AIP Journal of Applied Physics

On a point defect inside an idealized elastic sphere

Chih-Bing Ling and Charles R. Houska

Citation: Journal of Applied Physics **54**, 5123 (1983); doi: 10.1063/1.332734
View online: http://dx.doi.org/10.1063/1.332734
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/54/9?ver=pdfcov
Published by the AIP Publishing

On a point defect inside an idealized elastic sphere

Chih-Bing Ling[a] and Charles R. Houska[b]
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

(Received 12 July 1982; accepted for publication 25 April 1983)

This paper presents a method of solution for the displacement, stress, and strain due to a point defect located inside a sphere. The solution is represented by a Love stress function in spherical coordinates, which is biharmonic in character. Two axisymmetric types of the point defect are considered. One is treated as a center of dilatation and the other as a double force without moment, or a doublet, oriented axisymmetrically. The Love stress function for the point defect in an infinite solid is specified in each case by a single biharmonic function. The residual tractions on the surface of the sphere left by this function are annulled by superposing two series of biharmonic functions. When the Love stress function is determined, the displacement, stress, and strain can be derived straightforwardly.

PACS numbers: 71.55. — i, 61.70.Ey

INTRODUCTION

A point defect in a crystalline solid is usually treated as a center of dilatation[1,2] when it is spherically symmetric. The anisotropy of the solid is neglected in the sense that the solid is idealized as a homogeneous isotropic elastic medium. Such a treatment is satisfactory only when the solid is of infinite size. When it is of finite size or at least one of its dimensions is finite, residual tractions are left on its boundary. Such tractions vanish identically on a boundary located at infinity. In 1954, Eshelby[3] proposed to introduce an additional system of negative tractions to annul the residual tractions.

The determination of displacement or stress in a solid directly by integration when only the surface tractions are known generally presents mathematical difficulties. This was mentioned by Eshelby himself and also later by Dundurs and Guell.[4] The displacement, stress, and also the strain in the solid can be derived from one or more sets of biharmonic functions appropriate to the problem. Several approaches are available to achieve the purpose. The solution is considerably simplified whenever the problem possesses an axis of symmetry.

A center of dilatation is by nature a spherically symmetric elastic singularity, which can be resolved into three mutually orthogonal double forces without moment of equal strengths. Since each double force without moment can be treated separately, it is possible to construct a point defect of unequal strengths in orthogonal directions. For brevity, a double force without moment will henceforth be described as a *doublet*.

This paper presents a method of solution for the displacement, stress, and strain due to a point defect located inside an idealized sphere. Two types of the point defect are considered. One is treated as a center of dilatation and the other as a doublet oriented axisymmetrically. Both types possess an axis of symmetry. Love's theory of symmetric strain[5] is adapted in the solution and is represented by a biharmonic function commonly called a Love stress function. When such a function is determined, the displacement, stress, and strain in the sphere can be derived straightforwardly.

[a] Department of Mathematics.
[b] Department of Materials Engineering.

COORDINATES AND FORMULAS

Define a set of spherical coordinates (ρ, ϕ, θ) by

$$x = \rho \sin\phi \cos\theta,\quad y = \rho \sin\phi \sin\theta,\quad z = \rho\cos\theta, \tag{1}$$

where (x,y,z) is a set of cartesian coordinates with its origin at 0. Also let

$$\mu = \cos\phi. \tag{2}$$

In the axisymmetric case with the z axis as an axis of symmetry, the Love stress function is independent of θ. The formulas for the components of displacement, stress, and strain referred to spherical coordinates were given by Ling and Yang[6] some time ago. They are as follows:

(i) For the components of displacement,

$$u_\rho = \frac{1}{2G}\left[2(1-\nu)\mu\nabla^2 - \frac{\partial}{\partial\rho}\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\right]\chi,$$

$$u_\phi = \frac{1}{2G}(1-\mu^2)^{1/2}\left[-2(1-\nu)\nabla^2 + \frac{1}{\rho}\frac{\partial}{\partial\mu}\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\right]\chi; \tag{3}$$

