matrix tech-

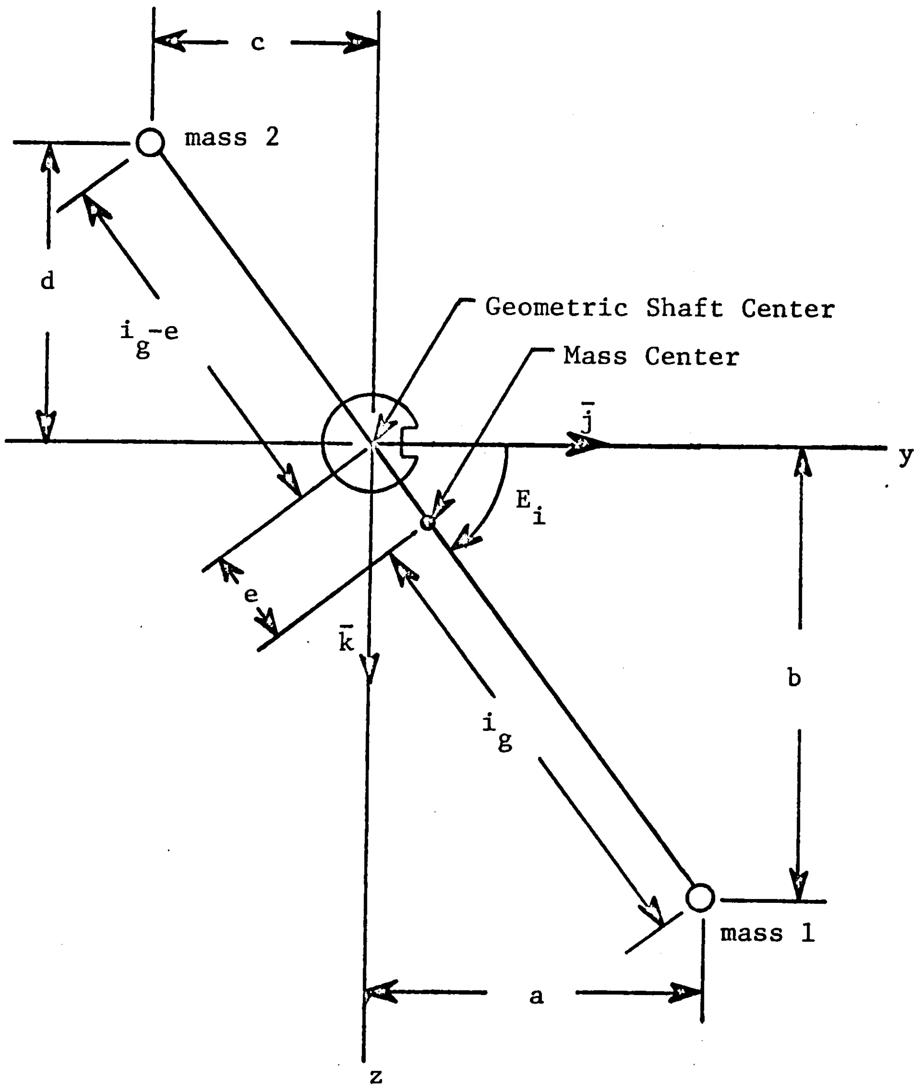
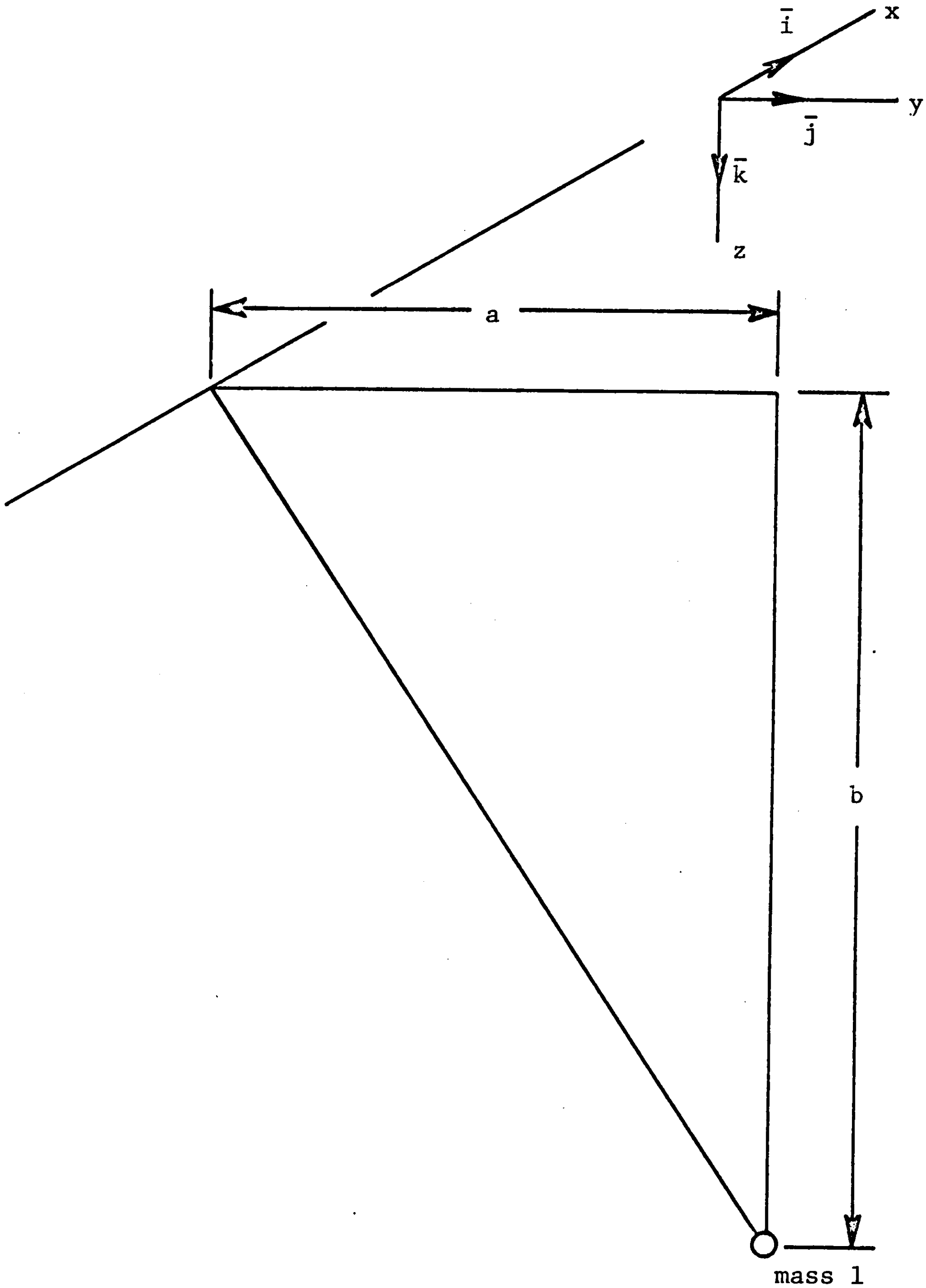


FIGURE 3. EQUIVALENT MODEL OF A DISK WITH AN ECCENTRIC MASS--BEFORE DEFLECTIONS

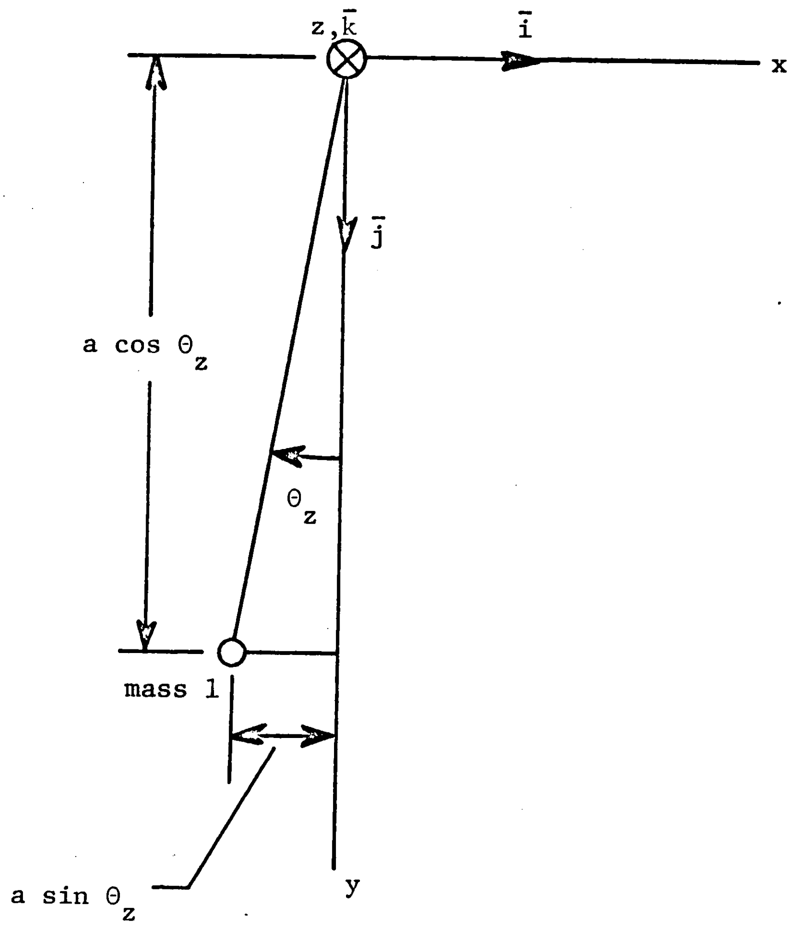
nique is to displace a system a small amount, both linearly and angularly, in the positive coordinate directions. This is a part of the procedure used to derive the equations of motion of any system. The angular displacements are done in the rotating coordinate system. In order to distinguish between the angular displacements in the fixed and moving reference frames some new subscripts will be introduced. The angular rotations about the x, y and z moving coordinate axes will be given the symbols ϕ_x, ψ_y and Θ_z, respectively. When the state variables ϕ, ψ and Θ are given without subscripts it should be assumed that they represent angular displacements about the fixed X, Y and Z coordinate axes, respectively. The order in which these displacements are performed is not important when small angles are considered. The linear displacements of the shaft center away from the undeflected shaft position are defined by the whirl orbit in the fixed coordinate system and will be discussed in the next section.

The angular displacements of the two masses about the shaft center are done in the following sequence: First the masses are rotated from their undeflected positions an amount Θ_z about the z axis, next the deflected masses are turned an amount ψ_y about the y axis and finally the masses are rotated through an angle ϕ_x about the x axis. Figure 4 presents a step by step representation of the above procedure as applied to mass one only. However, the same method can be used to find the deflected position of mass two. For convenience and ease of presentation the angle ϕ_i has been set equal to zero in Fig. 4. Also, only the plane wherein the specific angular displacement Θ_z, ψ_y or ϕ_x has an effect is



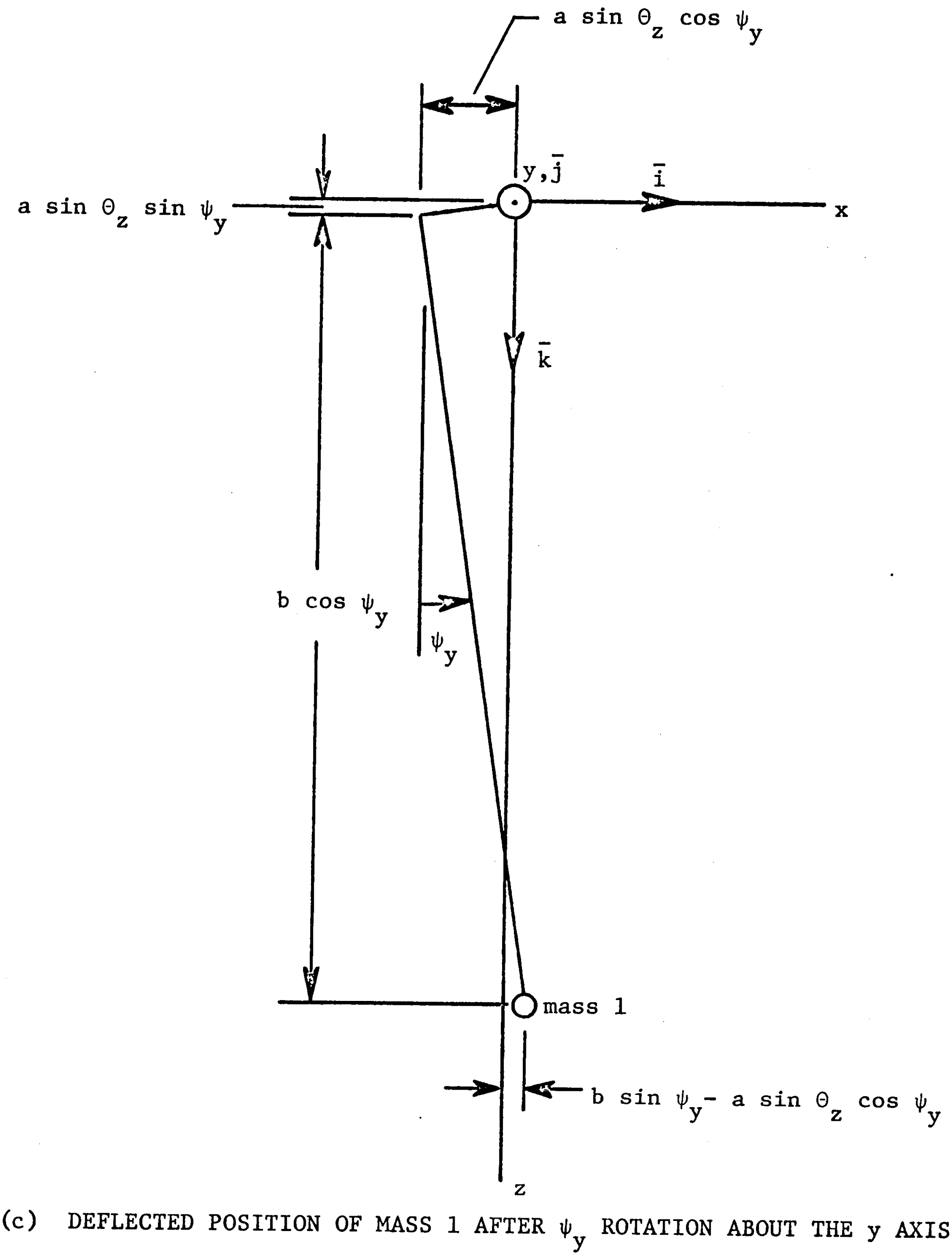
(a) UNDEFLECTED POSITION OF MASS 1

FIGURE 4. ANGULAR ROTATION SEQUENCE



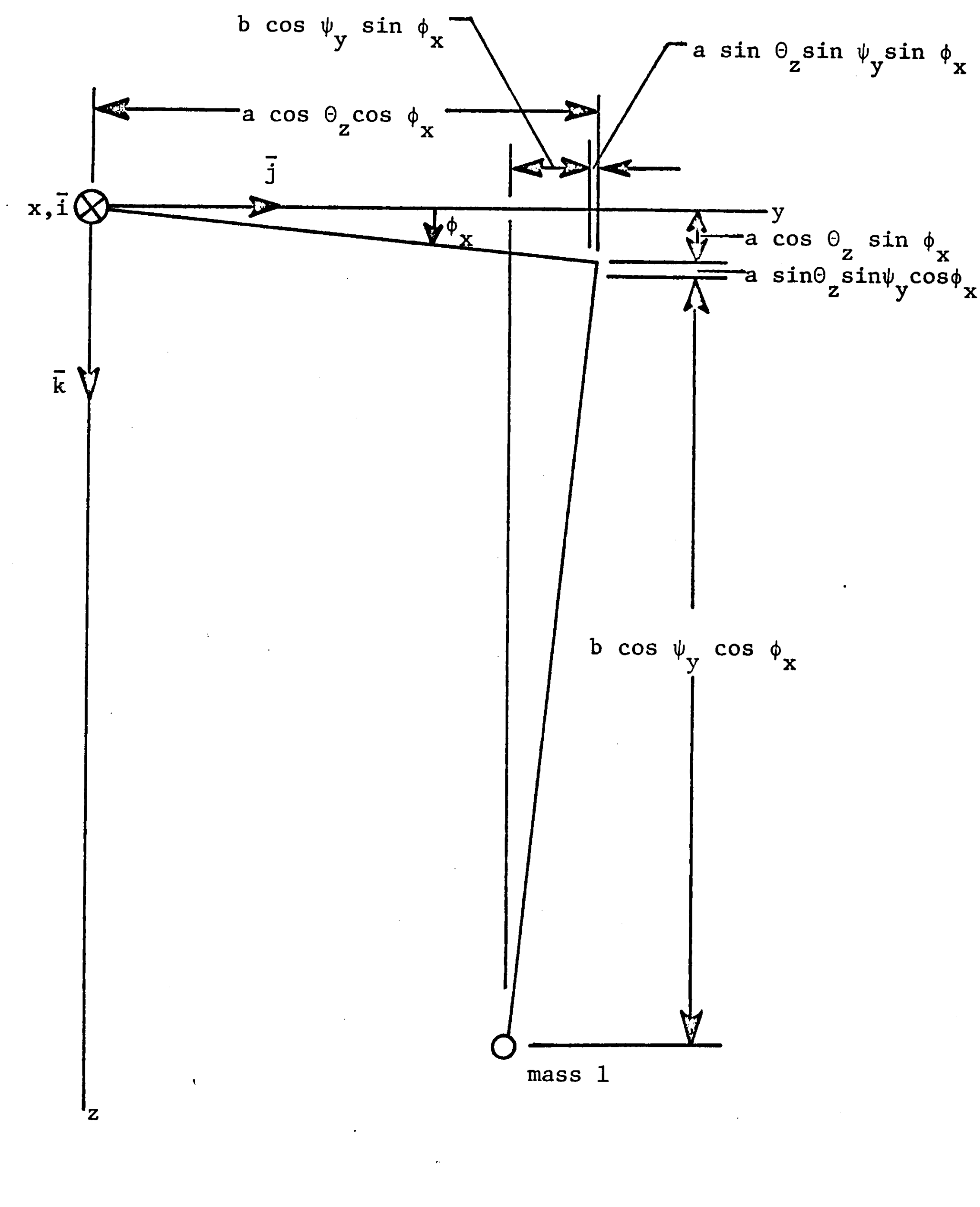
(b) DEFLECTED POSITION OF MASS 1 AFTER Θ_z ROTATION ABOUT THE z AXIS

FIGURE 4. ANGULAR ROTATION SEQUENCE



(c) DEFLECTED POSITION OF MASS 1 AFTER ψ_y ROTATION ABOUT THE y AXIS

FIGURE 4. ANGULAR ROTATION SEQUENCE



(d) DEFLECTED POSITION OF MASS 1 AFTER ϕ_x ROTATION ABOUT THE x AXIS

FIGURE 4. ANGULAR ROTATION SEQUENCE

shown. For example, Fig. 4(b) shows only the x-y plane, as no changes in the z components of the mass position vector occur as a result of the Θ_z rotation.

