Vehicle Routing for Emergency Evacuations

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(ABSTRACT)

This dissertation introduces and analyzes the Bus Evacuation Problem (BEP), a unique Vehicle Routing Problem motivated both by its humanitarian significance and by the routing and scheduling challenges of planning transit-based, regional evacuations. First, a variant where evacuees arrive at constant, location-specific rates is introduced. In this problem, a fleet of capacitated buses must transport all evacuees to a depot/shelter such that the last scheduled pick-up and the end of the evacuee arrival process occurs at a location-specific time. The problem seeks to minimize their accumulated waiting time, restricts the number of pick-ups on each location, and exploits efficiencies from service choice and from allowing buses to unload evacuees at the depot multiple times. It is shown that, depending on the problem instance, increasing the maximum number of pick-ups allowed may reduce both the fleet size requirement and the evacuee waiting time, and that, past a certain threshold, there exist a range of values that guarantees an efficient usage of the available fleet and equitable reductions in waiting time across pick-up locations. Second, an extension of the Ritter (1967) Relaxation Algorithm, which explores the inherent structure of problems with complicating variables and constraints, such as the aforementioned BEP variant, is presented. The modified algorithm allows problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions to be solved to optimality. Empirical studies

demonstrate the algorithm viability to solve large optimization problems. Finally, a two-stage stochastic formulation for the BEP is presented. Such variant assumes that all evacuees are at the pick-up locations at the onset of the evacuation, that the set of possible demands is provided, and, more importantly, that the actual demands become known once buses visit the pick-up locations for the first time. The effect of exploratory visits (sampling) and symmetry is explored, and the resulting insights used to develop an improved formulation for the problem. An iterative (dynamic) solution algorithm is proposed.

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Dedication

This work is dedicated to my loving wife, Carolina, to my daughter, Vivian, and to my parents, Rodinei and Telma. Without their love and unwavering support, I would not have been able to finish this monumental task.

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Contents

List of Figures

List of Tables

List of Abbreviations

BEP	Bus Evacuation Problem
BEP-CA	Bus Evacuation Problem with Constant Evacuee Arrival Rate
DARP	Dial-A-Ride Problem
D-TSP	Dynamic Traveling Salesman Problem
D-VRP	Dynamic Vehicle Routing Problem
LRP	Location Routing Problem
MCNFP	Minimum Cost Network Flow Problem
$MES(F)$	Minimum Exposure Schedule for the Service Parameter, F
m-TSP	m-route Traveling Salesman Problem
PDVRP	Pick-up and Delivery Vehicle Routing Problem
PVRP	Period Vehicle Routing Problem
PVRP-SC	Period Vehicle Routing Problem with Service Choice
SBRP	School Bus Routing Problem
SDVRP	Vehicle Routing Problem with Split Deliveries
TSP	Traveling Salesman Problem
VRP	Vehicle Routing Problem
VRPTW	Vehicle Routing Problem with Time Windows

Preface

This dissertation introduces and analyzes both a deterministic and a dynamic variant of the Bus Evacuation Problem (BEP) and presents related solution algorithms. Chapter 1 presents the motivation for this research, a description of the problem, a review of the related vehicle routing problem and transit-based evacuation literature, a summary of the contributions of this research, and a review of commonly used solution algorithms for the problem. Chapter 2 presents a manuscript entitled *Scheduling and Routing for a Bus-based Evacuation with Constant Evacuee Arrival Rate*, accepted for publication at *Transportation Science*. Chapter 3 presents a manuscript entitled *A Variant of the Ritter Relaxation Algorithm Applied to Problems with Mixed-Integer Subproblems and Linear or Quadratic Objective Functions*, whereas Chapter 4 presents a manuscript entitled *The Dynamic Bus Evacuation Problem with Stochastic Customers and Demand*, both currently under preparation for submission for peer review. All three manuscripts are coauthored by Dr. Douglas R. Bish, the committee chair for this dissertation. Finally, Chapter 5 summarizes the findings of this research and lists the future work needed on the area. Given the overlap of the literature review across manuscripts, the bibliography is consolidated after Chapter 5.

Chapter 1

Introduction

This Chapter discusses the humanitarian significance of transit-based regional evacuation planning and implementation, provides a description of the Bus Evacuation Problem (BEP), reviews the related Vehicle Routing Problem (VRP) and the modeling and optimization literature, and summarizes the contributions of this research as it relates to advances in problem formulation and solution methodology. Lastly, a review of commonly used solution algorithms for the problem is presented.

1.1 Motivation

Most of the planning emphasis for large-scale, regional evacuations is automobile-centric (U.S. Department of Homeland Security, 2006). While such is an extremely important aspect of disaster preparedness, there can be a large segment of the population without access to automobiles, and, consequently, without the means to leave the disaster area. For example, it is estimated that between 200,000 and 300,000 people in New Orleans (approximately 27% of its households) did not have access to any reliable means of personal transportation prior to Hurricane Katrina's landfall in 2005 (Wolshon, 2002; Hess and Gotham, 2007). On its aftermath, between 100,000 and 120,000 residents, most of which were poor and belonging to minorities, remained in the city, risk exposed, and later became the second wave of evacuees (Nigg et al., 2006). Similarly, Hess and Gotham (2007), while reviewing the evacuation plans from cities in the State of New York (excluding Long Island and the five counties comprising New York City), concluded that many of those cities either meet or exceed New

Orleans percentage of households without access to automobiles. In fact, 9.1% of all occupied housing units in the United States (approximately 10.5 million homes) currently lack any form of personal transportation (U.S. Census Bureau, 2011).

Unfortunately, not only is the size of the population lacking automobiles significant, but few regions are prepared for their mass evacuation. According to a survey conducted by the U.S. Department of Homeland Security (2006), only 9 States and 5 of the 75 largest urban centers of the United States sufficiently consider all modes of transportation in their evacuation plans. Similarly, Hess and Gotham (2007) determined that upstate New York, a rural area and site of many nuclear reactors, has a considerably large carless population, but is insufficiently prepared for their mass evacuation in case of natural or man-made disasters.

Finally, while buses are often the most readily available and the most flexible mode of public transport, their number is limited, and often insufficient to transport all evacuees to a safe destination in a single trip. In fact, the 500 buses available for the evacuation of New Orleans would only have sufficed provided their adequate routing during the 48 hours of advanced notice (Litman, 2006). Clearly, transit-based, regional evacuation plans capable of meeting the usually tight deadlines and of using the available resources efficiently are an essential component of disaster preparedness.

1.2 Problem Description

The BEP is a variant of the VRP motivated both by its humanitarian significance and by the routing and scheduling challenges involved in planning transit-based, regional evacuations. In summary, the BEP aims to identify a timetable of pick-ups and the routes for a fleet of buses to transport all carless evacuees of a region from a set of neighborhood pick-up locations (or bus stops) to one or more shelters, while optimizing some evacuation-specific objective function (often minimization of the evacuation completion time). Although this research focuses on bus-based evacuations, the concepts herein developed are readily applicable to any other mass transit vehicle, and, possibly, to alternative VRP settings.

Chapter 2 and Appendix B present multiple formulations for a BEP variant which seeks to minimize the amount time that evacuees must wait at the pick-up locations before boarding a transit vehicle. These formulations assume that evacuees arrive at constant, location-specific rates, from the start of the evacuation to a location-specific end-time (aiming to mitigate evacuation risks). Moreover, they allow vehicles to visit the depot multiple times to unload evacuees and include spatial-temporal vehicle synchronization of pick-up times. Such constraints coordinate the number of vehicles and the time when these depart the pick-up locations, ensure that these have enough free capacity to accommodate all evacuees gathered since the last scheduled pick-up time, and force at least one pick-up to take place on each location, scheduled for the end of the evacuee arrival process. Efficiencies from allowing fewer than a limiting number of pick-ups (service choice) are exploited. Consequently, the

problem ensures that every evacuee boards a transit vehicle on or before the evacuation end-time. It is shown that, depending on the problem instance, restricting the maximum number of pick-ups on each location may lead to an efficient usage of the available fleet, to more equitable pick-up schedules across locations, and to an improved problem tractability.

While the BEP is often solved during the planning phase of the evacuation, reducing its computation time is still desirable, as it allows larger problem instances to be assessed or the model sensitivity to the input parameters to be analyzed. Unfortunately, given its complexity, currently, only small instances of the aforementioned BEP variant can be solved to optimality under reasonable time. Motivated by the need to improve its solving time, Chapter 3 introduces an exact optimization algorithm based on a less known relaxation methodology originally developed by Ritter (1967). The algorithm was selected due to its ability to explore the inherent structure of problems with both complicating variables and constraints, such as the aforementioned BEP variant, without the need to compound multiple decomposition/relaxation strategies. The modifications proposed require fewer matrix manipulations and fewer numerical comparisons, thus leading to improved numerical stability. In addition, the modified algorithm is combined with the branch and bound algorithm, thus allowing bordered (or arrowhead) structure problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions to be solved to optimality. While the algorithm is not employed to solve actual instances of the aforementioned BEP variant, empirical studies demonstrate its viability to solve large optimization problems.

Finally, Chapter 4 presents a two-stage stochastic formulation for the BEP with uncertain

demand. The problem assumes that all evacuees are at the pick-up locations at the onset of the evacuation, that the fleet has homogeneous capacity, that the set of possible demands (scenarios) on each pick-up location are provided, and that these location-scenario demands are functions (potentially multiples) of the vehicle capacities. The first stage of the proposed two-stage stochastic formulation seeks the subsequent arc traversal for each vehicle, common across scenarios, whereas the second stage identifies the remaining optimal tours under each location-scenario demand realization, starting from the respective destination nodes identified by the first stage. The problem seeks the first stage decision (i.e., the subsequent arc traversal for each vehicle) that minimizes the expected total travel cost of the fleet, while ensuring that every evacuee is rescued under each location-scenario demand realization.

In addition, the problem assumes that any demand uncertainty is resolved upon visual observation, that is, once a bus visits the respective pick-up location for the first time. A solution algorithm that iteratively re-optimizes the problem as additional information becomes available is proposed. The algorithm terminates once every pick-up location has been visited, and, thus, once all uncertainty has been cleared. The effect of exploratory visits or sampling (i.e., visits to pick-up locations with the sole purpose of ascertain the realized demand) and of symmetrical tours is explored, and the resulting insights used to develop a more efficient problem formulation.

The next section presents a review of the related VRP and BEP literature.

1.3 Literature Review

1.3.1 The Vehicle Routing Problem

As aforementioned, the BEP is a variant of the VRP. In summary, the VRP aims to identify the routes for a fleet of vehicles that minimize the accumulated cost of each vehicle's arc traversals, while ensuring that every customer is visited exactly once and that every vehicle starts and ends its route at a depot (Toth and Vigo, 1998). Since the first related paper from Dantzig et al. (1954), the VRP literature has grown exponentially, evolving into specific instances that include multiple depots, capacitated vehicles, multiple trips per vehicle, split deliveries, time windows per customer, service choice, and backhauls. Eksioglu et al. (2009) present an overview of the VRP literature and propose a methodology for its classification, while Laporte (1992) presents three exact algorithms (direct enumeration, dynamic programming, and branch and bound) and four heuristics (the savings algorithm, the sweep algorithm, the k-degree center tree algorithm, and the tabu search algorithm) for solving the problem.

All BEP variants presented and analyzed in this research share characteristics with specific variations of the VRP such as the School Bus Routing Problem (SBRP) and the Dial-A-Ride Problem (DARP). However, the variation presented in Chapter 2 is, perhaps, more closely related to the Period Vehicle Routing Problem (PVRP), and, in particular, to the Period Vehicle Routing Problem with Service Choice (PVRP-SC), whereas the one presented in Chapter 4 is more closely related to the Stochastic Dynamic Vehicle Routing Problem

(stochastic D-VRP).

The SBRP, introduced by Newton and Thomas (1969), aims to find a school bus loading pattern and a timetable of pick-ups that minimize total operating costs and passenger travel times, while enforcing fleet size and bus capacity restrictions (Angel et al., 1972). Park and Kim (2010) present a comprehensive review of the SBRP and propose a classification methodology for the problem. Similar to the SBRP, the proposed BEP variants are constrained by bus loading capacities, the fact that all pick-up locations must be visited, that all passengers must be accommodated, and that vehicles must start and end their route at the depot (school). However, while the SBRP enforces a single pick-up by a single bus on each location, the proposed BEP variants allow split deliveries and multiple pick-ups at the same location. Moreover, while the SBRP restricts the overall drop-off time of students (based on the start time of classes), the BEP variant presented in Chapter 2 restricts the last pick-up time on each location (aiming to mitigate evacuation risks), whereas the one presented in Chapter 4 restricts neither the last pick-up or drop-off time.

The DARP, introduced by Wilson et al. (1971), generalizes other variations of the VRP such as the Pick-up and Delivery Vehicle Routing Problem (PDVRP) and the Vehicle Routing Problem with Time Windows (VRPTW). The problem seeks the routes and schedules for a fleet of homogeneous vehicles operating from a central depot to transport customers from a set of pick-up to a set of drop-off locations, while minimizing either service cost or customer waiting and riding time. Cordeau and Laporte (2003) describe the DARP, review the related literature, and provide a discussion on the modeling issues and future research

needs. Although the proposed BEP variants do not specifically incorporate service time, a customer-focused objective function, namely the minimization of evacuee waiting time at the various pick-up locations, is presented in Chapter 2.

The PVRP was first mathematically formulated by Christofides and Beasley (1984). In this problem, routes are constructed over multiple-days as opposed to the single-day case of the ordinary VRP. Each customer requires a fixed number of visits during the period, and, on each day, a set of capacitated buses becomes available to meet their demand. Francis et al. (2006) offer a variation of the PVRP by introducing service choice as a decision variable, namely the PVRP-SC. The problem seeks the routes for a fleet of vehicles operating from a central depot such that each customer is visited, at least, a minimum number of times in order to meet their demand. Although the problem also enforces a maximum number of visits before a predetermined end-time, operational efficiencies and increased service benefits may be gained by allowing customers to be visited more often than their minimum required frequencies. For instance, suppose that two customers must be served at least twice and no more than four times during a period of one workweek, and that a single vehicle is available and is limited to three visits per customer over the same period. Then, one solution would be to visit one customer on Monday and Tuesday and the other on Wednesday, Thursday and Friday. Despite its similarities with the spatial-temporal vehicle synchronization constraints of the BEP variant presented in Chapter 2, the PVRP-SC assumes that the set of potential arrival times is both discrete and finite. Conversely, the BEP variant of Chapter 2 allows visits to occur anytime before the evacuation end-time, thus extending the PVRP-SC to

continuous arrival times.

Finally, Chapter 4 presents a variation of the D-VRP applied to the BEP. The origins of the D-VRP can be traced to Wilson and Colvin (1977), who proposed a DARP variation in which customer requests for trips arrive dynamically (i.e., after vehicles begin their tour). According to Pillac et al. (2013), unlike traditional VRPs, the D-VRP is based on the premise that either the problem parameters change or their uncertainty is resolved after the originally planned tours begin to be implemented. Consequently, the problem allows factors initially unknown to the planner to be incorporated into the final solution, thus potentially reducing operational costs, improving customer service, and even reducing environmental impact (Pillac et al., 2013). Since the originally planned routes must be re-optimized to account for the new information, the D-VRP often consists of a sequence of static VRPs. The D-VRP literature is extensive, and the original problem has grown to include not only dynamic demand, but also dynamic travel time and fleet availability (See Larsen et al., 2007; Pillac et al., 2013, for a comprehensive review of the D-VRP literature).

Among the many variations of the D-VRP, the formulation presented in Chapter 4 more closely resembles the stochastic D-VRP. The stochastic D-VRP is similar to its deterministic counterpart in the sense that some of the parameters (in this case, the number of full bus loads needed to accommodate every evacuee present at a pick-up location) are unknown a priori and reveled dynamically during the route execution (in this case, as vehicles visit the pick-up locations for the first time). However, contrary to its deterministic counterpart, the problem exploits the known stochastic nature of demands to determine the arc traversals

more likely to result in better objective function values.

As aforementioned, the formulation presented in Chapter 4 accounts for the stochastic nature of the demand on each pick-up location by formulating the problem as a two-stage stochastic VRP. The origins of stochastic VRP with uncertain demand can be traced to Tillman (1969), who introduce the problem and propose a solution methodology based on a variation of the savings algorithm. Gendreau et al. (1996) present an overview of Stochastic VRPs and a review of the related literature, whereas Louveaux and Schultz (2003) identify several structural properties of stochastic integer programs and review the related solution techniques. Stochastic VRPs arise when some of the problem parameters are random (in this case, the demands), and, thus, must be accounted for in two stages: before and after they are disclosed. In the first stage, a planned solution is identified, and, in the second stage, the realization of the random variables become known and a recourse or corrective action must be taken. Consequently, the problem seeks the first stage decision that minimizes both its associated cost and the expected cost of the recourse action (Gendreau et al., 1996). The BEP variant presented in Chapter 4 follows a similar approach: first identifying the subsequent arc traversal for each vehicle, common across scenarios, and, later, the remaining tours under each possible demand realization, starting from the respective destination nodes identified by the first stage. Consequently, the problem seeks the first stage decision that minimizes both its associated travel time and the expected travel time under each possible location-scenario demand realization. While recourse costs are not explicitly modeled, they are accounted for in the proposed formulation as increased total travel times resulting from

poor first stage decisions.

Next, a comprehensive review of the modeling and optimization literature on transit-based evacuation planning is presented.

1.3.2 The Bus Evacuation Problem

As aforementioned, most of the modeling and optimization literature for large-scale, regional evacuations is automobile centric (see Xiongfei et al., 2010, for a survey of the automobile-based evacuation literature). In fact, research focused on transit-based evacuation planning is limited and fairly recent. This section presents a comprehensive review of the current, transit-based evacuation literature, emphasizing their modeling and solution methodology differences.

Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010) investigate the effect of different objective functions on a multi-modal (automobile and mass transit) evacuation problem. Abdelgawad et al. (2010b) propose a Vehicle Routing Problem with Split Deliveries (SPVRP) formulation for the BEP which incorporates multiple tours per vehicle and a last pick-up time restriction on each pick-up location. While the capacity of the vehicles is individualized, as to allow the use of different mass transit vehicle types (such as commuter and school buses), shelter capacities are not enforced. The authors consider three parameters of interest: total vehicle travel time, total evacuee waiting time at the various pick-up locations, and number of vehicles needed for the evacuation. These parameters are combined into four

objective functions via arbitrary weight coefficients: minimize the total travel time; minimize the total travel and waiting time; minimize the total travel time and the fleet size required; and minimize the total travel time, waiting time, and fleet size. The problem is solved via constraint programming, where multiple search strategies - including the Tabu search metaheuristic - and constraint propagation techniques are used to obtain good, feasible solutions to the problem. Their proposed framework is used to generate a transit-based evacuation plan for downtown Toronto, Canada. Abdelgawad et al. (2010a); Abdelgawad and Abdulhai (2010) propose a staged solution strategy for designing evacuation plans that integrate both automobile and mass transit vehicles. In their methodology, first, the automobile and mass transit demand is estimated, then, the automobile component is solved, and, finally, the resulting arc traversal times from the automobile component are used as input parameters for the mass transit component. The automobile component formulation is solved via the genetic algorithm metaheuristic and assesses three objective functions: minimize the total travel time, minimize the total waiting time to enter the network, and minimize an arbitrary, weighted combination of travel and waiting time. The mass transit component formulation and solution methodology follows the one presented in Abdelgawad et al. (2010b). Abdelgawad et al. (2010a) use their proposed framework to generate a transit-based evacuation plan for downtown Toronto, Canada, whereas Abdelgawad and Abdulhai (2010) extends the evacuation area to the entire city of Toronto.

Bish (2011) proposes and analyzes two alternative formulations for the BEP, both seeking to minimize the duration of the evacuation (as measured by the longest travel time across a

fleet of homogeneous capacity vehicles), while ensuring that all demand is satisfied and that bus and shelter capacity restrictions are enforced. Similar to Abdelgawad et al. (2010a,b), and Abdelgawad and Abdulhai (2010), the proposed formulations allow split deliveries and buses, initially parked at a depot, to travel between pick-up nodes and shelters, to visit multiple, intermediate nodes during their tour, and to visit shelters multiple times to unload evacuees. However, the formulations proposed by Bish (2011) do not have the same problems as the ones proposed Abdelgawad et al. (2010a,b), and Abdelgawad and Abdulhai (2010) which include non-linearities and the possibility of trivial solutions (e.g., having buses remain parked at the depot). Unfortunately, given the added complexity of the model, computation times grow quickly with problem size. Aiming to attenuate this issue, the second formulation disaggregates the optimal route construction and assignment, and assumes that a set of feasible routes is provided. In addition, the min-max objective function of the first formulation is modified to a lexicographic equivalent, which minimizes the travel time of every vehicle provided a limiting travel time parameter. Although such lexicographic min-max objective function eliminates some of the symmetries inherent to the problem, it failed to reduce the computation time on empirical studies. Finally, the author proposes two heuristic algorithms to solve the problem. The first heuristic employs a staged solution technique, where the movement of each bus (from the depot to a pick-up location, from a pick-up location to a shelter, and so on) is determined before the movement of the next bus starts. Once a feasible solution is identified, an algorithm involving route swapping and route reassignment searches for alternative feasible solutions with better objective function

values. The second heuristic relaxes the binary variables to identify preferred routes, and uses these routes, possibly along with any non-duplicates identified by the first heuristic, as a warm-start for the second formulation. While these heuristics were not coded for efficiency, they have shown promising reductions in computation time and were able to attain near-optimal solutions for small instances of the problem.

Chen and Chou (2009) propose an iterated approach for solving the BEP. First, the location of pick-ups and shelters that minimize infrastructure development cost and travel time is determined through a non-linear variation of the facility location problem. Once the location of bus stops and shelters is available, and their respective demand and capacity is known, the routes that minimize the total travel distance of the fleet are identified through a variation of the savings algorithm. Finally, these routes are simulated along with traffic data to determine the effect of congestion on the results obtained. The simulated arc traversal times are then used as input parameters for the next iteration. Unfortunately, while their solution framework is very efficient in terms of computation time, the proposed formulation does not allow split deliveries or multiple trips per vehicle, as the one proposed by Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010), and Bish (2011). Consequently, they unnecessarily restrict the feasible region, yielding particularly inefficient usage of mass transit vehicles and an increased need for pick-up locations. The authors use their proposed framework to generate a transit-based evacuation plan for the University of Maryland, College Park.

Deai et al. (2011) formulate the BEP as a VRPTW and uncapacitated shelters. The authors divide the evacuation time horizon into time windows, such that, on each time window, a predetermined number of buses becomes available and a certain number of evacuees arrive at the various pick-up locations (estimated via a logistic mobilization curve). Their formulation seeks the tour for each vehicle (from the shelters to the various pick-up locations, and back) that minimizes the total travel time of vehicles, while ensuring that each pick-up location is served, at least, by one vehicle. Nonetheless, their formulation does not require a pick-up to be performed on each time window and it does not carry unmet demands to subsequent time windows. Consequently, it allows a substantial number of evacuees to be left stranded. The authors propose solving the problem using a modified genetic algorithm, in which the selection probability of each chromosome is determined by simulated annealing. According to the authors, such modification has shown to improve the evolution speed of the chromosomes and to reduce the likelihood of termination at a local optimum. The authors use their formulation and solution methodology to generate a transit-based evacuation plan for a real network containing 19 pick-up locations and 4 shelters. Unfortunately, they did not specify the evacuation region studied.

Goerigk and Grün (2012) propose a deterministic and a stochastic formulation with no recourse for the BEP, which treat the problem as a scheduling problem on a bipartite graph rather than a VRP on a complete graph. Similar to Bish (2011), both formulations seek to minimize the duration of the evacuation, as measured by the longest travel time across vehicles. Interestingly, Goerigk and Grün (2012) formulations do not explicitly model evacuees,

but only transit vehicles. These, initially parked at a depot, travel to the various pick-up locations, and, from there, back and forth between pick-up locations and shelters until a certain number of visits to each pick-up location is performed. Shelter capacity restrictions are similarly enforced, by ensuring a maximum number of visits to each shelter. Such simplifying assumptions greatly improve the tractability of the problem, allowing considerably larger instances to be solved to optimality under reasonable amounts of time. However, they also ignore potential efficiencies pertaining shelter and vehicle capacity utilization, which may lead to overestimation on the resources needed for the evacuation. Their stochastic formulation further enhances their deterministic model by seeking the balance between the number of buses that depart immediately and those that remain stationed at a depot for a fixed amount of time until the number of evacuees on each pick-up location becomes known. Given the increased complexity deriving from multiple types of buses and from multiple scenarios, the authors suggest the use of the tabu search metaheuristic. The authors use their proposed formulation and solution methodology to generate transit-based evacuation plans for theoretical networks and for the city of Kaiserslautern, Germany.

Goerigk et al. (2013) propose and analyze different strategies for solving the deterministic formulation presented by Goerigk and Grün (2012). In particular, the authors propose three, greedy heuristics to identify lower bounds on the longest travel time across vehicles; four, greedy heuristics to identify upper bounds on the longest travel time across vehicles; and three branching rules and two fathoming rules for the branch and bound algorithm. The computation of all upper and lower bounds has polynomial time order of complexity, and

they all account for any known feasible tours, as long as these are connected and start from the bus yard, thus allowing their use within the branch and bound algorithm. The first lower bound heuristic assumes that buses can only visit pick-up locations with residual demand (i.e., still requiring evacuation services) from its nearest shelter (or bus yard, if this is their first trip). The second, reformulates the problem as a Minimum Cost Network Flow Problem (MCNFP), seeking to minimize the total travel time of the fleet rather than the longest travel time across vehicles. Finally, the third heuristic is based on the premise that, in a real evacuation setting, pick-up locations are close to each other whereas shelters are usually spread farther apart. Hence, the heuristic models all sources as a single pick-up node with demand equal to the sum of all individual demands. While all these bounds outperformed the ones obtained by the commercial optimization package IBM ILOG CPLEX, empirical studies have shown that the second lower bound is the strongest. In the first upper bound heuristic, a new set of round trips between a pick-up location with residual demand and its nearest shelter with residual supply is generated. Newly created tours are assigned to the bus with smallest total travel time, and the process repeats until all demand is satisfied. The second heuristic improves the first by assessing round trips in non-decreasing order of travel time. Since the model assumes that round trips are centered at the pick-up locations rather than shelters, on the third heuristic, the longest tour is added to the last trip of buses, as these will remain stationed at the shelters and not return to the pick-up locations. Finally, on the forth heuristic, tours are generated and assigned to buses concurrently. Similar to the empirical studies for the lower bounds, all these bounds outperformed the ones obtained by IBM ILOG

CPLEX, with the second one being the strongest. The first branching rule generates branches exhaustively, one for each bus, source, and sink with residual supply/demand. In the second, branches for each bus are generated according to their index. Finally, the third branching strategy is similar to the second, but, instead of using the variable index to determine the branching order, these are based on the smallest travel time across vehicles. The first fathoming rule eliminates any branches that result in symmetric results, whereas the second fathoming rule compares the results with the ones generated by the upper bounds in order to identify the longest travel time across vehicles. Branches leading to alternative solutions that do not alter such critical path are immediately fathomed. Combined, the proposed bounds and branching/fathoming rules were able to considerably reduce the optimality gap obtained by IBM ILOG CPLEX under the same time frame, in some instances from 200% to 20%.

Kulshrestha et al. (2012) propose and analyze a stochastic, Location Routing Problem (LRP) formulation with no recourse for the BEP which seeks to minimize the expected total travel time of the fleet. Vehicles are initially assigned to one of the possible pick-up locations, and, from there, they perform round trips to a set of capacitated shelters until all demand is satisfied or until a limiting travel time is reached. Although the formulation is conceptually similar to the one proposed by Goerigk and Grün (2012), it differs in one important aspect: vehicles are not allowed to move to a different pick-up location once they finish rescuing all evacuees from their initial assignment. In addition, while the problem assumes that a set of scenarios is provided, it limits the number of pick-up locations whose demand can differ from

the expected value. Finally, the model assumes that every evacuee arrives at their closest, open pick-up location before the onset of the evacuation. The authors propose solving the problem though a cutting plane algorithm, where possible evacuee demands are sequentially added to the original problem parameter set until the change in objective function value becomes negligible. The authors use their proposed formulation and solution methodology to generate a transit-based evacuation plan for Sioux Falls, SD.

Margulis et al. (2006) propose a formulation for the BEP which seeks to maximize the number of rescued evacuees (i.e., to minimize the number of stranded evacuees) before an overall evacuation end-time. Similar to Goerigk and Grün (2012) and Kulshrestha et al. (2012), their formulation also assumes that buses will perform round trips between pick-up locations and shelters, but, whereas Goerigk and Grün (2012) assume that buses are initially parked at a central depot and Kulshrestha et al. (2012) assume that buses are initially parked at the pick-up locations, their formulation assumes that buses are initially parked at the shelters. Unfortunately, their formulation does not include the flow conservation constraints needed to ensure the adequate vehicle routing through the problem network. In addition, while their model also accounts for delays pertaining to congestion and loading/unloading of evacuees, such are treated as problem parameters rather than decision variables. The authors use their proposed formulation to generate a transit-based evacuation plan for Miami-Dade County, FL.

Sayyady and Eksioglu (2010) propose a variation of the BEP, where a dispatch operator routes a fleet of buses, initially either stationed at a depot or traveling along their regular

routes, to transport evacuees from a set of predetermined pick-up locations to one or more uncapacitated shelters outside the evacuation zone. The problem seeks to minimize both the total travel time of vehicles and the number of stranded evacuees. Given the need for fast response time, the authors suggest restricting the optimization model to the movement between two nodes, eliminating similar paths from the feasible region (i.e., paths sharing multiple, common arcs). The problem is solved via the tabu search metaheuristic, results are communicated to the bus drivers en route, and a new iteration is performed once these drivers reach their destination and gather reliable demand data. Similar to Deai et al. (2011), the authors use a logistic mobilization curve to estimate the number of transit-dependent evacuees on each pick-up location upon arrival of the buses. In addition, arc traversal times are estimated via a commercial simulation package. The authors use their proposed formulation and solution methodology to generate an evacuation plan for Fort Worth, TX.

Song et al. (2009) propose a stochastic, LRP formulation for the BEP, which incorporates a reliability constraint to account for the uncertainty on the number of transit-dependent evacuees on each pick-up location. Similar to Deai et al. (2011), strict pick-up time windows are enforced, but, here, all evacuees are assumed to be at the pick-up locations at the onset of the evacuation. In addition, a maximum travel time per vehicle is enforced as a proxy for the overall completion of the evacuation. Their formulation simultaneously seeks the set of uncapacitated shelters that must open and the corresponding routes for a fleet of capacitated vehicles that minimize their total travel time. Given the difficult in solving this stochastic formulation under reasonable amounts of time, the authors propose a hybrid algorithm which

combines the genetic algorithm, neural networks, and the hill climbing method to identify good solutions for the problem. The authors use their proposed solution methodology to generate an evacuation plan for the region of Gulfport, MS.

Wei and Zhaodong (2010) propose a mixed-integer variation of the maximum flow problem for solving the multi-modal (cars and buses) evacuation problem. Their formulation seeks to minimize the total travel time required for vehicles to reach their destination, and assumes that car owners allow their vehicles to be used during the evacuation. Consequently, private vehicles become varying-capacity transit vehicles, a conceptually similar approach to the work of Abdelgawad et al. (2010a,b), and Abdelgawad and Abdulhai (2010). Nonetheless, the formulation proposed by Wei and Zhaodong (2010) does not consider the bus, car, and shelter capacities, but, rather, a maximum number of cars and buses available and a maximum number of vehicles that can occupy an arc (road) in the network without causing significant changes in the estimated travel time. The effect of congestion, however, is not explicitly incorporated into the model. Finally, their formulation ensures that the available budget is not exceeded. While the cost centers for the evacuation are not explicitly provided, it is reasonable to assume that the authors refer to the cost of gas, driver salaries, and renting automobiles for use as transit vehicles. The authors propose a heuristic to maximize the outflow of evacuees through the network by iteratively solving relaxed instances of the problem, assigning more and more vehicles to each route until significant changes in the estimated travel time are observed.

Zhang et al. (2010) propose a shortest path formulation for the multi-modal (pedestrians and buses) evacuation problem. Similar to Deai et al. (2011) and Sayyady and Eksioglu (2010), their model also incorporates a logistic mobilization curve to estimate the number of evacuees on each pick-up location. A distinct characteristic of their formulation is the assumption that pedestrians, while moving towards the various pick-up locations, alter the flow of vehicles through the network. Their formulation seeks is to minimize the total evacuation time (from both pedestrians and vehicles) and is solved via Dijkstra's algorithm. The authors use their proposed formulation and solution methodology to generate an evacuation plan for the south-central region of Beijing, China.

Finally, Zheng (2013) formulates the BEP as a VRPTW. Similar to Goerigk and Grün (2012), the proposed formulation does not explicitly model evacuees. Instead, the author assumes that the time windows for each pick-up location are sufficiently large to allow a single bus to be filled to capacity, but not so large as to exceed vehicle capacities or to encourage evacuees to leave the pick-up location in search for alternative evacuation strategies. In addition, shelter capacities are similarly modeled, that is, as function of the capacity of each bus. Consequently, the model directs a fleet of homogeneous buses, initially parked at a depot or en route, to a set of predetermined pick-up locations, such that a single bus must visit each location on each of its time windows, and, from there, return to one of the shelters. After the evacuation is concluded, buses are directed to a bus yard. Since vehicles are assumed to travel back and forth between pick-up locations and shelters until all evacuees are rescued, the problem network reduces to a bipartite graph, once again, a conceptually

similar framework to the one proposed by Goerigk and Grün (2012). The formulation seeks to minimize the total vehicle travel time between pick-up and drop-off locations, and, while it ignores potential efficiencies pertaining shelter and vehicle capacity utilization, it allows larger instances of the problem to be solved to optimality under reasonable amounts of time. Nonetheless, the author further suggests the use of Lagrangian relaxation in order to reduce the computation time. In particular, by relaxing both the constraints enforcing shelter capacity restrictions and the ones ensuring that a single bus visits each pick-up location on each of its time windows, the problem can be reformulated into a MCNFP, and, thus, can have its binary restrictions relaxed without affecting the integrality of the solution. Such reformulation consists in creating a new network where arcs represent visits from shelters to pick-up locations that meet the time window restrictions. Unfortunately, the larger memory requirements to store the additional arcs offset the benefits attained from the reformulation/relaxation. Empirical studies show that the performance of the algorithm (as compared to alternative solution methodologies) varied by problem instance.

Tables A.1 and A.2 in Appendix A summarize the transit-based evacuation modeling and optimization literature. Next, the contributions of this research, as it relates to advances in problem formulation and solution methodology, are presented.

1.3.3 Conclusions from the Literature Review

The BEP variants proposed and analyzed in this research have several distinguishing characteristics from those in the related literature (see Section 1.3.1 for a review of related VRPs and Section 1.3.2 for a review of the modeling and optimization literature on transit-based evacuation planning). First, the formulations presented in the related transit-based evacuation literature almost invariably assume that all evacuees are at the pick-up locations at the onset of the evacuation. In fact, only Deai et al. (2011), Sayyady and Eksioglu (2010), Zhang et al. (2010), and Zheng (2013) account for the logical, time-dependent, evacuee arrival behavior within their formulations. Deai et al. (2011) and Zheng (2013) formulate the BEP as a VRPTW, such that, on each time window, a new wave of evacuees arrive at the various pick-up locations and are transported to the depot or shelters before the start of the next time window. While this approach allows the linearization of any time-dependent demand function, it is overly restrictive in nature, as it does not account for the interaction between the arrival (or departure) time of buses and the arrival time of evacuees. Consequently, buses cannot be held back at the depot either to improve the objective function value or to attain a more equitable pick-up schedule across pick-up locations. Moreover, residual unmet demand from one time window cannot be added to the demand of a subsequent one, thus either leading to infeasibility or to an inefficient fleet usage. Alternatively, Sayyady and Eksioglu (2010) and Zhang et al. (2010) use a logistic mobilization curve (introduced by Jamei, 1984), to estimate the number of evacuees present on each pick-up location at the time buses are expected to arrive. There are two main issues relating to this approach. First, since the

chosen mobilization curve is non-linear in nature, it is not incorporated into their proposed formulations. Consequently, the approach has the same disadvantages as the use of time windows. Second, while a logistic function has been shown to accurately portray the rate in which automobiles enter the roadway network during emergency evacuations (Sorensen and Mileti, 1988), no research fitting evacuee arrival rates to known probability distributions could be located.

As aforementioned, Chapter 2 presents a BEP variant in which evacuees gather at the pick-up locations at a constant, location-specific rate, from the onset of the evacuation to a location-specific end-time. This arrival process more realistically portrays the true evacuee arrival behavior, and was used by Delgado et al. (2009), Puong and Wilson (2008), Zolfaghari et al. (2004), among other authors, to model the inflow of passengers at transit stations. Nonetheless, these authors only sought a pick-up schedule for a fleet of vehicles traveling on a fixed route. Delgado et al. (2009) propose a formulation to identify the pick-up schedule for a fleet of buses traveling on a transit corridor that minimizes passenger waiting and travel time; Puong and Wilson (2008) propose a formulation for urban rail transit lines that allows trains to wait at designated stations following a disruption in service; and Zolfaghari et al. (2004) propose a formulation and a heuristic that uses real-time information to determine holding times along a specific route.

Second, the same BEP variant coordinates the departure time of buses from the various pick-up locations in order to construct more equitable pick-up schedules. Such spatial-temporal vehicle synchronization is similar to the one employed by Mankowska et al. (2011), but,

whereas these authors assume that the number of vehicles simultaneously arriving at (or departing from) the customer nodes is fixed, in the BEP variant of Chapter 2, these depend upon the accumulated demand since the last scheduled pick-up time.

Third, similar to Abdelgawad et al. (2010a,b), and Abdelgawad and Abdulhai (2010), the same BEP variant enforces a last pick-up time on each pick-up location. Consequently, instead of minimizing some function of the overall duration of the evacuation, the formulations proposed in Chapter 2 and Appendix B seek to minimize the accumulated time that evacuees must wait at the pick-up locations before boarding a transit vehicle.

Unfortunately, such approach results in a non-linear (quadratic) objective function, and, when combined with the large number of binary variables required to coordinate the flow of vehicles through the problem network, into a considerable increase in computation time. In fact, the problem can only be solved to optimality for toy examples involving few buses, pick-ups, and pick-up locations. Nonetheless, since the problem has a bordered-structure, it can be solved by relaxation. The most commonly used decomposition/relaxation algorithms are Benders' Decomposition (Benders, 1962), Dantzig-Wolfe Decomposition (Dantzig and Wolfe, 1960), and Lagrangian Relaxation. However, none of these methods is able to solve bordered-structure problems directly. Instead, Chapter 3 explores a less known relaxation algorithm for bordered-structure problems developed by Ritter (1967). In particular, the Chapter introduces a branch and bound extension of the algorithm, which allows its use for solving problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions. The Chapter focuses on the theoretical aspects of the algorithm, and

while such is not employed to solve actual instances of the aforementioned BEP variant, empirical studies demonstrate its viability to solve large optimization problems.

Table 1.1 summarizes and maps the main characteristics the BEP variant presented in Chapter 2 to the related VRP and to the modeling and optimization literature on transit-based evacuation planning.

Table 1.1: Contributions of the BEP variant presented in Chapter 2

Feature	**Related Literature**	
	BEP-Specific	**Other**
Objective function (minimize the accumulated evacuee waiting time)	None Located	Delgado et al. (2009); Puong and Wilson (2008); Zolfaghari et al. (2004)
Spatial-temporal vehicle synchronization	None Located	Mankowska et al. (2011)
Constant evacuee arrival rate	None Located	Delgado et al. (2009); Puong and Wilson (2008); Zolfaghari et al. (2004)
Last pick-up time on each location	Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010)	VRPTW
Service choice	None Located	PVRP-SC

Feature	**Related Literature**	
	BEP-Specific	**Other**
Split deliveries	Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010); Bish (2011)	SDVRP
Use of a complete digraph satisfying the triangle inequality with symmetrical arc traversal times	Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010); Bish (2011)	SBRP
Multiple visits to the depot to unload evacuees	Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010); Bish (2011)	DARP, PDVRP

Although the introduction of a time-dependent, continuous and constant evacuee arrival process presents an important distinction from the modeling and optimization literature on transit-based evacuation planning, such formulation remains unrealistic, as the demands on each pick-up location are clearly stochastic in nature. In fact, only Goerigk and Grün (2012), Kulshrestha et al. (2012), and Song et al. (2009) account for such stochasticity within their formulations. Nonetheless, given the ensuing increase in complexity, multiple

simplifying assumptions to improve the problem tractability are needed. Both Goerigk and Grün (2012) and Kulshrestha et al. (2012) assume that the set of possible demands (scenarios) are a known problem parameters, but, whereas Goerigk and Grün (2012) assume that such scenarios have equally likely probability of realization, Kulshrestha et al. (2012) assume that, at least, an arbitrary number of pick-up locations must attain their highest demand scenario. Consequently, Kulshrestha et al. (2012), albeit indirectly, account for the different likelihoods of each scenario over the optimal route and scheduling of the fleet. Finally, Song et al. (2009) assume that evacuees arrive at the pick-up locations according to a logistic distribution. The authors include a reliability constraint, introduced by Charnes and Cooper (1963), within their formulation to ensure, under an arbitrary probability, that if the given pick-up location is selected for the evacuation, then enough vehicles will arrive to meet its demand. However, no justification was provided for the chosen distribution. Chapter 4 extends the BEP variant presented on Chapter 2 to account for the uncertainty on the number of evacuees present at each pick-up location, and explores its effect over the expected total travel time of the fleet. Nonetheless, in order to improve the problem tractability, similar to Abdelgawad et al. (2010a,b); Abdelgawad and Abdulhai (2010); Bish (2011); Chen and Chou (2009); Goerigk and Grün (2012); Goerigk et al. (2013); Kulshrestha et al. (2012); Margulis et al. (2006); Song et al. (2009), and Wei and Zhaodong (2010), the formulation assumes that all evacuees are at the pick-up locations at the onset of the evacuation; similar to Goerigk and Grün (2012) and Kulshrestha et al. (2012), that the possible demands (scenarios) are provided; and, similar to Goerigk and Grün (2012), that the demands are functions (possibly

multiples) of the bus capacities. The effect of uncertainty on the initial routing of vehicles is explored. It is assumed that any uncertainty is resolved once buses visit the pick-up locations for the first time.

Consequently, the proposed formulation can be classified as a two-stage stochastic D-VRP with stochastic customers and demands, that is a D-VRP which relies on the known stochastic nature of the demand in order to identify improved routes for the fleet, and which assumes that some of the demands are potentially zero. Although the D-VRP literature is extensive, its application to humanitarian logistics is somewhat scarce. In fact, no literature specifically focusing on transit-based evacuation planning was located. Brotcorne et al. (2003) and Gendreau et al. (2001) propose a D-VRP formulation for ambulance relocation during disaster relief operations, whereas Haghani and Yang (2007) and Yi and Özdamar (2007) propose a D-VRP formulation for supply deployment during such operations. Finally, Fajardo and Waller (2012) propose a D-VRP formulation for search-and-rescue operations. Although not focused on transit-based evacuation planning, the authors introduce, perhaps, the closest variant to the proposed problem formulation.

Fajardo and Waller (2012) introduce and analyze a single-vehicle, uncapacitated Stochastic Dynamic Traveling Salesman Problem (Stochastic D-TSP) formulation for rescuing disaster victims whose number and position is not known a priori. Their formulation shares several commonalities with the BEP variant presented in Chapter 4. First, their formulation assumes that a set of pick-up locations (i.e., a list of potential victim positions) and a set of potential demands within each location are provided. Second, the realized demand on each location

(i.e., whether any victims are present or not) only becomes known upon visual observation, that is, once vehicles visit the location for the first time. Consequently, the problem must be re-optimized as additional information becomes available. Third, the authors also propose a two-stage stochastic formulation, where, in the first stage, the subsequent arc traversal for the vehicle is identified, and, in the second stage, the remaining arc traversals leading it back to the depot under each possible location-scenario realization are identified. Finally, their formulation also seeks to minimize the expected total travel time, provided a set of scenario probabilities. Despite these similarities, their proposed formulation differs from the variation presented in Chapter 4 is some important aspects. First, the authors modeled the problem as a TSP. Consequently, efficiencies from using multiple vehicles and from allowing vehicles to return to the depot to unload evacuees (or disaster victims) multiple times are ignored. Second, their formulation focuses on a search-and-rescue operation rather than a transit-based evacuation setting. Consequently, their formulation ignores vehicle capacity restrictions, which are an important consideration of emergency evacuations. Finally, and more importantly, while the authors mention the potential benefits of sampling and symmetry, their formulation does not explicitly integrate these features in order to identify better routes for the vehicle.

Lastly, some of the most commonly used VRP solution algorithms are presented.

1.4 Solution Methodologies

This section presents some of the most commonly used VRP solution algorithms. Although such methodologies fall beyond the scope of this research, they are included here due to their historical significance and due to their overt use within the modeling and optimization literature on transit-based evacuation planning.

1.4.1 Exact Algorithms

1.4.1.1 The k-Degree Center Tree Algorithm

Christofides et al. (1981) propose an exact algorithm which relies on the properties of graphs for solving symmetric (i.e., problems whose arc traversal cost is the same regardless if the vehicle is traversing the arc from node i to node j or from node j to node i), single-depot VRPs. The methodology is based on the m-TSP problem, that is, the problem of identifying m routes for a single vehicle to visit all the nodes in the network. Figure 1.1 depicts an example solution for a m-TSP with 2 routes over an undirected, complete graph. For simplicity, the arc traversal costs are omitted.

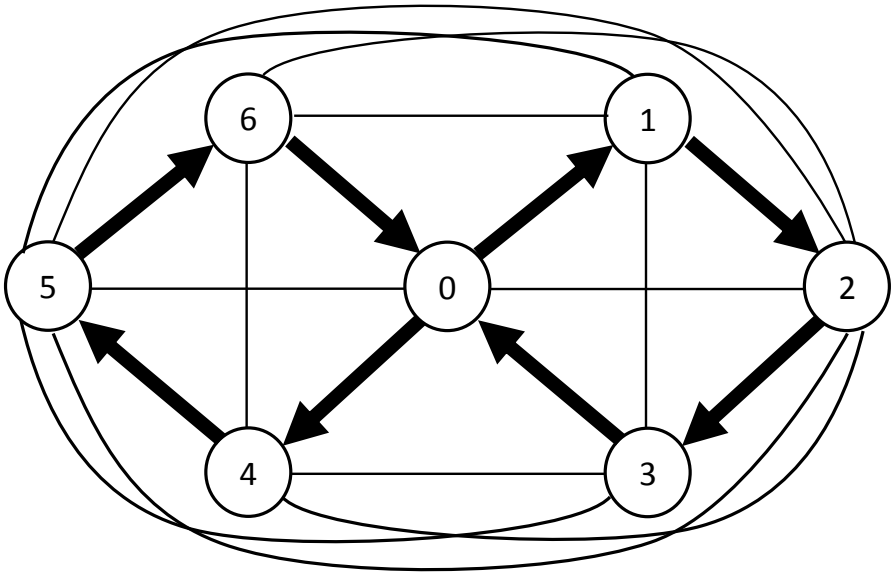

Figure 1.1: An example optimum solution for a m-TSP with 2 routes.

In summary, the k-degree center tree algorithm identifies the routes that minimize the total travel cost by building a spanning tree (i.e., a connected, acyclic subgraph spanning all nodes in the original graph) rooted at the depot (in the example presented in Figure 1.1, node 0), and, later, adding additional arcs between leafs and between leafs and the depot, such that the final route has m cycles (tours). The algorithm derives its name from the fact that the degree of the depot node, before the additional arcs are added, equals k. Consequently, the algorithm partitions the set of network arcs, $\mathcal{A}$, into four subsets:

- $\mathcal{A}_0$: The subset of arcs not belonging to the optimum solution;
- $\mathcal{A}_1$: The subset of arcs forming a *k-degree center tree*, that is, a spanning tree where the degree of the depot node, k, equals $2m - y$;

- $\mathcal{A}_2$: The subset of y arcs added between the leaves of the spanning tree and the depot node, where $0 \leq y \leq m$;

- $\mathcal{A}_3$: The subset of $m - y$ arcs added between the remaining leaves of the spanning tree.

Figure 1.2 depicts an example optimum partition for the network depicted in Figure 1.1. In this partition, $m = 2$, $y = 1$, and $k = 2m - y = 3$. The final m tours are attained by combining the arcs in sets $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$.

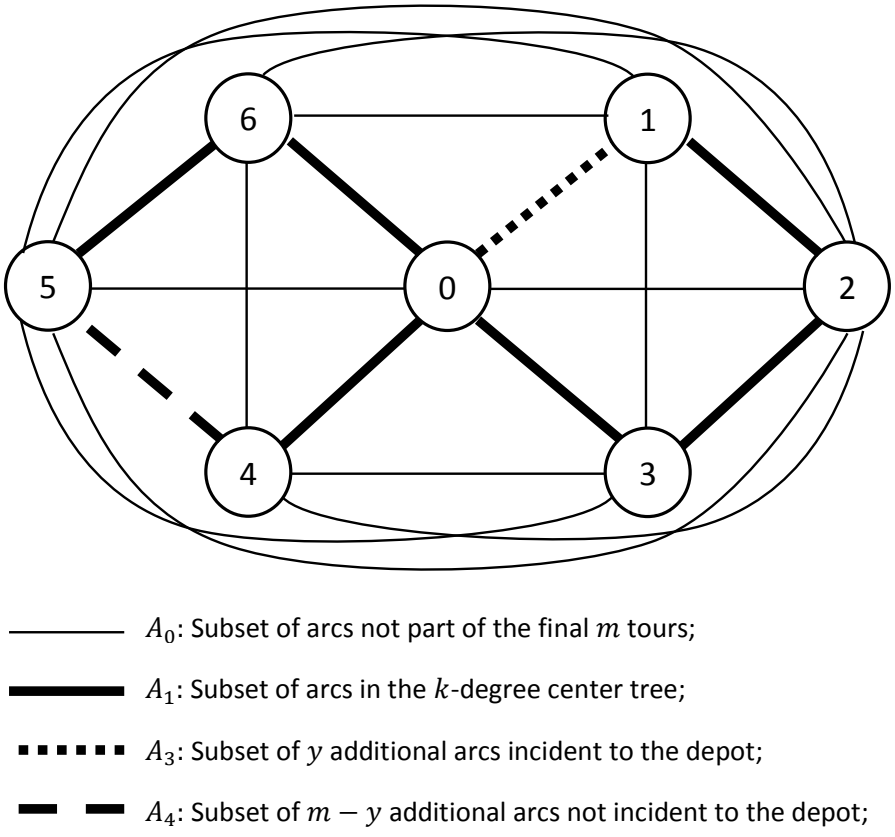

Figure 1.2: An example optimum partition for the k-degree center tree algorithm.

Now, let x_a^t be a binary decision variable that equals one if arc a belongs to partition $\mathcal{A}_t,\ t \in \{1,2,3\}$, and equals zero otherwise. Moreover, let $y \in \{y \in \mathbb{Z} | 0 \leq y \leq m\}$, such that the problem to minimize the total travel cost can be formulated as follows:

$$\text{Minimize} \quad \sum_{a \in \mathcal{A}} c_a (x_a^1 + x_a^2 + x_a^3) \tag{1.1}$$

$$\text{Subject to} \quad \sum_{a \in (S, \overline{S})} x_a^1 \geq 1, \quad \forall\ S \subseteq \mathcal{N}, |S| \geq 1 \tag{1.2}$$

$$\sum_{a \in \mathcal{A}_1} x_a^1 = 2m - y \tag{1.3}$$

$$\sum_{a \in \mathcal{A}} x_a^1 = |\mathcal{N}| - 1 \tag{1.4}$$

$$\sum_{a \in \mathcal{A}_1} x_a^2 = y \tag{1.5}$$

$$\sum_{a \in \mathcal{A} \setminus \mathcal{A}_1} x_a^3 = m - y \tag{1.6}$$

$$\sum_{a \in \mathcal{A}_i} (x_a^1 + x_a^2 + x_a^3) = 2, \quad i = 2, \cdots, |\mathcal{N}|. \tag{1.7}$$

Objective Function (1.1) minimizes the total cost of the arcs included in the final solution. Constraints (1.2)-(1.4) build the k-degree center tree. Constraint (1.2) prevents the inclusion of arcs leading to cycles in the spanning tree, (1.3) forces the degree of the depot, k, to equal $2m - y$, and (1.4) ensures that the resulting tree spans all nodes in the network. Constraint (1.5) forces the inclusion of y additional arcs connecting the leaves of the spanning tree back to the depot, and (1.6) forces the inclusion of $m - y$ additional arcs connecting the remaining leaves of the spanning tree. Finally, (1.7) forces the degree of every node (except the depot's) to equal 2. This constraint is similar to the flow conservation constraint in traditional TSP formulations. Next, Constraint (1.7) is relaxed via Lagrangian relaxation,

and (1.1) becomes:

$$\text{Minimize } \sum_{a\in\mathcal{A}} c_a(x_a^1 + x_a^2 + x_a^3) + \sum_{a\in\mathcal{A}}(\lambda_{\alpha(a)} + \lambda_{\beta(a)})(x_a^1 + x_a^2 + x_a^3) - 2\sum_{i=2}^{|\mathcal{N}|}\lambda_i$$

$$\Rightarrow \quad \text{Minimize } \sum_{t=1}^{3}\sum_{a\in\mathcal{A}}(c_a + \lambda_{\alpha(a)} + \lambda_{\beta(a)})x_a^t - 2\sum_{i=2}^{|\mathcal{N}|}\lambda_i,$$

Where, λ denotes the Lagrangean multiplier vector, and $\alpha(a)$ and $\beta(a)$ denote the nodes adjoined by arc $a \in \mathcal{A}$. Consequently, given an arbitrary y-value, the resulting problem can be decomposed into three subproblems, one for each subset of $\mathcal{A}$ except $\mathcal{A}_0$, and each with Objective Function:

$$\text{Minimize } \sum_{a\in\mathcal{A}}(c_a + \lambda_{\alpha(a)} + \lambda_{\beta(a)})x_a^t, \quad \forall\, t \in \{1, 2, 3\}. \tag{1.8}$$

Although these subproblems can be solved relatively quickly, they only return a lower bound on the objective function value for the original problem. Nonetheless, such lower bound can be used as a fathoming criterion during the ensuing branch and bound algorithm, thus reducing the overall computation time of the problem. Despite its improved performance and its historical significance, the k-degree center tree algorithm can only be used for symmetric, single-depot VRPs, and, thus, is not sufficiently robust for the wide array of VRP variations used within the modeling and optimization literature on transit-based evacuation planning.

1.4.1.2 Set Partitioning and Column Generation

The column generation algorithm was introduced by Dantzig and Wolfe (1960), but it was first applied to VRPs by Balinski and Quandt (1964). Consider the following VRP formula-

tion:

Let z_{ij} be a binary parameter that equals one if node $i \in \mathcal{N}$ is part of route $j \in J$ (where $\mathcal{N}$ is the set of all nodes in the network and J is the set of all possible cycles that include the depot), let c_j be the total cost of route j, and, finally, let x_j be a binary decision variable that equals 1 if route $j \in J$ is selected.

The problem can be formulated as follows:

$$\text{Minimize} \quad \sum_{j \in J} c_j x_j \tag{1.9}$$

$$\text{Subject to} \quad \sum_{j \in J} z_{ij} x_j = 1, \quad \forall\, i \in \mathcal{N} \backslash \{0\}, \tag{1.10}$$

Where, Objective Function (1.9) identifies the subset of routes from set J that incur the lowest cost, and Constraint (1.10) ensures that these routes serve each customer exactly once. Here, node 0 is the depot. The formulation holds from the fact that set J is finite, as any network with a finite number of nodes has a finite number of paths connecting these nodes.

Unfortunately, since the number of possible routes in set J grows exponentially with the number of customers, the ensuing problem may require a considerably long time to solve. In addition, it may be difficult to determine the feasibility of the route in capacitated VRPs, a particularly cumbersome problem when split deliveries are allowed. Nonetheless, the issue can be addressed by column generation.

In summary, the column generation algorithm decomposes the original problem into a relaxed

master problem, containing only a subset of the decision variables, and a subproblem. First, the relaxed master problem identifies the optimum dual vector, and, once such vector has been identified, a subproblem aiming to identify the corresponding entering variable, x_s, with minimum marginal cost (based on the identified dual vector) is generated and solved. If the objective function value of the subproblem is non-positive (i.e., if the marginal cost of every potential entering variable does not improve the objective function value of the master problem), then the solution is optimum. Otherwise, another iteration is performed. Since only a subset of potential variables is assessed on each iteration of the algorithm, the computation time is considerably reduced. See Bazaraa et al. (2005, p. 333-364) for a detailed description of the algorithm and for numerical examples.

Wilhelm (2001) reviews the use of the column generation algorithm for solving integer programming problems, including the VRPTW. According to the author, despite the reduced computation time, several issues remain. First, since the algorithm is based on the Revised Simplex algorithm (except for the identification of the entering column, which is performed within the subproblem, and, thus, can rely on a different solution methodology), it suffers from its the same disadvantages, foremost of which are the precision issues ensuing from the matrix inversion step. Second, the presence of degenerate constraints may lead to cycling or even stalling. Finally, current solution strategies may require a large number of subproblems to be solved for each entering column.

1.4.2 Heuristics

1.4.2.1 The Savings Algorithm

The savings algorithm, introduced by Clarke and Wright (1964), is a classical algorithm for solving capacitated VRPs. The algorithm is based on the method proposed by Dantzig and Ramser (1959), and it assumes that the number of vehicles available is unconstrained.

The algorithm is as follows (Laporte, 1992; Lysgaard, 1997):

1. Initialization: create a dummy circuit, C_i for each customer $i \in \mathcal{N}\backslash\{0\}$, and compute its associated cost, u_i, that is, the cost of traversing arcs $(0, i)$ and $(i, 0)$, where node 0, once again, denotes the depot (see Figure 1.3(a));

2. For each pair of routes C_i and C_j, $i \neq j$, created, compute the savings ensuing from their merger, that is, compute $S_{ij} = u_i + u_j - u_{ij}$, $\forall\ (i, j) \in \mathcal{A}$, where u_{ij} is the cost of the merged circuit;

3. Identify the highest savings feasible merger, that is, the merger not exceeding the vehicle capacity restriction (see Figure 1.3(b));

4. Repeat steps 2 and 3 until no more routes can be merged without violating the vehicle capacity restrictions.

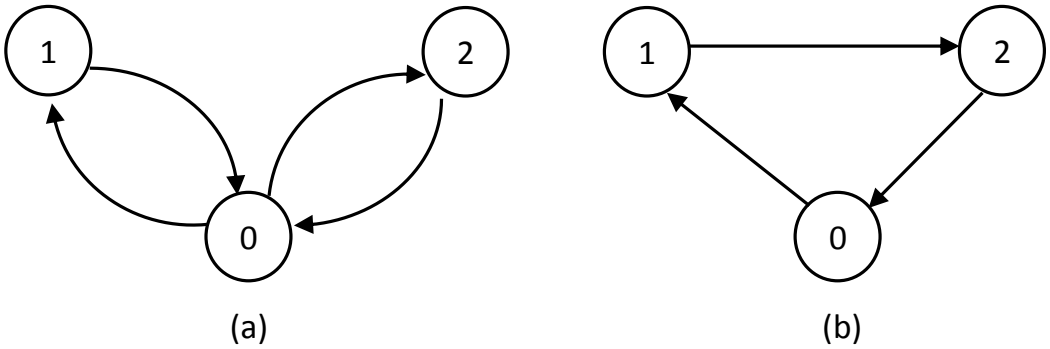

Figure 1.3: Conceptualization of the savings algorithm, before (a) and after (b) individual circuits are merged.

The savings algorithm has polynomial order of complexity, $O(n^2 \log n)$, but such can be further reduced by the use of appropriate data structures (Laporte, 1992). As aforementioned, Chen and Chou (2009) propose an iterated approach for solving the BEP. In their framework, once the location of bus stops and shelters has been determined, and their respective demand and capacity is known, the routes that minimize the total travel distance of the fleet are identified through a variation of the savings algorithm.

Notwithstanding its easy implementation and its fast convergence rate, the savings algorithm has several disadvantages hindering its application on practical settings. First, the algorithm does not consider a fixed cost per vehicle. While, this cost can easily be taken into account within the u_j parameters, such inclusion does not affect the final solution, as the computation of the savings cancel out their effect. Consequently, the algorithm does not penalize the use of extra vehicles, even if such leads to increased costs. In addition, the algorithm requires the unrestricted availability of vehicles, one for each route created. It is easy to envision

an example where the fleet size available allows a feasible solution to the problem, but is insufficient to meet the requirements of the savings algorithm. Finally, the algorithm is a heuristic, and, consequently, does not guarantee optimality. In fact, specific instances of the problem may yield particularly poor results.

1.4.2.2 The Sweep Algorithm

The sweep algorithm, introduced by Wren and Carr (1971), Wren and Holliday (1972), and Gillett and Miller (1974), relies on the fact that customer nodes are usually dispersed around the depot(s), and, thus, can be grouped into clusters (based on their polar coordinates relative to their closest depot), and assigned to individual vehicles. The algorithm is designed for capacitated VRPs with single or multiple-depots, and its steps are as follows:

1. Initialization: starting from some arbitrary x-y coordinates, label each customer (i.e., each node $j \in \mathcal{N}\backslash\{0\}$) as $L(j) = (r, \theta, -)$, where r denotes its closest depot and θ denotes its polar coordinate angle relative to its closest depot and to the selected x-y coordinate system;

2. Choose an unused vehicle k. Starting from the smallest θ-value among all unassigned customers near the depot where the vehicle is parked, assign customers to the vehicle until its capacity is exceeded. Update the label of each assigned customer with the vehicle index, s, that is, set $L(j) = (r, \theta, s)$;

3. Repeat step 2 until all customers are assigned to a vehicle. Optimize the ensuing TSP

for each cluster of customers assigned to a specific vehicle.

Figure 1.4 depicts an example solution for the sweep algorithm. While the computation time of this algorithm is similar to the one of the savings algorithm, the solutions obtained are generally better (i.e., closer to optimality). Nonetheless, the algorithm suffers from the same disadvantages, namely, the need for an unrestricted fleet size (potentially one for each customer), the failure to penalize the use of extra vehicles in the presence of fixed costs, and the generation of particular poor solutions for some instances of the problem. Observe that the set partitioning algorithm proposed by Balinski and Quandt (1964) is a natural extension of the sweep algorithm, as all possible routes within clusters of customers are added to the set S, and the problem becomes the minimization of the cost associated with the set partition (Laporte et al., 2000).

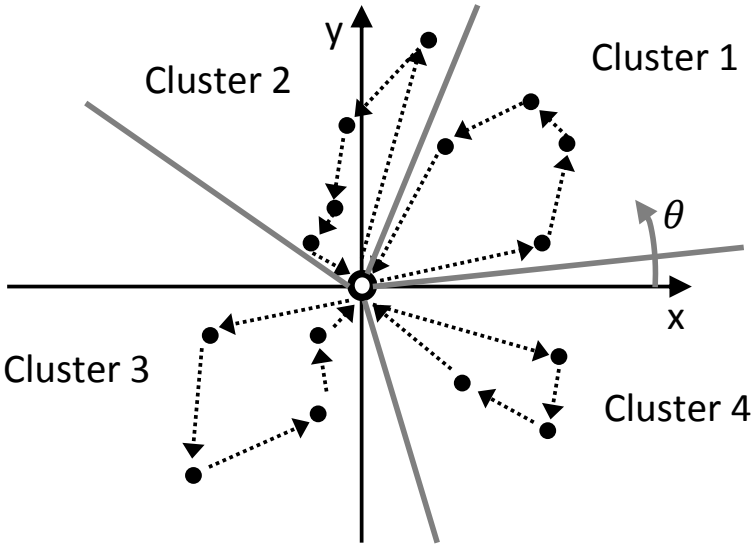

Figure 1.4: An example optimum solution for the Sweep Algorithm

1.4.3 Metaheuristics

1.4.3.1 Simulated Annealing

In the physical annealing process, a solid (such as a metal) is melted and its temperature is then slowly lowered to the vicinity of its freezing point, allowing its atoms to form a crystalline pattern (Aarts et al., 1997; Kirkpatrick et al., 1983). Metropolis et al. (1953) present an analytical, iterative algorithm to search for the thermal equilibrium of the system. In each step of the algorithm, the displacement of an atom is simulated and the resulting energy of the material is computed using tools from statistical mechanics. If the resulting change in energy of the material is non-positive (i.e., if the atom displacement led to cooling of the material), then the change is accepted. Conversely, if the change in energy of the material is strictly positive, then it may be accepted with an arbitrary probability. The new configuration is then used as the starting point for the next iteration. Since the algorithm allows intermediary configurations to have higher energies, it is less likely to converge to a local optimal solution.

The first use of the algorithm for identifying good solutions to combinatorial optimization problems can be attributed to Kirkpatrick et al. (1983). The authors extend the Metropolis et al. (1953) algorithm to account for annealing schedules, where the equilibrium "temperature" of the material is updated between runs of the Metropolis algorithm via an exponential curve that simulates its gradual cooling. Brooks and Morgan (1995) presents the following summary of the Simulation Annealing algorithm for optimization problems:

1. Initialization:

   - Set $k = 0$ and identify an initial feasible solution, x_0, either generated randomly or through another heuristic;
   - Compute the objective function value, $E(x_0)$, corresponding to x_0;
   - Arbitrarily select an initial, and potentially high, temperature, T_0, and a constant, u, sampled from an Uniform(0,1) distribution;

2. Identify the equilibrium solution for the current temperature, T:

   - Update the iteration number, that is, set $k = k + 1$;
   - Randomly identify an alternative solution, x_k, within an arbitrary neighborhood of the previous feasible solution (e.g., in a TSP, this step may consist of switching the order in which two nearby nodes are visited);
   - Compute the objective function value, $E(x_k)$, corresponding to x_k;
   - If $E(x_k) \leq E(x_{k-1}) - T \ln u$, then set x_k as the new solution;
   - Repeat this step until a certain criteria is met (e.g., $E(x_{k-1}) - E(x_k) < \epsilon$ for a certain number of iterations). The system is now in equilibrium with the temperature T;

3. Perform an annealing schedule, that is, after the equilibrium solution is identified, update the temperature by setting $T = k^n T_0$, where k is a constant between 0 and 1, exclusively, and n is the number of annealing schedules previously performed. Identify the equilibrium solution for the new temperature;

4. Repeat the annealing schedule step until a certain criteria is met (e.g., $E(x_{n-1}) - E(x_n) < \epsilon$ for a certain number of iterations). After the termination criteria has been met, the system is considered *frozen*.

The simulated annealing algorithm has been applied to a variety of settings and has shown ambivalent performance depending on the problem characteristics (Aarts et al., 1997). Several variations have been proposed to improve its performance (speed and convergence rate). These, however, are beyond the scope of this research. In the modified genetic algorithm proposed by Deai et al. (2011) to solve the BEP, simulated annealing is used to determine the selection probability of each chromosome.

1.4.3.2 Genetic Approaches

Genetic approaches, a subclass of evolutionary algorithms, are random searches inspired by the Darwinian evolutionary process (Larrañaga et al., 1999). In these algorithms, solutions for the optimization problem (not necessarily feasible solutions) are represented by strings, referred to as *chromosomes*. For instance, the solution $x_1 = 0,\ x_2 = 1$, and $x_3 = 1$ can be represented by the string "011." The algorithm simulates the evolution of this artificial population over time until a good string is produced. The term *Genetic Algorithm* can be attributed to Holland (1975), who consolidated the existing ideas of the time (namely the use of mutation associated with hill-climbing methods) and further developed the algorithm to accommodate recombination of selected individuals (Reeves, 2003). In summary, the algorithm is as follows (modified from Larrañaga et al., 1999, page 131, and Reeves, 2003,

page 59):

1. Initialization: Randomly generate a population of chromosomes.

2. Produce a new generation of chromosomes:

   (a) *Select* parents from the current population and *produce children* from these individuals via *crossover*;

   (b) *Mutate* these individuals;

   (c) Evaluate the *fitness* of the new generation of chromosomes (parents and offspring) and destroy unfit ones;

3. Repeat Step 2 until a certain termination criteria is satisfied (e.g, until a certain number of generations has been produced or no significant changes in the objective function value from the best chromosome has been observed in a certain number of generations).

Crossover is the process in which parent chromosomes are randomly combined to generate new offspring. Figure 1.5 exemplifies the crossover of two parent strings (P1 and P2) to generate two offspring (O1 and O2).

Figure 1.5: An example of crossover under the Genetic Algorithm.

In the **mutation** step, a subset of chromosomes is chosen randomly and some of their *genes* are arbitrarily changed. Figure 1.6 exemplifies the mutation of genes 3 and 5 of string O1 above.

Figure 1.6: An example of mutation under the Genetic Algorithm.

Finally, in the **selection** step, the fitness of the resulting population of individuals is evaluated, and strings considered unfit (e.g., infeasible solutions or with poor objective function values) are eliminated. By randomly combining, modifying, and eliminating unfit strings, the algorithm is able to identify good solutions to the problem. Genetic approaches have been the focus of considerable research over the past four decades and several improvements to the original algorithm have been suggested, including the use of multiple criteria for selection (multiple objective functions), use of matrices to represent the chromosomes as opposed to simple strings, use of variable chromosome population size, etc. As aforementioned, Abdelgawad et al. (2010a) and Abdelgawad and Abdulhai (2010) propose a staged solution procedure for the BEP that integrates the evacuation of both cars and buses. The genetic algorith is used to solve the automobile component of their model framework. Similarly, Deai et al. (2011) and Song et al. (2009) use modified genetic algorithms to solve their respective BEP formulations. Deai et al. (2011) integrate the genetic and the simulated annealing algorithms, whereas Song et al. (2009) integrate the genetic algorithm, neural networks, and the hill climbing method.

1.4.3.3 Tabu Search

The origins of the tabu search algorithm can be traced to Glover (1977), which used some of its elements to generate surrogate constraints for integer programming problems (i.e., inequalities implied by one or more constraints that eliminate non-integer portions of the feasible region). The algorithm, formalized by Glover (1986), starts by identifying an initial solution, x_1, and then, on each iteration, proceeding towards an improving direction until a certain criteria is satisfied, such as a maximum number of iterations has been reached or the problem has shown no improvement for several iterations (Laporte et al., 2000). Since successive solutions not necessarily have smaller objective function values, solutions that were recently examined are disallowed (or *tabued*) in order to prevent cycling (Glover, 1986). It is also important to note that intermediate solutions may not be feasible (Laporte, 1992).

The tabu search algorithm is readily applicable to a variety of settings. The following variation (modified from Laporte, 1992, page 356) seeks good solutions for the VRP. Throughout, let T denote the tabu list, R_r denote the subset of customer nodes visited by route r, such that each node belongs to exactly one route, x denote current solution set (i.e., the current set of routes $R_1, \cdots, R_m$ found), and $N_p(x)$ denote the set of nearby solutions around x (obtained, for instance, by switching the nodes visited between two routes). The algorithm is as follows:

1. Create circuits connecting the depot and each customer (i.e., let route j consist of arcs $(0, j) \cup (j, 0),\ \forall\ j \in \mathcal{N} \backslash \{0\}$) regardless if these routes are feasible or not. Let x denote

such solution, and let $F(x)$ denote its associated objective function value, that is, let $F(x) = \sum_r \sum_{(i,j)\in R_r} c_{ij}$. Set $T = \emptyset$;

2. Let $N_p(x)$ denote the set of all solutions in a neighborhood of radius p from the current solution x. If $N_p(x)\backslash T = \emptyset$, then go to step 3: all nearby solutions are tabued. Otherwise identify the least cost solution, $y \in N_p(x)\backslash T$, and set $x = y$ (i.e., update the best known solution);

3. If the stopping criteria has been met (e.g., if a maximum number of iterations has been reached or no significant improvement in the objective function value have been observed for a certain number of iterations), stop. Otherwise set $T = T \cup \{x\}$ and return to step 2.

According to Basu and Ghosh (2008), tabu search is widely used to solve practical combinatorial optimization problems. In addition, Gendreau et al. (1994) propose a modification which prevents the insertion of nodes far from the current solution and forces an assessment of variations of the final solution, thus improving the convergence rate and the quality of the final solution. As aforementioned, Abdelgawad et al. (2010b) propose solving their BEP formulation via constraint programming, where multiple search strategies - including the Tabu search metaheuristic - and constraint propagation techniques are used to obtain good, feasible solutions to the problem. In addition, one of the heuristics for solving the BEP proposed by Bish (2011) involves route swapping and reassignment, elements of the tabu search metaheuristic. Finally, Goerigk and Grün (2012) and Sayyady and Eksioglu (2010)

use the tabu search metaheuristic to solve their respective BEP formulations.

1.4.3.4 Ant Colony Optimization

As the name suggests, the ant colony optimization algorithm, introduced by Colorni et al. (1991), simulates the decision making process of ants as they find the most efficient trail from their nest to every available food source and back. In nature, ants deposit pheromones as they travel along a trail. The larger the amount of pheromone in a trail, the higher the probability that other ants will follow it, and, if they do, their own pheromone reinforces the existing trail. Since shorter, more direct routes are traversed more often, the amount of pheromones on these trails are more quickly reinforced, whereas the one in less direct routes slowly evaporates, becoming less likely to be followed by other ants. Nonetheless, the continuous random selection of trails by individual ants helps the colony to discover more efficient routes (Bell and McMullen, 2004). The algorithm was initially developed for identifying good solutions to the TSP, but it has since been modified and applied to a variety of other problems, including several variations of the VRP.