(ii) For the components of stress,

$$\sigma_\rho = \left[(2-\nu)\mu\frac{\partial}{\partial\rho} + \frac{\nu(1-\mu^2)}{\rho}\frac{\partial}{\partial\mu}\right]\nabla^2\chi - \frac{\partial^2}{\partial\rho^2}\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi;$$

$$\sigma_\phi = -(1-\nu)\left(\mu\frac{\partial}{\partial\rho} - \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\nabla^2\chi + \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{\mu}{\rho^2}\frac{\partial}{\partial\mu}\right) \times \left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi,$$

$$\sigma_\theta = \left(\nu\nabla^2 - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\mu}{\rho^2}\frac{\partial}{\partial\mu}\right)\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi,$$

$$\sigma_{\rho\phi} = (1-\mu^2)^{1/2}\left[-(1-\nu)\left(\frac{\partial}{\partial\rho} + \frac{\mu}{\rho}\frac{\partial}{\partial\mu}\right)\nabla^2\chi + \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\mu}\right)\left(\mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}\right)\chi\right]; \quad (4)$$

(iii) For the components of strain,

$$\epsilon_\rho = \frac{\partial\mu_\rho}{\partial\rho},$$
$$\epsilon_\phi = \frac{u_\rho}{\rho} - \frac{(1-\mu^2)^{1/2}}{\rho}\frac{\partial u_\phi}{\partial\mu},$$
$$\epsilon_\theta = \frac{u_\rho}{\rho} + \frac{u_\phi}{\rho}\cot\phi, \quad (5)$$
$$\epsilon_{\rho\phi} = \frac{\partial u_\phi}{\partial\rho} - \frac{u_\phi}{\rho} - \frac{(1-\mu^2)^{1/2}}{\rho}\frac{\partial u_\rho}{\partial\mu}.$$

The other components vanish identically. The expressions for σ_ϕ and σ_θ are here modified slightly for simplicity. ν is the Poisson ratio and G the modulus of rigidity. ∇^2 is a Laplace operator. For the axisymmetric case, it is given in spherical coordinates by

$$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\right]. \quad (6)$$

A CENTER OF DILATATION

Define a second set of spherical coordinates (ρ^*, ϕ^*, θ) by

$$x = \rho^*\sin\phi^*\cos\theta,$$
$$y = \rho^*\sin\phi^*\sin\theta, \quad (7)$$
$$z - c = \rho^*\cos\phi^*.$$

This set of spherical coordinates has its pole on the z axis at the point 0^* or $z = c$. Similarly, let

$$\mu^* = \cos\phi^*. \quad (8)$$

Consider a center of dilatation of strength s located on the z axis at the point 0^* inside a sphere $\rho \leqslant a$. Suppose that the Love stress function in question is composed of two parts as follows:

$$\chi = \chi_0 + \chi_1. \quad (9)$$

The first part is the Love stress function for the center of dilatation in an infinite solid. The second part is added to annul the residual tractions on the surface of the sphere left by the first part. We construct

$$\chi_0 = 2GsQ_0(\mu^*),$$
$$\chi_1 = \sum_{n=1}^{\infty} A_n\rho^n P_n(\mu) + \sum_{n=0}^{\infty} B_{n+2}\rho^{n+2}P_n(\mu), \quad (10)$$

where Q_0 is a Legendre function of the second kind of order zero and P_n a Legendre function of the first kind of degree n. A_n and B_n are parametric coefficients. Both parts are biharmonic.

From the first part χ_0, we find the following radial components of displacement and stress by Eqs. (3) and (4), respectively, referred to ρ^* and μ^*:

$$\mu_{\rho^*} = \frac{s}{\rho^{*2}}, \quad \sigma_{\rho^*} = -\frac{4Gs}{\rho^{*3}}. \quad (11)$$

With proper choice of units, the first relation defines s or the strength of the center of dilatation.

When $\rho > |c|$, χ_0 can be expanded into the series:

$$\chi_0 = 2Gs\left[Q_0(\mu) - \sum_{n=0}^{\infty}\frac{c^{n+1}}{(n+1)\rho^{n+1}}P_n(\mu)\right]. \quad (12)$$