Plane views of both of the deflected masses are given in Figs. 5, 6 and 7 for the x-y, x-z and y-z planes, respectively. The lengths ρ_{1x}, ρ_{1y} and ρ_{1z} shown in Figs. 5, 6 and 7 are the total x, y and z components, respectively, of the vector $\bar{\rho}_1$. This vector gives the distance, as measured in the rotating coordinate system, from the center of the shaft to the center of mass one. The vector $\bar{\rho}_2$ and its components ρ_{2x}, ρ_{2y} and ρ_{2z} have similar meanings with respect to the position of mass two.

In order to determine the component parts of vectors $\bar{\rho}_1$ and $\bar{\rho}_2$ it is necessary to refer to the plane views of the two masses given in Figs. 5, 6 and 7. For example, the component length ρ_{1x} is the distance from the shaft center to the center of mass one as measured parallel to the x axis in either the x-y or x-z planes. The total length ρ_{1x} is the vector sum of the x components, $a \sin \Theta \cos \psi$ and $b \sin \psi$. The remaining total component lengths, ρ_{1y}, ρ_{1z}, ρ_{2x}, ρ_{2y} and ρ_{2z} are also vector sums and are found in a similar manner.

For further clarification of the now deflected eccentric mass a view of the disk, with the key phasor shown, in the y-z plane is presented as Fig. 8.

3.2.4 SYSTEM MOTION

It can be concluded from the previous assumption of anisotropic

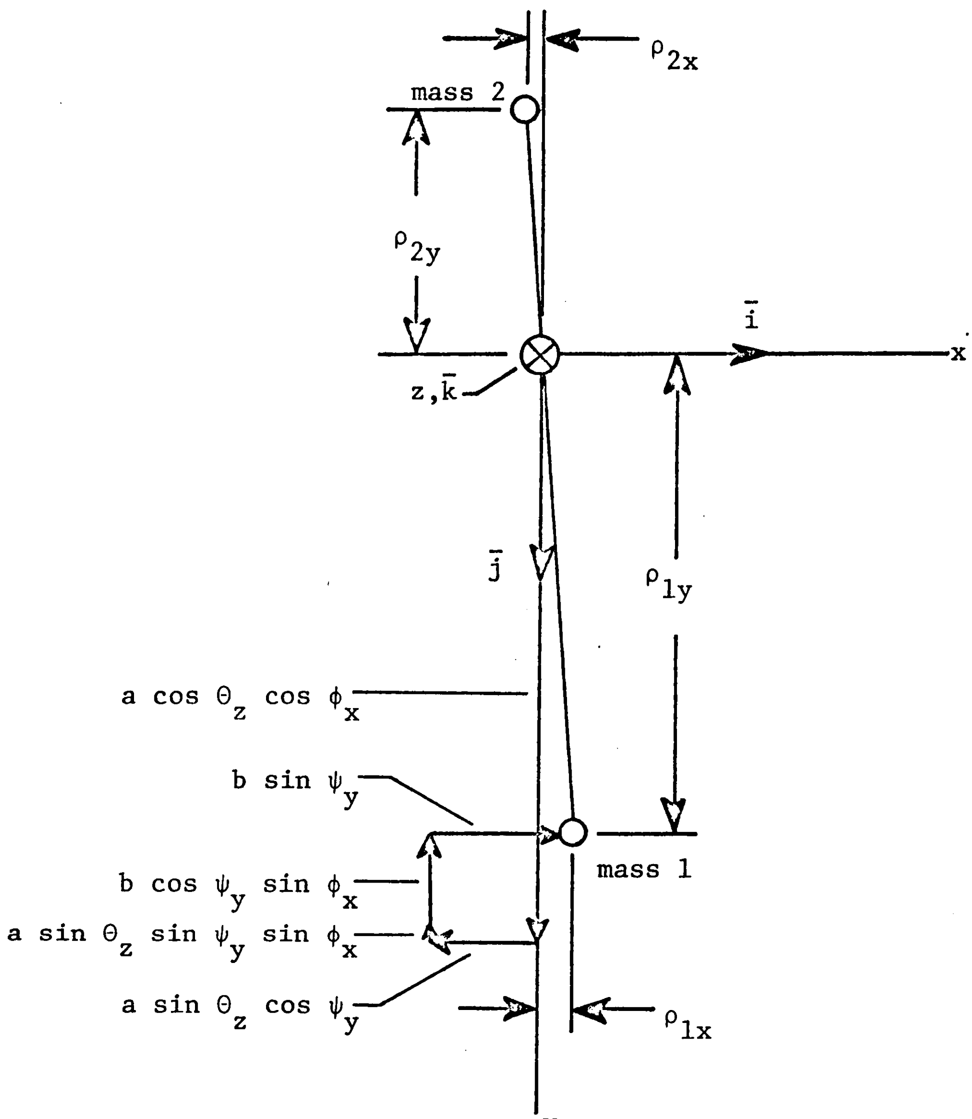


FIGURE 5. x-y PLANE VIEW--AFTER ALL DEFLECTIONS

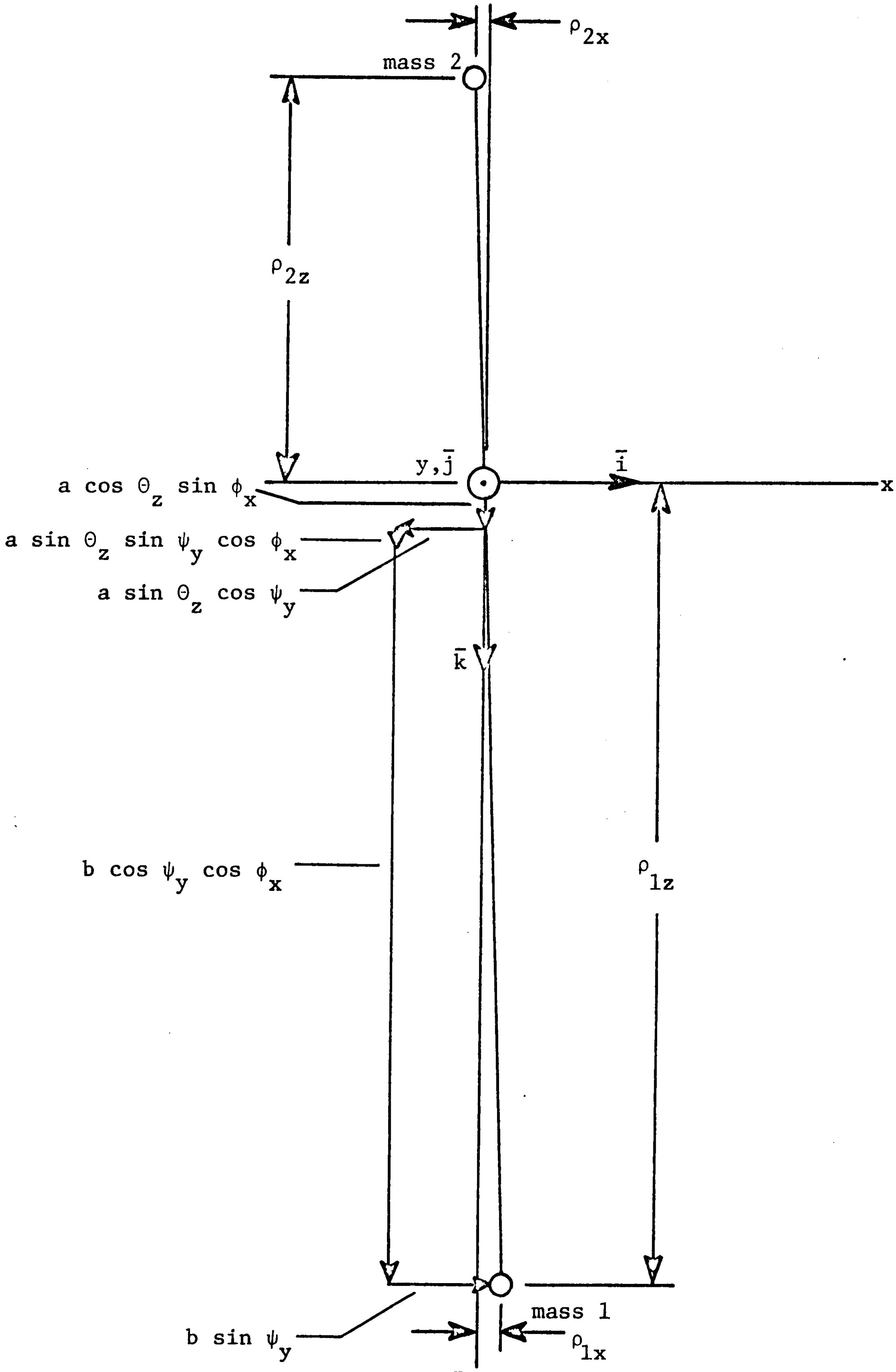


FIGURE 6. x-z PLANE VIEW--AFTER ALL DEFLECTIONS

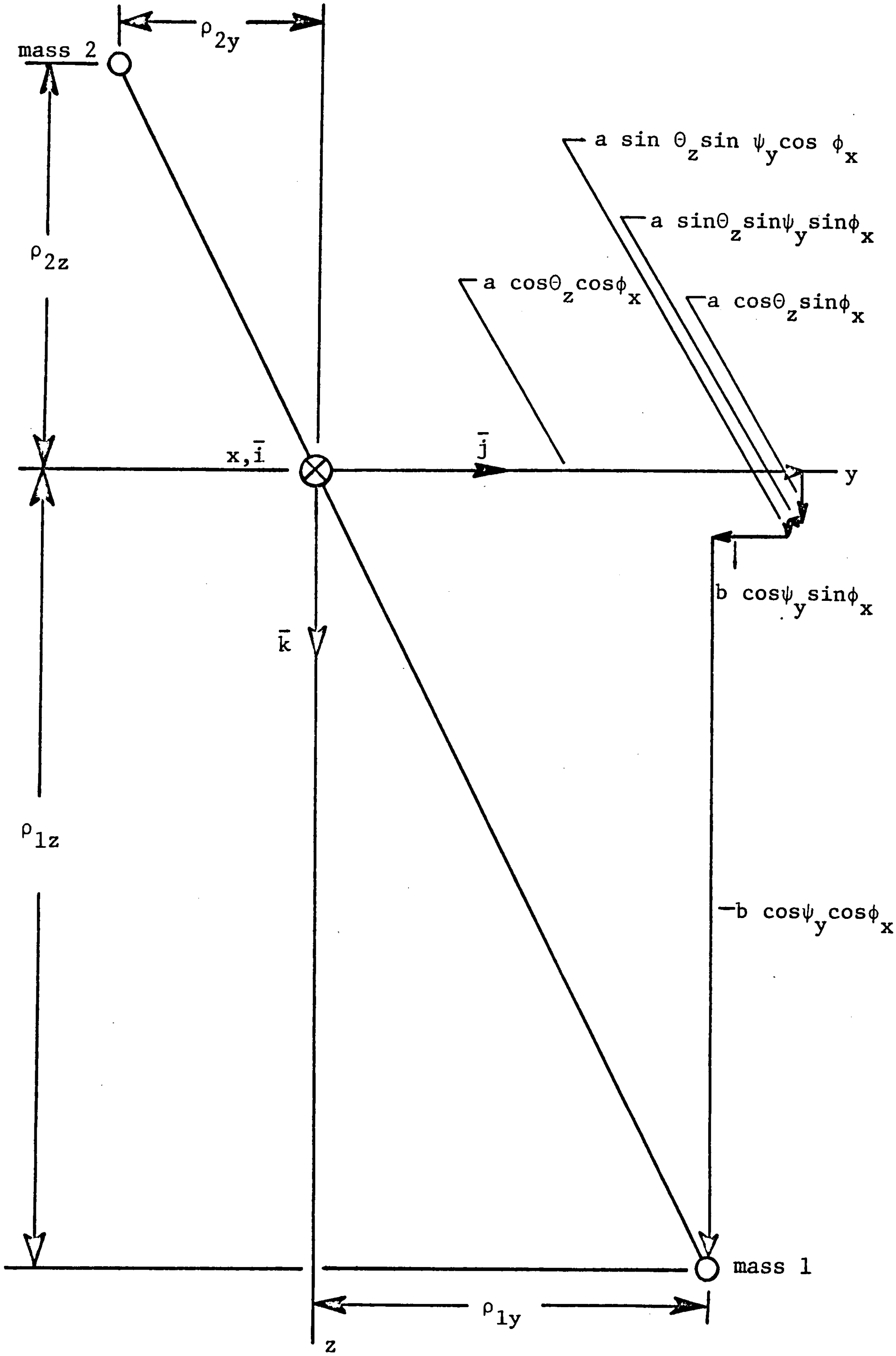


FIGURE 7. y-z PLANE VIEW--AFTER ALL DEFLECTIONS

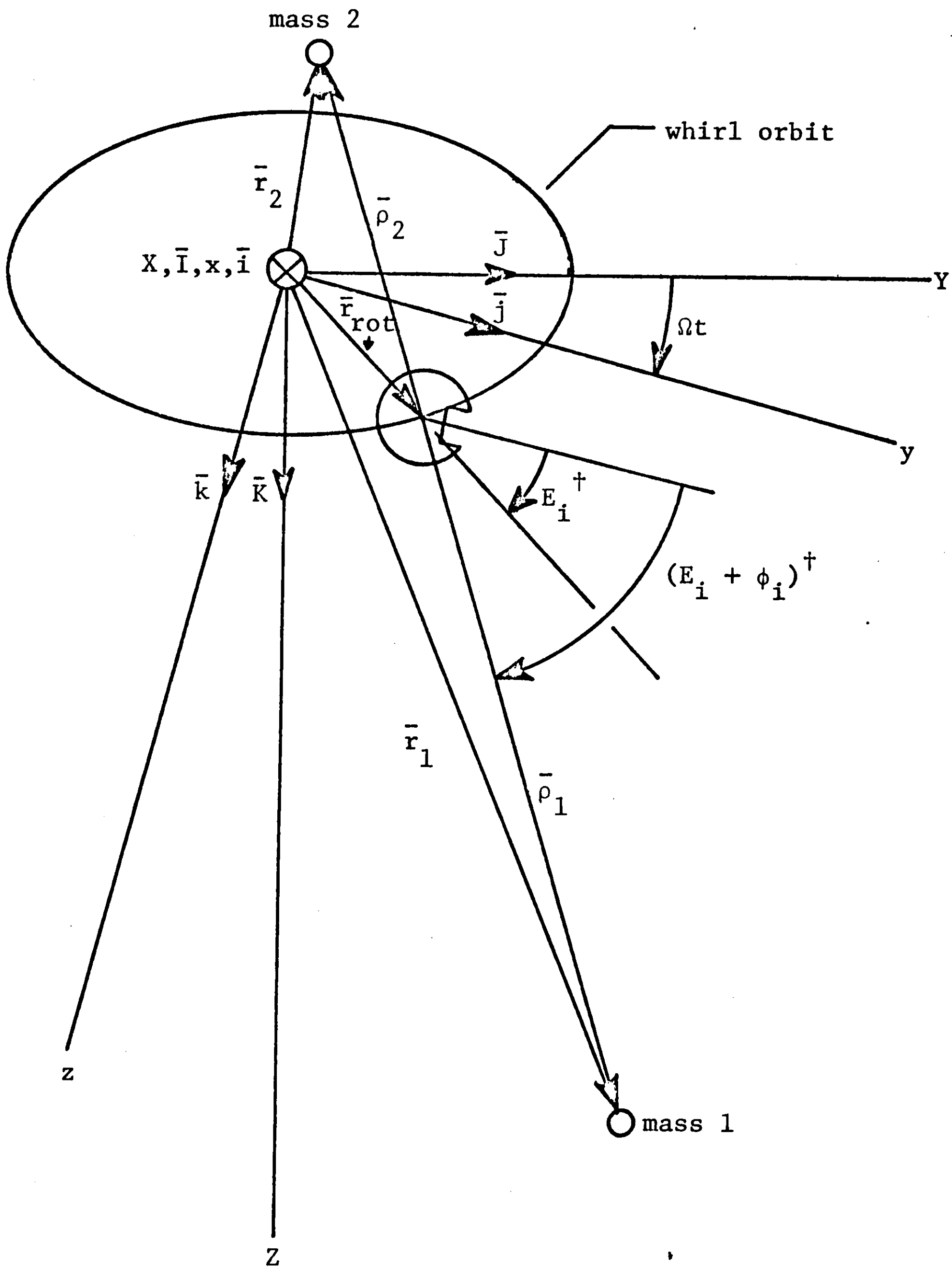


FIGURE 8. EQUIVALENT MODEL OF A DISK WITH AN ECCENTRIC MASS--AFTER ALL DEFLECTIONS

† E_i and $(E_i + \phi_i)$ are actually the y-z view of the true E_i and $(E_i + \phi_i)$ angles.