In the original algorithm, each of the n nodes (food sources and nest) has $|b_i(k)|,\ i = 1, \cdots, n$, ants, such that the total number of ants, m, equals $\sum_{i=1}^{n} |b_i(k)|$. As each ant starts its tour, the intensity of pheromone on arc $(i, j) \in \mathcal{A}$ at iteration k, $\tau_{ij}(k)$, is updated as follows:

$$\tau_{ij}(k) = \rho \cdot \tau_{ij}(k-1) + \Delta\tau_{ij}(k-1, k), \tag{1.11}$$

Where ρ is an arbitrary pheromone evaporation coefficient and $\Delta\tau_{ij}(k-1, k)$ is the amount

of pheromone deposited on arc (i, j) between iterations $k-1$ and k. Colorni et al. (1991) suggests that, if ant y traverses arc (i, j), then the amount of pheromone it deposits can either be a fixed, arbitrary quantity, Q, or a function of the travel distance between nodes i and j (e.g., $\Delta\tau_{ij}(k-1, k) = Q/c_{ij}$). The initial pheromone intensity on each path, $\tau_{ij}(0),\ \forall\ (i, j) \in \mathcal{A}$, can be chosen arbitrarily. Given the current amount of pheromone on each path, the desirability of node j starting from node i at iteration k, $p_{ij}(k)$, is computed as follows:

$$p_{ij}(k) = \frac{(\tau_{ij}(k))^{\alpha}/(c_{ij})^{\beta}}{\sum_{s:(i,s)\in\mathcal{A}}(\tau_{is}(k))^{\alpha}/(t_{is})^{\beta}}, \tag{1.12}$$

Where the arbitrary parameters α and β control the relative importance of selecting shorter travel distances versus highly traveled ones (i.e., paths with high pheromone intensities). In order to prevent subtours, a tabu list is associated with each ant such that visits to repeated food sources is disallowed. In summary, the algorithm is as follows (modified from Colorni et al., 1991, page 5):

1. Initialization:

   - Set $T = \infty$, $T^y = 0,\ \forall\ y = 1, \cdots, m,\ k = 1,\ \Delta\tau_{ij}(0, 1) = 0,\ \forall\ (i, j) \in \mathcal{A}$, and $L_y = \emptyset,\ \forall\ y = 1, \cdots, m$ (L_y is the tabu list associated with ant y);
   - Choose the parameters: $\tau_{ij}(0),\ \forall\ (i, j) \in \mathcal{A},\ \alpha,\ \beta$, and $b_i(0),\ \forall\ i = 1, \cdots, n$, the last one satisfying $\sum_{i=1}^{n} |b_i(0)| = m$;

2. For every node $i = 1, \cdots, n$ and for every ant $y \in b_i(k)$:

   (a) For every $j \notin L_y$, compute $p_{ij}(k)\ \forall\ j : (i, j) \in \mathcal{A}$ via Equation (1.12); set $p_{ij}(k) = 0$ otherwise. Set $j = \arg\max p_{ij}(k)$;

(b) Set $\Delta\tau_{ij}(k-1,k) = \Delta\tau_{ij}(k-1,k) + f(Q)$, where $f(Q) = Q$ or $f(Q) = Q/c_{ij}$, as aforementioned;

(c) Update $\tau_{ij}(k)$ via Equation (1.11), set $L_y = L_y \cup \{j\}$, and set $T^y = T^y + c_{ij}$;

3. Repeat step 2 until $L_y,\ \forall\ y = 1, \cdots, m$ includes every node in the network. Save the best solution found (i.e., set $T = \min\{T, \min_{y=1,\cdots,m} T^y\}$) and erase the tabu lists. Repeat for an arbitrary number of iterations.

Observe that the described implementation of the algorithm does not constitute a random search, as paths are selected deterministically. A random search, however, can be easily implemented by using the inverse CDF method to sample a path from the desirabilities not in the tabu list. The ant colony optimization algorithm initially yielded poor results when applied to VRPs. However, several modifications have been proposed since its introduction. According to (Bell and McMullen, 2004), such modifications have led to solutions within 1% of the known optimum for small VRP variations.

1.4.4 Summary of Solution Algorithms

This section introduced different algorithms that rely on the known structure of VRPs to reduce its computation time. Unfortunately, to date, none of these methods is capable or solving particularly large VRP instances (i.e., instances with many vehicles and customers) to optimality.

The k-degree center tree algorithm, introduced by Christofides et al. (1981), and the column

generation algorithm, introduced by Dantzig and Wolfe (1960), are presented as examples of exact optimization methods. These methods always identify the optimum solution to the problem, but, since they require the concomitant use of the branch and bound algorithm, they may take a particularly long time to solve. The k-degree center tree algorithm relies on the known properties of graphs to build a spanning tree rooted at the depot, and, later, identifies the set of additional arcs needed to generate a predetermined number of tours for the fleet. Conversely, Balinski and Quandt (1964) reason that any graph with a finite number of customers has a finite number of potential routes serving these customers. However, instead of enumerating every possibility, the problem can be solved by column generation, that is, by assessing the benefits of one route at a time.

The savings algorithm, introduced by Clarke and Wright (1964), and the sweep algorithm, whose origins can be traced to Wren and Carr (1971), Wren and Holliday (1972), and Gillett and Miller (1974), are presented as examples of heuristics. These algorithms are easy to implement and have fast convergence rate. However, they may also yield particularly poor results depending on the problem instance. The savings algorithm relies on the idea of consolidating existing routes by appending connecting arcs and removing redundant ones. The sweep algorithm uses the fact that customer nodes are usually dispersed around the depot, and, thus, can be grouped into clusters (based on their polar coordinates relative to their closest depot and to an arbitrary x-y coordinate system), and assigned to individual vehicles. In summary, the algorithm transforms the VRP into a m-TSP, a much more efficient formulation.

Finally, the simulated annealing, introduced by Metropolis et al. (1953) and first applied to combinatorial optimization problems by Kirkpatrick et al. (1983), the genetic algorithm, whose origin can be attributed to Holland (1975), the tabu search algorithm, introduced by Glover (1986), and the ant colony optimization algorithm, introduced by Colorni et al. (1991), are presented as examples of metaheuristics. Since these methods do not require the simultaneous use of the branch and bound algorithm, they are particularly suited to combinatorial optimization problems. As the name suggests, the simulated annealing algorithm simulates the physical annealing process using tools from statistical mechanics. The genetic algorithm, a subclass of evolutionary algorithms, is a random search method inspired by the Darwinian evolutionary process of crossover, mutation, and selection. The tabu search algorithm searches for nodes to be included or excluded from the route that lead to improvements in the objective function value without ever returning to a previous solution. Finally, the ant colony optimization algorithm, a particle swarm metaheuristic, simulates the decision process of ants (vehicles) traveling on a set of trails (arcs) in search for food sources (customers).

Chapter 2

Scheduling and Routing for a Bus-based Evacuation with Constant Evacuee Arrival Rate

Scheduling and Routing for a Bus-based Evacuation with Constant Evacuee Arrival Rate

Victor C. Pereira and Douglas R. Bish

(ABSTRACT)

This paper introduces a variant of the vehicle routing problem adapted for bus-based, regional evacuation planning, where evacuees arrive at predetermined pick-up locations at constant, location-specific rates. This arrival process more realistically portrays evacuee arrival behavior, presenting an important distinction from the current transit-based evacuation literature. In this problem, capacitated buses are routed, potentially multiple times, to transport all evacuees to a depot/shelter such that the last pick-up, and the end of the arrival process, occurs at a location-specific time, determined by risk considerations. The problem seeks to minimize waiting time at these locations (total exposure) and exploits efficiencies from allowing service choice. The structural properties of this problem guide the choice for two important strategic parameters: the maximum number of pick-ups allowed on each location and the fleet size. It is shown that, depending on the problem instance, increasing the maximum number of pick-ups may reduce both the fleet size requirement and the total exposure, and that, past a certain threshold, there exists a maximum number of pick-ups (or a range of potential values, if the problem cannot be solved to optimality) that guarantees an efficient usage of the available fleet and equitable reductions in exposure across pick-up locations.

Keywords: Evacuation planning; bus scheduling; vehicle routing problem; mixed-integer programming

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2.1 Introduction

Most of the planning emphasis for large-scale, regional evacuations is automobile-centric. While such planning is extremely important, there can be a large segment of the population without access to automobiles, and, consequently, without the means to leave the disaster area. For instance, it is estimated that between 200,000 and 300,000 residents of New Orleans did not have access to any reliable means of personal transportation prior to Hurricane Katrina's landfall in 2005 (Wolshon, 2002), and that between 100,000 and 120,000 people remained in the city, under unsafe conditions, after the evacuation (Nigg et al., 2006). According to Litman (2006), while the 500 buses available at the time were insufficient to evacuate all transit-dependent residents of New Orleans in a single trip, they would have sufficed, given the 48 hours of advanced notice, were it not for a lack of planning. Clearly, the development of safe and efficient, bus-based, regional evacuation plans is an important part of disaster preparedness.

This paper introduces, formulates, and analyzes the Bus Evacuation Problem with Constant Evacuee Arrival Rate (BEP-CA), a variant of the vehicle routing problem (VRP) adapted for bus-based, regional evacuation planning. In this problem, evacuees gather at the pick-up locations at constant rates, from the start of the evacuation to a location-specific end-time (dictated by risk considerations). The BEP-CA seeks a pick-up schedule at these locations and the corresponding routes for a fleet of buses in such a way that the total exposure is minimized (i.e., that the total evacuee waiting time at these locations is minimized). At each scheduled pick-up time, all evacuees present must be served and eventually transported to a depot/shelter. The formulation forces at least one pick-up to take place at each pick-up location, scheduled for the time when the arrival process of evacuees ends. Efficiencies from allowing buses to unload evacuees at the depot multiple times and from service choice, that is, from selecting the optimal number of pick-ups at each location (up to a maximum service level), are exploited.

The modeling research on bus-based evacuations is somewhat limited. Abdelgawad et al. (2010a,b) and Abdelgawad and Abdulhai (2010) investigate the effect of different objective functions on a multi-modal (automobile and mass transit) evacuation problem. Bish (2011) proposes and analyzes two alternative models for the multi-depot, multi-trip, bus-based evacuation problem: the first simultaneously identifies optimal route construction and assignment, while the second identifies the optimal route assignment from a set of feasible routes. Chen and Chou (2009) propose a model that determines pick-up and shelter locations that minimize infrastructure development cost, and, then, the optimal routes for

the evacuation of transit-dependent students of a university campus. Simulation is used to study the effect of vehicular contraflow on the results obtained. Deai et al. (2011) propose a VRP with time windows, such that, for each time window, a certain number of buses are available, a certain number of evacuees are at each pick-up location, and every location with non-zero demand must be served by exactly one vehicle. Goerigk and Grün (2012) propose a deterministic and a stochastic formulation without recourse in which vehicles perform round trips between pick-up locations and shelters until each location is visited a certain number of times. The stochastic formulation balances the number of buses that depart immediately with those that remain stationed at the depot waiting for the demand at each location to become known. Kulshrestha et al. (2012) propose and analyze a stochastic model without recourse that selects which pick-up locations to open and the corresponding routes for a fleet of buses such that their total travel time is minimized. Vehicles are initially assigned to one of the possible pick-up locations, and, from there, they perform round trips to a set of capacitated shelters until either all demand is satisfied or they reach a limiting travel time. The formulation assumes that a set of scenarios is provided, but it limits the number of pick-up locations whose demand can differ from the expected value. Sayyady and Eksioglu (2010) propose a model for the single-depot evacuation problem, which incorporates traffic flow dynamics from a simulation package and a logistic function to estimate the number of evacuees at each pick-up location. Song et al. (2009) propose a stochastic model, which selects optimal shelters from a set of candidate nodes and which accounts for the uncertainty on the number of transit-dependent evacuees at each pick-up location by introducing a re-

liability constraint. The model incorporates strict pick-up time windows, but assumes that all evacuees are at the pick-up locations at time zero. Wei and Zhaodong (2010) propose a model for solving the multi-modal (cars and buses) evacuation problem. Finally, Zheng (2013) proposes a formulation which, similar to Goerigk and Grün (2012), models demand in terms of the number of pick-ups required on each location, and which, similar to Deai et al. (2011) and Song et al. (2009), enforces strict pick-up time windows.

The BEP-CA has several characteristics that distinguish it from the related evacuation planning and VRP literature. First, the literature almost invariably assumes that all demand is ready, whereas the BEP-CA assumes that demand accumulates over time and under constant, location-specific rates. Deai et al. (2011) and Zheng (2013) incorporate time-dependent evacuee demand through the use of time windows, but such approach still assumes that demand is ready and static within each time window. More importantly, since time windows disassociate the routing and scheduling decisions, they may lead to infeasibility or to suboptimal solutions. Although not in a VRP setting, such a linear arrival process has been used to model transportation problems. For instance, Delgado et al. (2009) propose a model to identify the pick-up schedule for a fleet of buses traversing a transit corridor that minimizes passenger waiting and travel time; Puong and Wilson (2008) propose a model for urban rail transit lines that allows trains to wait at designated stations following a disruption in service; and Zolfaghari et al. (2004) propose a model that uses real-time information to determine holding times along a specific route. The second major distinguishing characteristic of the BEP-CA is its ability to limit the number, and to synchronize the time buses

perform pick-ups at each location, thus allowing the construction of more equitable pick-up schedules. Mankowska et al. (2011) study the use of spatial-temporal vehicle synchronization on transportation settings, but, contrary to the BEP-CA, the study focuses on general formulations for the VRP and assumes that the number of vehicles simultaneously arriving at the customer nodes is fixed, and, thus, independent of the demand accumulated since the last scheduled pick-up time. Third, instead of minimizing some function of the overall duration of the evacuation, the problem assumes that the evacuation end-time is fixed and seeks to minimize the time evacuees wait for transit vehicles, a customer-oriented objective function that mitigates risk both to evacuees and to drivers. Clearly, not only is the BEP-CA distinct from the modeling and optimization literature on transit-based evacuation planning, but it also has unique features that distinguish it from the VRP literature. In fact, its closest VRP variant is, perhaps, the period vehicle routing problem with service choice (PVRP-SC), introduced and mathematically formulated by Francis et al. (2006). Similar to the BEP-CA, the PVRP-SC seeks to identify the routes for a fleet of vehicles operating from a central depot such that each customer is visited, at most, a maximum number of times before a predetermined end-time. While a minimum number of visits to each customer (as to meet their demand) is also enforced, additional visits improve the objective function value. Nonetheless, the PVRP-SC assumes that the set of potential arrival times is both discrete and finite (e.g., once a day over a workweek), and while the remaining demand depends on previous pick-ups, such does not accumulate over time.

In summary, the contributions of this paper are: (i) introduction of a VRP model for transit-

based regional evacuation planning that incorporates a linear evacuee arrival process, a spatial-temporal synchronization of vehicles (and, thus, pick-up times), and a customer-oriented objective function to mitigate evacuation risk, features which, to our knowledge, have not been studied in either the evacuation or VRP literature; (ii) assessment of the marginal effect of the maximum service level (i.e., the number of pick-ups allowed at each pick-up location) and fleet size over the problem feasibility and the total exposure objective function; and, more importantly, (iii) identification of policies for selecting the maximum service level which guarantees the problem feasibility, an efficient use of the available fleet, and equitable reductions in exposure across pick-up locations. Instead of providing methodologies to find optimal or near-optimal solutions for problems of practical size, this research studies the structural properties of a difficult optimization problem, and, thus, identifies policies that can aid transit agencies in making practical decisions without having to perform tedious and time-consuming sensitivity analyses. It is shown that the dynamic relationship between the maximum service level and the fleet size is not only theoretically interesting, but can also lead to the development of more efficient transit-based, regional evacuation plans.

This paper is organized as follows: Section 2.2 introduces the modeling notation, presents a necessary condition for feasibility, and proposes a formulation for the BEP-CA; Section 2.3 explores the effect of maximum service level and fleet size over the problem feasibility and the total exposure objective function value, identifies and examines a lower bound on exposure for a maximum service level, and proposes upper and lower bounds on the fleet size required for feasibility; and, finally, Section 2.4 summarizes the findings of this research

and lists the future work needed on the area.

2.2 Modeling Framework

This section presents a formulation for the BEP-CA, including an evacuation-related objective function, and two scenarios for illustration. In addition, a necessary condition for feasibility is introduced and it is shown that the proposed objective function is strictly convex.

Consider a complete network (N, A), where N and A are the sets of nodes and arcs, respectively. The set N is composed of P, the set of pick-up locations, and node 0, the depot (which also functions as a shelter). Each arc $(i, j) \in A$ has an associated traversal time, t_{ij}, satisfying the triangle inequality, that is, $t_{ij} \leq t_{ik} + t_{kj},\ \forall\ (i, j), (i, k), (k, j) \in A$, and each pick-up node $j \in P$ has demand of $D_j L_j$ evacuees, where D_j denotes the evacuee arrival rate and L_j denotes both the last scheduled pick-up time and the end of the arrival process (note that the evacuation is assumed to begin at time zero). Moreover, V denotes the number of available buses, each with capacity of C evacuees, and F denotes the number of pick-ups allowed at a pick-up location (maximum service level). Finally, the non-policy related parameter T denotes the maximum number of arc traversals per vehicle. Since the BEP-CA allows service choice, F is not enforced. However, by synchronizing the time when vehicles visit each location, by ensuring that these have enough free capacity to accommodate all evacuees that arrived since the last scheduled pick-up time, and by forcing at least one pick-

up to take place at the end of the evacuee arrival process, the problem serves every evacuee even if fewer than F pick-ups take place at the location. The decision variables, objective function, and constraint set for the BEP-CA are as follows:

Table 2.1: Decision variables for the BEP-CA.

Variable	**Description**
x_{ij}^{mtf}	binary variable that equals 1 if arc (i,j) is the t^{th} arc traversed by vehicle m for the f^{th} pick-up at node j, else 0, $\forall\ (i,j) \in A$, $m = 1, \cdots, V$, $t = 1, \cdots, T(F)$, $f = 1, \cdots, F$;
b_j^{mtf}	number of evacuees from the f^{th} pick-up at node j assigned to (or, if j is the depot, released from) vehicle m on its t^{th} arc traversal, $\forall\ j \in N$, $m = 1, \cdots, V$, $t = 1, \cdots, T(F)$, $f = 1, \cdots, F$;
d^{mt}	time when vehicle m makes a pick-up after its t^{th} arc traversal, $\forall\ m = 1, \cdots, V$, $t = 0, \cdots, T(F)$. Let $d^{m0} = 0,\ \forall\ m = 1, \cdots, V$;
p_j^f	time when the f^{th} pick-up is scheduled for node j, $\forall\ j \in P$, $f = 0, \cdots, F$. Let $p_j^0 = 0$ and $p_j^F = L_j,\ \forall\ j \in P$.

For notational simplicity, the depot variables x_{i0}^{mtf} and b_0^{mtf} use the same index f as the ones for the pick-up nodes, but they are set to zero for pick-ups $f = 2, \cdots, F$ (i.e., let $x_{i0}^{mtf} = 0$ and $b_0^{mtf} = 0,\ \forall\ i : (i,0) \in A,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 2, \cdots, F$).

BEP-CA Formulation:

$$\text{Minimize} \quad \sum_{j\in P}\frac{D_j}{2}\sum_{f=1}^{F}(p_j^f - p_j^{f-1})^2 \tag{2.1}$$

Subject to

$$x_{ij}^{m1f} = 0, \quad \forall\ i \in P,\ j:(i,j)\in A,\ m = 1,\cdots,V,\ f = 1,\cdots,F \tag{2.2}$$

$$\sum_{i:(i,j)\in A}\sum_{f=1}^{F} x_{ij}^{mtf} = \sum_{k:(j,k)\in A}\sum_{f=1}^{F} x_{jk}^{m(t+1)f}, \quad \forall\ j \in P,\ m = 1,\cdots,V,\ t = 1,\cdots,T(F)-1 \tag{2.3}$$

$$\sum_{i:(i,0)\in A}\sum_{f=1}^{F} x_{i0}^{mtf} \geq \sum_{j:(0,j)\in A}\sum_{f=1}^{F} x_{0j}^{m(t+1)f}, \quad \forall\ m = 1,\cdots,V,\ t = 1,\cdots,T(F)-1 \tag{2.4}$$

$$x_{ij}^{mTf} = 0, \quad \forall\ j \in P,\ i:(i,j)\in A,\ m = 1,\cdots,V,\ f = 1,\cdots,F \tag{2.5}$$

$$\sum_{(i,j)\in A}\sum_{f=1}^{F} x_{ij}^{mtf} \leq 1, \quad \forall\ m = 1,\cdots,V,\ t = 1,\cdots,T(F) \tag{2.6}$$

$$d^{mt} \geq d^{m(t-1)} + \sum_{(i,j)\in A}\sum_{f=1}^{F} t_{ij}x_{ij}^{mtf}, \quad \forall\ m = 1,\cdots,V,\ t = 1,\cdots,T(F) \tag{2.7}$$

$$d^{mt} \geq p_j^f - M(1 - \sum_{i:(i,j)\in A} x_{ij}^{mtf}), \quad \forall\ j \in P,\ m = 1,\cdots,V,\ t = 1,\cdots,T(F),\ f = 1,\cdots,F \tag{2.8}$$

$$d^{mt} \leq p_j^f + M(1 - \sum_{i:(i,j)\in A} x_{ij}^{mtf}), \quad \forall\ j \in P,\ m = 1,\cdots,V,\ t = 1,\cdots,T(F),\ f = 1,\cdots,F \tag{2.9}$$

$$b_j^{mtf} \leq \sum_{i:(i,j)\in A} Cx_{ij}^{mtf}, \quad \forall\ j \in N,\ m = 1,\cdots,V,\ t = 1,\cdots,T(F),\ f = 1,\cdots,F \tag{2.10}$$

$$0 \leq \sum_{j\in P}\sum_{l=1}^{t}\sum_{f=1}^{F} b_j^{mlf} - \sum_{l=1}^{t}\sum_{f=1}^{F} b_0^{mlf} \leq C, \quad \forall\ m = 1,\cdots,V,\ t = 1,\cdots,T(F) \tag{2.11}$$

$$\sum_{m=1}^{V}\sum_{t=1}^{T(F)} b_j^{mtf} = D_j(p_j^f - p_j^{f-1}), \quad \forall\ j \in P,\ f = 1,\cdots,F. \tag{2.12}$$

Objective Function (2.1) minimizes the *total exposure*, that is, the weighted sum of the squared time intervals between successive pick-ups at each pick-up node. Since evacuees are assumed to gather at these nodes at a constant rate, D_j, and since the constraint set for the BEP-CA forces buses to pick-up all evacuees that have arrived since the last pick-up, (2.1) minimizes the total evacuee waiting time (i.e., the time between the arrival of each evacuee and the subsequent scheduled pick-up, in units of people $\times$ minute, for example). Constraints (2.2)-(2.6) ensure the construction of valid routes, including the elimination of subtours. Constraint (2.2) prevents a vehicle's first arc traversal ($t = 1$) from starting at a pick-up node. Constraint (2.3) forces a vehicle's $(t+1)^{th}$ arc traversal to start from node $j \in P$, if the vehicle's t^{th} arc traversal ended at node j. Constraint (2.4) is similar to (2.3), but it allows vehicles to end service at the depot. Constraint (2.5) forces a vehicle's $T(F)^{th}$ arc traversal, if used, to end at the depot, and, finally, (2.6) limits vehicles to one pick-up per arc traversal. Constraints (2.7)-(2.9) generate the pick-up schedule. Constraint (2.7) ensures that the travel time for a vehicle is appropriately incremented after each arc traversal, while (2.8) and (2.9) coordinate the arrival time of vehicles assigned to the f^{th} scheduled pick-up at node j. Here, let $M = \max_{j \in P}\{L_j\} + \max_{(i,j) \in A}\{t_{ij}\}$. Constraint (2.10) prevents a bus from picking-up (or dropping-off) evacuees from a node unless the bus is at the node, while (2.11) enforces the bus capacity restriction, allowing for multiple drop-offs at the depot. Constraint (2.12) ensures that every evacuee waiting at a pick-up node is picked-up at the next scheduled pick-up, and prevents higher index scheduled pick-ups from happening before lower index ones, that is, it forces $p_j^f \geq p_j^{f-1},\ \forall\ j \in P,\ f = 1 \cdots F$. Observe that the sequential pick-up

ordering restriction is only locally enforced, for instance, a vehicle can go from the second pick-up at node i to the first pick-up at node j. In addition, (2.12), along with the variable restriction $p_j^F = L_j,\ \forall\ j \in P$, ensure that every evacuee at pick-up node j is picked-up by time L_j. Finally, the binary restriction on the x-variables and the non-negativity restriction on the $b,\ d$, and p-variables are included.

As examples of the BEP-CA network structure, consider Scenarios 1 and 2 depicted in Figures 2.1(a) and 2.1(b), respectively, where buses have capacity of $C = 20$ evacuees. Observe that these networks are complete, and that all arc traversal times satisfy the triangle inequality.

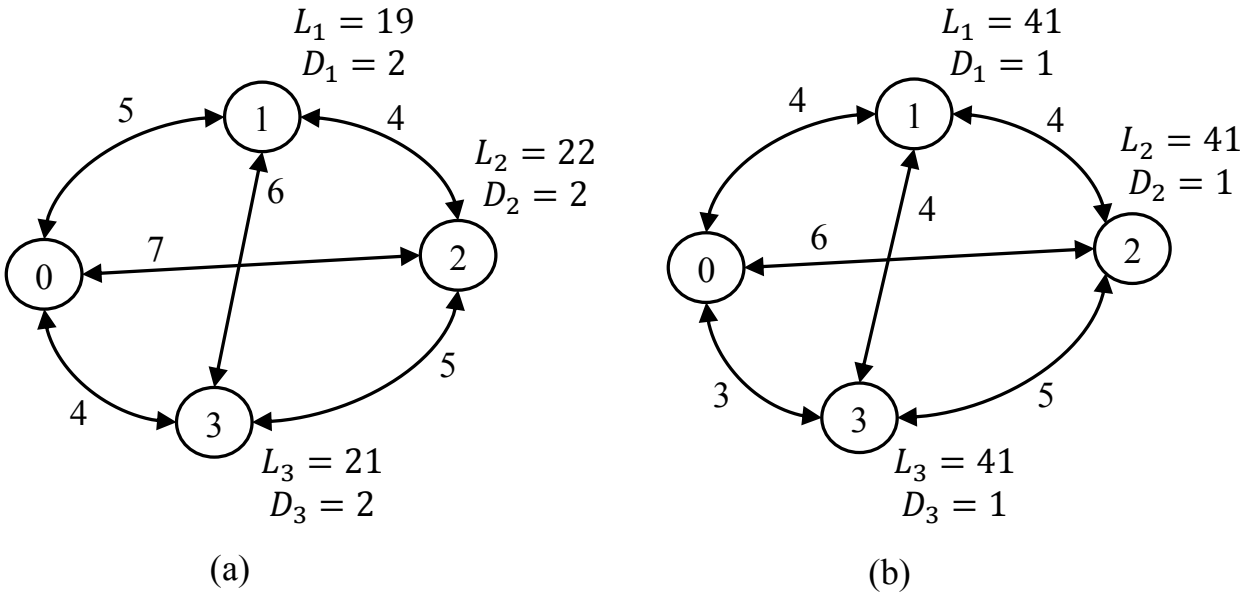


Figure 2.1: The networks for Scenarios 1 (a) and 2 (b).

This formulation has two important properties. First, Lemma 2.1 shows that (2.1) is strictly convex. Unfortunately, since the x-variables are binary, the resulting feasible region is non-convex. Therefore, despite the use of a strictly convex objective function, multiple optimal

solutions may exist for this problem, an intuitive finding considering its inheriting (and complex) symmetries. Nonetheless, since (2.1) is strictly convex, every local minimum solution is also a global minimum, and since this objective function is also twice differentiable, several standard algorithms can be used to identify the optimal solution for the problem. Second, Lemma 2.2 presents a necessary condition the problem feasibility. Note, however, that this condition is not sufficient, as it ignores (2.2)-(2.6), the route construction constraints; (2.10), the pick-up/drop-off constraint; and (2.11), the vehicle capacity constraint.

Lemma 2.1 *Objective Function (2.1) is strictly convex.*

Proof. See Appendix. ∎

Lemma 2.2 *The pick-up schedule* $0 = p_j^0 < t_{0j} \leq p_j^1 \leq \cdots \leq p_j^F = L_j,\ \forall\ j \in P$ *is a necessary condition for the BEP-CA feasibility.*

Proof. See Appendix. ∎

Next, Section 2.3 identifies and examines several properties of the BEP-CA.

2.3 Model Analysis

This section explores the effect of the maximum service level, F, and fleet size, V, over the problem feasibility and the total exposure objective function value. Specifically, it is shown

that (i) a feasible solution can always be obtained by increasing the fleet size provided the necessary condition stated on Lemma 2.2; that (ii) if a feasible solution exists, then continuing to increase the fleet size can result in a lower total exposure solution, though not beyond the fleet size required for a lower bound on exposure for a given F-parameter; that (iii) after this threshold is attained, a lower total exposure solution can only be obtained by increasing F, even if the subsequent lower bound on exposure is not attained; and that (iv) even though increasing F always results in a lower or identical total exposure, given the diminishing returns, the concomitant increase in problem size, and, potentially, non-equitable reductions in exposure across pick-up locations, it is not advantageous to pursue ever increasing values. In addition, upper and lower bounds on the fleet size required for feasibility and for such bound on exposure and a formulation and an upper bound on the maximum number of arc traversals per vehicle are proposed and examined.

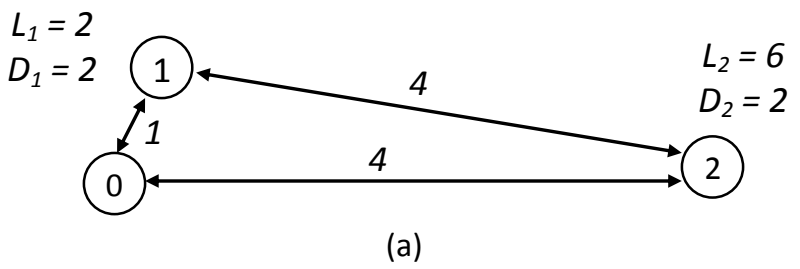
2.3.1 The Maximum Service Level

The maximum service level, F, can substantially affect the problem feasibility and the objective function value. This section explores the effect of this problem parameter and shows that there are detrimental effects resulting from the use of arbitrarily large F-values (i.e., from letting the model decide the frequency of visits on each pick-up location unrestrictedly). First, however, an important feature of the BEP-CA formulation, service choice, is presented.

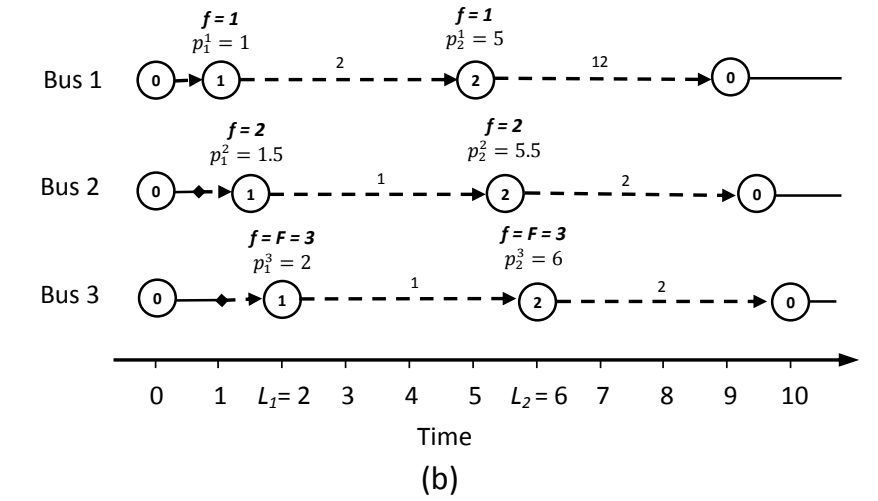
2.3.1.1 Service Choice.

The BEP-CA exploits efficiencies deriving from service choice, that is, from letting F be an upper bound on the number of pick-ups at each pick-up node. Since service choice increases the size of the feasible region, it allows certain instances of the problem to become feasible or to yield lower total exposure solutions. Example 2.1 presents one such instance.

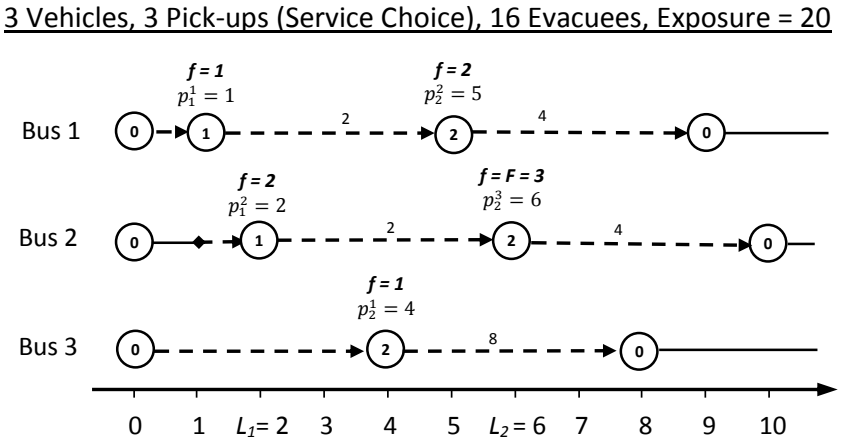
Example 2.1 *Consider the network displayed in Figure 2.2(a), with parameters* $F = 3$ *pick-ups,* $V = 3$ *buses, and* $C = 20$ *evacuees/bus. Depictions of the corresponding optimal pick-up schedules without service choice (at time* $p_1^1 = 1,\ p_1^2 = 1.5,\ p_1^3 = L_1 = 2$ *and* $p_2^1 = 5,\ p_2^2 = 5.5,\ p_2^3 = L_2 = 6$*) and with service choice (at time* $p_1^1 = 1,\ p_1^2 = L_1 = 2$ *and* $p_2^1 = 4,\ p_2^2 = 5,\ p_2^3 = L_2 = 6$*) are presented in Figures 2.2(b) and 2.2(c), respectively. In these figures, solid lines indicate periods when buses are idle, whereas dashed lines indicate periods when buses are moving. The number of evacuees on the bus, if any, is presented on top of the dashed lines. Although it would appear that the lowest total exposure solution is obtained when all* F *pick-ups take place, as these figures exemplify, enforcing all* F *pick-ups may result in a higher exposure solution than the one obtained from allowing service choice (27 versus 20). Moreover, note that if* F *is increased (i.e., if* $F \geq 4$*), if service choice is not allowed, and if the fleet size remains unchanged (i.e., if* $V = 3$ *buses), then the problem becomes infeasible.*



(a)



(b)



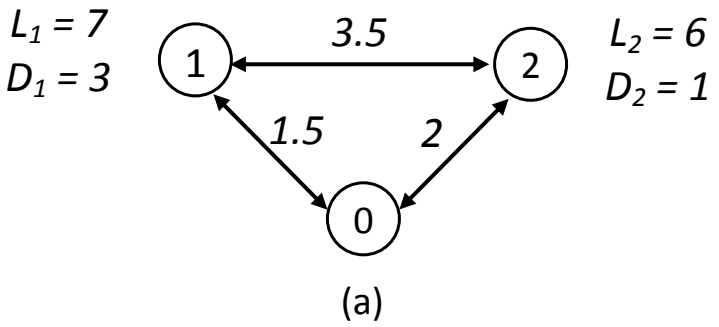
(c)

Figure 2.2: An example network (a) and representations of the corresponding optimal pick-up schedules without service choice (b), and with service choice (c).

Since the BEP-CA allows service choice, increasing F always results in either an identical or a lower total exposure objective function value. However, there are detrimental effects resulting from the use of arbitrarily large F-values. First, increasing F leads to an increase in problem size (particularly on the number of binary variables), and, consequently, in the time needed to obtain an optimal solution. Second, such increases yield diminishing returns (as it will be shown in Section 2.3.1.2). Finally, reductions in exposure, if any, may not be equitable across pick-up locations. The concept of non-equitable reductions in exposure is presented in Definition 2.1 and demonstrated in Example 2.2.

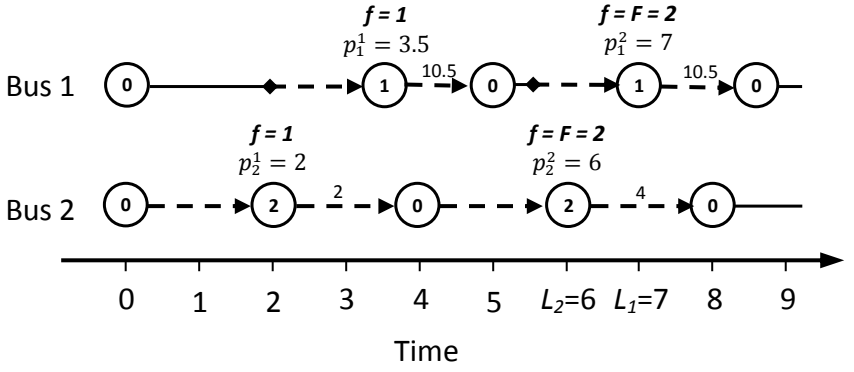
Definition 2.1 *A non-equitable reduction in exposure occurs when larger F-values lead to increases in exposure at one or more pick-up locations, even though the total exposure is reduced.*

Example 2.2 *Consider the network displayed in Figure 2.3(a), with parameters $V = 2$ buses and $C = 20$ evacuees/bus. Depictions of the corresponding optimal pick-up schedules for $F = 2$ (at time $p_1^1 = 3.5,\ p_1^2 = L_1 = 7$, and $p_2^1 = 2,\ p_2^2 = L_2 = 6$) and for $F = 3$ (at time $p_1^1 = 2,\ p_1^2 = 4,\ p_1^3 = L_1 = 7$ and $p_2^1 = L_2 = 6$) are presented in Figures 2.3(b) and 2.3(c), respectively. While increasing F from two to three led to a decrease in total exposure from 46.75 to 43.50, indicating that, on average, evacuees would need to wait less time for the next pick-up, such overall improvement was not equitable among pick-up nodes: while the contribution in exposure from pick-up node 1 decreased from 36.75 to 25.50 when F increased from two to three, the one from pick-up node 2 increased from 10.00 to 18.00.*



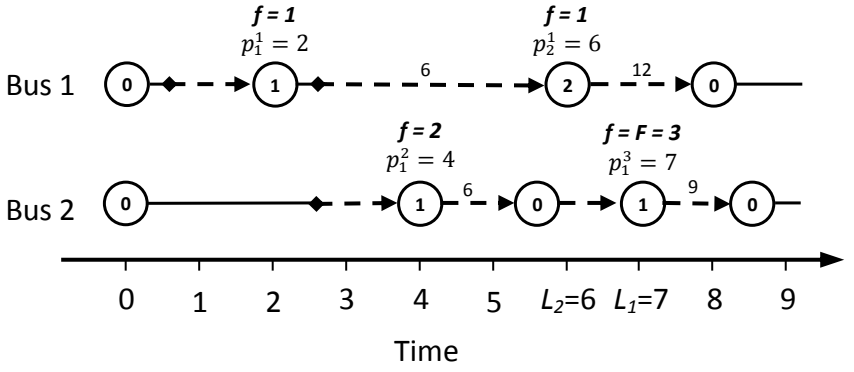
(a)

2 Vehicles, 2 Pick-ups, 27 Evacuees, Total Exposure = 46.75



(b)

2 Vehicles, 3 Pick-ups, 27 Evacuees, Total Exposure = 43.50



(c)

Figure 2.3: An example network (a) and representations of the corresponding optimal pick-up schedules for $F = 2$ (b) and $F = 3$ (c) pick-ups.

In addition, since the BEP-CA assumes that evacuees arrive linearly at the various pick-up locations, some individuals may need to wait longer under larger F-values, even in pick-up nodes having reduced exposure, higher frequency of visits, or, even, a reduction in the largest interval between bus arrivals. For instance, an individual that arrived on pick-up node 1 at time 3 would need to wait 0.5 time units for the next pick-up under $F = 2$ and 1 time unit under $F = 3$. These differences, however, are not related to the definition of equity used for the purposes of this research.

As demonstrated in Example 2.2, service choice, albeit often desirable, can also lead to non-equitable reductions in exposure across pick-up locations. While it is easy to enforce F pick-ups on each location, such a constraint, by itself, would not solve the issue due to the difficult in differentiating between relatively close scheduled pick-up times. For instance, suppose that $F = 2$, then the optimal pick-up schedule at a given location may consist of a single visit by two vehicles, say at time $L_j = 10$. If service choice is prevented, then the optimal pick-up schedule may consist of two visits by one vehicle each, say at times 9.9 and 10. Since both schedules are virtually identical, neither one addresses the issue. Generally, such can only be prevented by fixing the p-variables to an equitable pick-up schedule before the problem is optimized. However, by limiting the size of the feasible region, such approach may lead to infeasibility or worsen the objective function value. The next section introduces and examines the pick-up schedule resulting in the lowest exposure for a given F-value. Among other properties, it is shown that such pick-up schedule, when feasible, guarantees equitable reductions in exposure across pick-up locations.

The next section introduces and examines the pick-up schedule resulting in the lowest exposure for a given F-value. Among other properties, it is shown that such pick-up schedule, when feasible, guarantees equitable reductions in exposure across pick-up locations.

2.3.1.2 The Minimum Exposure Schedule.

The *Minimum Exposure Schedule* for a given F-parameter, $MES(F)$, is defined as follows:

Definition 2.2 *The $MES(F)$ is the pick-up schedule that minimizes (2.1) subject only to the necessary condition for feasibility presented in Lemma 2.2.*

The $MES(F)$ can be obtained by the following formulation:

$$\text{Minimize} \quad \sum_{j \in P} \sum_{f=1}^{F} \frac{1}{2} D_j {q_j^f}^2 \tag{2.13}$$

$$\text{subject to} \sum_{f=1}^{F} q_j^f = L_j, \quad \forall\ j \in P \tag{2.14}$$

$$q_j^1 \geq t_{0j}, \quad \forall\ j \in P \tag{2.15}$$

$$q_j^f \geq 0, \quad \forall\ j \in P,\ f = 1, \cdots, F, \tag{2.16}$$

where q_j^f denotes the interval between scheduled pick-up times at pick-up node j (i.e., $q_j^f = p_j^f - p_j^{f-1},\ \forall\ j \in P,\ f = 1, \cdots, F$). Objective Function (2.13) follows from (2.1), and Constraints (2.14)-(2.16) enforce the necessary condition presented in Lemma 2.2. Since this condition is not sufficient, the $MES(F)$ yields a lower bound on (2.1), and will only

be feasible provided that enough vehicles are available to meet the corresponding pick-up schedule (a sufficient condition for the problem feasibility, as it will be shown in Section 2.3.2). Proposition 2.1 presents the optimal solution for this formulation:

Proposition 2.1 *The* $MES(F)$ *is given by* $p_j^f = fL_j/F,\ \forall\ f = 1, \cdots, F$, *if* $0 < t_{0j} \leq L_j/F$ *or* $F = 1$, *and* $p_j^1 = t_{0j}$ *and* $p_j^f = t_{0j} + (f-1)(L_j - t_{0j})/(F-1),\ \forall\ f = 2, \cdots, F$, *if* $L_j/F < t_{0j} \leq L_j$ *and* $F \geq 2,\ \forall\ j \in P$.

Proof. See Appendix. ∎

Given the quadratic nature of (2.1), the total exposure is dominated by the largest interval between pick-ups (i.e., $\max_{j \in P} \max_{f=1,\cdots,F}\ \{p_j^f - p_j^{f-1}\}$). Consequently, the $MES(F)$ is obtained when all F pick-ups take place and when these are spaced evenly before evacuation end-time. Figure 2.4(a) illustrates this pattern for node 1, Scenario 1, and for $F = 2,\ F = 5$, and $F = 15$, whereas Figure 2.4(b) presents the corresponding contribution in exposure from this node.

When $F = 2$, $t_{01} < 9.5$ (L_1/F) and the $MES(2)$ for pick-up node 1 has of $p_1^1 = 9.5$ and $p_1^2 = L_1 = 19$. Under this pick-up schedule, the exposure from this node is $\int_0^{9.5-0} D_1 t\, \mathrm{d}t + \int_0^{19-9.5} D_1 t\, \mathrm{d}t = 180.5$ (as $D_1 = 2$). Conversely, when $F = 15$, $t_{01} > 1.27$ (L_1/F), and the $MES(15)$ for pick-up node 1 has $p_1^1 = t_{01} = 5$ and $p_1^f = p_1^{f-1} + 1,\ \forall\ f = 2, \cdots, 15$, that is, when the remaining 14 pick-ups are spaced evenly within the remaining time. Under this pick-up schedule, the exposure from this node is $\int_0^5 D_1 t\, \mathrm{d}t + \sum_{f=2}^{15} \int_0^1 D_1 t\, \mathrm{d}t = 39.0$.

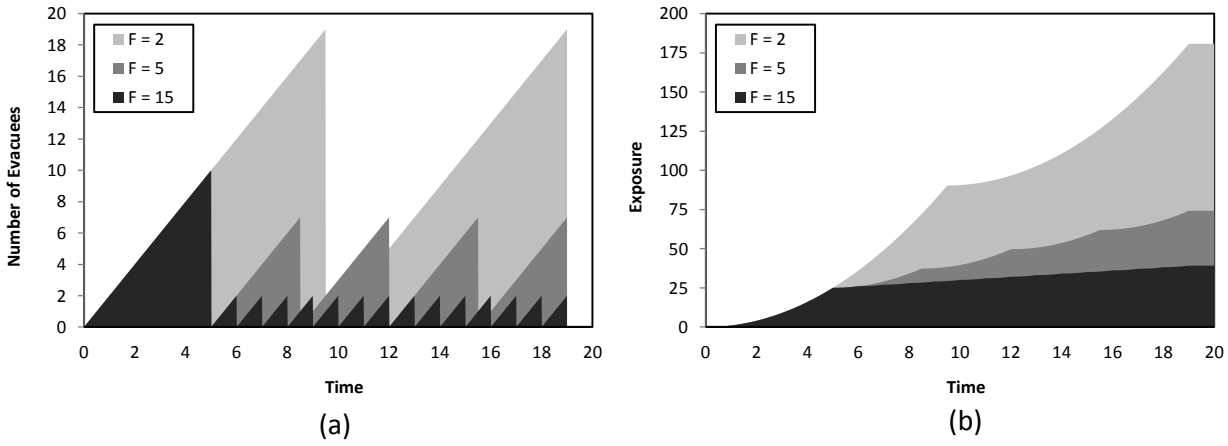


Figure 2.4: Evacuees waiting at pick-up node 1, Scenario 1 (a) and corresponding exposure (b) for the $MES(F)$ for $F = 2$, $F = 5$, and $F = 15$.

Next, Corollaries 2.1 and 2.2 show, respectively, that increasing F yields diminishing returns over the $MES(F)$ total exposure and that, as $F \mapsto \infty$, the $MES(F)$ total exposure converges to the sum of accumulated exposures for the first scheduled pick-up at each pick-up node, that is, $\sum_{j \in P} D_j t_{0j}^2/2$. This constant is independent of F, and, thus, serves both as the limiting exposure on the $MES(F)$ and as a greatest lower bound on (2.1).

Corollary 2.1 *The change in the $MES(F)$ total exposure is quadratic-decreasing with F.*

Proof. See Appendix. ∎

Corollary 2.2 *The $MES(F)$ total exposure converges to $\sum_{j \in P} D_j t_{0j}^2/2$, as $F \mapsto \infty$.*

Proof. See Appendix. ∎

Figure 2.5 depicts the $MES(F)$ total exposure as a function of F for Scenarios 1 and 2. Observe the quadratic-decreasing change in total exposure from increasing F (as expected

from Corollary 2.1), and that the $MES(F)$ total exposure is converging to the sum of accumulated exposures for the first pick-up at each pick-up node (as expected from Corollary 2.2).

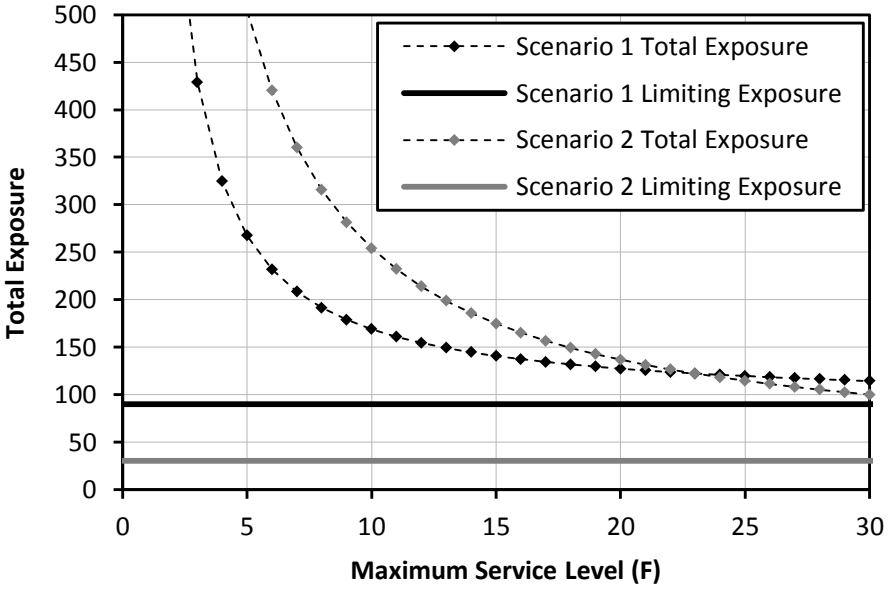


Figure 2.5: $MES(F)$ total exposure as a function of F and limiting exposure for Scenarios 1 and 2.

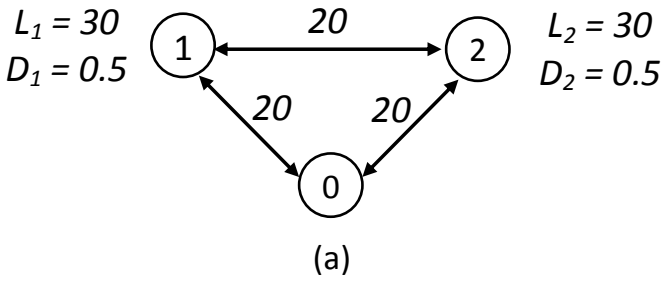
In fact, diminishing returns can be observed even between problem instances that do not attain the $MES(F)$, although such returns may not be quadratic-decreasing with F. Finally, as it will be shown in Section 2.3.2, such returns may only be realizable if accompanied by a concomitant increase in fleet size. Given the diminishing (and potentially unrealizable) returns from increasing F and the ensuing growth in problem complexity, it is not advantageous to increase the maximum service level boundlessly, especially as the total exposure approaches its greatest lower bound: $\sum_{j \in P} D_j t_{0j}^2/2 = 90.0$ and 30.5 for Scenarios 1 and 2,

respectively.

Since the $MES(F)$ yields a lower bound on (2.1) and since the BEP-CA allows service choice, whenever F is increased, the total exposure will remain between the one for the previous solution and the one corresponding to the $MES(F)$ for the new F-value. Example 2.3 presents one such instance.

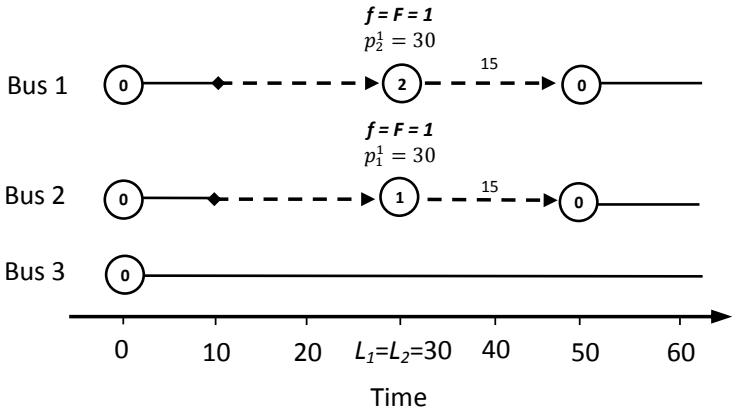
Example 2.3 *Consider the network displayed in Figure 2.6(a) with parameters* $V = 3$ *buses and* $C = 20$ *evacuees/bus. Depictions of the corresponding optimal pick-up schedules for* $F = 1$ *(at time* $p_1^1 = L_1 = 30$ *and* $p_2^1 = L_2 = 30$*) and* $F = 2$ *(at time* $p_1^1 = 20,\ p_1^2 = L_1 = 30$*, and* $p_2^1 = L_2 = 30$*) are presented in Figures 2.6(b) and 2.6(c), respectively. Observe that the pick-up schedule presented in Figure 2.6(b) corresponds to the* $MES(1)$*, whereas the one presented in Figure 2.6(c) does not correspond to the* $MES(2)$*, which would consist of* $p_1^1 = 20,\ p_1^2 = L_1 = 30$ *and* $p_2^1 = 20,\ p_2^2 = L_2 = 30$ *and have total exposure of 250. Nonetheless, increasing* F *by one led to a reduction in total exposure from 450 to 350.*

The next section examines the effect of fleet size over the problem feasibility and the total exposure objective function value.



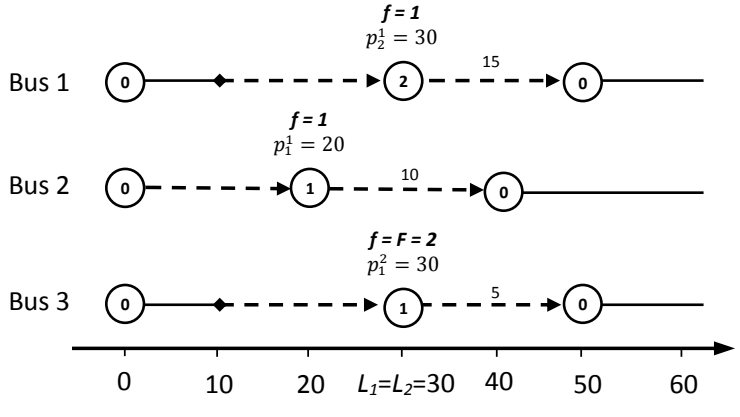
(a)

3 Vehicles, 1 Pick-up, 30 Evacuees, Total Exposure = 450



(b)

3 Vehicles, 2 Pick-ups, 30 Evacuees, Total Exposure = 350



(c)

Figure 2.6: An example network (a) and representations of the corresponding optimal pick-up schedules for $F = 1$ (b) and $F = 2$ (c) pick-ups.

2.3.2 Fleet Size

In this section, an alternative objective function to minimize fleet size, as well as upper and lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ and for a feasible solution to the BEP-CA are presented and examined.

Similar to the maximum service level, F, the fleet size parameter, V, can also influence feasibility and the total exposure objective function value. First, since the BEP-CA allows service choice, if enough vehicles are available, then it is feasible to pick-up all $\sum_{j \in P} D_j L_j$ evacuees in one pick-up. Consequently, a feasible solution can always be obtained by increasing the fleet size provided the necessary condition stated on Lemma 2.2. Moreover, if multiple pick-ups are feasible (i.e., if $F \geq 2$ and if enough vehicles are available to meet the extra pick-ups), then a lower exposure pick-up schedule may also be feasible. For instance, in the network for Scenario 1, 38 evacuees must be picked-up in node 1, 44 in node 2, and 42 in node 3 (for a total of 124 evacuees). Hence a feasible solution would be to send 2 buses to node 1 (arriving at time $p_1^1 = L_1 = 19$), 3 buses to node 2 (arriving at time $p_2^1 = L_2 = 22$), and 3 buses to node 3 (arriving at time $p_3^1 = L_3 = 21$) for a total exposure of 1286. However, as it will be shown later, the $MES(8)$ is feasible for the same number of vehicles, resulting in a total exposure of 191.4. Next, an alternative objective function to minimize the fleet size parameter, V, is introduced.

2.3.2.1 Minimum Fleet Size.

The minimum fleet size needed for a feasible solution can be obtained by the following objective function and the BEP-CA constraint set:

$$\text{Minimize} \quad \sum_{j:(0,j)\in P} \sum_{m=1}^{V} \sum_{f=1}^{F} x_{0j}^{m1f}. \tag{2.17}$$

Objective Function (2.17) holds by (2.3) and (2.4), which force vehicles to increase their arc traversal index sequentially. Therefore, vehicles that do not make a first arc traversal are not utilized. Observe that (2.1) and (2.17) can be combined into a single objective function through arbitrary weight coefficients. Nonetheless, since the relative importance of these functions is context-specific, its assessment lies outside the scope of this research. See Abdelgawad et al. (2010a,b) and Abdelgawad and Abdulhai (2010) for practical examples on the use of such function in evacuation settings.

Alternatively, (2.1) and (2.17) can be used sequentially, by first identifying the pick-up schedule that minimizes evacuee exposure (given an arbitrary V-parameter), and later the minimum number of vehicles required to meet this pick-up schedule. In fact, since the $MES(F)$ can be computed analytically, (2.17) can be used to determine the minimum number of vehicles required for a particular $MES(F)$ by fixing the respective p-variables. However, even under a fixed pick-up schedule (and the ensuing reduction in problem size), it is still difficult to obtain a solution for (2.17) subject to the BEP-CA constraint set. Consequently, the following upper and lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ and for a feasible solution to the BEP-CA are both theoretically and practically

relevant.

2.3.2.2 Upper Bound on the $MES(F)$ Fleet Size.

Proposition 2.2 presents an analytical upper bound on the fleet size needed for a feasible implementation of the $MES(F)$.

Proposition 2.2 *The upper bound on the fleet size needed for a feasible implementation of the $MES(F)$, $V_{UB}(F)$, is calculated as follows:*

$$V_{UB}(F) = \sum_{j \in P} \left(\left\lceil \frac{D_j \lambda_j(F)}{C} \right\rceil + \left(\min \left\{ \left\lceil \frac{2t_{0j}}{\tau_j(F)} \right\rceil, F \right\} - 1 \right) \left\lceil \frac{D_j \tau_j(F)}{C} \right\rceil \right), \tag{2.18}$$

where $\tau_j(F)$ and $\lambda_j(F)$ are, respectively, the smallest and largest intervals between scheduled pick-up times at pick-up node j under the $MES(F)$, that is, $\tau_j(F) = \min_{f=1,\cdots,F}\{p_j^f - p_j^{f-1}\} = \min\{L_j/F, (L_j - t_{0j})/(F-1)\}$ and $\lambda_j(F) = \max_{f=1,\cdots,F}\{p_j^f - p_j^{f-1}\} = \max\{L_j/F, t_{0j}\}$.

Proof. See Appendix. ∎

Since the $MES(F)$ yields a lower bound on (2.1), increasing the fleet size beyond what is required for its feasibility will never result in a lower exposure solution. Consequently, the fleet size obtained from Proposition 2.2 yields the lowest exposure for a given F-parameter. In addition, observe that $V_{UB}(F)$ is tight whenever the $MES(F)$ is achieved and such solution is attained by having vehicles dedicated to a single pick-up node. These properties are shown in Example 2.4.

Example 2.4 *Consider the network displayed in Figure 2.7(a) with parameter* $C = 20$ *evacuees/bus, and observe that this network has* $V_{UB}(F) = 2$ *buses for* $F = 2, \cdots, 5$ *pick-ups. Depictions of the corresponding* $MES(2)$ *(at time* $p_1^1 = 50$ *and* $p_1^2 = L_1 = 100$*) and* $MES(5)$ *(at time* $p_1^1 = 20,\ p_1^2 = 40,\ p_1^3 = 60,\ p_1^4 = 80$*, and* $p_1^5 = L_1 = 100$*) are presented in Figures 2.7(b) and 2.7(c), respectively. Observe that these pick-up schedules would still be optimal (for their respective* F*-parameters) even if more than two buses were available for the evacuation. Nonetheless, increasing* F *from 2 to 5 resulted in a decrease in total exposure from 1250 to 500, an improvement obtained without an increase in fleet size.*

Next, a lower bound on the fleet size needed for a feasible implementation of the $MES(F)$ is presented.

2.3.2.3 Lower Bound on the $MES(F)$ Fleet Size.

Proposition 2.3 presents an analytical lower bound on the fleet size needed for a feasible implementation of the $MES(F)$.

Proposition 2.3 *The lower bound on the fleet size needed for a feasible implementation of the* $MES(F)$*,* $V_{LB}(F)$*, is calculated as follows:*

$$V_{LB}(F) = \left\lceil \frac{F|P|}{T(F) - 1} \right\rceil \min_{j \in P} \left\{ \left\lceil \frac{D_j \tau_j(F)}{C} \right\rceil \right\}, \tag{2.19}$$

where $\tau_j(F)$*, once again, equals* $\min\{L_j/F, (L_j - t_{0j})/(F-1)\}$.

Proof. See Appendix. ∎

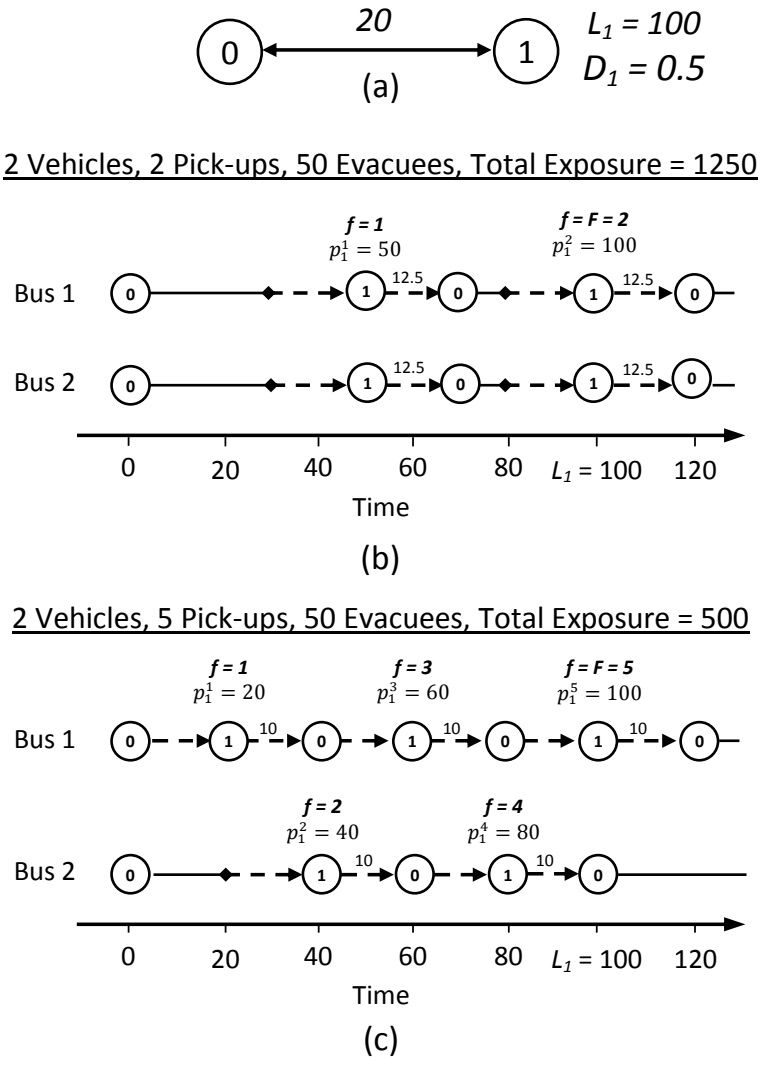


Figure 2.7: An example network (a) and representations of the corresponding $MES(F)$ for $F = 2$ (b) and $F = 5$ (c).

This bound corresponds to the product of two factors: a lower bound on the number of groups of buses simultaneously performing pick-ups under the $MES(F)$ and a lower bound on the number of buses needed in each group in order to pick-up all evacuees that arrived between scheduled pick-up times. Example 2.5 shows that, contrary to $V_{UB}(F)$, the lower

bound on the fleet size needed for a feasible implementation of the $MES(F)$ is not tight in general, even in instances that achieve the $MES(F)$ with vehicles dedicated to a single pick-up location.

Example 2.5 *Recall the network presented in Example 2.4, Figure 2.7(a) with parameter* $C = 20$ *evacuees/bus. Depictions of the corresponding* $MES(1)$ *(at time* $p_1^1 = L_1 = 100$*) and* $MES(3)$ *(at time* $p_1^1 = 33.3$, $p_1^2 = 66.7$, *and* $p_1^3 = L_1 = 100$*) are presented in Figures 2.8(a) and 2.8(b), respectively. When* $F = 1$, $T(1) = 2$ *and* $\tau_1(1) = 100 \Rightarrow V_{LB}(1) = 3$, *the exact fleet size needed for a feasible implementation of the* $MES(1)$. *However, when* $F = 3$, $T(3) = 6$ *and* $\tau_1(3) = 33.3 \Rightarrow V_{LB}(3) = 1$, *and, as depicted in Figure 2.8(b), at least two vehicles are needed for a feasible implementation of the* $MES(3)$ *when* $C = 20$ *evacuees/bus.*

Next, the proposed upper and lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ are used to derive important relationships between the number of available buses, the maximum service level, and the total exposure objective function value.

2.3.2.4 The $MES(F)$ Fleet Size.

This section explores the behavior of the exact fleet size needed for a feasible implementation of the $MES(F)$ based on its relationship with $V_{UB}(F)$ and $V_{LB}(F)$. Specifically, it is shown that there exists an interval of F-values such that smaller F-values result in the sub-utilization of the fleet available whereas larger F-values may result in non-equitable reductions in exposure across pick-up locations.

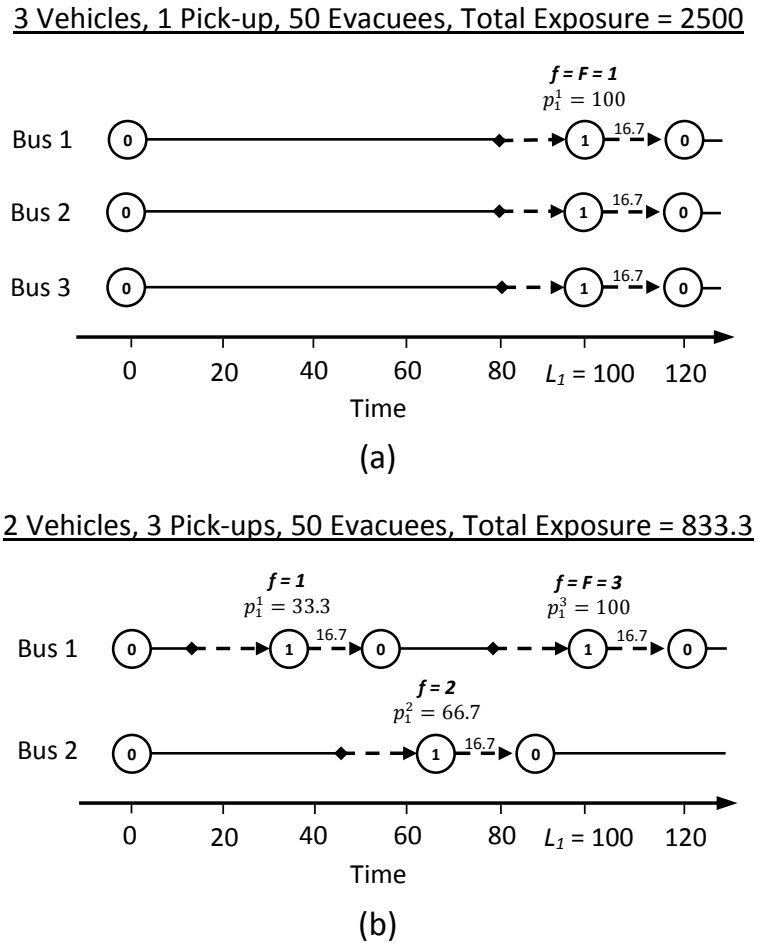


Figure 2.8: Representations of the $MES(F)$ for $F = 1$ (a) and $F = 3$ (b) corresponding to the network presented in Example 2.4, Figure 2.7(a).

As an example, Figure 2.9 depicts the upper and lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ and the fleet size obtained by (2.17) after fixing the p-variables to the $MES(F)$ by the maximum service level for Scenarios 1 and 2 (Figures 2.9(a) and 2.9(b), respectively).

Observe that the exact fleet size needed for a feasible implementation of the $MES(F)$, as well as $V_{UB}(F)$ and $V_{LB}(F)$, may not be monotonically increasing with F for all F-values.

As depicted in Figure 2.9(a), eight vehicles are required for a feasible implementation of the $MES(1)$ in Scenario 1, but only six are needed for the $MES(F)$ with $F = 2, \cdots, 5$. Similarly, as depicted in Figure 2.9(b), nine and six vehicles are required for a feasible implementation of the $MES(1)$ and $MES(2)$ in Scenario 2, respectively, but only three vehicles are needed for a feasible implementation of the $MES(3)$ and $MES(4)$. While this behavior, noted in Remark 2.1, was not observed on every network studied, it has important planning applications. Specifically, it shows that increasing the F-parameter may reduce both the total exposure and the fleet size requirement.

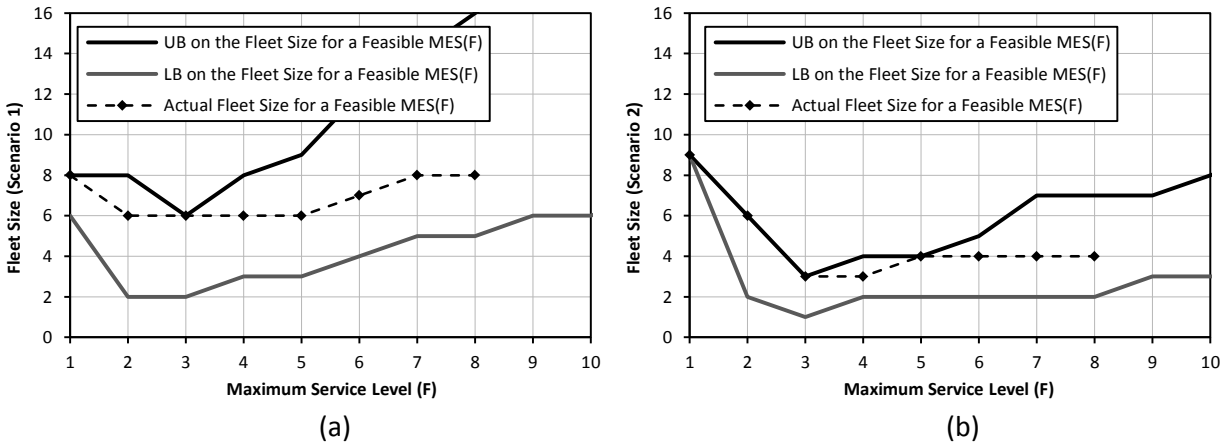


Figure 2.9: Upper bound, lower bound, and the exact fleet size needed for a feasible implementation of the $MES(F)$ by the maximum service level for Scenarios 1 (a) and 2 (b).

Remark 2.1 *The exact number of vehicles required for a feasible implementation of the $MES(F)$ may not increase monotonically with F. Consequently, it may be possible to obtain the $MES(F)$ for a larger F-value with fewer vehicles.*

This non-monotonic behavior is caused by two factors. First, under small F-values, vehicles

may spend a considerable amount of time idle (i.e., stationed at the depot or at some pick-up node). As F increases, the existing fleet is reused to meet the extra pick-ups and the excess idle time diminishes. Second, the increased F-parameter (and consequent reduced interval between scheduled pick-up times) results in a reduced number of evacuees waiting for each bus arrival, and, thus, in an smaller number of buses that need to arrive simultaneously in order to pick-up these evacuees. As F continues to increase, the excess idle time is depleted, vehicles starts to run well under capacity, and the fleet size needed for a feasible implementation of the $MES(F)$ once again starts to increase.

Clearly, the F-value (if one exists) where the fleet size needed for a feasible implementation of the $MES(F)$ starts to increase monotonically with F has important planning applications, as any smaller F-values result in higher total exposure solutions and underutilization of the available fleet. Unfortunately, since the BEP-CA formulation is combinatorial in nature, an analytical expression for such F-parameter may not exist. Instead, Propositions 2.4 and 2.5 show, respectively, that such thresholds do exist for $V_{UB}(F)$ and $V_{LB}(F)$.

Proposition 2.4 *For $F \geq F_{UB}$, $\{V_{UB}(F)\}$, the sequence of upper bounds on the fleet size needed for a feasible implementation of the $MES(F)$, increase monotonically with the maximum service level, F, where:*

$$F_{UB} = \max\left\{\max_{j\in P}\left\{\frac{L_j}{t_{0j}}\right\}, \frac{1}{C}\max_{j\in P}\left\{D_j(L_j - t_{0j})\right\} + 1\right\}. \tag{2.20}$$

Proof. See Appendix. ∎

Proposition 2.5 *For $F \geq F_{LB}$, $\{V_{LB}(F)\}$, the sequence of lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$, increase monotonically with the maximum service level, F, where:*

$$F_{LB} = \max\left\{\max_{j\in P}\left\{\frac{L_j}{t_{0j}}\right\}, \frac{1}{C}\max_{j\in P}\left\{D_j(L_j - t_{0j})\right\} + 1, \left\lfloor \frac{\max_{j\in P}\{L_j\}}{2\min_{(i,j)\in A}\{t_{ij}\}} + \frac{1}{2}\right\rfloor\right\}. \tag{2.21}$$

Proof. See Appendix. ∎

Proposition 2.6 shows that the lower bound on the fleet size needed for a feasible implementation of the $MES(F)$ diverges as $F \mapsto \infty$. Consequently, so does the actual fleet size needed for a feasible implementation of the $MES(F)$ and $V_{UB}(F)$.

Proposition 2.6 *$V_{LB}(F) \mapsto \infty$, as $F \mapsto \infty$.*

Proof. See Appendix. ∎

While F-values greater than or equal to F_{UB} and F_{LB} guarantee the monotonicity of $V_{UB}(F)$ and $V_{LB}(F)$, respectively, depending on the problem instance, such behavior may start from smaller F-values. For instance, Scenario 1 has $F_{UB} = F_{LB} = 21/4$, implying that $V_{UB}(F)$ and $V_{LB}(F)$ increase monotonically for $F \geq 6$. However, as shown in Figure 2.9(a), the monotonicity of $V_{UB}(F)$ and $V_{LB}(F)$ start from $F \geq 3$ and $F \geq 2$, respectively. Similarly, for Scenario 2, $F_{UB} = F_{LB} = 41/3$, but, as shown in Figure 2.9(b), the monotonicity of $V_{UB}(F)$ and $V_{LB}(F)$ start from $F \geq 3$.

Figure 2.10 presents a conceptualization of the upper and lower bounds on the fleet size

needed for a feasible implementation of the $MES(F)$ and the total exposure under the $MES(F)$ by maximum service level.