The following normal and tangential components of stress are derived from χ, or the sum of χ_0 and χ_1, for $\rho > |c|$:

$$\sigma_\rho = -\frac{4Gs}{\rho^3} - \sum_{n=0}^{\infty}\left\{\frac{2Gs(n+1)(n+2)c^n}{\rho^{n+3}} + \frac{2n+5}{2n+3}(n+1)(n^2-n-2-2\nu)B_{n+3}\rho^n + n(n-1)\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}B_{n+1}\right]\rho^{n-2}\right\}P_n(\mu),$$

$$\sigma_{\rho\phi} = (1-\mu^2)^{1/2}\sum_{n=1}^{\infty}\left\{-\frac{2Gs(n+2)c^n}{\rho^{n+3}} + \frac{2n+5}{2n+3}\times(n^2+2n-1+2\nu)B_{n+3}\rho^n + (n-1)\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}\times B_{n+1}\right]\rho^{n-2}\right\}P_n'(\mu). \quad (13)$$

The prime on P_n denotes a derivative. They are found with the aid of the following relations:

$$\frac{\partial}{\partial z} = \mu\frac{\partial}{\partial\rho} + \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu},$$
$$\frac{\partial}{\partial z}Q_0(\mu) = \frac{1}{\rho},$$
$$\frac{\partial}{\partial z}[\rho^n P_n(\mu)] = n\rho^{n-1}P_{n-1}(\mu), \quad (n \geqslant 1), \quad (14)$$
$$\frac{\partial}{\partial z}\left(\frac{P_n(\mu)}{\rho^{n+1}}\right) = -(n+1)\frac{P_{n+1}(\mu)}{\rho^{n+2}}, \quad (n \geqslant 0),$$

and also the recurrence formulas[7] for P_n. Consequently, the normal and tangential tractions on the surface of the sphere are annulled if we substitute $\rho = a$ into Eq. (13) and equate each coefficient of $P_n(\mu)$ as well as of $(1-\mu^2)^{1/2}P_n'(\mu)$ to zero. This leads to two sets of equations, of which the solution is

$$B_3 = \frac{12Gs}{5(1+\nu)a^3},$$
$$B_{n+3} = \frac{Gs(n+2)(2n+1)(2n+3)c^n}{(2n+5)(n^2+n+2n\nu+1+\nu)a^{2n+3}}, \quad (n \geqslant 1),$$
$$A_3 = \frac{16Gs(4-5\nu)}{15(1+\nu)a^3} - \frac{28Gsc^2}{(7+5\nu)a^5}, \quad (15)$$
$$A_{n+1} = \frac{2Gsn(2n-3)(3n-4n\nu+2-2\nu)c^{n-2}}{(n+1)(2n+1)(n^2-3n+2n\nu+3-3\nu)a^{2n-1}} - \frac{Gs(n+2)(2n+3)c^n}{(n^2+n+2n\nu+1+\nu)a^{2n+1}}, \quad (n \geqslant 3).$$

The preceding solution does not include A_1, A_2, and B_2. The term involving A_1 produces no effect on both displacement and stress and may therefore be omitted. The terms involving A_2 and B_2 produce no effect on stress but each gives a rigid body displacement in the z direction of amounts $-A_2/G$ and $B_2(5-6\nu)/G$, respectively. If the rigid body displacement of the sphere is zero, these terms are absent.

The components of displacement and stress are given below:

$$\begin{aligned} u_\rho = {} & \frac{s(\rho - c\mu)}{\rho^{*3}} - \frac{1}{2G}\sum_{n=0}^{\infty}\frac{2n-5}{2n+3}(n+1)(n-2+4\nu)B_{n+3}\rho^{n+1}P_n(\mu) \\ & - \frac{1}{2G}\sum_{n=2}^{\infty} n\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}B_{n+1}\right]\rho^{n-1}P_n(\mu), \\ u_\phi = {} & \frac{sc(1-\mu^2)^{1/2}}{\rho^{*3}} + \frac{1}{2G}(1-\mu^2)^{1/2}\sum_{n=1}^{\infty}\frac{2n-5}{2n+3}(n+5-4\nu)B_{n+3}\rho^{n+1}P_n'(\mu) \\ & + \frac{1}{2G}(1-\mu^2)^{1/2}\sum_{n=2}^{\infty}\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}B_{n+1}\right]\rho^{n-1}P_n'(\mu), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \sigma_\rho = {} & 2Gs\left(\frac{1}{\rho^{*3}} - \frac{3(\rho - c\mu)^2}{\rho^{*5}}\right) - \sum_{n=0}^{\infty}\left\{\frac{2n+5}{2n+3}(n+1)(n^2-n-2-2\nu)B_{n+3}\rho^n \right. \\ & \left. + n(n-1)\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}B_{n+1}\right]\rho^{n-2}\right\}P_n(\mu), \\ \sigma_\phi = {} & -2Gs\left(\frac{2}{\rho^{*3}} - \frac{3(\rho - c\mu)^2}{\rho^{*5}}\right) + \sum_{n=2}^{\infty}(n+1)A_{n+1}\rho^{n-2}[n(n-1)P_n(\mu) - P_{n-1}'(\mu)] \\ & + \sum_{n=0}^{\infty}B_{n+3}\rho^n\{(1-\nu)(2n+2)(2n+5)[P_n(\mu) - 2\mu P_{n-1}(\mu)] \\ & + (n+1)(n+2)[(n+2)P_n(\mu) + 2\mu P_{n+1}(\mu)] - (n+3)\mu P_n'(\mu)\}, \\ \sigma_\theta = {} & \frac{2Gs}{\rho^{*3}} + \sum_{n=2}^{\infty}(n+1)A_{n+1}\rho^{n-2}P_{n-1}'(\mu) + \sum_{n=0}^{\infty}B_{n+3}\rho^n\{[2\nu(2n+5) \\ & - (n+2)](n+1)P_n(\mu) + (n+3)\mu P_n'(\mu)\} \\ \sigma_{\rho\phi} = {} & -\frac{6Gsc(1-\mu^2)^{1/2}(\rho - c\mu)}{\rho^{*5}} + (1-\mu^2)^{1/2}\sum_{n=1}^{\infty}\left\{\frac{2n+5}{2n+3}(n^2+2n-1+2\nu)B_{n+3}\rho^n \right. \\ & \left. + (n+1)\left[(n+1)A_{n+1} - \frac{2(3n-4n\nu+2-2\nu)}{2n-1}B_{n+1}\right]\rho^{n-2}\right\}P_n'(\mu), \end{aligned} \tag{17}$$

where

$$\rho^* = (\rho^2 + c^2 - 2c\rho\mu)^{1/2}. \tag{18}$$

The sphere is assumed to undergo no rigid body displacement. For the sake of brevity, the components of strain are not shown. They can be found readily from Eq. (5) whenever needed.

A DOUBLET

Consider a doublet located on the z axis at the point 0^* or $z = c$ inside a sphere $\rho \geqslant a$. Let the doublet be oriented axisymmetrically in the z direction. Likewise, suppose that the required Love stress function is composed of two parts as follows:

$$\chi = \chi_0 + \chi_1. \tag{19}$$

To find the first part χ_0, consider a Love stress function in the form

$$\chi_0^* = K\rho^*, \tag{20}$$

where K is a constant. This function is biharmonic and gives the following radial components of displacement and stress referred to ρ^* and μ^*:

$$u_{\rho^*} = \frac{2K(1-\nu)\mu^*}{G\rho^*}, \quad \sigma_{\rho^*} = -\frac{2K(2-\nu)\mu^*}{\rho^{*2}}. \tag{21}$$

It gives a concentrated force P in the z direction at the point 0^* of amount[8]

$$P = 8\pi(1-\nu)K. \tag{22}$$

Let a concentrated force P/h be applied at the point 0^* or $z = c$ in the z direction and an equal and opposite concentrated force be applied at the point $z = c - h$ on the z axis. When h tends to zero while the concentrated force P remains constant, we obtain a doublet at the point 0^* oriented in the z direction. The corresponding Love stress function is

$$\begin{aligned} \chi_0 &= \lim_{h\to 0}\left\{\frac{K}{h}[\rho^2\sin^2\phi + (z-c)^2]^{1/2} \right. \\ &\quad \left. - \frac{K}{h}[\rho^2\sin^2\phi + (z-c+h)^2]^{1/2}\right\} \\ &= -K\mu^*, \end{aligned} \tag{23}$$

which gives the following radial components of displacement and stress referred to ρ^* and μ^*:

$$u_{\rho^*} = -\frac{K}{2G\rho^{*2}}[1-(5-4\nu)\mu^{*2}],$$

$$\sigma_{\rho^*} = \frac{2K}{\rho^{*3}}[1+\nu-(5-\nu)\mu^{*2}]. \tag{24}$$

Suppose that three mutually orthogonal doublets each of strength s are combined to form a center of dilatation of strength s. It can be shown that