bearing supports, that the shaft center at each discrete point along the shaft whirls in some elliptic path as viewed in the Y-Z plane of the stationary reference frame. Since the two masses may also have axial motion along the x axis, the total description of the whirl orbit is known as an ellipsoid. The dimensions of the whirl path may vary along the length of the rotor. The whirl orbit, of a discrete point along the rotor, represents the linear displacements of the shaft center in the fixed X, Y and Z directions as discussed in section 3.2.3. To show, as done in Figures 5, 6 and 7, these deflections in the rotating coordinate system a coordinate transformation must be performed.

In order to represent the total motion of the rotor as closely as possible, we allow each state variable to exist as the coefficient of a harmonic function of time. The form of each state variable used in the derivation of our point transfer matrix will be further explained in the next section.

3.3 STATE VARIABLE REPRESENTATION

We now wish to describe the form of each state variable used in the matrix derivation. This is done in the stationary coordinate system.

The following derivation of the response of a rotor undergoing whirl is taken from section 3.4 of Thomson [23]. Assume an idealized system in the fixed reference frame consisting of a disk of mass m located symmetrically on a shaft which is supported by two bearings.

The center of mass G of the disk is at a radial distance e from the geometric center S of the disk. The line connecting the axial centers of the bearings intersects the plane of the disk at 0 and the shaft geometric center is deflected a distance OS due to deflection of the bearings. Figure 9 illustrates the geometry of the system as described above. The coordinates of the shaft center are v and w and the coordinates of the mass center are $(v + e \cos \omega t)$ and $(w + e \sin \omega t)$. The equations of motion in the Y and Z directions assume viscous damping proportional to the velocity of the shaft center S. This is equivalent to saying that the bearings develop viscous damping forces as the rotor whirls. Since the rotor is whirling in an orbit with a constant radius OS it should be clear that the shaft can only sustain a mean bend and follow the mass unbalance in synchronous whirl. If OS represented shaft bend, as suggested by Thomson [23] in his derivation, then no damping could be included in the model because no radial component of velocity would exist. However, for the purpose of this derivation it is assumed that the deflections v and w are totally taken in the support bearings. Since the bearings are connected to the fixed coordinate system they will see dynamic velocities $\dot{v}$ and $\dot{w}$. This is not made clear in Thomson's work [23]. Thus the equations of motion are

$$m\ddot{v} + c\dot{v} + kv = me\omega^2 \cos \omega t$$

$$m\ddot{w} + c\dot{w} + kw = me\omega^2 \sin \omega t$$

The solutions of these equations are

$$v = R \cos (\omega t - \alpha)$$

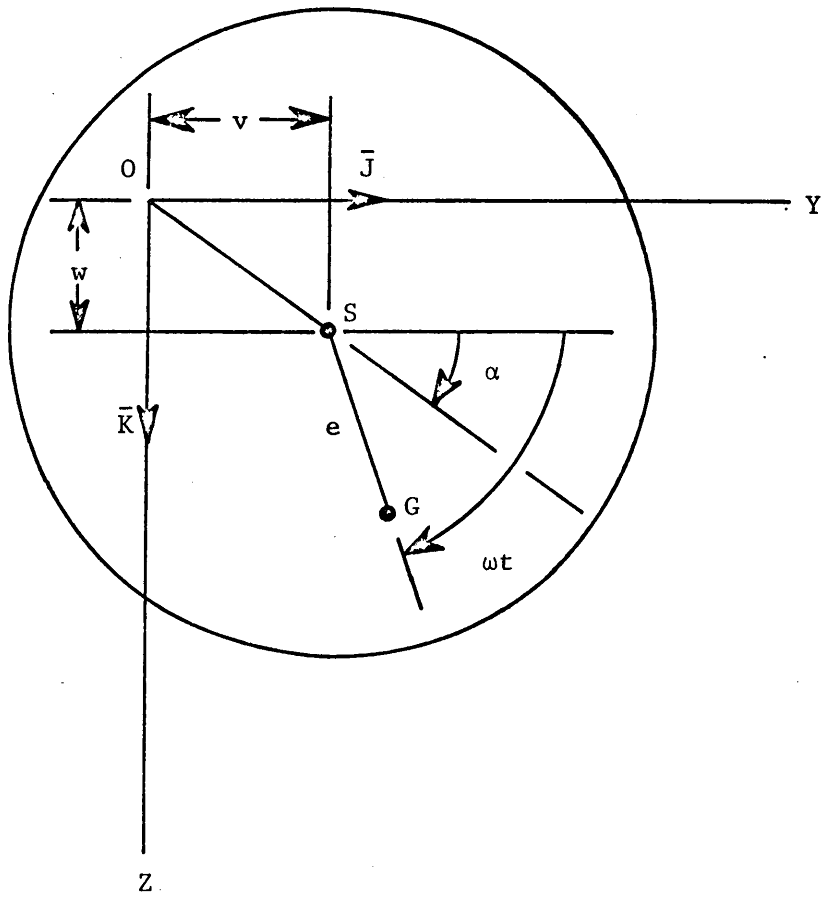


FIGURE 9. WHIRL OF A SHAFT DUE TO MASS UNBALANCE

AFTER THOMSON [23]

$$w = R \sin (\omega t - \alpha)$$

where

$$R = \frac{me\omega^2}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}, \text{ a constant at any speed } \omega$$

These can be rewritten using the angle-difference trigonometric relations to get

$$v = R \cos \alpha \cos \omega t + R \sin \alpha \sin \omega t$$

$$w = -R \sin \alpha \cos \omega t + R \cos \alpha \sin \omega t$$

These equations can then be rewritten as

$$v = v_c \cos \omega t + v_s \sin \omega t \tag{3-9}$$

$$w = w_c \cos \omega t + w_s \sin \omega t \tag{3-10}$$

where

$$v_c = R \cos \alpha \quad , \quad v_s = R \sin \alpha$$

$$w_c = -R \sin \alpha \ , \quad w_s = R \cos \alpha$$

The subscripts c and s refer to the type of sinusoidal function which follows, cosine and sine, respectively.

Equations (3-9) and (3-10) present the total form for the linear displacement beam state variables in the fixed coordinate system. It is now necessary to describe the other beam state variables in similar form. To do this, use is made of the first order differential equations relating the displacement, slope, moment and shear of a beam. Because these differential equations are independent of time the form of all of the beam state variables is exactly the same as shown in equations (3-9) and (3-10). Lund [24] finds this same form of the state variables of bending in his paper.

The form of the state variables of axial and torsional motion can be found quite easily. Vierck [25] solves the differential equations of motion for single-degree-of-freedom axial and torsional systems in section 2-3 and section 2-7, respectively.

The solution of the axial case has the form

$$u = C \sin \alpha \cos \omega t + C \cos \alpha \sin \omega t$$

which can be rewritten as

$$u = u_c \cos \omega t + u_s \sin \omega t \tag{3-11}$$

The solution given by Vierck for the torsional case is

$$\phi = D \sin \alpha \cos \omega t + D \cos \alpha \sin \omega t$$

or

$$\phi = \phi_c \cos \omega t + \phi_s \sin \omega t \tag{3-12}$$

Use is then made of time independent differential equations relating the state variables u and N as well as ϕ and T. The form of the axial force N and torque T state variables solution is the same as those given in equations (3-11) and (3-12), respectively.

We now have a complete representation of each state variable solution obtained from simple physical models and differential equations of elasticity. There are now two each of these original twelve state variables discussed in section 3.2.2. The transfer matrix sought, then should have twenty-four (24) rows and 24 columns. If, in addition, one allows for the presence of external forces, moments and torques one expects a transfer matrix whose dimensions are 25 by 25.

3.4 GENERAL PROCEDURE

After thoroughly describing, mathematically, the physical system involved, vector mechanics is used in the derivation of the point transfer matrix. In order to gain insight into the problem of an eccentric mass undergoing whirl one chooses to perform the derivation in the rotating coordinate system. Therefore, a final coordinate transformation is needed to get the equations into the fixed reference frame where they can ultimately be put into transfer matrix form. All of the state variables are those in the fixed coordinate system.

The following discussion of the procedure used is given as an aid to the reader. It is hoped that only minimal discussion of procedure will be necessary during the course of the derivation given in section 4.

The first step in the derivation is the development of displacement vectors $\bar{r}_1$ and $\bar{r}_2$ describing the deflected position of each mass relative to the undeflected shaft center. These vectors are split into two vector parts as shown in Fig. 8. The first part is the same for each mass. It gives the distance of the whirling shaft center from the undeflected shaft center and is termed $\bar{r}_{rot}$. This vector, $\bar{r}_{rot}$, is the description in the rotating reference frame of the whirl orbit. It is obtained by a coordinate transformation from the assumed form of the whirl orbit, $\bar{R}_{abs}$, in the fixed coordinate system. This form of the whirl orbit, $\bar{R}_{abs}$, comes from the combination of the form of the linear state variables, u, v and w. The second part of the vectors $\bar{r}_1$ and $\bar{r}_2$ gives the distance from the whirling shaft center to masses 1 and 2, respectively. The second portion of the vectors is given the corresponding symbol $\bar{\rho}_1$ and $\bar{\rho}_2$. They are formed, as previously discussed in section 3.2.3, from the plane views shown in Figs. 5, 6 and 7. Now, the vector sum of $\bar{r}_{rot}$ and $\bar{\rho}_1$ (or $\bar{\rho}_2$) results in the total displacement vector $\bar{r}_1$ (or $\bar{r}_2$).

The next step is the development of the forces acting on the masses. In this derivation only the forces due to inertia and gravity are considered. The inertia force of each mass is found by multiplying the mass by the negative of the mass acceleration. The mass of each lumped mass is known to be one-half the total of the lumped mass at that station. The acceleration of the mass is found by differentiating the displacement vector twice with respect to time. This is done separately for each mass because the displacement vectors will not be the same owing to the eccentricity. By working in the rotating coordinate system one will be able to identify the different types of accelerations; centripetal acceleration, acceleration relative to the rotating reference frame and Coriolis acceleration. It should be realized that the accelerations and ultimately the inertial forces could also be found in the absolute reference frame if one chose to do so, as described by Loewy and Piarulli [26]. The acceleration due to gravity is known to act on each mass. It is also known to be directed along the positive Z axis of the fixed coordinate system. Therefore, it is necessary to transform this acceleration into rotating coordinates (given the symbol $\bar{g}_{rot}$) and multiply by the appropriate mass. Thus the inertial force and the force due to gravity acting on each mass are now known. These forces must be summed to give the total force $\bar{f}_1$ and $\bar{f}_2$, in the rotating reference frame, acting on each mass. The process for finding $\bar{f}_1$ and $\bar{f}_2$ is written in equation form as

$$\bar{f}_1 = -\frac{m}{2}\ddot{\bar{r}}_1 + \frac{m}{2}\bar{g}_{rot} \tag{3-13}$$

$$\bar{f}_2 = -\frac{m}{2}\ddot{\bar{r}}_2 + \frac{m}{2}\bar{g}_{rot} \tag{3-14}$$

The total force acting on the system in the rotating reference frame is $\bar{f}_{tot}$. It is found by adding, vectorially, the forces $\bar{f}_1$ and $\bar{f}_2$.

The third step in the derivation is the determination of the moments acting on each mass. The vector equation for moments given by Pletta and Frederick [27] is

$$\bar{M} = \bar{r} \times \bar{F} \tag{3-15}$$

where $\bar{M}$ is the moment, $\bar{r}$ is a position vector and $\bar{F}$ is a force acting at the end of $\bar{r}$. Since the moments $\bar{m}_1$ and $\bar{m}_2$ about the whirling shaft center are desired the position vectors, $\bar{\rho}_1$ and $\bar{\rho}_2$, and the force vectors, $\bar{f}_1$ and $\bar{f}_2$, are used in equation (3-15). The general form of the moment equations for this case can be written as

$$\bar{m}_\ell = \bar{\rho}_\ell \times \bar{f}_\ell \tag{3-16}$$

or

$$\bar{m}_\ell = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \rho_{\ell x} & \rho_{\ell y} & \rho_{\ell z} \\ f_{\ell x} & f_{\ell y} & f_{\ell z} \end{vmatrix} \tag{3-17}$$

where the subscript ℓ denotes which mass, 1 or 2, is under consideration. The total moment, $\bar{m}_{tot}$, of the system in the rotating coordinate system is found by adding vectorwise the individual moments $\bar{m}_1$ and $\bar{m}_2$.

Upon completion of steps two and three above, one has the total system of forces and moments expressed in the rotating coordinate system. Therefore, the next step in the matrix derivation is to transform the equations for $\bar{f}_{tot}$ and $\bar{m}_{tot}$ into the stationary coordinate system to get the total moments, $\bar{M}$ and total forces, $\bar{F}$ in terms of the fixed reference frame.

The fifth, and last, step of the derivation involves putting the equations for $\bar{M}$ and $\bar{F}$ into matrix form. To do this one must realize the relations between the state variables and the unit vectors $\bar{I}$, $\bar{J}$ and $\bar{K}$. Table 1 on the following page shows which internal force and internal moment state variables correspond to which unit vector. Next $\bar{F}$ and $\bar{M}$ are used in the development of the sums of forces and of moments about the free body diagrams of the masses. $\bar{F}$ and $\bar{M}$ are balanced by internal shaft shears and moments. These equations are then separated

TABLE 1. UNIT VECTOR--STATE VARIABLE RELATIONS

	STATE VARIABLES	
UNIT VECTOR	INTERNAL FORCE	INTERNAL MOMENT
$\bar{I}$	AXIAL FORCE (N_{c1}, N_{s1})	TORQUE (T_{c1}, T_{s1})
$\bar{J}$	SHEAR FORCE (V_{Yc1}, V_{Ys1})	MOMENT (M_{Yc1}, M_{Ys1})
$\bar{K}$	SHEAR FORCE (V_{Zc1}, V_{Zs1})	MOMENT (M_{Zc1}, M_{Zs1})

into the conventional form of a transfer function equation, i.e., an output equals a transfer function times an input. These sets of equations are then rewritten into transfer matrix form.

With the preceeding discussion the reader is now ready to follow the derivation in section 4. Some insight into the problem of a whirling eccentric shaft (or disk) will also be included, as appropriate, during the course of the derivation.

