

### III. CONCLUSIONS

The generalized smoothed boundary condition to rough surface scatter appears capable of explaining quite adequately the observed scatter from the rough interface between water and an overlying thin plate, for  $ka < 1$ , i.e., in the "low-frequency" limit. The observations used in this paper were those reported by McClanahan<sup>2</sup> and McClanahan and Diachok<sup>1</sup> for a model consisting of a thin lucite plate with corrugations over water. This strongly suggests that the technique should be applied in the context of low-frequency Arctic data—for which a variety of other factors may also enter into play.

<sup>1</sup>T. P. McClanahan and O. I. Diachok, "Scattering and reflection of acoustic waves by striated surfaces depicting the Arctic Ocean ice cover," J.

Acoust. Soc. Am. Suppl. 1 **84**, S218 (1988).

<sup>2</sup>T. P. McClanahan, "Scattering and reflection of acoustic waves by striated surfaces depicting the Arctic Ocean ice cover," Thesis, Catholic University of America, Washington, DC (1988).

<sup>3</sup>I. Tolstoy, "Effect of roughness on subsonic flexural waves in a thin plate overlying a compressible fluid," J. Acoust. Soc. Am. **79**, 666–672 (1986).

<sup>4</sup>M. A. Biot, "Generalized boundary conditions for multiple scatter in acoustic reflection," J. Acoust. Soc. Am. **44**, 1616–1622 (1968).

<sup>5</sup>I. Tolstoy, "Smoothed boundary conditions, coherent low-frequency scatter, and boundary modes," J. Acoust. Soc. Am. **75**, 1–22 (1984).

<sup>6</sup>I. Tolstoy, "Long wavelength acoustical scatter from rough surfaces," in *Frontiers of Acoustics, Proceedings of the International School of Physics 'Enrico Fermi'*, edited by D. Sette (North Holland, Amsterdam, 1986), pp. 209–254.

<sup>7</sup>M. J. Lighthill, *Waves in Fluids* (Cambridge U.P., Cambridge, England, 1978).

<sup>8</sup>V. Twersky, "On scattering and reflection of sound from rough surfaces," J. Acoust. Soc. Am. **29**, 209–225 (1957).

## Remarks on "On the coefficient of nonlinearity $\beta$ in nonlinear acoustics" [J. Acoust. Soc. Am. **83**, 74–77 (1988)]

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Through an alternate analysis, the results of Hamilton and Blackstock [J. Acoust. Soc. Am. **83**, 74–77 (1988)] are reviewed and a new interpretation is presented. In this view, the nonlinearity of the momentum equation is examined, and its effect on the coefficient of nonlinearity is revealed.

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In their paper,<sup>1</sup> Hamilton and Blackstock suggest that the nonlinear term in the conservation of momentum equation plays no role in the coefficient of nonlinearity  $\beta = 1 + (B/2A)$ , and that this nonlinear term need not be included in the analysis arriving at the wave speed. The calculations presented here indicate that the term in question does play a vital role in both the coefficient of nonlinearity and the propagation speed. To illustrate this point, an alternate derivation is provided based on the method of characteristics including a shadow factor to follow the pertinent nonlinear term.

The salient points of the argument can be made using a one-dimensional analysis that generalizes directly to the full three-dimensional case. To this end, the equations of motion are written in the following form:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0, \quad (1a)$$

$$\frac{\partial v}{\partial t} + \alpha v \frac{\partial v}{\partial x} + \frac{a^2(\rho)}{\rho} \frac{\partial \rho}{\partial x} = 0, \quad (1b)$$

where

$$a(\rho) = \sqrt{\frac{dp}{d\rho}}$$

is the isentropic density-dependent sound speed,  $\rho$  is the density,  $p$  is the pressure,  $v$  is the particle velocity, and  $\alpha$  is the

shadow factor used to follow the influence of the nonlinear momentum term. It follows that if the coefficient of nonlinearity is independent of the momentum nonlinearity, then  $\beta$  will be independent of  $\alpha$ . It will also be shown that  $\alpha$  must be 1 in order to obtain the correct wave speed.

Having applied the method of characteristics to the system (1), the resulting solution is

$$d\rho + \rho \frac{(\lambda^\pm - v)}{a^2(\rho)} dv = 0 \quad \text{on} \quad \frac{dx}{dt} = \lambda^\pm, \quad (2)$$

where

$$\lambda^\pm = [(\alpha + 1)/2]v \pm \sqrt{(\alpha - 1)^2 v^2 + 4a^2/2}.$$

Keeping only first-order terms in  $v$ , the expression for  $\lambda$  becomes

$$\lambda^\pm = \frac{(\alpha + 1)}{2} v \pm \left( a_0 + \frac{B}{2A} \frac{a_0}{\rho_0} (\rho - \rho_0) + (\text{H.O.T.}) \right),$$

where the sound speed has been expanded about  $\rho_0$  and  $B/2A$  is as in Ref. 1. The nonlinear acoustic approximation is now employed by linearizing the compatibility condition and retaining the first nonlinear term in the sound speed. This process yields, after integrating the compatibility condition,