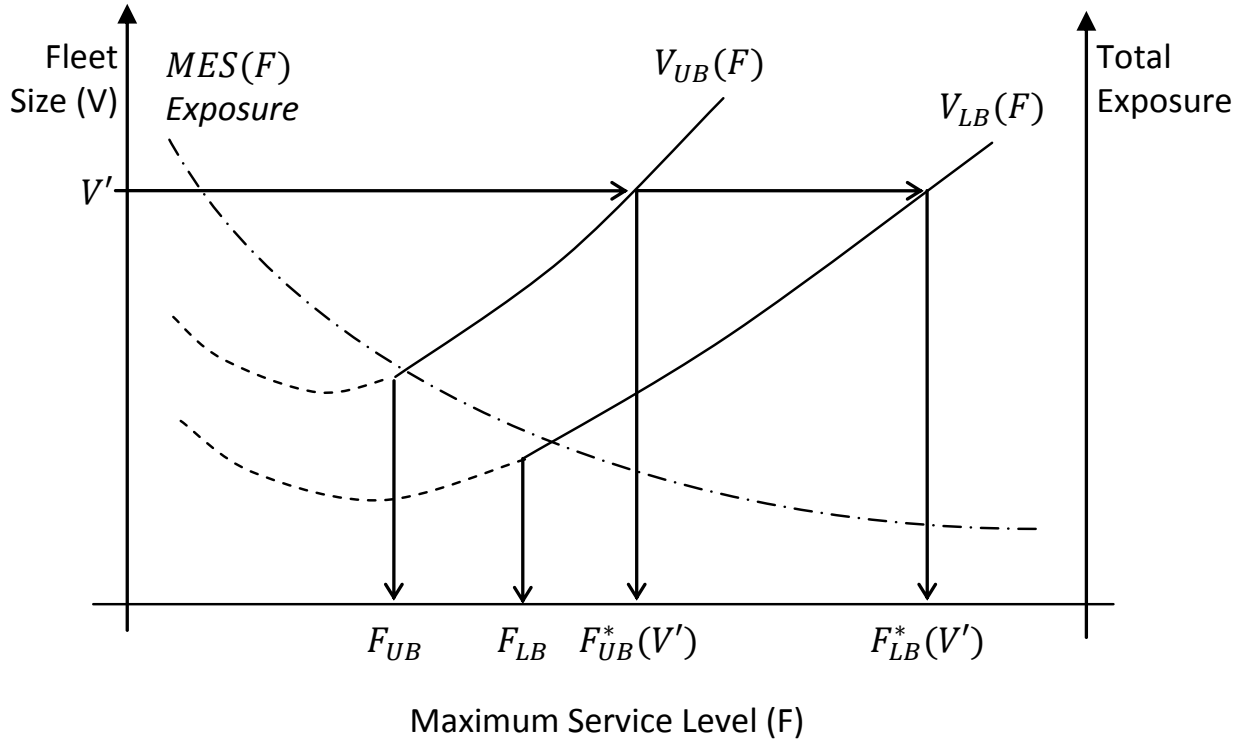


Figure 2.10: Conceptualization of the upper and lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ and its corresponding total exposure.

As shown in this figure, for a fixed (and sufficiently large) fleet size, V, there exist integers $F^*_{UB}(V)$ and $F^*_{LB}(V)$ corresponding, respectively, to the largest and smallest F-values satisfying $V_{UB}(F^*_{UB}(V)) = V_{LB}(F^*_{LB}(V)) = V$, that is:

$$F^*_{UB}(V) = \sup\{F \in \mathbb{N} \wedge F \geq F_{UB} : V_{UB}(F) \leq V\}, \text{ and} \tag{2.22}$$

$$F^*_{LB}(V) = \inf\{F \in \mathbb{N} \wedge F \geq F_{LB} : V_{LB}(F) \geq V\}. \tag{2.23}$$

Consequently, $F^*_{UB}(V)$ and $F^*_{LB}(V) + 1$ represent, respectively, the largest known F-value

for which a feasible $MES(F)$ is guaranteed and the smallest known F-value for which the $MES(F)$ is infeasible (provided the fleet size parameter, V). The relationship between F_{UB}, F_{LB}, $F^*_{UB}(V)$, and $F^*_{LB}(V)$ is stated in Lemma 2.3.

Lemma 2.3 *Either* $F_{UB} \leq F_{LB} \leq F^*_{UB}(V) \leq F^*_{LB}(V)$ *or* $F_{UB} \leq F^*_{UB}(V) \leq F_{LB} \leq F^*_{LB}(V)$.

Proof. See Appendix. ∎

As aforementioned, the parameters $F^*_{UB}(V)$ and $F^*_{LB}(V)$ have important planning applications. First, given a fixed fleet size, V, the maximum service level should equal, at least, $F^*_{UB}(V)$, as the $MES(F^*_{UB}(V))$ is always feasible and as the $MES(F)$ total exposure is quadratic-decreasing with F. Clearly, there exists a point F' (where $F^*_{UB}(V) \leq F' \leq F^*_{LB}(V)$), such that the $MES(F')$ is feasible, but the $MES(F'+1)$ is infeasible. Although the total exposure either continues to decrease or remains unchanged for $F > F'$, if the $MES(F)$ is infeasible, any reductions in total exposure from increased F may not be equitable across pick-up locations. Following the discussion in Section 2.3.1.1, if the $MES(F)$ is not achieved, then it is possible for the exposure from some pick-up nodes to increase even though the total exposure either decreases or remains unchanged.

As an example, consider Scenario 1 with a fleet size of $V = 12$ buses. Since the largest F-value satisfying $V_{UB}(F) = 12$ is 6 and the smallest F-value satisfying $V_{LB}(F) = 12$ is 19, it follows that $F^*_{UB}(12) = 6$ and $F^*_{LB}(12) = 19$. Consequently, for the given fleet size of 12 buses, the $MES(6)$ is feasible, and F should be set no smaller than 6 (for a total exposure of 232.0). In addition, since the $MES(20)$ is infeasible, setting F to any values larger than

or equal to 20 may result in non-equitable reductions in exposure across pick-up locations (assuming any is realizable with 12 vehicles). Since the $MES(19)$ total exposure equals 129.4, since the greatest lower bound on exposure for Scenario 1 equals $\sum_{j\in P} D_j t_{0j}^2/2 = 90.0$, and giving the diminishing returns in exposure from increased maximum service level, any improvements in exposure past $F = 19$ are likely to be small. The exact F'-value where the $MES(F')$ is feasible, but the $MES(F'+1)$ is infeasible, unfortunately, can only be identified by systematically searching for the largest feasible $MES(F)$ solution among $F = 7, \cdots, 19$. Next, an upper bound on the fleet size needed for a feasible solution to the BEP-CA is presented.

2.3.2.5 Upper Bound on the BEP-CA Fleet Size.

Proposition 2.7 presents an analytical upper bound on the fleet size needed for a feasible solution to the BEP-CA.

Proposition 2.7 *The upper bound on the fleet size needed for a feasible solution to the BEP-CA, $V_{UB}^*(F)$, is calculated as follows:*

$$V_{UB}^*(F) = \min_{f=1,\cdots,F}\{V_{UB}(f)\}. \tag{2.24}$$

Proof. See Appendix. ∎

As aforementioned, while (2.18) is a tight upper bound on the fleet size needed for a feasible implementation of the $MES(F)$ in instances where vehicles are dedicated to a single pick-up

node, it is not a tight estimate on the fleet size needed for a feasible solution to the BEP-CA. Since (2.24) is an extension of (2.18), it follows that (2.24) is also not tight, as illustrated in Example 2.6.

Example 2.6 *Recall the network presented in Example 2.4, Figure 2.7(a) with parameter* $C = 20$ *evacuees/bus. When* $F = 1$, $\tau_1(1) = \lambda_1(1) = 100$ *and* $V^*_{UB}(1) = V_{UB}(1) = 3$, *the exact fleet size needed for a feasible implementation of the* $MES(1)$, *as depicted in Figure 2.8(a) (with* $p^1_1 = L_1 = 100$*), and, thus, for a feasible solution to the BEP-CA. When* $F = 2$, $\tau_1(2) = \lambda_1(2) = 50$ *and* $V^*_{UB}(2) = V_{UB}(2) = 2$, *once again the exact fleet size needed for a feasible implementation of the* $MES(2)$, *as depicted in Figure 2.7(b) (with* $p^1_1 = 50$ *and* $p^2_1 = L_1 = 100$*), and, thus, for a feasible solution to the BEP-CA. However, when* $F = 3$, $\tau_1(3) = \lambda_1(3) = 33.3$ *and* $V^*_{UB}(3) = V_{UB}(3) = 2$, *but, as depicted in Figure 2.11, only one vehicle suffices for a feasible solution to the problem (with* $p^1_1 = 20$, $p^2_1 = 60$, *and* $p^3_1 = L_1 = 100$*).*

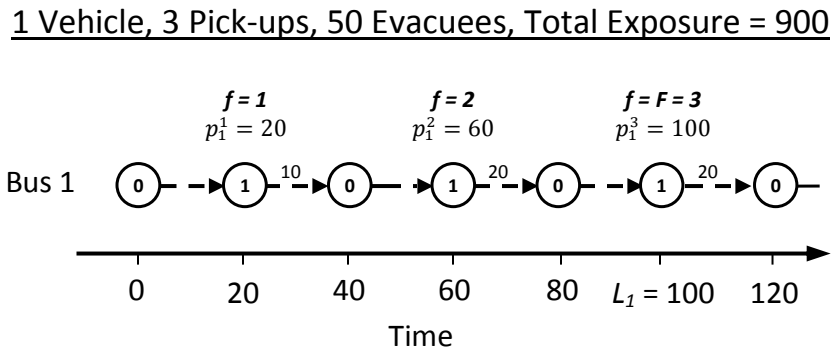


Figure 2.11: Representations of the optimal pick-up schedule for $F = 3$, $V = 1$, corresponding to the network presented in Example 2.4, Figure 2.7(a).

Corollary 2.3, based on the analytical upper bound introduced in Proposition 2.7, presents a sufficient condition for the feasibility of the BEP-CA.

Corollary 2.3 *Provided the necessary condition for feasibility presented in Lemma 2.2, a feasible solution to the BEP-CA can always be obtained with* $\sum_{j \in P} \lceil D_j L_j / C \rceil$ *or fewer vehicles.*

Proof. See Appendix. ∎

Next, a lower bound on the fleet size needed for a feasible solution to the problem is presented.

2.3.2.6 Lower Bound on the BEP-CA Fleet Size.

Archetti and Speranza (2012); Cordeau and Laporte (2006); Friedrich (2006); Letchford and Salazar-González (2006) and Stray (2010) propose $\lceil \sum_{j \in P} Q_j / C \rceil$, as an analytical lower bound on the fleet size needed for a feasible solution to the VRP with split deliveries, where Q_j denotes the total demand of pick-up node j (i.e., $Q_j = D_j L_j, \ \forall \ j \in P$). However, no literature suggesting a lower bound on the fleet size needed for a feasible solution to the VRP with split deliveries, service choice, and multiple tours per vehicle was located. Equation (2.25) presents one such bound:

$$V^*_{LB}(F) = \left\lceil \sum_{j \in P} \frac{D_j L_j}{C \lfloor T(F)/2 \rfloor} \right\rceil, \tag{2.25}$$

where $\lfloor T(F)/2 \rfloor$ is a proxy for the maximum number of times a vehicle can leave the depot. Consequently, this lower bound is similar to the one presented in the literature, except that

it also accounts for the possibility of multiple tours per vehicle. Corollary 2.4 presents a greatest lower bound on the fleet size needed for a feasible solution to the BEP-CA, that is, one that does not depend on the maximum service level.

Corollary 2.4 *A feasible solution to the BEP-CA (provided one exists) always demands*

$$\left\lceil \frac{\sum_{j\in P} D_j L_j}{C\lfloor(\theta+1)/2\rfloor} \right\rceil,$$

or more vehicles, where $\theta = \max_{j\in P}\{L_j\}/\min_{(i,j)\in A}\{t_{ij}\}$.

Proof. See Appendix. ∎

Figure 2.12 depicts the upper and lower bounds on the fleet size needed for a feasible solution to the BEP-CA and the fleet size obtained by (2.17) by maximum service level for Scenarios 1 and 2 (Figures 2.12(a) and 2.12(b), respectively).

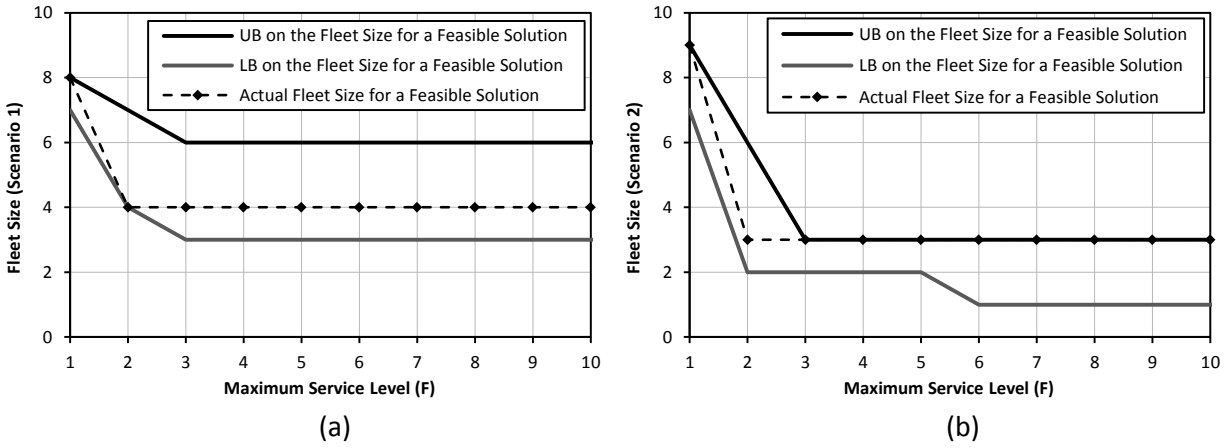


Figure 2.12: Upper bound, lower bound, and the exact fleet size needed for a feasible solution to the problem by the maximum service level for Scenarios 1 (a) and 2 (b).

Observe that $V^*_{UB}(F)$, $V^*_{LB}(F)$, and the exact fleet size needed for a feasible solution to the BEP-CA are monotonically decreasing with F. This behavior, noted in Remark 2.2, follows from the use of service choice (i.e., if a feasible solution exists, then additional pick-ups are only performed if they improve the objective function value):

Remark 2.2 *As the F-parameter increases, the number of vehicles required for a feasible solution for the BEP-CA monotonically decreases.*

Next, Figure 2.13 depicts the total exposure by fleet size and by maximum service level for Scenarios 1 and 2 (Figures 2.13(a) and 2.13(b), respectively).

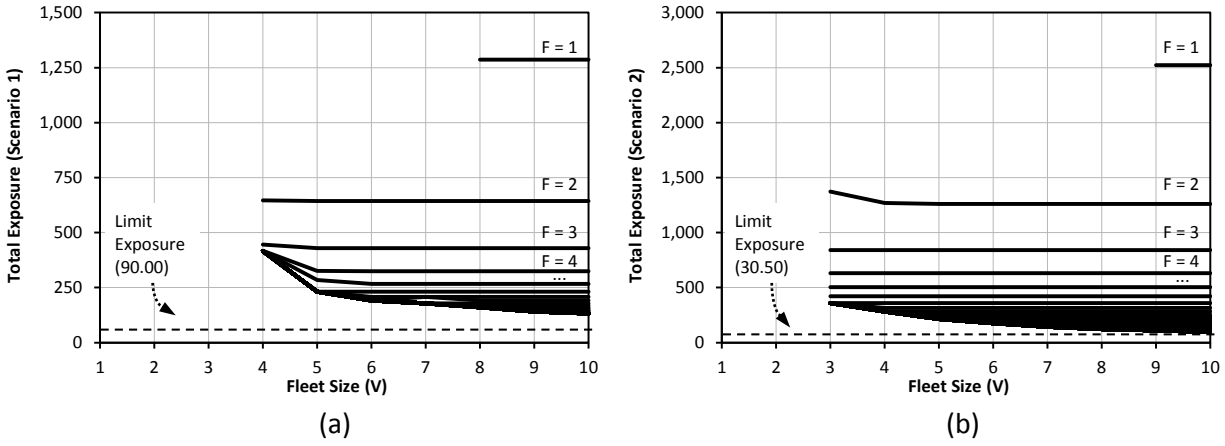


Figure 2.13: The minimum exposure by fleet size and by maximum service level for Scenarios 1 (a) and 2 (b) for $F = 1, \cdots, 4$. All remaining data is extrapolated.

Several important properties of the problem can be observed in this figure. First, since these scenarios satisfy the conditions presented in Lemma 2.2, a feasible solution can always be obtained by increasing the fleet size. For instance, Scenario 1 is always infeasible for $V = 1, \cdots, 3$ vehicles. However, if $F = 1$, then a feasible solution can be obtained with $V = 8$

vehicles, and if $F \geq 2$, then a feasible solution can be obtained with $V = 4$ vehicles. Similarly, Scenario 2 is always infeasible for $V = 1$ and $V = 2$ vehicles. However, if $F = 1$ then a feasible solution can be obtained with $V = 9$ vehicles, and if $F \geq 2$, then a feasible solution can be obtained with $V = 3$ vehicles. These results are also apparent from Figures 2.9 and 2.12. Second, the $MES(F)$ is feasible for a sufficiently large fleet size, but increasing the fleet size beyond the number of vehicles required for a feasible $MES(F)$ does not improve the objective function value (i.e., the level curves becomes flat for a sufficiently large fleet). Third, since the BEP-CA constraint set does not require all pick-ups to be performed, the contour lines depicted in Figure 2.13 are monotonically decreasing and never cross. Consequently, if a feasible solution exists, then the total exposure can be reduced by either increasing the maximum service level, the fleet size (although not beyond the fleet size needed for a feasible implementation of the $MES(F)$), or both. In every case, the objective function value will remain between the total exposure for the previous solution and the one corresponding to the $MES(F)$ for the new F-parameter. Moreover, note that as F increases, the contour lines approach each other, indicating the diminishing returns from increased F, and that the total exposure converges to the sum of accumulated exposures for the first pick-up at each pick-up node, $\sum_{j \in P}\{D_j t_{0j}^2/2\}$. Lastly, observe that while the fleet size needed for a feasible implementation of the $MES(F)$ first increased and later decreased with F, the one required for a feasible solution to the BEP-CA monotonically decreased. The next section presents and examines a formulation and an upper bound on the maximum number of arc traversals per vehicle for a given F-parameter, $T(F)$.

2.3.3 Upper Bound on the Maximum Number of Arc Traversals per Vehicle

While $T(F)$, the maximum number of arc traversals per vehicle for a given F-parameter, is not based on a policy decision, it may affect the model performance and results, as excessive values increase the complexity of the model, while insufficient values may either prevent its feasibility or yield suboptimal solutions. The following mixed-integer formulation identifies $T(F)$. The decision variables, objective function, and constraint set for this MIP are as follows:

Table 2.2: Decision variables for the $T(F)$ MIP.

Variable	**Description**
x_{ij}^t	binary variable that equals 1 if arc (i,j) is the t^{th} arc traversed by the vehicle, else 0, $\forall\ (i,j) \in A,\ t = 1, \cdots, T$, where T is a large positive integer (i.e., $T \gg T(F)$).
d^t	time when the vehicle finishes the t^{th} arc traversal. Let $d^0 = 0,\ \forall\ t = 0, \cdots, T$

$T(F)$ **Formulation:**

$$\text{Maximize}\ \ T(F) = \sum_{(i,j)\in A} \sum_{t=1}^{T} x_{ij}^t \tag{2.26}$$

subject to

$$\sum_{j:(0,j)\in A} x_{0j}^1 = 1 \tag{2.27}$$

$$\sum_{i:(i,j)\in A} x_{ij}^t = \sum_{k:(j,k)\in A} x_{jk}^{(t+1)}, \quad \forall\ j \in P,\ t = 1, \cdots, T-1 \tag{2.28}$$

$$\sum_{i:(i,0)\in A} x_{i0}^t \geq \sum_{j:(0,j)\in A} x_{0j}^{(t+1)}, \quad \forall\ t = 1, \cdots, T-1 \tag{2.29}$$

$$\sum_{i:(i,j)\in A} \sum_{t=1}^{T} x_{ij}^t \leq F, \quad \forall\ j \in P \tag{2.30}$$

$$d^t \geq d^{(t-1)} + \sum_{(i,j)\in A} t_{ij} x_{ij}^t, \quad \forall\ t = 1, \cdots, T \tag{2.31}$$

$$d^t \leq L_j + M(1 - \sum_{i:(i,j)\in A} x_{ij}^t), \quad \forall\ j \in P,\ t = 1, \cdots, T \tag{2.32}$$

$$d^t \geq L_j - M(F - \sum_{i:(i,j)\in A} \sum_{l=1}^{t} x_{ij}^l), \quad \forall\ j \in P,\ t = 1, \cdots, T. \tag{2.33}$$

Objective Function (2.26) maximizes the number of arc traversals that a vehicle can make. Constraints (2.27)-(2.29) ensure the construction of valid routes, including the elimination of subtours. Constraint (2.27) prevents the generation of simultaneous tours by forcing the vehicle's initial arc traversal to start from the depot. Constraint (2.28) forces the vehicle's $(t+1)^{th}$ arc traversal to start from node $j \in P$, if the vehicle's t^{th} arc traversal ended at node j. Constraint (2.29) is similar to (2.28), but it allows the vehicle to end service at the depot. Constraint (2.30) prevents the vehicle from visiting any pick-up node more than F times. Constraint (2.31) ensures that the travel time for the vehicle is appropriately incremented after each arc traversal. Constraints (2.32)-(2.33) ensure that the final visit to node j occurs at time $L_j,\ \forall\ j \in P$. Here, let $M = \max_{j\in P}\{L_j\}$. Finally, the binary restriction on the x-variables and the non-negativity restriction on the d-variables are included.

Since this formulation does not require the use of both binary and continuous variables to track the flow of multiple vehicles and evacuees through the network, it has much lower dimensionality than the BEP-CA, and, thus, is easier to solve. Nonetheless, the following greedy algorithm is shown to provide an upper bound on $T(F)$ in polynomial time:

1. Set $L = P$, $F^* = 0$, and $t^* = \min_{(i,j)\in A}\{t_{ij}\}$;
2. While $L \neq \{\emptyset\}$:
   (a) Set $j^* = \arg\min_{j\in L}\{L_j\}$;
   (b) Set $F^* = \min\{\lfloor L_{j^*}/2t^* + 1/2 \rfloor, F + F^*\}$;
   (c) Set $L = L/j^*$;
3. Set $T_{UB}(F) = 2F^*$,

where L denotes the set of unexplored pick-up nodes and F^* denotes the number of pick-ups performed on previously explored pick-up nodes. In summary, this procedure identifies the maximum number of arc traversals per vehicle possible such that neither the maximum service level nor the last pick-up times are violated. Since complete graphs have $|A| = |N|(|N|-1)/2$ arcs, the execution of this procedure is bounded by a polynomial of order $O(|N|^2)$, and, thus, is easier to solve than the corresponding MIP, which is combinatorial in nature. However, since this upper bound procedure assumes the smallest arc traversal time between nodes, depending on the problem instance, it can substantially overestimate the number of arc traversals per vehicle.

As an example of the algorithm, consider the network from Scenario 1, with $F = 3$.

1. Initialization: Set $L = \{1, 2, 3\}$, $F^* = 0$, and $t^* = 4$;
2. Iteration 1: Set $j^* = \{1\}$, $F^* = \min\{\lfloor L_1/8 + 1/2 \rfloor, 3 + 0\} = 2$, and $L = \{2, 3\}$;
3. Iteration 2: Set $j^* = \{3\}$, $F^* = \min\{\lfloor L_3/8 + 1/2 \rfloor, 3 + 2\} = 3$, and $L = \{2\}$;
4. Iteration 3: Set $j^* = \{2\}$, $F^* = \min\{\lfloor L_2/8 + 1/2 \rfloor, 3 + 3\} = 3$, and $L = \{\emptyset\}$;
5. Termination criteria met ($L = \{\emptyset\}$). Return $T_{UB}(F) = 6$ arc traversals.

Figure 2.14 depicts the results obtained from the MIP formulation, from the proposed upper bound procedure, and the minimum $T(F)$-parameter needed for a feasible implementation of the $MES(F)$ by the maximum service level for Scenarios 1 and 2 (Figures 2.14(a) and 2.14(b), respectively).

Observe that the minimum $T(F)$-parameter needed a feasible implementation of the $MES(F)$ oscillates as the F-parameter increases. Most likely, this non-monotonic behavior follows from the concurrent increase in fleet size, as every increment in the V-parameter lessens the number of arc traversals that each vehicle needs to perform in order to accommodate all pick-ups. Conversely, the maximum number of arc traversals per vehicle obtained from the MIP formulation presented in Section 2.2 and from the proposed upper bound procedure increase monotonically with F, thus leading to Remarks 2.3 and 2.4:

Remark 2.3 *The minimum $T(F)$-parameter needed for a feasible implementation of the $MES(F)$ may not increase monotonically with F.*

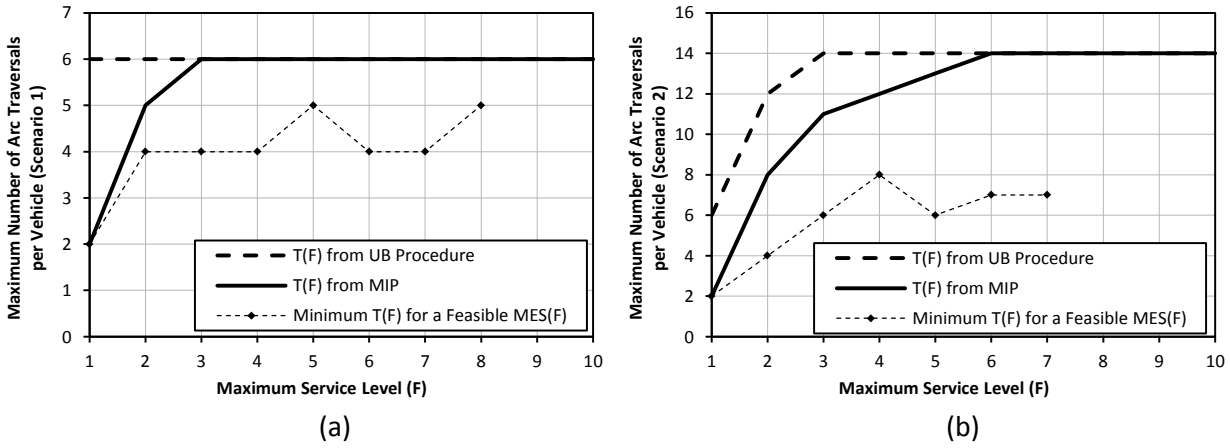


Figure 2.14: $T(F)$, as obtained from the MIP formulation, from the upper bound procedure, and the minimum value needed for a feasible implementation of the $MES(F)$ by maximum service level for Scenarios 1 (a) and 2 (b).

Remark 2.4 *The $T(F)$-parameter obtained from the MIP formulation presented in Section 2.2 and the upper bound procedure presented in this section increase monotonically with F.*

The next section summarizes the findings of this paper and lists directions for future research.

2.4 Conclusions and Further Research

This paper introduced the Bus Evacuation Problem with Constant Evacuee Arrival Rate (BEP-CA), a variant of the vehicle routing problem adapted for bus-based, regional evacuation planning. Several important properties of the model were identified. In particular, the effect of maximum service level (i.e., the maximum number of pick-ups allowed on each pick-up location) and fleet size over the problem feasibility and the total exposure objective

function value was examined. It was shown that, under a necessary condition for feasibility, if a feasible solution does not exist, then one can be obtained by increasing the fleet size; that if one exists, then the exposure can be reduced by either increasing the fleet size or the maximum service level; and that, after the $MES(F)$ is attained, the exposure can only be reduced by further increasing the maximum service level, but with diminishing returns. In addition, upper and lower bounds on the fleet size required for feasibility and for such bound on exposure, and an exact formulation and an upper bound on the maximum number of arc traversals per vehicle were proposed. More importantly, it was shown that, depending on the problem instance, it is possible to reduce both the fleet size requirement and the total exposure objective function, and that past a certain threshold, there is an optimal choice for the maximum service level (or a range of potential values, if the problem cannot be solved to optimality) that guarantees an efficient usage of the available fleet and equitable reductions in exposure across pick-up locations. Despite these advances, the literature focusing on bus-based evacuation is scarce and considerable research is warranted. First, while the assumption of a constant arrival rate presents an important distinction from the current literature on transit-based evacuation, it is still unrealistically portraying evacuee arrival behavior, as the true arrival process is probabilistic and will likely vary over time. Moreover, obtaining an optimal solution is computationally expensive. Therefore, before the BEP-CA can be implemented in networks of practical size, alternative solution methodologies, such as problem decomposition and use of metaheuristics, should be studied.

2.5 Appendix

This section presents the mathematical proofs of the propositions introduced in Sections 2.2 and 2.3.

Proof of Lemma 2.1. Consider the kernel of (2.1), $g_j : \mathbb{R}^F \mapsto \mathbb{R}$, where $g_j(p_j) = \sum_{f=1}^{F}(p_j^f - p_j^{f-1})^2$. Since g_j is quadratic, it is twice differentiable. Consequently, the Hessian matrix, the $F \times F$ matrix of second-order partial derivatives of g_j, is given by:

$$H(g_j) = \begin{bmatrix} \frac{\partial^2 g_j}{\partial^2 p_j^1} & \frac{\partial^2 g_j}{\partial p_j^1 \partial p_j^2} & \cdots & \frac{\partial^2 g_j}{\partial p_j^1 \partial p_j^F} \\ \frac{\partial^2 g_j}{\partial p_j^1 \partial p_j^2} & \frac{\partial^2 g_j}{\partial^2 p_j^2} & \cdots & \frac{\partial^2 g_j}{\partial p_j^2 \partial p_j^F} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g_j}{\partial p_j^1 \partial p_j^F} & \frac{\partial^2 g_j}{\partial p_j^2 \partial p_j^F} & \cdots & \frac{\partial^2 g_j}{\partial^2 p_j^F} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 & \ldots & 0 & 0 & 0 \\ -2 & 4 & -2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -2 & 4 & -2 \\ 0 & 0 & 0 & \ldots & 0 & -2 & 2 \end{bmatrix}.$$

Note that all elements of the matrix diagonal are strictly positive. Using the superdiagonalization algorithm (see Bazaraa et al., 2006, p. 120-122) inductively, it can be shown that $H(g_j)$ is positive definite, and, consequently, that g_j is strictly convex. Since (2.1) is the non-negatively weighted sum of strictly convex functions, it is also strictly convex. ∎

Proof of Lemma 2.2. The pick-up schedule $0 = p_j^0 < t_{0j} \leq p_j^1 \leq \cdots \leq p_j^F = L_j,\ \forall\ j \in P$ is a necessary condition for feasibility, as it satisfies: (i) the p-variable restrictions, that is, $p_j^0 = 0,\ p_j^F = L_j$, and $p_j^f \geq 0,\ \forall\ j \in P,\ f = 1, \cdots, F$; (ii) the sequential pick-up ordering restriction implied by (2.12), that is, $p_j^{f-1} \leq p_j^f,\ \forall\ j \in P,\ f = 1, \cdots, F$; and (iii) the first pick-up time restriction implied by (2.7)-(2.9) and the triangle inequality assumption, that

is $t_{0j} \leq p_j^1, \ \forall \ j \in P$. ∎

Proof of Proposition 2.1. Since (2.14)-(2.16) are linear and since (2.13) is the non-negatively weighted sum of strictly convex functions (as shown in the proof of Lemma 2.1), it follows that the corresponding formulation is separable into one subproblem per pick-up node. Therefore, it suffices to show that the proposed solution satisfies all KKT conditions for any of its subproblems, that is, primal feasibility, dual feasibility, and complementary slackness (see Bazaraa et al., 2006, p. 207-211). For any subproblem $j \in P$, if $0 < t_{0j} \leq L_j/F$ or if $F = 1$, then let $q_j^f = fL_j/F - (f-1)L_j/F \Rightarrow q_j^f = L_j/F, \ \forall \ f = 1, \cdots, F$, and if $L_j/F < t_{0j} \leq L_j$ and $F \geq 2$, then let $q_j^1 = t_{0j}$ and $q_j^f = [t_{0j} + (f-1)(L_j - t_{0j})/(F-1)] - [t_{0j} + (f-2)(L_j - t_{0j})/(F-1)] \Rightarrow q_j^f = (L_j - t_{0j})/(F-1), \ \forall \ f = 2, \cdots, F$. Observe that these values for q_j^f satisfy (2.14)-(2.16). Let δ_1, δ_2, and δ_3^f be the Lagrangian multipliers (dual variables) corresponding to (2.14)-(2.16), respectively, such that the dual feasibility and complementary slackness conditions can be expressed as:

$$\frac{\partial}{\partial q_j^f}\left[\sum_{f=1}^{F} \frac{1}{2} D_j {q_j^f}^2 + \delta_1 \left(\sum_{f=1}^{F} q_j^f - L_j\right) + \delta_2 \left(t_{0j} - q_j^1\right) - \delta_3^f q_j^f\right] = 0, \quad \forall \ f = 1, \cdots, F \tag{2.34}$$

$$\delta_2 \left(t_{0j} - q_j^1\right) = 0 \tag{2.35}$$

$$\delta_3^f q_j^f = 0, \ \forall \ f = 1, \cdots, F \tag{2.36}$$

$$\delta_1 \text{ unrestricted}, \quad \delta_2 \geq 0, \quad \delta_3^f \geq 0, \ \forall \ f = 1, \cdots, F. \tag{2.37}$$

Equation (2.34) simplifies to $q_j^1 = (\delta_2 + \delta_3^1 - \delta_1)/D_j$ and $q_j^f = (\delta_3^f - \delta_1)/D_j, \ \forall \ f = 2, \cdots, F$. Clearly, for $q_j^f = L_j/F, \ \forall \ f = 1, \cdots, F$, (2.34)-(2.37) are satisfied for $\delta_1 = -D_j L_j/F, \ \delta_2 = 0$, and $\delta_3^f = 0, \ \forall \ f = 1, \cdots, F$. Conversely, for $q_j^1 = t_{0j}$ and $q_j^f = (L_j - t_{0j})/(F-1), \ \forall \ f =$

$2, \cdots, F$, (2.34)-(2.37) are satisfied for $\delta_1 = -D_j(L_j - t_{0j})/(F-1)$, $\delta_2 = D_j(Ft_{0j} - L_j)/(F-1)$, and $\delta_3^f = 0, \ \forall \ f = 1, \cdots, F$. ∎

Proof of Corollary 2.1. For an arbitrary pick-up node j, suppose that $0 < t_{0j} \leq L_j/(F+1)$ or that $F = 1$, then by Proposition 2.1 the $MES(F)$ for pick-up node j has $p_j^f = fL_j/F, \ \forall \ f = 1, \cdots, F$, resulting in an exposure of:

$$\sum_{f=1}^{F} \frac{1}{2} D_j \left(p_j^f - p_j^{f-1} \right)^2 = \sum_{f=1}^{F} \frac{1}{2} D_j \left(f\frac{L_j}{F} - (f-1)\frac{L_j}{F} \right)^2 = \frac{D_j L_j^2}{2F}.$$

Therefore, increasing F by one results in a change in exposure of $D_j L_j^2/2(F+1) - D_j L_j^2/2F = -D_j L_j^2/2F(F+1)$. Similarly, if $L_j/F < t_{0j} \leq L_j$ and $F \geq 2$, then by Proposition 2.1, the $MES(F)$ for pick-up node j has $p_j^1 = t_{0j}$ and $p_j^f = t_{0j} + (f-1)(L_j - t_{0j})/(F-1), \ \forall \ f = 2, \cdots, F$, resulting in an exposure of:

$$\begin{aligned} \sum_{f=1}^{F} \frac{1}{2} D_j \left(p_j^f - p_j^{f-1} \right)^2 &= \frac{D_j}{2} \left(t_{0j}^2 + \sum_{f=2}^{F} \left(\frac{(f-1)(L_j - t_{0j})}{F-1} - \frac{(f-2)(L_j - t_{0j})}{F-1} \right)^2 \right) \\ &= \frac{D_j t_{0j}^2}{2} + \frac{D_j (L_j - t_{0j})^2}{2(F-1)}. \end{aligned}$$

Therefore, increasing F by one results in a change in exposure of $(D_j t_{0j}^2/2 + D_j(L_j - t_{0j})^2/2F) - (D_j t_{0j}^2/2 + D_j(L_j - t_{0j})^2/2(F-1)) = -D_j(L_j - t_{0j})^2/2F(F-1)$. Since, in both cases, the exposure for pick-up node j under the $MES(F)$ is quadratic-decreasing with F, it follows, inductively, that the $MES(F)$ total exposure is also quadratic-decreasing with the maximum service level. ∎

Proof of Corollary 2.2. On every node $j \in P$, as F increases, L_j/F decreases, and eventually becomes smaller than t_{0j}. Afterward, the $MES(F)$ is obtained when $p_j^1 = t_{0j}$

and $p_j^f = t_{0j} + (f-1)(L_j - t_{0j})/(F-1),\ \forall\ j \in P,\ f = 2, \cdots, F$. Under this schedule, (2.1) yields:

$$\begin{aligned}\sum_{j\in P}\sum_{f=1}^{F}\frac{1}{2}D_j(p_j^f - p_j^{f-1})^2 &= \sum_{j\in P}\frac{D_j}{2}\left(t_{0j}^2 + \sum_{f=2}^{F}\left(\frac{(f-1)(L_j-t_{0j})}{F-1} - \frac{(f-2)(L_j-t_{0j})}{F-1}\right)^2\right)\\ &= \sum_{j\in P}\frac{D_j}{2}\left(t_{0j}^2 + \frac{(L_j-t_{0j})^2}{F-1}\right)\\ &\mapsto \sum_{j\in P}\frac{D_j t_{0j}^2}{2},\ \text{as } F \mapsto \infty.\ \blacksquare\end{aligned}$$

Proof of Proposition 2.2. Here, it is assumed that vehicles are dedicated to a single pick-up node, their least efficient use, as potential savings ensuing from the inclusion of multiple pick-up nodes in each tour are ignored. Consider the two possible cases pertaining an arbitrary pick-up node j:

Case 1: $0 < t_{0j} \leq L_j/F$ **or** $F = 1$**.** By Proposition 2.1, the $MES(F)$ for pick-up node j has $p_j^f = fL_j/F,\ \forall\ f = 1, \cdots, F$. Therefore, $\tau_j(F) = \lambda_j(F) = L_j/F$, and each pick-up requires $\lceil D_j L_j/FC \rceil$ buses. In this case, either one or two groups of dedicated buses are required to make all scheduled pick-ups, that is, either $\lceil D_j L_j/FC \rceil$ or $2\lceil D_j L_j/FC \rceil$ buses are needed. When $0 < t_{0j} \leq L_j/2F$ or $F = 1$, only one group of buses is needed, as either a single pick-up needs to be performed or the time between scheduled pick-ups is sufficiently long for buses to return to the depot between pick-ups (i.e., $2t_{0j} \leq L_j/F$). Conversely, when $L_j/2F < t_{0j} \leq L_j/F$ and $F \geq 2$, each group of buses can only make every other scheduled pick-up, and two groups are needed.

Case 2: $L_j/F < t_{0j} \leq L_j$ **and** $F \geq 2$**.** By Proposition 2.1, the $MES(F)$ for pick-up node

j has $p_j^1 = t_{0j}$ and $p_j^f = t_{0j} + (f-1)(L_j - t_{0j})/(F-1),\ \forall\ f = 2, \cdots, F$. Therefore, $\tau_j(F) = (L_j - t_{0j})/(F-1),\ \lambda_j(F) = t_{0j}$, the first pick-up requires $\lceil D_j t_{0j}/C \rceil$ buses, and each subsequent pick-up requires $\lceil D_j(L_j - t_{0j})/(F-1)C \rceil$ buses. Under this pick-up schedule, vehicles also do not have time to return to the depot between pick-ups, and multiple groups of buses are needed. Specifically, this pick-up schedule requires a total of $\lceil D_j t_{0j}/C \rceil + (\min\{\lceil 2t_{0j}(F-1)/(L_j - t_{0j}) \rceil, F\} - 1)\lceil D_j(L_j - t_{0j})/(F-1)C \rceil$ vehicles, where $\min\{\lceil 2t_{0j}(F-1)/(L_j - t_{0j}) \rceil, F\}$ is the total number of groups of buses needed, namely the minimum between the ceiling of the round-trip time to node j divided by the interval between scheduled pick-up times, and F, an upper bound on the number of groups assuming that each group visits node j at least once.

In both cases, (2.18) corresponds to the exact number of vehicles required for a feasible implementation of the $MES(F)$ on pick-up node j, and, inductively, for the $MES(F)$. ∎

Proof of Proposition 2.3. First, note that $T(F) - 1$ is as an upper bound on the number of pick-ups per vehicle, assuming that each vehicle performs a single tour with a single returning arc to the depot. Second, $F|P|$ is the total number of distinct scheduled pick-up times under the $MES(F)$. Consequently, $V_{LB}(F)$ corresponds to the product of two factors: a lower bound on the number of groups of buses simultaneously performing pick-ups under the $MES(F)$ and a lower bound on the number of buses needed in each group in order to pick-up all evacuees that arrived between scheduled pick-up times. ∎

Proof of Proposition 2.4. Let $F \geq F_{UB}$. Since $F \geq L_j/t_{0j},\ \forall\ j \in P$, it follows that for

every pick-up node j:

$$t_{0j} \geq \frac{L_j}{F} \;\Rightarrow\; \lambda_j(F) = \max\left\{\frac{L_j}{F}, t_{0j}\right\} = t_{0j}$$

$$\frac{L_j - t_{0j}}{F-1} \leq \frac{L_j}{F} \;\Rightarrow\; \tau_j(F) = \min\left\{\frac{L_j}{F}, \frac{L_j - t_{0j}}{F-1}\right\} = \frac{L_j - t_{0j}}{F-1}.$$

Consequently,

$$V_{UB}(F) = \sum_{j\in P}\left(\left\lceil\frac{D_j t_{0j}}{C}\right\rceil + \left(\min\left\{\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil, F\right\} - 1\right)\left\lceil\frac{D_j(L_j - t_{0j})}{C(F-1)}\right\rceil\right).$$

In addition, since $F \geq D_j(L_j - t_{0j})/C + 1,\ \forall\ j \in P$, it follows that:

$$\left\lceil\frac{D_j(L_j - t_{0j})}{C(F-1)}\right\rceil = 1,\ \forall\, j \in P \Rightarrow V_{UB}(F) = \sum_{j\in P}\left(\left\lceil\frac{D_j t_{0j}}{C}\right\rceil + \min\left\{\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil, F\right\} - 1\right).$$

Moreover, for every pick-up node j where $0 < t_{0j} \leq L_j/3 \Rightarrow L_j \geq 3t_{0j}$, it follows that:

$$\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil \leq \frac{2t_{0j}(F-1)}{L_j - t_{0j}} + 1 \leq \frac{2t_{0j}(F-1)}{3t_{0j} - t_{0j}} + 1 = F$$

$$\Rightarrow \min\left\{\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil, F\right\} = \left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil,$$

And for every pick-up node j where $L_j/3 < t_{0j} \leq L_j \Rightarrow L_j < 3t_{0j}$, it follows that:

$$\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil \geq \frac{2t_{0j}(F-1)}{L_j - t_{0j}} > \frac{2t_{0j}(F-1)}{3t_{0j} - t_{0j}} = F - 1 \Rightarrow \min\left\{\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil, F\right\} = F.$$

For notational simplicity, let $\min\{\lceil 2t_{0j}(F-1)/(L_j - t_{0j})\rceil, F\} = F$, if $t_{0j} = L_j$, and let P^* denote the set of all pick-up nodes j such that $0 < t_{0j} \leq L_j/3$, therefore:

$$\Rightarrow\; V_{UB}(F) = \sum_{j\in P}\left\lceil\frac{D_j t_{0j}}{C}\right\rceil + \sum_{j\in P^*}\left\lceil\frac{2t_{0j}(F-1)}{L_j - t_{0j}}\right\rceil + F(|P| - |P^*|) - |P|.$$

Clearly, for any $F \geq F_{UB},\ V_{UB}(F) \leq V_{UB}(F+1)$, thus completing the proof. ∎

Before Proposition 2.5, lemma 2.4 shows that, for large F-values, $T(F)$ is constant and independent of F.

Lemma 2.4 *If* $F \geq \lfloor(\theta+1)/2\rfloor$*, then* $T(F) = \phi$*, a constant independent of* F*, such that* $\phi \leq 2\lfloor(\theta+1)/2\rfloor$ *and*

$$\theta = \frac{\max_{j\in P}\{L_j\}}{\min_{(i,j)\in A}\{t_{ij}\}}.$$

Proof. First, observe that $F \geq \lfloor(\theta+1)/2\rfloor \Rightarrow F \geq \lfloor(L_j + t^*)/2t^*\rfloor,\ \forall\ j \in P$, where $t^* = \min_{(i,j)\in A}\{t_{ij}\}$. Consequently, step 2(b) of the upper bound procedure on $T(F)$ reduces to $F^* = \lfloor(L_{j^*} + t^*)/2t^*\rfloor$. Since the algorithm assesses nodes from shortest to longest evacuation end-times, on the last iteration, $F^* = \lfloor(\max_{j\in P}\{L_j\} + t^*)/2t^*\rfloor = \lfloor(\theta+1)/2\rfloor$, and since $T_{UB}(F) = 2F^*$, it follows that $T(F) = \phi$, where ϕ is a constant independent of F and $\phi \leq 2\lfloor(\theta+1)/2\rfloor$. ∎

Proof of Proposition 2.5. Let $F \geq F_{LB}$. Since $F \geq L_j/t_{0j},\ \forall\ j \in P$, it follows that for every pick-up node j:

$$\frac{L_j - t_{0j}}{F-1} \leq \frac{L_j}{F} \ \Rightarrow\ \tau_j(F) = \min\left\{\frac{L_j}{F}, \frac{L_j - t_{0j}}{F-1}\right\} = \frac{L_j - t_{0j}}{F-1}$$
$$\Rightarrow V_{LB}(F) = \left\lceil \frac{F|P|}{T(F)-1}\right\rceil \min_{j\in P}\left\{\left\lceil \frac{D_j(L_j - t_{0j})}{C(F-1)}\right\rceil\right\}.$$

In addition, since $F \geq D_j(L_j - t_{0j})/C + 1,\ \forall\ j \in P$, it follows that:

$$\left\lceil \frac{D_j(L_j - t_{0j})}{C(F-1)}\right\rceil = 1,\ \forall\ j \in P\ \Rightarrow\ V_{LB}(F) = \left\lceil \frac{F|P|}{T(F)-1}\right\rceil.$$

Finally, since $F \geq \lfloor(\theta+1)/2\rfloor$, by Proposition 2.4, $T(F) = \phi$, where ϕ is a constant independent of F and $\phi \leq 2\lfloor(\theta+1)/2\rfloor$, consequently:

$$V_{LB}(F) = \left\lceil \frac{F|P|}{\phi-1}\right\rceil.$$

Clearly, for any $F \geq F_{LB},\ V_{LB}(F) \leq V_{LB}(F+1)$, thus completing the proof. ∎

Proof of Proposition 2.6. Here, it is shown that $\nexists\ M \in \mathbb{R}$ such that $V_{LB}(F) < M,\ \forall\ F \in \mathbb{N}$. Let $M \in \mathbb{R}$ be given, and let $F \geq F^*(M)$, where:

$$F^*(M) = \max\left\{F_{LB}, \frac{2M\lfloor(\theta+1)/2\rfloor}{|P|}\right\}.$$

Since $F \geq F_{LB}$, it follows, from the proof of Proposition 2.5, that:

$$V_{LB}(F) = \left\lceil \frac{F|P|}{\phi - 1} \right\rceil \Rightarrow V_{LB}(F) \geq \left\lceil \frac{F|P|}{2\lfloor(\theta+1)/2\rfloor} \right\rceil,\ \text{as } \phi \leq 2\lfloor(\theta+1)/2\rfloor.$$

In addition, since $F \geq 2M\lfloor(\theta+1)/2\rfloor/|P|$, it follows that $V_{LB}(F) \geq \lceil M \rceil \geq M$. Since for every arbitrary real number M, there exists a real number $F^*(M)$, such that $F \geq F^*(M)$ implies that $V_{LB}(F) \geq M$, and since, by Proposition 2.5, the sequence of lower bounds on the fleet size needed for a feasible implementation of the $MES(F)$ is monotonically increasing for $F \geq F_{LB}$, it follows that $V_{LB}(F) \mapsto \infty$, as $F \mapsto \infty$. ∎

Proof of Lemma 2.3. The relationship between F_{UB}, F_{LB}, $F^*_{UB}(V)$, and $F^*_{LB}(V)$ can be derived from their definitions. Since $V_{UB}(F) \geq V_{LB}(F),\ \forall\ F \in \mathbb{N}$, (2.22)-(2.23) imply that $F^*_{UB}(V) \leq F^*_{LB}(V)$, that $F_{UB} \leq F^*_{UB}(V)$, and that $F_{LB} \leq F^*_{LB}(V)$. Since the term $\lceil(\theta+1)/2\rceil \geq 0$ for $\theta = \max_{j\in P}\{L_j\}/\min_{(i,j)\in A}\{t_{ij}\}$, (2.20)-(2.21) imply that $F_{UB} \leq F_{LB}$. Altogether, either $F_{UB} \leq F_{LB} \leq F^*_{UB}(V) \leq F^*_{LB}(V)$ or $F_{UB} \leq F^*_{UB}(V) \leq F_{LB} \leq F^*_{LB}(V)$. ∎

Proof of Proposition 2.7. This bound corresponds to the smallest upper bound on the fleet size needed for a feasible implementation of the $MES(f)$ for $f = 1, \cdots, F$, and holds for two reasons. First, since (2.18) is an upper bound on the fleet size needed for a feasible

implementation of the $MES(F)$, it is also an upper bound on the fleet size needed for a feasible solution. Second, since the BEP-CA does not require all pick-ups to be performed, once the fleet size is sufficiently large for a feasible solution, increasing F will never result in a larger fleet size requirement. ∎

Proof of Corollary 2.3. First, observe that the upper bound on the fleet size needed for a feasible solution to the BEP-CA is monotonically decreasing with F, as:

$$\begin{aligned}\min_{f=1,\cdots,F}\{V_{UB}(F)\} &\geq \min\{\min_{f=1,\cdots,F}\{V_{UB}(F)\}, V_{UB}(F+1)\} \\ &= \min_{f=1,\cdots,F+1}\{V_{UB}(F)\}\end{aligned}$$

Consequently, the exact fleet size needed for a feasible solution to the BEP-CA is always smaller than or equal to $V_{UB}(1) = \sum_{j\in P}\lceil D_j L_j/C\rceil$. ∎

Proof of Corollary 2.4. Since $T(F)$ is monotonically increasing with F, the lower bound on the fleet size needed for a feasible solution to the BEP-CA, $V^*_{LB}(F)$, is monotonically decreasing with F. Moreover, by Proposition 2.4, for $F \geq \lfloor(\theta+1)/2\rfloor$, $T(F) \leq 2\lfloor(\theta+1)/2\rfloor$. Consequently, the exact fleet size needed for a feasible solution to the BEP-CA is always greater than or equal to:

$$\lceil\sum_{j\in P} D_j L_j/C\lfloor(\theta+1)/2\rfloor\rceil,$$

where $\theta = \max_{j\in P}\{L_j\}/\min_{(i,j)\in A}\{t_{ij}\}$. Hence completing the proof. ∎

Chapter 3

A Variant of the Ritter Relaxation Algorithm Applied to Problems with Mixed-Integer Subproblems and Linear or Quadratic Objective Functions

A Variant of the Ritter Relaxation Algorithm Applied to Problems with Mixed-Integer Subproblems and Linear or Quadratic Objective Functions

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(ABSTRACT)

This research introduces a variation of the Ritter (1967) Relaxation Algorithm able to solve bordered (or arrowhead) structure problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions without the need to compound multiple decomposition/relaxation strategies. Empirical studies demonstrate that the proposed algorithm is a viable alternative for solving difficult, bordered-structure, optimization problems.

Keywords: Mixed-Integer Quadratic Programming; Ritter Relaxation; Bordered Structure;

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3.1 Introduction

Bordered (or arrowhead) structure problems, that is, problems comprising both complicating variables and constraints, arise in a variety of settings, such as in the integrated aircraft routing and crew pairing problem (Sandhu and Klabjan, 2007) and in the non-linear portfolio

optimization problem (Gondzio and Grothey, 2007). Unfortunately, none of the commonly used decomposition/relaxation algorithms, such as Benders' Decomposition (Benders, 1962), Dantzig-Wolfe Decomposition (Dantzig and Wolfe, 1960), and Lagrangian Relaxation, is able to solve bordered-structure problems directly. This paper introduces an optimization algorithm based on a less known relaxation methodology originally developed by Ritter (1967). The algorithm explores the inherent structure of such problems in order to reduce its size (both in terms of decision variables and constraints), and, thus, the time needed to attain its optimum solution. The modifications proposed require fewer matrix manipulations and fewer numerical comparisons, thus leading to improved numerical stability. In addition, the modified algorithm is combined with the branch and bound algorithm, thus allowing problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions to be solved to optimality.

This paper is organized as follows: Section 3.2 describes the original Ritter (1967) Relaxation Algorithm and introduces some modifications that extend the algorithm to problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions; Section 3.3 presents and discusses the results of empirical studies; and, finally, Section 3.4 summarizes the findings of this paper and lists the future work needed on the area.

3.2 The Ritter (1967) Relaxation Algorithm

This section describes the original Ritter (1967) Relaxation Algorithm and introduces some modifications that extend the algorithm to problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions. Consider the following generic, linear programming problem:

$$\text{Minimize} \quad \sum_{j=1}^{p+1} c_j^t \, x_j \tag{3.1a}$$

$$\text{Subject to} \quad B_j \, x_j + D_j \, x_{p+1} \leq b_j, \ \forall \ j = 1, \cdots, p \tag{3.1b}$$

$$\sum_{j=1}^{p} A_j \, x_j + D_{p+1} \, x_{p+1} \leq b_{p+1} \tag{3.1c}$$

$$x_j \geq 0, \quad \forall \ j = 1, \cdots, p+1, \tag{3.1d}$$

Where $c_j, \ j = 1, \cdots, p+1$, are column vectors of n_j objective function coefficients; $B_j, \ j = 1, \cdots, p$, $A_j, \ j = 1, \cdots, p$, and $D_j, \ j = 1, \cdots, p+1$, are, respectively, $(m_j \times n_j)$, $(m_{p+1} \times n_j)$, and $(m_j \times n_{p+1})$ matrices of constraint coefficients; and $b_j, \ j = 1, \cdots, p+1$, are column vectors of m_j right hand side coefficients. Consequently, problem (3.1) has a bordered-structure with p subproblems, where (3.1c) is a set of complicating constraints and x_{p+1} is a set of complicating variables.

Throughout, the following notation will be used: $0_{(a \times b)}$ and $1_{(a \times b)}$ are $(a \times b)$ matrices of zeros and ones, respectively; e_i is an i^{th} unit column vector, that is, a column vector of zeros except for a one in the i^{th} position; and I_a are $(a \times a)$ identity matrices. In addition, the superscript t indicates the transpose of a matrix or vector and the power -1 indicates

the matrix inverse. Ritter (1967) proposes a relaxation algorithm for bordered-structure problems similar to (3.1). The algorithm assumes that the problem is in standard form. For simplicity, here, the slack variables are grouped with the existing problem variables as to maintain the notation used in (3.1) unchanged. The augmented c_j, B_j, and A_j, $j = 1, \cdots, p$, and D_{p+1} matrices are as follows:

$$c_j \equiv \begin{bmatrix} c_j \\ \\ 0_{(m_j \times 1)} \end{bmatrix} \quad B_j \equiv \begin{bmatrix} B_j & I_{m_j} \end{bmatrix} \quad A_j \equiv \begin{bmatrix} A_j & 0_{(m_{p+1} \times m_j)} \end{bmatrix} \quad D_{p+1} \equiv \begin{bmatrix} D_{p+1} & I_{m_{p+1}} \end{bmatrix}.$$

Next, the original Ritter (1967) Relaxation Algorithm is described.

3.2.1 Original Algorithm

In the original Ritter (1967) Relaxation Algorithm, if the problem has the structure of (3.1), then, for a given starting solution x_{p+1}, each subproblem $\left\{\min c_j^t x_j | B_j x_j = b_j - D_j x_{p+1},\ x_j \geq 0\right\},\ j = 1, \cdots, p$, has an initial basis, namely the slack variables used to place the problem in standard form. In this case, the algorithm partitions the decision variables of subproblem j into basic (set S_{j1}) and non-basic (set S_{j2}), and groups the corresponding columns of B_j and A_j and the rows of c_j into the disjoint sets B_{j1} and B_{j2}, A_{j1} and A_{j2}, and c_{j1} and c_{j2}. Ritter (1967) suggests using the optimal basis of each subproblem as set S_{j1} and all non-basic variables as set S_{j2}, but such approach does not guarantee a quicker convergence to the optimum solution and is problematic whenever the subproblem is infeasible or unbounded for the given x_{p+1} starting solution. In addition, an initial basis may not be readily available for subproblems containing equality constraints. In those cases,

Ritter (1967) proposes further augmenting each subproblem with $k_j = m_j - rank(B_j)$ artificial variables such that every $B_j,\ j = 1, \cdots, p$, matrix has full rank. Under this scheme, the D_j and $b_j,\ j = 1, \cdots, p+1$, matrices remain unchanged and the c_j, B_j, and $A_j,\ j = 1, \cdots, p$ matrices are augmented as follows:

$$c_j \equiv \begin{bmatrix} c_j \\ \\ M \cdot 1_{(k_j \times 1)} \end{bmatrix} \quad B_j \equiv \begin{bmatrix} B_j & e_1 & \cdots & e_{k_j} \end{bmatrix} \quad A_j \equiv \begin{bmatrix} A_j & 0_{(m_{p+1} \times k_j)} \end{bmatrix},$$

Where M is a sufficient large, positive constant aiming to penalize solutions that include artificial variables. Clearly, such scheme is similar to the Big-M Method. Nonetheless, as shown in Example 3.1, such approach may be problematic depending on the problem instance.

Example 3.1 *Let B_j, for some $j = 1, \cdots, p$, be as follows:*

$$B_j = \begin{bmatrix} 0.5 & 0.7 \\ \\ 0 & 0 \end{bmatrix}.$$

Observe that $rank(B_j) = 1$. Consequently, $k_j = m_j - rank(B_j) = 2 - 1 = 1$, and the augmented matrix, say B'_j, is as follows:

$$B'_j = \begin{bmatrix} B_j & e_1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.7 & 1 \\ \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(B'_j) = 1.$$

Naturally, an obvious alternative is to iteratively recompute the rank of B_j, adding artificial variables until the matrix has full rank. In Example 3.1, this approach would demand two

artificial variables:

$$B'_j = \begin{bmatrix} B_j & e_1 & e_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow rank(B'_j) = 2.$$

However, such approach may require the inclusion of more artificial variables than needed. In particular, the matrix presented in Example 3.1 requires a single artificial variable, namely:

$$B'_j = \begin{bmatrix} B_j & e_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow rank(B'_j) = 2.$$

Aiming to resolve such issues, the following algorithm simultaneously augments B_j, A_j and c_j and allocates the problem decision variables into sets S_{j1} or S_{j2}.

Algorithm 3.1 *A methodology to augment B_j, A_j and c_j, and to partition the problem decision variables into sets S_{j1} and S_{j2}.*

Requires: *The coefficients m_j and n_j; a $(m_j \times n_j)$ matrix B_j, a $(m_{p+1} \times n_j)$ matrix A_j, and a $(n_j \times 1)$ vector c_j.*

1 $B'_j \leftarrow rref(B_j)$

2 $row \leftarrow 1$

3 **for** $column \in \{1, \cdots, n_j\}$ **do**

4 **if** $row \leq m_j$ **then**

5 **if** $B'_j(row, column) = 1$ **then**

6 $row \leftarrow row + 1$

7 $S_{j1}.pushback(column)$

8 **else**

```
9            S_{j2}.pushback(column)
10         end if
11      else
12         S_{j2}.pushback(column)
13      end if
14   end for
15   if row = m_j + 1 and m_j = n_j then
16      k_j ← 1
17      B_j.addunitcolumn(k_j)
18      A_j.addzerocolumn()
19      c_j.pushback(M)
20      S_{j2}.pushback(n_j + k_j)
21   else
22      k_j ← 0
23      while row + k_j ≤ m_j do
24         k_j ← k_j + 1
25         B_j.addunitcolumn(n_j + k_j)
26         A_j.addzerocolumn()
27         c_j.pushback(M)
28         S_{j1}.pushback(n_j + k_j)
29      end while
```

30 **end if**

Returns: *A* $(m_j \times n_j + k_j)$*, full rank matrix* B_j*, a* $(m_{p+1} \times n_j + k_j)$ *matrix* A_j*, a* $(n_j + k_j \times 1)$ *vector* c_j*, a* $(m_j \times 1)$ *vector* S_{j1}*, and an* $(n_j - m_j + k_j \times 1)$ *vector* S_{j2}*.*

Row 1 creates a temporary matrix B'_j to store matrix B_j in reduced echelon form; rows 2-14 partition the subproblem decision variables into sets S_{j1} (basic variables) and S_{j2} (non-basic variables); and rows 15-30 augment B_j, A_j, and c_j with artificial variables. Specifically, rows 16-20 ensure that at least one variable is added to set S_{j2} (even if such is artificial) and rows 22-29 ensure that the B_j matrix has full rank by adding artificial variables to set S_{j1}. Observe that Algorithm 3.1 is polynomial in nature, as the Gaussian elimination algorithm, used to generate matrix B'_j in row 1, has worst-case computational complexity of $O(m_j^3)$.

Once the problem is placed in standard form and the partitioned sets B_{j1}, B_{j2}, A_{j1}, A_{j2}, c_{j1}, and c_{j2}, $j = 1, \cdots, p$, are available, (3.1) can be rewritten as:

$$\text{Minimize} \quad \sum_{j=1}^{p} c_{j1}^t \, x_{j1} + \sum_{j=1}^{p} c_{j2}^t \, x_{j2} + c_{p+1}^t \, x_{p+1} \tag{3.2a}$$

$$\text{Subject to} \quad B_{j1} \, x_{j1} + B_{j2} \, x_{j2} + D_j \, x_{p+1} = b_j, \ \forall \ j = 1, \cdots, p \tag{3.2b}$$

$$\sum_{j=1}^{p} A_{j1} \, x_{j1} + \sum_{j=1}^{p} A_{j2} \, x_{j2} + D_{p+1} \, x_{p+1} = b_{p+1} \tag{3.2c}$$

$$x_{j1}, \ x_{j2} \geq 0, \quad \forall \ j = 1, \cdots, p, \ x_{p+1} \geq 0. \tag{3.2d}$$

Since B_{j1} is a full rank, square matrix it must be non-singular. Consequently, by (3.2b), the x_{j1} decision variables can be computed as follows:

$$x_{j1} = B_{j1}^{-1} b_j - B_{j1}^{-1} B_{j2} x_{j2} - B_{j1}^{-1} D_j x_{p+1}, \ \forall \ j = 1, \cdots, p, \tag{3.3}$$

And (3.2), after some rearrangement of the terms, becomes:

$$\text{Minimize} \quad \sum_{j=1}^{p} \left(c_{j2}^t - c_{j1}^t B_{j1}^{-1} B_{j2}\right) x_{j2} + \left(c_{p+1}^t - \sum_{j=1}^{p} c_{j1}^t B_{j1}^{-1} D_j\right) x_{p+1} + \sum_{j=1}^{p} c_{j1}^t B_{j1}^{-1} b_j \tag{3.4a}$$

$$\text{Subject to} \quad \sum_{j=1}^{p} \left(A_{j2} - A_{j1} B_{j1}^{-1} B_{j2}\right) x_{j2} + \left(D_{p+1} - \sum_{j=1}^{p} A_{j1} B_{j1}^{-1} D_j\right) x_{p+1} = b_{p+1} - \sum_{j=1}^{p} A_{j1} B_{j1}^{-1} b_j \tag{3.4b}$$

$$x_{j2} \geq 0, \quad \forall\ j = 1, \cdots, p,\ x_{p+1} \geq 0. \tag{3.4c}$$

Observe that (3.4) does not encompass the non-negativity restriction on the $x_{j1},\ j = 1, \cdots, p$, decision variables. Now, for simplicity, let the following parameter substitutions be made:

$$d_j^t = c_{j2}^t - c_{j1}^t B_{j1}^{-1} B_{j2},\ \forall\ j = 1, \cdots, p \tag{3.5a}$$

$$d_{p+1}^t = c_{p+1}^t - \sum_{j=1}^{p} c_{j1}^t B_{j1}^{-1} D_j \tag{3.5b}$$

$$\alpha = \sum_{j=1}^{p} c_{j1}^t B_{j1}^{-1} b_j \tag{3.5c}$$

$$F_j = A_{j2} - A_{j1} B_{j1}^{-1} B_{j2},\ \forall\ j = 1, \cdots, p \tag{3.5d}$$

$$F_{p+1} = D_{p+1} - \sum_{j=1}^{p} A_{j1} B_{j1}^{-1} D_j \tag{3.5e}$$

$$f = b_{p+1} - \sum_{j=1}^{p} A_{j1} B_{j1}^{-1} b_j. \tag{3.5f}$$

Consequently, (3.4) can be written as the Reduced Problem:

$$\text{Minimize} \quad \sum_{j=1}^{p} d_j^t\, x_{j2} + d_{p+1}^t\, x_{p+1} + \alpha \tag{3.6a}$$

$$\text{Subject to} \quad \sum_{j=1}^{p} F_j\, x_{j2} + F_{p+1}\, x_{p+1} = f \tag{3.6b}$$

$$x_{j2} \geq 0, \quad \forall\ j = 1, \cdots, p,\ x_{p+1} \geq 0. \tag{3.6c}$$

Since x_{j2} and x_{p+1} can be used to compute the corresponding value for the x_{j1} decision variables via (3.3), it follows that the feasible region of the Reduced Problem encompasses the feasible region of the original problem. This, property, in fact, is formally proven in the next Section. Observe that the Reduced Problem is a relaxation of the original problem, as the non-negativity restriction on the $x_{j1},\ j = 1, \cdots, p$, decision variables are not enforced. In addition, some instances of (3.6) may be unbounded even if the optimal solution for (3.1) is finite. In order to avoid the ensuing computational problems, Ritter (1967) suggests including the constraint $\sum_{j=1}^{p} 1_{(1\times|S_{j2}|)} x_{j2} + 1_{(1\times n_{p+1})} x_{p+1} \leq M$ in the Reduced Problem. Note that, by letting x_{j2}^i and x_{p+1}^i denote, respectively, the i^{th} element of x_{j2} and x_{p+1}, such constraint can be rewritten as:

$$\sum_{j=1}^{p} \sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M. \tag{3.7}$$

After the Reduced Problem, along with (3.7), is generated and solved, its optimum solution, $\{x_{j2}^*,\ j = 1, \cdots, p,\ x_{p+1}^*\}$, is used to find the corresponding solution for the relaxed variables, $\{x_{j1}^*,\ j = 1, \cdots, p\}$, through Equation (3.3). If $x_{j1}^i \geq 0,\ \forall\ j = 1, \cdots, p,\ i = 1, \cdots, |S_{j1}|$, that is, if the current solution is feasible to the original problem, then the algorithm terminates with optimal solution $\{x_{j1}^*, x_{j2}^*,\ j = 1, \cdots, p,\ x_{p+1}^*\}$ stripped of any previously added non-basic artificial variables. As in the Big-M method, if the optimum solution has positive (basic) artificial variables, then the original problem is infeasible. Conversely, if there exists a subproblem j with a negative decision variable, say x_{j1}^i, then such can either be swapped

with a positive decision variable in set S_{j2}, say x_{j2}^k, or its non-negativity restriction can be explicitly enforced in the Reduced Problem through an additional constraint. Consider the following cases:

Case 1: Swap Elements. Suppose that, after the Reduced Problem is generated and solved and the corresponding solution for the $x_{j1},\ j = 1, \cdots, p$, decision variables are computed, the i^{th} element of set S_{j1} is negative and the k^{th} element of set S_{j2} is positive, that is, suppose that $x_{j1}^i < 0$ and $x_{j2}^k > 0$ for some subproblem $j = 1, \cdots, p$. Then, the relaxed variable x_{j1}^i may be assigned to set S_{j2} and the non-relaxed variable x_{j2}^k assigned to set S_{j1} as long as the i^{th} row, k^{th} column element of matrix $B_{j1}^{-1}B_{j2} \neq 0$. Such restriction ensures that the new B_{j1} matrix, generated after these decision variables are swapped, remains non-singular.

Case 2: Explicitly enforce the non-negativity restriction of x_{j1}^i. If no positive elements on S_{j2} exist or if the i^{th} row, k^{th} column element of matrix $B_{j1}^{-1}B_{j2} = 0,\ \forall\ i \in \{S_{j1} : x_{j1}^i < 0\}$ and $k \in \{S_{j2} : x_{j2}^k > 0\}$, then the non-negativity restriction on x_{j1}^i must be explicitly enforced in the Reduced Problem through an additional constraint, namely:

$$\begin{aligned} x_{j1}^i \geq 0 &\Leftrightarrow \{B_{j1}^{-1}b_j\}_i - \{B_{j1}^{-1}B_{j2}\}_i\, x_{j2} - \{B_{j1}^{-1}D_j\}_i\, x_{p+1} \geq 0 \\ &\Leftrightarrow \{B_{j1}^{-1}B_{j2}\}_i\, x_{j2} + \{B_{j1}^{-1}D_j\}_i\, x_{p+1} \leq \{B_{j1}^{-1}b_j\}_i, \end{aligned} \tag{3.8}$$

Where the notation $\{B_{j1}^{-1}B_{j2}\}_i$, for instance, indicates the i^{th} row of the matrix $B_{j1}^{-1}B_{j2}$. While any negative relaxed decision variable can have its non-negativity restriction enforced via (3.8), it is reasonable to enforce only the restriction of the most negative

element in set S_{j1}, as to maintain the size of the Reduced Problem small, and, thus, computationally tractable.

Observe that only one of these cases is applicable to each subproblem, as any changes in the problem structure results in a new optimum solution $\{x^*_{j1}, x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$, and, thus, in a new set of negative $\{x^*_{j1},\ j = 1, \cdots, p\}$, elements. Moreover, from Case 2, it is clear that the original problem can be solved in a single iteration by having all the relaxed non-negativity restrictions enforced through (3.8). Nonetheless, since such a large number of constraints would offset the benefits of the relaxation, improved solution times are attained when only the biding restrictions are maintained. In fact, subsequent iterations of the algorithm evaluate whether previously added constraints of this form can be removed. Suppose that, on a previous iteration of the algorithm, the non-negativity restriction of the i^{th} relaxed decision variable of the j^{th} subproblem was enforced as in Case 2. Consider the following cases:

Case 3: Constraint (3.8) is non-binding. If, for a current optimum solution to the Reduced Problem, $\{x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$, (3.8) is non-binding, that is, if $\{B^{-1}_{j1}B_{j2}\}_i x^*_{j2} + \{B^{-1}_{j1}D_j\}_i x^*_{p+1} < \{B^{-1}_{j1}b_j\}_i \Leftrightarrow x^i_{j1} > 0$, then the constraint can be removed without affecting the current solution $\{x^*_{j1}, x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$.

Case 4: Constraint (3.8) is biding, but the respective x^i_{j1} decision variable can be swapped with a positive element in x_{j2}. If, for a current optimum solution to the Reduced Problem, $\{x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$, (3.8) is binding, that is, if $\{B^{-1}_{j1}B_{j2}\}_i x^*_{j2} +$

$\{B_{j1}^{-1}D_j\}_i x_{p+1}^* = \{B_{j1}^{-1}b_j\}_i \Leftrightarrow x_{j1}^i = 0$, then the constraint can be removed if there exists a strictly positive element k in set S_{j2}, such that the i^{th} row, k^{th} column element of matrix $B_{j1}^{-1}B_{j2} \neq 0$. Since the swap operation only reorganizes the decision variables into sets S_{j1} and S_{j2}, removing (3.8) from the Reduced Problem does not alter its current solution.

In summary, the original Ritter (1967) Relaxation Algorithm proceeds as follows:

1. For each subproblem $j = 1, \cdots, p$, augment the B_j, A_j, and c_j matrices with artificial variables in order to ensure that B_j has full rank, and, at least, $m_j + 1$ columns;

2. For each subproblem $j = 1, \cdots, p$, partition the columns of B_j and A_j and the rows of c_j into sets B_{j1} and B_{j2}, A_{j1} and A_{j2}, and c_{j1} and c_{j2} such that B_{j1} is non-singular. Note that the previous step and the decision variable partitioning into sets S_{j1} and S_{j2} may be accomplished simultaneously, for instance, through Algorithm 3.1;

3. Given the partitions B_{j1} and B_{j2}, A_{j1} and A_{j2}, and c_{j1} and c_{j2}, $j = 1, \cdots, p$, compute parameters (3.5) and use such parameters, along with Constraint (3.7), to generate the Reduced Problem (3.6). Solve the Reduced Problem. If no feasible solution to the Reduced Problem exists, then the original problem is infeasible. Terminate;

4. Given the optimum solution to the Reduced Problem, $\{x_{j2}^*,\ j = 1, \cdots, p,\ x_{p+1}^*\}$, use (3.3) to compute the corresponding solution to the relaxed variables, $\{x_{j1}^*,\ j = 1, \cdots, p\}$. If all relaxed variables are non-negative, terminate:

- If any of the artificial variables added in Step 1 are positive (basic), then the original problem is either infeasible or unbounded;
- If all artificial variables equal zero, but Constraint (3.7) is binding, then the original problem is unbounded; otherwise
- The solution attained is optimal to the original problem;

5. If any Constraint of the form (3.8) have been added on previous iterations, then evaluate whether those may be removed. If the constraint is non-binding, then remove it as in Case 3; otherwise, search set S_{j2} for a positive decision variable x_{j2}^{k}, such that the i^{th} row, k^{th} column element of matrix $B_{j1}^{-1}B_{j2} \neq 0$. If such variable exists, then swap them as in Case 4 and remove (3.8); otherwise maintain the constraint unchanged;

6. For each subproblem $j = 1, \cdots, p$, if there exists a negative element in set S_{j1} and a positive element in set S_{j2} (i.e., if $x_{j1}^{i} < 0$ and $x_{j2}^{k} > 0$), such that the i^{th} row, k^{th} column element of matrix $B_{j1}^{-1}B_{j2} \neq 0$, then swap these variables as in Case 1; otherwise, explicitly enforce the non-negativity restriction of the most negative variable in S_{j1} as in Case 2. Return to Step 3.

Next, a modification to the Ritter (1967) Relaxation Algorithm able to solve bordered-structure problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions is presented.