4. MATRIX DERIVATION

4.1 DISPLACEMENT VECTORS IN THE ROTATING COORDINATE SYSTEM

4.1.1 WHIRL ORBIT

The first step in the derivation is the description of the whirl orbit in the absolute coordinate system. This is accomplished by adding vectorially the assumed form of the linear displacement state variables u, v and w as discussed in section 3.2.4. This results in the equation for $\bar{R}_{abs}$ which can be written as

$$\bar{R}_{abs} = (u_{c1}\cos\Omega t + u_{s1}\sin\Omega t)\bar{I} + (v_{c1}\cos\Omega t + v_{s1}\sin\Omega t)\bar{J}$$
$$+ (w_{c1}\cos\Omega t + w_{s1}\sin\Omega t)\bar{K} \qquad (4\text{-}1)$$

A similar form for a planar elliptical whirl path has been used by Lund [24]. Equation (3-7) is now used to transform $\bar{R}_{abs}$ into the rotating coordinate system. This gives the equation for $\bar{r}_{rot}$ which is

$$\bar{r}_{rot} = (u_{c1}\cos\Omega t + u_{s1}\sin\Omega t)\bar{i} \qquad (4\text{-}2)$$
$$+ (\tfrac{1}{2}v_{c1} + \tfrac{1}{2}w_{s1} + \tfrac{1}{2}v_{c1}\cos 2\Omega t - \tfrac{1}{2}w_{s1}\cos 2\Omega t + \tfrac{1}{2}v_{s1}\sin 2\Omega t + \tfrac{1}{2}w_{c1}\sin 2\Omega t)\bar{j}$$
$$+ (-\tfrac{1}{2}v_{s1} + \tfrac{1}{2}w_{c1} + \tfrac{1}{2}v_{s1}\cos 2\Omega t + \tfrac{1}{2}w_{c1}\cos 2\Omega t - \tfrac{1}{2}v_{c1}\sin 2\Omega t + \tfrac{1}{2}w_{s1}\sin 2\Omega t)\bar{k}$$

In the $y(\bar{j})$ and $z(\bar{k})$ parts of this equation there is a mean value corresponding to an average whirl bow in a shaft and an oscillatory value at 2 times Ωt. This 2 times value might be responsible for shaft fatigue if it results in a varying value of shaft bow. However, such elliptical orbits are usually caused by asymmetrical support stiffness of bearings. Therefore, only when the shaft response at the bearing position and at some other point along the shaft differ by a varying amount (not a constant) can the shaft experience dynamic shears and moments at two times the whirl rate.

4.1.2 MASS POSITIONS

The position vector components ρ_{1x}, ρ_{1y}, ρ_{1z}, ρ_{2x}, ρ_{2y} and ρ_{2z} must be determined from the plane views given in Figs. 5, 6 and 7. These equations are

$$\rho_{1x} = - a \sin \Theta_z \cos \psi_y + b \sin \psi_y$$

$$\rho_{1y} = a \cos \Theta_z \cos \phi_x - a \sin \Theta_z \sin \psi_y \sin \phi_x - b \cos \psi_y \sin \phi_x$$

$$\rho_{1z} = a \cos \Theta_z \sin \phi_x + a \sin \Theta_z \sin \psi_y \cos \phi_x + b \cos \psi_y \cos \phi_x$$

$$\rho_{2x} = c \sin \Theta_z \cos \psi_y - d \sin \psi_y$$

$$\rho_{2y} = - c \cos \Theta_z \cos \phi_x + c \sin \Theta_z \sin \psi_y \sin \phi_x + d \cos \psi_y \sin \phi_x$$

$$\rho_{2z} = - c \cos \Theta_z \sin \phi_x - c \sin \Theta_z \sin \psi_y \cos \phi_x - d \cos \psi_y \cos \phi_x$$

These components are now summed vectorially to give the total position vectors $\bar{\rho}_1$ and $\bar{\rho}_2$. These equations are

$$\bar{\rho}_1 = (- a \sin \Theta_z \cos \psi_y + b \sin \psi_y)\bar{i} \tag{4-3}$$

$$+ (a \cos \Theta_z \cos \phi_x - a \sin \Theta_z \sin \psi_y \sin \phi_x - b \cos \psi_y \sin \phi_x)\bar{j}$$

$$+ (a \cos \Theta_z \sin \phi_x + a \sin \Theta_z \sin \psi_y \cos \phi_x + b \cos \psi_y \cos \phi_x)\bar{k}$$

$$\bar{\rho}_2 = (c \sin \Theta_z \cos \psi_y - d \sin \psi_y)\bar{i} \tag{4-4}$$

$$+ (- c \cos \Theta_z \cos \phi_x + c \sin \Theta_z \sin \psi_y \sin \phi_x + d \cos \psi_y \sin \phi_x)\bar{j}$$

$$+ (- c \cos \Theta_z \sin \phi_x - c \sin \Theta_z \sin \psi_y \cos \phi_x - d \cos \psi_y \cos \phi_x)\bar{k}$$

The limitation to small angular deflections is used to make these vectors linear. It must be realized that the sine of a small angle is approximately equal to the angle and the cosine of a small angle is approximately equal to one. Upon completion of the above steps, any terms which have the product of two or more state variables are eliminated. These two steps result in the following linear representations of the total position vectors $\bar{\rho}_1$ and $\bar{\rho}_2$.

$$\bar{\rho}_1 = (- a \Theta_z + b\psi_y)\bar{i} + (a - b\phi_x)\bar{j} + (a\phi_x + b)\bar{k} \tag{4-5}$$

$$\bar{\rho}_2 = (c\Theta_z - d\psi_y)\bar{i} + (- c + d\phi_x)\bar{j} + (- c\phi_x - d)\bar{k} \tag{4-6}$$

A slight complication now arises. The angular deflections Θ_z, ψ_y and ϕ_x used in equations (4-5) and (4-6) are angular variables in the rotating coordinate system and are not the angular state variables used in the fixed reference frame. To eliminate this problem a coordinate transformation is performed on the assumed form of Θ, ψ and ϕ in the absolute reference frame to the rotating reference frame. The x, y and z components are then separated as shown below.

$$\phi_x = (\phi_{c1}\cos\Omega t + \phi_{s1}\sin\Omega t)\bar{i}$$

$$\psi_y = (\tfrac{1}{2}\psi_{c1} + \tfrac{1}{2}\Theta_{s1} + \tfrac{1}{2}\psi_{c1}\cos 2\Omega t - \tfrac{1}{2}\Theta_{s1}\cos 2\Omega t + \tfrac{1}{2}\psi_{s1}\sin 2\Omega t + \tfrac{1}{2}\Theta_{c1}\sin 2\Omega t)\bar{j}$$

$$\Theta_z = (-\tfrac{1}{2}\psi_{s1} + \tfrac{1}{2}\Theta_{c1} + \tfrac{1}{2}\psi_{s1}\cos 2\Omega t + \tfrac{1}{2}\Theta_{c1}\cos 2\Omega t - \tfrac{1}{2}\psi_{c1}\sin 2\Omega t + \tfrac{1}{2}\Theta_{s1}\sin 2\Omega t)\bar{k} \quad (4\text{-}7)$$

Now, these values are placed in the equations, (4-5) and (4-6), for the total position vectors $\bar{\rho}_1$ and $\bar{\rho}_2$. This results in

$$\bar{\rho}_1 = (\tfrac{1}{2}a\psi_{s1} - \tfrac{1}{2}a\Theta_{c1} - \tfrac{1}{2}a\psi_{s1}\cos 2\Omega t - \tfrac{1}{2}a\Theta_{c1}\cos 2\Omega t + \tfrac{1}{2}a\psi_{c1}\sin 2\Omega t - \tfrac{1}{2}a\Theta_{s1}\sin 2\Omega t$$

$$+ \tfrac{1}{2}b\psi_{c1} + \tfrac{1}{2}b\Theta_{s1} + \tfrac{1}{2}b\psi_{c1}\cos 2\Omega t - \tfrac{1}{2}b\Theta_{s1}\cos 2\Omega t + \tfrac{1}{2}b\psi_{s1}\sin 2\Omega t + \tfrac{1}{2}b\Theta_{c1}\sin 2\Omega t)\bar{i}$$

$$+ (a - b\phi_{c1}\cos\Omega t - b\phi_{s1}\sin\Omega t)\bar{j} + (a\phi_{c1}\cos\Omega t + a\phi_{s1}\sin\Omega t + b)\bar{k} \quad (4\text{-}8)$$

$$\bar{\rho}_2 = (-\tfrac{1}{2}c\psi_{s1} + \tfrac{1}{2}c\Theta_{c1} + \tfrac{1}{2}c\psi_{s1}\cos 2\Omega t + \tfrac{1}{2}c\Theta_{c1}\cos 2\Omega t - \tfrac{1}{2}c\psi_{c1}\sin 2\Omega t + \tfrac{1}{2}c\Theta_{s1}\sin 2\Omega t$$

$$- \tfrac{1}{2}d\psi_{c1} - \tfrac{1}{2}d\Theta_{s1} - \tfrac{1}{2}d\psi_{c1}\cos 2\Omega t + \tfrac{1}{2}d\Theta_{s1}\cos 2\Omega t - \tfrac{1}{2}d\psi_{s1}\sin 2\Omega t - \tfrac{1}{2}d\Theta_{c1}\sin 2\Omega t)\bar{i}$$

$$+ (-c + d\phi_{c1}\cos\Omega t + d\phi_{s1}\sin\Omega t)\bar{j} + (-c\phi_{c1}\cos\Omega t - c\phi_{s1}\sin\Omega t - d)\bar{k} \quad (4\text{-}9)$$

Equations (4-8) and (4-9) are the position vectors in the rotating coordinate system, using transformed state variables from the fixed coordinate system, from the center of the shaft to the center of masses 1 and 2, respectively.

4.1.3 TOTAL DISPLACEMENT VECTORS

In order to arrive at the displacement vector $\bar{r}_1$ (or $\bar{r}_2$), which describes the deflected position of mass 1 (or 2) relative to the undeflected shaft center, it is necessary to sum vectorially $\bar{r}_{rot}$ and $\bar{\rho}_1$ (or $\bar{\rho}_2$). These summations are written vectorially as

$$\bar{r}_1 = \bar{r}_{rot} + \bar{\rho}_1 \tag{4-10}$$

$$\bar{r}_2 = \bar{r}_{rot} + \bar{\rho}_2 \tag{4-11}$$

4.2 FORCES IN THE ROTATING COORDINATE SYSTEM

4.2.1 INERTIAL FORCES

With the position of masses 1 and 2 known, it is now possible to determine the accelerations and then the inertial forces of each mass. Equations (4-8) and (4-2) are substituted into equation (4-10) which is then differentiated twice with respect to time to give the acceleration of mass 1, $\ddot{\bar{r}}_1$. This differentiation process produces terms such as $\dot{\bar{i}}$, $\dot{\bar{j}}$ and $\dot{\bar{k}}$ which are the velocities of the tips of the unit vectors. They can be written as

$$\begin{aligned} \dot{\bar{i}} &= \omega \times \bar{i} \\ \dot{\bar{j}} &= \omega \times \bar{j} \\ \dot{\bar{k}} &= \omega \times \bar{k} \end{aligned} \tag{4-12}$$

where ω is the angular velocity of the moving coordinate system relative to the fixed system. Recalling that the rotating coordinate system is turning at a constant Ω radians per second in the Y-Z plane of the fixed system one can write for this case

$$\omega = \Omega\bar{i} + 0\bar{j} + 0\bar{k} \tag{4-13}$$

Equation (4-13) can then be inserted into equation (4-12) to arrive at the final form of the velocities of the tips of the unit vectors

$$\dot{\bar{i}} = 0$$

$$\dot{\bar{j}} = \Omega\bar{k} \tag{4-14}$$

$$\dot{\bar{k}} = -\Omega\bar{j}$$

The actual differentiation process can now be performed resulting in the following expanded equation for $\ddot{\bar{r}}_1$, that is, the equation before like terms are collected.

$$\begin{aligned}\ddot{\bar{r}}_1 = &(-\Omega^2 u_{c1}\cos\Omega t-\Omega^2 u_{s1}\sin\Omega t-2b\Omega^2\psi_{c1}\cos 2\Omega t-2b\Omega^2\psi_{s1}\sin 2\Omega t\\
&- 2b\Omega^2\Theta_{c1}\sin 2\Omega t+2b\Omega^2\Theta_{s1}\cos 2\Omega t-2a\Omega^2\psi_{c1}\sin 2\Omega t+2a^2\psi_{s1}\cos 2\Omega t\\
&+ 2a\Omega^2\Theta_{c1}\cos 2\Omega t+2a\Omega^2\Theta_{s1}\sin 2\Omega t)\bar{i}\\
&- \Omega^2(\tfrac{1}{2}v_{c1}+\tfrac{1}{2}w_{s1}+\tfrac{1}{2}v_{c1}\cos 2\Omega t-\tfrac{1}{2}w_{s1}\cos 2\Omega t+\tfrac{1}{2}v_{s1}\sin 2\Omega t+\tfrac{1}{2}w_{c1}\sin 2\Omega t\\
&+ a-b\phi_{c1}\cos\Omega t-b\phi_{s1}\sin\Omega t)\bar{j}\\
&+ (-2\Omega^2 v_{c1}\cos 2\Omega t+2\Omega^2 w_{s1}\cos 2\Omega t-2\Omega^2 v_{s1}\sin 2\Omega t-2\Omega^2 w_{c1}\sin 2\Omega t\\
&+ b\Omega^2\phi_{c1}\cos\Omega t+b\Omega^2\phi_{s1}\sin\Omega t)\bar{j}\\
&- 2\Omega(-\Omega v_{s1}\sin 2\Omega t-\Omega w_{c1}\sin 2\Omega t-\Omega v_{c1}\cos 2\Omega t+\Omega w_{s1}\cos 2\Omega t\\
&- a\Omega\phi_{c1}\sin\Omega t+a\Omega\phi_{s1}\cos\Omega t)\bar{j}\\
&- \Omega^2(-\tfrac{1}{2}v_{s1}+\tfrac{1}{2}w_{c1}+\tfrac{1}{2}v_{s1}\cos 2\Omega t+\tfrac{1}{2}w_{c1}\cos 2\Omega t-\tfrac{1}{2}v_{c1}\sin 2\Omega t+\tfrac{1}{2}w_{s1}\sin 2\Omega t\\
&+ a\phi_{c1}\cos\Omega t+a\phi_{s1}\sin\Omega t+b)\bar{k}\end{aligned}$$

$$+ (-2\Omega^2 v_{s1}\cos 2\Omega t - 2\Omega^2 w_{c1}\cos 2\Omega t + 2\Omega^2 v_{c1}\sin 2\Omega t - 2\Omega^2 w_{s1}\sin 2\Omega t$$

$$- a\Omega^2 \phi_{c1}\cos\Omega t - a\Omega^2 \phi_{s1}\sin\Omega t)\bar{k}$$

$$+ 2\Omega(-\Omega v_{c1}\sin 2\Omega t + \Omega w_{s1}\sin 2\Omega t + \Omega v_{s1}\cos 2\Omega t + \Omega w_{c1}\cos 2\Omega t$$

$$+ b\Omega\phi_{c1}\sin\Omega t - b\Omega\phi_{s1}\cos\Omega t)\bar{k} \qquad (4\text{-}15)$$

The x component of $\ddot{\bar{r}}_1$ is the acceleration of mass 1 in the x coordinate direction relative to the moving reference frame. Equation (4-15) has the y and z vector components, $\bar{j}$ and $\bar{k}$, respectively. Each is divided into three parts. Each part has a different meaning attached. The first part of both vector components corresponds to the centripetal acceleration of mass 1 resulting from the angular velocity of the rotating reference frame. The second part represents the acceleration of mass 1 in the y or z coordinate direction relative to the moving frame. It can be thought of as the acceleration of mass 1 as seen by an observer rotating with the moving reference frame. The third, and final part of the y and z vector components is called the Coriolis acceleration of mass 1. It is a result of the interaction of the rotation of the moving frame and the change of position of mass 1 relative to the moving frame. Due to the assumption of a constant angular velocity of the rotating coordinate system there is no part of the y and/or z vector components which can be labelled as solely tangential acceleration. However, the assumed form of the ϕ state variable, $\phi = \phi_{c1} \cos\Omega t + \phi_{s1} \sin \Omega t$, allows the existence of such acceleration. Throughout the acceleration development ϕ has been differentiated and thus the tangential acceleration effects must be contained within some or all of the other three parts in a slightly disguised form.

A similar equation for the acceleration of mass 2 can be found by substituting equations (4-9) and (4-2) into equations (4-11) and twice differentiating the result with respect to time.

Once the accelerations of masses 1 and 2 are determined it is a simple matter to get the inertial forces. It is necessary to multiply the mass by the negative of the mass's acceleration.

4.2.2 GRAVITATIONAL FORCE

In addition to the inertial forces acting on each mass is the force due to gravity. This force is due to the acceleration of gravity which is known to be directed along the positive Z direction in the absolute reference frame. Upon transformation of this acceleration into the rotating coordinate system it is necessary to multiply by the appropriate mass to get the force due to gravity acting on each mass. The acceleration in the rotating reference frame, $\bar{g}_{rot}$, has the form

$$\bar{g}_{rot} = (g \sin\Omega t)\bar{j} + (g \cos\Omega t)\bar{k} \tag{4-16}$$

where g is the acceleration of gravity in the absolute reference frame.