$$K = Gs/(1-2\nu). \tag{25}$$

The first part χ_0 in Eq. (23) can be expanded for $\rho > |c|$ into the series:

$$\chi_0 = -K\mu - K\sum_{n=2}^{\infty}\frac{nc^{n-1}}{(2n-1)\rho^{n-1}}P_n(\mu) + K\sum_{n=0}^{\infty}\frac{(n+2)c^{n+1}}{(2n+3)\rho^{n+1}}P_n(\mu), \tag{26}$$

from which the following normal and tangential components of stress are found:

$$\sigma_\rho = -\frac{4K(1-2\nu)}{3\rho^3} - \frac{12Kc(1-4\nu)\mu}{5\rho^4} + K\sum_{n=2}^{\infty}\left(\frac{(n+1)^2(n+2)(n-2+4\nu)c^2}{(2n+3)\rho^2} - \frac{n(n-1)(n^2+3n-2\nu)}{2n-1}\right)\frac{c^{n-2}}{\rho^{n+1}}P_n(\mu),$$

$$\sigma_\rho = -K(1-\mu^2)^{1/2}\left[\frac{6(1-4\nu)c}{5\rho^4} + \sum_{n=2}^{\infty}\left(-\frac{(n+1)(n+2)(n-2+4\nu)c^2}{(2n+3)\rho^2} + \frac{(n-1)(n^2-2+2\nu)}{2n-1}\right)\frac{c^{n-2}}{\rho^{n+1}}P'_n(\mu)\right]. \tag{27}$$

The second part χ_1 is constructed as follows:

$$\chi_1 = \sum_{n=3}^{\infty}A_n\rho^nP_n(\mu) + \sum_{n=1}^{\infty}B_{n+2}\rho^{n+2}P_n(\mu), \tag{28}$$

which is the same as in Eq. (10), except that the terms involving A_1, A_2, and B_2 are now absent. This implies that the sphere undergoes no rigid body displacement. The components of stress derived from χ_1 are the same as the corresponding parts in Eq. (13). The residual normal and tangential tractions on the surface of sphere are therefore annulled if we substitute $\rho = a$ into the expressions for σ_ρ and $\sigma_{\rho\phi}$ derived from χ, or the sum of χ_0 and χ_1, and equate each coefficient of $P_n(\mu)$ as well as of $(1-\mu^2)^{1/2}P'_n(\mu)$ to zero. Likewise, two sets of equations are obtained. Their solution is

$$B_3 = \frac{2K(1-2\nu)}{5(1+\nu)a^3},$$

$$B_4 = \frac{3K(1-4\nu)c}{7(1+\nu)a^5},$$

$$B_{n+3} = \frac{(n+2)K}{2(2n+5)}\left(\frac{n(n-1)(2n+3)}{n^2+n+2n\nu+1+\nu} - \frac{(n+1)(2n+1)(n-2+4\nu)c^2}{(n^2+n+2n\nu+1+\nu)a^2}\right)\times\frac{c^{n-2}}{a^{2n+1}}, \quad (n\geqslant 2),$$

$$A_{n+1} = \frac{2(3n-4n\nu+2-2\nu)}{(n+1)(2n-1)}B_{n+1} - \frac{2n+5}{2n+1}B_{n+3}a^2 - \frac{2K(n^2+n-2n\nu+1-\nu)c^{n-2}}{(n+1)(2n-1)(2n+1)a^{2n-1}}, \quad (n\geqslant 2). \tag{29}$$

The following components of displacement and stress are found from the first part χ_0:

$$u_\rho = \frac{K}{G}\left(\frac{2(1-\nu)(\rho\mu-c)\mu+\rho(1-\mu^2)}{\rho^{*3}} - \frac{3\rho^2(\rho-c\mu)(1-\mu^2)}{\rho^{*5}}\right),$$

$$u_\phi = -\frac{K}{G}(1-\mu^2)^{1/2}\left(\frac{2(1-2\nu)(\rho\mu-c)-\rho\mu}{\rho^{*3}} - \frac{3c\rho^2(1-\mu^2)}{2\rho^{*5}}\right),$$