3.2.2 Modified Algorithm

Recall that the Ritter (1967) Relaxation Algorithm enforces violated non-negativity restrictions either by swapping elements between the relaxed and non-relaxed sets (as in Case 1) or by adding constraints of the form (3.8) into the Reduced Problem (as in Case 2). In fact, the algorithm favors the swap operation, as constraints of the form (3.8) are only added to the Reduced Problem if all non-relaxed variables equal zero or if strictly positive elements in S_{j2} exist, but swapping them with negative elements in S_{j1} results in singular B_{j1} matrices. Moreover, previously added constraints of the form (3.8) are removed whenever they are non-binding (as in Case 3) or the non-negativity restriction of their referenced decision variables can be enforced by swapping (as in Case 4). By favoring the swap operation, the Ritter (1967) Relaxation Algorithm is able to maintain the size of the Reduced Problem small, and, thus, relatively quick to solve. Nonetheless, the increased number of matrix inversions and multiplications involved in the swap operation not only offset the reduced computation time attained by the relaxation, but may also lead to considerable machine-precision issues. In this Section, a variation of the Ritter (1967) Relaxation Algorithm, which relies solely on the addition and removal of constraints from the Reduced Problem, is presented.

First, however, some important modifications to the problem definition are warranted. Recall that the Ritter (1967) Relaxation Algorithm requires the problem to be placed in standard form. While such operation is often performed during the initialization of many commonly used solution algorithms, such as Simplex and Revised Simplex, the inclusion of such large number of slack variables increases the complexity of the Reduced Problem, thus possibly

slowing its solution time (e.g., if alternative solution methodologies, such as Network Simplex and Interior Point Methods, are used to solve the Reduced Problem). Naturally, such transformation is essential for the algorithm, as placing the original problem in standard form allows an algebraic solution to $x_{j1},\ j = 1, \cdots, p$, decision variables to be derived, which, in turn, allows such decision variables to be eliminated from the Reduced Problem by substitution. Nonetheless, other than (3.7), all the structural constraints of the Reduced Problem generated on the first iteration of the Ritter (1967) Relaxation Algorithm follow from the complicating constraints in the original problem, and, as such, do not need to be placed in standard form. By avoiding the inclusion of the slack variables corresponding to these constraints, depending on the solution methodology used to solve the Reduced Problem, it is possible to improve the performance of the Ritter (1967) Relaxation Algorithm.

In addition, recall that the original algorithm was developed for continuous (convex) problems, whereas many important optimization problems contain integer decision variables. Since the swap operation enforces violated non-negativity restrictions of the decision variables in the relaxed set by placing them in the non-relaxed set (where their non-negativity restriction remains enforced), it stands to reason that such operation could also allow integer restrictions to be similarly enforced. Unfortunately, while such is indeed the case for the element entering set S_{j2}, it is not the case, in general, for the element leaving S_{j2} and entering S_{j1}, for, after the Reduced Problem is reoptimized, any new (integer) solution $\{x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$ results in a new (possibly non-integer) solution $\{x^*_{j1},\ j = 1, \cdots, p\}$. As a result, the swap operation not only reduces the performance of the algorithm and leads

to considerable machine-precision issues, but it also leads to cycling and possibly stalling when attempting to solve problems with integer or mixed-integer subproblems. Nonetheless, if only constraints of the form (3.8) are added to the Reduced Problem, then the integer restriction of the decision variables in the relaxed set can be enforced in a scheme similar to the branch and bound algorithm. Consider the following generic, mixed-integer problem:

$$\text{Minimize} \quad \sum_{j=1}^{p+1} c_j^t \, x_j \tag{3.9a}$$

$$\text{Subject to} \quad B_j \, x_j + I_{m_j} \, y_j + D_j \, x_{p+1} = b_j, \ \forall \ j = 1, \cdots, p \tag{3.9b}$$

$$\sum_{j=1}^{p} Aeq_j \, x_j + Deq \, x_{p+1} = beq \tag{3.9c}$$

$$\sum_{j=1}^{p} Aineq_j \, x_j + Dineq \, x_{p+1} \leq bineq \tag{3.9d}$$

$$x_j \in (\mathbb{R}_+ \times \mathbb{Z}_+), \quad \forall \ j = 1, \cdots, p+1 \tag{3.9e}$$

$$y_j \in \mathbb{R}_+, \quad \forall \ j = 1, \cdots, p, \tag{3.9f}$$

Where y_j are the slack variables corresponding to Constraint (3.1b); Aeq_j, $Aineq_j$, Deq, and $Dineq$ are, respectively, $(m_{eq} \times n_j)$, $(m_{ineq} \times n_j)$, $(m_{eq} \times n_{p+1})$, and $(m_{ineq} \times n_{p+1})$ matrices of constraint coefficients; and beq and $bineq$ are column vectors of m_{eq} and m_{ineq} right hand side coefficients, respectively. The notation $(\mathbb{R}_+ \times \mathbb{Z}_+)$ indicates the non-negative, mixed-integer nature of the problem. Although (3.9) allows for the possibility of mixed-integer subproblems and parses the complicating constraints of the original problem into equalities and inequalities, the inheriting bordered-structure of (3.1) remains unchanged.

In general, the Modified Ritter (1967) Relaxation Algorithm relies on the same steps of the

original algorithm save for the swap operation. Consequently, the non-negativity restriction of any negative variable in the relaxed set is enforced solely by constraints of the form (3.8). However, contrary to the original algorithm where previously added constraints of this form may be removed if they are non-binding, here, such constraints are only removed provided they are redundant. Moreover, since the problem decision variables are no longer swapped between the relaxed and non-relaxed sets, the corresponding $B_{j1},\ B_{j2},\ A_{j1},\ A_{j2},\ c_{j1}$, and $c_{j2},\ j = 1, \cdots, p$, partitions remain unchanged throughout the execution of the algorithm. Hence, the following constants can be computed a priori and reused as needed:

$$\overline{B}_j = B_{j1}^{-1}\, B_{j2}, \quad \forall\ j = 1, \cdots, p \tag{3.10a}$$

$$\overline{D}_j = B_{j1}^{-1}\, D_j, \quad \forall\ j = 1, \cdots, p \tag{3.10b}$$

$$\overline{b}_j = B_{j1}^{-1}\, b_j, \quad \forall\ j = 1, \cdots, p. \tag{3.10c}$$

Consequently, (3.3) simplifies to:

$$\begin{aligned} x_{j1} &= B_{j1}^{-1} b_j - B_{j1}^{-1} B_{j2}\, x_{j2} - B_{j1}^{-1} D_j\, x_{p+1},\ \forall\ j = 1, \cdots, p \\ &= \overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1},\ \forall\ j = 1, \cdots, p, \end{aligned} \tag{3.11}$$

Equation (3.8) simplifies to:

$$\begin{aligned} x_{j1}^i \geq 0 &\Leftrightarrow \{B_{j1}^{-1} B_{j2}\}_i\, x_{j2} + \{B_{j1}^{-1} D_j\}_i\, x_{p+1} \leq \{B_{j1}^{-1} b_j\}_i \\ &\Leftrightarrow \{\overline{B}_j\}_i\, x_{j2} + \{\overline{D}_j\}_i\, x_{p+1} \leq \{\overline{b}_j\}_i, \end{aligned} \tag{3.12}$$

And (3.5), after allowing for the possibility of both equality and inequality complicating constraints, change to:

$$d_j^t = c_{j2}^t - c_{j1}^t\, \overline{B}_j,\ \forall\ j = 1, \cdots, p \tag{3.13a}$$

$$d^t_{p+1} = c^t_{p+1} - \sum_{j=1}^{p} c^t_{j1}\,\overline{D}_j \tag{3.13b}$$

$$\alpha = \sum_{j=1}^{p} c^t_{j1}\,\overline{b}_j \tag{3.13c}$$

$$F_j = Aeq_{j2} - Aeq_{j1}\,\overline{B}_j,\ \forall\ j = 1, \cdots, p \tag{3.13d}$$

$$F_{p+1} = Deq - \sum_{j=1}^{p} Aeq_{j1}\,\overline{D}_j \tag{3.13e}$$

$$f = beq - \sum_{j=1}^{p} Aeq_{j1}\,\overline{b}_j \tag{3.13f}$$

$$G_j = Aineq_{j2} - Aineq_{j1}\,\overline{B}_j,\ \forall\ j = 1, \cdots, p \tag{3.13g}$$

$$G_{p+1} = Dineq - \sum_{j=1}^{p} Aineq_{j1}\,\overline{D}_j \tag{3.13h}$$

$$g = bineq - \sum_{j=1}^{p} Aineq_{j1}\,\overline{b}_j. \tag{3.13i}$$

After the appropriate parameter substitutions and the inclusion of Constraint (3.7), the resulting Reduced Problem is as follows:

$$\text{Minimize} \quad \sum_{j=1}^{p} d^t_j\,x_{j2} + d^t_{p+1}\,x_{p+1} + \alpha \tag{3.14a}$$

$$\text{Subject to} \quad \sum_{j=1}^{p} F_j\,x_{j2} + F_{p+1}\,x_{p+1} = f \tag{3.14b}$$

$$\sum_{j=1}^{p} G_j\,x_{j2} + G_{p+1}\,x_{p+1} \leq g \tag{3.14c}$$

$$\sum_{j=1}^{p}\sum_{i=1}^{|S_{j2}|} x^i_{j2} + \sum_{i=1}^{n_{p+1}} x^i_{p+1} \leq M \tag{3.14d}$$

$$x_{j2} \in (\mathbb{R}_+ \times \mathbb{Z}_+),\ \ \forall\ j = 1, \cdots, p,\ x_{p+1} \in (\mathbb{R}_+ \times \mathbb{Z}_+). \tag{3.14e}$$

As aforementioned, constraints of the form (3.8), or, now (3.12), not only can be used to

enforce non-negativity restrictions, but also the integer restriction of any decision variable in the relaxed set. For instance, suppose that, on some iteration of the Modified Ritter (1967) Relaxation Algorithm, the i^{th} relaxed element of the j^{th} subproblem attains a fractional (but positive) value a, then, following a scheme similar to the branch and bound algorithm, two new subproblems are generated and solved, namely:

Branch I of the Reduced Problem:

$$x_{j1}^{i} \leq \lfloor a \rfloor \Leftrightarrow \{\bar{b}_j\}_i - \{\overline{B}_j\}_i \, x_{j2} - \{\overline{D}_j\}_i \, x_{p+1} \leq \lfloor a \rfloor$$

$$\Leftrightarrow -\{\overline{B}_j\}_i \, x_{j2} - \{\overline{D}_j\}_i \, x_{p+1} \leq \lfloor a \rfloor - \{\bar{b}_j\}_i. \tag{3.15a}$$

Branch II of the Reduced Problem:

$$x_{j1}^{i} \geq \lceil a \rceil \Leftrightarrow \{\bar{b}_j\}_i - \{\overline{B}_j\}_i \, x_{j2} - \{\overline{D}_j\}_i \, x_{p+1} \geq \lceil a \rceil$$

$$\Leftrightarrow \{\overline{B}_j\}_i \, x_{j2} + \{\overline{D}_j\}_i \, x_{p+1} \leq -\lceil a \rceil + \{\bar{b}_j\}_i. \tag{3.15b}$$

Observe that no consideration has been given to the integer restriction, if any, of the decision variables in set $S_{j2},\ j = 1, \cdots, p$, as those are enforced within the Reduced Problem. Consequently, the modified algorithm disaggregates the branch and bound tree that would have been produced while solving the original problem into two separate trees: one enforced by constraints of the form (3.15), and one enforced by whichever algorithm is used to solve the Reduced Problem. As a result, the same concerns pertaining to the branch and bound algorithm, such as branching strategy and choice of branching variable, also apply to the modified algorithm. These, however, are beyond the scope of this research.

Moreover, recall that the modified algorithm only allows previously added constraints of

the form (3.12), or, for that matter, (3.15), to be removed from the Reduced Problem provided they are redundant (and not if they are simply non-binding). Such is an important consideration, for it preserves the dichotomy of the branch and bound search, and thus prevents cycling and, possibly, stalling.

Lastly, on any iteration of the modified algorithm, if one of the decision variables in the relaxed set becomes negative, then an additional constraint of the form (3.12) is added to the Reduced Problem and such is reoptimized before any other subsequent branches are created. A similar approach is followed, for instance, to enforce the binary restriction of any relaxed decision variables with values greater than one.

Next, a quadratic variation of the problem is presented. Consider problem (3.9), but, instead of the linear Objective Function (3.9a), suppose that such problem has the quadratic objective function {Minimize $0.5\, x^t\, H\, x + c^t\, x$}, where H denotes its Hessian matrix (i.e., the $|x| \times |x|$ matrix of its second-order partial derivatives). Following the notation used in (3.9), let H be partitioned into $(p+1)^2$ submatrices $H_{jk},\ j = 1, \cdots, p+1,\ k = 1, \cdots, p+1$:

$$\text{Minimize} \quad \frac{1}{2} \sum_{j=1}^{p+1} \sum_{k=1}^{p+1} x_j^t H_{jk} x_k + \sum_{j=1}^{p+1} c_j^t\, x_j. \tag{3.16}$$

Now, following an scheme similar to the one used to derive (3.13), let the decision variables of each subproblem $j = 1, \cdots, p$, be partitioned into sets S_{j1} and S_{j2}, and the corresponding columns of B_j and A_j and the rows of c_j be grouped into the disjoint sets B_{j1} and B_{j2}, A_{j1} and A_{j2}, and c_{j1} and c_{j2}. Moreover, let each of the $H_{jk},\ j = 1, \cdots, p,\ k = 1, \cdots, p$, submatrices be partitioned into four additional submatrices, $H_{jk11},\ H_{jk12},\ H_{jk21}$, and H_{jk22};

let the rows of $H_{j(p+1)},\ j = 1, \cdots, p$, be partitioned into two additional submatrices, $H_{j(p+1)1}$ and $H_{j(p+1)2}$; let the columns of $H_{(p+1)k},\ k = 1, \cdots, p$, be partitioned into two additional submatrices, $H_{(p+1)k1}$ and $H_{(p+1)k2}$; and let submatrix $H_{(p+1)(p+1)}$ remain intact, such that H_{jklm} is $(|S_{jl}| \times |S_{km}|)$, $H_{j(p+1)l}$ is $(|S_{jl}| \times n_{p+1})$, $H_{(p+1)km}$ is $(n_{p+1} \times |S_{km}|)$, and $H_{(p+1)(p+1)}$ is $(n_{p+1} \times n_{p+1})$, $\forall\ j = 1, \cdots, p,\ k = 1, \cdots, p,\ l = 1, 2,\ m = 1, 2$. Figure 3.1 depicts such partitioning scheme.

$$
H = \left[\begin{array}{cc:cc:c:cc:c:c}
H_{1111} & H_{1112} & H_{1211} & H_{1212} & \cdots & H_{1k11} & H_{1k12} & \cdots & H_{1(p+1)1} \\
H_{1121} & H_{1122} & H_{1221} & H_{1222} & & H_{1k21} & H_{1k22} & & H_{1(p+1)2} \\
\hdashline
H_{2111} & H_{2112} & H_{2211} & H_{2212} & \cdots & H_{2k11} & H_{2k12} & \cdots & H_{2(p+1)1} \\
H_{2121} & H_{2122} & H_{2221} & H_{2222} & & H_{2k21} & H_{2k22} & & H_{2(p+1)2} \\
\hdashline
\vdots & & \vdots & & \ddots & \vdots & & \ddots & \vdots \\
\hdashline
H_{j111} & H_{j112} & H_{j211} & H_{j212} & \cdots & H_{jk11} & H_{jk12} & \cdots & H_{j(p+1)1} \\
H_{j121} & H_{j122} & H_{j221} & H_{j222} & & H_{jk21} & H_{jk22} & & H_{j(p+1)2} \\
\hdashline
\vdots & & \vdots & & \ddots & \vdots & & \ddots & \vdots \\
\hdashline
H_{(p+1)11} & H_{(p+1)12} & H_{(p+1)21} & H_{(p+1)22} & \cdots & H_{(p+1)k1} & H_{(p+1)k2} & \cdots & H_{(p+1)(p+1)}
\end{array}\right]
$$

Figure 3.1: A representation of the scheme used to partition the original Hessian matrix.

Consequently, matrix H_{jklm} corresponds to the second-order partial derivative of the original problem objective function with respect to the decision variables in sets S_{jl} and S_{km}; $H_{j(p+1)l}$ corresponds to the second-order partial derivative with respect to the decision variables in set S_{jl} and to the set of complicating variables; and $H_{(p+1)km}$ to the second-order partial derivative with respect to the complicating variables and to the variables in set $S_{km},\ \forall\ j =$

$1, \cdots, p,\ k = 1, \cdots, p,\ l = 1, 2,\ m = 1, 2$. As a result, (3.16) becomes:

$$\text{Minimize}\ \ \frac{1}{2}\left(\sum_{j=1}^{p}\sum_{k=1}^{p}\begin{bmatrix} x_{j1}^t & x_{j2}^t \end{bmatrix}\begin{bmatrix} H_{jk11} & H_{jk12} \\ H_{jk21} & H_{jk22} \end{bmatrix}\begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix} + \right.$$

$$+\sum_{j=1}^{p}\begin{bmatrix} x_{j1}^t & x_{j2}^t \end{bmatrix}\begin{bmatrix} H_{j(p+1)1} \\ H_{j(p+1)2} \end{bmatrix} x_{p+1} + \sum_{k=1}^{p} x_{p+1}^t \begin{bmatrix} H_{(p+1)k1} & H_{(p+1)k2} \end{bmatrix}\begin{bmatrix} x_{k1} \\ x_{k2} \end{bmatrix} +$$

$$\left. x_{p+1}^t H_{(p+1)(p+1)} x_{p+1}\right) + \sum_{j=1}^{p}\begin{bmatrix} c_{j1}^t & c_{j2}^t \end{bmatrix}\begin{bmatrix} x_{j1} \\ x_{j2} \end{bmatrix} + c_{p+1}^t x_{p+1}.$$

Or, after fully expanding the matrix operations:

$$\text{Minimize}\ \ \frac{1}{2}\left(\sum_{j=1}^{p}\sum_{k=1}^{p}\left(x_{j1}^t H_{jk11} x_{k1} + x_{j2}^t H_{jk21} x_{k1} + x_{j1}^t H_{jk12} x_{k2} + x_{j2}^t H_{jk22} x_{k2}\right) + \right.$$

$$\sum_{j=1}^{p}\left(x_{j1}^t H_{j(p+1)1} x_{p+1} + x_{j2}^t H_{j(p+1)2} x_{p+1}\right) + \sum_{k=1}^{p}\left(x_{p+1}^t H_{(p+1)k1} x_{k1} + x_{p+1}^t H_{(p+1)k2} x_{k2}\right) +$$

$$\left. + x_{p+1}^t H_{(p+1)(p+1)} x_{p+1}\right) + \sum_{j=1}^{p}\left(c_{j1}^t x_{j1} + c_{j2}^t x_{j2}\right) + c_{p+1}^t x_{p+1}. \quad (3.17)$$

Now, by (3.11), $x_{j1} = \overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1},\ \forall\ j = 1, \cdots, p$. Consequently, (3.17) can be rewritten as:

$$\text{Minimize}\ \ \frac{1}{2}\left(\sum_{j=1}^{p}\sum_{k=1}^{p}\left(\left(\overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1}\right)^t H_{jk11}\left(\overline{b}_k - \overline{B}_k\, x_{k2} - \overline{D}_k\, x_{p+1}\right) + \right.\right.$$

$$+ x_{j2}^t H_{jk21}\left(\overline{b}_k - \overline{B}_k\, x_{k2} - \overline{D}_k\, x_{p+1}\right) + \left(\overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1}\right)^t H_{jk12} x_{k2} + x_{j2}^t H_{jk22} x_{k2}\Big) +$$

$$\sum_{j=1}^{p}\left(\left(\overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1}\right)^t H_{j(p+1)1} x_{p+1} + x_{j2}^t H_{j(p+1)2} x_{p+1}\right) +$$

$$+\sum_{k=1}^{p}\left(x_{p+1}^t H_{(p+1)k1}\left(\overline{b}_k - \overline{B}_k\, x_{k2} - \overline{D}_k\, x_{p+1}\right) + x_{p+1}^t H_{(p+1)k2} x_{k2}\right) +$$

$$\left. + x_{p+1}^t H_{(p+1)(p+1)} x_{p+1}\right) + \sum_{j=1}^{p}\left(c_{j1}^t\left(\overline{b}_j - \overline{B}_j\, x_{j2} - \overline{D}_j\, x_{p+1}\right) + c_{j2}^t x_{j2}\right) + c_{p+1}^t x_{p+1},$$

Or, after some index substitutions and some rearrangement of the terms, as:

$$\begin{aligned}
\text{Minimize } & \frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{p} x_{j2}^{t}\left(H_{jk22}-\overline{B}_{j}^{t}H_{jk12}-H_{jk21}\overline{B}_{k}+\overline{B}_{j}^{t}H_{jk11}\overline{B}_{k}\right)x_{k2}+\\
& \frac{1}{2}\sum_{j=1}^{p} x_{j2}^{t}\left(H_{j(p+1)2}-\overline{B}_{j}^{t}H_{j(p+1)1}+\sum_{k=1}^{p}\left(\overline{B}_{j}^{t}H_{jk11}-H_{jk21}\right)\overline{D}_{k}\right)x_{p+1}+\\
& \frac{1}{2}\sum_{k=1}^{p} x_{p+1}^{t}\left(H_{(p+1)k2}-H_{(p+1)k1}\overline{B}_{k}+\sum_{j=1}^{p}\overline{D}_{j}^{t}\left(H_{jk11}\overline{B}_{k}-H_{jk12}\right)\right)x_{k2}+\\
& \frac{1}{2}x_{p+1}^{t}\left(H_{(p+1)(p+1)}-\sum_{j=1}^{p}\left(\overline{D}_{j}^{t}H_{j(p+1)1}+H_{(p+1)j1}\overline{D}_{j}\right)+\sum_{j=1}^{p}\sum_{k=1}^{p}\overline{D}_{j}^{t}H_{jk11}\overline{D}_{k}\right)x_{p+1}+\\
& \sum_{j=1}^{p}\left(c_{j2}^{t}-c_{j1}^{t}\overline{B}_{j}+\frac{1}{2}\sum_{k=1}^{p}\overline{b}_{k}^{t}\left(H_{jk21}^{t}-H_{jk11}^{t}\overline{B}_{j}+H_{kj12}-H_{kj11}\overline{B}_{j}\right)\right)x_{j2}+\\
& \left(c_{p+1}^{t}-\sum_{j=1}^{p}\left(c_{j1}^{t}\overline{D}_{j}-\frac{1}{2}\overline{b}_{j}^{t}\left(H_{j(p+1)1}+H_{(p+1)j1}^{t}\right)+\frac{1}{2}\sum_{k=1}^{p}\overline{b}_{j}^{t}\left(H_{jk11}+H_{kj11}^{t}\right)\overline{D}_{k}\right)\right)x_{p+1}+\\
& \sum_{j=1}^{p}\left(c_{j1}^{t}\overline{b}_{j}+\frac{1}{2}\sum_{k=1}^{p}\overline{b}_{j}^{t}H_{jk11}\overline{b}_{k}\right).
\end{aligned}$$

Now, let the following parameter substitutions be performed:

$$L_{jk}=H_{jk22}-\overline{B}_{j}^{t}H_{jk12}-H_{jk21}\overline{B}_{k}+\overline{B}_{j}^{t}H_{jk11}\overline{B}_{k},\ \forall\ j=1,\cdots,p,\ k=1,\cdots,p \tag{3.18a}$$

$$L_{j(p+1)}=H_{j(p+1)2}-\overline{B}_{j}^{t}H_{j(p+1)1}+\sum_{k=1}^{p}\left(\overline{B}_{j}^{t}H_{jk11}-H_{jk21}\right)\overline{D}_{k},\ \forall\ j=1,\cdots,p \tag{3.18b}$$

$$L_{(p+1)k}=H_{(p+1)k2}-H_{(p+1)k1}\overline{B}_{k}+\sum_{j=1}^{p}\overline{D}_{j}^{t}\left(H_{jk11}\overline{B}_{k}-H_{jk12}\right),\ \forall\ k=1,\cdots,p \tag{3.18c}$$

$$L_{(p+1)(p+1)}=H_{(p+1)(p+1)}-\sum_{j=1}^{p}\left(\overline{D}_{j}^{t}H_{j(p+1)1}+H_{(p+1)j1}\overline{D}_{j}\right)+\sum_{j=1}^{p}\sum_{k=1}^{p}\overline{D}_{j}^{t}H_{jk11}\overline{D}_{k} \tag{3.18d}$$

$$d_{j}^{t}=c_{j2}^{t}-c_{j1}^{t}\overline{B}_{j}+\frac{1}{2}\sum_{k=1}^{p}\overline{b}_{k}^{t}\left(H_{jk21}^{t}-H_{jk11}^{t}\overline{B}_{j}+H_{kj12}-H_{kj11}\overline{B}_{j}\right),\ \forall\ j=1,\cdots,p \tag{3.18e}$$

$$d_{p+1}^{t}=c_{p+1}^{t}-\sum_{j=1}^{p}\left(c_{j1}^{t}\overline{D}_{j}-\frac{1}{2}\overline{b}_{j}^{t}\left(H_{j(p+1)1}+H_{(p+1)j1}^{t}\right)+\frac{1}{2}\sum_{k=1}^{p}\overline{b}_{j}^{t}\left(H_{jk11}+H_{kj11}^{t}\right)\overline{D}_{k}\right) \tag{3.18f}$$

$$\alpha = \sum_{j=1}^{p} \left(c_{j1}^t \overline{b}_j + \frac{1}{2} \sum_{k=1}^{p} \overline{b}_j^t H_{jk11} \overline{b}_k \right), \tag{3.18g}$$

Such that (3.16) can be rewritten as:

$$\text{Minimize } \frac{1}{2} \left(\sum_{j=1}^{p} \sum_{k=1}^{p} x_{j2}^t L_{jk} x_{k2} + \sum_{j=1}^{p} x_{j2}^t L_{j(p+1)} x_{p+1} + \sum_{k=1}^{p} x_{p+1}^t L_{(p+1)k} x_{k2} + x_{p+1}^t L_{(p+1)(p+1)} x_{p+1} \right) + \sum_{j=1}^{p} d_j^t x_{j2} + d_{p+1}^t x_{p+1} + \alpha$$

Interestingly, since $L_{jk},\ j = 1, \cdots, p+1,\ k = 1, \cdots, p+1$, form a partition of a larger matrix, L, it follows that the objective function of the Reduced Problem is also quadratic in nature:

$$\text{Minimize } \frac{1}{2} z^t L z + d^t z + \alpha, \tag{3.19}$$

Where $z^t = [x_{12}^t, \cdots, x_{j2}^t, \cdots, x_{p2}^t, x_{p+1}^t]$. The remaining parameter substitutions, (3.13d)-(3.13i), used to obtain the initial Reduced Problem remain unchanged. Observe that if the objective function of the original problem is linear, then its Hessian matrix will be a $(\sum_{j=1}^{p+1} n_j \times \sum_{j=1}^{p+1} n_j)$ zero matrix. As a result, $L_{jk},\ j = 1, \cdots, p+1,\ k = 1, \cdots, p+1$ will also be zero matrices and (3.18) simplifies to (3.13). Interestingly, observe that the Hessian matrix does not need to be augmented with artificial variables in order to ensure the existence of an initial basic feasible solution to each subproblem, as such is already guaranteed by the inclusion of large M coefficients into the c-vector. Nonetheless, given the ensuing change in problem dimensionality, such a matrix must be augmented by new rows and columns of zeros.

In summary, the Modified Ritter (1967) Relaxation Algorithm proceeds as follows:

1. For each subproblem $j = 1, \cdots, p$, augment the B_j, A_j, and c_j matrices with artificial variables in order to ensure that B_j has full rank, and, at least, $m_j + 1$ columns. If the objective function is quadratic, then also augment the Hessian matrix with corresponding rows and columns of zeros;

2. For each subproblem $j = 1, \cdots, p$, partition the columns of B_j and A_j and the rows of c_j into sets B_{j1} and B_{j2}, A_{j1} and A_{j2}, and c_{j1} and c_{j2} such that B_{j1} is non-singular. Once again, note that the previous step and the decision variable partitioning into sets S_{j1} and S_{j2} may be accomplished simultaneously, for instance, through Algorithm 3.1;

3. Use these partitions to compute parameters (3.10) and coefficients (3.13). If the original problem has a quadratic objective function, then substitute the computation of (3.13a)-(3.13c) by variation (3.18). Generate the initial Reduced Problem along with Constraint (3.7);

4. Now, for a minimization problem, set $x^*_{IP} = \emptyset,\ z^*_{IP} = \infty$, and $NL = \{0\}$, where x^*_{IP} denotes the incumbent (best known) integer-valued solution, z^*_{IP} denotes its objective function value, NL denotes the list of unexplored nodes (Reduced Problems), and Reduced Problem 0 denotes the Reduced Problem generated on Step 3;

5. If $NL = \emptyset$, then terminate:

   - If $x^*_{IP} = \emptyset$, then the original problem has no feasible integer solutions;
   - If $x^*_{IP} \neq \emptyset$ but Constraint (3.7) is binding, then the original problem is unbounded; otherwise,

- The incumbent solution x^*_{IP} is optimal with objective function value z^*_{IP};

6. Remove Reduced Problem r from set NL. Solve the problem;

7. If Reduced Problem r is infeasible (or has no feasible integer solutions), then fathom node r due to infeasibility and return to Step 5. Otherwise, proceed to Step 8;

8. If Reduced Problem r has a feasible, integer solution, but its objective function value is larger than or equal to z^*_{IP}, then fathom node r due to bounding and return to Step 5. Otherwise, proceed to Step 9;

9. Given the optimum solution to Reduced Problem r, $\{x^*_{j2},\ j = 1, \cdots, p,\ x^*_{p+1}\}$, and parameters (3.10), use (3.11) to compute the corresponding solution to the relaxed variables, $\{x^*_{j1},\ j = 1, \cdots, p\}$;

10. For each subproblem $j = 1, \cdots, p$, determine if at least one element in set S_{j1} has a violated non-negativity restriction. If so, explicitly enforce the non-negativity restriction of the most violated element in S_{j1} by adding a constraint of the form (3.12). Reoptimize Reduced Problem r and return to Step 7. Conversely, if the non-negativity restriction of all relaxed decision variables is satisfied, proceed to Step 11;

11. If any of the artificial variables added in Step 1 are positive (basic), then fathom node r due to infeasibility and return to Step 5. Otherwise, proceed to Step 12;

12. For each subproblem $j = 1, \cdots, p$, determine if at least one element in set S_{j1} has a violated upper bound restriction (e.g., $x^i_{j1} > 1$ with x_{j1} binary). If so, explicitly

enforce the upper bound restriction of the most violated element in S_{j1} by adding a constraint of the form (3.15a) with the appropriate choice for a (e.g., let $a = 1$ for binary variables). Reoptimize Reduced Problem r and return to Step 7. Conversely, if the upper bound restriction of all relaxed decision variables is satisfied, proceed to Step 13;

13. If the integer restriction of all relaxed variables is satisfied, then fathom node r due to integrality. If its objective function value is smaller than z^*_{IP}, then update z^*_{IP} and x^*_{IP} accordingly, remove all Reduced Problems from NL with lower bound objective function value greater than or equal to the new z^*_{IP} value, and return to Step 5. Otherwise, proceed to Step 14;

14. If some of the relaxed variables have violated integer restrictions, then use a branching rule and Constraints of the form (3.15) to create two new Reduced Problems. Remove any redundant constraints of form (3.12) or (3.15) from these problems, add both problems to NL (setting their lower bound objective function value to the objective function value of the current Reduced Problem), and return to Step 5;

Observe that the algorithm only evaluates whether or not the problem is infeasible due to the presence of positive artificial variables in its basis (Step 11) after it determines that the non-negativity restriction of all relaxed decision variables is satisfied (Step 10). In general, the presence of positive artificial variables indicates infeasibility, as the arbitrarily large M-coefficient discourages such variables from entering the basis. However, since increases in

the objective function value resulting from such variables may be offset by the presence of negative basic variables (artificial or not), it follows that these, by themselves, do not imply infeasibility. In fact, since the relaxed set may also contain artificial variables, since their non-negativity restriction is relaxed, since the transformations used to attain the Reduced Problem do not alter the objective function of the original problem (other than reducing its dimensionality), and since M is arbitrary large, it follows that the Reduced Problem will (for a minimization problem) favor large negative values for such artificial variables. Therefore, artificial variables must be treated as regular decision variables until the non-negativity restriction of all the elements in the relaxed set are satisfied. Once it is determined that such problems are indeed infeasible, the node is fathomed, as any child nodes resulting from the inclusion of additional constraints of the form (3.15) will also be infeasible. Note that such fathoming criterion follows directly from the Big-M method, for feasibility is not prevented by basic artificial variables that equal zero: it is possible for the Reduced Problem to have an alternate basis representation in which such variables are non-basic. Moreover, since violated upper bound restrictions do not alter these conclusions, improved efficiencies are attained when these are evaluated - and corrected via the inclusion of constraints of the form (3.15a) with the appropriate choice for a - later in the algorithm (Step 12).

In addition, since the basic structure of the initial Reduced Problem remains unaffected throughout the execution of the algorithm, its parameters do not need to be recomputed. Therefore, only new constraints of the form (3.12) and (3.15) must be allocated to NL, thus significantly diminishing both the number computations and the memory required. In

addition, observe that the removal of redundant constraints within Step 14 is unnecessary, as only their right hand side coefficients must be updated to ensure the dichotomy nature of the Reduced Problems resulting from a branching decision. Lastly, observe that these modifications to the original Ritter (1967) Relaxation Algorithm do not preclude it from finding the optimal solution to the original problem, if one exists, in a finite number of iterations, as asserted by the following propositions:

Proposition 3.1 *If the original problem feasible region is bounded, then such is a subset of the Reduced Problem feasible region generated on the first iteration of the Modified Ritter (1967) Relaxation Algorithm.*

Proof. See Appendix. ∎

Corollary 3.1 *If the original problem feasible region is unbounded, then its intersection with a sufficiently small ε-neighborhood around the origin is a subset of the Reduced Problem feasible region generated on the first iteration of the Modified Ritter (1967) Relaxation Algorithm.*

Proof. See Appendix. ∎

Corollary 3.2 *If the original problem has a finite optimum solution, then, at least on the first iteration of the Modified Ritter (1967) Relaxation Algorithm, such solution is an element of the Reduced Problem feasible region.*

Proof. See Appendix. ∎

Corollary 3.3 *If the original problem has a finite optimum solution, then, at least on the first iteration of the Modified Ritter (1967) Relaxation Algorithm, the Reduced Problem has a finite, though possibly different, optimum solution.*

Proof. See Appendix. ∎

Proposition 3.2 *An optimum solution to the Reduced Problem is incumbent if and only if it is feasible to the original problem.*

Proof. See Appendix. ∎

Proposition 3.3 *If the original problem has an optimum solution, then such is also the final incumbent solution to the Modified Ritter (1967) Relaxation Algorithm.*

Proof. See Appendix. ∎

Proposition 3.4 *The Modified Ritter (1967) Relaxation Algorithm converges to the optimum solution to the original problem, if one exists, in a finite number of iterations.*

Proof. See Appendix. ∎

The next Section presents and discusses the results of empirical studies.

3.3 Empirical Studies

This Section empirically explores the performance of the Modified Ritter (1967) Relaxation Algorithm for solving somewhat larger problem instances. All computational runs were performed on a PC with two quad-core, 2.13 GHz Intel Xeon processors and 4.00 GB RAM running on a 64-Bit operating system. The algorithm was coded with Mathworks Matlab version R2012a and the Reduced Problems generated on each iteration were optimized via IBM ILOG CPLEX Optimization Studio version 12.4.

All instances assessed have a bordered-structure. Their constraint matrix, right hand side, and objective function coefficients (including the coefficients of the Hessian matrix) were randomly drawn from a discrete uniform distribution on the interval [-5,5]. The diagonal elements of the Hessian matrix were later adjusted to enforce its positive definitiveness, and the right hand side coefficients later adjusted to ensure the existence of at least one feasible solution. Since all variables are assumed to be either non-negative or binary, since all instances seek to minimize a quadratically-constrained problem, and since a feasible solution exists, each problem instance has a finite optimum solution (though not necessarily unique or equal to the enforced feasible solution). Moreover, each example comprises 20 complicating constraints and 15 complicating variables, and each of its subproblems comprises 10 constraints and 15 variables. Except for the complicating constraints, all remaining constraints are equalities. In addition, excluding the diagonal elements, which are strictly positive, an approximate 20% density (percent non-zero elements) was enforced for the Hessian ma-

trix. Finally, approximately 10% of all decision variables (complicating or not) are binary. Table 3.2 presents the wall-clock time (in seconds) needed by the Modified Ritter (1967) Relaxation Algorithm and IBM ILOG CPLEX, under its default settings, to converge to the optimal solution to the problem. Observe that IBM ILOG CPLEX does not make use of any decomposition/relaxation strategies. Although the instances assessed have a relatively small number of decision variables and constraints, they demand considerable computational effort, an expected result from the quadratic nature of their objective functions.

Table 3.2: Wall-clock time (seconds) comparison of the Modified Ritter (1967) Relaxation Algorithm and IBM ILOG CPLEX for various bordered-structure, MIQP problems.

Problem Size				**Mod. Ritter Relaxation Algorithm**	**IBM ILOG CPLEX**
Subproblems	**Variables**		**Density (%)**		
	C	**B**			
1	29	1	89.9	< 1	< 1
10	148	17	28.9	< 1	< 1
20	276	39	16.1	9	3
30	421	44	11.2	10	6 *
40	552	63	8.6	58	92 *
50	693	72	6.9	43	26 *
60	833	82	5.8	88	430 *

Problem Size				**Mod. Ritter Relaxation Algorithm**	**IBM ILOG CPLEX**
Subproblems	**Variables**		**Density**		
	C	**B**	**(%)**		
70	963	102	5.0	311	408 *
80	1100	115	4.4	2344	12220 *

As indicated in Table 3.2, even though the Modified Ritter (1967) Relaxation algorithm was not coded for efficiency, its performance - at least for the problem instances assessed - is comparable to the one attained by IBM ILOG CPLEX. Such is a particularly interesting result, as it shows that the overhead computations needed to generate the initial Reduced Problem had minimum effect over its performance. In addition, the algorithm is largely affected by the same concerns pertaining to the branch and bound algorithm, namely branching strategy and choice of branching variable. Further development of the source code (such as use of strong branching and proper memory allocation) is needed before the algorithm is able to systematically outperform commercial optimization packages such as IBM ILOG CPLEX. Such modifications, however, lie outside the scope of this research. Finally, given the quadratic nature of the objective function, even relatively small problem instances (instances involving as little as 44 binary and 421 continuous variables and 320 constraints) already resulted in approximate (within the default tolerance limits) integer solutions. Such instances are indicated in Table 3.2 by the notation "*". Nonetheless, for the problem instances assessed, the solution of the Modified Ritter (1967) Relaxation Algorithm was virtually identical to

the one of IBM ILOG CPLEX and often obtained under faster computation times, thus clearly demonstrating its viability for solving solving large, bordered-structure optimization problems. The next section summarizes the findings of this paper and lists directions for future research.

3.4 Conclusions and Future Research

This paper presents an extension of the Ritter (1967) Relaxation Algorithm for solving bordered-structure problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions. In summary, the algorithm relaxes some of the non-negativity and integer restrictions in order to produce a smaller and easier to solve problem, namely the Reduced Problem. In the original algorithm, the relaxed non-negativity restrictions are enforced sequentially either by swapping decision variables between the relaxed and non-relaxed sets (and recomputing the Reduced Problem parameters) or by adding new constraints to the Reduced Problem. The algorithm terminates once the optimum solution to the Reduced Problem also satisfies the non-negativity restriction of all relaxed variables. The proposed variation is similar to the original Ritter (1967) Relaxation Algorithm, except that it does not allow swapping of decision variables between the relaxed and non-relaxed sets. Hence, the modified algorithm relies solely on the inclusion of constraints into the Reduced Problem in order to enforce the non-negativity (and, now, the integer restriction) of the relaxed variables. By preventing the swapping operation from taking place, the mod-

ified algorithm requires fewer matrix manipulations and fewer numerical comparisons, and, thus, has improved numerical stability. Perhaps even more important, given the dichotomous nature of the added constraints, it is shown that the algorithm can be combined with the branch and bound algorithm in order to solve problems with integer or mixed-integer subproblems.

However, considerable research is needed before such an algorithm is able to systematically outperform commercial optimization packages such as IBM ILOG CPLEX. First, the same concerns pertaining to the branch and bound algorithm, namely branching strategy and choice of branching variable, also apply to the proposed algorithm. In addition, since a considerable amount of computational effort is dedicated to the generation of constraints enforcing the branching and the upper and lower bound restrictions, and since these are incorporated sequentially into the Reduced Problem, it may be possible to further improve the efficiency of the algorithm by the concomitant use of Gomory cuts. Alternatively, since the branch and bound computations are so demanding, it may be advantageous to generate and add all boundary restrictions to the Reduced Problem a priori, thus ensuring that each subproblem generated enforces, at least, the LP relaxation of the original problem. Finally, since the algorithm is designed for bordered-structure problems, it may be applicable to seemingly structureless problems, but which can be mapped into an structured problem through rearrangement of their constraint coefficients. See Aykanat et al. (2004), Duff and Scott (2005), Ferris and Horn (1998), Fragnière et al. (2000), Pinar and Aykanat (1996), and Pinar et al. (1996) for different algorithms to reorganize the coefficients of the constraint

matrix into exploitable structures.

3.5 Appendix

This section presents the mathematical proofs of the propositions introduced in Section 3.2.

Proof of Proposition 3.1. Let $\mathcal{X}_O$ be the original problem feasible region and let $\mathcal{X}_R^1$ be the Reduced Problem feasible region generated on the first iteration of the Modified Ritter (1967) Relaxation Algorithm, that is, let:

$$\mathcal{X}_O \equiv \left\{ x \in (\mathbb{R}_+ \times \mathbb{Z}_+) \,\middle|\, B_j x_j + D_j x_{p+1} = b_j,\ \forall\ j = 1, \cdots, p,\ \sum_{j=1}^{p} Aeq_j\, x_j + Deq\, x_{p+1} = beq, \right.$$
$$\left. \sum_{j=1}^{p} Aineq_j\, x_j + Dineq\, x_{p+1} \leq bineq \right\}$$

$$\mathcal{X}_R^1 \equiv \left\{ x_{j2},\ \forall\ j = 1, \cdots, p,\ x_{p+1} \in (\mathbb{R}_+ \times \mathbb{Z}_+) \,\middle|\, \sum_{j=1}^{p} F_j x_{j2} + F_{p+1} x_{p+1} = f, \right.$$
$$\left. \sum_{j=1}^{p} G_j x_{j2} + G_{p+1} x_{p+1} \leq g,\ \sum_{j=1}^{p} \sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M \right\},$$

Where the notation $(\mathbb{R}_+ \times \mathbb{Z}_+)$ indicates the non-negative, mixed-integer nature of the problem. For simplicity, the slack variables (if any) used to place constraint $B_j x_j + D_j x_{p+1} = b_j$ in standard form are grouped with $x_j,\ j = 1, \cdots, p$. Let M be a sufficiently large, positive constant preventing $\mathcal{X}_R^1$ from being unbounded. Now, using the parameter substitutions (3.13) and some rearrangement of the terms, constraint $\sum_{j=1}^{p} F_j x_{j2} + F_{p+1} x_{p+1} = f$ can be rewritten as $\sum_{j=1}^{p} \left(Aeq_{j1} \left(\overline{b}_j - \overline{B}_j x_{j2} - \overline{D}_j x_{p+1} \right) + Aeq_{j2}\, x_{j2} \right) + Deq\, x_{p+1} = beq$ and constraint $\sum_{j=1}^{p} G_j x_{j2} + G_{p+1} x_{p+1} \leq g$ can be rewritten as

$\sum_{j=1}^{p}\left(Aineq_{j1}\left(\overline{b}_j - \overline{B}_j x_{j2} - \overline{D}_j x_{p+1}\right) + Aineq_{j2}\, x_{j2}\right) + Dineq\, x_{p+1} \leq bineq$. After employing the variable substitution (3.11) and realizing that x_{j1} and x_{j2}, are mutually exclusive partitions of the j^{th} subproblem decision variables, it follows that:

$$\mathcal{X}_O \cap \left\{ x_{j2},\ \forall\ j = 1, \cdots, p,\ x_{p+1} \in \mathbb{R} \left| \sum_{j=1}^{p} \sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M \right. \right\} =$$

$$\mathcal{X}_R^1 \cap \{x_{j1} \in (\mathbb{R}_+ \times \mathbb{Z}_+),\ \forall\ j = 1, \cdots, p\}$$

Now, since the original problem feasible region is bounded, any partial summation of its decision variables is finite. Consequently, for M sufficiently large, the constraint $\sum_{j=1}^{p}\sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M$ is redundant to the original problem, that is: $\mathcal{X}_O \cap \{x_{j2},\ \forall\ j = 1, \cdots, p,\ x_{p+1} \in \mathbb{R} | \sum_{j=1}^{p}\sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M\} = \mathcal{X}_O$. As a result, $\mathcal{X}_O = \mathcal{X}_R^1 \cap \{x_{j1} \in (\mathbb{R}_+ \times \mathbb{Z}_+),\ \forall\ j = 1, \cdots, p\} \Rightarrow \mathcal{X}_O \subseteq \mathcal{X}_R^1$. ∎

Proof of Corollary 3.1. Let $\mathcal{X}_O$ and $\mathcal{X}_R^1$ be defined as in the proof of Proposition 3.1, and let $N_\varepsilon(0)$ be a sufficiently small ε-neighborhood around the origin. For notational simplicity, let $Q \equiv \{x_{j2},\ \forall\ j = 1, \cdots, p,\ x_{p+1} \in \mathbb{R} | \sum_{j=1}^{p}\sum_{i=1}^{|S_{j2}|} x_{j2}^i + \sum_{i=1}^{n_{p+1}} x_{p+1}^i \leq M\}$ and let $R \equiv \{x_{j1} \in (\mathbb{R}_+ \times \mathbb{Z}_+),\ \forall\ j = 1, \cdots, p\}$, that is, let Q be the polyhedral set preventing $\mathcal{X}_R^1$ from being unbounded and let R be the polyhedral set corresponding to the non-negativity and integer restrictions, if any, of the relaxed decision variables. Now, since $\mathcal{X}_O$ is unbounded and since $\mathcal{X}_R^1$ is bounded, in general, $\mathcal{X}_O \not\subseteq \mathcal{X}_R^1$. However, since $\mathcal{X}_R^1$ is a relaxation of $\mathcal{X}_O$, it remains true that $(\mathcal{X}_O \cap Q) = (\mathcal{X}_R^1 \cap R)$, and, thus, that $(\mathcal{X}_O \cap N_\varepsilon(0) \cap Q) = (\mathcal{X}_R^1 \cap N_\varepsilon(0) \cap R)$. However, for ε sufficiently small, $N_\varepsilon(0) = (N_\varepsilon(0) \cap Q)$ and $\mathcal{X}_R^1 = (\mathcal{X}_R^1 \cap N_\varepsilon(0))$. Therefore:

$$(\mathcal{X}_O \cap N_\varepsilon(0) \cap Q) = \left(\mathcal{X}_R^1 \cap N_\varepsilon(0) \cap R\right)$$

$\Rightarrow (\mathcal{X}_O \cap N_\varepsilon(0)) = \left(\mathcal{X}_R^1 \cap R\right)$

$\Rightarrow (\mathcal{X}_O \cap N_\varepsilon(0)) \subseteq \mathcal{X}_R^1$. ∎

Proof of Corollary 3.2. Let $\mathcal{X}_O$ and $\mathcal{X}_R^1$ be defined as in the proof of Proposition 3.1, let $N_\varepsilon(0)$ be a sufficiently small ε-neighborhood around the origin, and let y_O be the optimum solution to the original problem. If $\mathcal{X}_O$ is bounded, then, by Proposition 3.1, $\mathcal{X}_O \subseteq \mathcal{X}_R^1$. Consequently, $y_O \in \mathcal{X}_O \Rightarrow y_O \in \mathcal{X}_R^1$. Conversely, if $\mathcal{X}_O$ is unbounded, then, by Corollary 3.1, $(\mathcal{X}_O \cap N_\varepsilon(0)) \subseteq \mathcal{X}_R^1$. Since y_O is finite, $y_O \in N_\varepsilon(0)$. Together, $y_O \in \mathcal{X}_O,\ y_O \in N_\varepsilon(0) \Rightarrow y_O \in (\mathcal{X}_O \cap N_\varepsilon(0)) \Rightarrow y_O \in \mathcal{X}_R^1$. ∎

Proof of Corollary 3.3. Let $\mathcal{X}_O$ and $\mathcal{X}_R^1$ be defined as in the proof of Proposition 3.1, and let y_O and y_R^1 denote, respectively, the optimum solution to the original and to the initial Reduced Problem. First, note that (3.7) ensures that $\mathcal{X}_R^1$ is bounded. Now, from Corollary 3.2, $y_O \in \mathcal{X}_O \Rightarrow y_O \in \mathcal{X}_R^1 \Rightarrow \mathcal{X}_R^1 \neq \emptyset$. Since the Reduced Problem is non-empty and bounded, it must have a finite optimum solution y_R^1. Observe that no restrictions on the uniqueness of y_O and y_R^1 are imposed. In addition, the converse relationship may not be true, that is, $y_R^1 \in \mathcal{X}_R^1 \not\Rightarrow y_R^1 \in \mathcal{X}_O$, since, in general, $\mathcal{X}_R^1 \not\subseteq \mathcal{X}_O$. Hence, it is possible for the initial Reduced Problem to have an optimum solution, but for such solution to violate the non-negativity and/or integer restriction of some of the relaxed variables. ∎

Proof of Proposition 3.2. Let $\mathcal{X}_O$ and y_O be, respectively, the feasible region of the original problem and its optimum solution; and let $\mathcal{X}_R^k$ and y_R^k be, respectively, the feasible region of the Reduced Problem solved on the k^{th} iteration of the Modified Ritter (1967)

Relaxation Algorithm and its optimum solution. Here, it is shown that $y_R^k \in \mathcal{X}_O \Leftrightarrow y_R^k$ incumbent:

($\Longrightarrow$) If $y_R^k \in \mathcal{X}_O$, then y_R^k satisfies all constraints of the original problem, including those that have been relaxed in the Reduced Problem. As per Step 13 of the Modified Ritter (1967) Relaxation Algorithm, the Reduced Problem is fathomed, and y_R^k is saved as an incumbent solution.

($\Longleftarrow$) Here, the contrapositive is proven, that is, it is shown that an optimum solution to the Reduced Problem that is not feasible to the original problem cannot be an incumbent solution to the Modified Ritter (1967) Relaxation Algorithm. Suppose that $y_R^k \notin \mathcal{X}_O$. Since $y_R^k \in \mathcal{X}_R^k$, it follows that y_R^k satisfies all constraints of the current Reduced Problem, and, thus, that it satisfies all constraints in the original problem except, perhaps, for some integer and/or boundary restrictions which have been relaxed. In fact, since $y_R^k \in \mathcal{X}_R^k$, but $y_R^k \notin \mathcal{X}_O$, at least one of its components, say the i^{th} element of the j^{th} subproblem, must violate some of these restrictions (e.g., $x_{j1}^i < 0$ or $x_{j1}^i \notin \mathbb{Z}$). Observe that the violated element cannot be a decision variable in the non-relaxed partition (i.e., $x_{j2},\ j = 1, \cdots, p$) or in the set of complicating variables (i.e., x_{p+1}), as, otherwise, $y_R^k \notin \mathcal{X}_R^k$. Now, if a boundary restriction is violated, for instance if $x_{j1}^i < 0$, then constraint $\{\overline{B}_j\}_i\, x_{j2} + \{\overline{D}_j\}_i\, x_{p+1} \leq \{\overline{b}_j\}_i$ is added to the Reduced Problem and such is reoptimized as per Steps 10 and 12 of the Modified Ritter (1967) Relaxation Algorithm. Conversely, if an integer restriction is violated, for instance if $x_{j1}^i = a$, where a is some fractional value, then the algorithm removes

the current Reduced Problem from set NL and adds two new Reduced Problems, each one with a mutually exclusive constraint: $-\{\overline{B}_j\}_i\, x_{j2} - \{\overline{D}_j\}_i\, x_{p+1} \leq \lfloor a \rfloor - \{\bar{b}_j\}_i$ and $\{\overline{B}_j\}_i\, x_{j2} + \{\overline{D}_j\}_i\, x_{p+1} \leq -\lceil a \rceil + \{\bar{b}_j\}_i$ as per Step 14 of the Modified Ritter (1967) Relaxation Algorithm. Whichever the case, such violations trigger the inclusion of additional constraints that eliminate y_R^k from the feasible region of the new Reduced Problem(s). Therefore, y_R^k cannot be an incumbent solution to the algorithm. ∎

Proof of Proposition 3.3. Let $\mathcal{X}_O$, $\mathcal{X}_R^k$ and y_O be defined as in the proof of Proposition 3.2. First, from Corollary 3.2, if y_O exists, then $y_O \in \mathcal{X}_R^1$. Therefore, at least on the first iteration of the Modified Ritter (1967) Relaxation Algorithm, the feasible region of the Reduced Problem contains the optimum solution to the original problem. Since the algorithm performs a dichotomous, exhaustive search following a scheme similar to the branch and bound algorithm, on its k^{th} iteration, either the optimum solution to the original problem has already been identified and saved as an incumbent solution or at least one of the Reduced Problems maintain the optimum solution to the original problem within its feasible region. Since the algorithm terminates in a finite number of iterations, as it will be shown in the proof of Proposition 3.4, it will eventually identify the optimum solution to the original problem as an incumbent solution. Finally, since, by Proposition 3.2, all incumbent solutions are feasible to the original problem, since the transformations used to generate the Reduced Problem do not alter the objective function of the original problem (other than reducing its dimensionality), and since the algorithm outputs the one with the best objective function value, it follows that the Modified Ritter (1967) Relaxation Algorithm will always identify

the optimum solution to the original problem if one exists. ∎

Proof of Proposition 3.4. As shown in the proof of Proposition 3.3, a new iteration of the modified algorithm takes place whenever one of the relaxed variables, say the i^{th} element of the j^{th} subproblem, violates a boundary or integer restriction (e.g., $x_{j1}^i < 0$ or $x_{j1}^i \notin \mathbb{Z}$). Since such violations are eliminated via the inclusion of extra constraints in the Reduced Problem, since such constraints are only removed when they become redundant, and since there is a finite number of such possible constraints (as there are a finite number of elements in set $x_{j1},\ j = 1, \cdots, p$; as the algorithm employs a dichotomous search strategy; and as the feasible region of the Reduced Problem is bounded), it follows that the algorithm must terminate in a finite number of iterations. ∎

Chapter 4

The Dynamic Bus Evacuation Problem with Stochastic Customers and Demand

The Dynamic Bus Evacuation Problem with Stochastic Customers and Demand

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(ABSTRACT)

This research introduces a two-stage stochastic formulation and an iterative (dynamic) algorithm for the Bus Evacuation Problem, a unique Vehicle Routing Problem motivated both by its humanitarian significance and by the challenges of planning and implementing transit-based, regional evacuations. In this problem, a fleet of capacitated buses must transport all transit-dependent individuals from a set of pick-up locations to a depot/shelter with minimum total travel time. The problem assumes that a set of possible demand scenarios is available, and that the exact number of evacuees on each location, if any, becomes known once a bus visits it for the first time. The effect of exploratory visits (sampling) and symmetry is explored, and the resulting insights used to develop a deterministic formulation for the problem. While, at least in a dynamic context, these formulations are equivalent, the deterministic variation has considerably improved tractability, preserves the desired sampling behavior, and employs constraints that break some of the problem symmetries in favor of arc traversals leading to reduced total travel times. Although this research focuses on evacuation planning/implementation, the concepts herein developed are readily applicable to other transportation settings such as supply deployment and search and rescue operations.

Keywords: Evacuation Planning; Dynamic Vehicle Routing Problem; Stochastic

Programming; Uncertain Demand

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4.1 Introduction

The Bus Evacuation Problem (BEP) is a variant of the Vehicle Routing Problem (VRP), where a fleet of capacitated buses, initially parked at a depot, must transport transit-dependent individuals from a set of predetermined pick-up locations to one or more shelters. The problem arises in regional evacuation settings, where some of the residents either lack reliable means of personal transportation, as in the evacuation of New Orleans preceding the landfall of Hurricane Katrina, or lack enough time for alternative evacuation methods, such as in the evacuation of urban clusters facing bomb threats.

This research presents and analyzes a two-stage stochastic formulation for the BEP in which the number of evacuees on each pick-up location, if any, is revealed dynamically during the course of the evacuation. The problem forces every evacuee to be transported to a depot/shelter, while minimizing the expected total travel time of a homogeneous fleet of buses. The first stage of the proposed stochastic formulation identifies the subsequent arc traversal for each vehicle, whereas the second stage identifies their remaining arc traversals provided a set of equal-probability scenarios. In this research, the term scenario refers to a set of potential demand realizations, one value for each pick-up location. Consequently, the

problem does not require demands to be independent of each other, an important distinction from most of the stochastic VRP literature. In addition, it is assumed that all evacuees are waiting at these locations at the onset of the evacuation. These, however, are not explicitly modeled. Instead, demands are treated as a percentage (possibly multiples) of the vehicle capacity. The problem assumes that the exact number of evacuees (or, rather, full bus loads) waiting on each pick-up location is only revealed after a vehicle visits it for the first time. A iterative algorithm, which reoptimizes the problem as new information becomes available, is presented. Finally, structural properties of the problem, namely sampling and symmetry, are identified and used to generate an alternative, deterministic formulation, able to attain better solutions under considerably smaller computation times. Sampling refers to exploratory visits to pick-up locations aiming to ascertain its demand, and, thus, to eliminate future round trips between said location and the depot. Symmetry refers to the problem default inability to distinguish between seemingly identical arc traversals, but which may result in distinct objective function values depending on the realized demands at the destination nodes.

The modeling and optimization literature on transit-based evacuation planning/implementation is scarce and fairly recent. In fact, with the exception of Goerigk and Grün (2012), Kulshrestha et al. (2012), and Song et al. (2009), no related literature accounting for the stochastic nature of evacuee demand has been located. Goerigk and Grün (2012) propose a deterministic and a stochastic formulation with no recourse for the BEP, both seeking to minimize the longest travel time across vehicles (a proxy for the evacuation completion

time). Similar to the proposed formulation, the authors do not explicitly model evacuees, treating demands as a function of the vehicle capacity. Vehicles, initially parked at a depot, travel to the various pick-up locations, and, from there, back and forth between pick-up locations and shelters until each location has been visited enough times to accommodate all its evacuees. Their stochastic formulation seeks to balance the number of buses that depart immediately with those that remain stationed at the depot until the number of evacuees on each pick-up location becomes known. Kulshrestha et al. (2012) propose and analyze a stochastic, Location Routing Problem (LRP) formulation for the BEP that minimizes the expected total travel time of the fleet. In their formulation, vehicles are initially assigned one of the possible pick-up locations, and, from there, they perform round trips to a set of capacitated shelters until either all demand is satisfied or a limiting travel time is reached. Although their formulation is conceptually similar to the one proposed by Goerigk and Grün (2012), it differs in one important aspect: vehicles are not allowed to move to a different pick-up location once they finish rescuing all evacuees from their initial assignment. In addition, while the problem assumes that a set of scenarios is provided, it limits the number of pick-up locations which demands can differ from the expected value. Finally, Song et al. (2009) propose a stochastic, LRP formulation for the BEP that incorporates a reliability constraint (based on the logistic distribution) to account for the uncertainty on the number of transit-dependent evacuees on each pick-up location. Their formulation enforces strict pick-up time windows, but assumes that all evacuees are at the pick-up locations at the onset of the evacuation. In addition, a maximum travel time per vehicle is enforced as a

proxy for the overall completion time of the evacuation. Their formulation simultaneously seeks the set of uncapacitated shelters that must open and the corresponding routes for a fleet of capacitated vehicles that minimize their total travel time.

The stochastic formulation presented in this research shares several commonalities with the ones proposed by Kulshrestha et al. (2012) and, in particular, by Goerigk and Grün (2012). However, there are some important differences. First, both studies assume that vehicles can only perform round trips between pick-up locations and shelters, whereas the proposed formulation assumes a complete network. Second, the formulation proposed by Goerigk and Grün (2012) assumes that the amount of time that vehicles must wait at the depot for additional information is a known problem parameter, whereas, in the proposed formulation, the realized demand on each pick-up location only becomes available once a bus visits it for the first time. Third, both studies assume that vehicles cannot be rerouted, whereas the proposed solution methodology reoptimizes the problem as additional information becomes available. Finally, both studies assume independence of the possible demand realizations, whereas the proposed formulation does not.

In summary, the proposed formulation can be classified as a two-stage stochastic Dynamic Vehicle Routing Problem (D-VRP) with stochastic customers and demands, that is a D-VRP which relies on the known stochastic nature of demands to identify improved routes for the fleet and which considers the possibility that some of these demands equal zero. Although the assumption of zero-demand at any pick-up location may not seem realistic for most real-world evacuation settings, it is well suited for modeling LRP variations of the BEP, such as

the ones proposed by Kulshrestha et al. (2012) and Song et al. (2009).

The origins of the D-VRP can be traced to Wilson and Colvin (1977), who propose a variation of the Dial-A-Ride Problem in which customer requests for trips arrive dynamically (i.e., after vehicles begun their tour). According to Pillac et al. (2013), unlike traditional VRPs, the D-VRP is based on the premise that either the problem parameters change or their uncertainty is resolved after the originally planned tours start to be implemented. Consequently, the D-VRP allows factors initially unknown to the planner to be incorporated into the final solution, thus potentially reducing operational costs, improving customer service, and even reducing environmental impact (Pillac et al., 2013). Since the originally planned routes must be reoptimized to account for the new information, the D-VRP consists of a sequence of static VRPs (see Larsen et al., 2007; Pillac et al., 2013, for a comprehensive review of the D-VRP literature). Among its many variations, the proposed formulation more closely resembles the stochastic D-VRP. Such variant is similar to its deterministic counterpart in the sense that some of the parameters (in this case, the number of full bus loads needed to accommodate every evacuee waiting at a pick-up location) are unknown a priori and reveled dynamically during the implementation of the routes (in this case, as vehicles visit the pick-up locations for the first time). However, contrary to its deterministic counterpart, the problem exploits the known stochastic nature of demands to determine the arc traversals more likely to result in better objective function values.

Although the D-VRP literature is extensive, its application to humanitarian logistics is somewhat scarce. In fact, no literature specifically focusing on transit-based evacuation planning

was located. Brotcorne et al. (2003) and Gendreau et al. (2001) propose a D-VRP formulation for ambulance relocation during disaster relief operations, whereas Haghani and Yang (2007) and Yi and Özdamar (2007) propose a D-VRP formulation for supply deployment during such operations. Finally, Fajardo and Waller (2012) introduce and analyze a single-vehicle, uncapacitated Stochastic Dynamic Traveling Salesman Problem (D-TSP) formulation for rescuing disaster victims whose number and position is unknown a priori. Their formulation shares several commonalities with the proposed BEP variation. First, their formulation assumes that a set of pick-up locations (i.e., a list of potential victim positions) and a set of potential demands within each location are provided. Second, the realized demand on each location (i.e., whether any victims are present or not) only becomes known upon visual observation, that is, once vehicles visit the location for the first time. Third, the authors also propose a two-stage stochastic formulation, where, on the first stage, the subsequent arc traversal for the vehicle is identified, and, on the second stage, the remaining arc traversals leading it back to the depot under each possible demand realization are identified. Finally, their formulation also seeks to minimize the expected total travel time, as a function of the demand probabilities. Despite these similarities, their formulation differs from the proposed BEP variation in some important aspects. First, the authors modeled the problem as a D-TSP. Consequently, efficiencies from using multiple vehicles and from allowing vehicles to return to the depot to unload evacuees (or disaster victims) multiple times are ignored. Second, their formulation focuses on search-and-rescue operations. Consequently, it ignores vehicle capacity restrictions and the possibility of split deliveries, important considerations

on a evacuation setting. Finally, and more importantly, while the authors mention the potential benefits of sampling and symmetry, their formulation does not explicitly integrate these features in order to identify improved routes for the fleet.

In summary, the contributions of this paper are: (i) introduction of a dynamic, two-stage stochastic formulation that extends the BEP to the case of uncertain demand; (ii) development of an algorithm that iteratively solves the problem as additional information becomes available; (iii) examination of the effect of sampling (i.e., the use of exploratory visits to determine the demand on each pick-up location) and symmetry over the expected total travel time of the fleet; and (iv) development of an alternative, deterministic formulation that incorporates these insights to attain better routes for the fleet while considerably improving the problem tractability. Observe that the concepts herein developed are readily applicable to any other mass transit vehicle, and, possibly, to alternative VRP settings such as supply deployment and search and rescue operations.

This paper is organized as follows: Section 4.2 presents the proposed formulation and an iterative algorithm that reoptimizes the problem as additional information becomes available; Section 4.3 explores the effect of sampling and symmetry over the expected total travel time of the fleet and uses these insights to develop an improved formulation for the problem; and, finally, Section 4.4 summarizes the main findings of this paper and lists the future work needed on the area.

4.2 Modeling Framework

This section presents the modeling notation, a two-stage stochastic D-VRP formulation for the BEP with stochastic customers and demand, an upper bound on the maximum number of arc traversals per vehicle (a support parameter used within the proposed formulation), and an iterative algorithm that reoptimizes the problem as additional information becomes available.

Consider the complete, directed network (N, A), where N and A represent the set of nodes and arcs, respectively. The set N consists of set P, the set of pick-up locations (each one serving a neighborhood requiring evacuation services), and node 0, the depot, which also functions as a shelter. Each arc $(i, j) \in A$ has an associated arc traversal time, t_{ij}, satisfying the triangle inequality, that is, $t_{ij} \leq t_{ik} + t_{kj},\ \forall\ (i, j),\ (i, k)$, and $(k, j) \in A$. For simplicity, let $t_{ij} = t_{ji},\ \forall\ i, j \in N$. In addition, let S be the set of possible scenarios, let P^* be the set of pick-up locations with uncertain demand (i.e., not yet visited by any vehicle), let R^* be the set of previously traversed arcs, and let C^* be the set of previously performed pick-ups. Finally, let D_j^s be the total demand of node j under scenario s, let V be the number of available buses, and let T be the maximum number of arc traversals per vehicle. The decision variables, objective function, and constraint set for the problem are as follows:

4.2.1 Two-Stage Stochastic Formulation

Table 4.1: Decision variables for the two-stage stochastic formulation.

Variable	Description
x_{ij}^{mts}	binary variable that equals 1 if arc (i,j) is the t^{th} arc traversed by vehicle m under scenario s, else 0, $\forall\ (i,j) \in A$, $m = 1, \cdots, V$, $t = 1, \cdots, T$, $s \in S$;
y_j^{mts}	percent capacity of vehicle m consumed at pick-up node j following its t^{th} arc traversal under scenario s, $\forall\ j \in P$, $m = 1, \cdots, V$, $t = 1, \cdots, T$, $s \in S$. Let $0 \leq y_j^{mts} \leq 1,\ \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T,\ s \in S$.

Proposed Two-Stage Stochastic Formulation:

$$\text{Minimize} \quad \frac{1}{|S|} \sum_{(i,j)\in A} \sum_{m=1}^{V} \sum_{t=1}^{T} \sum_{s\in S} t_{ij} x_{ij}^{mts} \tag{4.1}$$

Subject to

$$x_{ij}^{m1s} = 0, \quad \forall\ i \in P,\ j : (i,j) \in A,\ m = 1, \cdots, V,\ s \in S \tag{4.2}$$

$$\sum_{i:(i,j)\in A} x_{ij}^{mts} = \sum_{k:(j,k)\in A} x_{jk}^{m(t+1)s}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T-1,\ s \in S \tag{4.3}$$

$$\sum_{i:(i,0)\in A} x_{i0}^{mts} \geq \sum_{k:(0,k)\in A} x_{0k}^{m(t+1)s}, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T-1,\ s \in S \tag{4.4}$$

$$x_{ij}^{mTs} = 0, \quad \forall\ j \in P,\ i : (i,j) \in A,\ m = 1, \cdots, V,\ s \in S \tag{4.5}$$

$$\sum_{(i,j)\in A} x_{ij}^{mts} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T,\ s \in S \tag{4.6}$$

$$y_j^{mts} \leq \sum_{i:(i,j)\in A} x_{ij}^{mts}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots T,\ s \in S \tag{4.7}$$

$$\sum_{j\in P}\sum_{l=1}^{t} y_j^{mls} - \sum_{i:(i,0)\in A}\sum_{l=1}^{t} x_{i0}^{mls} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T,\ s \in S \tag{4.8}$$

$$\sum_{m=1}^{V}\sum_{t=1}^{T} y_j^{mts} = D_j^s, \quad \forall\ j \in P,\ s \in S \tag{4.9}$$

$$x_{ij}^{mts} = x_{ij}^{mtu}, \quad \forall\ (i,j) \in A,\ m = 1, \cdots, V,\ t = t^*,\ s \in S,\ u \in S,\ s \neq u \tag{4.10}$$

$$\sum_{i:(i,j)\in A}\sum_{m=1}^{V}\sum_{t=1}^{T} x_{ij}^{mts} \geq 1, \quad \forall\ j \in P,\ s \in S \tag{4.11}$$

$$x_{ij}^{mts} = 1, \quad \forall\ \{(i,j), m, t\} \in R^*,\ s \in S \tag{4.12}$$

$$y_j^{mts} = g, \quad \forall\ \{j, m, t, g\} \in C^*,\ s \in S. \tag{4.13}$$

Objective Function (4.1) minimizes the expected total travel time of the fleet assuming that all scenarios have equally likely probability of realization. Constraints (4.2)-(4.6) prevent the construction of invalid routes, including the elimination of subtours. Constraint (4.2) prevents buses from starting their tour from a pick-up location. If a vehicle's t^{th} arc traversal ends at a pick-up location, then (4.3) forces the vehicle's $(t+1)^{th}$ arc traversal to start from such location. Constraint (4.4) is similar to (4.3), but it allows a vehicle's final arc traversal to end at the depot. Constraint (4.5) forces the vehicle's T^{th} arc traversal, if used, to end at the depot, and (4.6) limits vehicles to one arc traversal a time. Constraints (4.7)-(4.9) enforce the vehicle capacity restrictions. Constraint (4.7) prevents evacuees from boarding a transit vehicle unless such is visiting their respective pick-up location; (4.8) enforces the bus capacity restriction, allowing for multiple drop-offs at the depot; and (4.9) ensures that the demand

corresponding to each scenario is met. Although the y-variables may take fractional values, it is often disadvantageous for a vehicle to leave a pick-up location with a partial load while some of the evacuees remain behind. Aiming to avoid data inconsistencies in the accompanying algorithm, here, it is assumed that the remaining capacity of the vehicle is exhausted before its departure from the respective pick-up location. Constraints (4.10)-(4.11) enforce the first and second stage decisions for this stochastic formulation. Constraint (4.10) ensures that the $(t^*)^{th}$ arc traversal of each vehicle is common across scenarios, and (4.11) forces every pick-up location to be visited at least once, even under scenarios with zero-demand at the respective pick-up location. Such Constraint is needed, for the realized demands remain uncertain until the pick-up locations are visited for the first time. Constraints (4.12)-(4.13) ensure that previously performed routes and pick-ups remain unchanged as new information becomes available. Constraint (4.12) fixes all previously traversed arcs for the fleet, and, thus, fixes their current position in the network, whereas (4.13) fixes the percent capacity of each vehicle consumed on all previously performed pick-ups. Since these constraints are only defined for $\{(i,j),m,t \ : \ (i,j) \in A, \ m = 1,\cdots,V, \ t = 1,\cdots,T\} \in R^*$ and $\{j,m,t,g \ : \ j \in P, \ m = 1,\cdots,V, \ t = 1,\cdots,T, \ g \in \mathbb{R}_+\} \in C^*$ and since these sets are modified on each iteration of the proposed algorithm, only previously traversed arcs and previously performed pick-ups are restricted. Initially, let $R^* = \emptyset$, $C^* = \emptyset$, $P^* = P$, and $t^* = 1$. Finally, the logical binary restriction on the x-variables and the non-negativity restriction on the y-variables are included.

While T, the maximum number of arc traversals per vehicle, is not based on a policy decision,

it may affect the model performance and results, as excessive values increase its complexity, while insufficient values may either prevent its feasibility or yield suboptimal solutions. Proposition 4.1 presents an upper bound on this parameter:

Proposition 4.1 *The upper bound on the maximum number of arc traversals per vehicle for the two-stage stochastic formulation, T_{UB}, is calculated as follows:*

$$T_{UB} = |P^*| + 1 + 2 \max_{s \in S} \left\lceil \frac{1}{V} \max \left(\sum_{j \in P} \lceil D_j^s \rceil - 1, 0 \right) \right\rceil . \tag{4.14}$$

Proof. This bound corresponds to the number of arc traversals needed both to sample the demand of nodes in P^*, and to transport evacuees to the depot under the worst-case (highest demand) scenario realization. The term $|P^*| + 1$ corresponds to the number of arc traversals needed for a traveling salesman tour through the pick-up locations with uncertain demand, whereas the remainder portion of the equation corresponds to the average number of arc traversals per vehicle needed to rescue all evacuees via round-trips to the depot. It is assumed that one pick-up is performed during the execution of the traveling salesman tour, unless, of course, the demands for all those pick-up locations equal zero. ∎

Next, an iterative algorithm, which reoptimizes the problem as additional information becomes available, is presented.

4.2.2 Iterative Algorithm for the Stochastic Formulation

Figure 4.1 presents a flowchart of the proposed algorithm.