4.2.3 TOTAL FORCE

The total forces $\bar{f}_1$ and $\bar{f}_2$ in the rotating reference frame acting on each mass is found by summing the inertial and gravitational forces of each mass. Equations (3-13) and (3-14) give the equational form of the processes described in section 4.2.1 and in section 4.2.2. These equations are repeated below for convenience

$$\bar{f}_1 = -\tfrac{1}{2}\,m\,\ddot{\bar{r}}_1 + \tfrac{1}{2}\,m\,\bar{g}_{rot} \tag{3-13}$$

$$\bar{f}_2 = -\tfrac{1}{2}\,m\,\ddot{\bar{r}}_2 + \tfrac{1}{2}\,m\,\bar{g}_{rot} \tag{3-14}$$

The total force $\bar{f}_{tot}$ acting on the two masses is the vector sum of $\bar{f}_1$ and $\bar{f}_2$. This is written as

$$\begin{aligned}
\bar{f}_{tot} = {}& \tfrac{1}{2}m[2\Omega^2u_{c1}\cos\Omega t+2\Omega^2u_{s1}\sin\Omega t+2(b-d)\Omega^2\psi_{c1}\cos2\Omega t+2(b-d)\Omega^2\psi_{s1}\sin2\Omega t \\
&+2(b-d)\Omega^2\Theta_{c1}\sin2\Omega t+2(d-b)\Omega^2\Theta_{s1}\cos2\Omega t+2(a-c)\Omega^2\psi_{c1}\sin2\Omega t \\
&+2(c-a)\Omega^2\psi_{s1}\cos2\Omega t+2(c-a)\Omega^2\Theta_{c1}\cos2\Omega t+2(c-a)\Omega^2\Theta_{s1}\sin2\Omega t]\bar{i} \\
&+\tfrac{1}{2}m[\Omega^2v_{c1}+\Omega^2w_{s1}+\Omega^2v_{c1}\cos2\Omega t-\Omega^2w_{s1}\cos2\Omega t+\Omega^2v_{s1}\sin2\Omega t+\Omega^2w_{c1}\sin2\Omega t \\
&+(a-c)\Omega^2+2(d-b)\Omega^2\phi_{c1}\cos\Omega t+2(d-b)\Omega^2\phi_{s1}\sin\Omega t+2(c-a)\Omega^2\phi_{c1}\sin\Omega t \\
&+2(a-c)\Omega^2\phi_{s1}\cos\Omega t+2g\sin]\bar{j} \\
&+\tfrac{1}{2}m[-\Omega^2v_{s1}+\Omega^2w_{c1}+\Omega^2v_{s1}\cos2\Omega t+\Omega^2w_{c1}\cos2\Omega t-\Omega^2v_{c1}\sin2\Omega t \\
&+\Omega^2w_{s1}\sin2\Omega t+(b-d)\Omega^2+2(d-b)\Omega^2\phi_{c1}\sin\Omega t+2(b-d)\Omega^2\phi_{s1}\cos\Omega t \\
&+2(a-c)\Omega^2\phi_{c1}\cos\Omega t+2(a-c)\Omega^2\phi_{s1}\sin\Omega t+2g\cos\Omega t]\bar{k}
\end{aligned} \tag{4-17}$$

This is the total force as described in the rotating coordinate system acting on both masses. This force must be balanced by the internal shears within the shafting cross section.

4.3 MOMENTS IN THE ROTATING COORDINATE SYSTEM

4.3.1 MOMENTS FOR INDIVIDUAL MASSES

The position vectors $\bar{\rho}_1$ and $\bar{\rho}_2$ found in section 4.1.2, equations (4-8) and (4-9), and the force vectors $\bar{f}_1$ and $\bar{f}_2$ found in section 4.2.3, equations (3-13) and (3-14), are now used to find the moments, $\bar{m}_1$ and $\bar{m}_2$, about the whirling shaft center. To find the moments $\bar{m}_1$ and $\bar{m}_2$ use is made of equation (3-16), with ℓ, the mass number, set to 1 and 2, respectively. These cross products can be rewritten to be

$$\begin{aligned}\bar{m}_1&=(\rho_{1y}f_{1z}-\rho_{1z}f_{1y})\bar{i}-(\rho_{1x}f_{1z}-\rho_{1z}f_{1x})\bar{j} + (\rho_{1x}f_{1y}-\rho_{1y}f_{1x})\bar{k}\\ \bar{m}_2&=(\rho_{2y}f_{2z}-\rho_{2z}f_{2y})\bar{i}-(\rho_{2x}f_{2z}-\rho_{2z}f_{2x})\bar{j} + (\rho_{2x}f_{2y}-\rho_{2y}f_{2x})\bar{k}\end{aligned} \qquad (4\text{-}18)$$

In the multiplications indicated in equation (4-18), any product of two state variables is assumed to be negligible and as such, is set equal to zero. This is in keeping with the procedure used in the description of $\bar{\rho}_1$ and $\bar{\rho}_2$ given in section 4.1.2 and with the original assumption of small displacements made in section 3.2.1. There would be little value in writing the final equations for $\bar{m}_1$ and $\bar{m}_2$ individually.

4.3 TOTAL MOMENTS

The total moment, $\bar{m}_{tot}$, is obtained by the vector summation of the moments $\bar{m}_1$ and $\bar{m}_2$. The equation for $\bar{m}_{tot}$ is

$$\bar{m}_{tot}=\tfrac{1}{2}m[\tfrac{1}{2}(c-a)\Omega^2v_{s1}+\tfrac{1}{2}(a-c)\Omega^2w_{c1}+\tfrac{1}{2}(a-c)\Omega^2v_{s1}\cos2\Omega t+\tfrac{1}{2}(a-c)\Omega^2w_{c1}\cos2\Omega t$$

$$+\tfrac{1}{2}(c-a)\Omega^2v_{c1}\sin2\Omega t+\tfrac{1}{2}(a-c)\Omega^2w_{s1}\sin2\Omega t+(a^2+b^2+c^2+d^2)\Omega^2\phi_{c1}\cos\Omega t$$

$$+(a^2+b^2+c^2+d^2)\Omega^2\phi_{s1}\sin\Omega t+\tfrac{1}{2}(d-b)\Omega^2v_{c1}+\tfrac{1}{2}(d-b)\Omega^2w_{s1}+\tfrac{1}{2}(d-b)\Omega^2v_{c1}\cos2\Omega t$$

$$+\tfrac{1}{2}(b-d)\Omega^2w_{s1}\cos2\Omega t+\tfrac{1}{2}(d-b)\Omega^2v_{s1}\sin2\Omega t+\tfrac{1}{2}(d-b)\Omega^2w_{c1}\sin2\Omega t$$

$$+(a-c)g\cos\Omega t+\tfrac{1}{2}(c-a)g\phi_{c1}\sin2\Omega t+\tfrac{1}{2}(c-a)g\phi_{s1}+\tfrac{1}{2}(a-c)g\phi_{s1}\cos2\Omega t$$

$$+(d-b)g\sin\Omega t+\tfrac{1}{2}(d-b)g\phi_{c1}+\tfrac{1}{2}(d-b)g\phi_{c1}\cos2\Omega t+\tfrac{1}{2}(d-b)g\phi_{s1}\sin2\Omega t]\bar{i}$$

$$+\tfrac{1}{2}m[-\tfrac{1}{2}(b^2+d^2)\Omega^2\psi_{c1}+\frac{3}{2}(b^2+d^2)\Omega^2\psi_{c1}\cos2\Omega t+\frac{3}{2}(b^2+d^2)\Omega^2\psi_{s1}\sin2\Omega t$$

$$+\frac{3}{2}(b^2+d^2)\Omega^2\Theta_{c1}\sin2\Omega t-\frac{3}{2}(b^2+d^2)\Omega^2\Theta_{s1}\cos2\Omega t-\tfrac{1}{2}(b^2+d^2)\Omega^2\Theta_{s1}$$

$$+\frac{3}{2}(ab+cd)\Omega^2\psi_{c1}\sin2\Omega t-\frac{3}{2}(ab+cd)\Omega^2\psi_{s1}\cos2\Omega t-\tfrac{1}{2}(ab+cd)\Omega^2\psi_{s1}$$

$$-\frac{3}{2}(ab+cd)\Omega^2\Theta_{c1}\cos2\Omega t-\frac{3}{2}(ab+cd)\Omega^2\Theta_{s1}\sin2\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{c1}$$

$$+(b-d)\Omega^2u_{c1}\cos\Omega t+(b-d)\Omega^2u_{s1}\sin\Omega t+\frac{3}{4}(d-b)g\psi_{c1}\cos\Omega t+\tfrac{1}{4}(d-b)g\psi_{c1}\cos3\Omega t$$

$$+\tfrac{1}{4}(d-b)g\psi_{s1}\sin3\Omega t+\tfrac{1}{4}(d-b)g\psi_{s1}\sin\Omega t+\tfrac{1}{4}(d-b)g\Theta_{c1}\sin3\Omega t+\tfrac{1}{4}(d-b)g\Theta_{c1}\sin\Omega t$$

$$+\tfrac{1}{4}(d-b)g\Theta_{s1}\cos\Omega t+\tfrac{1}{4}(b-d)g\Theta_{s1}\cos3\Omega t+\tfrac{1}{4}(c-a)g\psi_{c1}\sin3\Omega t+\tfrac{1}{4}(c-a)g\psi_{c1}\sin\Omega t$$

$$+\tfrac{1}{4}(c-a)g\psi_{s1}\cos\Omega t+\tfrac{1}{4}(a-c)g\psi_{s1}\cos3\Omega t+\frac{3}{4}(a-c)g\Theta_{c1}\cos\Omega t+\tfrac{1}{4}(a-c)g\Theta_{c1}\cos3\Omega t$$

$$+\tfrac{1}{4}(a-c)g\Theta_{s1}\sin3\Omega t+\tfrac{1}{4}(a-c)g\Theta_{s1}\sin\Omega t]\bar{j}$$

$$+\tfrac{1}{2}m[-\frac{3}{2}(a^2+c^2)\Omega^2\psi_{c1}\sin2\Omega t+\tfrac{1}{2}(a^2+c^2)\Omega^2\psi_{s1}+\frac{3}{2}(a^2+c^2)\Omega^2\psi_{s1}\cos2\Omega t$$

$$-\tfrac{1}{2}(a^2+c^2)\Omega^2\Theta_{c1}+\frac{3}{2}(a^2+c^2)\Omega^2\Theta_{c1}\cos2\Omega t+\frac{3}{2}(a^2+c^2)\Omega^2\Theta_{s1}\sin2\Omega t$$

$$+\tfrac{1}{2}(ab+cd)\Omega^2\psi_{c1}-\tfrac{3}{2}(ab+cd)\Omega^2\psi_{c1}\cos 2\Omega t-\tfrac{3}{2}(ab+cd)\Omega^2\psi_{s1}\sin 2\Omega t$$

$$-\tfrac{3}{2}(ab+cd)\Omega^2\Theta_{c1}\sin 2\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{s1}+\tfrac{3}{2}(ab+cd)\Omega^2\Theta_{s1}\cos 2\Omega t$$

$$(c-a)\Omega^2 u_{c1}\cos\Omega t+(c-a)\Omega^2 u_{s1}\sin\Omega t+\tfrac{1}{4}(b-d)g\psi_{c1}\sin\Omega t+\tfrac{1}{4}(b-d)g\psi_{c1}\sin 3\Omega t$$

$$+\tfrac{1}{4}(b-d)g\psi_{s1}\cos\Omega t+\tfrac{1}{4}(d-b)g\psi_{s1}\cos 3\Omega t+\tfrac{1}{4}(b-d)g\Theta_{c1}\cos\Omega t+\tfrac{1}{4}(d-b)g\Theta_{c1}\cos 3\Omega t$$

$$+\tfrac{3}{4}(b-d)g\Theta_{s1}\sin\Omega t+\tfrac{1}{4}(d-b)g\Theta_{s1}\sin 3\Omega t+\tfrac{1}{4}(a-c)g\psi_{c1}\cos\Omega t+\tfrac{1}{4}(c-a)g\psi_{c1}\cos 3\Omega t$$

$$+\tfrac{3}{4}(a-c)g\psi_{s1}\sin\Omega t+\tfrac{1}{4}(c-a)g\psi_{s1}\sin 3\Omega t+\tfrac{1}{4}(c-a)g\Theta_{c1}\sin\Omega t+\tfrac{1}{4}(c-a)g\Theta_{c1}\sin 3\Omega t$$

$$+\tfrac{1}{4}(c-a)g\Theta_{s1}\cos\Omega t+\tfrac{1}{4}(a-c)g\Theta_{s1}\cos 3\Omega t]\bar{k} \qquad (4\text{-}19)$$

Equation (4-19) then is the total moment about the shaft center due to both masses in the rotating coordinate system.

4.4 TOTAL FORCES AND MOMENTS IN THE STATIONARY COORDINATE SYSTEM

The next step in the derivation is to transform the equations for $\bar{f}_{tot}$ and $\bar{m}_{tot}$ into the stationary coordinate system to get the total forces $\bar{F}$ and total moments $\bar{M}$ as described in the fixed reference frame. This is done by using the transformation equation (3-8) on equations (4-17) and (4-19). This procedure results in the following equation for $\bar{F}$.

$$\bar{F}=\tfrac{1}{2}m[2\Omega^2 u_{c1}\cos\Omega t+2\Omega^2 u_{s1}\sin\Omega t+2(b-d)\Omega^2\psi_{c1}\cos 2\Omega t$$

$$+2(b-d)\Omega^2\psi_{s1}\sin 2\Omega t+2(b-d)\Omega^2\Theta_{c1}\sin 2\Omega t+2(d-b)\Omega^2\Theta_{s1}\cos 2\Omega t$$

$$+2(a-c)\Omega^2\psi_{c1}\sin 2\Omega t+2(c-a)\Omega^2\psi_{s1}\cos 2\Omega t+2(c-a)\Omega^2\Theta_{c1}\cos 2\Omega t$$

$$+2(c-a)\Omega^2\theta_{s1}\sin2\Omega t]\bar{I}$$

$$+\tfrac{1}{2}m[2\Omega^2v_{c1}\cos\Omega t+2\Omega^2v_{s1}\sin\Omega t+(a-c)\Omega^2\cos\Omega t+(d-b)\Omega^2\sin\Omega t$$

$$+2(d-b)\Omega^2\phi_{c1}\cos2\Omega t+2(d-b)\Omega^2\phi_{s1}\sin2\Omega t+2(c-a)\Omega^2\phi_{c1}\sin2\Omega t$$

$$+2(a-c)\Omega^2\phi_{s1}\cos2\Omega t]\bar{J}$$

$$+\tfrac{1}{2}m[2\Omega^2w_{c1}\cos\Omega t+2\Omega^2w_{s1}\sin\Omega t+(b-d)\Omega^2\cos\Omega t+(a-c)\Omega^2\sin\Omega t$$

$$+2(d-b)\Omega^2\phi_{c1}\sin2\Omega t+2(b-d)\Omega^2\phi_{s1}\cos2\Omega t+2(a-c)\Omega^2\phi_{c1}\cos2\Omega t$$

$$+2(a-c)\Omega^2\phi_{s1}\sin2\Omega t+2g]\bar{K} \qquad (4\text{-}20)$$

The final form of the equation for the total moments in the absolute reference frame follows.