$$[(\rho - \rho_0)/\rho_0] \pm v/a_0 = R_\pm, \quad (3)$$

where  $R_+$  and  $R_-$  are the Riemann invariants. To represent

a simple right-moving wave, set  $R_-$  to zero and then the integrated expression (3) is equivalent to Hamilton and Blackstock's Eq. (11) for a right-moving progressive plane wave. Having dropped the higher-order terms in the propagation speed and then using the compatibility condition for a simple right-moving wave, one sees that

$$\begin{aligned}\frac{dx}{dt} &= \frac{(\alpha + 1)}{2} v + a_0 + \frac{B}{2A} \frac{a_0}{\rho_0} (\rho - \rho_0), \\ \frac{dx}{dt} &= a_0 + \left( \frac{(\alpha + 1)}{2} + \frac{B}{2A} \right) v,\end{aligned}\quad (4)$$

which shows that the wave speed is dependent on  $\alpha$ , indicating that the nonlinear momentum term *does* have an effect. Further, the effect is present in the coefficient preceding  $v$ , which is the nonlinearity coefficient. The correct wave speed demands that  $\alpha$  be 1; then

$$\frac{dx}{dt} = a_0 + \left( 1 + \frac{B}{2A} \right) v,$$

as in Ref. 1. This analysis also shows that the constant term 1 in the nonlinearity coefficient arises from nonlinearities in *both* the mass and momentum equations, not just in the mass equation.

The critical role of the momentum nonlinearity also becomes evident after a detailed review of Hamilton and Blackstock's original derivation. It is easily demonstrated that the origin of the kinetic energy in the Lagrangian term of Eq. (8) of Ref. 1 is just the momentum nonlinearity in the exact equations. The right-hand side of Eq. (8) vanishes only if a precise cancellation between the momentum nonlinearity, as represented by the kinetic energy, and the pressure potential energy takes place.

It must be emphasized that the issue here is one of interpretation and that the basic results described in Ref. 1 must be regarded as sound in all respects. In particular, the use of the linearized momentum equation [Eq. (13) of Ref. 1] is entirely appropriate. However, the point made here is that the linear nature of the momentum equation is due to a cancellation of the nonlinear pressure work term by the convective nonlinearity rather than some linearization scheme.

<sup>1</sup>M. F. Hamilton and D. T. Blackstock, "On the coefficient of nonlinearity  $\beta$  in nonlinear acoustics," *J. Acoust. Soc. Am.* **83**, 74–77 (1988).

## On the linearity of the momentum equation for progressive plane waves of finite amplitude

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This Letter is primarily in response to the Letter to the Editor by G. M. Tarkenton entitled "Remarks on 'On the coefficient of nonlinearity  $\beta$  in nonlinear acoustics'" [*J. Acoust. Soc. Am.* **88**, 2024–2025 (1990)]. It is reaffirmed that for progressive plane waves, the momentum equation contributes nothing at second order to the expressions for the coefficient of nonlinearity and finite amplitude propagation speed.

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We are grateful to G. M. Tarkenton<sup>1</sup> for pointing out that our comment in Ref. 2 about the origin of the convection term in the expression for  $\beta$  might be misconstrued by readers. Following Eq. (14) (in Ref. 2), we claim that "To order  $\epsilon^2$  (for progressive plane waves), conservation of momentum contributes nothing to the coefficient of nonlinearity." The parameter  $\epsilon$  characterizes the dimensionless magnitude of the acoustical field variables. Tarkenton asserts that "*the nonlinear term* (emphasis added) in the conservation of momentum equation ...does play a vital role in both the coefficient of nonlinearity and the propagation speed." The analysis in question is that for progressive plane waves. The momentum equation is

$$\rho \left( \frac{\partial v}{\partial t} + av \frac{\partial v}{\partial x} \right) + \frac{\partial p}{\partial x} = 0, \quad (1)$$

where all quantities are defined in Ref. 1. For ease of com-

parison, we use Tarkenton's notation, including the shadow factor  $\alpha$  introduced to keep track of the convection term. [However, we note that to obtain the form of the momentum equation given by Eq. (1b) of Ref. 1, Tarkenton used a thermodynamic relation to eliminate the pressure in favor of the density, a modification that is discussed below.]

The momentum equation contains *two* nonlinear terms of order  $\epsilon^2$ , not just the convection term investigated by Tarkenton. The two terms are easily identified by writing Eq. (1), correct to order  $\epsilon^2$ , as follows:

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} = \frac{(p - p_0)}{\rho_0^2 a_0^2} \frac{\partial(p - p_0)}{\partial x} - av \frac{\partial v}{\partial x}, \quad (2)$$

where  $p_0$ ,  $\rho_0$ , and  $a_0$  are the ambient values of the pressure, density, and sound speed, respectively. Since for progressive waves, correct to order  $\epsilon^2$ , the first nonlinear term is equal to  $v(\partial v/\partial x)$ , Eq. (2) becomes