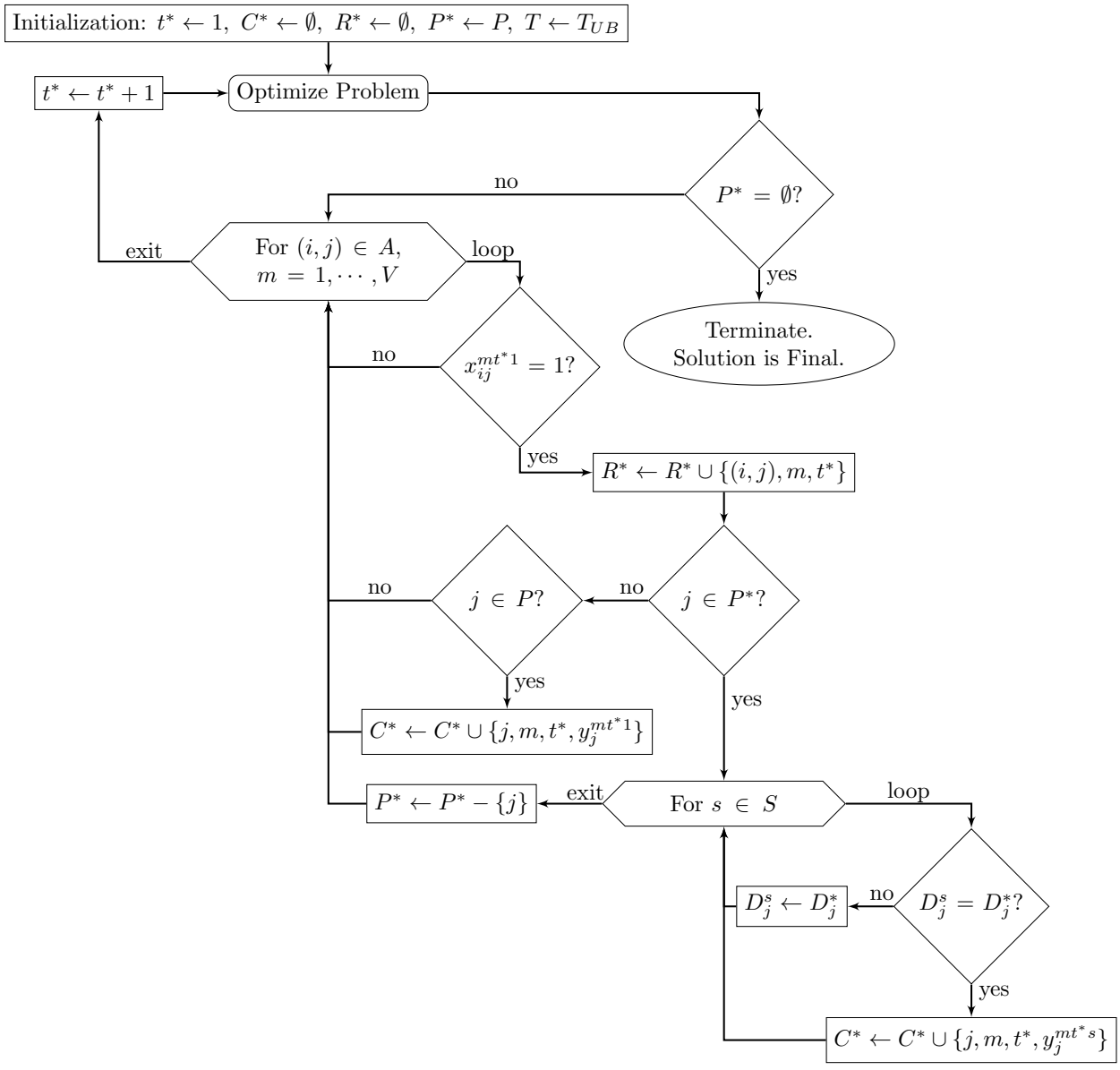

Figure 4.1: Flowchart of the iterative algorithm for the stochastic formulation.

The algorithm proceeds as follows. First, the parameters t^*, C^*, R^*, and P^* are initialized. Recall that t^* corresponds to the first stage decision of the stochastic formulation, that is, to the subsequent arc traversal of each vehicle; that C^* corresponds to the set of previously performed pick-ups; and that R^* corresponds to the set of previously traversed arcs. The parameter P^* is not part of the two-stage stochastic formulation, but corresponds to the set of pick-up locations that have not yet been visited, and, thus, that have uncertain demand. Since the demands on each pick-up location are updated as additional information becomes available, the upper bound on the maximum number of arc traversals per vehicle, T_{UB}, decreases monotonically with t^* (except, of course, when the worst-case scenario is realized, in which case T_{UB} remains unchanged). Consequently, the initial T_{UB}-parameter is sufficiently large for all iterations of the algorithm, and does not need to be recomputed in order to prevent conflicts with previously added elements in C^* and R^*. Now, on each iteration of the algorithm, the problem is reoptimized; the $(t^*)^{th}$ arc traversal of each vehicle, if any, is saved into set R^*; and its percent capacity consumed at the destination node (if such is a pick-up location) is saved into set C^*. If the destination node is a pick-up location that has not yet been visited, then such is removed from set P^* and all its scenarios updated with its realized demand, D_j^*. Unfortunately, unrealized scenarios cannot be removed from the problem parameter set in order to maintain the integrity of the decision variables indexes. Although somewhat inefficient, by updating the value of all non-realized demands to D_j^*, the algorithm ensures that future pick-ups at this pick-up location meet its realized demand regardless of the scenario realizations at alternative pick-up locations. If all pick-up locations

have been visited (i.e., if $P^* = \emptyset$), then, after one last optimization, the solution attained is final and the algorithm terminates. Otherwise, t^* is incremented by one and a new iteration is performed. Example 4.1 shows the implementation of the algorithm for an example network with three pick-up locations and three scenarios.

Example 4.1 *Consider the example network presented in Figure 4.2. Observe that the network is a complete digraph with symmetrical arc traversal times that satisfy the triangle inequality. For simplicity, let* $V = 1$ *bus. Suppose that there are three possible scenarios: scenario 1 has* $D_1^1 = 1,\ D_2^1 = 1$, *and* $D_3^1 = 1$*; scenario 2 has* $D_1^2 = 0,\ D_2^2 = 1$, *and* $D_3^2 = 2$*; and scenario 3 has* $D_1^3 = 0,\ D_2^3 = 2$, *and* $D_3^3 = 1$.

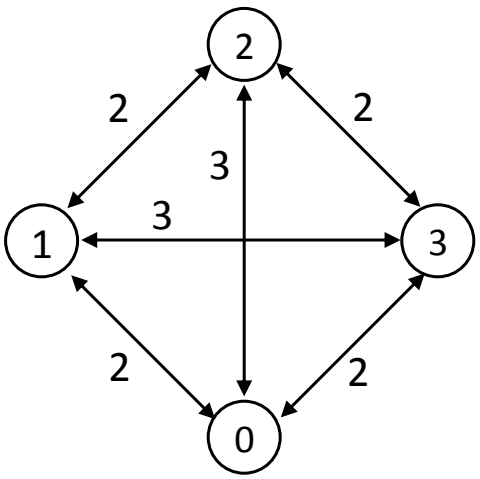

Figure 4.2: An example network for the algorithm implementation.

Naturally, the optimal solution for this problem is trivial under the assumption of perfect information, that is, under the assumption that the exact number of pick-ups required on each pick-up location is a known problem parameter. For instance, suppose that scenario 2 is realized, that is, suppose that $D_1^* = 0,\ D_2^* = 1$, *and* $D_3^* = 2$. *Under the assumption of*

perfect information, the optimal solution consists of the route $0 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$*, with a total travel time of 14 time units. In this problem, however, the assumption of perfect information is relaxed. Consequently, on the first iteration of the algorithm,* $t^* = 1$*,* $C^* = \emptyset$*,* $R^* = \emptyset$*,* $P^* = \{1, 2, 3\}$*, and* $T = 8$*. The corresponding optimum solution for each possible scenario after the first iteration of the algorithm is presented in Table 4.2.*

Table 4.2: Solution for each scenario of Example 4.1 on the first iteration of the algorithm.

Scenario	**Demands**			**Optimal Route**	**Obj. Value**
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$	$\mathbf{D_3^s}$		
$s = 1$	*1*	*1*	*1*	$0 \to 1 \to 0 \to 2 \to 0 \to 3 \to 0$	*14*
$s = 2$	*0*	*1*	*2*	$0 \to 1 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$	*15*
$s = 3$	*0*	*2*	*1*	$0 \to 1 \to 2 \to 0 \to 2 \to 0 \to 3 \to 0$	*17*

Observe that every pick-up location is visited at least once, even under scenarios with zero-demand. Since arc $(0, 1)$ *is the first, common arc traversal across scenarios (as enforced by the first stage decision), such is saved into set* R^**, that is,* $R^* \overset{set}{=} \{\{(0, 1), 1, 1\}\}$*. Now, suppose that the realized demand at node 1 equals zero (i.e., suppose that* $D_1^* = 0$*), a demand realization that corresponds either to scenario 2 or 3. Since the demand at this location is now known, the location is removed from set* P^**. In addition, set* C^* *is updated with the y-value corresponding to the realized scenario (i.e., either* y_1^{112} *or* y_1^{113}*, both necessarily equal*

to zero), that is, $C^* \overset{set}{=} \{\{1,1,1,0\}\}$. *Finally,* t^* *is incremented by one,* D_1^1 *is set to 0, and the problem is reoptimized. The corresponding optimum solution for each possible scenario after the second iteration of the algorithm is presented in Table 4.3.*

Table 4.3: Solution for each scenario of Example 4.1 on the second iteration of the algorithm.

Scenario	**Demands**			**Optimal Route**	**Obj. Value**
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$	$\mathbf{D_3^s}$		
$s=1$	*0*	*1*	*1*	$0 \to 1 \to 2 \to 0 \to 3 \to 0$	*11*
$s=2$	*0*	*1*	*2*	$0 \to 1 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$	*15*
$s=3$	*0*	*2*	*1*	$0 \to 1 \to 2 \to 0 \to 2 \to 0 \to 3 \to 0$	*17*

Recall that arc $(0,1)$ *is enforced within* R^*. *Since the bus is empty, it will naturally visit pick-up node 2, as such is the closest pick-up location with known positive demand. Consequently, arc* $(1,2)$, *the second, common arc traversal across scenarios, is saved into set* R^*, *that is,* $R^* \overset{set}{=} \{\{(0,1),1,1\},\{(1,2),1,2\}\}$. *Now, suppose that the realized demand at node 2 equals one full bus load, a demand realization that corresponds either to scenario 1 or 2. Since the demand at this pick-up location is now known, such is removed from set* P^*. *In addition, set* C^* *is updated with the y-value corresponding to the realized scenario (i.e., either* y_2^{121} *or* y_2^{122}, *both necessarily equal to one), that is,* $C^* \overset{set}{=} \{\{1,1,1,0\},\{2,1,2,1\}\}$. *Finally,* t^* *is incremented by one,* D_2^3 *is set to 1, and the problem is reoptimized. The*

corresponding optimum solution for each possible scenario after the third iteration of the algorithm is presented in Table 4.4.

Table 4.4: Solution for each scenario of Example 4.1 on the third iteration of the algorithm.

Scenario	**Demands**			**Optimal Route**	**Obj. Value**
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$	$\mathbf{D_3^s}$		
$s = 1$	*0*	*1*	*1*	$0 \to 1 \to 2 \to 0 \to 3 \to 0$	*11*
$s = 2$	*0*	*1*	*2*	$0 \to 1 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$	*15*
$s = 3$	*0*	*1*	*1*	$0 \to 1 \to 2 \to 0 \to 3 \to 0$	*11*

The algorithm proceeds until pick-up node 3 is visited for the first time, in which case $P^* = \emptyset$. *After another optimization, the algorithm terminates, with final solution* $0 \to 1 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$ *and objective function value of 15 time units. As expected, such objective function value is larger than the total travel time attained under the assumption of perfect information (i.e., 14 time units), but smaller than the expected travel time of the initial stochastic formulation (i.e.,* $(1/3) \times (14 + 15 + 17) = 15.33$ *time units), that is, smaller than the expected travel time before the routes were corrected to account for the realized demands.*

Observe that the proposed dynamic variation of the BEP requires the explicit use of the arc traversal subscript. Suppose, instead, that only the tours (from and to the depot) were indexed. Since each tour requires, at least, two arc traversals, the resulting problem would

have considerably fewer variables and constraints, and, thus, improved tractability. However, such approach may also lead to the construction of subtours within previously performed routes. For instance, consider a network with three pick-up locations (nodes 1, 2, and 3) and that a single bus is available for the evacuation. Now, suppose that, after two iterations of the algorithm, the resulting route (saved into set R^*) is $0 \rightarrow 1 \rightarrow 2$. At this point, the vehicle is stationed at pick-up location 2, and only the demand at pick-up node 3 remains uncertain. Logically, the vehicle may either be routed through arc $(2, 3)$ or arc $(2, 0)$. However, if only the tours were indexed, then the model may choose to modify the previously performed route to accommodate a visit to pick-up node 3 starting from pick-up node 1, that is, it may choose to enforce the route $0 \rightarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 2$, an impossibility.

The next section explores important properties of the proposed BEP variation, and leverages these insights into an improved, deterministic formulation for the problem.

4.3 Problem Analysis

This Section explores important properties of the problem. It is shown that the proposed stochastic formulation relies on exploratory visits (or sampling) to pick-up locations that have not yet been visited in order to reduce the expected total travel time of the fleet, even though, in some instances of the problem, such approach actually results in larger objective function values. In addition, it is shown that, depending on the realized demand at the destination nodes, symmetrical tours may not produce symmetrical results. These insights

are used to develop a deterministic formulation for the problem (and accompanying iterative algorithm) which has improved tractability, preserves the sampling behavior, and employs symmetry breaking constraints to improve the final routes for the fleet.

Consider the example network presented in Figure 4.3. Although theoretical, such example illustrates several problem properties, and will be repeatedly referenced throughout this research.

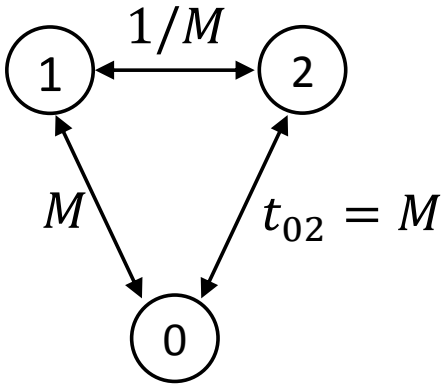

Figure 4.3: A conceptual network for the exemplification of sampling and symmetry.

First, the importance of exploratory visits or sampling is discussed.

4.3.1 The Effect of Sampling

The concept of sampling is presented in Definition 4.1 and its importance is demonstrated in Example 4.2.

Definition 4.1 *Sampling occurs when a vehicle is routed to a pick-up location to assert its demand and not to pick-up evacuees.*

Example 4.2 *Consider the network depicted in Figure 4.3, where the demand at pick-up node 1 is known, say* $D_1^* = 1$*, and the one at node 2 either equals zero or one full bus load. Thus, there are two possible scenarios: scenario 1 has* $D_1^1 = 1$ *and* $D_2^1 = 0$*, and scenario 2 has* $D_1^2 = 1$ *and* $D_2^2 = 1$*. For simplicity, suppose that a single bus is available for the evacuation. Since the demand at node 1 equals 1 full bus load, it is reasonable for the vehicle to traverse arc* $(0, 1)$ *and to perform a pick-up at the node. However, once the bus reaches node 1, it may either return to the depot to unload evacuees or visit node 2 without performing a pick-up (as the bus is already full). If the bus returns to the depot to unload evacuees before proceeding to node 2, then its final route,* $0 \to 1 \to 0 \to 2 \to 0$*, satisfies both scenarios and has objective function value of* $4M$*.*

Conversely, if the bus is used to sample the demand at node 2, then two possibilities arise. If $D_2^* = 0$*, then its final route,* $0 \to 1 \to 2 \to 0$*, has objective function value of* $2M + 1/M$*, whereas if* $D_2^* = 1$*, then its final route,* $0 \to 1 \to 2 \to 0 \to 2 \to 0$*, has objective function value of* $4M + 1/M$*. Consequently, the expected total travel time of the bus under the assumption of sampling equals* $(1/2) * [(2M + 1/M) + (4M + 1/M)] = 3M + 1/M$*. Since, for* M *sufficiently large,* $3M + 1/M < 4M$*, the optimal routing strategy will include a sampling visit to node 2 even though, in 50% of the cases, this strategy results in a larger total travel time.*

Clearly, sampling is beneficial whenever the value of the additional information offsets the cost (time) of the ensuing arc traversal. Nonetheless, sampling will only lead to smaller expected total travel times when the number of pick-ups needed on each pick-up location is potentially zero, as demonstrated in the following example.

Example 4.3 *Consider the network depicted in Figure 4.3 with* $D_1^1 = 1,\ D_2^1 = 1,\ D_1^2 = 1,$ *and* $D_2^2 = 2$. *Observe that, contrary to Example 4.2, here, the demand at pick-up location 2 cannot equal zero. Let* $V = 1$ *vehicle, and let such vehicle be routed to pick-up node 1. Suppose that, after reaching node 1, the bus returns to the depot to unload evacuees before visiting node 2. If* $D_2^* = 1$, *then its final route,* $0 \to 1 \to 0 \to 2 \to 0$, *has objective function value of* $4M$, *whereas if* $D_2^* = 2$, *then its final route,* $0 \to 1 \to 0 \to 2 \to 0 \to 2 \to 0$, *has objective function value of* $6M$. *Consequently, the expected total travel time under the assumption that the vehicle will return to the depot to unload evacuees after reaching node 1 (i.e., without sampling the demand at node 2) equals* $(1/2) * (4M + 6M) = 5M$.

Conversely, suppose that, after reaching node 1, the bus is used to sample the demand at node 2. If $D_2^* = 1$, *then its final route,* $0 \to 1 \to 2 \to 0 \to 2 \to 0$, *has objective function value of* $4M + 1/M$, *whereas if* $D_2^* = 2$, *then its final route,* $0 \to 1 \to 2 \to 0 \to 2 \to 0 \to 2 \to 0$, *has objective function value of* $6M + 1/M$. *Consequently, the expected total travel time under the assumption that the vehicle will sample the demand at node 2 before returning to the depot equals* $(1/2) * [(4M + 1/M) + (6M + 1/M)] = 5M + 1/M$. *Since, for any* $M > 0$, $5M < 5M + 1/M$, *the optimal routing strategy consists in returning to the depot directly, without sampling the demand at node 2.*

As demonstrated in Examples 4.2 and 4.3, sampling the demand at a pick-up location is only advantageous if such, potentially, equals zero. Otherwise, sampling only determines the number of round trips required between the depot and said pick-up location, instead of determining if a round trip is required. These properties are formalized in Propositions 4.2

and 4.3.

Proposition 4.2 *If the demand at a given pick-up location is potentially zero, then a lower expected total travel time may be attained by sampling the demand at this location.*

Proof. See Example 4.2. ∎

Proposition 4.3 *If the demand at a given pick-up location is never zero, then a lower expected total travel time cannot be attained by sampling the demand at this location.*

Proof. See Example 4.3. ∎

Next, the effect of symmetry is investigated.

4.3.2 The Effect of Symmetry

Here, the term symmetry refers to the problem default inability to distinguish between seemingly identical first stage decisions (i.e., between subsequent arc traversals for each vehicle leading to identical expected total travel times), but which may result in distinct objective function values depending on the realized demands at the destination nodes. The effect of symmetry is demonstrated in Example 4.4.

Example 4.4 *Consider the network depicted in Figure 4.3 with* $D_1^1 = 1,\ D_2^1 = 0,\ D_1^2 = 1,$ *and* $D_2^2 = 1$. *Once again, suppose that a single bus is available for the evacuation, and consider the ensuing symmetrical solutions presented in Table 4.5.*

Table 4.5: Symmetrical solutions for the network presented in Figure 4.3.

Scenario	**Demand**		**Optimal Route**	**Obj. Value**
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$		
$s = 1$	*1*	*0*	$0 \to 1 \to 2 \to 0$	*2M + 1/M*
$s = 2$	*1*	*1*	$0 \to 1 \to 0 \to 2 \to 0$	*4M*
$s = 1$	*1*	*0*	$0 \to 2 \to 1 \to 0$	*2M + 1/M*
$s = 2$	*1*	*1*	$0 \to 2 \to 0 \to 1 \to 0$	*4M*

Clearly, Objective Function (4.1) is unable to differentiate between these solutions, as both yield the same expected total travel time, $3M + 1/2M$*. However, given the set of possible scenarios, it is advantageous to start the evacuation by routing the vehicle through arc* $(0, 2)$ *instead of arc* $(0, 1)$*. Suppose that the bus is initially routed to node 1, and, then, used to sample the demand at node 2. If, upon its arrival at node 2, the bus finds that* $D_2^* = 0$*, then its final route,* $0 \to 1 \to 2 \to 0$*, has objective function value of* $2M + 1/M$*, whereas if it finds that* $D_2^* = 1$*, then its final route,* $0 \to 1 \to 2 \to 0 \to 2 \to 0$*, has objective function value of* $4M + 1/M$*. Consequently, by routing the bus to node 1 first, its expected total travel time equals* $(1/2) * [(2M + 1/M) + (4M + 1/M)] = 3M + 1/M$*. Conversely, suppose that the bus is initially routed to node 2. If, upon its arrival at the node, it finds that* $D_2^* = 0$*, then its final route,* $0 \to 2 \to 1 \to 0$*, has objective function value of* $2M + 1/M$*, whereas if it finds that* $D_2^* = 1$*, then its final route,* $0 \to 2 \to 0 \to 1 \to 0$*, has objective function*

value of $4M$*. Consequently, by routing the bus to node 2 first, its expected total travel time equals* $(1/2) * [(2M + 1/M) + (4M)] = 3M + 1/2M$*, a smaller value than the one attained by routing the vehicle to pick-up node 1 first.*

Interestingly, the difference between these symmetrical tours follows from an additional sampling visit to pick-up node 2, which does not take place when arc $(0, 2)$ is the first one traversed. This property is formalized in Proposition 4.4.

Proposition 4.4 *Even though the problem formulation assumes symmetrical arc traversal times, symmetrical routes may not yield equal travel times. Better solutions may be attained by prioritizing visits to pick-up locations that have, potentially, zero-demand.*

Proof. See Example 4.4. ∎

Next, an alternative, deterministic formulation that leverages these sampling and symmetry properties is presented.

4.3.3 Deterministic Formulation

Recall that the first stage of the proposed two-stage stochastic formulation identifies the subsequent arc traversal (common across scenarios) for each vehicle, whereas the second stage identifies the remaining portion of their tour for each possible scenario realization. However, since the problem is re-optimized as additional information becomes available, individual outcomes of the second stage decision are redundant, and, thus, unnecessary. Consequently,

at least in a dynamic context, the previous two-stage stochastic formulation can be modified to a deterministic equivalent, with arbitrary demands that are updated dynamically during the course of the evacuation. Since scenarios are no longer explicitly modeled, the resulting formulation has fewer variables and constraints, and, thus, is considerably more efficient.

However, such approach may prevent sampling from occurring, for, as shown in Examples 4.2 and 4.3 and formalized in Propositions 4.2 and 4.3, exploratory visits to pick-up locations are only advantageous (and, thus, may only take place) when their demand is potentially zero. Consequently, in order to preserve sampling, demands must be set to their lowest realizable value (best-case scenario). Such concept is formalized in Proposition 4.5.

Proposition 4.5 *Replacing the uncertain demands at each pick-up location with their worst-case scenario reduces the problem size and the ensuing computation time. However, such strategy also negates sampling, thus potentially leading to worse routing strategies.*

Proof. Consider the network depicted in Figure 4.3 with $D_1^1 = 1,\ D_2^1 = 0,\ D_1^2 = 1$, and $D_2^2 = 1$. Once again, suppose that a single bus is available for the evacuation. Under the worst-case strategy, D'_2, the updated demand at node 2, is set to $\max\{0, 1\} = 1$. Since the model assumes that at least one pick-up is needed at node 2, there is no need to sample its actual demand. Consequently, arc $(1, 2)$ will never be traversed, even though, as demonstrated in Example 4.2, such strategy reduces the expected total travel time of the fleet from $4M$ to $3M + 1/M$, and, in 50% of the cases, leads to an actual solution as low as $2M + 1/M$. ∎

Such approach is counterintuitive, as it is common practice to assume the worst-case scenario

on each pick-up location in an attempt to improve the robustness of the problem, that is, in an attempt to increase the likelihood that the chosen routes will satisfy any demand realization. Here, however, setting the demands to their worst-case scenario on each pick-up location merely leads to an upper bound on the overall travel time of the fleet, while potentially worsening the final solution.

In summary, the deterministic formulation eliminates the scenario subscripts from the x and y-variables and fixes the demands to their lowest realizable value on each pick-up location, even if the resulting demands conflict with the set of possible scenarios (i.e., even if none of the defined scenarios allow these demands to occur simultaneously). Moreover, the effect of symmetry can be leveraged by preventing buses from leaving the depot towards pick-up locations with uncertain, but positive, demand without a future return arc traversal from such node. By prioritizing visits to pick-up locations that have, potentially, zero-demand, such constraint breaks some of the problem symmetries in favor of arc traversals more likely to reduce the total travel time of the fleet. Consequently, the deterministic formulation has improved tractability, preserves the desired sampling behavior, and is able to break problem symmetries in favor of arc traversals leading to improved final routes. Its decision variables, objective function, and constraint set are as follows.

Table 4.6: Decision variables for the deterministic formulation.

Variable	**Description**
x_{ij}^{mt}	binary variable that equals 1 if arc (i,j) is the t^{th} arc traversed by vehicle m, else 0, $\forall\ (i,j) \in A,\ m = 1, \cdots, V,\ t = 1, \cdots, T$;
y_j^{mt}	percent capacity of vehicle m consumed at pick-up node j following its t^{th} arc traversal, $\forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T$. Let $0 \leq y_j^{mt} \leq 1,\ \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T$.

Proposed Deterministic Formulation:

$$\text{Minimize} \quad \sum_{(i,j)\in A} \sum_{m=1}^{V} \sum_{t=1}^{T} t_{ij} x_{ij}^{mt} \tag{4.15}$$

Subject to

$$x_{ij}^{m1} = 0, \quad \forall\ i \in P,\ j : (i,j) \in A,\ m = 1, \cdots, V \tag{4.16}$$

$$\sum_{i:(i,j)\in A} x_{ij}^{mt} = \sum_{k:(j,k)\in A} x_{jk}^{m(t+1)}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T-1 \tag{4.17}$$

$$\sum_{i:(i,0)\in A} x_{i0}^{mt} \geq \sum_{k:(0,k)\in A} x_{0k}^{m(t+1)}, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T-1 \tag{4.18}$$

$$x_{ij}^{mT} = 0, \quad \forall\ j \in P,\ i : (i,j) \in A,\ m = 1, \cdots, V \tag{4.19}$$

$$\sum_{(i,j)\in A} x_{ij}^{mt} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T \tag{4.20}$$

$$y_j^{mt} \leq \sum_{i:(i,j)\in A} x_{ij}^{mt}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots T \tag{4.21}$$

$$\sum_{j\in P}\sum_{l=1}^{t} y_j^{ml} - \sum_{i:(i,0)\in A}\sum_{l=1}^{t} x_{i0}^{ml} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T \tag{4.22}$$

$$\sum_{m=1}^{V}\sum_{t=1}^{T} y_j^{mt} = D'_j, \quad \forall\ j \in P \tag{4.23}$$

$$\sum_{i:(i,j)\in A}\sum_{m=1}^{V}\sum_{t=1}^{T} x_{ij}^{mt} \geq 1, \quad \forall\ j \in P^* \tag{4.24}$$

$$D'_j(\sum_{l=t+1}^{T} x_{j0}^{ml} - x_{0j}^{mt}) \geq 0, \quad \forall\ j \in P^*,\ m = 1, \cdots, V,\ t = 1, \cdots, T-1 \tag{4.25}$$

$$x_{ij}^{mt} = 1, \quad \forall\ \{(i,j), m, t\} \in R^* \tag{4.26}$$

$$y_j^{mt} = g, \quad \forall\ \{j, m, t, g\} \in C^*. \tag{4.27}$$

Objective Function (4.15) minimizes the total travel time of the fleet under the best-case scenario (i.e., the lowest possible demand realization) on each pick-up location. Constraints (4.16)-(4.20) are similar to (4.2)-(4.6), preventing the construction of invalid routes, including the elimination of subtours. Constraints (4.21)-(4.23) are similar to (4.7)-(4.9), enforcing the vehicle capacity restrictions. Note that (4.23), contrary to (4.9), only ensures that the best-case demand, D'_j, among the set of remaining scenarios, S^*, is met. Let $D'_j = \min_{s:(j,s)\in S^*} D_j^s,\ \forall\ j \in P^*$ and $D'_j = D_j^*,\ \forall\ j \in (P \cap (P^*)^c)$. Constraint (4.10) is no longer necessary, since every routing decision is now equal across scenarios. Constraint (4.24) is similar to (4.11), but it is now defined only for pick-up locations with uncertain demand. Since elements from S^* are now iteratively removed, it may be possible to infer the demand at some pick-up locations without an initial visit. Consequently, (4.24) allows pick-up lo-

cations with known demand equal to zero to be skipped, while visits to pick-up locations with known demand greater than zero remain enforced by (4.23). Constraint (4.25) prevents buses from leaving the depot towards pick-up locations with uncertain, but positive, demand under the best-case scenario, unless such is also accompanied by a future return arc traversal. Therefore, this constraint breaks some of the problem symmetries by prioritizing visits to pick-up locations that have, potentially, zero-demand. In addition, since this constraint only affects pick-up locations in set P^*, such is eliminated as the algorithm progresses (i.e., as $P^* \mapsto \emptyset$). Consequently, any round trips between the depot and said location enforced by this constraint are relaxed before vehicles are actually forced to perform them. Constraints (4.26)-(4.27) are similar to (4.12)-(4.13), ensuring that the previously performed routes and pick-ups remain unchanged as new information becomes available. Finally, the logical binary restriction on the x-variables and the non-negativity restriction on the y-variables are included. This formulation has considerably fewer variables and constraints than the two-stage stochastic variation, with $O(|P|^2\,V\,T)$ binary variables instead of $O(|P|^2\,V\,T\,|S|)$, and $O(|P|\,V\,T)$ constraints instead of $O(|P|\,V\,T\,|S|)$. Next, Proposition 4.6 presents an upper bound on the T-parameter for this formulation:

Proposition 4.6 *The upper bound on the maximum number of arc traversals per vehicle for the deterministic formulation, T'_{UB}, is calculated as follows:*

$$T'_{UB} = |P^*| + 1 + 2\left\lceil \frac{1}{V} \max\left(\sum_{j \in P} \lceil D'_j \rceil - 1, 0 \right) \right\rceil . \tag{4.28}$$

Proof. See proof of Proposition 4.1. ∎

Next, the iterative algorithm for the deterministic variation of the BEP is presented.

4.3.4 Iterative Algorithm for the Deterministic Formulation

Figure 4.4 presents a flowchart of the proposed algorithm. Generally, the proposed algorithm is similar to the one presented for the stochastic variation of the BEP. Nonetheless, there are some important distinctions. First, the previous algorithm identifies the optimal y-values directly from the optimization model, specifically from the lowest index variable with demand corresponding to the realized scenario. Here, however, these must to be computed analytically, as the optimization model merely identifies the y-value corresponding to the lowest possible demand realization at the pick-up location. Specifically, let $y_j^{mt^*}$ be computed as follows:

Proposition 4.7 *The percent capacity of vehicle m consumed at pick-up node j following its $(t^*)^{th}$ arc traversal is computed as follows:*

$$y_j^{mt^*} = y_j^{mt^*} + \min\left\{1 - \sum_{n\in P}\sum_{l=t'^m+1}^{t^*} y_n^{ml}, D_j^* - \sum_{l=1}^{m} y_j^{lt^*}\right\}, \quad \forall\ j \in P,\ m = 1, \cdots, V \tag{4.29}$$

Proof. Equation (4.29) corresponds to the minimum of two terms: the remaining capacity of the vehicle and the remaining realized demand at the pick-up location. The remaining capacity of the vehicle is computed via the term $1-\sum_{n\in P}\sum_{l=t'^m+1}^{t^*} y_n^{ml}+y_j^{mt^*}$, which accounts for the capacity consumed during the current tour (i.e., between its last visit to the depot, t'^m, and the current arc traversal, t^*). The remaining realized demand at the pick-up location

is computed via the term $D_j^* - \sum_{l=1}^{m} y_j^{lt^*} + y_j^{mt^*}$, which accounts for any pick-ups by alternate, lower index vehicles also visiting said location on their $(t^*)^{th}$ arc traversal. Observe that the term $y_j^{mt^*}$ is part of both subtractions, and, thus, must be reinserted into the equation. ∎

Second, the algorithm eliminates any scenarios that are no longer feasible from the problem parameter set. Recall that such was not possible in the two-stage stochastic formulation, as the decision variables were indexed for all elements in S. Third, since infeasible scenarios are eliminated from the problem parameter set, D'_j, the demand under the best-case scenario, must be recomputed for the remaining elements in P^*. Finally, any pick-up location with a single realizable scenario is eliminated from P^*, even if such location has not yet been visited. Consequently, the algorithm allows pick-up locations with known zero-demand to be skipped, while still ensuring that every evacuee boards a transit vehicle.

In summary, the algorithm proceeds as follows. First, the parameters t^*, C^*, R^*, and P^* are initialized. Recall that t^* corresponds to the current iteration of the algorithm, and, thus, to the subsequent arc traversal of each vehicle; that C^* corresponds to the set of previously performed pick-ups; that R^* corresponds to the set of previously traversed arcs; and that P^* corresponds to the set of pick-up locations with uncertain demand. Observe that the parameter P^* is now part of the proposed formulation. In addition, let $t'^m,\ m = 1, \cdots, V$, denote the last arc traversal in which the m^{th} vehicle visited the depot, and let z_1, z_2, and T' be support parameters. On each iteration of the proposed algorithm, the T-parameter is updated. Since the algorithm eliminates non-realizable scenarios from the set of possible demand realizations, it may avoid visiting pick-up locations which have known demand

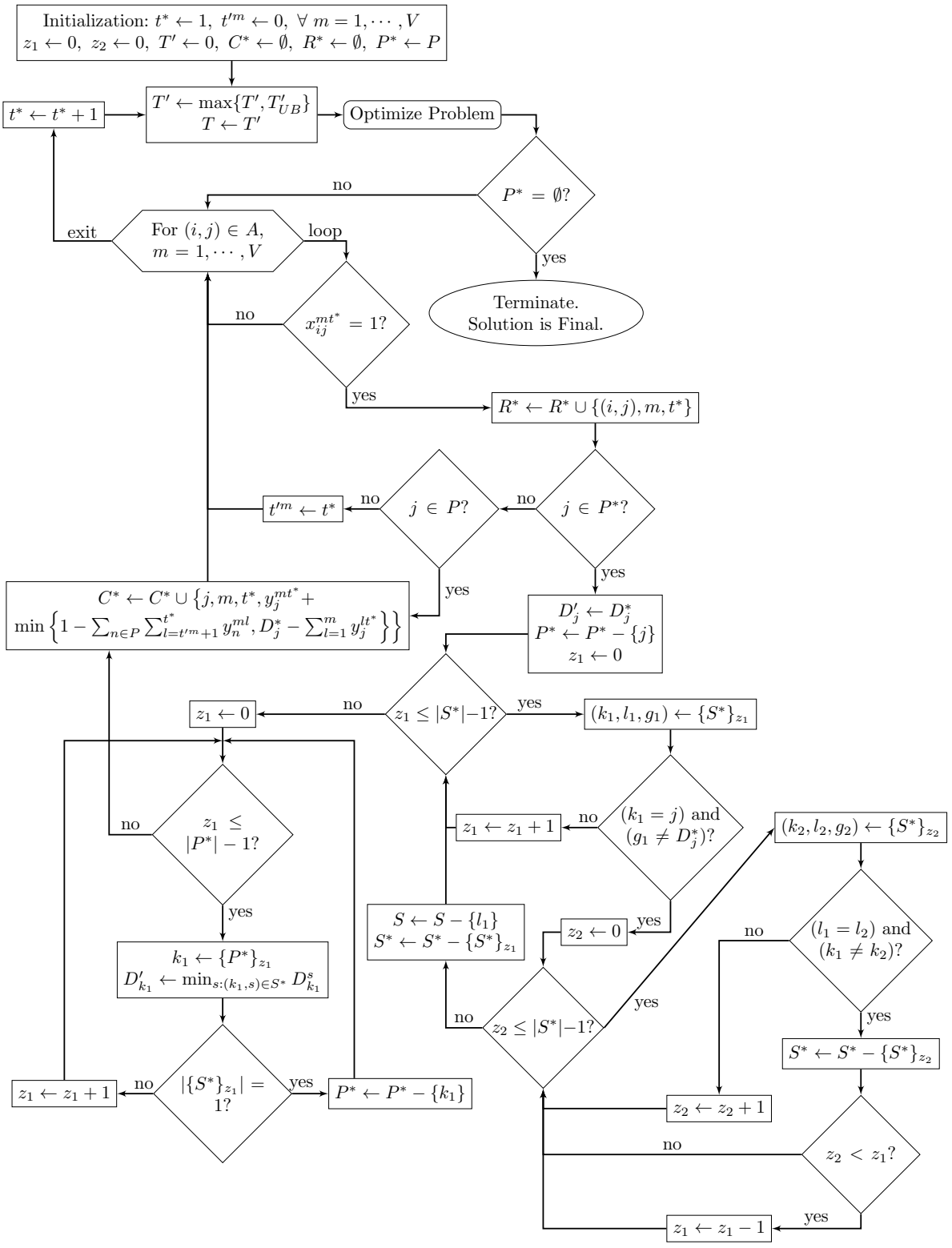

Figure 4.4: Flowchart of the iterative algorithm for the deterministic formulation.

equal to zero. Consequently, T'_{UB} may not change monotonically with t^*. In order to prevent conflicts with previously added elements in R^* and C^*, the monotonically increasing behavior of T is enforced via the support parameter T'. The problem is reoptimized; the $(t^*)^{th}$ arc traversal of each vehicle, if any, is saved into set R^*; and its percent capacity occupied at the destination node (if such is a pick-up location) is saved into set C^*. Observe that the realized y-value is now computed via (4.29). If the destination node is a pick-up location that has not yet been visited, then such is removed from set P^* and its demand set to D^*_j. In addition, any infeasible scenarios are removed from set S^* and the lowest demand among the remaining scenarios, D'_j, is recomputed for each pick-up node $j \in P^*$. Conversely, if the destination node is the depot, then the corresponding t'^m parameter is set to t^*. The algorithm terminates once $P^* = \emptyset$, in which case the solution attained by the optimization problem is final. Otherwise, t^* is incremented by one and a new iteration is performed. Example 4.5 shows the implementation of the algorithm for the same theoretical network used to exemplify the first algorithm.

Example 4.5 *Once again, consider the example network presented in Figure 4.2 with* $V = 1$ *bus. Provided the scenarios presented in Table 4.2, the lowest realizable demand on each pick-up location equals* $D'_1 = 0,\ D'_2 = 1,$ *and* $D'_3 = 1$. *Since* $D'_2,\ D'_3 > 0$ *and since* $V = 1$, *by (4.25),* $x^{1t}_{0j} \leq \sum_{l=t+1}^{T} x^{1l}_{j0},\ \forall\ j \in \{2,3\},\ t = 1, \cdots, T-1$, *forcing any traversals through arcs* $(0,2)$ *and* $(0,3)$ *to be followed by at least one future traversal through arcs* $(2,0)$ *and* $(3,0)$, *respectively. Since such round trips do not reduce the expected total travel time of the fleet, arc* $(0,1)$ *is preferred. While the same arc was initially traversed in Example 4.1, it*

is important to observe that such occurred merely by chance, as, for example, the algorithm was unable to distinguish between the tours $0 \to 1 \to 2 \to 0$ *and* $0 \to 2 \to 1 \to 0$. *Here, however, this symmetry is broken in favor of arc traversals leading to pick-up locations with, potentially, zero demand (i.e., node 1).*

Now, suppose that the realized demand at node 1, once again, equals zero (i.e., suppose that $D_1^* = 0$*). Since the demand at this location is now known, the location is removed from set* P^*. *In addition, sets* R^* *and* C^* *are updated to reflect the realized arc traversal and pick-up, respectively, that is,* $R^* \overset{set}{=} \{\{(0,1),1,1\}\}$ *and* $C^* \overset{set}{=} \{\{1,1,1,0\}\}$*; and scenario 1 is removed from the problem parameter set. Such change, however, does not affect the best-case scenario demands for the remaining nodes in* P^*. *Finally,* t^* *is incremented by one, and the problem is reoptimized.*

As in Example 4.1, the bus is directed to pick-up node 2. Now, suppose that the realized demand at node 2 equals one full bus load (i.e., suppose that $D_2^* = 1$*). In this case,* $R^* \overset{set}{=} \{\{(0,1),1,1\},\{(1,2),1,2\}\}$, $C^* \overset{set}{=} \{\{1,1,1,0\},\{2,1,2,1\}\}$, *and* $P^* \overset{set}{=} \{3\}$. *However, provided this demand realization, scenario 3 cannot take place, and is removed from* S^*. *Consequently, the only realizable demand at pick-up node 3 is* $D_3^* = 2$. *Since the demand at this location is now known, the location is removed from set* P^*, t^* *is incremented by one, and the problem is reoptimized. Since* P^* *is now empty, the algorithm terminates, with final solution* $0 \to 1 \to 2 \to 0 \to 3 \to 0 \to 3 \to 0$ *and objective function value of 15 time units. Observe that, here, the final solution was attained in two iterations, whereas, previously, four iterations were needed.*

Interestingly, since the problem starts with the premise of the best-case scenario on each pick-up location and since additional arc traversals are iteratively added to accommodate the realized demand in excess of the original assumption, the objective function value attained by the algorithm increases monotonically with t^*. Consequently, this algorithm starts from a possibly infeasible route and iteratively adds arc traversals to it until the realized demand on every pick-up location is accommodated. Conversely, the previous algorithm starts from a set of feasible routes, one for each scenario, and modify them (possibly leading some of these to infeasibility during intermediary iterations), until they all converge to a single route that meets the realized demand on each pick-up location. The next Section compares these formulations and accompanying algorithms in terms of tractability and solution quality.

4.3.5 Algorithm Comparison

As indicated by Pillac et al. (2013), improving the solution time of the model is particularly important in D-VRPs given the potentially large frequency of requests and the urgency of the requests. The same premise holds true for dynamic variations of the BEP, as buses must remain parked at the pick-up locations waiting for further instructions. This section empirically explores the performance of both formulations and related algorithms for solving a somewhat larger problem instance. All computational runs were performed using IBM ILOG CPLEX Optimization Studio version 12.4 on a PC with two quad-core, 2.13 GHz Intel Xeon processors and 4.00 GB RAM running on a 64-Bit operating system.

Consider the example network depicted in Figure 4.5 with five pick-up locations scattered around an evacuation zone, possibly the vicinity of a building under bomb threat. The problem requires every evacuee to be transported to a depot/shelter, but their exact number, if any, is only revealed after an initial visit by a vehicle. The set of possible demand realizations is presented in Table 4.7.

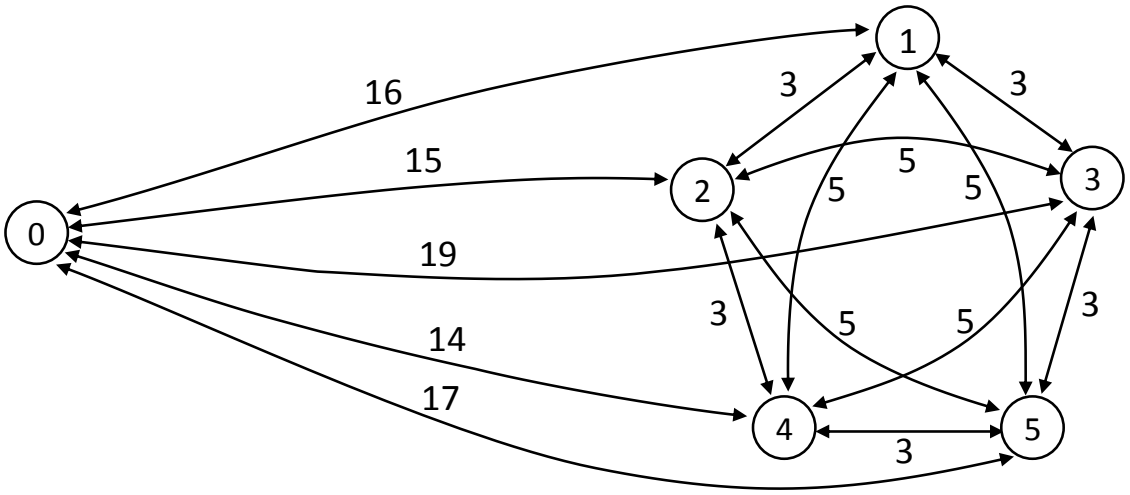

Figure 4.5: An example network for the algorithm comparison.

Table 4.7: Example scenarios for the network depicted in Figure 4.5.

	Demand				
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$	$\mathbf{D_3^s}$	$\mathbf{D_4^s}$	$\mathbf{D_5^s}$
$s = 1$	0	2	0	4	1
$s = 2$	0	2.5	1.5	4	1
$s = 3$	0	3	0	3	1
$s = 4$	0	3.2	1.5	3.3	1
$s = 5$	0	4	0	2	1
$s = 6$	0	4	1.5	2.5	1

	Demand				
	$\mathbf{D_1^s}$	$\mathbf{D_2^s}$	$\mathbf{D_3^s}$	$\mathbf{D_4^s}$	$\mathbf{D_5^s}$
$s = 7$	1	2	0	4	0
$s = 8$	1	2.5	1.5	4	0
$s = 9$	1	3	0	3	0
$s = 10$	1	3.2	1.5	3.3	0
$s = 11$	1	4	0	2	0
$s = 12$	1	4	1.5	2.5	0

For simplicity, let $V = 2$ buses and suppose that Scenario 7 is realized. Under the second algorithm (i.e., the one for the deterministic formulation), $D'_1 = 0,\ D'_2 = 2,\ D'_3 = 0,\ D'_4 = 2,$ and $D'_5 = 0$. Bus 1 is initially routed to pick-up node 4 and bus 2 to pick-up node 1. Upon their arrival, it is determined that $D_1^* = 1$ and that $D_4^* = 4$. Both buses are loaded to capacity and the corresponding arc traversals and pick-ups are saved to sets R^* and C^*, that is, $R^* \overset{set}{=} \{\{(0,4),1,1\},\{(0,1),2,1\}\}$ and $C^* \overset{set}{=} \{\{4,1,1,1\},\{1,2,1,1\}\}$. Since the demand at these pick-up locations is now known, these are removed from set P^*, that is, $P^* \overset{set}{=} \{2,3,5\}$. Moreover, provided the realized demand at pick-up node 1, it is clear that only scenarios 7-12 are possible, and, provided the realized demand at pick-up node 4, that only scenarios 1, 2, 7, and 8 are possible. Therefore, scenarios 1-6 and 9-12 can be removed from the problem

parameter set, $D'_1 \overset{set}{=} 1$, and $D'_4 \overset{set}{=} 4$. The remaining problem parameters, other than t^*, remain unchanged. On the second iteration, the first bus is routed to the depot to unload evacuees and the second bus is used to sample the realized demand at pick-up node 3. Upon its arrival, it is determined that $D^*_3 = 0$. Consequently, scenario 8 is no longer possible, and is removed from the problem parameter set. The D'_j demands are set to the demand realizations of scenario 7, $R^* \overset{set}{=} \{\{(0,4),1,1\},\{(0,1),2,1\},\{(4,0),1,2\},\{(1,3),2,2\}\}$, $C^* \overset{set}{=} \{\{4,1,1,1\},\{1,2,1,1\},\{3,2,2,0\}\}$, t^* is incremented by one, and the problem is reoptimized. Since P^* is now empty, the algorithm terminates. Under the final solution, bus 1 is routed though arcs $0 \to 4 \to 0 \to 4 \to 0 \to 4 \to 0$, transporting three full bus loads from pick-up node 4 to the depot, and bus 2 is routed through arcs $0 \to 1 \to 3 \to 0 \to 2 \to 0 \to 4 \to 0 \to 2 \to 0$, transporting a full bus load from pick-up nodes 1 and 4 and two full bus loads from pick-up node 2 to the depot. Under this routing strategy, the realized total travel time equals 210 time units. Note that pick-up node 5 was never visited, as the algorithm correctly inferred that its realized demand equals zero.

Both algorithms have worst-case complexity of polynomial order: the first algorithm has $O(|P|^4\,V\,|S|)$, whereas the second has $O(|P|^5\,V\,|S|^2)$. While more cumbersome, the second algorithm is much more efficient, as it allows the use of a considerably smaller problem formulation, an important consideration since even small variations of the classic VRP are known to be *NP-Hard* (Toth and Vigo, 2002). In fact, the first iteration of the second algorithm required 9.9 seconds, the second 4.6 seconds, and the last optimization 2.6 seconds. A solution for the first algorithm could not be obtained, as it exceeded the computer memory

capacity on its very first iteration, after 12 hours and 33 minutes of computation and with an optimality gap of 18.11%.

Alternative empirical experiments attained similar results. Consider an instance of the example network depicted in Figure 4.5, with $V = 2$ buses, and involving only 3 scenarios: the first scenario has demands $D_1^1 = 0$, $D_2^1 = 2$, $D_3^1 = 0.5$, $D_4^1 = 1.5$, and $D_5^1 = 1$; the second, $D_1^2 = 0$, $D_2^2 = 2$, $D_3^2 = 0$, $D_4^2 = 2$, and $D_5^2 = 1$; and the third, $D_1^3 = 1$, $D_2^3 = 1.5$, $D_3^3 = 0.5$, $D_4^3 = 2$, and $D_5^3 = 0$. Suppose that scenario 1 is realized. The second algorithm was able to attain the final solution in one iteration, under 3.6 seconds, and an additional 2.6 seconds for the last optimization. Conversely, the first algorithm required 5 iterations, with 102 hours and 19 minutes for the first iteration, 4 minutes and 34 seconds for the second, 36 seconds for the third, 2 seconds for the fourth, 1.5 seconds for the fifth, and an additional second for the last optimization. Final objective function values also differed, with 157 for the first algorithm and 155 for the second, which, interestingly, is the optimum solution for the problem under the assumption of perfect information. The next section summarizes the main findings of this paper and presents directions for future research.

4.4 Conclusion and Future Research

This research introduced a two-stage stochastic formulation and an iterative algorithm for solving a variation of the BEP in which the realized demands on each pick-up location are

only revealed during the course of the evacuation. The effect of sampling and symmetry over the expected total travel time of the fleet is explored, and these insights leveraged into an alternative, deterministic formulation. Such formulation has considerably improved tractability, preserves the desired sampling behavior, and employs constraints that break some of the problem symmetries in favor of arc traversals leading to reduced total travel times. Despite such advances, both variations rely on simplifying assumptions that diminish their usefulness to practitioners. First, they do not account for the time-dependent arrival behavior of evacuees, and, instead, assume that these are at the pick-up locations at the onset of the evacuation. While such assumption is realistic for no-notice evacuation settings (such as the evacuation of urban clusters facing bomb threats), it fails to consider situations in which enough time is available for adequate planning and implementation (such as a region-wide evacuation preceding the arrival of a hurricane). Second, they neither limit nor attempt to minimize the duration of the evacuation. While it is relatively easy to incorporate any measure of completion time into the proposed formulations, further research is needed to determine if the properties of sampling and symmetry (and, thus, the alternative deterministic formulation) remain valid. Third, they do not balance the benefits of holding buses at the depot or at the pick-up locations until other vehicles have reached their respective destinations. By eliminating unnecessary visits to pick-up locations, such delay may reduce the expected total travel time of the fleet, and, perhaps, the duration of the evacuation. Fourth, the assumption that all scenarios have equally likely probability of realization is unrealistic for most real-world evacuation settings. As shown by Fajardo and Waller (2012),

it is possible to derive the probabilities of each scenario realization directly from a Markov Chain, at least in a D-TSP context. Nonetheless, even if such probabilities were available, it is likely that the realized demands at the pick-up locations would fail to match any of the predetermined scenarios. Further research is needed to accommodate fuzzy variations of the problem, capable of adjusting the demand (and the likelihood) of the remaining scenarios as new information becomes available.

Chapter 5

Conclusions

This dissertation introduces and analyzes important variations of the Bus Evacuation Problem, a unique VRP motivated both by its humanitarian significance and by the routing and scheduling challenges of planning transit-based, regional evacuations. Although this research focuses on evacuation planning and implementation, the concepts herein developed are readily applicable to many other transportation settings, within and outside the humanitarian logistics ambit.

Chapter 2 presents and examines several structural properties of a particularly complex variant of the BEP. The problem assumes that evacuees gather at the pick-up locations at constant rates, from the start of the evacuation to a location-specific end-time (dictated by risk considerations), and seeks a pick-up schedule at these locations, and the corresponding routes for a fleet of buses, in such a way that their total waiting time (exposure) is minimized. At each scheduled pick-up time, all evacuees present must be served and eventually transported to a depot/shelter. The formulation forces at least one pick-up to take place at each pick-up location, scheduled for the time when the arrival process of evacuees ends. Efficiencies from allowing buses to unload evacuees at the depot multiple times and from service choice, that is, from allowing the model to select the optimal number of pick-ups at each location (up to a maximum service level), are exploited. It is shown that, under certain conditions, if a feasible solution does not exist, then one can be obtained by increasing the fleet size; that if one exists, then the exposure can be reduced by either increasing the fleet size or the maximum service level; and that, after the minimum exposure schedule for a given service level is attained, the exposure can only be reduced by further increasing the maxi-

mum service level, but with diminishing returns. In addition, upper and lower bounds on the fleet size required for feasibility and for such bound on exposure, and an exact formulation and an upper bound on the maximum number of arc traversals per vehicle, were proposed. More importantly, it is shown that, depending on the problem instance, it is possible to reduce both the fleet size requirement and the amount of time that evacuees must wait for rescue, and that, past a certain threshold, there is an optimal choice for the maximum service level (or a range of potential values, if the problem cannot be solved to optimality under a reasonable amount of time) that guarantees an efficient usage of the available fleet and equitable reductions in exposure across pick-up locations. Appendix B presents alternative formulations for such BEP variant.

Chapter 3 introduces an exact optimization algorithm based on a less known relaxation methodology originally developed by Ritter (1967). The algorithm was selected due to its ability to explore the inherent structure of problems with both complicating variables and constraints, such as the aforementioned BEP variant, without the need to compound multiple decomposition/relaxation strategies. In summary, the algorithm relaxes some of the non-negativity and integer restrictions in order to produce a smaller and easier to solve problem, namely the Reduced Problem. In the original algorithm, the relaxed non-negativity restrictions are enforced sequentially either by swapping decision variables between the relaxed and non-relaxed sets (and recomputing the Reduced Problem parameters) or by adding new constraints to the Reduced Problem. The algorithm terminates once the optimum solution to the Reduced Problem also satisfies the non-negativity restriction of all relaxed variables.

The proposed variation is similar to the original algorithm, except that it precludes swapping of decision variables between the relaxed and non-relaxed sets. Hence, the modified algorithm relies solely on the inclusion of constraints into the Reduced Problem in order to enforce the non-negativity (and, now, the integer restriction) of the relaxed variables. By preventing the swapping operation from taking place, the modified algorithm requires fewer matrix manipulations and fewer numerical comparisons, and, thus, has improved numerical stability. Perhaps even more important, given the dichotomous nature of the added constraints, the modified algorithm can be combined with the branch and bound algorithm in order to allow problems with integer or mixed-integer subproblems to be solved to optimality. Appendix C presents an example problem which is solved algebraically via the original and modified algorithms. Such example problem is then modified to an integer, mixed-integer, and, later, mixed-integer with a quadratic objective function variation and resolved using the modified algorithm. Appendix D presents the MathWorks Matlab source code for the original and modified algorithms. Empirical studies demonstrate its viability to solve large, bordered-structure, optimization problems.

Finally, Chapter 4 introduces a two-stage stochastic formulation and an iterative (dynamic) algorithm for the Bus Evacuation Problem. In this problem, a fleet of capacitated buses must transport all transit-dependent individuals from a set of pick-up locations to a depot/shelter with minimum total travel time. The problem assumes that a set of equal-probability demand scenarios is available, and that the exact number of evacuees on each location, if any, becomes known once a bus visits it for the first time. The effect of exploratory visits (sampling) and

symmetry is explored, and the resulting insights used to develop a deterministic formulation for the problem. While, at least in a dynamic context, these formulations are equivalent, the deterministic variation has considerably improved tractability, preserves the desired sampling behavior, and employs constraints that break some of the problem symmetries in favor of arc traversals leading to reduced total travel times. Appendix E presents the IBM ILOG CPLEX source code for solving both the two-stage and the deterministic formulations.

Despite these advances, the modeling and optimization literature on bus-based evacuation planning and implementation is scarce and considerable research remains needed. First, the variant presented in Chapter 2 assumes that evacuees arrive at the pick-up locations at a constant rate. While such is an important distinction from the related literature, it is still unrealistically portraying the true evacuee arrival behavior, which is probabilistic in nature and is likely to vary over time. Conversely, while the problem variant presented in Chapter 4 assumes stochastic demand, it relies on simplifying assumptions that diminish its resemblance to real-life evacuation settings. First, the formulations presented in this Chapter do not account for the time-dependent arrival behavior of evacuees, as the ones presented in Chapter 2 and Appendix B. Instead, they assume that all evacuees are at the pick-up locations upon the start of the evacuation. While such assumption is realistic for modeling no-notice evacuation problems, such as the evacuation of urban clusters facing bomb threats, it fails to consider situations in which enough time is available for its adequate planning and implementation, such as a region-wide evacuation preceding the arrival of a hurricane. Second, the formulations presented in Chapter 4 neither attempt to minimize the duration of

the evacuation nor limit its completion time. While it is relatively easy to incorporate some measure of completion time into the proposed formulations, further research is needed to determine if the properties of sampling and symmetry (and, thus, the alternative deterministic formulation proposed) remain valid. Third, these formulations do not balance the benefits of holding buses at the depot or at the pick-up locations until other vehicles have reached their respective destinations. By waiting for the demand at these locations to unravel, unnecessary sampling visits may be avoided, thus leading to a reduction in the expected total travel time of the fleet, and, perhaps, in the duration of the evacuation. In addition, while it is common practice to use a scenario-based approach to model stochastic problems, the assumption that these have equally likely probability of realization is unrealistic. Varying probabilities can be attained from a Markov Chain (provided that the stochastic process is memoryless and follows a known probability distribution) and incorporated into the model via weight coefficients for the objective function. Nonetheless, in most real-world evacuation settings, it is unlikely that the realized demands at the pick-up locations would match any of the predetermined scenarios. Further research is needed to accommodate fuzzy variations of the problem, capable of adjusting the demand (and the likelihood) of the remaining scenarios as new information becomes available.

In addition, it remains computationally expensive to obtain an optimal solution to the BEP variant presented in Chapter 2. Currently, the problem can only be solved to optimality for toy examples involving few buses, pick-ups, and pick-up locations. As aforementioned, the Modified Ritter (1967) Relaxation Algorithm presented in Chapter 3 is a general purpose

algorithm, capable of solving bordered-structure problems with linear, integer, or mixed-integer subproblems and with linear or quadratic objective functions. However, further research is needed to determine the algorithm viability to solve such a BEP variant. In particular, it may be possible to leverage inheriting structural properties of the problem into the solution algorithm. First, the existence of non-singular matrices on each subproblem (i.e., within the constraint set coordinating the tour for each vehicle) would eliminate the need to augment the original problem with artificial variables, and, thus, unnecessary computations. Second, it may be advantageous to set the relaxed decision variables to the tours leading to the minimum exposure schedule for a given service level. As shown in Chapter 2, such a pick-up schedule, albeit sometimes infeasible, presents a lower bound on the objective function value, and, thus, may expedite the fathoming of unprofitable branches during the branch and bound algorithm. Finally, since the complicating constraints are equalities, they can be used to eliminate the complicating variables from the original problem by simple substitution. As a result, the inheriting bordered-structure of the problem would change to a block diagonal or angular structure (as demonstrated in Appendix B), where the subproblem structure (i.e., the decision variables and constraints controlling the routing, scheduling, and loading restrictions for a vehicle) remain unchanged, and the complicating constraints change to the allocation of evacuees to individual vehicles. Although such formulation has a simpler structure, empirical studies have demonstrated that the ensuing increase in the number of non-zero coefficients (problem density) leads to an increase in solution time when no decomposition/relaxation strategy is used. Nonetheless, such formulation allows the problem

to be solved by Dantzig-Wolfe Decomposition (Dantzig and Wolfe, 1960), or, in turn, its dual problem solved by Benders' Decomposition (Benders, 1962). Research is needed to determine whether such algorithms, when used to solve such a dual angular formulation of the aforementioned BEP variant, are able to attain faster computation times than the Modified Ritter (1967) Relaxation Algorithm applied the original formulation.

Finally, considerable research is needed to improve the Modified Ritter (1967) Relaxation Algorithm itself. As indicated in Chapter 3, the same concerns pertaining to the branch and bound algorithm, namely as branching strategy and choice of branching variable, also apply to the proposed algorithm. Consequently, further development of the source code presented in Appendix D.2 is needed before the algorithm is able to systematically outperform commercial optimization packages such as IBM ILOG CPLEX. In addition, since a considerable amount of computational effort is dedicated to the generation of constraints enforcing the branching and the upper and lower bound restrictions, and since these are incorporated sequentially into the Reduced Problem, it may be possible to further improve the efficiency of the algorithm by the concomitant use of Gomory cuts. Finally, since the algorithm is designed for bordered-structure problems, it may be applicable to seemingly structureless problems, but which can be mapped into an structured problem through rearrangement of their constraint coefficients. See Aykanat et al. (2004), Duff and Scott (2005), Ferris and Horn (1998), Fragnière et al. (2000), Pinar and Aykanat (1996), and Pinar et al. (1996) for multiple algorithms to reorganize the coefficients of the constraint matrix into exploitable structures.

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Appendix A

Summary of the Bus Evacuation Problem Literature Review

Table A.1: Summary of the Bus Evacuation Problem literature review

Reference	Objective Function	Scope	Route Restrictions	Time Restrictions	Shelter Capacities
Abdelgawad et al. (2010a), Abdelgawad et al. (2010b), and Abdelgawad and Abdulhai (2010)	Multi-objective with arbitrary weight coefficients	Automobile and mass transit	Multiple-tour	Network last pick-up time	Unrestricted
Bish (2011)	Minimize the longest travel time across vehicles	Mass transit	Multiple-tour	None	Restricted
Chen and Chou (2009)	Minimize the total travel time of vehicles	Mass transit	Single-tour	None	Unrestricted
Deai et al. (2011)	Minimize the total travel time of vehicles	Mass transit	Single round-trip on each time window	Time window for pick-ups	Unrestricted
Goerigk and Grün (2012)	Minimize the longest travel time across vehicles	Mass transit	Round-trip between pick-up locations and shelters	None	Restricted

Reference	Objective Function	Scope	Route Restrictions	Time Restrictions	Shelter Capacities
Goerigk et al. (2013)	Minimize the longest travel time across vehicles	Mass transit	Round-trip between pick-up locations and shelters	None	Restricted
Kulshrestha et al. (2012)	Minimize the total travel time of vehicles	Mass transit	Round-trip between pick-up locations and shelters	Maximum run-time of each vehicles	Restricted
Margulis et al. (2006)	Maximize the number of rescued evacuees	Mass transit	Round-trip between pick-up locations and shelters	Network last delivery time	Restricted
Sayyady and Eksioglu (2010)	Minimize the total travel time of vehicles and the number of stranded evacuees	Mass transit	Single-tour	None	Unrestricted
Song et al. (2009)	Minimize the total travel time of vehicles	Mass transit	Single-tour	Time window for pick-ups and vehicle-specific	Unrestricted

Reference	Objective Function	Scope	Route Restrictions	Time Restrictions	Shelter Capacities
Wei and Zhaodong (2010)	Minimize the total travel time of vehicles	Automobile and mass transit	Maximum flow of vehicles through the network	None	Unrestricted
Zhang et al. (2010)	Minimize the total travel time of vehicles	Pedestrian and mass transit	Single-tour	None	Unrestricted
Zheng (2013)	Minimize the total travel time of vehicles between pick-up and drop-off locations	Mass transit	Round-trip between pick-up locations and shelters	Time window for pick-ups	Restricted

Table A.2: Summary of the Bus Evacuation Problem literature review (continued)

Reference	Initial Vehicle Location	Arc Travel Times	Demand	Solution Method	Stochastic Model
Abdelgawad et al. (2010a), Abdelgawad et al. (2010b), and Abdelgawad and Abdulhai (2010)	Bus yard	Problem parameter	Problem parameter	Constraint programming	N/A
Bish (2011)	Bus yard	Problem parameter	Problem parameter	Column generation with routes generated heuristically	N/A
Chen and Chou (2009)	Shelters	Estimated via simulation	Problem parameter	Clarke & Wright's savings algorithm	N/A
Deai et al. (2011)	Shelters, but availability is time window dependent	Problem parameter	Estimated by a logistic mobilization curve	Hybrid genetic/simulated annealing algorithm	N/A
Goerigk and Grün (2012)	Bus yard	Problem parameter	Problem parameter	Tabu Search Metaheuristic	Stochastic model with no recourse

Reference	Initial Vehicle Location	Arc Travel Times	Demand	Solution Method	Stochastic Model
Goerigk et al. (2013)	Bus yard	Problem parameter	Problem parameter	Heuristic upper and lower bounds coupled with a modified branch and bound algorithm	N/A
Kulshrestha et al. (2012)	Pick-up locations	Problem parameter	Problem parameter	Cutting plane algorithm	Stochastic model with no recourse
Margulis et al. (2006)	Shelters	Problem parameter	Problem parameter	None	N/A
Sayyady and Eksioglu (2010)	Bus yard or en route	Estimated via simulation	Estimated by a logistic mobilization curve	Tabu Search Metaheuristic	N/A
Song et al. (2009)	Shelters or en route	Problem parameter	Problem parameter	Hybrid algorithm which combines the genetic algorithm, neural networks, and the hill climbing method	Reliability constraint

Reference	Initial Vehicle Location	Arc Travel Times	Demand	Solution Method	Stochastic Model
Wei and Zhaodong (2010)	Pick-up locations	Problem parameter. Also assumes maximum road capacity	Problem parameter	Relaxation strategy	N/A
Zhang et al. (2010)	Shelters	Probabilistic effect of pedestrian traffic on mass transit vehicle speed	Estimated by a logistic mobilization curve	Dijkstra's algorithm	N/A
Zheng (2013)	Bus yard or en route	Problem parameter	Problem parameter	Lagrangian relaxation coupled with reformulation into MCNFP	N/A

Appendix B

Alternative Formulations for the Bus Evacuation Problem with Constant Evacuee Arrival Rate

This section reiterates the modeling notation and introduces alternative formulations to the BEP variant introduced in Chapter 2.

B.1 Original Formulation in Standard Form

Consider the complete network (N, A), where N and A are the set of nodes and arcs, respectively. The set N consists of set P, the set of pick-up locations (each one located on a neighborhood requiring evacuation services), and node 0, the depot, which, for simplicity, also functions as a shelter. Each arc $(i, j) \in A$ has an associated arc traversal time, t_{ij}, satisfying the triangle inequality, that is, $t_{ij} \leq t_{ik} + t_{kj},\ \forall\ (i, j),\ (i, k)$, and $(k, j) \in A$. Each pick-up location $j \in P$ has an associated constant evacuee arrival rate, D_j, and a last scheduled pick-up time, L_j, which also corresponds to the end of the evacuee arrival process on this node. Consequently, each pick-up location has total demand of $D_j L_j$ evacuees. Finally, let F be the maximum number of scheduled pick-ups on each pick-up location (service level); let $T(F)$ be an upper bound on the number of arc traversals for each vehicle; and let V be the number of buses available for the evacuation, each with capacity for C evacuees. Table B.1 presents the decision variables for the BEP variant introduced in Chapter 2.

Table B.1: Decision variables for the BEP variant introduced in Chapter 2.

Variable	**Description**
x_{ij}^{mtf}	binary variable that equals 1 if arc (i,j) is the t^{th} arc traversed by vehicle m for the f^{th} pick-up at node j, else 0, $\forall\ (i,j) \in A$, $m = 1, \cdots, V$, $t = 1, \cdots, T(F)$, $f = 1, \cdots, F$;
b_j^{mtf}	number of evacuees from the f^{th} pick-up at node j assigned to (or, if j is the depot, released from) vehicle m on its t^{th} arc traversal, $\forall\ j \in N$, $m = 1, \cdots, V$, $t = 1, \cdots, T(F)$, $f = 1, \cdots, F$;
d^{mt}	time when vehicle m makes a pick-up after its t^{th} arc traversal, $\forall\ m = 1, \cdots, V$, $t = 0, \cdots, T(F)$;
p_j^f	time when the f^{th} pick-up is scheduled for node j, $\forall\ j \in P$, $f = 1, \cdots, F$.

In addition, Table B.2 presents the slack variables used to place the formulation introduced in Chapter 2 in standard form, and, thus, in a format compatible with the Modified Ritter (1967) Relaxation Algorithm. Observe that such a formulation is presented for reference purposes only, as the algorithm was not used to solve actual instances of the problem. Since the Modified Ritter (1967) Relaxation Algorithm was not coded for efficiency, and since it is affected by the same concerns pertaining to the branch and bound algorithm, such as branching strategy and choice of branching variable, further development of the source code presented in Appendix D.2 is needed before the algorithm is able to outperform commercial

optimization packages such as IBM ILOG CPLEX.

Table B.2: Slack variables for the formulation introduced in Chapter 2.

Variable	**Description**
$C3^{mt}$	slack variable that equals 1 if vehicle m visits and parks at the depot after its t^{th} arc traversal, else 0, $\forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) - 1$;
$C5^{mt}$	slack variable that equals 1 if vehicle m does not perform a pick-up after its t^{th} arc traversal (i.e., if the vehicle either visits the depot or bypasses a pick-up node without stopping), else 0, $\forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F)$;
$C6^{mt}$	slack variable corresponding to the amount of time vehicle m remains idle after its t^{th} arc traversal, $\forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F)$;
$C7_j^{mtf}$	slack variable corresponding to the delay between the departure time of vehicle m from node j after its t^{th} arc traversal and the f^{th} scheduled pick-up time for this node, assuming that such a vehicle performs this pick-up, else unrestricted, $\forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F$;
$C8_j^{mtf}$	slack variable corresponding to the lead time between vehicle m departure from node j after its t^{th} arc traversal and the f^{th} scheduled pick-up time for this node, assuming that such a vehicle is performs this pick-up, else unrestricted, $\forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F$;

Variable	Description
$C9_j^{mtf}$	slack variable corresponding to the remaining capacity of vehicle m assuming that such a vehicle is empty and that it performs the f^{th} scheduled pick-up at node j after its t^{th} arc traversal, $\forall\ j \in N,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F$;
$C10^{mt}$	slack variable corresponding to the used capacity of vehicle m after its t^{th} arc traversal, $\forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F)$;
$C11^{mt}$	slack variable corresponding to the remaining capacity of vehicle m after its t^{th} arc traversal, $\forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F)$

Chapter 2 Formulation in Standard Form:

$$\text{Minimize} \quad \sum_{j \in P} \frac{D_j}{2}(p_j^1)^2 + \sum_{j \in P} \frac{D_j}{2} \sum_{f=2}^{F} (p_j^f - p_j^{f-1})^2 \tag{B.1}$$

Subject to

$$x_{ij}^{m1f} = 0, \quad \forall\ i \in P,\ j : (i,j) \in A,\ m = 1, \cdots, V,\ f = 1, \cdots, F \tag{B.2}$$

$$\sum_{i:(i,j)\in A} \sum_{f=1}^{F} x_{ij}^{mtf} - \sum_{k:(j,k)\in A} \sum_{f=1}^{F} x_{jk}^{m(t+1)f} = 0, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) - 1 \tag{B.3}$$

$$\sum_{i:(i,0)\in A} \sum_{f=1}^{F} x_{i0}^{mtf} - \sum_{j:(0,j)\in A} \sum_{f=1}^{F} x_{0j}^{m(t+1)f} - C3^{mt} = 0, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) - 1 \tag{B.4}$$

$$x_{ij}^{mTf} = 0, \quad \forall\ j \in P,\ i : (i,j) \in A,\ m = 1, \cdots, V,\ f = 1, \cdots, F \tag{B.5}$$

$$x_{i0}^{mtf} = 0, \quad \forall\, i : (i,0) \in A,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 2, \cdots, F \tag{B.6}$$

$$\sum_{(i,j)\in A} \sum_{f=1}^{F} x_{ij}^{mtf} + C5^{mt} = 1, \quad \forall\, m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.7}$$

$$d^{m0} = 0, \quad \forall\, m = 1, \cdots, V \tag{B.8}$$

$$\sum_{(i,j)\in A} \sum_{f=1}^{F} t_{ij} x_{ij}^{mtf} + d^{m(t-1)} - d^{mt} + C6^{mt} = 0, \quad \forall\, m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.9}$$

$$\sum_{i:(i,j)\in A} M\, x_{ij}^{mtf} - d^{mt} + C7_j^{mtf} + p_j^f = M, \quad \forall\, j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F \tag{B.10}$$

$$\sum_{i:(i,j)\in A} M\, x_{ij}^{mtf} + d^{mt} + C8_j^{mtf} - p_j^f = M, \quad \forall\, j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F \tag{B.11}$$

$$b_0^{mtf} = 0, \quad \forall\, m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 2, \cdots, F \tag{B.12}$$

$$\sum_{i:(i,j)\in A} C x_{ij}^{mtf} - b_j^{mtf} - C9_j^{mtf} = 0, \quad \forall\, j \in N,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F),\ f = 1, \cdots, F \tag{B.13}$$

$$\sum_{l=1}^{t} \sum_{f=1}^{F} b_0^{mlf} - \sum_{j\in P} \sum_{l=1}^{t} \sum_{f=1}^{F} b_j^{mlf} + C10^{mt} = 0, \quad \forall\, m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.14}$$

$$\sum_{l=1}^{t} \sum_{f=1}^{F} b_0^{mlf} - \sum_{j\in P} \sum_{l=1}^{t} \sum_{f=1}^{F} b_j^{mlf} - C11^{mt} = -C, \quad \forall\, m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.15}$$

$$\sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mt1} - D_j\, p_j^1 = 0, \quad \forall\, j \in P \tag{B.16}$$

$$\sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mtf} + D_j\, p_j^{f-1} - D_j\, p_j^f = 0, \quad \forall\, j \in P,\ f = 2, \cdots, F \tag{B.17}$$

$$p_j^F = L_j, \quad \forall\, j \in P. \tag{B.18}$$

Although some of the slack variables for the above formulation can only take binary values, given the nature of the original decision variables and constraint set, such restrictions do not need to be explicitly enforced.

As indicated in Chapter 2, Objective Function (B.1) minimizes the *total exposure*, that is, the weighted sum of squared time intervals between successive pick-ups at each pick-up location. Since evacuees are assumed to arrive at these locations at a constant rate and since they must board a vehicle at the subsequent scheduled pick-up time, (B.1) results from the expression:

$$\left\{ \text{Minimize} \quad \sum_{j \in P} \sum_{f=1}^{F} \int_{0}^{p_j^f - p_j^{f-1}} D_j t \, \mathrm{d}t \;\middle|\; p_j^0 = 0 \right\},$$

Measured in units of *people* $\times$ *minutes*, for example. As also indicated in Chapter 2, Constraints (B.2)-(B.7) control the flow of vehicles through the problem network, including the elimination of subtours; (B.8)-(B.11) and (B.18) generate a valid pick-up schedule, including the synchronization of vehicle departure times from each pick-up location (let $M = \max_{j \in P}\{L_j\} + \max_{(i,j) \in A}\{t_{ij}\}$); and (B.12)-(B.17) enforce the vehicle loading and capacity restrictions. Observe that such formulation has a bordered-structure, where p, the pick-up schedule decision variables, correspond to the set of complicating variables; Constraints (B.16)-(B.18) correspond to the set of complicating constraints; and each subproblem corresponds to the set of decision variables and constraints controlling the routing, scheduling, and loading restrictions for a vehicle. Next, alternative formulations for the BEP variant introduced in Chapter 2 are presented.