$$\bar{M}=\tfrac{1}{2}m[\tfrac{1}{2}(c-a)\Omega^2v_{s1}+\tfrac{1}{2}(a-c)\Omega^2w_{c1}+\tfrac{1}{2}(a-c)\Omega^2v_{s1}\cos2\Omega t+\tfrac{1}{2}(a-c)\Omega^2w_{c1}\cos2\Omega t$$

$$+\tfrac{1}{2}(c-a)\Omega^2v_{c1}\cos2\Omega t+\tfrac{1}{2}(a-c)\Omega^2w_{s1}\sin2\Omega t+(a^2+b^2+c^2+d^2)\Omega^2\phi_{c1}\cos\Omega t$$

$$+(a^2+b^2+c^2+d^2)\Omega^2\phi_{s1}\sin\Omega t+\tfrac{1}{2}(d-b)\Omega^2v_{c1}+\tfrac{1}{2}(d-b)\Omega^2w_{s1}+\tfrac{1}{2}(d-b)\Omega^2v_{c1}\cos2\Omega t$$

$$+\tfrac{1}{2}(b-d)\Omega^2w_{s1}\cos2\Omega t+\tfrac{1}{2}(d-b)\Omega^2v_{s1}\sin2\Omega t+\tfrac{1}{2}(d-b)\Omega^2w_{c1}\sin2\Omega t$$

$$+(a-c)g\cos\Omega t+\tfrac{1}{2}(c-a)g\phi_{c1}\sin2\Omega t+\tfrac{1}{2}(c-a)g\phi_{s1}+\tfrac{1}{2}(a-c)g\phi_{s1}\cos2\Omega t$$

$$+(d-b)g\sin\Omega t+\tfrac{1}{2}(d-b)g\phi_{c1}+\tfrac{1}{2}(d-b)g\phi_{c1}\cos2\Omega t+\tfrac{1}{2}(d-b)g\phi_{s1}\sin2\Omega t]\bar{I}$$

$$+\tfrac{1}{2}m[\tfrac{1}{4}(b^2+d^2)\Omega^2\psi_{c1}\cos\Omega t+\tfrac{3}{4}(a^2+c^2)\Omega^2\psi_{c1}\cos\Omega t+\tfrac{3}{4}(b^2+d^2)\Omega^2\psi_{s1}\sin\Omega t$$

$$+\tfrac{1}{4}(a^2+c^2)\Omega^2\psi_{s1}\sin\Omega t+\tfrac{3}{4}(b^2+d^2)\Omega^2\theta_{c1}\sin\Omega t+\tfrac{5}{4}(a^2+c^2)\Omega^2\theta_{c1}\sin\Omega t$$

$$-\tfrac{5}{4}(b^2+d^2)\Omega^2\theta_{s1}\cos\Omega t-\tfrac{3}{4}(a^2+c^2)\Omega^2\theta_{s1}\cos\Omega t-\tfrac{1}{2}(ab+cd)\Omega^2\psi_{c1}\sin\Omega t$$

$$-\tfrac{1}{2}(ab+cd)\Omega^2\psi_{s1}\cos\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{c1}\cos\Omega t-\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{s1}\sin\Omega t$$

$$+\frac{3}{4}(b^2+d^2)\Omega^2\psi_{c1}\cos 3\Omega t-\frac{3}{4}(a^2+c^2)\Omega^2\psi_{c1}\cos 3\Omega t+\frac{3}{4}(b^2+d^2)\Omega^2\psi_{s1}\sin 3\Omega t$$

$$-\frac{3}{4}(a^2+c^2)\Omega^2\psi_{s1}\sin 3\Omega t+\frac{3}{4}(b^2+d^2)\Omega^2\Theta_{c1}\sin 3\Omega t-\frac{3}{4}(a^2+c^2)\Omega^2\Theta_{c1}\sin 3\Omega t$$

$$-\frac{3}{4}(b^2+d^2)\Omega^2\Theta_{s1}\cos 3\Omega t+\frac{3}{4}(a^2+c^2)\Omega^2\Theta_{s1}\cos 3\Omega t+\frac{3}{2}(ab+cd)\Omega^2\psi_{c1}\sin 3\Omega t$$

$$-\frac{3}{2}(ab+cd)\Omega^2\psi_{s1}\cos 3\Omega t-\frac{3}{2}(ab+cd)\Omega^2\Theta_{c1}\cos 3\Omega t-\frac{3}{2}(ab+cd)\Omega^2\Theta_{s1}\sin 3\Omega t$$

$$+\tfrac{1}{2}(b-d)\Omega^2 u_{c1}+\tfrac{1}{2}(b-d)\Omega^2 u_{c1}\cos 2\Omega t-\tfrac{1}{2}(c-a)\Omega^2 u_{c1}\sin 2\Omega t-\tfrac{1}{2}(c-a)\Omega^2 u_{s1}$$

$$+\tfrac{1}{2}(c-a)\Omega^2 u_{s1}\cos 2\Omega t+\tfrac{1}{2}(b-d)\Omega^2 u_{s1}\sin 2\Omega t+\tfrac{1}{2}(d-b)g\psi_{c1}+\tfrac{1}{2}(d-b)g\psi_{c1}\cos 2\Omega t$$

$$+\tfrac{1}{2}(d-b)g\psi_{s1}\sin 2\Omega t+\tfrac{1}{2}(d-b)g\Theta_{c1}\sin 2\Omega t+\tfrac{1}{2}(d-b)g\Theta_{s1}+\tfrac{1}{2}(b-d)g\Theta_{s1}\cos 2\Omega t$$

$$+\tfrac{1}{2}(c-a)g\psi_{c1}\sin 2\Omega t+\tfrac{1}{2}(c-a)g\psi_{s1}+\tfrac{1}{2}(a-c)g\psi_{s1}\cos 2\Omega t$$

$$+\tfrac{1}{2}(a-c)g\Theta_{c1}+\tfrac{1}{2}(a-c)g\Theta_{c1}\cos 2\Omega t+\tfrac{1}{2}(a-c)g\Theta_{s1}\sin 2\Omega t]\bar{J}$$

$$+\tfrac{1}{2}m[-\frac{5}{4}(b^2+d^2)\Omega^2\psi_{c1}\sin\Omega t-\frac{3}{4}(a^2+c^2)\Omega^2\psi_{c1}\sin\Omega t+\frac{3}{4}(b^2+d^2)\Omega^2\psi_{s1}\cos\Omega t$$

$$+\frac{5}{4}(a^2+c^2)\Omega^2\psi_{s1}\cos\Omega t+\frac{3}{4}(b^2+d^2)\Omega^2\Theta_{c1}\cos\Omega t+\frac{1}{4}(a^2+c^2)\Omega^2\Theta_{c1}\cos\Omega t$$

$$+\tfrac{1}{4}(b^2+d^2)\Omega^2\Theta_{s1}\sin\Omega t+\frac{3}{4}(a^2+c^2)\Omega^2\Theta_{s1}\sin\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\psi_{c1}\cos\Omega t$$

$$-\tfrac{1}{2}(ab+cd)\Omega^2\psi_{s1}\sin\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{c1}\sin\Omega t+\tfrac{1}{2}(ab+cd)\Omega^2\Theta_{s1}\cos\Omega t$$

$$+\frac{3}{4}(b^2+d^2)\Omega^2\psi_{c1}\sin 3\Omega t-\frac{3}{4}(a^2+c^2)\Omega^2\psi_{c1}\sin 3\Omega t-\frac{3}{4}(b^2+d^2)\Omega^2\psi_{s1}\cos 3\Omega t$$

$$+\frac{3}{4}(a^2+c^2)\Omega^2\psi_{s1}\cos 3\Omega t-\frac{3}{4}(b^2+d^2)\Omega^2\Theta_{c1}\cos 3\Omega t+\frac{3}{4}(a^2+c^2)\Omega^2\Theta_{c1}\cos 3\Omega t$$

$$-\frac{3}{4}(b^2+d^2)\Omega^2\Theta_{s1}\sin 3\Omega t+\frac{3}{4}(a^2+c^2)\Omega^2\Theta_{s1}\sin 3\Omega t-\frac{3}{2}(ab+cd)\Omega^2\psi_{c1}\cos 3\Omega t$$

$$-\frac{3}{2}(ab+cd)\Omega^2\psi_{s1}\sin 3\Omega t-\frac{3}{2}(ab+cd)\Omega^2\Theta_{c1}\sin 3\Omega t+\frac{3}{2}(ab+cd)\Omega^2\Theta_{s1}\cos 3\Omega t$$

$$+\tfrac{1}{2}(c-a)\Omega^2 u_{c1}+\tfrac{1}{2}(c-a)\Omega^2 u_{c1}\cos 2\Omega t+\tfrac{1}{2}(b-d)\Omega^2 u_{c1}\sin 2\Omega t$$

$$+\tfrac{1}{2}(b-d)\Omega^2 u_{s1}-\tfrac{1}{2}(b-d)\Omega^2 u_{s1}\cos 2\Omega t+\tfrac{1}{2}(c-a)\Omega^2 u_{s1}\sin 2\Omega t]\bar{K} \quad (4\text{-}21)$$

4.5 PLACEMENT OF FORCES AND MOMENTS INTO MATRIX FORM

In this final step of the matrix derivation equations (4-20) and (4-21) are placed in a matrix. Use is made of the relations between the unit vectors in the fixed reference frame and the internal force and internal moment state variables as presented in Table 1. Before inserting the equations for $\bar{F}$ and $\bar{M}$ in the sums of forces and of moments equations, a conversion to a more recognizable form of the terms in the equations for $\bar{F}$ and $\bar{M}$ is performed. This structural change is accomplished by inserting the following forms for the lengths a, b, c and d which are easily obtained from Fig. 3.

$$\begin{aligned} a &= (i_g + e)\cos E_i \\ b &= (i_g + e)\sin E_i \\ c &= (i_g - e)\cos E_i \\ d &= (i_g - e)\sin E_i \end{aligned} \quad (4\text{-}22)$$

The combinations of these lengths which occur in equations (4-20) and (4-21) are given as

$$\begin{aligned} a - c &= 2e\cos E_i \\ b - d &= 2e\sin E_i \\ a^2 + c^2 &= (i_g{}^2 + e^2) + (i_g{}^2 + e^2)\cos 2E_i \end{aligned}$$

$$b^2 + d^2 = (i_g^{\,2} + e^2) - (i_g^{\,2} + e^2)\cos 2\,E_i \tag{4-23}$$
$$ab + cd = (i_g^{\,2} + e^2)\sin 2\,E_i$$
$$a^2 + b^2 + c^2 + d^2 = 2(i_g^{\,2} + e^2)$$

Now, by inserting equation (4-23) into equations (4-20) and (4-21) and then placing the resulting equations into the conventional form of a transfer function equation, the final transfer matrix may be assembled.

The force equilibrium equation for the axial forces is

$$\Sigma F_X = 0$$

Upon rearranging this equation becomes

$$\begin{aligned} N^R_{c1}\cos\Omega t + N^R_{s1}\sin\Omega t &= N^L_{c1}\cos\Omega t + N^L_{s1}\sin\Omega t - m\Omega^2 u_{c1}\cos\Omega t \\ &-m\Omega^2 u_{s1}\sin\Omega t + (-2me\Omega^2\psi_{c1}\sin E_i + 2me\Omega^2\Theta_{s1}\sin E_i \\ &+2me\Omega^2\psi_{s1}\cos E_i + 2me\Omega^2\Theta_{c1}\cos E_i)\cos 2\Omega t + (-2me\Omega^2\psi_{s1}\sin E_i \\ &-2me\Omega^2\Theta_{c1}\sin E_i - 2me\Omega^2\psi_{c1}\cos E_i + 2me\Omega^2\Theta_{s1}\cos E_i)\sin 2\Omega t \end{aligned} \tag{4-24}$$

The following equation is used for the equilibrium of forces in the Y direction

$$\Sigma F_Y = 0$$

which, after rearranging can be written as

$$\begin{aligned} -V^R_{Yc1}\cos\Omega t - V^R_{Ys1}\sin\Omega t &= -\,V^L_{Yc1}\cos\Omega t - V^L_{Ys1}\sin\Omega t \\ &+(m\Omega^2 v_{c1} + me\Omega^2\cos E_i)\cos\Omega t + (m\Omega^2 v_{s1} - me\Omega^2\sin E_i)\sin\Omega t \end{aligned}$$

$$+(-2me\Omega^2\phi_{c1}\sin E_i + 2me\Omega^2\phi_{s1}\cos E_i)\cos 2\Omega t$$

$$+(-2me\Omega^2\phi_{c1}\cos E_i - 2me\Omega^2\phi_{s1}\sin E_i)\sin 2\Omega t \qquad (4\text{-}25)$$

Rearranging the equation

$$\Sigma F_Z = 0$$

results in the following equation for the equilibrium of forces in the Z direction.

$$V^R_{Zc1}\cos\Omega t + V^R_{Zs1}\sin\Omega t = V^L_{Zc1}\cos\Omega t + V^L_{Zs1}\sin\Omega t$$

$$+(-m\Omega^2 w_{c1} - me\Omega^2\sin E_i)\cos\Omega t + (-m\Omega^2 w_{s1} - me\Omega^2\cos E_i)\sin\Omega t$$

$$+(-2me\Omega^2\phi_{c1}\cos E_i - 2me\Omega^2\phi_{s1}\sin E_i)\cos 2\Omega t$$

$$+(2me\Omega^2\phi_{c1}\sin E_i - 2me\Omega^2\phi_{s1}\cos E_i)\sin 2\Omega t - mg \qquad (4\text{-}26)$$

The torque equilibrium equation

$$\Sigma T = 0$$

is reordered into transfer function form to give the following equation for torsional load states.

$$T^R_{c1}\cos\Omega t+T^R_{s1}\sin\Omega t=T^L_{c1}\cos\Omega t+T^L_{s1}\sin\Omega t$$

$$+[-m\Omega^2(i_g^{\,2}+e^2)\phi_{c1}-meg\cos E_i]\cos\Omega t+[-m\Omega^2(i_g^{\,2}+e^2)\phi_{s1}+meg\sin E_i]\sin\Omega t$$

$$+[\tfrac{1}{2}me\Omega^2 v_{s1}\cos E_i-\tfrac{1}{2}me\Omega^2 w_{c1}\cos E_i+\tfrac{1}{2}me\Omega^2 v_{c1}\sin E_i+\tfrac{1}{2}me\Omega^2 w_{s1}\sin E_i$$

$$+\tfrac{1}{2}meg\phi_{s1}\cos E_i+\tfrac{1}{2}meg\phi_{c1}\sin E_i]$$

$$+[-\tfrac{1}{2}me\Omega^2 v_{s1}\cos E_i-\tfrac{1}{2}me\Omega^2 w_{c1}\cos E_i+\tfrac{1}{2}me\Omega^2 v_{c1}\sin E_i-\tfrac{1}{2}me\Omega^2 w_{s1}\sin E_i$$

$$-\tfrac{1}{2}meg\phi_{s1}\cos E_i+\tfrac{1}{2}meg\phi_{c1}\sin E_i]\cos 2\Omega t$$

$$+[\tfrac{1}{2}me\Omega^2 v_{c1}\cos E_i-\tfrac{1}{2}me\Omega^2 w_{s1}\cos E_i+\tfrac{1}{2}me\Omega^2 v_{s1}\sin E_i+\tfrac{1}{2}me\Omega^2 w_{c1}\sin E_i$$

$$+\tfrac{1}{2}meg\phi_{c1}\cos E_i+\tfrac{1}{2}meg\phi_{s1}\sin E_i]\sin 2\Omega t \qquad (4\text{-}27)$$

The radius of gyration, i_g, above is related to I_x, the rotational moment of inertia about the x axis by the equation

$$i_g = \sqrt{\frac{I_x}{m}}$$

The reordering of the moment equilibrium equation about the y axis

$$\Sigma M_y = 0$$

results in the equation

$$M^R_{Yc1}\cos\Omega t+M^R_{Ys1}\sin\Omega t=M^L_{Yc1}\cos\Omega t+M^L_{Ys1}\sin\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[-\tfrac{1}{2}\psi_{c1}-\tfrac{1}{4}\psi_{c1}\cos 2E_i+\Theta_{s1}-\tfrac{1}{4}\Theta_{s1}\cos 2E_i+\tfrac{1}{4}\psi_{s1}\sin 2E_i$$

$$-\tfrac{1}{4}\Theta_{c1}\sin 2E_i]\cos\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[-\tfrac{1}{2}\psi_{s1}+\tfrac{1}{4}\psi_{s1}\cos 2E_i-\Theta_{c1}-\tfrac{1}{4}\Theta_{c1}\cos 2E_i+\tfrac{1}{4}\psi_{c1}\sin 2E_i$$

$$+\tfrac{1}{4}\Theta_{s1}\sin 2E_i]\sin\Omega t$$

$$+[-\tfrac{1}{2}me\Omega^2 u_{c1}\sin E_i-\tfrac{1}{2}me\Omega^2 u_{s1}\cos E_i+\tfrac{1}{2}meg\psi_{c1}\sin E_i+\tfrac{1}{2}meg\Theta_{s1}\sin E_i$$

$$+\tfrac{1}{2}meg\psi_{s1}\cos E_i-\tfrac{1}{2}meg\Theta_{c1}\cos E_i]$$

$$+[-\tfrac{1}{2}me\Omega^2 u_{c1}\sin E_i+\tfrac{1}{2}me\Omega^2 u_{s1}\cos E_i+\tfrac{1}{2}meg\psi_{c1}\sin E_i$$

$$-\tfrac{1}{2}meg\psi_{s1}\cos E_i-\tfrac{1}{2}meg\Theta_{c1}\cos E_i-\tfrac{1}{2}meg\Theta_{s1}\sin E_i]\cos 2\Omega t$$

$$+[-\tfrac{1}{2}me\Omega^2 u_{c1}\cos E_i - \tfrac{1}{2}me\Omega^2 u_{s1}\sin E_i + \tfrac{1}{2}meg\psi_{c1}\cos E_i$$

$$+\tfrac{1}{2}meg\psi_{s1}\sin E_i + \tfrac{1}{2}meg\Theta_{c1}\sin E_i - \tfrac{1}{2}meg\Theta_{s1}\cos E_i]\sin 2\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[\tfrac{3}{4}\psi_{c1}\cos 2E_i - \tfrac{3}{4}\Theta_{s1}\cos 2E_i + \tfrac{3}{4}\psi_{s1}\sin 2E_i + \tfrac{3}{4}\Theta_{c1}\sin 2E_i]\cos 3\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[\tfrac{3}{4}\psi_{s1}\cos 2E_i + \tfrac{3}{4}\Theta_{c1}\cos 2E_i - \tfrac{3}{4}\psi_{c1}\sin 2E_i + \tfrac{3}{4}\Theta_{s1}\sin 2E_i]\sin 3\Omega t \quad (4\text{-}28)$$