$$\sigma_\rho = \frac{2K}{\rho^{*3}}[1+\nu+(1-2\nu)\mu^2] - \frac{3K}{\rho^{*5}}\begin{Bmatrix}2(2-\nu)(\rho\mu-c)(\rho-c\mu)\mu \\ +[5\rho^2-2(2+\nu)c\rho\mu+2\nu c^2](1-\mu^2)\end{Bmatrix} + \frac{15K\rho^2(\rho-c\mu)^2(1-\mu^2)}{\rho^{*7}},$$

$$\sigma_\phi = -\frac{2K(1-2\nu)\mu^2}{\rho^{*3}} + \frac{6K}{\rho^{*5}}\times\begin{bmatrix}(1-\nu)(\rho\mu-c)(\rho\mu-2c\mu^2+c) \\ +\rho(3\rho-2c\mu)(1-\mu^2)\end{bmatrix} - \frac{15K\rho^2(\rho-c\mu)^2(1-\mu^2)}{\rho^{*7}},$$

$$\sigma_\theta = \frac{2K(1+\nu)}{\rho^{*3}} - \frac{3K}{\rho^{*5}}\begin{bmatrix}2\nu(\rho\mu-c)^2 \\ +(3\rho^2-3c\rho\mu-c\mu)(1-\mu^2)\end{bmatrix},$$

$$\sigma_{\rho\phi} = K(1-\mu^2)^{1/2}\left\{\frac{2\nu}{\rho^{*3}} + \frac{6}{\rho^{*5}}\times\begin{bmatrix}(1-\nu)(\rho\mu-c)(\rho-2c\mu) \\ -\rho(\rho\mu+c-2c\mu^2)\end{bmatrix} + \frac{15c\rho^2(\rho-c\mu)(1-\mu^2)}{\rho^{*7}}\right\}, \tag{30}$$

where ρ^* is given before in Eq. (18). The components of displacement and stress from the second part χ_1 are the same as the corresponding parts in Eqs. (16) and (17). The components of strain are given by Eq. (5).

THE PARTICULAR CASE $c = 0$

In the particular case $c = 0$, the point defect is located at the center of the sphere. The solutions are considerably simplified. For a center of dilatation of strength s,

TABLE I. μ_ρ/s and μ_ϕ/s on $\rho = a$ due to a center of dilatation for $a = 2$ and $\nu = \frac{1}{4}$.

ϕ	Values of μ_ρ/s				Values of μ_ϕ/s			
	$c=0$	$c=\frac{1}{4}$	$c=\frac{1}{2}$	$c=\frac{3}{4}$	$c=0$	$c=\frac{1}{4}$	$c=\frac{1}{2}$	$c=\frac{3}{4}$
0°	−0.1500	−0.0218	0.3127	1.0679	0	0	0	0
15	−0.1500	−0.0321	0.2550	0.8413	0	−0.0307	−0.1193	−0.3275
30	−0.1500	−0.0593	0.1153	0.3708	0	−0.0544	−0.1951	−0.4795
45	−0.1500	−0.0952	−0.0386	−0.0333	0	−0.0669	−0.2122	−0.4516
60	−0.1500	−0.1304	−0.1571	−0.2633	0	−0.0677	−0.1831	−0.3313
75	−0.1500	−0.1583	−0.2242	−0.3481	0	−0.0594	−0.1305	−0.1922
90	−0.1500	−0.1759	−0.2462	−0.3428	0	−0.0460	−0.0743	−0.0725
105	−0.1500	−0.1836	−0.2370	−0.2908	0	−0.0314	−0.0271	0.0131
120	−0.1500	−0.1838	−0.2108	−0.2208	0	−0.0186	0.0052	0.0622
135	−0.1500	−0.1797	−0.1792	−0.1514	0	−0.0092	0.0213	0.0781
150	−0.1500	−0.1743	−0.1508	−0.0945	0	−0.0035	0.0230	0.0675
165	−0.1500	−0.1700	−0.1316	−0.0575	0	−0.0009	0.0143	0.0385
180	−0.1500	−0.1684	−0.1248	−0.0448	0	0	0	0

TABLE II. μ_ρ/s and μ_ϕ/s on $\rho = a$ due to a doublet for $a = 2$ and $\nu = \frac{1}{4}$.