B.2 Alternate Formulation I

Consider the following, alternative formulation for the BEP, where x_{ij}^{mt} is a binary decision variable that equals 1 if vehicle m traverses arc (i,j) on its t^{th} tour, else 0, $\forall\ (i,j) \in A$, $m = 1, \cdots, V$, $t = 1, \cdots, T(F)$, and y_j^{mf} is a binary decision variable that equals 1 if vehicle m performs the f^{th} scheduled pick-up at node j, else 0, $\forall\ j \in P$, $m = 1, \cdots, V$, $f = 1, \cdots, F$. Except for $T(F)$, which now represents an upper bound on the number of times a vehicle can return to the depot to unload evacuees (number of tours), the problem notation remains unchanged. The new constraint set is given by:

Constraint Set for Alternate Formulation I

$$\sum_{j:(0,j)\in A} x_{0j}^{mt} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.19}$$

$$\sum_{i:(i,j)\in A} x_{ij}^{mt} = \sum_{k:(j,k)\in A} x_{jk}^{mt}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.20}$$

$$\sum_{i:(i,0)\in A} x_{i0}^{m(t-1)} \geq \sum_{k:(0,k)\in A} x_{0k}^{mt}, \quad \forall\ m = 1, \cdots, V,\ t = 2, \cdots, T(F) \tag{B.21}$$

$$d_j^{mt} \geq d_0^{mt} + (t_{0j} + M)x_{0j}^{mt} - M, \quad \forall\ j : (0,j) \in A,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.22}$$

$$d_j^{mt} \geq d_i^{mt} + (t_{ij} + L_i)x_{ij}^{mt} - L_i, \quad \forall\ i \in P,\ j \in P,\ i \neq j,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.23}$$

$$d_0^{mt} \geq d_i^{m(t-1)} + (t_{i0} + L_i)x_{i0}^{m(t-1)} - L_i, \quad \forall\ i : (i,0) \in A,\ m = 1, \cdots, V,\ t = 2, \cdots, T(F) \tag{B.24}$$

$$\sum_{f=1}^{F} y_j^{mf} = \sum_{i:(i,j)\in A} \sum_{t=1}^{T(F)} x_{ij}^{mt}, \quad \forall\ j \in P,\ m = 1, \cdots, V \tag{B.25}$$

$$p_j^f \geq d_j^{mt} - L_j(1 - y_j^{mf}), \quad \forall\ j \in P,\ m = 1, \cdots, V,\ f = 1, \cdots, F,\ t = 1, \cdots, T(F) \tag{B.26}$$

$$p_j^f \leq d_j^{mt} + L_j(1 - y_j^{mf}), \quad \forall\ j \in P,\ m = 1, \cdots, V,\ f = 1, \cdots, F,\ t = 1, \cdots, T(F) \tag{B.27}$$

$$\sum_{j \in P} \sum_{f=1}^{F} b_j^{mtf} \leq C, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.28}$$

$$\sum_{f=1}^{F} b_j^{mtf} \leq C \sum_{i:(i,j) \in A} x_{ij}^{mt}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.29}$$

$$\sum_{t=1}^{T(F)} b_j^{mtf} \leq C y_j^{mf}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ f = 1, \cdots, F \tag{B.30}$$

$$\sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mt1} = D_j p_j^1, \quad \forall\ j \in P \tag{B.31}$$

$$\sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mtf} = D_j (p_j^f - p_j^{f-1}), \quad \forall\ j \in P,\ f = 2, \cdots, F. \tag{B.32}$$

Objective Function (B.1) remains unchanged. Constraints (B.19)-(B.24) are similar to (B.2)-(B.7), controlling the flow of vehicles through the problem network, including the elimination of subtours. Constraint (B.19) prevents the generation of simultaneous tours; (B.20) prevents vehicles from finishing the evacuation process while parked at a pick-up location; and (B.21) enforces the sequential ordering of tours by preventing buses from starting new tours unless their previous one is completed. Constraints (B.22)-(B.24) increment the time when vehicles leave a pick-up location following a visit to the depot, leave a pick-up location following a visit to another pick-up location, and leave the depot following a visit to a pick-up location, respectively. Here, let $M = \max_{i:(i,0) \in A}\{L_i + t_{i0}\}$. Constraints (B.25)-(B.27) construct the pick-up schedule at each pick-up location, including the synchronization of vehicle departure times. Constraint (B.25) maps each scheduled pick-up to a visit by one or more buses, thus ensuring that, at most, F scheduled pick-ups take place on each pick-up location.

Constraints (B.26)-(B.27) map vehicle departure times to scheduled pick-up times and enforce a sequential pick-up schedule (i.e., they imply that $p_j^f \geq p_j^{f-1},\ \forall\ j \in P,\ f = 1, \cdots, F$). Finally, (B.28) enforces vehicle capacity restrictions; (B.29)-(B.30) prevent evacuees from boarding vehicles unless these are visiting the corresponding pick-up location; and (B.31)-(B.32) ensure that every new arrival boards a vehicle on the next scheduled pick-up time. In addition, Constraints (B.8) and (B.18) and the logical binary and non-negativity restrictions are included.

In the formulation introduced in Chapter 2, a variable index tracks each arc traversal through the problem network. Therefore, the formulation allows buses to visit the same pick-up location multiple times during the same tour, that is, it allows the construction of transit corridors. Such phenomenon is not necessarily problematic, and, in fact, may even be desired during the operation of real-life transit systems. However, in evacuation settings, such phenomenon may increase the average riding time of evacuees, as shown in Example B.1.

Example B.1 *Consider the network displayed in Figure B.1(a), with parameters* $F = 2$ *pick-ups,* $V = 1$ *bus, and* $C = 20$ *evacuees/bus. Figures B.1(b) and B.1(c) depict, respectively, the optimal pick-up schedules assuming that the construction of transit corridors is allowed (at time* $p_1^1 = 1,\ p_1^2 = L_1 = 5,\ p_2^1 = 2,\ p_2^2 = L_2 = 4,$ *and* $p_3^1 = L_3 = 3$*) and not allowed (at time* $p_1^1 = L_1 = 5,\ p_2^1 = L_2 = 4,$ *and* $p_3^1 = L_3 = 3$*). In these figures, solid lines are used to represent time periods when buses are idle, whereas dashed lines to represent periods when buses are moving. The number of evacuees on each bus, if any, is presented on top of the dashed lines. Since the formulation prevents the construction of transit corridors, it*

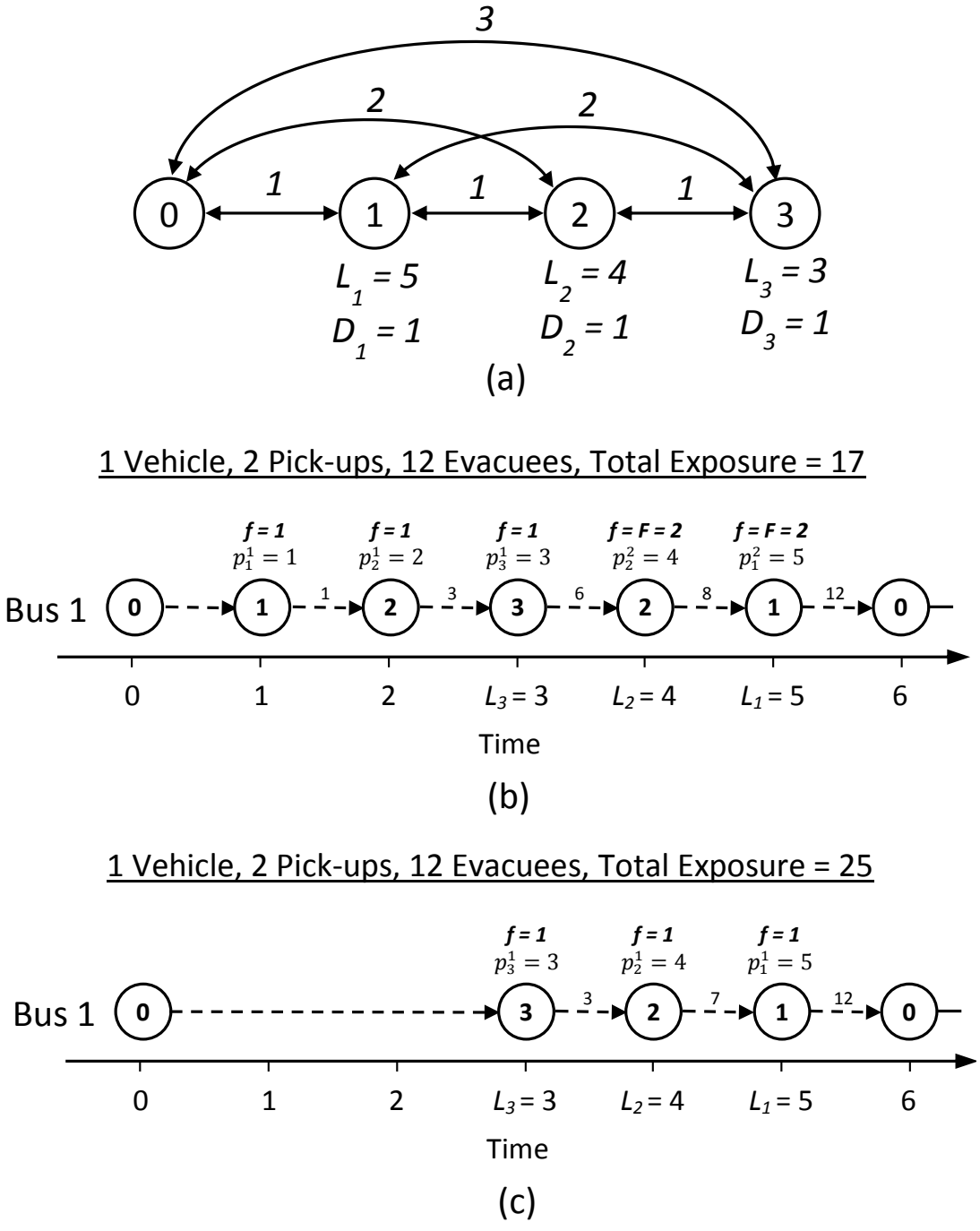

Figure B.1: An example network (a), a representation of the corresponding optimal pick-up schedule for $F = 2$ pick-ups assuming the possibility of transit corridors (b) and assuming that these are disallowed (c).

restricts the problem feasible region, thus potentially leading to infeasibility or to larger objective function values. In this example, the proposed formulation led to an increase in objective function value from 17 to 25 people × minutes. Nonetheless, such increase was accompanied by a concomitant reduction in total riding time, from 30 to 22 people × minutes.

Finally, observe that such formulation still maintains the bordered-structure of the original formulation, where p, the pick-up schedule decision variables, correspond to the set of complicating variables; Constraints (B.31)-(B.32) and (B.18) correspond to the set of complicating constraints; and each subproblem corresponds to the set of decision variables and constraints controlling the routing, scheduling, and loading restrictions for a vehicle.

B.3 Alternate Formulation II

While the formulation proposed in Section B.2 considerably restricts the search space for the problem, such can be further reduced by noting, from (B.31)-(B.32), that:

$$\begin{cases} p_j^1 = \frac{1}{D_j} \sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mt1}, \quad \forall\ j \in P, \\ p_j^f - p_j^{f-1} = \frac{1}{D_j} \sum_{m=1}^{V} \sum_{t=1}^{T(F)} b_j^{mtf}, \quad \forall\ j \in P,\ f = 2, \cdots, F \end{cases}$$

$$\Leftrightarrow p_j^f = \frac{1}{D_j} \sum_{m=1}^{V} \sum_{l=1}^{f} \sum_{t=1}^{T(F)} b_j^{mtl}, \quad \forall\ j \in P,\ f = 1, \cdots, F$$

Consequently, such constraints and the respective decision variables can be eliminated by performing the appropriate substitutions. In the new formulation, (B.8), (B.19)-(B.25), and (B.28)-(B.30) remain unchanged; (B.26)-(B.27) are replaced by (B.34)-(B.35); (B.31)-(B.32)

are removed; and Constraint $p_j^f = L_j,\ \forall\ j \in P$, is replaced by (B.36). Finally, Objective Function (B.1) is replaced by (B.33).

Constraint Set for Alternate Formulation II

$$\text{Minimize} \quad \sum_{j\in P}\sum_{f=1}^{F}\frac{1}{2\,D_j}\Big(\sum_{m=1}^{V}\sum_{t=1}^{T(F)} b_j^{mtf}\Big)^2 \tag{B.33}$$

subject to

$$\sum_{n=1}^{V}\sum_{l=1}^{f}\sum_{s=1}^{T(F)} b_j^{nsl} \geq D_j d_j^{mt} - D_j L_j(1-y_j^{mf}),\ \ \forall\ j\in P,\ m=1,\cdots,V,$$

$$f=1,\cdots,F,\ t=1,\cdots,T(F) \tag{B.34}$$

$$\sum_{n=1}^{V}\sum_{l=1}^{f}\sum_{s=1}^{T(F)} b_j^{nsl} \leq D_j d_j^{mt} + D_j L_j(1-y_j^{mf}),\ \ \forall\ j\in P,\ m=1,\cdots,V,$$

$$f=1,\cdots,F,\ t=1,\cdots,T(F) \tag{B.35}$$

$$\sum_{m=1}^{V}\sum_{f=1}^{F}\sum_{t=1}^{T(F)} b_j^{mtf} = D_j L_j,\ \ \forall\ j\in P \tag{B.36}$$

Note that the implied constraint $p_j^f \geq p_j^{f-1},\ \forall\ j \in P,\ f = 1,\cdots,F$ remains enforced by the proposed variable substitution. Objective Function (B.33) minimizes the weighted squared number of evacuees boarding the transit vehicles on each scheduled pick-up time. Since evacuees are assumed to arrive linearly, (B.33) is a proxy for the total exposure. Constraint (B.34)-(B.35) ensure that every evacuee present at a pick-up location boards a vehicle on the subsequent scheduled pick-up time, and (B.36) ensures that all evacuees are rescued. On the previous formulation, (B.36) is implied by (B.31)-(B.32), which ensure that every new arrival boards a vehicle on the next scheduled pick-up time, and by the constraint $p_j^F = L_j,\ \forall\ j \in P$, which forces the last scheduled pick-up time to coincide with

the end of the evacuee arrival process. Interestingly, by eliminating the pick-up scheduling variables, the inheriting bordered-structure of the problem change to a block diagonal or angular structure, where each subproblem remains the tour for one of the available buses, and the complicating constraints change to the allocation of evacuees to individual vehicles. Although such formulation has a simpler structure, empirical studies have demonstrated that the ensuing increase in the number of non-zero coefficients (problem density) leads to an increase in solution time. Nonetheless, such formulation allows the problem to be solved by Dantzig-Wolfe Decomposition (Dantzig and Wolfe, 1960), or, in turn, its dual problem solved by Benders' Decomposition (Benders, 1962). Research is needed to determine whether such algorithms, when used to solve such a dual angular formulation of the aforementioned BEP variant, are able to attain faster computation times than the Modified Ritter (1967) Relaxation Algorithm applied the original problem.

B.4 Alternate Formulation III

Alternatively, consider the following formulation which, similar to the one introduced in Chapter 2, combines the decision variables x_{ij}^{mt} and y_j^{mf} into a single, binary decision variable, z_{ij}^{mtf}, that equals 1 if arc (i, j) is traversed by vehicle m on its t^{th} tour for the f^{th} pick-up at node j, else 0, $\forall\ (i,j) \in A,\ m = 1, \cdots, V,\ f = 1, \cdots, F,\ t = 1, \cdots, T(F)$. For notational simplicity, the depot variables z_{i0}^{mtf} use the same index f as the ones for the pick-up nodes, but they are set to zero for pick-ups $f = 2, \cdots, F$ (i.e., let $z_{i0}^{mtf} = 0,\ \forall\ i : (i,0) \in A,\ m =$

$1, \cdots, V,\ f = 2, \cdots, F,\ t = 1, \cdots, T(F)$). Similar to the previous alternate formulations, the subscript t denotes the tour index rather than the arc traversal index, and the construction of transit corridors is prevented by the proposed constraint set.

Constraint Set for Alternate Formulation III

$$\sum_{j:(0,j)\in A} \sum_{f=1}^{F} z_{0j}^{mtf} \leq 1, \quad \forall\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.37}$$

$$\sum_{i:(i,j)\in A} \sum_{f=1}^{F} z_{ij}^{mtf} = \sum_{k:(j,k)\in A} \sum_{f=1}^{F} z_{jk}^{mtf}, \quad \forall\ j \in P,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.38}$$

$$\sum_{i:(i,0)\in A} z_{i0}^{m1(t-1)} \geq \sum_{k:(0,k)\in A} \sum_{f=1}^{F} z_{0k}^{mtf}, \quad \forall\ m = 1, \cdots, V,\ t = 2, \cdots, T(F) \tag{B.39}$$

$$d_j^{mt} \geq d_0^{mt} + (t_{0j} + M) \sum_{f=1}^{F} z_{0j}^{mtf} - M, \quad \forall\ j : (0,j) \in A,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.40}$$

$$d_j^{mt} \geq d_i^{mt} + (t_{ij} + L_i) \sum_{f=1}^{F} z_{ij}^{mtf} - L_i, \quad \forall\ i \in P,\ j \in P,\ i \neq j,\ m = 1, \cdots, V,\ t = 1, \cdots, T(F) \tag{B.41}$$

$$d_0^{mt} \geq d_i^{m(t-1)} + (t_{i0} + L_i) z_{i0}^{m(t-1)1} - L_i, \quad \forall\ i : (i,0) \in A,\ m = 1, \cdots, V,\ t = 2, \cdots, T(F) \tag{B.42}$$

$$p_j^f \geq d_j^{mt} - L_j (1 - \sum_{i:(i,j)\in A} z_{ij}^{mtf}), \quad \forall\ j \in P,\ m = 1, \cdots, V,\ f = 1, \cdots, F,\ t = 1, \cdots, T(F) \tag{B.43}$$

$$p_j^f \leq d_j^{mt} + L_j (1 - \sum_{i:(i,j)\in A} z_{ij}^{mtf}), \quad \forall\ j \in P,\ m = 1, \cdots, V,\ f = 1, \cdots, F,\ t = 1, \cdots, T(F) \tag{B.44}$$

$$\sum_{j\in P}\sum_{f=1}^{F} b_j^{mtf} \leq C, \quad \forall\, m = 1,\cdots,V,\ t = 1,\cdots,T(F) \tag{B.45}$$

$$b_j^{mtf} \leq \sum_{i:(i,j)\in A} C z_{ij}^{mtf}, \quad \forall\, j \in P,\ m = 1,\cdots,V,\ f = 1,\cdots,F,\ t = 1,\cdots,T(F) \tag{B.46}$$

$$\sum_{m=1}^{V}\sum_{t=1}^{T(F)} b_j^{mt1} = D_j p_j^1, \quad \forall\, j \in P \tag{B.47}$$

$$\sum_{m=1}^{V}\sum_{t=1}^{T(F)} b_j^{mtf} = D_j (p_j^f - p_j^{f-1}), \quad \forall\, j \in P,\ f = 2,\cdots,F. \tag{B.48}$$

The formulation is similar to the one proposed in Section B.2. Objective Function (B.1) remains unchanged. Constraint (B.37) prevents the generation of simultaneous tours; (B.38) prevents vehicles from finishing the evacuation process while parked at a pick-up location; and (B.39) enforces the sequential ordering of tours by preventing buses from starting new tours unless their previous one is completed. Constraints (B.40)-(B.42) increment the time when vehicles leave a pick-up location following a visit to the depot, leave a pick-up location following a visit to another pick-up location, and leave the depot following a visit to a pick-up location, respectively. Once again, let $M = \max_{i:(i,0)\in A}\{L_i + t_{i0}\}$. Constraints (B.43)-(B.44) construct the pick-up schedule at each pick-up location. Constraint (B.45) enforces vehicle capacity restrictions; (B.46) prevents evacuees from boarding vehicles unless these are visiting the corresponding pick-up location; and (B.47)-(B.48) ensure that every new arrival boards a vehicle on the next scheduled pick-up time. In addition, Constraints (B.8) and (B.18) and the logical binary and non-negativity restrictions are included. Similar to the formulation presented in Chapter 2, such formulation has a bordered-structure, where p, the pick-up schedule decision variables, correspond to the set of complicating variables;

Constraints (B.18) and (B.47)-(B.48) correspond to the set of complicating constraints; and each subproblem corresponds to the set of decision variables and constraints controlling the routing, scheduling, and loading restrictions for a vehicle.

Table B.3 shows a comparison of problem sizes for the formulations presented here and the one introduced in Chapter 2. Empirical studies have demonstrated that the formulation introduced in Chapter 2 attains the fastest computation times when no decomposition/relaxation strategies are employed.

Table B.3: Comparison of the alternate formulations for the BEP by problem size.

Criteria	**Chapter 2**	**Proposed Formulations**	
		I & III	II
Binary Variables	$O(\|P\|^2VF)$	$O(\|P\|^2VF)$	$O(\|P\|^2VF)$
Continuous Variables	$O(\|P\|VF)$	$O(\|P\|VF)$	$O(\|P\|VF)$
Constraints	$O(\|P\|^2VF)$	$O(\|P\|^2VF)$	$O(\|P\|^2VF)$
Non-zero Coefficients	$O(\|P\|^2VF)$	$O(\|P\|^2VF)$	$O(\|P\|^2V^2F^2)$

Appendix C

Numerical Examples of the Original and Modified Ritter (1967) Relaxation Algorithms

C.1 Example Problem

This appendix illustrates the differences between the original and the modified Ritter (1967) Relaxation Algorithms. Consider the following, bordered (or arrowhead) structure, linear programming problem. Observe that the problem is already in standard form and that all decision variables are non-negative.

$$
\begin{array}{lrcrcrcrcrcrcrcrcl}
\text{Minimize} & x^1 & + & \frac{1}{2}\,x^2 & + & \frac{3}{4}\,x^3 & + & \frac{3}{8}\,x^4 & - & x^5 & - & x^6 & + & \frac{3}{2}\,x^7 & & \\
\text{Subject to} & 2\,x^1 & & & + & x^3 & & & & & & & + & x^7 & + & 2\,x^8 & = & 11 \\
 & & & x^2 & + & 2\,x^3 & & & & & & & - & x^7 & + & x^8 & = & 8 \\
 & & & & & & + & \frac{1}{3}\,x^4 & & & + & \frac{2}{3}\,x^6 & + & x^7 & & & = & 5 \\
 & & & & & & & & & x^5 & + & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & = & 9 \\
 & \frac{1}{2}\,x^1 & + & x^2 & & & + & x^4 & + & x^5 & & & - & x^7 & + & x^8 & = & 8 \\
 & x^1 & , & x^2 & , & x^3 & , & x^4 & , & x^5 & , & x^6 & , & x^7 & , & x^8 & \geq & 0
\end{array}
$$

Let x^7 and x^8 be complicating decision variables, and let the last structural constraint be the only complicating constraint. Consequently, the problem has two subproblems: the first involving $x^1,\ x^2$, and x^3, and the second, $x^4,\ x^5$, and x^6. First, the problem is solved via the original Ritter (1967) Relaxation Algorithm.

C.2 A Numerical Example of the Ritter (1967) Relaxation Algorithm

In this Section, a numerical example of the original Ritter (1967) Relaxation Algorithm is presented.

Initialization

Using the notation of (3.1), it follows that:

$$c_1^t = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \quad c_2^t = \begin{bmatrix} \frac{3}{8} & -1 & -1 \end{bmatrix}, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad b_3 = 8.$$

Since both B_1 and B_2 are full rank matrices (i.e., since $rank(B_1) = rank(B_2) = 2$) and since both matrices have strictly more columns than rows, it is not necessary to augment this problem with artificial variables in order to ensure the existence of non-singular submatrices B_{11} and B_{21}. Now, let the decision variables, x, and the corresponding objective function,

right-hand side, and constraint coefficient matrices be partitioned as follows:

$$x_{11} \equiv \{x^1, x^2\}, \quad x_{12} \equiv \{x^3\}, \quad x_{21} \equiv \{x^4, x^5\}, \quad x_{22} \equiv \{x^6\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad c_{12}^t = \tfrac{3}{4}, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \quad B_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}, \quad A_{12} = 0, \quad A_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad A_{22} = 0, \quad D_3 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad b_3 = 8,$$

Where x_{jk} denotes the j^{th} subproblem, k^{th} subset of decision variables. Recall that $k = 1$ corresponds to the subset of relaxed variables (i.e., variables excluded from the Reduced Problem) and that $k = 2$ corresponds to the subset of variables that maintained their non-negativity restriction (i.e., variables included in the Reduced Problem). Consequently, c_{21}^t, for instance, represents the set of objective function coefficients of the relaxed variables in the second subproblem.

Iteration 1 - Current lower bound = $-\infty$

First, the Reduced Problem parameters are computed via equations (3.5):

$$d_1^t = c_{12}^t - c_{11}^t \, B_{11}^{-1} \, B_{12}$$

$$\Rightarrow d_1^t = \frac{3}{4} - \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow d_1^t = -\frac{3}{4}$$

$$d_2^t = c_{22}^t - c_{21}^t \, B_{21}^{-1} \, B_{22}$$

$$\Rightarrow d_2^t = -1 - \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}$$

$$\Rightarrow d_2^t = \frac{5}{4}$$

$$d_3^t = c_3^t - c_{11}^t \, B_{11}^{-1} \, D_1 - c_{21}^t \, B_{21}^{-1} \, D_2$$

$$\Rightarrow d_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\Rightarrow d_3^t = \begin{bmatrix} \frac{7}{8} & -\frac{5}{2} \end{bmatrix}$$

$$\alpha = c_{11}^t \, B_{11}^{-1} \, b_1 + c_{21}^t \, B_{21}^{-1} \, b_2$$

$$\Rightarrow \alpha = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 8 \end{bmatrix} + \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \alpha = \frac{49}{8}$$

$$F_1 = A_{12} - A_{11} \, B_{11}^{-1} \, B_{12}$$

$$\Rightarrow F_1 = 0 - \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow F_1 = -\frac{9}{4}$$

$$F_2 = A_{22} - A_{21} \, B_{21}^{-1} \, B_{22}$$

$$\Rightarrow F_2 = 0 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}$$

$$\Rightarrow F_2 = -5$$

$$F_3 = D_3 - A_{11}\, B_{11}^{-1}\, D_1 - A_{21}\, B_{21}^{-1}\, D_2$$

$$\Rightarrow F_3 = \begin{bmatrix} -1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\Rightarrow F_3 = \begin{bmatrix} -\frac{15}{4} & \frac{1}{2} \end{bmatrix}$$

$$\bar{b} = b_3 - A_{11}\, B_{11}^{-1}\, b_1 - A_{21}\, B_{21}^{-1}\, b_2$$

$$\Rightarrow \bar{b} = 8 - \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \bar{b} = -\frac{107}{4}.$$

Consequently, the initial Reduced Problem is:

$$\begin{array}{lrcrcrcrcl} \text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} \\ \text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} \\ & x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M \\ & x^3 & , & x^6 & , & x^7 & , & x^8 & \geq & 0, \end{array}$$

Where M is a sufficiently large constant aiming to prevent unboundedness in the Reduced Problem. Here, let $M = 1000$. The optimal solution for the Reduced Problem, $(x_{12}^*, x_{22}^*, x_3^*)$, is: $x^3 = 2107/11$, $x^6 = 0$, $x^7 = 0$, and $x^8 = 8893/11$, with objective function value

$-189963/88 \approx -2158.670$. Such objective function value corresponds to the new lower bound on the objective function value for the original (non-relaxed) problem. The corresponding optimal x_{11}^* and x_{21}^* decision variables are attained via (3.3):

$$x_{11}^* = B_{11}^{-1}\, b_1 - B_{11}^{-1}\, B_{12}\, x_{12}^* - B_{11}^{-1}\, D_1\, x_3^*$$

$$\Rightarrow x_{11}^* = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\frac{2107}{11} \right) - \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8893}{11} \end{bmatrix} \right) = \begin{bmatrix} -\frac{9886}{11} \\ -\frac{13019}{11} \end{bmatrix}$$

$$\Rightarrow x^1 = -\frac{9886}{11},\ x^2 = -\frac{13019}{11}$$

$$x_{21}^* = B_{21}^{-1}\, b_2 - B_{21}^{-1}\, B_{22}\, x_{22}^* - B_{21}^{-1}\, D_2\, x_3^*$$

$$\Rightarrow x_{21}^* = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 5 \\ 9 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix} (0) - \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8893}{11} \end{bmatrix} \right) = \begin{bmatrix} 15 \\ \frac{8992}{11} \end{bmatrix}$$

$$\Rightarrow x^4 = 15,\ x^5 = \frac{8992}{11}.$$

Since both elements of set x_{11} are negative, a new iteration is needed. Observe that the constraint used to prevent unboundedness in the Reduced Problem is binding.

Iteration 2 - Current lower bound = -2158.670

If the i^{th} element of x_{11}^* is negative, if the j^{th} element of x_{12}^* is positive, and if the i^{th} row, j^{th} column of $B_{11}^{-1}B_{12} \neq 0$, then those elements can be swapped without producing a singular B_{11} submatrix. Currently, both elements of x_{11}^* are negative and the only element of x_{12}^* is

positive. Now, consider matrix $B_{11}^{-1}B_{12}$:

$$B_{11}^{-1}B_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}.$$

Since all conditions are satisfied, both elements are suitable candidates for swapping. Note that swapping is only allowed between elements within the relaxed and non-relaxed subsets of the same subproblem (in this case, between elements in sets x_{11} and x_{12}). Let the first element of x_{11} (i.e., x^1) be swapped with the only element of x_{12} (i.e., x^3). Consequently, the decision variables, x, and the corresponding objective function, right-hand side, and constraint coefficient matrices are partitioned as follows:

$$x_{11} \equiv \{x^2, x^3\}, \quad x_{12} \equiv \{x^1\}, \quad x_{21} \equiv \{x^4, x^5\}, \quad x_{22} \equiv \{x^6\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \quad c_{12}^t = 1, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \quad B_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{12} = \tfrac{1}{2}, \quad A_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad A_{22} = 0, \quad D_3 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad b_3 = 8.$$

Observe that only the matrices related to the first subproblem, $c_{11}^t,\ c_{12}^t,\ B_{11},\ B_{12},\ A_{11}$, and A_{12}, changed. Provided the new problem partitioning, the corresponding Reduced Problem parameters are as follows (for simplicity, matrix operations are omitted):

$$d_1^t = c_{12}^t - c_{11}^t\, B_{11}^{-1}\, B_{12} = \frac{3}{2}$$

$$d_2^t = c_{22}^t - c_{21}^t\, B_{21}^{-1}\, B_{22} = \frac{5}{4}$$

$$d_3^t = c_3^t - c_{11}^t\, B_{11}^{-1}\, D_1 - c_{21}^t\, B_{21}^{-1}\, D_2 = \begin{bmatrix} \frac{13}{8} & -1 \end{bmatrix}$$

$$\alpha = c_{11}^t\, B_{11}^{-1}\, b_1 + c_{21}^t\, B_{21}^{-1}\, b_2 = -\frac{17}{8}$$

$$F_1 = A_{12} - A_{11}\, B_{11}^{-1}\, B_{12} = \frac{9}{2}$$

$$F_2 = A_{22} - A_{21}\, B_{21}^{-1}\, B_{22} = -5$$

$$F_3 = D_3 - A_{11}\, B_{11}^{-1}\, D_1 - A_{21}\, B_{21}^{-1}\, D_2 = \begin{bmatrix} -\frac{3}{2} & 5 \end{bmatrix}$$

$$\bar{b} = b_3 - A_{11}\, B_{11}^{-1}\, b_1 - A_{21}\, B_{21}^{-1}\, b_2 = -2.$$

Resulting in the new Reduced Problem:

$$\begin{array}{lrcrcrcrcl} \text{Minimize} & \frac{3}{2}\,x^1 & + & \frac{5}{4}\,x^6 & + & \frac{13}{8}\,x^7 & - & x^8 & - & \frac{17}{8} \\ \text{Subject to} & \frac{9}{2}\,x^1 & - & 5\,x^6 & - & \frac{3}{2}\,x^7 & + & 5\,x^8 & = & -2 \\ & x^1 & + & x^6 & + & x^7 & + & x^8 & \leq & M \\ & x^1 & , & x^6 & , & x^7 & , & x^8 & \geq & 0. \end{array}$$

Observe that the coefficients corresponding to the x_{22} decision variables remain unchanged, as no alteration has been performed to the set. The optimal solution for the Reduced Problem, $(x_{12}^*, x_{22}^*, x_3^*)$, is: $x^1 = 0,\ x^6 = 2/5,\ x^7 = 0$, and $x^8 = 0$, with objective function value $-13/8 = -1.625$. The corresponding optimal x_{11}^* and x_{21}^* decision variables are, once again, attained via (3.3):

$$x_{11}^* = B_{11}^{-1}\, b_1 - B_{11}^{-1}\, B_{12}\, x_{12}^* - B_{11}^{-1}\, D_1\, x_3^*$$

$$\Rightarrow x_{11}^{*} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} (0) - \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -14 \\ 11 \end{bmatrix}$$

$$\Rightarrow x^2 = -14,\ x^3 = 11$$

$$x_{21}^{*} = B_{21}^{-1}\, b_2 - B_{21}^{-1}\, B_{22}\, x_{22}^{*} - B_{21}^{-1}\, D_2\, x_3^{*}$$

$$\Rightarrow x_{21}^{*} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 5 \\ 9 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix} \left(\frac{2}{5} \right) - \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{71}{5} \\ \frac{39}{5} \end{bmatrix}$$

$$\Rightarrow x^4 = \frac{71}{5},\ x^5 = \frac{39}{5}.$$

Since the first element of set x_{11} is negative, a new iteration is needed. Observe that the constraint used to prevent unboundedness in the Reduced Problem is no longer binding.

Iteration 3 - Current lower bound = -1.625

Once again, it is necessary to determine if the non-negativity restriction of the first element in set x_{11} (i.e., x^2), can be enforced by swapping. Unfortunately, since x^1, the only element in set x_{12}, currently equals zero, such operation is not possible. Consequently, its non-negativity restriction must be enforced through an additional constraint in the Reduced Problem. By (3.8), it follows that:

$$x^2 \geq 0 \Leftrightarrow \{B_{11}^{-1}\, B_{12}\}_1\, x_{12} + \{B_{11}^{-1}\, D_1\}_1\, x_3 \leq \{B_{11}^{-1}\, b_1\}_1$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}_1 x^1 + \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 8 \end{bmatrix} \right\}_1$$

$\Rightarrow -4\,x^1 - 3\,x^7 - 3\,x^8 \leq -14.$

Resulting in the new Reduced Problem:

$$\begin{array}{lrcrcrcrcrcl}
\text{Minimize} & \frac{3}{2}\,x^1 & + & \frac{5}{4}\,x^6 & + & \frac{13}{8}\,x^7 & - & x^8 & - & \frac{17}{8} & \\
\text{Subject to} & \frac{9}{2}\,x^1 & - & 5\,x^6 & - & \frac{3}{2}\,x^7 & + & 5\,x^8 & = & -2 & \\
& -4\,x^1 & & & - & 3\,x^7 & - & 3\,x^8 & \leq & -14 & \leftarrow\ x^2 \geq 0 \\
& x^1 & + & x^6 & + & x^7 & + & x^8 & \leq & M & \\
& x^1 & , & x^6 & , & x^7 & , & x^8 & \geq & 0. &
\end{array}$$

Observe that since no variables were exchanged between sets, the problem parameters do not need to be recomputed. The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^1 = 0,\ x^6 = 76/15,\ x^7 = 0$, and $x^8 = 14/3$, with objective function value $-11/24 \approx -0.458$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.3):

$$x^*_{11} = B^{-1}_{11}\,b_1 - B^{-1}_{11}\,B_{12}\,x^*_{12} - B^{-1}_{11}\,D_1\,x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} (0) - \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{14}{3} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$$

$$\Rightarrow x^2 = 0,\ x^3 = \frac{5}{3}$$

$$x^*_{21} = B^{-1}_{21}\,b_2 - B^{-1}_{21}\,B_{22}\,x^*_{22} - B^{-1}_{21}\,D_2\,x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 5 \\ 9 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix} \left(\frac{76}{15} \right) - \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{14}{3} \end{bmatrix} \right) = \begin{bmatrix} \frac{73}{15} \\ -\frac{23}{15} \end{bmatrix}$$

$$\Rightarrow x^4 = \frac{73}{15},\ x^5 = -\frac{23}{15}.$$

Since the second element of set x_{21} is negative, a new iteration is needed.

Iteration 4 - Current lower bound = -0.458

In order to maintain the size of the Reduced Problem small, the algorithm evaluates whether any constraints added in previous iterations remain needed. Currently, $x^2 = -4\,x^1 - 3\,x^7 - 3\,x^8 + 14 = -4(0) - 3(0) - 3(14/3) + 14 = 0$, implying that the constraint is binding. Note that such computation is unnecessary since the previous iteration already determined that $x^2 = 0$. If $x^2 \neq 0$ (or, more specifically, if $x^2 > 0$, as the constraint enforces $x^2 \geq 0$), then such constraint could be removed from the Reduced Problem without affecting the current solution. Nonetheless, this constraint may still be eliminated provided that a suitable candidate for swapping between the elements in sets x_{11} and x_{12}, exists. Unfortunately, since x^1, the only element in set x_{12}, currently equals zero, a swapping operation is not possible, and the constraint cannot be removed from the Reduced Problem.

Next, the algorithm determines whether the second element in x_{21}, namely, x^5, which is currently negative, can be swapped with a positive element in set x_{22}. Currently x^6, the only element in set x_{22}, is positive. Now, consider matrix $B_{21}^{-1}B_{22}$:

$$B_{21}^{-1}B_{22} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since the second row, first column of $B_{21}^{-1}B_{22} \neq 0$, the decision variables x^5 and x^6 can be swapped without producing a singular B_{21} submatrix. Consequently, the decision variables, x, and the corresponding objective function, right-hand side, and constraint coefficient

matrices are partitioned as follows:

$$x_{11} \equiv \{x^2, x^3\}, \quad x_{12} \equiv \{x^1\}, \quad x_{21} \equiv \{x^4, x^6\}, \quad x_{22} \equiv \{x^5\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \quad c_{12}^t = 1, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$\Rightarrow \quad B_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{12} = \tfrac{1}{2}, \quad A_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{22} = 1, \quad D_3 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad b_3 = 8.$$

Once again, observe that only the matrices related to the second subproblem, c_{21}^t, c_{22}^t, B_{21}, B_{22}, A_{21}, and A_{22} changed. Provided the new problem partitioning, the corresponding Reduced Problem parameters are as follows (for simplicity, matrix operations are omitted):

$$d_1^t = c_{12}^t - c_{11}^t \, B_{11}^{-1} \, B_{12} = \frac{3}{2}$$

$$d_2^t = c_{22}^t - c_{21}^t \, B_{21}^{-1} \, B_{22} = -\frac{5}{12}$$

$$d_3^t = c_3^t - c_{11}^t \, B_{11}^{-1} \, D_1 - c_{21}^t \, B_{21}^{-1} \, D_2 = \begin{bmatrix} \frac{17}{12} & -\frac{7}{12} \end{bmatrix}$$

$$\alpha = c_{11}^t \, B_{11}^{-1} \, b_1 + c_{21}^t \, B_{21}^{-1} \, b_2 = \frac{13}{8}$$

$$F_1 = A_{12} - A_{11} \, B_{11}^{-1} \, B_{12} = \frac{9}{2}$$

$$F_2 = A_{22} - A_{21} \, B_{21}^{-1} \, B_{22} = \frac{5}{3}$$

$$F_3 = D_3 - A_{11} \, B_{11}^{-1} \, D_1 - A_{21} \, B_{21}^{-1} \, D_2 = \begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \end{bmatrix}$$

$$\bar{b} = b_3 - A_{11} \, B_{11}^{-1} \, b_1 - A_{21} \, B_{21}^{-1} \, b_2 = 13.$$

Since the swapping operation did not involve any elements of the first subproblem, the parameters for the previously added constraint (currently enforcing the non-negativity restriction of $x^2 \in x_{11}$) do not need to be recomputed, resulting in the new Reduced Problem:

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & \frac{3}{2}\,x^1 & - & \frac{5}{12}\,x^5 & + & \frac{17}{12}\,x^7 & - & \frac{7}{12}\,x^8 & + & \frac{13}{8} \\
\text{Subject to} & \frac{9}{2}\,x^1 & + & \frac{5}{3}\,x^5 & - & \frac{2}{3}\,x^7 & + & \frac{10}{3}\,x^8 & = & 13 \\
& -4\,x^1 & & & - & 3\,x^7 & - & 3\,x^8 & \leq & -14 \quad \leftarrow \quad x^2 \geq 0 \\
& x^1 & + & x^5 & + & x^7 & + & x^8 & \leq & M \\
& x^1 & , & x^5 & , & x^7 & , & x^8 & \geq & 0.
\end{array}
$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^1 = 0,\ x^5 = 0,\ x^7 = 23/36$, and $x^8 = 145/36$, with objective function value $13/72 \approx 0.181$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.3):

$$
\begin{aligned}
& x^*_{11} = B^{-1}_{11}\,b_1 - B^{-1}_{11}\,B_{12}\,x^*_{12} - B^{-1}_{11}\,D_1\,x^*_3 \\
\Rightarrow & x^*_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 11 \\ 8 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} (0) - \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{23}{36} \\ \frac{145}{36} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{83}{36} \end{bmatrix} \\
\Rightarrow & x^2 = 0,\ x^3 = \frac{83}{36} \\
& x^*_{21} = B^{-1}_{21}\,b_2 - B^{-1}_{21}\,B_{22}\,x^*_{22} - B^{-1}_{21}\,D_2\,x^*_3 \\
\Rightarrow & x^*_{21} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} 5 \\ 9 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0) - \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{23}{36} \\ \frac{145}{36} \end{bmatrix} \right) = \begin{bmatrix} \frac{83}{18} \\ \frac{305}{72} \end{bmatrix} \\
\Rightarrow & x^4 = \frac{83}{18},\ x^6 = \frac{305}{72}.
\end{aligned}
$$

Since all elements of sets x_{11} and x_{21} are non-negative, the algorithm terminates with optimal solution: $x^1 = 0,\ x^2 = 0,\ x^3 = 83/36,\ x^4 = 83/18,\ x^5 = 0,\ x^6 = 305/72,\ x^7 = 23/36$, and $x^8 = 145/36$, and objective function value of $13/72 \approx 0.181$.

Next, it is shown that the original problem can be solved without the swapping operation. Although the operation helps maintain the size of the Reduced Problem small, restricting the number of constraints enforcing the non-negativity restriction of the relaxed variables also demands considerably more computations, thus leading to numerical instabilities.

C.3 A Numerical Example of the Ritter (1967) Relaxation Algorithm Without the Swapping Operation

In this Section, a numerical example for a variation of the Ritter (1967) Relaxation Algorithm, relying solely on the inclusion/exclusion of constraints from the Reduced Problem, is presented.

Initialization

Consider the same linear programming problem assessed earlier, but, for notational consistency, let the problem parameters be defined as in (3.9). Similar to the original algorithm, let the decision variables, x, and the corresponding objective function, right-hand side, and constraint coefficient matrices be partitioned as follows:

$$x_{11} \equiv \{x^1, x^2\}, \quad x_{12} \equiv \{x^3\}, \quad x_{21} \equiv \{x^4, x^5\}, \quad x_{22} \equiv \{x^6\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad c_{12}^t = \tfrac{3}{4}, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\Rightarrow \quad B_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Aeq_{11} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}, \quad Aeq_{12} = 0, \quad Aeq_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad Aeq_{22} = 0,$$

$$Deq = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad beq = 8$$

Since the algorithm does not rely on the swapping operation, such problem parameters remain unchanged throughout its execution. Now, let the following parameters, computed via Equations (3.10), be defined. Such constants will be reused throughout the execution of the algorithm:

$$\overline{B}_1 = B_{11}^{-1} B_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

$$\overline{B}_2 = B_{21}^{-1} B_{22} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\overline{D}_1 = B_{11}^{-1} D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix}$$

$$\overline{D}_2 = B_{21}^{-1} D_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\overline{b}_1 = B_{11}^{-1} b_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix}$$

$$\overline{b}_2 = B_{21}^{-1} b_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

Iteration 1 - Current lower bound = $-\infty$

First, the Reduced Problem parameters are computed via (3.13). Except for a few notational differences, these computations are identical to the ones performed on the first iteration of

the original Ritter (1967) Relaxation Algorithm. Hence, they are omitted for brevity. The initial Reduced Problem is as follows:

$$\begin{array}{lrcrcrcrcr}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \geq & 0,
\end{array}$$

Where M, once again, is a sufficiently large constant aiming to prevent unboundedness in the Reduced Problem. Here, let $M = 1000$. The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 2107/11,\ x^6 = 0,\ x^7 = 0$, and $x^8 = 8893/11$, with objective function value $-189963/88 \approx -2158.670$. Such objective function value corresponds to the new lower bound on the objective function value for the original (non-relaxed) problem. The corresponding optimal x^*_{11} and x^*_{21} decision variables are attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \left(\frac{2107}{11}\right) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8893}{11} \end{bmatrix} = \begin{bmatrix} -\frac{9886}{11} \\ -\frac{13019}{11} \end{bmatrix}$$

$$\Rightarrow x^1 = -\frac{9886}{11},\ x^2 = -\frac{13019}{11}$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (0) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{8893}{11} \end{bmatrix} = \begin{bmatrix} 15 \\ \frac{8992}{11} \end{bmatrix}$$

$$\Rightarrow x^4 = 15,\ x^5 = \frac{8992}{11}.$$

Since both elements of set x_{11} are negative, a new iteration is needed. Observe that the constraint used to prevent unboundedness in the Reduced Problem is binding.

Iteration 2 - Current lower bound = -2158.670

In the original algorithm, a test is performed to determine whether any variables can be swapped between the relaxed and non-relaxed sets. Here, however, such operation is not performed. Instead, the non-negativity restriction on violated decision variables is enforced by new constraints added to the Reduced Problem. Since added constraints may also affect the current solution of alternative decision variables (unintentionally enforcing their non-negativity restriction), as a rule-of-thumb, let only the restriction on the most violated decision variable within each subproblem be enforced, in this case, x^2. Now, by (3.12):

$$x^2 \geq 0 \Leftrightarrow \{\overline{B}_1\}_2\, x_{12} + \{\overline{D}_1\}_2\, x_3 \leq \{\overline{b}_1\}_2$$

$$\Rightarrow \left\{ \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \right\}_2 x^3 + \left\{ \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} \right\}_2$$

$$\Rightarrow 2\,x^3 - x^7 + x^8 \leq 8.$$

Resulting in the new Reduced Problem:

$$
\begin{array}{lrcrcrcrcll}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow\ x^2 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \geq & 0. &
\end{array}
$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 0,\ x^6 = 0,\ x^7 = 123/13$, and $x^8 = 227/13$, with objective function value $-117/4 = -29.25$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$
\begin{aligned}
& x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3 \\
\Rightarrow & x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (0) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{123}{13} \\ \frac{227}{13} \end{bmatrix} = \begin{bmatrix} -\frac{217}{13} \\ 0 \end{bmatrix} \\
\Rightarrow & x^1 = -\frac{217}{13},\ x^2 = 0 \\
& x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3 \\
\Rightarrow & x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (0) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{127}{13} \\ \frac{227}{13} \end{bmatrix} = \begin{bmatrix} -\frac{174}{13} \\ \frac{565}{26} \end{bmatrix} \\
\Rightarrow & x^4 = -\frac{174}{13},\ x^5 = \frac{565}{26}.
\end{aligned}
$$

Since the first element of sets x_{11} and x_{21} are negative, a new iteration is needed.

Iteration 3 - Current lower bound = -29.25

Now, in the original Ritter (1967) Relaxation Algorithm, previously added constraints can be removed from the Reduced Problem if they are not binding or if they are binding but a suitable candidate for swapping exists. Here, however, constraints are only eliminated provided they are redundant. Consequently, the restriction $x^2 \geq 0$ cannot be removed from the Reduced Problem.

Next, the constraints enforcing the non-negativity restriction on the first element of sets x_{11} and x_{21} (i.e., x^1 and x^4, respectively) are generated and added to the Reduced Problem. Once again, by (3.12):

$$
\begin{aligned}
x^1 \geq 0 \Leftrightarrow & \{\overline{B}_1\}_1 \, x_{12} + \{\overline{D}_1\}_1 \, x_3 \leq \{\overline{b}_1\}_1 \\
& \Rightarrow \left\{ \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \right\}_1 x^3 + \left\{ \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} \right\}_1 \\
& \Rightarrow \frac{1}{2} \, x^3 + \frac{1}{2} \, x^7 + x^8 \leq \frac{11}{2} \\
x^4 \geq 0 \Leftrightarrow & \{\overline{B}_2\}_1 \, x_{22} + \{\overline{D}_2\}_1 \, x_3 \leq \{\overline{b}_2\}_1 \\
& \Rightarrow \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}_1 x^6 + \left\{ \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} 15 \\ 9 \end{bmatrix} \right\}_1 \\
& \Rightarrow 2 \, x^6 + 3 \, x^7 \leq 15.
\end{aligned}
$$

Resulting in the new Reduced Problem:

$$
\begin{array}{lrcrcrcrcrcll}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow & x^2 \geq 0 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow & x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 15 & \leftarrow & x^4 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & & \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \geq & 0. & &
\end{array}
$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 5/3,\ x^6 = 76/15,\ x^7 = 0$, and $x^8 = 14/3$, with objective function value $-11/24 \approx -0.458$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$
\begin{aligned}
& x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3 \\
\Rightarrow & x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \left(\frac{5}{3}\right) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{14}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\Rightarrow & x^1 = 0,\ x^2 = 0 \\
& x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3 \\
\Rightarrow & x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \left(\frac{76}{15}\right) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{14}{3} \end{bmatrix} = \begin{bmatrix} \frac{73}{15} \\ -\frac{23}{15} \end{bmatrix} \\
\Rightarrow & x^4 = \frac{73}{15},\ x^5 = -\frac{23}{15}.
\end{aligned}
$$

Since the second element of set x_{21} is negative, a new iteration is needed.

Iteration 4 - Current lower bound = -0.458

While non-binding, the constraint enforcing the non-negativity restriction of the first element of set x_{21} (i.e., x^4) is maintained. Next, the constraint enforcing the non-negativity restriction on the second element of set x_{21} (i.e., x^5) is generated and added to the Reduced Problem. Once again, by (3.12):

$$x^5 \geq 0 \Leftrightarrow \{\overline{B}_2\}_2\, x_{22} + \{\overline{D}_2\}_2\, x_3 \leq \{\bar{b}_2\}_2$$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}_2 x^6 + \left\{ \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} 15 \\ 9 \end{bmatrix} \right\}_2$$

$$\Rightarrow 3\,x^6 + \frac{1}{2}\,x^7 - x^8 \leq 9.$$

Resulting in the new Reduced Problem:

$$\begin{array}{lrcrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow & x^2 \geq 0 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow & x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 15 & \leftarrow & x^4 \geq 0 \\
& & & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & \leq & 9 & \leftarrow & x^5 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & & \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \geq & 0. & &
\end{array}$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 83/36$, $x^6 =$

$305/72,\ x^7 = 23/36$, and $x^8 = 145/36$, with objective function value $13/72 \approx 0.181$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \left(\frac{83}{36}\right) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{23}{36} \\ \frac{145}{36} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x^1 = 0,\ x^2 = 0$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \left(\frac{305}{72}\right) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{23}{36} \\ \frac{145}{36} \end{bmatrix} = \begin{bmatrix} \frac{83}{18} \\ 0 \end{bmatrix}$$

$$\Rightarrow x^4 = \frac{83}{18},\ x^5 = 0.$$

Since all elements of sets x_{11} and x_{21} are non-negative, the algorithm terminates with optimal solution: $x^1 = 0,\ x^2 = 0,\ x^3 = 83/36,\ x^4 = 83/18,\ x^5 = 0,\ x^6 = 305/72,\ x^7 = 23/36$, and $x^8 = 145/36$, and objective function value of $13/72 \approx 0.181$, the same solution identified by the original Ritter (1967) Relaxation Algorithm.

By avoiding the swapping operation, the proposed variation is able to solve the problem with considerably fewer computations, and, thus, with improved numerical stability. Unfortunately, such benefit is attained at the cost of an increased number of constraints in the Reduced Problem. Next, it is shown that an integer solution for the Reduced Problem automatically results in an integer solution to the original problem.

C.4 A Numerical, Integer Example of the Ritter (1967) Relaxation Algorithm Without the Swapping Operation

In this Section, a numerical, integer example of the Ritter (1967) Relaxation Algorithm, relying solely on the inclusion/exclusion of constraints from the Reduced Problem, is presented.

Initialization

Consider the same problem assessed earlier, but, now, let $x \in \mathbb{Z}_+$, that is, let the problem decision variables be non-negative integers. Once again, let the decision variables, x, and the corresponding objective function, right-hand side, and constraint coefficient matrices be partitioned as follows:

$$x_{11} \equiv \{x^1, x^2\}, \quad x_{12} \equiv \{x^3\}, \quad x_{21} \equiv \{x^4, x^5\}, \quad x_{22} \equiv \{x^6\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad c_{12}^t = \frac{3}{4}, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\Rightarrow \quad B_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Aeq_{11} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}, \quad Aeq_{12} = 0, \quad Aeq_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad Aeq_{22} = 0,$$

$$Deq = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad beq = 8.$$

In addition, let the following constants be defined:

$$\overline{B}_1 = B_{11}^{-1} B_{12} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}, \quad \overline{B}_2 = B_{21}^{-1} B_{22} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\overline{D}_1 = B_{11}^{-1} D_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix}, \quad \overline{D}_2 = B_{21}^{-1} D_2 = \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\overline{b}_1 = B_{11}^{-1} b_1 = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix}, \quad \overline{b}_2 = B_{21}^{-1} b_2 = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

Iteration 1 - Current lower bound = $-\infty$

First, the Reduced Problem parameters are computed via (3.13). Except for a few notational differences, these computations are identical to the ones performed on the first iteration of the original Ritter (1967) Relaxation Algorithm. Hence, they are omitted for brevity. The initial Reduced Problem is as follows:

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \in & \mathbb{Z}_+.
\end{array}
$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 190,\ x^6 = 0,\ x^7 = 1,$ and $x^8 = 809$, with objective function value -2158. Such objective function value corresponds to the new lower bound on the objective function value for the original (non-relaxed) problem. The corresponding optimal x^*_{11} and x^*_{21} decision variables are attained via (3.11):

$$
\Rightarrow x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3
$$

$$
\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (190) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 809 \end{bmatrix} = \begin{bmatrix} -899 \\ -1180 \end{bmatrix}
$$

$$
\Rightarrow x^1 = -899,\ x^2 = -1180
$$

$$
x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3
$$

$$
\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (0) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 809 \end{bmatrix} = \begin{bmatrix} 12 \\ \frac{1635}{2} \end{bmatrix}
$$

$$
\Rightarrow x^4 = 12,\ x^5 = \frac{1635}{2}.
$$

Since both elements of set x_{11} are negative, a new iteration is needed. Observe that the constraint used to prevent unboundedness in the Reduced Problem is binding.

Iteration 2 - Current lower bound = -2158

Next, the constraint enforcing the non-negativity restriction on the second element of set x_{11} (i.e., x^2) is generated and added to the Reduced Problem. A constraint could also be generated for the first element, but, as a rule-of-thumb, here, the operation is performed only for the most negative element on each subproblem. Now, by (3.12):

$$x^2 \geq 0 \Leftrightarrow \{\overline{B}_1\}_2\, x_{12} + \{\overline{D}_1\}_2\, x_3 \leq \{\bar{b}_1\}_2$$

$$\Rightarrow \left\{ \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \right\}_2 x^3 + \left\{ \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} \right\}_2$$

$$\Rightarrow 2\,x^3 - x^7 + x^8 \leq 8.$$

Resulting in the new Reduced Problem:

$$\begin{array}{lrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 \qquad \leftarrow\ x^2 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \in & \mathbb{Z}_+.
\end{array}$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 0,\ x^6 = 0,\ x^7 = 9,$ and $x^8 = 14$, with objective function value -21. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \bar{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x_{11}^* = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (0) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \end{bmatrix}$$

$$\Rightarrow x^1 = -13,\ x^2 = 3$$

$$x_{21}^* = \overline{b}_2 - \overline{B}_2\, x_{22}^* - \overline{D}_2\, x_3^*$$

$$\Rightarrow x_{21}^* = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (0) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} -12 \\ \frac{37}{2} \end{bmatrix}$$

$$\Rightarrow x^4 = -12,\ x^5 = \frac{37}{2}.$$

Since the first element of sets x_{11} and x_{21} are negative, a new iteration is needed.

Iteration 3 - Current lower bound = -21

First, it is necessary to assert whether any previously added constraints can be removed from the Reduced Problem. Contrary to the original Ritter (1967) Relaxation Algorithm, here, constraints can only be removed if they are redundant. Clearly, such is not the case for the constraint associated with x^2.

Next, the constraints enforcing the non-negativity restriction on first element of sets x_{11} and x_{21} (i.e., x^1 and x^4, respectively) are generated and added to the Reduced Problem.

$$x^1 \geq 0 \Leftrightarrow \{\overline{B}_1\}_1\, x_{12} + \{\overline{D}_1\}_1\, x_3 \leq \{\overline{b}_1\}_1$$

$$\Rightarrow \left\{ \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \right\}_1 x^3 + \left\{ \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} \right\}_1$$

$$\Rightarrow \frac{1}{2}\,x^3 + \frac{1}{2}\,x^7 + x^8 \leq \frac{11}{2}$$

$x^4 \geq 0 \Leftrightarrow \{\overline{B}_2\}_1\, x_{22} + \{\overline{D}_2\}_1\, x_3 \leq \{\overline{b}_2\}_1$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ \\ 3 \end{bmatrix} \right\}_1 x^6 + \left\{ \begin{bmatrix} 3 & 0 \\ \\ \frac{1}{2} & -1 \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} 15 \\ \\ 9 \end{bmatrix} \right\}_1$$

$$\Rightarrow 2\,x^6 + 3\,x^7 \leq 15.$$

Resulting in the new Reduced Problem:

$$\begin{array}{lrcrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow\ x^2 \geq 0 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow\ x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 15 & \leftarrow\ x^4 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \in & \mathbb{Z}_+. &
\end{array}$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 2,\ x^6 = 4,\ x^7 = 1,$ and $x^8 = 3$, with objective function value 3. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \\ 2 \end{bmatrix} (2) - \begin{bmatrix} \frac{1}{2} & 1 \\ \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}$$

$$\Rightarrow x^1 = 1,\ x^2 = 2$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ \\ 3 \end{bmatrix} (4) - \begin{bmatrix} 3 & 0 \\ \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ \\ -\frac{1}{2} \end{bmatrix}$$

$$\Rightarrow x^4 = 4,\ x^5 = -\frac{1}{2}.$$

Since the second element of set x_{21} is negative, a new iteration is needed.

Iteration 4 - Current lower bound = 3

Next, the constraints enforcing the non-negativity restriction on the second element of set x_{21} (i.e., x^5) are generated and added to the Reduced Problem. Once again, by (3.12):

$$x^5 \geq 0 \Leftrightarrow \{\overline{B}_2\}_2\, x_{22} + \{\overline{D}_2\}_2\, x_3 \leq \{\overline{b}_2\}_2$$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ \\ 3 \end{bmatrix} \right\}_2 x^6 + \left\{ \begin{bmatrix} 3 & 0 \\ \\ \frac{1}{2} & -1 \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} 15 \\ \\ 9 \end{bmatrix} \right\}_2$$

$$\Rightarrow 3\, x^6 + \frac{1}{2}\, x^7 - x^8 \leq 9.$$

Resulting in the new Reduced Problem:

$$
\begin{array}{lrcrcrcrcrcll}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow & x^2 \geq 0 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow & x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 15 & \leftarrow & x^4 \geq 0 \\
& & & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & \leq & 9 & \leftarrow & x^5 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & & \\
& x^3 & , & x^6 & , & x^7 & , & x^8 & \in & \mathbb{Z}_+. & &
\end{array}
$$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 3,\ x^6 = 1,\ x^7 = 4,$ and $x^8 = 0$, with objective function value $69/8 = 8.625$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (3) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\Rightarrow x^1 = 2,\ x^2 = 6$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (1) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow x^4 = 1,\ x^5 = 4.$$

Since all elements of sets x_{11} and x_{21} are non-negative and integer, the algorithm terminates with optimal solution: $x^1 = 2,\ x^2 = 6,\ x^3 = 3,\ x^4 = 1,\ x^5 = 4,\ x^6 = 1,\ x^7 = 4$, and $x^8 = 0$, and objective function value of $69/8 = 8.625$.

Observe that the integer restriction on the relaxed variables was direct consequence of an integer solution to the Reduced Problem, that is, given that $x^3 = 3,\ x^6 = 1,\ x^7 = 4$, and $x^8 = 0$, the original problem simplifies to:

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\text{Minimize} & x^1 & + & \frac{1}{2}\,x^2 & + & \frac{3}{4}\,(3) & + & \frac{3}{8}\,x^4 & - & x^5 & - & (1) & + & \frac{3}{2}(4) & & \\
\text{Subject to} & 2\,x^1 & & & + & (3) & & & & & & & + & (4) & + & 2\,(0) & = & 11 \\
 & & & x^2 & + & 2(3) & & & & & & & - & (4) & + & (0) & = & 8 \\
 & & & & & & + & \frac{1}{3}\,x^4 & & & + & \frac{2}{3}(1) & + & (4) & & & = & 5 \\
 & & & & & & & & & x^5 & + & 3(1) & + & \frac{1}{2}(4) & - & (0) & = & 9 \\
 & \frac{1}{2}\,x^1 & + & x^2 & & & + & x^4 & + & x^5 & & & - & (4) & + & (0) & = & 8 \\
 & x^1 & , & x^2 & & & , & x^4 & , & x^5 & & & & & & & \in & \mathbb{Z}_+,
\end{array}
$$

Or, more specifically, to:

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & x^1 & + & \frac{1}{2}\,x^2 & + & \frac{3}{8}\,x^4 & - & x^5 & + & \frac{29}{4} \\
\text{Subject to} & 2\,x^1 & & & & & & & = & 4 \\
& & & x^2 & & & & & = & 6 \\
& & & & & \frac{1}{3}\,x^4 & & & = & \frac{1}{3} \\
& & & & & & & x^5 & = & 4 \\
& \frac{1}{2}\,x^1 & + & x^2 & + & x^4 & + & x^5 & = & 12 \\
& x^1 & , & x^2 & , & x^4 & , & x^5 & \in & \mathbb{Z}_+.
\end{array}
$$

Clearly, the last structural constraint (i.e., the complicating constraint) is redundant, and the system of equations has the unique solution $x^1 = 2,\ x^2 = 6,\ x^4 = 1$, and $x^5 = 4$, with objective function value of $69/8 = 8.625$. Next, a more interesting version of the problem, namely its application to mixed-integer problems is exemplified.

C.5 A Numerical, Mixed-Integer Example of the Ritter (1967) Relaxation Algorithm Without the Swapping Operation

In this Section, a numerical, mixed-integer example of the Ritter (1967) Relaxation Algorithm, relying solely on the inclusion/exclusion of constraints from the Reduced Problem, is presented.

Initialization

Consider the same problem assessed earlier, but, now, let $x^1,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6 \in \mathbb{Z}_+$ and let $x^7,\ x^8 \in \mathbb{R}_+$, that is, let all variables but the complicating variables be non-negative integers. Once again, let the decision variables, x, and the corresponding objective function, right-hand side, and constraint coefficient matrices be partitioned as follows:

$$x_{11} \equiv \{x^1, x^2\}, \quad x_{12} \equiv \{x^3\}, \quad x_{21} \equiv \{x^4, x^5\}, \quad x_{22} \equiv \{x^6\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad c_{12}^t = \frac{3}{4}, \quad c_{21}^t = \begin{bmatrix} \frac{3}{8} & -1 \end{bmatrix}, \quad c_{22}^t = -1, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$\Rightarrow \quad B_{21} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{2}{3} \\ 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$Aeq_{11} = \begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix}, \quad Aeq_{12} = 0, \quad Aeq_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad Aeq_{22} = 0,$$

$$Deq = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad beq = 8.$$

In addition, let the following constants be defined:

$$\overline{B}_1 = B_{11}^{-1} B_{12} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}, \quad \overline{B}_2 = B_{21}^{-1} B_{22} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\overline{D}_1 = B_{11}^{-1} D_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix}, \quad \overline{D}_2 = B_{21}^{-1} D_2 = \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$\overline{b}_1 = B_{11}^{-1} b_1 = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix}, \quad \overline{b}_2 = B_{21}^{-1} b_2 = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

Iteration 0 - Current lower bound = 0.181

For simplicity, let the algorithm start from the last Reduced Problem attained in Section C.3, except for the integer restriction on x^3 and x^6 (i.e., on the variables in sets x_{12} and x_{22}):

$$
\begin{array}{lrcrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 8 & \leftarrow & x^2 \geq 0 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow & x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 15 & \leftarrow & x^4 \geq 0 \\
& & & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & \leq & 9 & \leftarrow & x^5 \geq 0 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & & \\
& x^3 & , & x^6 & & & & & \in & \mathbb{Z}_+ & & \\
& & & & & x^7 & , & x^8 & \in & \mathbb{R}_+. & &
\end{array}
$$

The optimum solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 2,\ x^6 = 4,\ x^7 = 9/8,$ and $x^8 = 63/16$, with objective function value $49/64 \approx 0.766$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are attained via (3.11):

$$
\begin{aligned}
& x^*_{11} = \bar{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3 \\
\Rightarrow & x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (2) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{9}{8} \\ \frac{63}{16} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{19}{16} \end{bmatrix} \\
\Rightarrow & x^1 = 0,\ x^2 = \frac{19}{16} \\
& x^*_{21} = \bar{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3 \\
\Rightarrow & x^*_{21} = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (4) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{9}{8} \\ \frac{63}{16} \end{bmatrix} = \begin{bmatrix} \frac{29}{8} \\ \frac{3}{8} \end{bmatrix}
\end{aligned}
$$

$$\Rightarrow x^4 = \frac{29}{8},\ x^5 = \frac{3}{8}.$$

Since the second element of set x_{11} and both elements of set x_{21} are fractional, a new iteration is needed.

Iteration 1 - Current lower bound = 0.766

Following the same approach as the branch and bound algorithm, valid inequalities can be added to the Reduced Problem to eliminate portions of the feasible region not containing integer solutions. Suppose that the algorithm branches on x^2. Using (3.15), constraint $x^2 \leq 1$ is enforced via:

$$x^2 \leq 1 \Leftrightarrow -\{\overline{B}_1\}_2\, x_{12} - \{\overline{D}_1\}_2\, x_3 \leq 1 - \{\bar{b}_1\}_2$$

$$\Rightarrow -\left\{\begin{bmatrix} \frac{1}{2} \\ \\ 2 \end{bmatrix}\right\}_2 x^3 - \left\{\begin{bmatrix} \frac{1}{2} & 1 \\ \\ -1 & 1 \end{bmatrix}\right\}_2 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq 1 - \left\{\begin{bmatrix} \frac{11}{2} \\ \\ 8 \end{bmatrix}\right\}_2$$

$$\Rightarrow -2\,x^3 + x^7 - x^8 \leq -7,$$

Whereas, constraint $x^2 \geq 2$ is enforced via:

$$x^2 \geq 2 \Leftrightarrow \{\overline{B}_1\}_2\, x_{12} + \{\overline{D}_1\}_2\, x_3 \leq -2 + \{\bar{b}_1\}_2$$

$$\Rightarrow \left\{\begin{bmatrix} \frac{1}{2} \\ \\ 2 \end{bmatrix}\right\}_2 x^3 + \left\{\begin{bmatrix} \frac{1}{2} & 1 \\ \\ -1 & 1 \end{bmatrix}\right\}_2 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq -2 + \left\{\begin{bmatrix} \frac{11}{2} \\ \\ 8 \end{bmatrix}\right\}_2$$

$$\Rightarrow 2\,x^3 - x^7 + x^8 \leq 6.$$

Note that the restriction $x^2 \geq 0$ is now redundant for the branch containing constraint $2\,x^3 - x^7 + x^8 \leq 6$, and, thus, can be eliminated from the Reduced Problem. Now, if constraint $x^2 \leq 1 \Rightarrow -2\,x^3 + x^7 - x^8 \leq -7$ is added to the Reduced Problem, then its optimum solution, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 3,\ x^6 = 3,\ x^7 = 7/4$, and $x^8 = 25/8$, with objective function value $43/32 \approx 1.344$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are computed via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{11}{2} \\ \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \\ 2 \end{bmatrix} (3) - \begin{bmatrix} \frac{1}{2} & 1 \\ \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} \\ \\ \frac{25}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \frac{5}{8} \end{bmatrix}$$

$$\Rightarrow x^1 = 0,\ x^2 = \frac{5}{8}$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} 15 \\ \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ \\ 3 \end{bmatrix} (3) - \begin{bmatrix} 3 & 0 \\ \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} \\ \\ \frac{25}{8} \end{bmatrix} = \begin{bmatrix} \frac{15}{4} \\ \\ \frac{9}{4} \end{bmatrix}$$

$$\Rightarrow x^4 = \frac{15}{4},\ x^5 = \frac{9}{4}.$$

Conversely, if constraint $x^2 \geq 2 \Rightarrow 2\,x^3 - x^7 + x^8 \leq 6$ is added to the Reduced Problem, then its optimum solution, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 1,\ x^6 = 4,\ x^7 = 7/4$, and $x^8 = 33/8$, with objective function value $51/32 \approx 1.594$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x_{11}^* = \begin{bmatrix} \frac{11}{2} \\ 8 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} (1) - \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} \\ \frac{33}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{29}{8} \end{bmatrix}$$

$$\Rightarrow x^1 = 0,\ x^2 = \frac{29}{8}$$

$$x_{21}^* = \overline{b}_2 - \overline{B}_2\, x_{22}^* - \overline{D}_2\, x_3^*$$

$$\Rightarrow x_{21}^* = \begin{bmatrix} 15 \\ 9 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} (4) - \begin{bmatrix} 3 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{7}{4} \\ \frac{33}{8} \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow x^4 = \frac{7}{4},\ x^5 = \frac{1}{4}.$$

The initial branch and bound tree is presented in Figure C.1.

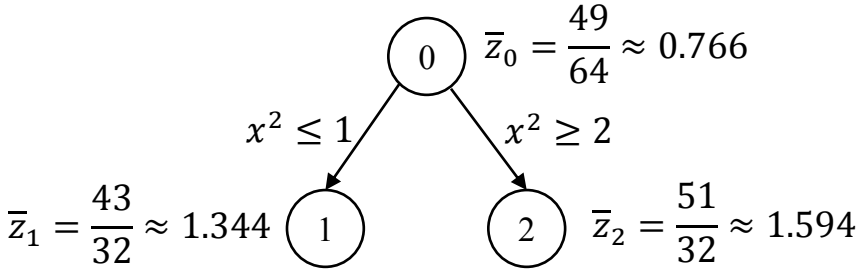

Figure C.1: First iteration of the branch and bound algorithm.

Since neither branch of the tree results in an integer solution, neither can be fathomed, and a new iteration is needed.

Iteration 2 - Current lower bound = 1.344

In order to maintain this example somewhat brief, instead of branching on every possible decision variable, a depth-first approach will be used for the branch leading to the optimum

integer solution. Note, however, that the procedure is the same regardless of the branch and bound variation used.

Suppose that the algorithm branches on node 2, and, specifically, on x^5. Following the same procedure described earlier, the constraint $x^5 \geq 1$ can be enforced in the Reduced Problem via the constraint $3\,x^6 + 0.5\,x^7 - x^8 \leq 8$, whereas the constraint $x^5 \leq 0$ can be enforced in the Reduced Problem via the constraint $-3\,x^6 - 0.5\,x^7 + x^8 \leq -9$.

The optimum solution for the Reduced Problem with the constraint $x^5 \geq 1 \Rightarrow 3\,x^6 + 0.5\,x^7 - x^8 \leq 8$, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 2$, $x^6 = 3$, $x^7 = 19/8$, and $x^8 = 53/16$, with objective function value $139/64 \approx 2.172$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are: $x^1 = 0$, $x^2 = 49/16$, $x^4 = 15/8$, and $x^5 = 17/8$. The new branch and bound tree is presented in Figure C.2.

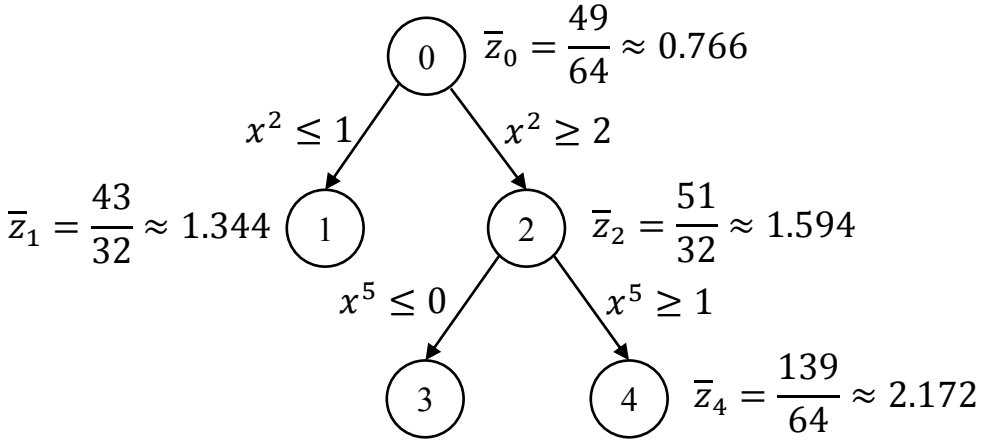

Figure C.2: Second iteration of the branch and bound algorithm.

Since some of the relaxed variables remain fractional, a new iteration is needed.

Iteration 3 - Current lower bound = 1.344

Suppose that the algorithm branches on node 4, and, specifically, on x^2. Following the same procedure described earlier, the constraint $x^2 \geq 4$ can be enforced in the Reduced Problem via the constraint $2\,x^3 - x^7 + x^8 \leq 4$, whereas the constraint $x^2 \leq 3$ can be enforced in the Reduced Problem via the constraint $-2\,x^3 + x^7 - x^8 \leq -5$.