In this equation i_g is related to the rotational moment of inertia about the y axis, I_y, by the equation

$$i_g = \sqrt{\frac{I_y}{m}}$$

The final equation for the equilibrium of moments about the Z axis is

$$\Sigma M_Z = 0$$

After rearranging, this equation becomes

$$M^R_{Zc1}\cos\Omega t + M^R_{Zs1}\sin\Omega t = M^L_{Zc1}\cos\Omega t + M^L_{Zs1}\sin\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[-\psi_{s1} - \tfrac{1}{4}\psi_{s1}\cos 2E_i - \tfrac{1}{2}\Theta_{c1} + \tfrac{1}{4}\Theta_{c1}\cos 2E_i$$

$$-\tfrac{1}{4}\psi_{c1}\sin 2E_i - \tfrac{1}{4}\Theta_{s1}\sin 2E_i]\cos\Omega t$$

$$+m\Omega^2(i_g{}^2+e^2)[\psi_{c1} - \tfrac{1}{4}\psi_{c1}\cos 2E_i - \tfrac{1}{2}\Theta_{s1} - \tfrac{1}{4}\Theta_{s1}\cos 2E_i$$

$$+\tfrac{1}{4}\psi_{s1}\sin 2E_i - \tfrac{1}{4}\Theta_{c1}\sin 2E_i]\sin\Omega t$$

$$+[\tfrac{1}{2}me\Omega^2 u_{c1}\cos E_i - \tfrac{1}{2}me\Omega^2 u_{s1}\sin E_i] + [\tfrac{1}{2}me\Omega^2 u_{c1}\cos E_i + \tfrac{1}{2}me\Omega^2 u_{s1}\sin E_i]\cos 2\Omega t$$

$$+[-\tfrac{1}{2}me\Omega^2 u_{c1}\sin E_i + \tfrac{1}{2}me\Omega^2 u_{s1}\cos E_i]\sin 2\Omega t$$

$$+m\Omega^2(i_g^{\,2}+e^2)[-\frac{3}{4}\psi_{s1}\cos 2E_i-\frac{3}{4}\theta_{c1}\cos 2E_i+\frac{3}{4}\psi_{c1}\sin 2E_i-\frac{3}{4}\theta_{s1}\sin 2E_i]\cos 3\Omega t$$

$$+m\Omega^2(i_g^{\,2}+e^2)[\frac{3}{4}\psi_{c1}\cos 2E_i-\frac{3}{4}\theta_{s1}\cos 2E_i+\frac{3}{4}\psi_{s1}\sin 2E_i+\frac{3}{4}\theta_{c1}\sin 2E_i]\sin 3\Omega t \quad (4\text{-}29)$$

In equation (4-29) i_g is related to the rotational moment of inertia about the z axis, I_z, by the equation

$$i_g = \sqrt{\frac{I_z}{m}}$$

Due to the assumption of a circular shaft and a circular disk made herein, the rotational moments of inertia I_y and I_z are equal. This results in the radius of gyrations used in equations (4-28) and (4-29) also being equal.

The transfer matrix for the case of one times running speed, Ωt, can now be assembled from equations (4-24) through (4-29), inclusive. This matrix will contain only those elements from the above mentioned equations which contain the terms $\sin\Omega t$ or $\cos\Omega t$. This final extended point transfer matrix is given in Fig. 10 which is inside the pocket located on the inside back cover of this thesis. The value zero is assigned to any empty block in this figure. Further discussion of the terms in equations (4-24) through (4-29) which are not included in Fig. 10 will be given in a subsequent section.

5. DISCUSSION OF RESULTS

5.1 VERIFICATION OF 1x† MATRIX ELEMENTS

Now that the desired matrix form has been obtained, it is necessary to check the matrix for correctness. This will be done using several methods. One method involves reducing the matrix, via simplifying assumptions and coordinate transformations, to a form which can be compared with an accepted form from the literature. The other method of checking will involve deriving matrix elements assuming there is no cross coupling effects. By using one or the other of these techniques, the total matrix will be verified.

In the matrix derivation process it was assumed that the mass, disk or model of the shaft at a station, was rigid. Due to this assumption it is obvious that all of the deflections, linear and angular are the same to the left and right of the station. This then accounts for the ones (1) appearing in the principal diagonal elements corresponding to the deflection state variables.

The matrix elements describing axial motion will be checked first. Starting with the equation for axial motion given by Pestel and Leckie [21] on page 53

$$N^R = N^L - m\Omega^2 u$$

and substituting the assumed form of the state variables u and N given in section 3.3 of this work

†The symbol 1x will be used to describe whirling at $1\Omega t$. Similar meanings will be given for 2x, 3x, etc.

$$u = u_{c1}\cos\Omega t + u_{s1}\sin\Omega t$$

$$N = N_{c1}\cos\Omega t + N_{s1}\sin\Omega t$$

the following equation for axial motion is obtained.

$$N^R_{c1}\cos\Omega t + N^R_{s1}\sin\Omega t = N^L_{c1}\cos\Omega t + N^L_{s1}\sin\Omega t$$

$$-m\Omega^2 u_{c1}\cos\Omega t - m\Omega^2 u_{s1}\sin\Omega t \qquad (5\text{-}1)$$

This equation, when put in matrix form, gives the same matrix entries for axial motion as those shown in Fig. 10.

The matrix entries for torsional motion will now be substantiated. There are two possible effects of torsional motion assumed here, inertial torque and torque due to gravity acceleration. The inertial torque is due to the oscillation of the masses by an amount ϕ about the X axis. Using the normal procedure of displacing the masses a small amount ϕ in the positive X direction the following equation for inertial torques can be written.

$$T_{in} = -\frac{m}{2}(i_g - e)^2\ddot{\phi} - \frac{m}{2}(i_g + e)^2\ddot{\phi}$$

Reference has been made to Fig. 3 in order to arrive at the lever arms $(i_g - e)$ and $(i_g + e)$ used in this equation. However, the figure is now viewed from the absolute reference frame where it is assumed that the rotating reference frame has rotated an amount Ωt about the X axis. Now, the assumed form of the response of the state variable ϕ

$$\phi = \phi_{c1}\cos\Omega t + \phi_{s1}\sin\Omega t$$

is differentiated twice with respect to time and then substituted into the simplified form of the previous equation. This results in

$$T_{in} = m\Omega^2(i_g^{\,2}+e^2)\phi_{c1}\cos\Omega t+m\Omega^2(i_g^{\,2}+e^2)\phi_{s1}\sin\Omega t \tag{5-2}$$

The force and then the torque due to gravity can also be found by viewing Fig. 3 from the fixed reference frame. An equation for the gravitational torque is

$$T_g = -\frac{m}{2}g(i_g-e)\cos(E_i+\Omega t)+\frac{m}{2}g(i_g+e)\cos(E_i+\Omega t)$$

which can be rewritten as

$$T_g = mge\ \cos(E_i+\Omega t)$$

The final form of the torque due to gravity can be obtained by using the angle-sum trigonometric relations. The final form is

$$T_g = mge\ \cos E_i\cos\Omega t - mge\ \sin E_i\sin\Omega t \tag{5-3}$$

The inertial and gravitational torques can now be put into the rearranged transfer function equation form of the equation

$$\Sigma T = 0$$

By using the assumed form of the state variable response for T, this becomes

$$\begin{aligned} T^R_{c1}\cos\Omega t+T^R_{s1}\sin\Omega t &= T^L_{c1}\cos\Omega t+T^L_{s1}\sin\Omega t \\ &+[m\Omega^2(i_g^{\,2}+e^2)\phi_{c1}+mge\ \cos E_i]\cos\Omega t \\ &+[m\Omega^2(i_g^{\,2}+e^2)\phi_{s1}-mge\ \sin E_i]\sin\Omega t \end{aligned} \tag{5-4}$$

This equation, when put in matrix form, gives the same matrix entries for torsional motion as those given in Fig. 10.

As just seen, the verification of the axial and torsional motion sections of Fig. 10 has been done in the stationary coordinate system. The remaining terms will be verified by transforming, via equations (3-7) or (3-8), from one reference frame to the other. This is because forms of the point transfer matrix are known for certain cases in terms of rotating coordinate system variables.

The remaining terms in the matrix involve the flexural vibrations of a beam. Only the terms in the M_{Zc1}, M_{Zs1}, V_{Yc1} and V_{Ys1} rows will be checked in detail. The remaining matrix elements can be verified using a similar procedure. Pestel and Leckie [22] on page 127 give the following equation for the shear forces as seen in the rotating reference frame.

$$-V_y^R = -V_y^L + m\Omega^2 v_y + me\Omega^2 \cos E_i \tag{5-5a}$$

$$V_z^R = V_z^L - m\Omega^2 w_z - me\Omega^2 \sin E_i \tag{5-5b}$$

In their Catalogue of Transfer Matrices, Pestel and Leckie [22] give the following entries for a point mass whirling at Ωt in the rotating coordinate system

$$M_y^R = M_y^L + m\Omega^2 (i_g{}^2 + e^2)\psi_y \tag{5-6a}$$

$$M_z^R = M_z^L + m\Omega^2 (i_g{}^2 + e^2)\Theta_z \tag{5-6b}$$

In their derivation Pestel and Leckie assume circular whirl. Therefore, in order to compare the results of Fig. 10 with those of Pestel and Leckie the same assumption is now made and will be applied to the matrix of Fig. 10. There are two cases of circular whirl which will be treated within this verification procedure. These are when the unbalance lies solely along the y axis ($E_i = 0$) and when the unbalance is along the z axis ($E_i = \frac{1}{2}\pi$) of the rotating reference frame. These two cases are chosen strictly for convenience and are in no way restrictive to this procedure.

First, the case where E_i is equal to zero will be treated. It is obvious that the only motion which will result in the moving frame will be along the y axis (linear displacement and shear) and about the z axis (angular displacement and moment). All other state variables in the moving frame will be zero.

The linear displacements in the y and z directions are given by the $\bar{j}$ and $\bar{k}$ components, respectively, of the $\bar{r}_{rot}$ vector, equation (4-2). These are

$$v_y = \tfrac{1}{2}v_{c1}+\tfrac{1}{2}w_{s1}+\tfrac{1}{2}v_{c1}\cos 2\Omega t-\tfrac{1}{2}w_{s1}\cos 2\Omega t+\tfrac{1}{2}v_{s1}\sin 2\Omega t+\tfrac{1}{2}w_{c1}\sin 2\Omega t \tag{5-7a}$$

$$w_z = -\tfrac{1}{2}v_{s1}+\tfrac{1}{2}w_{c1}+\tfrac{1}{2}v_{s1}\cos 2\Omega t+\tfrac{1}{2}w_{c1}\cos 2\Omega t-\tfrac{1}{2}v_{c1}\sin 2\Omega t+\tfrac{1}{2}w_{s1}\sin 2\Omega t \tag{5-7b}$$

The angular displacements about the y and z axes are given by the second and third equations, respectively, of equation (4-7). These are repeated below

$$\psi_y = \tfrac{1}{2}\psi_{c1}+\tfrac{1}{2}\Theta_{s1}+\tfrac{1}{2}\psi_{c1}\cos 2\Omega t-\tfrac{1}{2}\Theta_{s1}\cos 2\Omega t+\tfrac{1}{2}\psi_{s1}\sin 2\Omega t+\tfrac{1}{2}\Theta_{c1}\sin 2\Omega t \tag{5-8a}$$

$$\Theta_z = -\tfrac{1}{2}\psi_{s1}+\tfrac{1}{2}\Theta_{c1}+\tfrac{1}{2}\psi_{s1}\cos 2\Omega t+\tfrac{1}{2}\Theta_{c1}\cos 2\Omega t-\tfrac{1}{2}\psi_{c1}\sin 2\Omega t+\tfrac{1}{2}\Theta_{s1}\sin 2\Omega t \tag{5-8b}$$

Now, from the assumption of circular whirl it is known that the angular displacement about the y axis, ψ_y, given by equation (5-8a) is zero. For this to be true, it must be that

$$
\begin{aligned}
&\tfrac{1}{2}\psi_{c1} + \tfrac{1}{2}\Theta_{s1} = 0 \\
&\tfrac{1}{2}\psi_{c1}\cos 2\Omega t - \tfrac{1}{2}\Theta_{s1}\cos 2\Omega t = 0 \\
&\tfrac{1}{2}\psi_{s1}\sin 2\Omega t + \tfrac{1}{2}\Theta_{c1}\sin 2\Omega t = 0
\end{aligned}
\tag{5-9}
$$

One solution of these three simultaneous equations is

$$\psi_{c1} = \Theta_{s1} = 0 \tag{5-10a}$$

and

$$\psi_{s1} = -\Theta_{c1} \tag{5-10b}$$

When equations (5-10a) and (5-10b) are substituted into equation (5-8b), the following alternate description for the angular displacement about the z axis when $E_i = 0$ is found.

$$\Theta_z = \Theta_{c1}$$

Since the angular displacement and moment about the z axis are related by the beam differential equations the analogous form for the moment about z becomes

$$M_z = M_{Zc1}$$

By inserting the relations given in equations (5-10a) and (5-10b) for $E_i = 0$ into the M_{Zc1} row of Fig. 10, one gets the following

$$M^R_{Zc1}\cos\Omega t = M^L_{Zc1}\cos\Omega t - \tfrac{1}{4}m\Omega^2(i_g{}^2+e^2)\Theta_{c1}\cos\Omega t + 0 + 0$$

$$+ 1\tfrac{1}{4}m\Omega^2(i_g{}^2+e^2)\Theta_{c1}\cos\Omega t$$

Now by simplifying this equation, cancelling common factors and substituting for M_{Zc1} and Θ_{c1} the equation for the moments about the z axis of the rotating coordinate system becomes

$$M^R_z = M^L_z + m\Omega^2(i_g{}^2+e^2)\Theta_z \tag{5-11}$$

Equation (5-11) is the same as equation (5-6b) originally given by Pestel and Leckie.

From the assumption of circular whirl and $E_i = 0$ it is known that the linear displacement along the z axis, w_z, given by equation (5-7b) is zero. This leads to the following set of simultaneous equations

$$\begin{aligned} -\tfrac{1}{2}\,v_{s1} + \tfrac{1}{2}\,w_{c1} &= 0 \\ \tfrac{1}{2}\,v_{s1}\cos 2\Omega t + \tfrac{1}{2}\,w_{c1}\cos 2\Omega t &= 0 \\ -\tfrac{1}{2}\,v_{c1}\sin 2\Omega t + \tfrac{1}{2}\,w_{s1}\sin 2\Omega t &= 0 \end{aligned} \tag{5-12}$$

A solution for this set of equations is

$$v_{s1} = w_{c1} = 0 \tag{5-13a}$$

and

$$v_{c1} = w_{s1} \tag{5-13b}$$

By substituting equations (5-13a) and (5-13b) into equation (5-7a), the following alternate description for v_y when $E_i = 0$ results

$$v_y = v_{c1}$$

Recalling that the state variables for linear displacement and shear are related by the beam first order differential equations it is clear that a similar form

$$V_y = V_{Yc1}$$

for the shear along the y axis results. The following equation arises as a result of inserting equations (5-13a) and (5-13b) into the shear row, V_{Yc1}, shown in Fig. 10.

$$- V^R_{Yc1}\cos\Omega t = - V^L_{Yc1}\cos\Omega t + m\Omega^2 v_{c1}\cos\Omega t + me\Omega^2\cos E_i\cos\Omega t$$

Now, by substituting the rotating coordinate state variable equivalents of $- V_{Yc1}$ and v_{c1} one gets, after common term cancellation

$$- V^R_y = - V^L_y + m\Omega^2 v_y + me\Omega^2\cos E_i \tag{5-14}$$

When placed in matrix form this gives the same results as equation (5-5a) which is given by Pestel and Leckie. This then completes the verification for the case where $E_i = 0$. The unbalance will now be assumed to lie solely along the z axis ($E_i = \frac{1}{2}\pi$).

For the case where $E_i = \frac{1}{2}\pi$ the motion which can occur in the rotating reference frame will be along the z axis (linear displacement and shear) and about the y axis (angular displacement and moment). All other state variables in the moving reference frame will have a value of zero.