ϕ	Values of μ_ρ/s				Values of μ_ϕ/s			
	$c=0$	$c=\frac{1}{4}$	$c=\frac{1}{2}$	$c=\frac{3}{4}$	$c=0$	$c=\frac{1}{4}$	$c=\frac{1}{2}$	$c=\frac{3}{4}$
0°	2.1253	3.0267	4.4928	7.0582	0	0	0	0
15	1.8631	2.5186	3.4319	4.6450	−0.5862	−0.8582	−1.2982	−2.0291
30	1.1470	1.2495	1.1454	0.5735	−1.0153	−1.3479	−1.7384	−2.0758
45	0.1688	−0.2021	−0.8290	1.6626	−1.1723	−1.3160	−1.3017	−0.9819
60	−0.8094	−0.3018	−1.7836	2.0789	−1.0153	−0.8630	−0.5060	0.0271
75	−1.5255	−1.7906	−1.8511	−1.6820	−0.5862	−0.2215	0.2123	0.6063
90	−1.7876	−1.6846	−1.4163	−1.0772	0	0.3848	0.6807	0.8399
105	−1.5255	−1.1557	−0.7897	−0.4958	0.5862	0.8135	0.8952	0.8648
120	−0.8094	−0.4171	0.1553	−0.0063	1.0153	1.0089	0.9104	0.7741
135	0.1688	0.3418	0.3918	0.3760	1.1723	0.9771	0.7855	0.6197
150	1.1470	0.9779	0.8042	0.6493	1.0153	0.7591	0.5689	0.4295
165	1.8631	1.3954	1.0588	0.8135	0.5862	0.4122	0.2973	0.2193
180	2.1253	1.5403	1.1447	0.8682	0	0	0	0

$$\chi = 2GsQ_0(\mu) + A_3\rho^3P_3(\mu) + B_3\rho^3P_1(\mu), \tag{31}$$

where

$$A_3 = \frac{16(4-5\nu)Gs}{15(1+\nu)a^3}, \quad B_3 = \frac{12Gs}{5(1+\nu)a^3}. \tag{32}$$

For a doublet of strength s oriented axisymmetrically about the z axis,

$$\chi = -\frac{Gs\mu}{1-2\nu} + A_3\rho^3P_3(\mu) + A_5\rho^5P_5(\mu) + B_3\rho^3P_1(\mu) + B_5\rho^5P_3(\mu), \tag{33}$$

where

$$A_3 = \frac{2Gs}{45a^3}\left(\frac{4(4-5\nu)}{1+\nu} - \frac{25(7-\nu^2)}{(1-2\nu)(7+5\nu)}\right),$$
$$A_5 = \frac{16(7-9\nu)Gs}{45(1-2\nu)(7+5\nu)a^5}, \tag{34}$$
$$B_3 = \frac{2Gs}{5(1+\nu)a^3},$$
$$B_5 = \frac{28Gs}{9(1-2\nu)(7+5\nu)a^5}.$$

In either case, the rigid body displacement is assumed to be zero.

NUMERICAL EXAMPLES

The components of displacement on the surface of the sphere $\rho = a$ are computed in both cases for the following values:

$$a = 2, \quad \nu = \tfrac{1}{4}, \quad c = 0(\tfrac{1}{4})\tfrac{3}{4}. \tag{35}$$

The sphere is assumed to undergo no rigid body displacement. The results are shown in Tables I and II.

ACKNOWLEDGMENTS

The authors wish to acknowledge VPI & SU sponsorship in support of National Science Foundation Grant No. DMR-8000933.

[1]H. Ekstein, Phys. Rev. **68**, 120 (1945).
[2]K. Huang, Proc. R. Soc. London Ser. A **190**, 102 (1947).
[3]D. Eshelby, J. Appl. Phys. **25**, 255 (1954).
[4]J. Dundurs and D. L. Guell, *Developments in Theoretical and Applied Mechanics* (Pergamon, New York, 1965), Vol. II.
[5]A. E. H. Love, *Mathematical Theory of Elasticity* (Dover, New York, 1944), 4th edition.
[6]C. B. Ling and K. L. Yang, J. Appl. Mech. **18**, A367 (1951).
[7]T. M.MacRobert, *Spherical Harmonics* (Pergamon, New York, 1967), 3rd edition.
[8]S. Timoshenko and J. N. Goodier, *Theory of Elasticity* (McGraw–Hill, New York, 1951), 2nd edition.