The optimum solution for the Reduced Problem with the constraint $x^2 \geq 4 \Rightarrow 2\,x^3 - x^7 + x^8 \leq 4$, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 1,\ x^6 = 3,\ x^7 = 3$, and $x^8 = 7/2$, with objective function value 3. The corresponding optimal x^*_{11} and x^*_{21} decision variables are: $x^1 = 0,\ x^2 = 11/2,\ x^4 = 0$, and $x^5 = 2$. The new branch and bound tree is presented in Figure C.3.

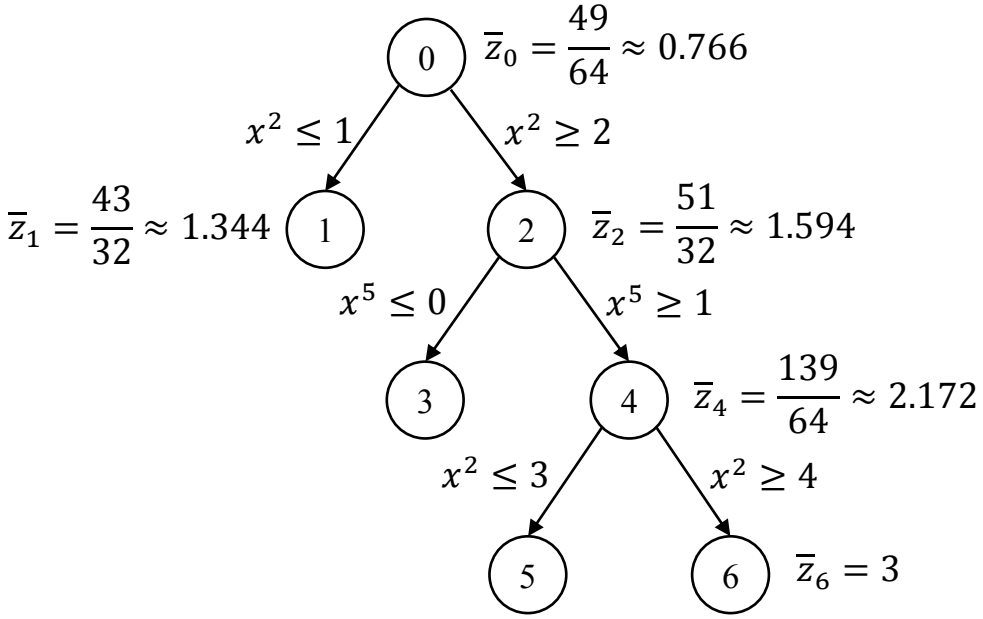

Figure C.3: Third iteration of the branch and bound algorithm.

Since x^2 remain fractional, a new iteration is needed.

Iteration 4 - Current lower bound = 1.344

Suppose that the algorithm branches on node 6, and, specifically, on x^2, as this decision variable is the only remaining fractional variable. Following the same procedure described earlier, the constraint $x^2 \geq 6$ can be enforced in the Reduced Problem via the constraint $2\,x^3 - x^7 + x^8 \leq 2$, whereas the constraint $x^2 \leq 5$ can be enforced in the Reduced Problem via the constraint $-2\,x^3 + x^7 - x^8 \leq -3$.

The optimum solution for the Reduced Problem with the constraint $x^2 \geq 6 \Rightarrow 2\,x^3 - x^7 + x^8 \leq 2$, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 0,\ x^6 = 3,\ x^7 = 29/8$, and $x^8 = 59/16$, with objective function value 245/64. The corresponding optimal x^*_{11} and x^*_{21} decision variables are: $x^1 = 0,\ x^2 = 127/16,\ x^4 = -15/8$, and $x^5 = 15/8$. The new branch and bound tree is presented in Figure C.4.

Here, however, the algorithm cannot proceed, as the relaxed variable x^4 is negative. Thus, in order to enforce its non-negativity restriction, the following constraint is added to the Reduced Problem: $x^4 \geq 0 \Rightarrow 2\,x^6 + 3\,x^7 \leq 15$. The optimum solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 1,\ x^6 = 3,\ x^7 = 38/13$, and $x^8 = 38/13$, with objective function value 35/8. The corresponding optimal x^*_{11} and x^*_{21} decision variables are: $x^1 = 8/13,\ x^2 = 6,\ x^4 = 3/13$, and $x^5 = 19/13$. The new branch and bound tree is presented in Figure C.5.

Since all decision variables, except for x^2, remain fractional, a new iteration is needed.

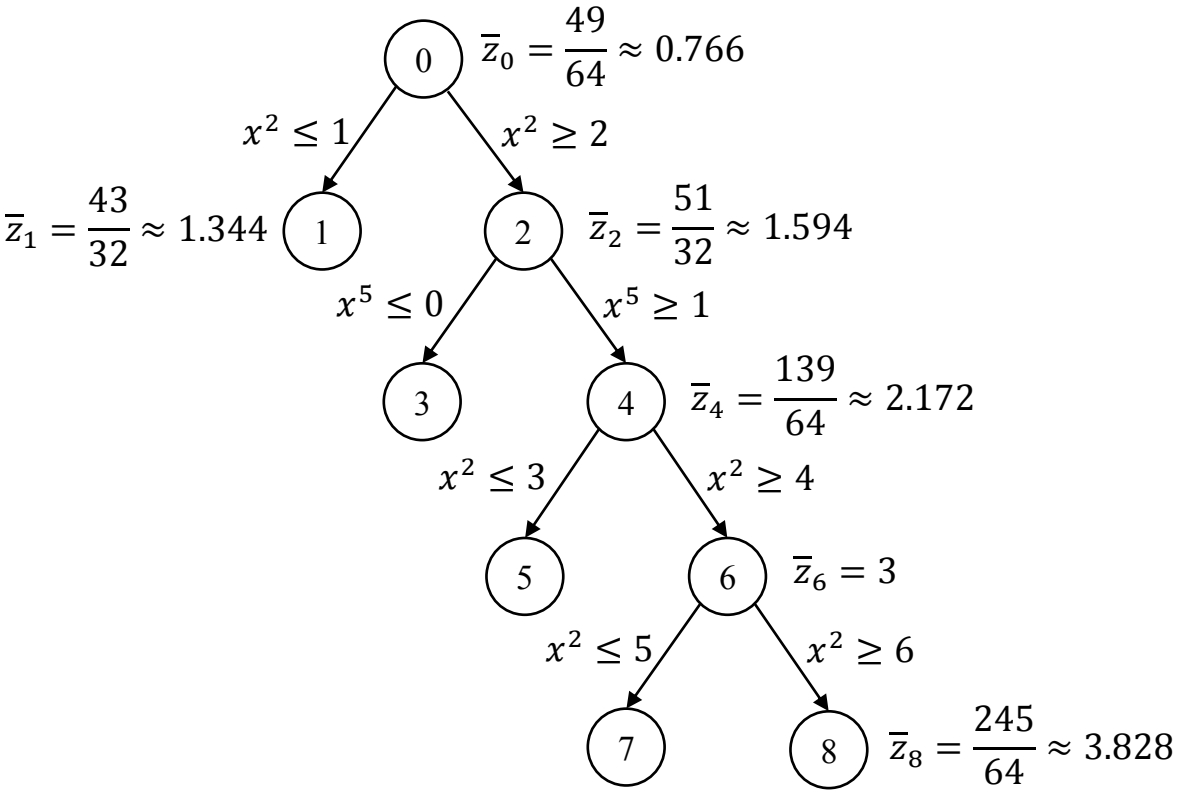

Figure C.4: Fourth iteration of the branch and bound algorithm.

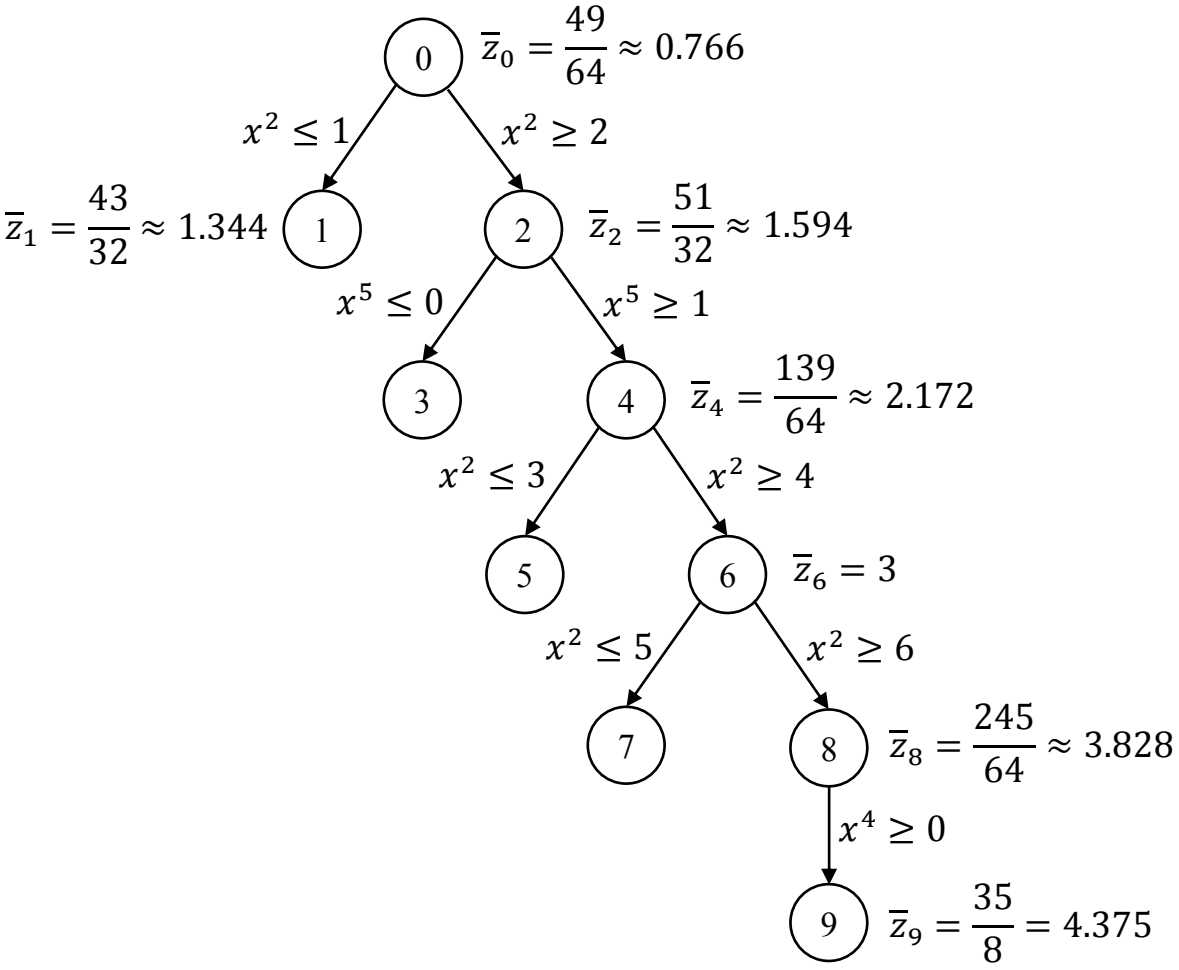

Figure C.5: Fourth iteration of the branch and bound algorithm, after a new non-negativity restriction is enforced.

Iteration 5 - Current lower bound = 1.344

Suppose that the algorithm branches on node 9, and, specifically, on x^4, as this decision variable is closest to an integer solution. Following the same procedure described earlier, the constraint $x^4 \geq 1$ can be enforced in the Reduced Problem via the constraint $2\,x^6 + 3\,x^7 \leq 14$, whereas the constraint $x^4 \leq 0$ can be enforced in the Reduced Problem via the constraint $-2\,x^6 - 3\,x^7 \leq 15$. The Reduced Problem (after the redundant constraints are removed) including the constraint $x^4 \geq 1 \Rightarrow 2\,x^6 + 3\,x^7 \leq 14$ is as follows:

$$
\begin{array}{lrcrcrcrcrcl}
\text{Minimize} & -\frac{3}{4}\,x^3 & + & \frac{5}{4}\,x^6 & + & \frac{7}{8}\,x^7 & - & \frac{5}{2}\,x^8 & + & \frac{49}{8} & & \\
\text{Subject to} & -\frac{9}{4}\,x^3 & - & 5\,x^6 & - & \frac{15}{4}\,x^7 & + & \frac{1}{2}\,x^8 & = & -\frac{107}{4} & & \\
& 2\,x^3 & & & - & x^7 & + & x^8 & \leq & 2 & \leftarrow & x^2 \geq 6 \\
& \frac{1}{2}\,x^3 & & & + & \frac{1}{2}\,x^7 & + & x^8 & \leq & \frac{11}{2} & \leftarrow & x^1 \geq 0 \\
& & & 2\,x^6 & + & 3\,x^7 & & & \leq & 14 & \leftarrow & x^4 \geq 1 \\
& & & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & \leq & 8 & \leftarrow & x^5 \geq 1 \\
& x^3 & + & x^6 & + & x^7 & + & x^8 & \leq & M & & \\
& x^3 & , & x^6 & & & & & \in & \mathbb{Z}_+ & & \\
& & & & & x^7 & , & x^8 & \in & \mathbb{R}_+. & &
\end{array}
$$

The optimum solution for this Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^3 = 3,\ x^6 = 1,\ x^7 = 4$, and $x^8 = 0$, with objective function value $69/8 = 8.625$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are: $x^1 = 2,\ x^2 = 6,\ x^4 = 1$, and $x^5 = 4$. Since all integer-restricted decision variables are integer, the branch is fathomed. The final branch and bound tree is

presented in Figure C.6. In this example, the integer solution attained corresponds to the optimum solution to the problem. In reality, however, the remaining branches would still need to be evaluated for possibly better integer solutions.

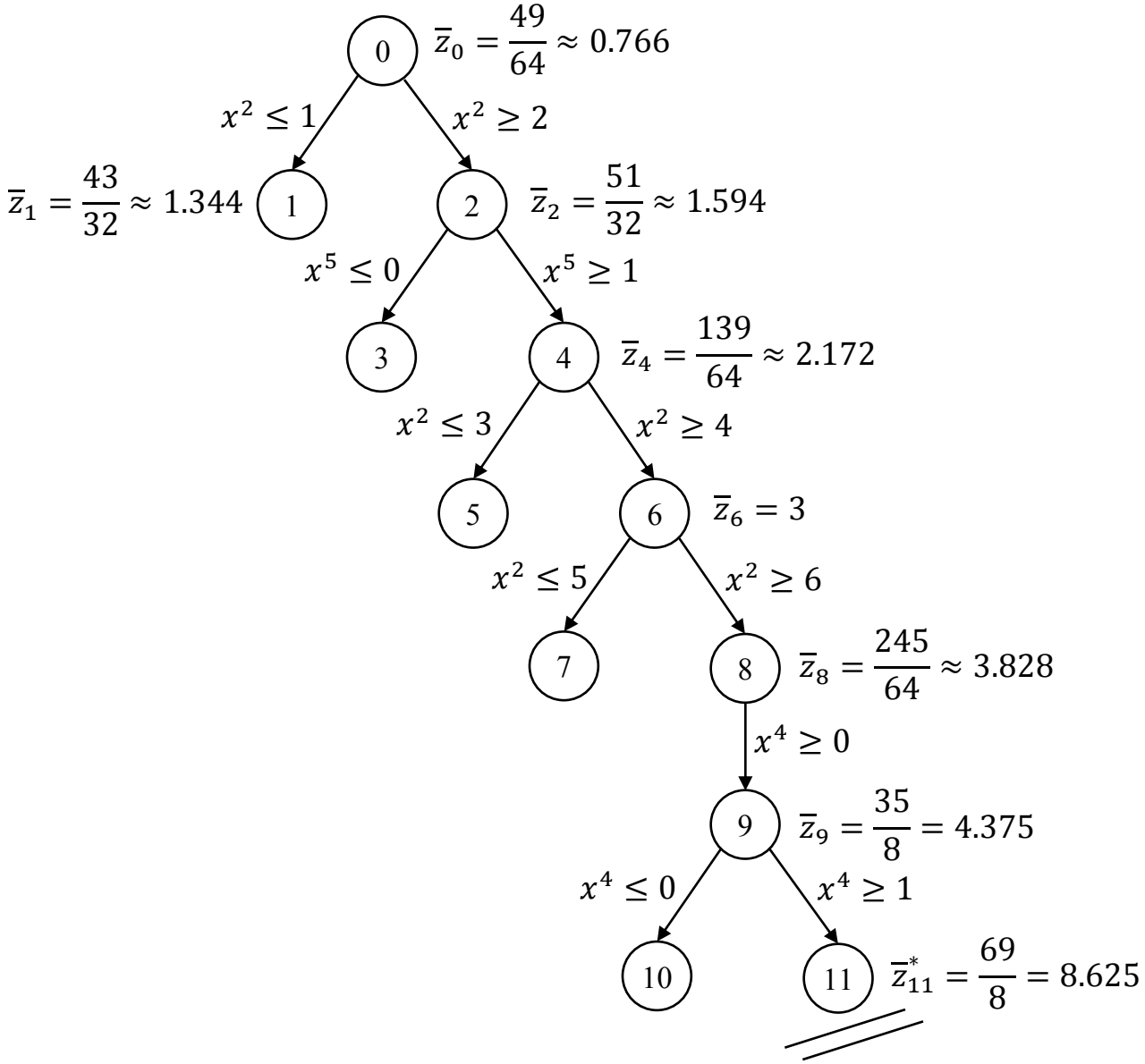

Figure C.6: Fifth iteration of the branch and bound algorithm.

C.6 A Numerical, Mixed-Integer, Quadratic Example of the Ritter (1967) Relaxation Algorithm Without the Swapping Operation

In this Section, a numerical, mixed-integer example of the Ritter (1967) Relaxation Algorithm, relying solely on the inclusion/exclusion of constraints from the Reduced Problem, is presented. Consider the following, bordered (or arrowhead) structure, mixed-integer programming problem. As before, let $x^1,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6 \in \mathbb{Z}_+$ and let $x^7,\ x^8 \in \mathbb{R}_+$, that is, let all variables but the complicating variables be non-negative integers. Observe that such example problem differs from the ones assessed in previous sections, as it includes a quadratic objective function and both complicating equality and inequality constraints.

$$
\begin{array}{lrcrcrcrcrcrcrcrcrcl}
\text{Minimize} & (x^1 & + & \frac{1}{2}\,x^2 & + & \frac{3}{4}\,x^3 & + & \frac{3}{8}\,x^4 & - & x^5 & - & x^6 & + & \frac{3}{2}\,x^7)^2 & + & & & \\
 & x^1 & + & \frac{1}{2}\,x^2 & + & \frac{3}{4}\,x^3 & + & \frac{3}{8}\,x^4 & - & x^5 & - & x^6 & + & \frac{3}{2}\,x^7 & & & & \\
\text{Subject to} & 2\,x^1 & & & + & x^3 & & & & & & & + & x^7 & + & 2\,x^8 & = & 11 \\
 & & & x^2 & + & 2\,x^3 & & & & & & & - & x^7 & + & x^8 & = & 8 \\
 & & & & & & + & \frac{1}{3}\,x^4 & & & + & \frac{2}{3}\,x^6 & + & x^7 & & & = & 5 \\
 & & & & & & & & & x^5 & + & 3\,x^6 & + & \frac{1}{2}\,x^7 & - & x^8 & = & 9 \\
 & & & x^2 & + & x^3 & & & + & x^5 & & & - & x^7 & - & x^8 & = & 0 \\
 & x^1 & + & x^2 & & & + & x^4 & & & & & + & x^7 & + & x^8 & = & 14 \\
 & \frac{1}{2}\,x^1 & + & x^2 & & & + & \frac{1}{2}\,x^4 & + & x^5 & & & - & x^7 & + & x^8 & \leq & 15 \\
 & -x^1 & + & x^2 & - & x^3 & + & x^4 & - & x^5 & + & x^6 & - & x^7 & + & x^8 & \leq & 15 \\
 & x^1 & , & x^2 & , & x^3 & , & x^4 & , & x^5 & , & x^6 & & & & & \in & \mathbb{Z}_+ \\
 & & & & & & & & & & & & & x^7 & , & x^8 & \in & \mathbb{R}_+
\end{array}
$$

Initialization

Since the subproblem constraint coefficients are identical to those in previous examples, it follows that the decision variables could be similarly partitioned. However, for completeness, suppose that a different partitioning scheme is selected upon initialization, namely:

$$x_{11} \equiv \{x^1, x^3\}, \quad x_{12} \equiv \{x^2\}, \quad x_{21} \equiv \{x^5, x^6\}, \quad x_{22} \equiv \{x^4\}, \quad x_3 \equiv \{x^7, x^8\}$$

$$c_{11}^t = \begin{bmatrix} 1 & \frac{3}{4} \end{bmatrix}, \quad c_{12}^t = \tfrac{1}{2}, \quad c_{21}^t = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad c_{22}^t = \tfrac{3}{8}, \quad c_3^t = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & 3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \quad Aeq_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Aeq_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Aeq_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Aeq_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$Deq = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad beq = \begin{bmatrix} 0 \\ 14 \end{bmatrix}$$

$$Aineq_{11} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{bmatrix}, \quad Aineq_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad Aineq_{21} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$Aineq_{22} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad Dineq = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad bineq = \begin{bmatrix} 15 \\ 15 \end{bmatrix}.$$

The Hessian matrix, the (8×8) matrix of second-order partial derivatives of the problem objective function, is similarly partitioned:

$$H = \begin{bmatrix} \frac{\partial^2}{\partial^2 x^1} & \frac{\partial^2}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2}{\partial x^1 \partial x^8} \\ \frac{\partial^2}{\partial x^1 \partial x^2} & \frac{\partial^2}{\partial^2 x^2} & \cdots & \frac{\partial^2}{\partial x^2 \partial x^8} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x^1 \partial x^8} & \frac{\partial^2}{\partial x^2 \partial x^8} & \cdots & \frac{\partial^2}{\partial^2 x^8} \end{bmatrix} = \begin{bmatrix} 2 & 1 & \frac{3}{2} & \frac{3}{4} & -2 & -2 & 3 & 0 \\ 1 & \frac{1}{2} & \frac{3}{4} & \frac{3}{8} & -1 & -1 & \frac{3}{2} & 0 \\ \frac{3}{2} & \frac{3}{4} & \frac{9}{8} & \frac{9}{16} & -\frac{3}{2} & -\frac{3}{2} & \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{3}{8} & \frac{9}{16} & \frac{9}{32} & -\frac{3}{4} & -\frac{3}{4} & \frac{9}{8} & 0 \\ -2 & -1 & -\frac{3}{2} & -\frac{3}{4} & 2 & 2 & -3 & 0 \\ -2 & -1 & -\frac{3}{2} & -\frac{3}{4} & 2 & 2 & -3 & 0 \\ 3 & \frac{3}{2} & \frac{9}{4} & \frac{9}{8} & -3 & -3 & \frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_{1111} = \begin{bmatrix} 2 & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{8} \end{bmatrix}, \quad H_{1112} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}, \quad H_{1121} = \begin{bmatrix} 1 & \frac{3}{4} \end{bmatrix}, \quad H_{1122} = \frac{1}{2}$$

$$H_{1211} = \begin{bmatrix} -2 & -2 \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}, \quad H_{1212} = \begin{bmatrix} \frac{3}{4} \\ \frac{9}{16} \end{bmatrix}, \quad H_{1221} = \begin{bmatrix} -1 & -1 \end{bmatrix}, \quad H_{1222} = \frac{3}{8}$$

$$H_{131} = \begin{bmatrix} 3 & 0 \\ \frac{9}{4} & 0 \end{bmatrix}, \quad H_{132} = \begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}$$

$$H_{2111} = \begin{bmatrix} -2 & -\frac{3}{2} \\ -2 & -\frac{3}{2} \end{bmatrix}, \quad H_{2112} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad H_{2121} = \begin{bmatrix} \frac{3}{4} & \frac{9}{16} \end{bmatrix}, \quad H_{2122} = \frac{3}{8}$$

$$H_{2211} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad H_{2212} = \begin{bmatrix} -\frac{3}{4} \\ -\frac{3}{4} \end{bmatrix}, \quad H_{2221} = \begin{bmatrix} -\frac{3}{4} & -\frac{3}{4} \end{bmatrix}, \quad H_{2222} = \frac{9}{32}$$

$$H_{231} = \begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}, \quad H_{232} = \begin{bmatrix} \frac{9}{8} & 0 \end{bmatrix}$$

$$H_{311} = \begin{bmatrix} 3 & \frac{9}{4} \\ 0 & 0 \end{bmatrix}, \quad H_{312} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}, \quad H_{321} = \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}, \quad H_{322} = \begin{bmatrix} \frac{9}{8} \\ 0 \end{bmatrix}$$

$$H_{33} = \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

As before, let the following constants be defined:

$$\overline{B}_1 = B_{11}^{-1} B_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$$

$$\overline{B}_2 = B_{21}^{-1} B_{22} = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\overline{D}_1 = B_{11}^{-1} D_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\overline{D}_2 = B_{21}^{-1} D_2 = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ \frac{3}{2} & 0 \end{bmatrix}$$

$$\overline{b}_1 = B_{11}^{-1} b_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 4 \end{bmatrix}$$

$$\overline{b}_2 = B_{21}^{-1} b_2 = \begin{bmatrix} 0 & \frac{2}{3} \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{27}{2} \\ \frac{15}{2} \end{bmatrix}.$$

Next, the Reduced Problem parameters are computed via equations (3.13) and, since the objective function is quadratic, (3.18). Matrix operations are omitted for brevity.

$$L_{jk} = H_{jk22} - \overline{B}_j^t H_{jk12} - H_{jk21}\overline{B}_k + \overline{B}_j^t H_{jk11}\overline{B}_k,\ \forall\ j = 1, \cdots, p,\ k = 1, \cdots, p$$

$$\Rightarrow L_{11} = \frac{9}{32}, \quad L_{12} = -\frac{15}{32}, \quad L_{21} = -\frac{15}{32}, \quad L_{22} = \frac{25}{32}$$

$$L_{j(p+1)} = H_{j(p+1)2} - \overline{B}_j^t H_{j(p+1)1} + \sum_{k=1}^{p} \left(\overline{B}_j^t H_{jk11} - H_{jk21} \right) \overline{D}_k,\ \forall\ j = 1, \cdots, p$$

$$\Rightarrow L_{13} = \begin{bmatrix} -\frac{33}{32} & -\frac{51}{32} \end{bmatrix}, \quad L_{23} = \begin{bmatrix} \frac{55}{32} & \frac{85}{32} \end{bmatrix}$$

$$L_{(p+1)k} = H_{(p+1)k2} - H_{(p+1)k1}\overline{B}_k + \sum_{j=1}^{p} \overline{D}_j^t \left(H_{jk11}\overline{B}_k - H_{jk12} \right),\ \forall\ k = 1, \cdots, p$$

$$\Rightarrow L_{31} = \begin{bmatrix} -\frac{33}{32} \\ -\frac{51}{32} \end{bmatrix}, \quad L_{32} = \begin{bmatrix} \frac{55}{32} \\ \frac{85}{32} \end{bmatrix}$$

$$L_{(p+1)(p+1)} = H_{(p+1)(p+1)} - \sum_{j=1}^{p} \left(\overline{D}_j^t H_{j(p+1)1} + H_{(p+1)j1}\overline{D}_j \right) + \sum_{j=1}^{p}\sum_{k=1}^{p} \overline{D}_j^t H_{jk11}\overline{D}_k$$

$$\Rightarrow L_{33} = \begin{bmatrix} \frac{121}{32} & \frac{187}{32} \\ \frac{187}{32} & \frac{289}{32} \end{bmatrix}$$

$$d_j^t = c_{j2}^t - c_{j1}^t\overline{B}_j + \frac{1}{2}\sum_{k=1}^{p} \overline{b}_k^t \left(H_{jk21}^t - H_{jk11}^t\overline{B}_j + H_{kj12} - H_{kj11}\overline{B}_j \right),\ \forall\ j = 1, \cdots, p$$

$$\Rightarrow d_1^t = \frac{39}{4}, \quad d_2^t = -\frac{65}{4}$$

$$d_{p+1}^t = c_{p+1}^t - \sum_{j=1}^{p} \left(c_{j1}^t\overline{D}_j - \frac{1}{2}\overline{b}_j^t \left(H_{j(p+1)1} + H_{(p+1)j1}^t \right) + \frac{1}{2}\sum_{k=1}^{p} \overline{b}_j^t \left(H_{jk11} + H_{kj11}^t \right) \overline{D}_k \right)$$

$$\Rightarrow d_3^t = \begin{bmatrix} -\frac{143}{4} & -\frac{221}{4} \end{bmatrix}$$

$$\alpha = \sum_{j=1}^{p} \left(c_{j1}^t\overline{b}_j + \frac{1}{2}\sum_{k=1}^{p} \overline{b}_j^t H_{jk11}\overline{b}_k \right)$$

$$\Rightarrow \alpha = \frac{675}{4}$$

$$F_j = Aeq_{j2} - Aeq_{j1}\,\overline{B}_j,\ \forall\ j = 1, \cdots, p$$

$$\Rightarrow F_1 = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{5}{4} \end{bmatrix}, \quad F_2 = \begin{bmatrix} \frac{3}{2} \\ \\ 1 \end{bmatrix}$$

$$F_{p+1} = Deq - \sum_{j=1}^{p} Aeq_{j1}\,\overline{D}_j$$

$$\Rightarrow F_3 = \begin{bmatrix} \frac{7}{2} & -\frac{1}{2} \\ \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$f = beq - \sum_{j=1}^{p} Aeq_{j1}\,\overline{b}_j$$

$$\Rightarrow f = \begin{bmatrix} \frac{19}{2} \\ \\ \frac{21}{2} \end{bmatrix}$$

$$G_j = Aineq_{j2} - Aineq_{j1}\,\overline{B}_j,\ \forall\ j = 1, \cdots, p$$

$$\Rightarrow G_1 = \begin{bmatrix} \frac{9}{8} \\ \\ \frac{5}{4} \end{bmatrix}, \quad G_2 = \begin{bmatrix} 2 \\ \\ -1 \end{bmatrix}$$

$$G_{p+1} = Dineq - \sum_{j=1}^{p} Aineq_{j1}\,\overline{D}_j$$

$$\Rightarrow G_3 = \begin{bmatrix} \frac{21}{8} & \frac{13}{8} \\ \\ -\frac{25}{4} & \frac{5}{4} \end{bmatrix}$$

$$g = bineq - \sum_{j=1}^{p} Aineq_{j1}\,\overline{b}_j$$

$$\Rightarrow g = \begin{bmatrix} \frac{107}{4} \\ \\ \frac{3}{2} \end{bmatrix}.$$

After merging the $L_{jk},\ j = 1, \cdots, 3,\ k = 1, \cdots, 3$ matrices into a larger, Hessian matrix, it follows that the objective function of the Reduced Problem is given by:

$$\text{Minimize} \quad \frac{1}{2} \begin{bmatrix} x^2 & x^4 & x^7 & x^8 \end{bmatrix} \begin{bmatrix} \frac{9}{32} & -\frac{15}{32} & -\frac{33}{32} & -\frac{51}{32} \\ -\frac{15}{32} & \frac{25}{32} & \frac{55}{32} & \frac{85}{32} \\ -\frac{33}{32} & \frac{55}{32} & \frac{121}{32} & \frac{187}{32} \\ -\frac{51}{32} & \frac{85}{32} & \frac{187}{32} & \frac{289}{32} \end{bmatrix} \begin{bmatrix} x^2 \\ x^4 \\ x^7 \\ x^8 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{39}{4} & -\frac{65}{4} & -\frac{143}{4} & -\frac{221}{4} \end{bmatrix} \begin{bmatrix} x^2 \\ x^4 \\ x^7 \\ x^8 \end{bmatrix} + \frac{675}{4},$$

Or, after some simplification:

$$\text{Minimize} \quad \left(-\frac{3}{8}x^2 + \frac{5}{8}x^4 + \frac{11}{8}x^7 + \frac{17}{8}x^8 \right)^2 + \frac{39}{4}x^2 - \frac{65}{4}x^4 - \frac{143}{4}x^7 - \frac{221}{4}x^8 + \frac{675}{4}.$$

Consequently, the initial Reduced Problem is as follows:

$$
\begin{array}{lrcrcrcrcl}
\text{Minimize} & \left(-\frac{3}{8}\,x^2 \right. & + & \frac{5}{8}\,x^4 & + & \frac{11}{8}\,x^7 & + & \left.\frac{17}{8}\,x^8\right)^2 & + & \\
 & \frac{39}{4}\,x^2 & - & \frac{65}{4}\,x^4 & - & \frac{143}{4}\,x^7 & - & \frac{221}{4}\,x^8 & + & \frac{675}{4} \\
\text{Subject to} & \frac{1}{2}\,x^2 & + & \frac{3}{2}\,x^4 & + & \frac{7}{2}\,x^7 & - & \frac{1}{2}\,x^8 & = & \frac{19}{2} \\
 & \frac{5}{4}\,x^2 & + & x^4 & + & \frac{1}{4}\,x^7 & + & \frac{1}{4}\,x^8 & = & \frac{21}{2} \\
 & \frac{9}{8}\,x^2 & + & 2\,x^4 & + & \frac{21}{8}\,x^7 & + & \frac{13}{8}\,x^8 & \leq & \frac{107}{4} \\
 & \frac{5}{4}\,x^2 & - & x^4 & - & \frac{25}{4}\,x^7 & + & \frac{5}{4}\,x^8 & \leq & \frac{3}{2} \\
 & x^2 & + & x^4 & + & x^7 & + & x^8 & \leq & M \\
 & x^2 & , & x^4 & & & & & \in & \mathbb{Z}_+ \\
 & & & & & x^7 & , & x^8 & \in & \mathbb{R}_+.
\end{array}
$$

Iteration 1 - Current lower bound = $-\infty$

The optimal solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^2 = 4,\ x^4 = 4,\ x^7 = 9/8$, and $x^8 = 39/8$, with objective function value $-247/1024 \approx -0.241$. Such objective function value corresponds to the new lower bound on the objective function value for the original (non-relaxed) problem. The corresponding optimal x^*_{11} and x^*_{21} decision variables are attained via (3.11):

$$
\begin{aligned}
& x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3 \\
\Rightarrow & x^*_{11} = \begin{bmatrix} \frac{7}{2} \\ 4 \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (4) - \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{9}{8} \\ \frac{39}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{8} \end{bmatrix} \\
\Rightarrow & x^1 = 0,\ x^3 = \frac{1}{8}
\end{aligned}
$$

$$x^*_{21} = \bar{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} -\frac{27}{2} \\ \\ \frac{15}{2} \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \\ \frac{1}{2} \end{bmatrix} (4) - \begin{bmatrix} -4 & -1 \\ \\ \frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{9}{8} \\ \\ \frac{39}{8} \end{bmatrix} = \begin{bmatrix} \frac{15}{8} \\ \\ \frac{61}{16} \end{bmatrix}$$

$$\Rightarrow x^5 = \frac{15}{8},\ x^6 = \frac{61}{16}.$$

Since the second element of set x_{11} and both elements of set x_{21} are fractional, a new iteration is needed.

Iteration 2 - Current lower bound = -0.241

Following the same approach as the branch and bound algorithm, valid inequalities can be added to the Reduced Problem to eliminate portions of the feasible region not containing integer solutions. Suppose that the algorithm branches on x^3. Using (3.15), constraint $x^3 \leq 0$ is enforced via:

$$x^3 \leq 0 \Leftrightarrow -\{\overline{B}_1\}_2\, x_{12} - \{\overline{D}_1\}_2\, x_3 \leq 0 - \{\bar{b}_1\}_2$$

$$\Rightarrow -\left\{ \begin{bmatrix} -\frac{1}{4} \\ \\ \frac{1}{2} \end{bmatrix} \right\}_2 x^2 - \left\{ \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq 0 - \left\{ \begin{bmatrix} \frac{7}{2} \\ \\ 4 \end{bmatrix} \right\}_2$$

$$\Rightarrow -\frac{1}{2}\, x^2 + \frac{1}{2}\, x^7 - \frac{1}{2}\, x^8 \leq -4,$$

Whereas, constraint $x^3 \geq 1$ is enforced via:

$$x^3 \geq 1 \Leftrightarrow \{\overline{B}_1\}_2\, x_{12} + \{\overline{D}_1\}_2\, x_3 \leq -1 + \{\bar{b}_1\}_2$$

$$\Rightarrow \left\{ \begin{bmatrix} -\frac{1}{4} \\ \\ \frac{1}{2} \end{bmatrix} \right\}_2 x^2 + \left\{ \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}_2 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq -1 + \left\{ \begin{bmatrix} \frac{7}{2} \\ \\ 4 \end{bmatrix} \right\}_2$$

$$\Rightarrow \frac{1}{2}\, x^2 - \frac{1}{2}\, x^7 + \frac{1}{2}\, x^8 \leq 3.$$

Now, if constraint $x^3 \leq 0 \Rightarrow -\frac{1}{2}\, x^2 + \frac{1}{2}\, x^7 - \frac{1}{2}\, x^8 \leq -4$ is added to the Reduced Problem, then its optimum solution, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^2 = 6,\ x^4 = 1,\ x^7 = 9/4$, and $x^8 = 23/4$, with objective function value $57/256 \approx 0.223$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, attained via (3.11):

$$x^*_{11} = \overline{b}_1 - \overline{B}_1\, x^*_{12} - \overline{D}_1\, x^*_3$$

$$\Rightarrow x^*_{11} = \begin{bmatrix} \frac{7}{2} \\ \\ 4 \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ \\ \frac{1}{2} \end{bmatrix} (6) - \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{9}{4} \\ \\ \frac{23}{4} \end{bmatrix} = \begin{bmatrix} -1 \\ \\ -\frac{3}{4} \end{bmatrix}$$

$$\Rightarrow x^1 = -1,\ x^3 = -\frac{3}{4}$$

$$x^*_{21} = \overline{b}_2 - \overline{B}_2\, x^*_{22} - \overline{D}_2\, x^*_3$$

$$\Rightarrow x^*_{21} = \begin{bmatrix} -\frac{27}{2} \\ \\ \frac{15}{2} \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \\ \frac{1}{2} \end{bmatrix} (1) - \begin{bmatrix} -4 & -1 \\ \\ \frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{9}{4} \\ \\ \frac{23}{4} \end{bmatrix} = \begin{bmatrix} \frac{11}{4} \\ \\ \frac{29}{8} \end{bmatrix}$$

$$\Rightarrow x^5 = \frac{11}{4},\ x^6 = \frac{29}{8}.$$

Conversely, if constraint $x^3 \geq 1 \Rightarrow \frac{1}{2}\, x^2 - \frac{1}{2}\, x^7 + \frac{1}{2}\, x^8 \leq 3$ is added to the Reduced Problem, then its optimum solution, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^2 = 2,\ x^4 = 7,\ x^7 = 0$, and $x^8 = 4$, with objective function value $33/64 \approx 0.516$. The corresponding optimal x^*_{11} and x^*_{21} decision

variables are, once again, computed via (3.11):

$$x_{11}^* = \overline{b}_1 - \overline{B}_1\, x_{12}^* - \overline{D}_1\, x_3^*$$

$$\Rightarrow x_{11}^* = \begin{bmatrix} \frac{7}{2} \\ 4 \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \end{bmatrix} (2) - \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x^1 = 1,\ x^3 = 1$$

$$x_{21}^* = \overline{b}_2 - \overline{B}_2\, x_{22}^* - \overline{D}_2\, x_3^*$$

$$\Rightarrow x_{21}^* = \begin{bmatrix} -\frac{27}{2} \\ \frac{15}{2} \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} (7) - \begin{bmatrix} -4 & -1 \\ \frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow x^5 = 1,\ x^6 = 4.$$

Since all integer-restricted decision variables are integer, the branch is fathomed. The solution $x^1 = 1,\ x^2 = 2,\ x^3 = 1,\ x^4 = 7,\ x^5 = 1,\ x^6 = 4,\ x^7 = 0$, and $x^8 = 4$ is saved as the new incumbent solution. The initial branch and bound tree is presented in Figure C.7.

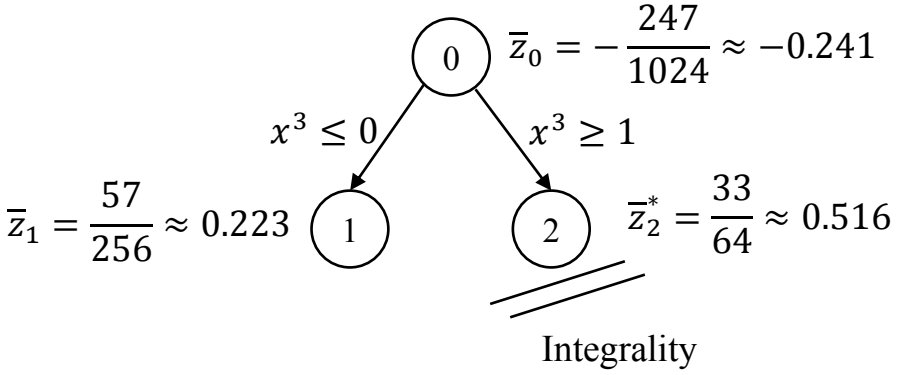

Figure C.7: First iteration of the branch and bound algorithm.

Iteration 3 - Current lower bound $= 0.223$, Incumbent Obj. $= 0.516$

As before, the algorithm would branch on node 1, as such is the only node still active. However, since both elements of set x_{11} are negative, first a constraint enforcing the non-negativity restriction of x^1 is generated and added to the Reduced Problem. Observe that a similar constraint could be generated for x^3, but, as a rule-of-thumb, here, the operation is performed only for the most negative element on each subproblem. Now, by (3.12):

$$
\begin{aligned}
&x^1 \geq 0 \Leftrightarrow \{\overline{B}_1\}_1 \, x_{12} + \{\overline{D}_1\}_1 \, x_3 \leq \{\bar{b}_1\}_1 \\
&\qquad \Rightarrow \left\{ \begin{bmatrix} -\frac{1}{4} \\ \\ \frac{1}{2} \end{bmatrix} \right\}_1 x^2 + \left\{ \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}_1 \begin{bmatrix} x^7 \\ \\ x^8 \end{bmatrix} \leq \left\{ \begin{bmatrix} \frac{7}{2} \\ \\ 4 \end{bmatrix} \right\}_1 \\
&\qquad \Rightarrow -\frac{1}{4} \, x^2 + \frac{3}{4} \, x^7 + \frac{3}{4} \, x^8 \leq \frac{7}{2}.
\end{aligned}
$$

The optimum solution for the Reduced Problem, $(x^*_{12}, x^*_{22}, x^*_3)$, is: $x^2 = 7,\ x^4 = 0,\ x^7 = 19/8,$ and $x^8 = 37/8$, with objective function value $979/159 \approx 6.157$. The corresponding optimal x^*_{11} and x^*_{21} decision variables are, once again, computed via (3.11):

$$
\begin{aligned}
&x^*_{11} = \bar{b}_1 - \overline{B}_1 \, x^*_{12} - \overline{D}_1 \, x^*_3 \\
\Rightarrow &x^*_{11} = \begin{bmatrix} \frac{7}{2} \\ \\ 4 \end{bmatrix} - \begin{bmatrix} -\frac{1}{4} \\ \\ \frac{1}{2} \end{bmatrix} (7) - \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{19}{8} \\ \\ \frac{37}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ -\frac{5}{8} \end{bmatrix} \\
\Rightarrow &x^1 = 0,\ x^3 = -\frac{5}{8} \\
&x^*_{21} = \bar{b}_2 - \overline{B}_2 \, x^*_{22} - \overline{D}_2 \, x^*_3
\end{aligned}
$$

$$\Rightarrow x_{21}^{*} = \begin{bmatrix} -\frac{27}{2} \\ \\ \frac{15}{2} \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ \\ \frac{1}{2} \end{bmatrix} (0) - \begin{bmatrix} -4 & -1 \\ \\ \frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{19}{8} \\ \\ \frac{37}{8} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} \\ \\ \frac{63}{16} \end{bmatrix}$$

$$\Rightarrow x^5 = \frac{5}{8},\ x^6 = \frac{63}{16}.$$

Since the second element of set x_{11} is negative, it would appear that another iteration is needed. However, note that the current objective function value for the node is higher than the objective function value for the incumbent solution. Consequently, the node can be fathomed due to bounding. The final branch and bound tree is presented in Figure C.8. Note that the incumbent integer solution attained on the previous iteration corresponds to the optimum solution to the problem.

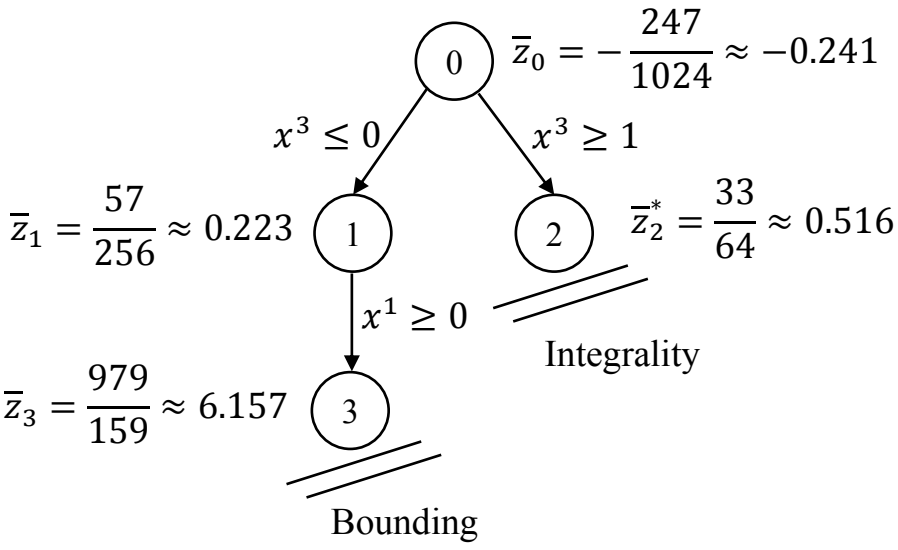

Figure C.8: First iteration of the branch and bound algorithm, after the non-negativity restriction on x^1 is enforced.

Appendix D

The Ritter (1967) Relaxation Algorithm - Source Code: MathWorks Matlab version R2012a (7.14.0.739)

D.1 Original Algorithm

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function solves problems of the type:                                  %
% Min    c{1}'*x{1}+c{2}'*x{2} + ... + c{p}'*x{p}+c{p+1}'*x{p+1}              %
% S.t.   B{1}*x{1}                                  + D{1}*x{p+1} == b{1}     %
%                  B{2}*x{2} +                      + D{2}*x{p+1} == b{2}     %
%                                 ...                                         %
%                                          B{p}*x{p} + D{p}*x{p+1} == b{p}     %
%        A{1}*x{1} + A{2}*x{2} + ... + A{p}*x{p} + D{p+1}*x{p+1} == b{p+1} %
%        x{1}, x{2}, ... , x{p}, x{p+1} >= 0                                  %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Input Parameters:                                                           %
% c{i} - a n{i}-column vector of obj. func. coefficients, i = 1,...,p+1       %
% A{i} - a m{p+1} x n{i} matrix of constraint coefficients, i = 1,...,p       %
% B{i} - a m{i} x n{i} matrix of constraint coefficients, i = 1,...,p         %
% D{i} - a m{i} x n{p+1} matrix of constraint coefficients, i = 1,...,p+1 %
% b{i} - a m{i}-column vector of RHS coefficients, i = 1,...,p+1              %
%                                                                             %
% Output Paramters:                                                           %
% x{i} - a n{i}-column vector of decision variables, i = 1,...,p+1            %
% fval - obj. fun. value                                                      %
% nIterations - Number of iterations from the Ritter algorithm                %
% sMessage - Optimization status (optimal, unbounded, infeasible, etc.        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%$
function [sOutput] = Ritter(A,B,D,b,c,bDispOutput)

%Initialize the global parameters
nInf = 10^3;
nPrecision = 10^(-9);

%Initialize the master problem property structure
```

```
31 tempNumberSubproblems = size(B,1);
32 if (tempNumberSubproblems == 0)
33     error('Ritter:InvalidParam','The problem has no subproblems');
34 end
35 tempNumberCompConstr = size(A{1},1);
36 if (tempNumberCompConstr == 0)
37     error('Ritter:InvalidParam','Missing complicating constraints');
38 end
39 tempNumberCompVars = size(c{tempNumberSubproblems+1},1);
40 if (tempNumberCompVars == 0)
41     error('Ritter:InvalidParam','Missing complicating variables');
42 end
43 mMasterProps = struct('nSubproblems',tempNumberSubproblems,...
44     'nComplicatingVars',tempNumberCompVars,...
45     'nCells',tempNumberSubproblems+1,...
46     'nComplicatingConstr',tempNumberCompConstr,...
47     'mReducedProbVars',[zeros(1,tempNumberSubproblems),tempNumberCompVars],...
48     'nTotalReducedProbVars',tempNumberCompVars,...
49     'bUpdateRedProbParam',true,...
50     'bRedProbObjIncreases',false,...
51     'bRedProbObjDecreases',false,...
52     'bHasBasicArtificials',false);
53
54 %Release the support variables
55 clear tempNumberSubproblems tempNumberCompConstr tempNumberCompVars;
56
57 %Initialize the enforced contraints property structure
58 mEnforcedConstraints = struct('nTotalExplicitConstraints',0,...
59     'nSubproblem',[],...
60     'nElement',[],...
61     'nRowReference',[]);
62
```

```
63 %Initialize the return parameters property structure
64 sOutput = struct('x',[],...
65     'nObjValue',-Inf,...
66     'nIterations',0,...
67     'nSolutionTime',clock,...
68     'sResult','',...
69     'sDescription','');
70
71 %Initialize the subproblem property structure
72 mSubProps = cell(mMasterProps.nSubproblems,1);
73
74 %Initialize the subproblem element property structure
75 mElementProps = cell(mMasterProps.nSubproblems,1);
76
77 %Initialize the subproblem solution property structure
78 mSubSolProps = cell(mMasterProps.nSubproblems,1);
79
80 %Initialize the partitioned submatrices:
81 %variables with their non-negativity restriction relaxed --> set 1;
82 %variables with their non-negativity restriction enforced --> set 2;
83 AA = cell(mMasterProps.nSubproblems,2);
84 BB = cell(mMasterProps.nSubproblems,2);
85 cc = cell(mMasterProps.nSubproblems,2);
86
87 %Initialize the reduced problem parameters
88 d = cell(mMasterProps.nCells,1);
89 F = cell(1,mMasterProps.nCells);
90
91 %Initialize the reduced problem support parameters
92 iBB = cell(mMasterProps.nSubproblems,1);
93 iBb = cell(mMasterProps.nSubproblems,1);
94 iBD = cell(mMasterProps.nSubproblems,1);
```

```
95
96  %For each subproblem
97  for i = 1:mMasterProps.nSubproblems
98
99      %Populate the i^{th} subproblem property structure
100     [tempRows,tempCols] = size(B{i});
101     tempArtificial = max(max(tempRows-tempCols+1,0),tempRows-rank(B{i}));
102     while (tempArtificial <= tempRows)
103         if (rank([B{i},eye(tempRows,tempArtificial)]) == tempRows)
104             break;
105         else
106             tempArtificial = tempArtificial+1;
107         end
108     end
109     mSubProps{i} = struct('nRealVariables',tempCols,...
110         'nArtificialVariables',tempArtificial,...
111         'nTotalVariables',tempCols+tempArtificial,...
112         'nConstraints',tempRows,...
113         'nVariablesSet1',tempRows,...
114         'nVariablesSet2',tempCols+tempArtificial-tempRows,...
115         'bUpdateRedProbParam',true,...
116         'nNonNegCase',0);
117
118     %Initialize the i^{th} subproblem element property structure
119     tempData = zeros(mSubProps{i}.nTotalVariables,1);
120     mElementProps{i} = struct('idxReference',tempData,...
121         'bIsRelaxed',boolean(tempData),...
122         'bIsArtificial',boolean(tempData),...
123         'bEnforceNonNeg',boolean(tempData),...
124         'bSwapCol',boolean(tempData),...
125         'bIsEnforced',boolean(tempData));
126
```

```
127     %Initialize the i^{th} subproblem solution property structure
128     mSubSolProps{i} = struct('nNegative',0,...
129         'idxNegatives',zeros(mSubProps{i}.nVariablesSet1,1),...
130         'idxMostNeg',0,...
131         'nMostNegativeValue',Inf,...
132         'nPositive',0,...
133         'idxPositives',zeros(mSubProps{i}.nVariablesSet2,1));
134
135     %Update the total number of variables in set 2
136     mMasterProps.mReducedProbVars(i) = mSubProps{i}.nVariablesSet2;
137     mMasterProps.nTotalReducedProbVars = ...
138         mMasterProps.nTotalReducedProbVars+mSubProps{i}.nVariablesSet2;
139
140     if mSubProps{i}.nArtificialVariables > 0
141         %If needed, augment the i^{th} subproblem A, B and c matrices /
142         %vectors to ensure the existance of a non-singular matrix
143         B{i} = [B{i},eye(mSubProps{i}.nConstraints,mSubProps{i}.nArtificialVariables)];
144         c{i} = [c{i};nInf*ones(mSubProps{i}.nArtificialVariables,1)];
145         A{i} = [A{i},...
146             zeros(mMasterProps.nComplicatingConstr,mSubProps{i}.nArtificialVariables)];
147
148         %Flag the created variables as artificial
149         mElementProps{i}.bIsArtificial(mSubProps{i}...
150             .nRealVariables+1:mSubProps{i}.nTotalVariables) = ...
151             ones(mSubProps{i}.nArtificialVariables,1);
152     end
153
154     %Idenfify a starting basis
155     tempData = rref(B{i});
156     n1 = 0;
157     n2 = 0;
158
```

```
159     %For each element in the i^{th} subproblem
160     for j = 1:mSubProps{i}.nTotalVariables
161
162         %Determine if the variable belongs to set 1 or 2 (i.e., determine
163         %if its non-negativity restriction should be relaxed or not)
164         if (n1 < mSubProps{i}.nVariablesSet1)
165             mElementProps{i}.bIsRelaxed(j) = boolean(tempData(n1+1,j) == 1);
166         else
167             mElementProps{i}.bIsRelaxed(j) = false;
168         end
169
170         %Partition the i^{th} subproblem constraint set elements into the
171         %relaxed and non-relaxed sets
172         if (mElementProps{i}.bIsRelaxed(j))
173
174             %Add the variable to set 1
175             AA{i,1} = [AA{i,1},A{i}(:,j)];
176             BB{i,1} = [BB{i,1},B{i}(:,j)];
177             cc{i,1} = [cc{i,1};c{i}(j)];
178
179             %Update counters
180             n1 = n1+1;
181             mElementProps{i}.idxReference(j) = n1;
182         else
183
184             %Add the variable to set 2
185             AA{i,2} = [AA{i,2},A{i}(:,j)];
186             BB{i,2} = [BB{i,2},B{i}(:,j)];
187             cc{i,2} = [cc{i,2};c{i}(j)];
188
189             %Update counters
190             n2 = n2+1;
```

```
191             mElementProps{i}.idxReference(j) = n2;
192
193         end %End the IsRelaxed if-statement
194     end %End the i^{th} suproblem element loop
195 end %End the subproblem loop
196
197 %Release the support variables
198 clear tempRows tempCols tempArtificial tempData;
199
200 %Initialize the global flags
201 bTerminate = false;
202 bIsUnbounded = false;
203 bIsInfeasible = false;
204
205 %Instantiate the complicating constraint matrix
206 G = ones(1,mMasterProps.nTotalReducedProbVars);
207 H = nInf;
208
209 %Generate and solve the reduced problem iteratively
210 while (~bTerminate)
211
212     %Lower the HasBasicArtificials flag
213     if (mMasterProps.bRedProbObjDecreases)
214         bIsInfeasible = true;
215         break;
216     else
217         mMasterProps.bHasBasicArtificials = false;
218     end
219
220     %Update the iteration number
221     sOutput.nIterations = sOutput.nIterations + 1;
222
```

```
223     %Display output
224     if (bDispOutput)
225         disp(['Iteration:',32,num2str(sOutput.nIterations),char(10),...
226             'Lower Bound:',32,num2str(sOutput.nObjValue),char(10)]);
227     end
228
229     %Reinitialize the reduced problem parameters, if needed
230     if (mMasterProps.bUpdateRedProbParam)
231         d{mMasterProps.nCells} = c{mMasterProps.nCells};
232         F{mMasterProps.nCells} = D{mMasterProps.nCells};
233         bbar = b{mMasterProps.nCells};
234         alpha = 0;
235     end
236
237     %For each subproblem
238     for i = 1:mMasterProps.nSubproblems
239
240         %Case I: Swap basis elements
241         if (mSubProps{i}.nNonNegCase == 1)
242
243             %Reset counters
244             idxSet = zeros(2,2);
245             n1 = 0;
246             n2 = 0;
247
248             %For each element in the i^{th} subproblem
249             for j = 1:mSubProps{i}.nTotalVariables
250
251                 switch true
252                     case (mElementProps{i}.bIsRelaxed(j)) && (mElementProps{i}.bSwapCol(j))
253
254                         %Update counter
```

```
255                         n2 = n2+1;
256
257                         %Save the current and destination column references
258                         idxSet(1,1) = mElementProps{i}.idxReference(j);
259                         idxSet(1,2) = n2;
260
261                         %Update the remaining flags
262                         mElementProps{i}.idxReference(j) = n2;
263                         mElementProps{i}.bIsRelaxed(j) = false;
264                         mElementProps{i}.bSwapCol(j) = false;
265
266                     case (~mElementProps{i}.bIsRelaxed(j)) && ...
267                             (mElementProps{i}.bSwapCol(j))
268
269                         %Update counter
270                         n1 = n1+1;
271
272                         %Save the current and destination column references
273                         idxSet(2,1) = mElementProps{i}.idxReference(j);
274                         idxSet(2,2) = n1;
275
276                         %Update the remaining flags
277                         mElementProps{i}.idxReference(j) = n1;
278                         mElementProps{i}.bIsRelaxed(j) = true;
279                         mElementProps{i}.bSwapCol(j) = false;
280
281                     case (mElementProps{i}.bIsRelaxed(j)) && ...
282                             (~mElementProps{i}.bSwapCol(j))
283
284                         %Update counter
285                         n1 = n1+1;
286
```

```
287                         %Update the column reference
288                         mElementProps{i}.idxReference(j) = n1;
289
290                     case (~mElementProps{i}.bIsRelaxed(j)) && ...
291                             (~mElementProps{i}.bSwapCol(j))
292
293                         %Update counter
294                         n2 = n2+1;
295
296                         %Update the column reference
297                         mElementProps{i}.idxReference(j) = n2;
298
299                 end %End switch
300             end %End the i^{th} suproblem element loop
301
302             %Save and delete the columns of the partitioned B-matrices;
303             %Then swap them, such that they are at the correct order
304             tDataSet1 = BB{i,1}(:,idxSet(1,1));
305             tDataSet2 = BB{i,2}(:,idxSet(2,1));
306             BB{i,1}(:,idxSet(1,1)) = [];
307             BB{i,2}(:,idxSet(2,1)) = [];
308             BB{i,1} = [BB{i,1}(:,1:idxSet(2,2)-1),tDataSet2,...
309                 BB{i,1}(:,idxSet(2,2):mSubProps{i}.nVariablesSet1-1)];
310             BB{i,2} = [BB{i,2}(:,1:idxSet(1,2)-1),tDataSet1,...
311                 BB{i,2}(:,idxSet(1,2):mSubProps{i}.nVariablesSet2-1)];
312
313             %Save and delete the columns of the partitioned A-matrices;
314             %Then swap them, such that they are at the correct order
315             tDataSet1 = AA{i,1}(:,idxSet(1,1));
316             tDataSet2 = AA{i,2}(:,idxSet(2,1));
317             AA{i,1}(:,idxSet(1,1)) = [];
318             AA{i,2}(:,idxSet(2,1)) = [];
```

```
319             AA{i,1} = [AA{i,1}(:,1:idxSet(2,2)-1),tDataSet2,...
320                 AA{i,1}(:,idxSet(2,2):mSubProps{i}.nVariablesSet1-1)];
321             AA{i,2} = [AA{i,2}(:,1:idxSet(1,2)-1),tDataSet1,...
322                 AA{i,2}(:,idxSet(1,2):mSubProps{i}.nVariablesSet2-1)];
323
324             %Save and delete the columns of the partitioned c-vectors;
325             %Then swap them, such that they are at the correct order
326             tDataSet1 = cc{i,1}(idxSet(1,1));
327             tDataSet2 = cc{i,2}(idxSet(2,1));
328             cc{i,1}(idxSet(1,1)) = [];
329             cc{i,2}(idxSet(2,1)) = [];
330             cc{i,1} = [cc{i,1}(1:idxSet(2,2)-1);tDataSet2;...
331                 cc{i,1}(idxSet(2,2):mSubProps{i}.nVariablesSet1-1)];
332             cc{i,2} = [cc{i,2}(1:idxSet(1,2)-1);tDataSet1;...
333                 cc{i,2}(idxSet(1,2):mSubProps{i}.nVariablesSet2-1)];
334
335             %Lower the non-negativity case flag
336             mSubProps{i}.nNonNegCase = 0;
337
338             %Release the support variables
339             clear idxSet tDataSet1 tDataSet2;
340
341         end %End Case I
342
343         %Check if updates are needed
344         if (mSubProps{i}.bUpdateRedProbParam)
345
346             %Recompute the support parameters for the i^{th} subproblem
347             iBB{i} = BB{i,1}\BB{i,2};
348             iBb{i} = BB{i,1}\b{i};
349             iBD{i} = BB{i,1}\D{i};
350
```

```
351             %Update the d and F matrices
352             d{i} = cc{i,2}-(transpose(iBB{i})*cc{i,1});
353             F{i} = AA{i,2}-(AA{i,1}*iBB{i});
354
355             %Lower the update flag
356             mSubProps{i}.bUpdateRedProbParam = false;
357
358         end
359
360         %Check if updates are needed
361         if (mMasterProps.bUpdateRedProbParam)
362
363             %Recompute the reduced problem parameters
364             d{mMasterProps.nCells} = d{mMasterProps.nCells}-(transpose(iBD{i})*cc{i,1});
365             F{mMasterProps.nCells} = F{mMasterProps.nCells}-(AA{i,1}*iBD{i});
366             bbar = bbar-(AA{i,1}*iBb{i});
367             alpha = alpha+(transpose(cc{i,1})*iBb{i});
368
369             %Lastly, lower the update flag
370             if (i == mMasterProps.nSubproblems)
371                 mMasterProps.bUpdateRedProbParam = false;
372             end
373         end %End the complicating vars reduced problem parameters update
374
375         %Case II - Explicitly enforce the non-negativity restriction
376         if (mSubProps{i}.nNonNegCase == 2)
377
378             %For each element in the i^{th} subproblem
379             for j = 1:mSubProps{i}.nTotalVariables
380
381                 %Only relaxed variables can have their non-negativity
382                 %restriction explicitly enforced
```

```
383                 if (mElementProps{i}.bEnforceNonNeg(j))
384
385                     %Update the Reduced Problem
386                     G = [G;...
387                         zeros(1,sum(mMasterProps.mReducedProbVars(1,1:i-1))),...
388                         iBB{i}(mElementProps{i}.idxReference(j),:),...
389                         zeros(1,mMasterProps.nTotalReducedProbVars...
390                         -mMasterProps.nComplicatingVars...
391                         -sum(mMasterProps.mReducedProbVars(1,1:i))),...
392                         iBD{i}(mElementProps{i}.idxReference(j),:)];
393                     H = [H;iBb{i}(mElementProps{i}.idxReference(j))];
394
395                     %Add a new element to the enforced contraints property
396                     %structure
397                     mEnforcedConstraints.nTotalExplicitConstraints = ...
398                         mEnforcedConstraints.nTotalExplicitConstraints+1;
399                     mEnforcedConstraints.nSubproblem = ...
400                         [mEnforcedConstraints.nSubproblem;i];
401                     mEnforcedConstraints.nElement = ...
402                         [mEnforcedConstraints.nElement;j];
403                     mEnforcedConstraints.nRowReference = ...
404                         [mEnforcedConstraints.nRowReference;...
405                         mEnforcedConstraints.nTotalExplicitConstraints+1];
406
407                     %Raise the IsEnforced flag
408                     mElementProps{i}.bIsEnforced(j) = true;
409
410                     %Lower the non-negativity case flag
411                     mElementProps{i}.bEnforceNonNeg(j) = false;
412                     mSubProps{i}.nNonNegCase = 0;
413
414                     %Exit the loop. There can be at most one explicit
```

```
415                     %non-negativity constraint per subproblem
416                     break;
417
418                 end %End the EnforNonNeg if-statement
419             end %End the i^{th} suproblem element loop
420         end %End Case II
421     end %End the subproblem loop
422
423     %Solve the r^th Reduced Problem:
424     %{min d*x | rConstraints*x <= rRHS, F*x = bbar, 0 <= x <= []}
425     try
426         [x2,fval,nExitFlag] = cplexlp(cell2mat(d),G,H,cell2mat(F),bbar,...
427             zeros(mMasterProps.nTotalReducedProbVars,1),[]);
428     catch sError
429         throw(sError);
430     end
431
432     if (nExitFlag ~= 1)
433         %Check for infeasibility: if the reduced problem cannot be solved
434         %to optimality, neither can the original problem.
435
436         bIsInfeasible = true;
437         bTerminate = true;
438
439         %Call the next iteration
440         continue;
441
442     end
443
444     %Compare the current and previous objective function increases
445     mMasterProps.bRedProbObjIncreases = (fval + alpha > sOutput.nObjValue + nPrecision);
446     mMasterProps.bRedProbObjDecreases = (fval + alpha < sOutput.nObjValue - nPrecision);
```

```
447
448     %Save the new objective function value
449     sOutput.nObjValue = fval+alpha;
450
451     %Check if the unboundedness constraint is binding
452     bIsUnbounded = (sum(x2) > nInf - nPrecision);
453
454     %Parse the results of the optimization problem into cells according
455     %to the number of variables in the reduced problem
456     x1 = cell(mMasterProps.nSubproblems,1);
457     x2 = mat2cell(x2,mMasterProps.mReducedProbVars,1);
458
459     %Raise the termination flag. It may have to be lowered again later
460     bTerminate = true;
461
462     %For each subproblem
463     for i = 1:mMasterProps.nSubproblems
464
465         %Compute the solution to the relaxed variables
466         x1{i} = iBb{i}-(iBB{i}*x2{i})-(iBD{i}*x2{mMasterProps.nCells});
467
468         %For each element in the subproblem
469         for j = 1:mSubProps{i}.nTotalVariables
470
471             %Determine if there are any basic artificial variables
472             if (mElementProps{i}.bIsArtificial(j))
473                 if ((mElementProps{i}.bIsRelaxed(j)) && ...
474                         (x1{i}(mElementProps{i}.idxReference(j)) > nPrecision)) || ...
475                     ((~mElementProps{i}.bIsRelaxed(j)) && ...
476                     (x2{i}(mElementProps{i}.idxReference(j)) > nPrecision))
477
478                     %Raise the HasBasicArtificialsFlag
```

```
479                     mMasterProps.bHasBasicArtificials = true;
480                 end
481             end
482
483             if (mElementProps{i}.bIsRelaxed(j))
484
485                 %If the relaxed decision variable is negative, add
486                 %it to the negative list
487                 if (x1{i}(mElementProps{i}.idxReference(j)) < -nPrecision)
488
489                     %Not all non-negativity restrictions were satisfied
490                     %Lower the termination flag
491                     bTerminate = false;
492
493                     %Add the variable to the negative list
494                     mSubSolProps{i}.nNegative = mSubSolProps{i}.nNegative+1;
495                     mSubSolProps{i}.idxNegatives(mSubSolProps{i}.nNegative) = j;
496
497                     %Update the most negative variable found so far.
498                     if (x1{i}(mElementProps{i}.idxReference(j)) < ...
499                             mSubSolProps{i}.nMostNegativeValue)
500                         mSubSolProps{i}.idxMostNeg = j;
501                         mSubSolProps{i}.nMostNegativeValue = ...
502                             x1{i}(mElementProps{i}.idxReference(j));
503                     end
504                 end %End IsRelaxed && IsNegative if-statement
505
506             else
507                 %If the reduced problem decision variable is positive,
508                 %add it to the positive list
509                 if (x2{i}(mElementProps{i}.idxReference(j)) > nPrecision)
510                     mSubSolProps{i}.nPositive = mSubSolProps{i}.nPositive+1;
```

```
511                     mSubSolProps{i}.idxPositives(mSubSolProps{i}.nPositive) = j;
512                 end
513
514             end %End the ~IsRelaxed && ~IsNegative if-statement
515         end %End the i^{th} subproblem element loop
516
517         %Remove unused row elements from the lists to clear memory
518         mSubSolProps{i}.idxNegatives(mSubSolProps{i}.nNegative+1:end) = [];
519         mSubSolProps{i}.idxPositives(mSubSolProps{i}.nPositive+1:end) = [];
520
521     end %End the subproblem loop
522
523     %Deal with the explicit non-negativity constraints
524     if (mMasterProps.bRedProbObjIncreases && ...
525             (mEnforcedConstraints.nTotalExplicitConstraints > 0))
526
527         %Reset counter
528         z1 = 0;
529
530         %For each enforced constraint
531         while (z1 < mEnforcedConstraints.nTotalExplicitConstraints)
532
533             %Increment counter
534             z1 = z1+1;
535
536             %Lower the support flags
537             bAllowRemoval = false;
538
539             if (mElementProps{mEnforcedConstraints.nSubproblem(z1)}....
540                     bIsRelaxed(mEnforcedConstraints.nElement(z1)))
541
542                 %Save the reference to the k^{th} row
```

```
543                 n1 = mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
544                     .idxReference(mEnforcedConstraints.nElement(z1));
545
546                 if (x1{mEnforcedConstraints.nSubproblem(z1)}(n1) > nPrecision)
547                     %CASE 3: Raise the removal flag if the explicit
548                     %non-negativity restriction is non-binding
549                     bAllowRemoval = true;
550
551                 elseif (mSubProps{mEnforcedConstraints.nSubproblem(z1)}.nNonNegCase == 0);
552                     %CASE 4: The restriction is binding. Scan the
553                     %reduced problem solution for a potential element
554                     %to leave the basis so that this one may enter
555
556                     %Loop through all elements again
557                     for n = 1:mSubProps{mEnforcedConstraints...
558                             .nSubproblem(z1)}.nTotalVariables
559                         if (~mElementProps{mEnforcedConstraints...
560                                 .nSubproblem(z1)}.bIsRelaxed(n))
561
562                             %Save the reference to the q^{th} column
563                             n2 = mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
564                                 .idxReference(n);
565
566                             %If we find a positive element in set 1
567                             %which will not lead to a singular matrix
568                             if (x2{mEnforcedConstraints.nSubproblem(z1)}(n2) ...
569                                     > nPrecision) && ((iBB{mEnforcedConstraints...
570                                     .nSubproblem(z1)}(n1,n2) < -nPrecision) || ...
571                                     (iBB{mEnforcedConstraints.nSubproblem(z1)}(n1,n2) ...
572                                     > nPrecision))
573
574                                 %Raise the respective swap basis flags
```

```
575                                 mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
576                                     .bSwapCol(mEnforcedConstraints.nElement(z1)) = true;
577                                 mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
578                                     .bSwapCol(n) = true;
579
580                                 %Update the remaining flags
581                                 bAllowRemoval = true;
582                                 mSubProps{mEnforcedConstraints.nSubproblem(z1)}...
583                                     .nNonNegCase = 1;
584                                 mSubProps{mEnforcedConstraints.nSubproblem(z1)}...
585                                     .bUpdateRedProbParam = true;
586                                 mMasterProps.bUpdateRedProbParam = true;
587
588                                 %Exit Loop
589                                 break;
590                             end
591                         end
592                     end %end i^{th} subproblem element loop
593                 end %End IsPositive if-statement
594             else
595                 %Save the reference to the q^{th} column
596                 n2 = mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
597                     .idxReference(mEnforcedConstraints.nElement(z1));
598
599                 %In order to prevent cycling, only allow the removal of
600                 %explicit non-negativity  constraints associated with
601                 %positive x_2j elements, when improvements are observed
602                 %in the reduced problem objective function value.
603                 if (mMasterProps.bRedProbObjIncreases) && ...
604                         (x2{mEnforcedConstraints.nSubproblem(z1)}(n2) > nPrecision)
605                     bAllowRemoval = true;
606                 end
```

```
607             end %End IsRelaxed if-statement
608             if (bAllowRemoval)
609
610                 %Remove the explicit non-negativity constraint and
611                 %right-hand-side
612                 G(mEnforcedConstraints.nRowReference(z1),:) = [];
613                 H(mEnforcedConstraints.nRowReference(z1)) = [];
614
615                 %Reindex all existing references
616                 for z2 = 1:mEnforcedConstraints.nTotalExplicitConstraints
617                     if ((z2 ~= z1) && ...
618                             (mEnforcedConstraints.nRowReference(z2) > ...
619                             mEnforcedConstraints.nRowReference(z1)))
620
621                         %Update the reference to the explicit non-
622                         %negativity constraint
623                         mEnforcedConstraints.nRowReference(z2) = ...
624                             mEnforcedConstraints.nRowReference(z2)-1;
625                     end
626                 end %End explicit constraints loop
627
628                 %Lower the IsEnforced flag
629                 mElementProps{mEnforcedConstraints.nSubproblem(z1)}...
630                     .bIsEnforced(mEnforcedConstraints.nElement(z1)) = false;
631
632                 %Delete the tagged reference to the explicit non-
633                 %negativity constraint
634                 mEnforcedConstraints.nSubproblem(z1) = [];
635                 mEnforcedConstraints.nElement(z1) = [];
636                 mEnforcedConstraints.nRowReference(z1) = [];
637                 mEnforcedConstraints.nTotalExplicitConstraints = ...
638                     mEnforcedConstraints.nTotalExplicitConstraints-1;
```