The angular displacement, Θ_z, given by equation (5-8b) is zero since there are no loads to cause deflection in that direction. This will result in a set of three simultaneous equations

$$
\begin{aligned}
-\tfrac{1}{2}\,\psi_{s1} + \tfrac{1}{2}\,\Theta_{c1} &= 0 \\
\tfrac{1}{2}\,\psi_{s1}\cos 2\Omega t + \tfrac{1}{2}\,\Theta_{c1}\cos 2\Omega t &= 0 \qquad (5\text{-}15) \\
-\tfrac{1}{2}\,\psi_{c1}\sin 2\Omega t + \tfrac{1}{2}\,\Theta_{s1}\sin 2\Omega t &= 0
\end{aligned}
$$

which has the possible solution

$$\psi_{s1} = \Theta_{c1} = 0 \qquad (5\text{-}16a)$$

and

$$\psi_{c1} = \Theta_{s1} \qquad (5\text{-}16b)$$

The result of substituting equations (5-16a) and (5-16b) into equation (5-8a) is an alternate description of the angular displacement ψ_y for the case when $E_i = \tfrac{1}{2}\pi$.

$$\psi_y = \Theta_{s1}$$

The analogous form for the moment about the y axis is

$$M_y = M_{Zs1}$$

By inserting equations (5-16a) and (5-16b) for $E_i = \tfrac{1}{2}\pi$ into the M_{Zs1} row of Fig. 10, one gets the following

$$
\begin{aligned}
M^R_{Zs1}\sin\Omega t = M^L_{Zs1}\sin\Omega t &+ 0 - \tfrac{1}{4}m\Omega^2(i_g{}^2+e^2)\Theta_{s1}\sin\Omega t + 0 \\
&+ 1\tfrac{1}{4}\, m\Omega^2(i_g{}^2+e^2)\Theta_{s1}\sin\Omega t
\end{aligned}
$$

which after simplifying, cancelling common factors and substituting for M_{Zs1} and Θ_{s1} becomes

$$M_y^R = M_y^L + m\Omega^2(i_g{}^2+e^2)\psi_y \tag{5-17}$$

This then is the equation for the moments about the y axis of the rotating coordinate system which is the same as that given by Pestel and Leckie, given here as equation (5-6a).

The assumption of circular whirl for the case where $E_i = \frac{1}{2}\pi$ forces the linear displacement along the y rotating axis, v_y, to be zero. This leads to a set of three simultaneous equations

$$\begin{aligned}
&\tfrac{1}{2}\, v_{c1} + \tfrac{1}{2}\, w_{s1} = 0 \\
&\tfrac{1}{2}\, v_{c1}\cos 2\Omega t - \tfrac{1}{2}\, w_{s1}\cos 2\Omega t = 0 \\
&\tfrac{1}{2}\, v_{s1}\sin 2\Omega t + \tfrac{1}{2}\, w_{c1}\sin 2\Omega t = 0
\end{aligned} \tag{5-18}$$

A solution for this set of equations is

$$v_{c1} = w_{s1} = 0 \tag{5-19a}$$

and

$$v_{s1} = -\, w_{c1} \tag{5-19b}$$

By inserting the relations given in equations (5-19a) and (5-19b) into equation (5-7b) the following alternate form of the state variable w_z when $E_i = \frac{1}{2}\pi$ results.

$$w_z = -\, v_{s1}$$

The beam relations are then used to arrive at the alternate form

$$V_z = - V_{Ys1}$$

of the shear in the z direction of the rotating frame. Substituting equations (5-19a) and (5-19b) into the $- V_{Ys1}$ row of Fig. 10 results in

$$- V^R_{Ys1}\sin\Omega t = - V^L_{Ys1}\sin\Omega t + m\Omega^2 v_{s1}\sin\Omega t - me\Omega^2 \sin E_i \sin\Omega t$$

After replacing $- V_{Ys1}$ by V_z and v_{s1} by $- w_z$ and cancelling common factors the following equation for shear in the z direction results.

$$V^R_z = V^L_z - m\Omega^2 w_z - me\Omega^2 \sin E_i \qquad (5\text{-}20)$$

This is the same equation as that given by Pestel and Leckie, given here as equation (5-5b).

As previously mentioned a similar procedure will verify the matrix elements in the M_{Yc1}, M_{Ys1}, V_{Zc1} and V_{Zs1} rows of Fig. 10. However, this will not be done here.

5.2 DISCUSSION OF REMAINING TERMS

The terms in equations (4-24) through (4-29) which are not included in the transfer matrix given in Fig. 10 will now be discussed. First, this discussion will concern those terms involving harmonic sine or cosine functions which have an argument of $2\Omega t$ or $3\Omega t$. These terms will be called 2x and 3x extended forces. This explains why each state variable in the preceeding matrix derivation was given the subscript 1. With this subscript each of the 2x and 3x extended terms are readily identifiable as having originated in the 1x derivation. Therefore, before performing subsequent higher harmonic analyses one knows the values of these extension terms upon completion of a 1x analysis.

The higher harmonic analyses can take two forms. The first type of analysis is performed when, after doing an analysis for whirl at Ωt, the analyst decides that there are no significant effects of whirl above one times running speed. To do this type of analysis a matrix for straight line vibration, such as that given by Pestel and Leckie [22] on page 386, is used in conjunction with the appropriate extended forcing function terms from the 1x whirl case. Appropriate transformations of these extensions should be made in order to make their application to Pestel and Leckie's matrix valid. If, however, the analyst wishes to do a 2x whirl analysis the matrix given in Fig. 10 is used with the following two modifications; Ω is replaced by 2Ω in the total 1x matrix (including the extended column) and the 2x extension terms from the 1x derivation are added to the extended column of the 2x matrix. The assumption of superposition is used when deriving higher harmonic whirl matrices. For example, both the state variable responses and the rotating coordinate system are moving at 2Ω in the 2x whirl case. The proper solution for this 2x whirl case can not be obtained by letting the state variable responses and the rotating coordinate system move at different rates.

In order to do a complete analysis, the principle of superposition is used once again. Upon completion of a 1x whirl analysis and subsequent 2x and 3x analyses, either straight line vibration or whirl, the total time response (including all 1x whirl effects) for a state variable is determined by a summation of each response. This is given equationally for the linear time response in the Z direction as

$$w = w_{c1}\cos\Omega t + w_{s1}\sin\Omega t + w_{c2}\cos 2\Omega t + w_{s2}\sin 2\Omega t$$
$$+ w_{c3}\cos 3\Omega t + w_{s3}\sin 3\Omega t$$

The remaining state variables have the same form for their total time responses.

The only terms from equations (4-24) through (4-29) which still have not been explained are classified as static extended forcing functions. This is because these terms are not multiplied by harmonic functions, sine or cosine. Therefore, they are to be inserted in an extended forcing function column of a static point transfer matrix. The only other terms present in such a point matrix are those in the principal diagonal elements of the matrix. All these principal diagonal terms have the value of unity. The reason for this is that all terms with an Ω multiplier go to zero when Ω is zero, as there is no motion present for the masses.

6. CONCLUSIONS

It is now possible, using the methods outlined by Pilkey [6], to do a static, stability and dynamic response analysis for a rotor-bearing system undergoing whirl. In such analyses use is made of the point transfer matrix developed herein, the field matrix for a whirling shaft given by Pestel and Leckie [22] and the point transfer matrix for a hydrodynamic bearing also given by Pestel and Leckie [22]. Some modification of the matrices given by Pestel and Leckie is necessary in order to have a consistent form of the three transfer matrices. A switch to the form of the state variables used in the development of the point mass transfer matrix is necessary. In addition to the variable change, a coordinate transformation should be done to both matrices given by Pestel and Leckie to get them into the absolute reference frame.

The possibility exists to analytically generate a spectrum analysis using the same three transfer matrices. The method used to do this analysis will now be briefly outlined. The same procedure is used in order to obtain a spectral analysis for all of the state variables. However, as an illustration only the analysis for the spectral components of the linear displacement in the Z direction whirling at Ω radians per second will be performed. First, an analysis is performed at the running speed Ω in order to find the response of the state variables w_{c1} and w_{s1}. Now, these two responses are combined mathematically (not a simple summation) to give the total response w_1 at the running speed Ω. Next, due to the extended forcing functions occurring at 2Ω, another analysis (either whirl or straight line vibration) must be performed at 2 times running speed. This will result in the two state variable responses, w_{c2} and w_{s2}, which are then combined to give the total response, w_2, at 2Ω. A final analysis must be performed at 3 times running speed because of the extended terms originating in the 1x whirl case. From this analysis the total response, w_3, at 3 times running speed can be obtained. A graph of amplitude versus frequency can now be obtained. The three magnitudes w_1, w_2 and w_3 are now plotted at the values of Ω, 2Ω and 3Ω, respectively. This is the equivalent of the spectral plot which would be obtained by a Real Time Analyzer or Fast Fourier Transform machine doing an experimental analysis of the shaft motion as seen by a displacement probe which is anchored or referenced to the absolute coordinate system.

One significance of the work done here is the presence of the state variables ϕ_{c1} and ϕ_{s1} in the extended column of the shear force rows for the 2Ω matrix. What this indicates is the possible coupling of torsional oscillations and lateral shears, in both the Y and Z directions. Thus, if torsional oscillations occur when a rotor is traversed through a speed range, dynamic shear forces develop at 2Ω. These dynamic shear forces can then excite lateral motions of the shaft. Therefore, if the lateral response at 2 times running speed increases when the rotor traverses a speed range this could indicate torsional response at 1Ω. However, in addition to the above mentioned coupling mechanism there are other known sources of dynamic forces (flexible mechanical coupling, universal joints, etc.) which can cause a lateral response at 2 times running speed. The significance of this coupling lies in the ability to monitor, via lateral motion transducers, the torsional motion of a rotor-bearing system. Thus, it may be possible to eliminate costly torque transducer systems which are often very complicated and as such unreliable.

7. RECOMMENDATIONS

7.1 FURTHER THEORETICAL WORK ON MASS MATRIX

In section 5.2 superposition was used to get a solution for cases of higher order whirl. Superposition was then used to obtain the total time response for all state variables by a simple summation process. It is thought that a more complete analysis can be obtained by using the concept of Fourier series, which is also dependent upon the principle of superposition. If the form of the state variables is described by a Fourier series such as

$$w = w_{c0} + w_{c1}\cos\Omega t + w_{s1}\sin\Omega t + w_{c2}\cos 2\Omega t + w_{s2}\sin 2\Omega t + \ldots$$

and if a coordinate transformation is performed, a total time analysis could be done in one step which will generate appropriate transfer matrices for 1x, 2x, 3x, ... However, this procedure would require a very large effort if carried out by hand. Such a long and complicated derivation is, however, well suited to a digital computer solution if done by an algebraic manipulating program package such as FORMAC.

A logical modification of the derivation given herein is the extension to the case of nonsynchronous precession. Upon completion of this work a check could be performed on the new matrix to see if it reduces to the matrix given in Fig. 10 when an assumption of synchronous precession is made.

Due to the assumption of superposition the developed synchronous precession transfer matrix, and ultimately the expanded nonsynchronous precession plus static sag transfer matrix, can be used in many applications other than simple disks. After developing the specific extended forcing function column this point transfer matrix can be applied to such varied dynamic situations as; geared systems, axial flow turbines and radial flow pumps. For example, the possible coupling effects due to the torsional-lateral interaction of gear teeth need to be investigated. In torsionally branched systems, gears have accounted for the coupling of the lateral vibrations in one shaft "a" to the lateral vibrations of another remote shaft "b". In fact, the lateral vibrations in shaft "a" were occurring at the frequency of shaft "b" but with an amplitude several times the amplitude of shaft "b" at the frequency of shaft "b".

7.2 OTHER FURTHER THEORETICAL WORK

A more extensive and complete point transfer matrix for hydrodynamic bearings is needed. Lund [29] expands the transfer matrix given by Pestel and Leckie [22] to include cross-coupled stiffness and damping effects of the bearings. The form of the state variables used by Lund and those used in the mass matrix derivation herein is identical. The only modifications necessary are the expansion of Lund's terms to include axial and torsional motion and any coupling effects which may be present. Pestel and Leckie do refer to another paper written by Pestel [30] which also provides for more realistic bearing properties. However, the same modifications as before are needed in order to be compatible with the mass matrix. As previously mentioned in section 5.2, the field matrix for a whirling elastic shaft given by Pestel and Leckie [22] must also be made compatible to the mass matrix developed within this thesis.

7.3 EXPERIMENTAL WORK

After compatible bearing and shaft matrices are developed an analysis of a physical rotor-bearing system should be performed using the new mass matrix. The same physical system should then be studied using a Real Time or Fast Fourier Transform Analyzer. If the predictions of the theoretical analysis are largely confirmed by the experimental work then the work done within this thesis is substantiated. Then, if a related problem arises in the future an analyst can feel confident in the results obtained when using this matrix. If, however, the experimental results and the theoretical predictions do not agree then it is necessary to expand the scope and nature of the model. This would involve using the theoretical work discussed in section 7.1.

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*This citation can not be found in the cited reference. Citation 30 was taken from Pestel and Leckie's book [22]. Pestel must be contacted to obtain this reference.

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THE DEVELOPMENT OF A DYNAMICALLY COUPLED AXIAL-TORSIONAL-LATERAL POINT TRANSFER MATRIX FOR A WHIRLING ECCENTRIC MASS

by

Robert Carter Johnstone

(ABSTRACT)

A point transfer matrix for a whirling eccentric mass was developed. Two right handed cartesian coordinate systems, a fixed and a rotating, were used in the derivation. The state variables and the elliptical whirl orbit were described relative to the fixed coordinate system. The rotating coordinate system was used to describe the equivalent model of the mass. The effects of both inertial forces and gravitational forces were included. The inertial forces contained the effects of; centripetal acceleration, acceleration relative to a moving reference frame, Coriolis acceleration and a disguised form of tangential acceleration.

The transfer matrix which resulted from this derivation was verified. To verify the lateral motion equations the assumption of a circular whirl orbit in the fixed reference frame was made. The axial and torsional motion equations were verified by assuming no coupling effects.

Terms were found which had harmonic forms at twice and three times running speed. Static load terms were also found as a result of the synchronous whirl analysis. These static, 2 times and 3 times

terms are extended forcing functions which can be used in static, 2 times and 3 times running speed analyses, respectively.

	u_{C1}	N_{C1}	u_{S1}	N_{S1}	ϕ_{C1}	T_{C1}	ϕ_{S1}	T_{S1}	v_{C1}	θ_{C1}	M_{ZC1}	$-V_{YC1}$	v_{S1}	θ_{S1}	M_{ZS1}	$-V_{YS1}$	$-w_{C1}$	ψ_{C1}	M_{YC1}	V_{ZC1}	$-w_{S1}$	ψ_{S1}	M_{YS1}	V_{ZS1}	EXTENDED
u_{C1}	1																								
N_{C1}	$-m\Omega^2$	1																							
u_{S1}			1																						
N_{S1}			$-m\Omega^2$	1																					
ϕ_{C1}					1																				
T_{C1}					$-m(e^2+i_g^2)\Omega^2$	1																			$-m e g \cos E$
ϕ_{S1}							1																		
T_{S1}							$-m(e^2+i_g^2)\Omega^2$	1																	$m e g \sin E$
v_{C1}									1																
θ_{C1}										1															
M_{ZC1}										$-\frac{1}{2}m(e^2+i_g^2)\Omega^2$ $+\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$	1			$-\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$-\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$-m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$			
$-V_{YC1}$									$m\Omega^2$			1													$m e \Omega^2 \cos E$
v_{S1}													1												
θ_{S1}														1											
M_{ZS1}										$-\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$-\frac{1}{2}m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$	1			$m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$				$\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$			
$-V_{YS1}$													$m\Omega^2$			1									$-m e \Omega^2 \sin E$
$-w_{C1}$																	1								
ψ_{C1}																		1							
M_{YC1}										$-\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$				$-\frac{1}{2}m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$	1			$\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$			
V_{ZC1}																	$m\Omega^2$			1					$-m e \Omega^2 \sin E$
$-w_{S1}$																					1				
ψ_{S1}																						1			
M_{YS1}										$-m(e^2+i_g^2)\Omega^2$ $-\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$				$\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$\frac{1}{4}m(e^2+i_g^2)\Omega^2\sin 2E$				$-\frac{1}{2}m(e^2+i_g^2)\Omega^2$ $+\frac{1}{4}m(e^2+i_g^2)\Omega^2\cos 2E$	1		
V_{ZS1}																					$m\Omega^2$			1	$-m e \Omega^2 \cos E$
EXTENDED																									1

THE SYMBOL i_g IS USED AS A GENERAL INDICATOR FOR THE RADIUS OF GYRATION ABOUT AN AXIS. FOR FURTHER INFORMATION REFER TO SECTION 4.5 IN THE TEXT.

FIGURE 10. EXTENDED POINT TRANSFER MATRIX FOR A WHIRLING ECCENTRIC MASS IN AN ABSOLUTE COORDINATE SYSTEM (1 TIMES RUNNING SPEED)