```
639
640                 %Decrease counter since a row was eliminated
641                 z1 = z1-1;
642
643             end
644         end %End while
645     end %End Cases 3 to 5
646
647     if (bTerminate)
648         %Call the next iteration
649         continue;
650     end
651
652     %For each subproblem
653     for i = 1:mMasterProps.nSubproblems
654
655         %Determine if any non-zero entry in the iBB submatrix exits
656         bElementFound = false;
657
658         %If the i^{th} subproblem has at least one negative entry
659         if ((mSubSolProps{i}.nNegative >= 1) && (mSubProps{i}.nNonNegCase == 0))
660             if (mSubSolProps{i}.nPositive >= 1)
661                 for n1 = 1:mSubSolProps{i}.nNegative
662                     for n2 = 1:mSubSolProps{i}.nPositive
663                         tempData = iBB{i}(...
664                             mElementProps{i}...
665                             .idxReference(mSubSolProps{i}.idxNegatives(n1)),...
666                             mElementProps{i}.idxReference(mSubSolProps{i}...
667                             .idxPositives(n2)));
668                         if ((tempData < -nPrecision) || (tempData > nPrecision))
669                             %A non-zero entry in the iBB submatrix exists
670
```

```
671                             %flag the corresponding variables in the
672                             %negative and positive sets for swap at the
673                             %next iteration
674                             mElementProps{i}.bSwapCol(mSubSolProps{i}...
675                                 .idxNegatives(n1)) = true;
676                             mElementProps{i}.bSwapCol(mSubSolProps{i}...
677                                 .idxPositives(n2)) = true;
678
679                             %Update the appropriate flags
680                             bElementFound = true;
681                             mSubProps{i}.nNonNegCase = 1;
682                             mSubProps{i}.bUpdateRedProbParam = true;
683                             mMasterProps.bUpdateRedProbParam = true;
684
685                             %Exit the loop
686                             break;
687
688                         end
689                     end %End pos. elements (iBB{i} submatrix cols) loop
690                     if (bElementFound)
691                         %If we found variables to swap, them move to the
692                         %next subproblem
693                         break;
694                     end
695                 end %End negative elements (iBB{i} submatrix rows) loop
696             end
697             if (~bElementFound)
698                 %If the iBB{i} submatrix contains only zero entries,
699                 %flag the most negative variable to be added as a hard
700                 %constraint in the reduced problem
701                 mElementProps{i}.bEnforceNonNeg(mSubSolProps{i}.idxMostNeg) = true;
702
```

```
703                 %Update the appropriate flags
704                 mSubProps{i}.nNonNegCase = 2;
705
706             end %End most negative if-statement
707         end %End has negative elements if-statement
708
709         %Reset the number of negative and positive values counters
710         mSubSolProps{i}.nNegative = 0;
711         mSubSolProps{i}.idxNegatives = zeros(mSubProps{i}.nVariablesSet1,1);
712         mSubSolProps{i}.idxMostNeg = 0;
713         mSubSolProps{i}.nMostNegativeValue = Inf;
714         mSubSolProps{i}.nPositive = 0;
715         mSubSolProps{i}.idxPositives = zeros(mSubProps{i}.nVariablesSet2,1);
716
717     end %End subproblem loop
718 end %End main while loop
719
720 %Release the support variables
721 clear bTerminate bDispOutput mSubSolProps mEnforcedConstraints nInf n1 n2 ...
722     nPrecision AA BB cc iBB iBb iBD G H d F bbar alpha nExitFlag fval ...
723     z1 z2 nSubproblem bAllowRemoval bElementFound tempData n;
724
725 %Output Results
726 switch true
727     case (bIsUnbounded) && (~bIsInfeasible)
728         sOutput.sResult = 'unbounded';
729         sOutput.sDescription = 'The original problem is unbounded';
730
731     case (bIsInfeasible || mMasterProps.bHasBasicArtificials)
732         sOutput.sResult = 'infeasible';
733         sOutput.sDescription = 'The original problem is infeasible';
734
```

```
735     otherwise
736         sOutput.sResult = 'optimal';
737         sOutput.sDescription = 'Program Finished. Optimal solution found';
738 
739         %Output the final solution
740         sOutput.x = cell(mMasterProps.nCells,1);
741 
742         %For each subproblem
743         for i = 1:mMasterProps.nSubproblems
744 
745             %Instantiate the solution set
746             sOutput.x{i} = zeros(mSubProps{i}.nRealVariables,1);
747 
748             %For each element in the i^{th} subproblem
749             n = 0;
750             for j = 1:mSubProps{i}.nTotalVariables
751 
752                 if (~mElementProps{i}.bIsArtificial(j))
753 
754                     %Update counter
755                     n = n + 1;
756 
757                     %Merge the relaxed and non-relaxed sets
758                     if (mElementProps{i}.bIsRelaxed(j))
759                         sOutput.x{i}(n) = x1{i}(mElementProps{i}.idxReference(j));
760                     else
761                         sOutput.x{i}(n) = x2{i}(mElementProps{i}.idxReference(j));
762                     end
763 
764                 end %End IsArtificial if-statement
765             end %End the i^{th} subproblem element loop
766         end %End the subproblem loop
```

```
767
768         %Include final value of complicating variables
769         sOutput.x{mMasterProps.nCells,1} = x2{mMasterProps.nCells,1};
770
771 end %End switch
772
773 %Compute the elapsed time
774 sOutput.nSolutionTime = etime(clock,sOutput.nSolutionTime);
775
776 %Release the remaining support variables
777 clear i j x1 x2 bIsInfeasible bIsUnbounded mMasterProps mSubProps ...
778     mElementProps;
779 end
```

D.2 Modified Algorithm

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function solves problems of the type:                                     %
% Min    0.5 x' H x + c' x                                                       %
% S.t.   B{1}*x{1}                                + D{1}*x{p+1} == b{1}          %
%                    B{2}*x{2} +                  + D{2}*x{p+1} == b{2}          %
%                                ...                                              %
%                                       B{p}*x{p} + D{p}*x{p+1} == b{p}          %
%        Aeq{1}*x{1} + ... + Aeq{p}*x{p} + Deq*x{p+1} == beq                     %
%        Aineq{1}*x{1} + ... + Aineq{p}*x{p} + Dineq*x{p+1} <= bineq             %
%        x{1}, x{2}, ... , x{p}, x{p+1} in (R_+ \times Z_+)                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Input Parameters:                                                              %
% H{ij} - a n_j x n_k matrix of 2nd-order partial derivs., i,j = 1..p+1          %
% c{i} - a n{i}-column vector of obj. func. coefficients, i = 1,...,p+1          %
% Aeq{i} - a m_eq x n{i} matrix of constraint coefficients, i = 1,...,p          %
% Aineq{i} - a m_ineq x n{i} matrix of constr. coefficients, i = 1,...,p         %
% B{i} - a m{i} x n{i} matrix of constraint coefficients, i = 1,...,p            %
% D{i} - a m{i} x n{p+1} matrix of constraint coefficients, i = 1,...,p          %
% Deq - a m_eq x n{p+1} matrix of constraint coefficients                        %
% Dineq - a m_ineq x n{p+1} matrix of constraint coefficients                    %
% b{i} - a m{i}-column vector of RHS coefficients, i = 1,...,p                   %
% beq - a m_eq-column vector of RHS coefficients                                 %
% bineq - a m_ineq-column vector of RHS coefficients                             %
%                                                                                %
% Output Paramters:                                                              %
% x{i} - a n{i}-column vector of decision variables, i = 1,...,p+1               %
% fval - obj. fun. value                                                         %
% nIterations - Number of iterations from the Ritter algorithm                   %
% sMessage - Optimization status (optimal, unbounded, infeasible, etc.           %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%$
```

```
31 function [sOutput] = modRitterQuadratic(H,c,B,D,b,Aeq,Deq,beq,Aineq,...
32     Dineq,bineq,cType,nOutputIterationInterval)
33
34 %Initialize the global parameters
35 nInf = 10^10;
36 nPrecision = 10^(-10);
37
38 %Initialize the master problem property structure
39 nSubproblems = size(B,1);
40 if (nSubproblems == 0)
41     error('Ritter:InvalidParam','The problem has no subproblems');
42 end
43 nComplicatingConstrEq = size(Aeq{1},1);
44 nComplicatingConstrIneq = size(Aineq{1},1);
45 if (nComplicatingConstrEq + nComplicatingConstrIneq == 0)
46     error('Ritter:InvalidParam','Missing complicating constraints');
47 end
48 nComplicatingVars = size(c{nSubproblems+1},1);
49 if (nComplicatingVars == 0)
50     error('Ritter:InvalidParam','Missing complicating variables');
51 end
52 nCells = nSubproblems+1;
53 mReducedProbVars = [zeros(1,nSubproblems),nComplicatingVars];
54 nTotalReducedProbVars = nComplicatingVars;
55
56 %Initialize the enforced contraints property structure
57 mEnforcedConstraints = struct('nTotalExplicitConstraints',0,...
58     'nSubproblem',[],...
59     'nElement',[],...
60     'nRowReference',[],...
61     'bUpperBoundRestriction',[]);
62
```

```
63 %Initialize the return parameters property structure
64 sOutput = struct('x',[],...
65     'nObjValue',Inf,...
66     'nIterations',0,...
67     'nSolutionTime',clock,...
68     'nRedProbTime',0,...
69     'sResult','',...
70     'sDescription','');
71
72 %Initialize the subproblem property structure
73 mSubProps = cell(nSubproblems,1);
74
75 %Initialize the subproblem element property structure
76 mElementProps = cell(nSubproblems,1);
77
78 %Initialize the partitioned submatrices:
79 %variables with their non-negativity restriction relaxed --> set 1;
80 %variables with their non-negativity restriction enforced --> set 2;
81 AAeq = cell(nSubproblems,2);
82 AAineq = cell(nSubproblems,2);
83 BB = cell(nSubproblems,2);
84 cc = cell(nSubproblems,2);
85 HH = cell(nCells);
86 for i = 1:nSubproblems
87     for j = 1:nSubproblems
88         HH{i,j} = cell(2);
89     end
90 end
91 for i = 1:nSubproblems
92     HH{i,nCells} = cell(2,1);
93 end
94 for j = 1:nSubproblems
```

```
95      HH{nCells,j} = cell(1,2);
96  end
97
98  %Initialize the Reduced Problem parameters
99  L = cell(nCells);
100 d = cell(nCells,1);
101 F = cell(1,nCells);
102 f = [];
103 G = cell(1,nCells);
104 g = [];
105 x2Type = cell(nCells,1);
106
107 %Save unpartitioned elements into the appropriate matrix/vectors
108 x2Type{nCells} = cType{nCells};
109 HH{nCells,nCells} = H{nCells,nCells};
110
111 %Initialize the Reduced Problem support parameters
112 iBB = cell(nSubproblems,1);
113 iBb = cell(nSubproblems,1);
114 iBD = cell(nSubproblems,1);
115
116 %For each subproblem
117 nTotalArtificial = 0;
118 for i = 1:nSubproblems
119
120     %Populate the i^{th} subproblem property structure
121     [tempRows,tempCols] = size(B{i});
122     tempArtificial = max(max(tempRows-tempCols+1,0),tempRows-rank(B{i}));
123     while (tempArtificial <= tempRows)
124         if (rank([B{i},eye(tempRows,tempArtificial)]) == tempRows)
125             break;
126         else
```

```
127             tempArtificial = tempArtificial+1;
128         end
129     end
130     nTotalArtificial = nTotalArtificial + tempArtificial;
131     mSubProps{i} = struct('nRealVariables',tempCols,...
132         'nArtificialVariables',tempArtificial,...
133         'nTotalVariables',tempCols+tempArtificial,...
134         'nConstraints',tempRows,...
135         'nVariablesSet1',tempRows,...
136         'nVariablesSet2',tempCols+tempArtificial-tempRows);
137
138     %Populate the i^{th} subproblem element property structure
139     tempData = zeros(mSubProps{i}.nTotalVariables,1);
140     tempArtificialString = num2str(zeros(tempArtificial,1));
141     for n = 1:tempArtificial
142         tempArtificialString(n) = 'C';
143     end
144     mElementProps{i} = struct('idxReference',tempData,...
145         'bIsRelaxed',boolean(tempData),...
146         'bIsArtificial',boolean(tempData),...
147         'sType',[cType{i};tempArtificialString]);
148
149     %Update the total number of variables in set 2
150     mReducedProbVars(i) = mSubProps{i}.nVariablesSet2;
151     nTotalReducedProbVars = nTotalReducedProbVars+mSubProps{i}.nVariablesSet2;
152
153     if mSubProps{i}.nArtificialVariables > 0
154         %If needed, augment the i^{th} subproblem Aeq, Aineq, and B matrices
155         %and c vectors to ensure the existance of a non-singular matrix
156         B{i} = [B{i},eye(mSubProps{i}.nConstraints,mSubProps{i}.nArtificialVariables)];
157         c{i} = [c{i};nInf*ones(mSubProps{i}.nArtificialVariables,1)];
158
```

```
159         H{i,i} = [H{i,i},zeros(mSubProps{i}.nRealVariables,...
160             mSubProps{i}.nArtificialVariables)];
161         H{i,i} = [H{i,i};zeros(mSubProps{i}.nArtificialVariables,...
162             mSubProps{i}.nTotalVariables)];
163         for j = 1:nSubproblems
164             if (i ~= j)
165                 n = size(H{i,j},2);
166                 H{i,j}=[H{i,j};zeros(mSubProps{i}.nArtificialVariables,n)];
167                 H{j,i}=[H{j,i},zeros(n,mSubProps{i}.nArtificialVariables)];
168             end;
169         end
170         H{nCells,i}=[H{nCells,i},...
171             zeros(nComplicatingVars,mSubProps{i}.nArtificialVariables)];
172         H{i,nCells}=[H{i,nCells};...
173             zeros(mSubProps{i}.nArtificialVariables,nComplicatingVars)];
174         Aeq{i} = [Aeq{i},zeros(nComplicatingConstrEq,mSubProps{i}.nArtificialVariables)];
175         Aineq{i} = [Aineq{i},...
176             zeros(nComplicatingConstrIneq,mSubProps{i}.nArtificialVariables)];
177
178         %Flag the created variables as artificial
179         mElementProps{i}.bIsArtificial(...
180             mSubProps{i}.nRealVariables+1:mSubProps{i}.nTotalVariables) = ...
181             ones(mSubProps{i}.nArtificialVariables,1);
182     end
183
184     %Preallocate partitioned matrices for speed
185     x2Type{i} = num2str(zeros(mSubProps{i}.nVariablesSet2,1));
186     if (nComplicatingConstrEq >= 1)
187         AAeq{i,1} = zeros(nComplicatingConstrEq,mSubProps{i}.nVariablesSet1);
188         AAeq{i,2} = zeros(nComplicatingConstrEq,mSubProps{i}.nVariablesSet2);
189     end
190     if (nComplicatingConstrIneq >= 1)
```

```
191         AAineq{i,1} = zeros(nComplicatingConstrIneq,mSubProps{i}.nVariablesSet1);
192         AAineq{i,2} = zeros(nComplicatingConstrIneq,mSubProps{i}.nVariablesSet2);
193     end
194     BB{i,1} = zeros(mSubProps{i}.nConstraints,mSubProps{i}.nVariablesSet1);
195     BB{i,2} = zeros(mSubProps{i}.nConstraints,mSubProps{i}.nVariablesSet2);
196     cc{i,1} = zeros(mSubProps{i}.nVariablesSet1,1);
197     cc{i,2} = zeros(mSubProps{i}.nVariablesSet2,1);
198     for j = 1:i
199         HH{i,j}{1,1} = zeros(mSubProps{i}.nVariablesSet1,mSubProps{j}.nVariablesSet1);
200         HH{i,j}{1,2} = zeros(mSubProps{i}.nVariablesSet1,mSubProps{j}.nVariablesSet2);
201         HH{i,j}{2,1} = zeros(mSubProps{i}.nVariablesSet2,mSubProps{j}.nVariablesSet1);
202         HH{i,j}{2,2} = zeros(mSubProps{i}.nVariablesSet2,mSubProps{j}.nVariablesSet2);
203         if (i ~= j)
204             HH{j,i}{1,1} = zeros(mSubProps{j}.nVariablesSet1,mSubProps{i}.nVariablesSet1);
205             HH{j,i}{1,2} = zeros(mSubProps{j}.nVariablesSet1,mSubProps{i}.nVariablesSet2);
206             HH{j,i}{2,1} = zeros(mSubProps{j}.nVariablesSet2,mSubProps{i}.nVariablesSet1);
207             HH{j,i}{2,2} = zeros(mSubProps{j}.nVariablesSet2,mSubProps{i}.nVariablesSet2);
208         end
209     end
210     HH{i,nCells}{1} = zeros(mSubProps{i}.nVariablesSet1,nComplicatingVars);
211     HH{i,nCells}{2} = zeros(mSubProps{i}.nVariablesSet2,nComplicatingVars);
212     HH{nCells,i}{1} = zeros(nComplicatingVars,mSubProps{i}.nVariablesSet1);
213     HH{nCells,i}{2} = zeros(nComplicatingVars,mSubProps{i}.nVariablesSet2);
214
215     %Idenfify a starting basis
216     tempData = rref(B{i});
217     tempRow = 1;
218     n1 = 0;
219     n2 = 0;
220
221     %For each element in the i^{th} subproblem
222     for tempCol = 1:mSubProps{i}.nTotalVariables
```

```
223
224         %Determine if the variable belongs to set 1 or 2 (i.e., determine
225         %if its non-negativity restriction should be relaxed or not)
226         if (tempRow <= mSubProps{i}.nVariablesSet1)
227             if (tempData(tempRow,tempCol) == 1)
228                 tempRow = tempRow + 1;
229                 mElementProps{i}.bIsRelaxed(tempCol) = true;
230             else
231                 mElementProps{i}.bIsRelaxed(tempCol) = false;
232             end
233         else
234             mElementProps{i}.bIsRelaxed(tempCol) = false;
235         end
236
237         %Partition the i^{th} subproblem constraint set elements into the
238         %relaxed and non-relaxed sets
239         if (mElementProps{i}.bIsRelaxed(tempCol))
240
241             %Update counters
242             n1 = n1 + 1;
243             mElementProps{i}.idxReference(tempCol) = n1;
244
245             %Add the variable to set 1
246             if (nComplicatingConstrEq >= 1)
247                 AAeq{i,1}(:,n1) = Aeq{i}(:,tempCol);
248             end
249             if (nComplicatingConstrIneq >= 1)
250                 AAineq{i,1}(:,n1) = Aineq{i}(:,tempCol);
251             end
252             BB{i,1}(:,n1) = B{i}(:,tempCol);
253             cc{i,1}(n1) = c{i}(tempCol);
254
```

```
255             %Loop through previous subproblems
256             for j = 1:i
257
258                 %Reset auxiliary counters
259                 n3 = 0;
260                 n4 = 0;
261
262                 %Loop through their decision variables
263                 for k = 1:mSubProps{j}.nTotalVariables
264
265                     if (mElementProps{j}.bIsRelaxed(k))
266                         n3 = n3 + 1;
267                         HH{i,j}{1,1}(n1,n3) = H{i,j}(tempCol,k);
268                         HH{j,i}{1,1}(n3,n1) = H{j,i}(k,tempCol);
269                     else
270                         n4 = n4 + 1;
271                         HH{i,j}{1,2}(n1,n4) = H{i,j}(tempCol,k);
272                         HH{j,i}{2,1}(n4,n1) = H{j,i}(k,tempCol);
273                     end
274
275                     %Exit if we surpassed the current decision variable
276                     if (i == j) && (k >= tempCol)
277                         break;
278                     end
279                 end
280             end
281             HH{i,nCells}{1}(n1,:) = H{i,nCells}(tempCol,:);
282             HH{nCells,i}{1}(:,n1) = H{nCells,i}(:,tempCol);
283         else
284             %Update counters
285             n2 = n2 + 1;
286             mElementProps{i}.idxReference(tempCol) = n2;
```

```
287
288             %Add the variable to set 2
289             if (nComplicatingConstrEq >= 1)
290                 AAeq{i,2}(:,n2) = Aeq{i}(:,tempCol);
291             end
292             if (nComplicatingConstrIneq >= 1)
293                 AAineq{i,2}(:,n2) = Aineq{i}(:,tempCol);
294             end
295             BB{i,2}(:,n2) = B{i}(:,tempCol);
296             cc{i,2}(n2) = c{i}(tempCol);
297
298             %Loop through previous subproblems
299             for j = 1:i
300
301                 %Reset auxiliary counters
302                 n3 = 0;
303                 n4 = 0;
304
305                 %Loop through their decision variables
306                 for k = 1:mSubProps{j}.nTotalVariables
307
308                     if (mElementProps{j}.bIsRelaxed(k))
309                         n3 = n3 + 1;
310                         HH{i,j}{2,1}(n2,n3) = H{i,j}(tempCol,k);
311                         HH{j,i}{1,2}(n3,n2) = H{j,i}(k,tempCol);
312                     else
313                         n4 = n4 + 1;
314                         HH{i,j}{2,2}(n2,n4) = H{i,j}(tempCol,k);
315                         HH{j,i}{2,2}(n4,n2) = H{j,i}(k,tempCol);
316                     end
317
318                     %Exit if we surpassed the current decision variable
```

```
319                         if (i == j) && (k >= tempCol)
320                             break;
321                         end
322                     end
323                 end
324                 HH{i,nCells}{2}(n2,:) = H{i,nCells}(tempCol,:);
325                 HH{nCells,i}{2}(:,n2) = H{nCells,i}(:,tempCol);
326
327                 %Assert the appropriate data type
328                 x2Type{i}(n2,1) = mElementProps{i}.sType(tempCol);
329
330             end %End the IsRelaxed if-statement
331         end %End the i^{th} suproblem element loop
332     end %End the subproblem loop
333
334     %Release the support variables;
335     clear tempRow tempRows tempCol tempCols tempArtificial ...
336         tempArtificialString tempData n1 n2 n3 n4;
337
338     %Initialize the Reduced Problem parameters
339     d{nCells} = c{nCells};
340     alpha = 0;
341     if (nComplicatingConstrEq >= 1)
342         F{nCells} = Deq;
343         f = beq;
344     end
345     if (nComplicatingConstrIneq >= 1)
346         G{nCells} = Dineq;
347         g = bineq;
348     end
349
350     %For each subproblem
```

```
351 for i = 1:nSubproblems
352
353     %Compute the support parameters for the i^{th} subproblem
354     iBB{i} = BB{i,1}\BB{i,2};
355     iBb{i} = BB{i,1}\b{i};
356     iBD{i} = BB{i,1}\D{i};
357
358     %Compute the Reduced Problem parameters
359     d{i} = cc{i,2}-transpose(iBB{i})*cc{i,1};
360     d{nCells} = d{nCells}-transpose(iBD{i})*cc{i,1};
361     alpha = alpha+transpose(cc{i,1})*iBb{i};
362
363     if(nComplicatingConstrEq >= 1)
364         F{i} = AAeq{i,2}-AAeq{i,1}*iBB{i};
365         F{nCells} = F{nCells}-AAeq{i,1}*iBD{i};
366         f = f-AAeq{i,1}*iBb{i};
367     end
368     if(nComplicatingConstrIneq >= 1)
369         G{i} = AAineq{i,2}-AAineq{i,1}*iBB{i};
370         G{nCells} = G{nCells}-AAineq{i,1}*iBD{i};
371         g = g-AAineq{i,1}*iBb{i};
372     end
373 end
374
375 %Include the quadratic variations
376 L{nCells,nCells}=HH{nCells,nCells};
377 for j = 1:nSubproblems
378     L{j,nCells} = HH{j,nCells}{2}-transpose(iBB{j})*HH{j,nCells}{1};
379     L{nCells,j} = HH{nCells,j}{2}-HH{nCells,j}{1}*iBB{j};
380     L{nCells,nCells} = L{nCells,nCells}-transpose(iBD{j})*HH{j,nCells}{1}...
381         -HH{nCells,j}{1}*iBD{j};
382     d{nCells} = d{nCells} + (1/2)*(HH{nCells,j}{1}+transpose(HH{j,nCells}{1}))*iBb{j};
```

```
383     for k = 1:nSubproblems
384         L{j,k} = HH{j,k}{2,2}-transpose(iBB{j})*HH{j,k}{1,2}-...
385             HH{j,k}{2,1}*iBB{k}+transpose(iBB{j})*HH{j,k}{1,1}*iBB{k};
386         L{j,nCells} = L{j,nCells}+(transpose(iBB{j})*HH{j,k}{1,1}-HH{j,k}{2,1})*iBD{k};
387         L{nCells,j} = L{nCells,j}+transpose(iBD{k})*(HH{k,j}{1,1}*iBB{j}-HH{k,j}{1,2});
388         L{nCells,nCells} = L{nCells,nCells}+transpose(iBD{j})*HH{j,k}{1,1}*iBD{k};
389         d{j}=d{j}+(1/2)*(HH{j,k}{2,1}-transpose(iBB{j})*HH{j,k}{1,1}+...
390             transpose(HH{k,j}{1,2})-transpose(iBB{j})*transpose(HH{k,j}{1,1}))*iBb{k};
391         d{nCells} = d{nCells}-(1/2)*transpose(iBD{k})*(HH{k,j}{1,1}+...
392             transpose(HH{j,k}{1,1}))*iBb{j};
393         alpha = alpha + (1/2)*transpose(iBb{j})*HH{j,k}{1,1}*iBb{k};
394     end
395 end
396
397 %Release support variables
398 clear nComplicatingConstrEq nComplicatingConstrIneq Aeq Aineq Deq Dineq ...
399     beq bineq B c D H b cType;
400
401 %Generate the Reduced Problem structural constraints
402 d = cell2mat(d);
403 F = cell2mat(F);
404 G = [cell2mat(G);ones(1,nTotalReducedProbVars)];
405 g = [g;nInf];
406 x2Type = horzcat(transpose(cell2mat(x2Type)));
407
408 %Unfortunately, small rounding errors propagate to the CPLEX engine. To
409 %prevent such issues, force L = L'
410 L = cell2mat(L);
411 L = triu(L,0)+transpose(triu(L,1));
412
413 %Initialize the branch and bound global parameters
414 x1Incumbent = [];
```

```
415 x2Incumbent = [];
416 bLBViolated = false;
417 bUBViolated = false;
418 nOutputIteration = 1;
419 nOutputLargeIteration = 1;
420
421 %NL STRUCTURE:
422 %Added Constraints | Added RHS | Added Constraints Properties | LowerBound
423 NL = {[],[],mEnforcedConstraints,Inf};
424 nOutputNLSize = 1;
425
426 %Set default CPLEX options
427 options = cplexoptimset;
428 options.Diagnostics = 'off';
429
430 %Output status
431 disp ('Iteration | Nodes Left | Incumbent');
432
433 %Generate and solve the Reduced Problem iteratively
434 while ((nOutputNLSize > 0) || (bLBViolated) || (bUBViolated))
435
436     %Reset global flags
437     bFathomInfeasible = false;
438
439     %Retrieve the first Reduced Problem. Here a FIFO approach is followed.
440     %The algorithm, however, can be modified to follow a different
441     %branching strategy, if so desired.
442     if ((~bLBViolated) && (~bUBViolated))
443         rConstraints = NL{1,1};
444         rRHS = NL{1,2};
445         mEnforcedConstraints = NL{1,3};
446         NL(1,:) = [];
```

```
447
448         %Update the iteration number
449         sOutput.nIterations = sOutput.nIterations + 1;
450         nOutputNLSize = nOutputNLSize - 1;
451
452         %Output status
453         if (nOutputLargeIteration == 10*nOutputIteration)
454             nNLClass = whos('NL');
455             disp (strcat(['*** Time Elapsed: ',' ',...
456                 sprintf('%.2f',etime(clock,sOutput.nSolutionTime)),...
457                 ' seconds | Tree Size: ',' ',...
458                 sprintf('%.2f',nNLClass.bytes/(1024^3)),' GB ***']));
459             disp ('Iteration | Nodes Left | Incumbent');
460             nOutputLargeIteration = 1;
461             clear nNLClass;
462         else
463             nOutputLargeIteration = nOutputLargeIteration+1;
464         end
465
466         if (nOutputIteration == nOutputIterationInterval)
467
468             %Update counter
469             nOutputIteration = 1;
470
471             %Display output
472             if (sOutput.nObjValue == Inf)
473                 disp (strcat([num2str(sOutput.nIterations),' ',...
474                     num2str(nOutputNLSize),' -']));
475             else
476                 disp (strcat([num2str(sOutput.nIterations),' ',...
477                     num2str(nOutputNLSize),' ',num2str(sOutput.nObjValue)]));
478             end
```

```
479         else
480             %Update counter
481             nOutputIteration = nOutputIteration+1;
482         end
483     end
484
485     %Solve the r^th Reduced Problem:
486     %    min        0.5*x'*L*x+f*x
487     %    st.        Aineq*x <= bineq
488     %               Aeq*x    = beq
489     %               lb <= x <= ub
490     %               x belongs to BICSN
491     %x = cplexmiqp(L,f,Aineq,bineq,Aeq,beq,sostype,sosind,soswt,lb,ub,ctype,x0,options)
492     tic;
493     try
494         [x2,fval,~,sCplexOutput] = cplexmiqp(L,d,[G;rConstraints],[g;rRHS],F,f,...
495             [],[],[],zeros(nTotalReducedProbVars,1),[],x2Type,[],options);
496     catch sError
497         throw(sError);
498     end
499
500     %Compute the elapsed time
501     sOutput.nRedProbTime = sOutput.nRedProbTime+toc;
502
503     %If the Reduced Problem cannot be solved to optimality, neither can the
504     %original problem. Fathom the node due to infeasibility
505     if ((sCplexOutput.cplexstatus ~= 1) && (sCplexOutput.cplexstatus ~= 101) && ...
506             (sCplexOutput.cplexstatus ~= 102))
507
508         %Lower the boundary violation flags
509         bLBViolated = false;
510         bUBViolated = false;
```

```
511
512         %Call the next iteration
513         continue;
514     end
515
516     %Update the objective function value (i.e., add alpha)
517     fval = fval+alpha;
518
519     %Check if the objective function value for the current Reduced Problem
520     %is larger than the objective function value for the incumbent
521     %solution. If so, fathom node.
522     if (fval >= sOutput.nObjValue)
523
524         %Release flags
525         bLBViolated = false;
526         bUBViolated = false;
527
528         %Call the next iteration
529         continue;
530     end
531
532     %Parse the results of the optimization problem into cells according
533     %to the number of variables in the Reduced Problem
534     x1 = cell(nSubproblems,1);
535     x2 = mat2cell(x2,mReducedProbVars,1);
536
537     %Lower the boundary violation flags
538     bLBViolated = false;
539     bUBViolated = false;
540
541     %For each subproblem
542     for i = 1:nSubproblems
```

```
543
544         %Compute the solution to the relaxed variables
545         x1{i} = iBb{i}-(iBB{i}*x2{i})-(iBD{i}*x2{nCells});
546
547         %Reset the counters/flags
548         bSubLBViolations = false;
549         idxLBViolation = -1;
550         valLBViolation = nInf;
551
552         bSubUBViolations = false;
553         idxUBViolation = -1;
554         valUBViolation = -nInf;
555
556         %For each element in the i^{th} subproblem
557         for j = 1:mSubProps{i}.nTotalVariables
558             if (mElementProps{i}.bIsRelaxed(j))
559
560                 %Check if the variable is negative
561                 if (x1{i}(mElementProps{i}.idxReference(j)) < -nPrecision)
562
563                     %Raise the lower bound violation flag
564                     bSubLBViolations = true;
565                     bLBViolated = true;
566
567                     %Not all non-negativity restrictions were satisfied
568                     %Update the most negative variable found so far.
569                     if (x1{i}(mElementProps{i}.idxReference(j)) < valLBViolation)
570                         idxLBViolation = mElementProps{i}.idxReference(j);
571                         valLBViolation = x1{i}(mElementProps{i}.idxReference(j));
572                     end
573
574                     %Check if the variable is binary and has a value
```

```
575                     %greater than one
576                 elseif ((x1{i}(mElementProps{i}.idxReference(j)) > 1+nPrecision) && ...
577                         strcmp(mElementProps{i}.sType(j),'B'))
578
579                     %Raise the upper bound violation flag
580                     bSubUBViolations = true;
581                     bUBViolated = true;
582
583                     %Not all binary restrictions were satisfied
584                     %Update the most positive variable found so far.
585                     if (x1{i}(mElementProps{i}.idxReference(j)) > valUBViolation)
586                         idxUBViolation = mElementProps{i}.idxReference(j);
587                         valUBViolation = x1{i}(mElementProps{i}.idxReference(j));
588                     end
589
590                 end % End IsNegative if-statement
591
592             end %End IsRelaxed if-statement
593         end %End the i^{th} subproblem element loop
594
595         if (bSubLBViolations)
596
597             %Update the current Reduced Problem
598             rConstraints(end+1,:) = [zeros(1,sum(mReducedProbVars(1,1:i-1))),...
599                 iBB{i}(idxLBViolation,:),zeros(1,nTotalReducedProbVars...
600                 -nComplicatingVars-sum(mReducedProbVars(1,1:i))),...
601                 iBD{i}(idxLBViolation,:)];
602             rRHS(end+1,1) = iBb{i}(idxLBViolation);
603
604             %Add a new element to the enforced contraints property structure
605             mEnforcedConstraints.nTotalExplicitConstraints = ...
606                 mEnforcedConstraints.nTotalExplicitConstraints+1;
```

```
607             mEnforcedConstraints.nSubproblem(end+1,1) = i;
608             mEnforcedConstraints.nElement(end+1,1) = idxLBViolation;
609             mEnforcedConstraints.nRowReference(end+1,1) = ...
610                 mEnforcedConstraints.nTotalExplicitConstraints;
611             mEnforcedConstraints.bUpperBoundRestriction(end+1,1) = false;
612
613         elseif (bSubUBViolations)
614
615             %Update the current Reduced Problem
616             rConstraints(end+1,:) = [zeros(1,sum(mReducedProbVars(1,1:i-1))),...
617                 -iBB{i}(idxUBViolation,:),zeros(1,nTotalReducedProbVars...
618                 -nComplicatingVars-sum(mReducedProbVars(1,1:i))),...
619                 -iBD{i}(idxUBViolation,:)];
620             rRHS(end+1,1) = 1-iBb{i}(idxUBViolation);
621
622             %Add a new element to the enforced contraints property structure
623             mEnforcedConstraints.nTotalExplicitConstraints = ...
624                 mEnforcedConstraints.nTotalExplicitConstraints+1;
625             mEnforcedConstraints.nSubproblem(end+1,1) = i;
626             mEnforcedConstraints.nElement(end+1,1) = idxUBViolation;
627             mEnforcedConstraints.nRowReference(end+1,1) = ...
628                 mEnforcedConstraints.nTotalExplicitConstraints;
629             mEnforcedConstraints.bUpperBoundRestriction(end+1,1) = true;
630
631         end %End HasNegatives if-statement
632     end %End the subproblem loop
633
634     %Call the next iteration, if needed
635     if (bLBViolated)
636         continue;
637     end
638
```

```
639     %Look for positive (basic) artificial variables
640     if (nTotalArtificial > 0)
641         n = 0;
642         for i = 1:nSubproblems
643             if (mSubProps{i}.nArtificialVariables > 0)
644                 for j = 1:mSubProps{i}.nTotalVariables
645                     if mElementProps{i}.bIsArtificial(j)
646
647                         %Determine if the node should be fathomed
648                         if mElementProps{i}.bIsRelaxed(j)
649                             bFathomInfeasible = ...
650                                 (x1(mElementProps{i}.idxReference(j)) > nPrecision);
651                         else
652                             bFathomInfeasible = ...
653                                 (x2(mElementProps{i}.idxReference(j)) > nPrecision);
654                         end
655
656                         %Update the artificial variable counter
657                         n = n + 1;
658
659                         %Exit if done
660                         if (bFathomInfeasible) || (n > nTotalArtificial)
661                             break;
662                         end
663
664                     end %End IsArtificial if-statement
665                 end %End the i^{th} subproblem element loop
666
667                 %Exit if done
668                 if (bFathomInfeasible) || (n > nTotalArtificial)
669                     break;
670                 end
```

```
671
672             end %End HasArtificials if-statement
673         end %End the subproblem loop
674     end
675
676     %Fathom the node due to infeasibility
677     if (bFathomInfeasible)
678
679         %Lower the boundary violation flags
680         bLBViolated = false;
681         bUBViolated = false;
682
683         %Call the next iteration
684         continue;
685     end
686
687     %Call the next iteration, if needed
688     if (bUBViolated)
689         continue;
690     end
691
692     %Start the branch and bound component
693     %Raise the FathomIntegrality flag
694     bFathomIntegrality = true;
695
696     %For each subproblem
697     for i = 1:nSubproblems
698
699         %For each element in the i^{th} subproblem
700         for j = 1:mSubProps{i}.nTotalVariables
701
702             %Check if the element is integer
```

```
703             if mElementProps{i}.bIsRelaxed(j)
704                 if (strcmp(mElementProps{i}.sType(j),'B') || ...
705                         strcmp(mElementProps{i}.sType(j),'I'))
706                     n = abs(x1{i}(mElementProps{i}.idxReference(j)));
707                     if (((ceil(n)-n) > nPrecision) && ...
708                             ((n-floor(n)) > nPrecision))
709                         bFathomIntegrality = false;
710                         break;
711                     end %End ~IsInteger if-statement
712                 end %End IsIntegerRestriced if-statement
713             end %End IsRelaxed if-statement
714
715         end %End the i^{th} subproblem element loop
716         if (~bFathomIntegrality)
717             break;
718         end
719     end %End the subproblem loop
720
721     if (bFathomIntegrality)
722         if (fval < sOutput.nObjValue)
723
724             %Save the new integer objective function value
725             sOutput.nObjValue = fval;
726             x1Incumbent = x1;
727             x2Incumbent = x2;
728
729             %Loop through the Reduced Problem list and fathom nodes with
730             %lower bound greater than or equal to fval
731             if (nOutputNLSize >= 1)
732                 if (NL{1,4} >= sOutput.nObjValue)
733                     %Since the NL is ordered, if the objective function
734                     %value of its first element is larger than or equal to
```

```
735                     %the objective function value of the incumbent solution,
736                     %then fathom the entire list.
737                     NL = [];
738                     nOutputNLSize = 0;
739
740                 elseif (NL{nOutputNLSize,4} >= sOutput.nObjValue)
741                     %If the objective function value of the incumbent solution
742                     %lies between the objective function value of NL's
743                     %first and last elements, then perform a dichotomous
744                     %search for the starting element to be fathomed
745                     nPosStart = 1;
746                     nPosEnd = nOutputNLSize;
747                     nPos = floor((1+nOutputNLSize)/2);
748                     while ((nPosEnd-nPosStart) > 1)
749                         if (NL{nPos,4} > sOutput.nObjValue)
750                             nPosEnd = nPos;
751                             nPos = floor((nPosStart+nPosEnd)/2);
752                         elseif (NL{nPos,4} < sOutput.nObjValue)
753                             nPosStart = nPos;
754                             nPos = ceil((nPosStart+nPosEnd)/2);
755                         else
756                             break;
757                         end
758                     end
759                     nPos = nPosEnd;
760                     NL(nPos:nOutputNLSize,:) = [];
761                     nOutputNLSize = nPos-1;
762                 end
763             end
764         end %End HasBetterObjFnValue if-statement
765
766         %Call the next iteration
```

```
767         continue;
768     else
769
770         %Before generating two new problems, check if the corresponding
771         %constraint is redundant. If so, simply update the RHS
772         bPreviousLowerBound = false;
773         bPreviousUpperBound = false;
774
775         %For each previously added constraint
776         %NOTE: i and j are the indices of the subproblem and element
777         %with the violated integer restriction
778         lRHS = rRHS;
779         uRHS = rRHS;
780         for n = 1:mEnforcedConstraints.nTotalExplicitConstraints
781
782             %If a redundant constraint exists
783             if (mEnforcedConstraints.nSubproblem(n) == i) && ...
784                     (mEnforcedConstraints.nElement(n) == ...
785                     mElementProps{i}.idxReference(j))
786                 if (mEnforcedConstraints.bUpperBoundRestriction(n))
787
788                     %Raise PreviousUpperBoundFlag
789                     bPreviousUpperBound = true;
790
791                     %A previous "<=" type constraint exists. The constraint
792                     %is now redundant. Update the RHS accordingly.
793                     uRHS(mEnforcedConstraints.nRowReference(n)) = ...
794                         floor(x1{i}(mElementProps{i}.idxReference(j))) -...
795                         iBb{i}(mElementProps{i}.idxReference(j));
796                 else
797
798                     %Raise PreviousLowerBoundFlag
```

```
799                     bPreviousLowerBound = true;
800
801                     %A previous ">=" type constraint exists. The constraint
802                     %is now redundant. Update the RHS accordingly.
803                     lRHS(mEnforcedConstraints.nRowReference(n)) = ...
804                         -ceil(x1{i}(mElementProps{i}.idxReference(j)))+ ...
805                         iBb{i}(mElementProps{i}.idxReference(j));
806                 end
807
808
809                 %Exit if already found all redundant constraints
810                 if (bPreviousLowerBound) && (bPreviousUpperBound)
811                     break;
812                 end
813
814             end %End IsRedundant if-statement
815         end %End the previously added constraints loop
816
817         %Search the node list for the best node placement
818         if (nOutputNLSize == 0)
819             %If NL is empty, append the node
820             nPos = 1;
821         elseif (NL{nOutputNLSize,4} <= fval)
822             %If NL is nonempty and the bound on its last element is smaller
823             %than the current bound, add the node to the back of the list
824             nPos = nOutputNLSize+1;
825         elseif (NL{1,4} >= fval)
826             %If NL is nonempty and the bound on its first element is larger
827             %than the current bound, add the node to the front of the list
828             nPos = 1;
829         else
830             %If NL is nonempty and the current bound lies between the
```

```
831             %bounds for the first and last elements of the list, use a
832             %dichotomous search strategy to find the optimum node placement
833             nPosStart = 1;
834             nPosEnd = nOutputNLSize;
835             nPos = floor((1+nOutputNLSize)/2);
836             while ((nPosEnd-nPosStart) > 1)
837                 if (NL{nPos,4} > fval)
838                     nPosEnd = nPos;
839                     nPos = floor((nPosStart+nPosEnd)/2);
840                 elseif (NL{nPos,4} < fval)
841                     nPosStart = nPos;
842                     nPos = ceil((nPosStart+nPosEnd)/2);
843                 else
844                     break;
845                 end
846             end
847             nPos = nPosEnd;
848         end
849
850         %Set the constraint properties, if needed
851         if (~bPreviousLowerBound) || (~bPreviousUpperBound)
852
853             %Create a temporaty structure to store the constraint properties
854             clear tempEnforcedConstraints
855             tempEnforcedConstraints = mEnforcedConstraints;
856
857             %Set the new constraint properties
858             tempEnforcedConstraints.nTotalExplicitConstraints = ...
859                 tempEnforcedConstraints.nTotalExplicitConstraints+1;
860             tempEnforcedConstraints.nSubproblem(end+1,1) = i;
861             tempEnforcedConstraints.nElement(end+1,1) = ...
862                 mElementProps{i}.idxReference(j);
```

```
863             tempEnforcedConstraints.nRowReference(end+1,1) = ...
864                 tempEnforcedConstraints.nTotalExplicitConstraints;
865         end
866
867         %Add the 2 new Reduced Problems to the NodeList set
868         NL = [NL(1:nPos-1,:);{[],[],[],fval};{[],[],[],fval};NL(nPos:nOutputNLSize,:)];
869         nOutputNLSize = nOutputNLSize+2;
870
871         if (bPreviousLowerBound)
872             %Add the existing constraint set and the update RHS vector
873             NL{nPos,1} = rConstraints;
874             NL{nPos,2} = lRHS;
875             NL{nPos,3} = mEnforcedConstraints;
876         else
877             %Append a new constraint into the existing constraint set
878             NL{nPos,1} = [rConstraints;zeros(1,sum(mReducedProbVars(1,1:i-1))),...
879                 iBB{i}(mElementProps{i}.idxReference(j),:),...
880                 zeros(1,nTotalReducedProbVars-nComplicatingVars...
881                 -sum(mReducedProbVars(1,1:i))),...
882                 iBD{i}(mElementProps{i}.idxReference(j),:)];
883
884             NL{nPos,2} = [rRHS;-ceil(x1{i}(mElementProps{i}.idxReference(j)))+ ...
885                 iBb{i}(mElementProps{i}.idxReference(j))];
886
887             tempEnforcedConstraints.bUpperBoundRestriction(end+1,1) = false;
888             NL{nPos,3} = tempEnforcedConstraints;
889         end
890
891         if (bPreviousUpperBound)
892             %Add the existing constraint set and the update RHS vector
893             NL{nPos+1,1} = rConstraints;
894             NL{nPos+1,2} = uRHS;
```

```
895             NL{nPos+1,3} = mEnforcedConstraints;
896         else
897             %Append a new constraint into the existing constraint set
898             NL{nPos+1,1} = [rConstraints;...
899                 zeros(1,sum(mReducedProbVars(1,1:i-1))),...
900                 -iBB{i}(mElementProps{i}.idxReference(j),:),...
901                 zeros(1,nTotalReducedProbVars-nComplicatingVars...
902                 -sum(mReducedProbVars(1,1:i))),...
903                 -iBD{i}(mElementProps{i}.idxReference(j),:)];
904
905             NL{nPos+1,2} = [rRHS;floor(x1{i}(mElementProps{i}.idxReference(j)))- ...
906                 iBb{i}(mElementProps{i}.idxReference(j))];
907
908             tempEnforcedConstraints.bUpperBoundRestriction(end+1,1) = true;
909             NL{nPos+1,3} = tempEnforcedConstraints;
910         end
911     end %End IsFathomIntegrality if-statement
912 end %End main while loop
913
914 %Release the support variables
915 clear AAeq AAineq BB HH cc iBB iBb iBD L d F f G g alpha x1 x2 NL fval k ...
916     idxLBViolation valLBViolation idxUBViolation valUBViolation ...
917     bSubLBViolations bLBViolated bSubUBViolations bUBViolated x2Type ...
918     bFathomInfeasible bFathomIntegrality rConstraints rRHS nTotalArtificial ...
919     mEnforcedConstraints options bPreviousLowerBound bPreviousUpperBound ...
920     lRHS uRHS nOutputNLSize tempEnforcedConstraints nPos nPosStart nPosEnd ...
921     nComplicatingVars nTotalReducedProbVars mReducedProbVars nOutputIteration ...
922     sCplexOutput nOutputIterationInterval;
923
924 %Output Results
925 switch true
926     case sOutput.nObjValue == Inf
```

```
927         sOutput.sResult = 'infeasible';
928         sOutput.sDescription = 'The original problem is infeasible';
929 
930     case (sum(cell2mat(x2Incumbent)) > nInf - nPrecision)
931         sOutput.sResult = 'unbounded';
932         sOutput.sDescription = 'The original problem is unbounded';
933 
934     otherwise
935         sOutput.sResult = 'optimal';
936         sOutput.sDescription = 'Program Finished. Optimal solution found';
937 
938         %Instantiate the solution set
939         sOutput.x = cell(nCells,1);
940 
941         %For each subproblem
942         for i = 1:nSubproblems
943 
944             %Initialize solution matrix
945             sOutput.x{i} = zeros(mSubProps{i}.nRealVariables,1);
946 
947             %For each element in the i^{th} subproblem
948             n = 0;
949             for j = 1:mSubProps{i}.nTotalVariables
950                 if (~mElementProps{i}.bIsArtificial(j))
951 
952                     %Update counter
953                     n = n + 1;
954 
955                     %Merge the relaxed and non-relaxed sets
956                     if (mElementProps{i}.bIsRelaxed(j))
957                         sOutput.x{i}(n) = x1Incumbent{i}(mElementProps{i}.idxReference(j));
958                     else
```

```
959                             sOutput.x{i}(n) = x2Incumbent{i}(mElementProps{i}.idxReference(j));
960                         end
961
962                     end %End IsArtificial if-statement
963                 end %End the i^{th} subproblem element loop
964             end %End the subproblem loop
965
966             %Include final value of complicating variables
967             sOutput.x{nCells,1} = x2Incumbent{nCells,1};
968
969 end %End switch
970
971 %Compute the elapsed time
972 sOutput.nSolutionTime = etime(clock,sOutput.nSolutionTime);
973 disp(sOutput)
974
975 %Release the remaining support variables
976 clear i j n mSubProps mElementProps nInf nPrecision x1Incumbent x2Incumbent ...
977     nCells nSubproblems;
978 end %End Function
```

Appendix E

Iterative Algorithm for the D-VRP Variation of the BEP - Source Code: IBM ILOG CPLEX Optimization Studio version 12.5

E.1 Two-Stage Stochastic Formulation

E.1.1 Model File (BaseModel.mod)

```
1  // MODEL FILE
2
3  // Import Network Parameters
4  {string} S = ... ;    // Set of Depots
5  {string} P = ... ;    // Set of Pick-up Locations
6  {string} N =(S union P) ;  // Set of Nodes
7  {string} C = ... ;    // Set of Scenarios
8
9  tuple Arcs {string orig; string dest;}; // Set of Arcs
10 {Arcs} A with orig in N, dest in N = ... ;
11
12 int V = ... ;    // Fleet Size Available
13 range VRange = 1..V;
14
15 float time[A] = ... ; // Arc Traversal Times
16 float Dmnd[P][C] = ... ; // Possible Demand Realizations
17
18 // Determine the Upper Bound on T
19 int T = ...;
20 range TRange = 1..T;
21
22 int ConstrainedT = ...;
23
24 // Set of Previous Arc Traversals
25 tuple rStar {Arcs arc; int vehicle; int traversal;};
26 {rStar} R_Star with arc in A, vehicle in VRange, traversal in TRange = ...;
27
28 // Set of Previous Visits to Pick-up Locations
```

```
29 tuple cStar {string dest; int vehicle; int traversal; float load;};
30 {cStar} C_Star with dest in P, vehicle in VRange, traversal in TRange = ...;
31
32 // Declare Decision Variables
33 dvar int+ x[<i,j> in A, m in 1..V, t in 1..T, c in C] in 0..1;
34 dvar float+ y[j in P, m in 1..V, t in 1..T, c in C] in 0..1;
35
36 // Objective Function
37 minimize (1/card(C))*sum(<i,j> in A, m in 1..V, t in 1..T, c in C) time[<i,j>]*x[<i,j>,m,t,c
       ];
38
39 // Constraint Set
40 subject to {
41    forall (i in P, <i,j> in A, m in 1..V, c in C) x[<i,j>,m,1,c] == 0;
42    forall (j in P, m in 1..V, t in 1..(T-1), c in C) sum(<i,j> in A) x[<i,j>,m,t,c] == sum(<
          j,k> in A) x[<j,k>,m,t+1,c];
43  forall (j in S, m in 1..V, t in 1..(T-1), c in C) sum(<i,j> in A) x[<i,j>,m,t,c] >= sum(<j,
       k> in A) x[<j,k>,m,t+1,c];
44  forall (j in P, <i,j> in A, m in 1..V, c in C) x[<i,j>,m,T,c] == 0;
45  forall (m in 1..V, t in 1..T, c in C) sum(<i,j> in A) x[<i,j>,m,t,c] <= 1;
46  forall (j in P, m in 1..V, t in 1..T, c in C) y[j,m,t,c] <= sum(<i,j> in A) x[<i,j>,m,t,c];
47  forall (m in 1..V, t in 1..T, c in C) sum(j in P, l in 1..t) y[j,m,l,c] - sum(j in S, <i,j>
       in A, l in 1..t) x[<i,j>,m,l,c] <= 1;
48  forall (<i,j> in A, m in 1..V, c in C, u in C) x[<i,j>,m,ConstrainedT,c] == x[<i,j>,m,
       ConstrainedT,u];
49  forall (j in P, c in C) sum(m in 1..V, t in 1..T) y[j,m,t,c] == Dmnd[j,c];
50  forall (j in P, c in C) sum(<i,j> in A, m in 1..V, t in 1..T) x[<i,j>,m,t,c] >= 1;
51  forall (r in R_Star, c in C) x[r.arc,r.vehicle,r.traversal,c] == 1;
52  forall (r in C_Star, c in C) y[r.dest,r.vehicle,r.traversal,c] == r.load;
53  }
```

E.1.2 Data File (BaseModel.dat)

```
1  // DATA FILE
2
3  // Network Data
4  SheetConnection myData("BEP-Probing.xlsx") ;
5
6  S from SheetRead(myData,"Comparison!Depot");
7  P from SheetRead(myData,"Comparison!Nodes");
8  A from SheetRead(myData,"Comparison!Arcs");
9  time from SheetRead(myData,"Comparison!Time");
10 C from SheetRead(myData,"Comparison!Scenarios");
11 Dmnd from SheetRead(myData,"Comparison!Demand");
12
13 // Analytical Data
14 V = 2;
15
16 // Dynamically Changed Data
17 T = 10;
18 ConstrainedT = 1;
19 R_Star = {};
20 C_Star = {};
```

E.1.3 Iterative Algorithm (BaseModelLoop.mod)

```
1  main {
2
3  // Support Variables
4  var overall_start_time = new Date();
5  var overall_end_time;
6  var start_time;
7  var end_time;
8
9  var a;
10 var m;
11 var z1;
12 var z2;
13 var z3;
14 var bFound;
15 var nLoad;
16 var bError = 0;
17
18 // Initialize CPLEX Model
19 var MyCplex = new IloCplex();
20 var MyModelDefs = new IloOplModelDefinition(new IloOplModelSource("BaseModel.mod"));
21 var MyResult = new IloOplModel(MyModelDefs,MyCplex);
22 var MyCurrentResult;
23
24 MyResult.addDataSource(new IloOplDataSource("BaseModel.dat"));
25 var MyDataElements = MyResult.dataElements;
26
27 // Initialize P^*, the set of pick-up locations with uncertain demand
28 var PStar = new Array(Opl.card(MyResult.P));
29 z2 = 0;
30 for(z1 in MyResult.P) {
31  PStar[z2] = z1;
```

```
32   z2++;
33  }
34
35  // Actual Realized Demands
36  var DStar = new Array(1,2,0,4,0);
37
38  // Set T to Tub
39  z1 = -Infinity;
40  for (a in MyDataElements.C) {
41   z2 = -1;
42   for (m in MyDataElements.P) {
43       z2 = z2 + Opl.ceil(MyDataElements.Dmnd[m][a]);
44   }
45   z1 = Opl.maxl(z1,Opl.ceil(Opl.maxl(z2,0)/MyDataElements.V));
46  }
47  MyDataElements.T = PStar.length+1+2*z1;
48
49  // Master Loop
50  while(PStar.length >= 0) {
51
52   // Generate a Temporary File to Store the Optimization Model
53   MyCurrentResult = new IloOplModel(MyModelDefs,MyCplex);
54
55   // Update Data Elements
56   MyCurrentResult.addDataSource(MyDataElements);
57   MyCurrentResult.generate();
58
59   // Record Run Start Time
60   start_time = new Date();
61
62   // Solve and Test for Feasibility
63   if (MyCplex.solve()) {
```

```
64
65    // Record Run End-Time
66     end_time = new Date();
67
68    // Invoke the PostProcess Method
69    MyCurrentResult.postProcess();
70
71    // Exit if All Uncertainty Resolved
72    if (PStar.length == 0) break;
73
74     // Output Iteration Data to the Script Log Window
75    writeln("Iteration: ",MyDataElements.ConstrainedT);
76    writeln("Expected Total Travel Time: ",MyCplex.getObjValue());
77    writeln("Solution Time: ",(end_time.getTime()-start_time.getTime())/1000," seconds");
78    writeln();
79    writeln("Bus","\t","Traversal","\t","Arc","\t","Loading");
80
81    for(a in MyCurrentResult.A){
82     for(m=1;m<=MyCurrentResult.V;m++){
83      if(MyCurrentResult.x[a][m][MyDataElements.ConstrainedT][Opl.first(MyDataElements.C)] ==
            1) {
84
85          nLoad = "-"
86
87       // Save the Common Arc Traversal to Set R^*
88       MyDataElements.R_Star.add(a,m,MyDataElements.ConstrainedT);
89
90       // Determine if the Destination Node is in Set P^*
91       bFound = 0;
92       for (z1=0;z1<=PStar.length-1;z1++) {
93        if (PStar[z1] == a.dest) {
94         bFound = 1;
```

```
95          break;
96         }
97        }
98        if (bFound == 1) {
99
100        // Set the Respective Location-Scenarios with the Realized Demand
101        for (z2 in MyCurrentResult.C) {
102
103           // Find the Corresponding DStar Element
104           for (z3=0;z3<=JStar.length-1;z3++) {
105              if (z2.dest == JStar[z3]) {
106               if(MyDataElements.Dmnd[a.dest][z2] == DStar[z3]) {
107                // Add Destination to Set C_Star
108                nLoad = MyCurrentResult.y[a.dest][m][MyDataElements.ConstrainedT][z2];
109            MyDataElements.C_Star.add(a.dest,m,MyDataElements.ConstrainedT,nLoad);
110           } else {
111                MyDataElements.Dmnd[a.dest][z2] = DStar[z3];
112               }
113               break;
114              }
115           }
116        }
117
118        // Remove the Corresponding Element from Set P^*
119        delete PStar[z1];
120
121       } else {
122
123        // Determine if the destination node is in Set P
124        for (z1 in MyDataElements.P) {
125         if (a.dest == z1) {
126          // Add Destination to Set C_Star
```

```
127           nLoad = MyCurrentResult.y[a.dest][m][MyDataElements.ConstrainedT][Opl.first(
                  MyDataElements.C)];
128           MyDataElements.C_Star.add(a.dest,m,MyDataElements.ConstrainedT,nLoad);
129           break;
130          }
131         }
132        }
133
134        // Output Arc Traversal Data to the Script Log Window
135        writeln(m,"\t",MyDataElements.ConstrainedT,"\t",a,"\t",nLoad);
136        }
137       }
138    }
139
140    // Update Solution Timer
141    end_time = new Date();
142    writeln();
143    writeln("Solution + Processing Time: ",(end_time.getTime()-start_time.getTime())/1000," 
           seconds");
144    writeln();
145    writeln("----------------------------------------");
146    writeln();
147
148    // Increment t^*
149    MyDataElements.ConstrainedT = MyDataElements.ConstrainedT +1;
150
151  } else {
152
153     // Output Error Flag to the Script Log Window
154    end_time = new Date();
155    writeln();
156    writeln("An Error Occurred");
```

```
157     writeln();
158     writeln("Time: ",(end_time.getTime()-start_time.getTime())/1000," seconds");
159     writeln("Best Objective Function Value: ",MyCplex.GetBestObjValue());
160     writeln("MIP Gap = ",(MyCplex.getBestObjValue()-MyCplex.getObjValue())/(1e-10+MyCplex.
            getObjValue()));
161
162     bError = 1;
163     break;
164   }
165 }
166
167 // Record Run Overall Completion Time
168 overall_end_time = new Date();
169
170 // Output Final Routes to the Script Log Window
171 if (bError == 0) {
172   writeln("Run Finished. Final Tour:");
173   writeln()
174   writeln("Bus","\t","Traversal","\t","Arc","\t","Loading");
175   for(m=1;m<=MyCurrentResult.V;m++){
176     for(z1=1;z1<=MyCurrentResult.T;z1++) {
177       for(a in MyCurrentResult.A) {
178       if(MyCurrentResult.x[a][m][z1][Opl.first(MyCurrentResult.C)] == 1) {
179         bFound = 0;
180         for(z2 in MyDataElements.P) {
181             if(z2 == a.dest) {
182               bFound = 1;
183               break;
184           }
185         }
186         if(bFound == 1) {
```

```
187             writeln(m,"\t",z1,"\t",a,"\t",MyCurrentResult.y[a.dest][m][z1][Opl.first(
                    MyCurrentResult.C)]);
188           } else {
189               writeln(m,"\t",z1,"\t",a,"\t","-");
190           }
191       }
192     }
193   }
194  }
195   writeln();
196   writeln("Realized Total Travel Time: ",MyCplex.getObjValue());
197   writeln("Solution + Processing Time: ",(overall_end_time.getTime()-start_time.getTime())
          /1000," seconds");
198   writeln();
199   writeln("Overall Solution Time: ",(overall_end_time.getTime()-overall_start_time.getTime())
          /1000," seconds");
200 }
201 // Release Remaining Objects
202 MyCplex.end();
203 MyModelDefs.end();
204 MyResult.end();
205 MyDataElements.end();
206 MyCurrentResult.end();
207 }
```

E.2 Deterministic Formulation

E.2.1 Model File (EnhancedModel.mod)

```
// MODEL FILE

// Import Network Parameters
{string} S = ... ;    // Set of Depots
{string} P = ... ;    // Set of Pick-up Locations
{string} N =(S union P) ;  // Set of Nodes
{string} PStar = ... ;       // Set of Unvisited Pick-up Locations
{string} C = ... ;    // Set of Scenarios

tuple Arcs {string orig; string dest;}; // Set of Arcs
{Arcs} A with orig in N, dest in N = ... ;

tuple Demand {string dest; string scenario; float Value;}; // Set of Demands
{Demand} D = ... ;

int V = ... ;    // Fleet Size Available
range VRange = 1..V;

float time[A] = ... ; // Arc Traversal Times
float Dmnd[P] = ... ; // Lowest Possible Demand Realizations
int ConstrainedT = ...; // Iteration Number

// Determine the Upper Bound on T
int T = ... ;
range TRange = 1..T;

// Set of Previous Arc Traversals
tuple rStar {Arcs arc; int vehicle; int traversal;};
```

```
29 {rStar} R_Star with arc in A, vehicle in VRange, traversal in TRange = ...;
30
31 // Set of Previous Visits to Pick-up Locations
32 tuple cStar {string dest; int vehicle; int traversal; float load;};
33 {cStar} C_Star with dest in P, vehicle in VRange, traversal in TRange = ...;
34
35 // Declare Decision Variables
36 dvar int+ x[<i,j> in A, m in 1..V, t in 1..T] in 0..1;
37 dvar float+ y[j in P, m in 1..V, t in 1..T] in 0..1;
38
39 // Objective Function
40 minimize sum(<i,j> in A, m in 1..V, t in 1..T) time[<i,j>]*x[<i,j>,m,t];
41
42 // Constraint Set
43 subject to {
44    forall (i in P, <i,j> in A, m in 1..V) x[<i,j>,m,1] == 0;
45    forall (j in P, m in 1..V, t in 1..(T-1)) sum(<i,j> in A) x[<i,j>,m,t] == sum(<j,k> in A)
          x[<j,k>,m,t+1];
46  forall (j in S, m in 1..V, t in 1..(T-1)) sum(<i,j> in A) x[<i,j>,m,t] >= sum(<j,k> in A) x
      [<j,k>,m,t+1];
47  forall (j in P, <i,j> in A, m in 1..V) x[<i,j>,m,T] == 0;
48  forall (m in 1..V, t in 1..T) sum(<i,j> in A) x[<i,j>,m,t] <= 1;
49  forall (j in P, m in 1..V, t in 1..T) y[j,m,t] <= sum(<i,j> in A) x[<i,j>,m,t];
50  forall (m in 1..V, t in 1..T) sum(j in P, l in 1..t) y[j,m,l] - sum(j in S, <i,j> in A, l
      in 1..t) x[<i,j>,m,l] <= 1;
51  forall (j in P) sum(m in 1..V, t in 1..T) y[j,m,t] == Dmnd[j];
52  forall (j in PStar) sum(<i,j> in A, m in 1..V, t in 1..T) x[<i,j>,m,t] >= 1;
53  forall (i in S, j in PStar, m in 1..V, t in 1..(T-1)) Dmnd[j]*(sum(l in (t+1)..T) x[<j,i>,m
      ,l] - x[<i,j>,m,t]) >= 0;
54  forall (r in R_Star) x[r.arc,r.vehicle,r.traversal] == 1;
55  forall (r in C_Star) y[r.dest,r.vehicle,r.traversal] == r.load;
56  }
```

E.2.2 Data File (EnhancedModel.dat)

```
1  // DATA FILE
2
3  // Network Data
4  SheetConnection myData("BEP-Probing.xlsx") ;
5
6  S from SheetRead(myData,"Comparison!Depot");
7  P from SheetRead(myData,"Comparison!Nodes");
8  A from SheetRead(myData,"Comparison!Arcs");
9  time from SheetRead(myData,"Comparison!Time");
10 C from SheetRead(myData,"Comparison!Scenarios");
11 D from SheetRead(myData,"Comparison!D");
12 Dmnd from SheetRead(myData,"Comparison!MinDemand");
13
14 // Analytical Data
15 V = 2;
16
17 // Dynamically Changed Data
18 T = 10;
19 PStar from SheetRead(myData,"Comparison!Nodes");
20 ConstrainedT = 1;
21 R_Star = {};
22 C_Star = {};
```

E.2.3 Iterative Algorithm (EnhancedModelLoop.mod)

```
1  main {
2
3  // Support Variables
4  var overall_start_time = new Date();
5  var overall_end_time;
6  var start_time;
7  var end_time;
8
9  var a;
10 var m;
11 var z1;
12 var z2;
13 var z3;
14 var bFound;
15 var nLoad;
16 var nLoad1;
17 var nLoad2;
18 var bError = 0;
19 var tempT = 0;
20
21 // Initialize CPLEX Model
22 var MyCplex = new IloCplex();
23 var MyModelDefs = new IloOplModelDefinition(new IloOplModelSource("EnhancedModel.mod"));
24 var MyResult = new IloOplModel(MyModelDefs,MyCplex);
25 var MyCurrentResult;
26
27 MyResult.addDataSource(new IloOplDataSource("EnhancedModel.dat"));
28 var MyDataElements = MyResult.dataElements;
29 MyResult.end();
30
31 // Actual Realized Demands
```

```
32 var JStar = new Array(Opl.card(MyDataElements.P));
33 z2 = 0;
34 for(z1 in MyDataElements.P) {
35  JStar[z2] = z1;
36  z2++;
37 }
38 var DStar = new Array(1,2,0,4,0);
39
40 // Initialize t'[m], the m^th Vehicle Last Visit to Depot
41 var tPrime = new Array(MyDataElements.V);
42 z2 = 0;
43 for(z2=0;z2<=MyDataElements.V-1;z2++) {
44  tPrime[z2] = 0;
45 }
46
47 // Master Loop
48 while(Opl.card(MyDataElements.PStar) >= 0) {
49
50  // Set T to max(T',Tub')
51  z1 = -1;
52  for (z2 in MyDataElements.P) {
53   z1 = z1 + Opl.ceil(MyDataElements.Dmnd[z2]);
54  }
55  if (z1 < 0) z1 = 0;
56  tempT = Opl.maxl(tempT,Opl.card(MyDataElements.PStar)+1+2*Opl.ceil(z1/MyDataElements.V));
57  MyDataElements.T = tempT;
58
59  // Generate a Temporary File to Store the Optimization Model
60  MyCurrentResult = new IloOplModel(MyModelDefs,MyCplex);
61
62  // Update Data Elements
63  MyCurrentResult.addDataSource(MyDataElements);
```

```
64  MyCurrentResult.generate();
65
66  // Record Run Start Time
67  start_time = new Date();
68
69  // Solve and Test for Feasibility
70  if (MyCplex.solve()) {
71
72   // Record Run End-Time
73    end_time = new Date();
74
75   // Invoke the PostProcess Method
76   MyCurrentResult.postProcess();
77
78   // Exit if All Uncertainty Resolved
79   if (Opl.card(MyDataElements.PStar) == 0) break;
80
81    // Output Iteration Data to the Script Log Window
82   writeln("Iteration: ",MyDataElements.ConstrainedT);
83   writeln("Expected Total Travel Time: ",MyCplex.getObjValue());
84   writeln("Solution Time: ",(end_time.getTime()-start_time.getTime())/1000," seconds");
85   writeln();
86   writeln("Bus","\t","Traversal","\t","Arc","\t","Loading");
87
88   for(a in MyDataElements.A){
89    for(m=1;m<=MyDataElements.V;m++){
90     if(MyCurrentResult.x[a][m][MyDataElements.ConstrainedT] == 1) {
91
92         nLoad = "-"
93
94      // Save the Common Arc Traversal to Set R^*
95      MyDataElements.R_Star.add(a,m,MyDataElements.ConstrainedT);
```

```
96
97        // Determine if the Destination Node is in Set P^*
98        bFound = 0;
99        for (z1 in MyDataElements.PStar) {
100        if (z1 == a.dest) {
101         bFound = 1;
102         break;
103        }
104       }
105       if (bFound == 1) {
106
107        // Find the Corresponding DStar Element
108        for (z2=0;z2<=JStar.length-1;z2++) {
109           if (JStar[z2] == a.dest) {
110              // Set the Respective Location-Scenario with the Realized Demand
111          MyDataElements.Dmnd[a.dest] = DStar[z2];
112              break;
113             }
114          }
115
116        // Remove the Corresponding Element from Sets P^* and D
117        MyDataElements.PStar.remove(z1);
118        z1 = 0;
119        while (z1 <= Opl.card(MyDataElements.D)-1) {
120
121         // Retrieve the Data
122         if (a.dest == Opl.item(MyDataElements.D,z1).dest) {
123          if (MyDataElements.Dmnd[a.dest] != Opl.item(MyDataElements.D,z1).Value) {
124
125           // Remove all Related <k,s> Elements from Set D
126           z2 = 0;
127           while (z2 <= Opl.card(MyDataElements.D)-1) {
```

```
128             if (Opl.item(MyDataElements.D,z1).dest != Opl.item(MyDataElements.D,z2).dest) {
129                 if (Opl.item(MyDataElements.D,z1).scenario == Opl.item(MyDataElements.D,z2).
                        scenario) {
130             MyDataElements.D.remove(Opl.item(MyDataElements.D,z2));
131             if (z2 < z1) z1--; // Correct the Index of the First Set
132            } else {
133             z2++;
134            }
135           } else {
136            z2++;
137           }
138          }
139
140          // Remove the Respective s Element from Set C
141          MyDataElements.C.remove(Opl.item(MyDataElements.D,z1).scenario);
142
143          // Remove the Respective <j,s> Element from Set D
144          MyDataElements.D.remove(Opl.item(MyDataElements.D,z1));
145
146         } else {
147          z1++;
148         }
149        } else {
150         z1++;
151        }
152       }
153
154       // Finally, Recompute the Minimum Demand Across Pick-up Locations
155       z1 = 0;
156       while (z1 <= Opl.card(MyDataElements.PStar)-1) {
157
158          MyDataElements.Dmnd[Opl.item(MyDataElements.PStar,z1)] = Infinity;
```

```
159           z3 = 0;
160          for (z2 in MyDataElements.D) {
161           if (Opl.item(MyDataElements.PStar,z1) == z2.dest) {
162            if (MyDataElements.Dmnd[Opl.item(MyDataElements.PStar,z1)] >= z2.Value) {
163               MyDataElements.Dmnd[Opl.item(MyDataElements.PStar,z1)] = z2.Value;
164            }
165            z3++;
166           }
167          }
168          if (z3 == 1) {
169           MyDataElements.PStar.remove(Opl.item(MyDataElements.PStar,z1));
170          } else {
171             z1++;
172          }
173         }
174
175        } else {
176
177         // Determine if the Destination Node is in Set P
178         bFound = 0;
179         for (z1 in MyDataElements.P) {
180          if (a.dest == z1) {
181           bFound = 1;
182           break;
183          }
184         }
185         if (bFound == 0) {
186          // Update tPrime'[m], Note: m = 0..(V-1)
187          tPrime[m-1] = MyDataElements.ConstrainedT;
188         }
189        } // End j in P^*
190
```

```
191       // Manually Compute the Loading at the Vehicle
192       if (bFound == 1) {
193        nLoad1 = 1;
194        for (z2 in MyDataElements.P) {
195         for (z3=tPrime[m-1]+1;z3<=MyDataElements.ConstrainedT;z3++) {
196          nLoad1 = nLoad1 - MyCurrentResult.y[z2][m][z3];
197         }
198        }
199        nLoad2 = MyDataElements.Dmnd[a.dest];
200        for (z2=1;z2<=m;z2++) {
201         nLoad2 = nLoad2 - MyCurrentResult.y[a.dest][z2][MyDataElements.ConstrainedT];
202        }
203        nLoad = MyCurrentResult.y[a.dest][m][MyDataElements.ConstrainedT]+Opl.minl(nLoad1,
               nLoad2);
204        MyDataElements.C_Star.add(a.dest,m,MyDataElements.ConstrainedT,nLoad);
205       }
206
207       // Output Arc Traversal Data to the Script Log Window
208       writeln(m,"\t",MyDataElements.ConstrainedT,"\t",a,"\t",nLoad);
209
210       } // End x_ij^mtf == 1?
211      } // Next m = 1,...,V
212    } // Next (i,j) in A
213
214    // Update Solution Timer
215    end_time = new Date();
216    writeln();
217    writeln("Solution + Processing Time: ",(end_time.getTime()-start_time.getTime())/1000,"
           seconds");
218    writeln();
219    writeln("----------------------------------------");
220    writeln();
```

```
221
222     // Increment t^*
223     MyDataElements.ConstrainedT = MyDataElements.ConstrainedT +1;
224
225     // Release the MyCurrentResult Object, before reinitiating it
226     MyCurrentResult.end();
227
228   } else {
229
230       // Output Error Flag to the Script Log Window
231     end_time = new Date();
232     writeln();
233     writeln("An Error Occurred");
234     writeln();
235     writeln("Time: ",(end_time.getTime()-start_time.getTime())/1000," seconds");
236     writeln("Best Objective Function Value: ",MyCplex.GetBestObjValue());
237     writeln("MIP Gap = ",(MyCplex.getBestObjValue()-MyCplex.getObjValue())/(1e-10+MyCplex.
          getObjValue()));
238
239     bError = 1;
240     break;
241
242   } // End Optimize
243 } // Next Loop
244
245 // Record Run Overall Completion Time
246 overall_end_time = new Date();
247
248 // Output Final Routes to the Script Log Window
249 if (bError == 0) {
250   writeln("Run Finished. Final Tour:");
251   writeln()
```

```
252   writeln("Bus","\t","Traversal","\t","Arc","\t","Loading");
253   for(m=1;m<=MyDataElements.V;m++){
254     for(z1=1;z1<=MyCurrentResult.T;z1++) {
255       for(a in MyDataElements.A) {
256       if(MyCurrentResult.x[a][m][z1] == 1) {
257         bFound = 0;
258         for(z2 in MyDataElements.P) {
259             if(z2 == a.dest) {
260               bFound = 1;
261               break;
262           }
263         }
264         if(bFound == 1) {
265             writeln(m,"\t",z1,"\t",a,"\t",MyCurrentResult.y[a.dest][m][z1]);
266           } else {
267               writeln(m,"\t",z1,"\t",a,"\t","-");
268           }
269       }
270     }
271    }
272   }
273   writeln();
274   writeln("Realized Total Travel Time: ",MyCplex.getObjValue());
275   writeln("Solution + Processing Time: ",(overall_end_time.getTime()-start_time.getTime())
          /1000," seconds");
276   writeln();
277   writeln("Overall Solution Time: ",(overall_end_time.getTime()-overall_start_time.getTime())
          /1000," seconds");
278 }
279 // Release Remaining Objects
280 MyCplex.end();
281 MyModelDefs.end();
```

```
282 MyDataElements.end();
283 MyCurrentResult.end();
284 }
```