

Linear inviscid wave propagation in a waveguide having a single boundary discontinuity: Part II: Application

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The method of match asymptotic expansions MMAE, is used to analyze wave propagation in two problem geometries. The acoustic pressure is evaluated for a waveguide having a single discontinuity in wall slope and a waveguide having a right-angle bend. A two-port representation of the fluid motion across the discontinuity for each problem is tabulated. A uniformly valid expression for the pressure for each problem is given.

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INTRODUCTION

In Part I of this paper we outlined a theoretical method for obtaining a uniformly valid approximation for the pressure in a waveguide having a single boundary discontinuity. It was shown that the dynamic behavior of the enclosed fluid can be parametrized, for long acoustic wavelength, by the ratio of two length scales H_0 and L_0 . This ratio, H_0/L_0 is equal to ϵ . H_0 is the typical height of the duct and L_0 represents the typical wall wavelength. The solution for the acoustic pressure wave was obtained using the method of matched asymptotic expansion, MMAE.

In this paper, the analysis of two problems utilizing the results outlined in Part I will be presented. These problems will serve to illustrate the applications of the general theoretical development given Part I. Section I will be devoted to the analysis of wave propagation in uniform waveguide coupled to a waveguide with slowly varying height. At the junction of the two waveguides there is a discontinuity in wall slope. We will show that the nondimensional acoustic impedance across the discontinuity in wall slope is $-i\epsilon^2 kh'(0^+)/3 + O(\epsilon^3)$. The composite solution for the pressure will also be given. In Sec. II we will examine how acoustic waves propagate in a waveguide which has a right angle bend. The composite solution for the pressure as well as a two-port representation for the fluid motion across the junction will be given. The z dependence of all acoustic variables will be suppressed following the method outlined in Part I.¹ The three-dimensionality of the solution can be regained by simply multiplying the results for composite solution by $\cos \beta_m z$.

I. ACOUSTIC WAVE PROPAGATION IN A WAVEGUIDE HAVING A SINGLE WALL SLOPE DISCONTINUITY

In this section the propagation of long wavelength acoustic waves through two waveguides coupled through a junction which introduces a discontinuity in wall slope will be examined. The particular geometry to be investigated will be a guide of uniform height connected to a guide having a slowly varying height (see Fig. 1). Despite the geometrical simplicity of the problem, evaluation of the acoustic impedance across the junction, to the author's knowledge, has not been discussed in the literature.

A plane acoustic pressure wave of amplitude A^- and

harmonic time dependence $e^{-i\omega T}$ is launched at $X = -\infty$. As a result of the discontinuity in wall slope at X equal to zero, part of the incident acoustic pressure is reflected at the junction. The remaining portion is transmitted across the junction. The perturbation parameter, ϵ , will be defined as the typical value of the wall slope at $X = 0^+$. We will show that the nondimensional acoustic impedance across the junction is $-i\epsilon^2 kh'(0^+)/3$ to order ϵ^3 .

A. Approximate solution in the wave regions

In the frequency range where $\omega H_0/2c \ll 1$ we expect the pressure to be a slowly varying function Y for $X > 0$ and independent of Y for $X < 0$. The variation of the acoustic pressure in Y is tied to the magnitude of the variation in the duct height, $O(H_0)$, compared to the typical wall wavelength $O(L_0)$. Using the length scales, H_0 and L_0 the behavior of the fluid in the duct can be parametrized by a single small parameter ϵ , where $\epsilon = H_0/L_0$. ϵ will be taken to be of the order of the typical wall slope of the upper wall.

Following the nondimensionalization scheme outlined in Part I¹

$$u = U/U_0, \quad v = V/(U_0\epsilon), \quad w = W/U_0,$$

and

$$p = P/(\omega \rho_0 L_0 U_0).$$

For the coordinates the nondimensionalization is

$$x = X/L_0, \quad z = Z/L_0,$$

and

$$y = Y/H_0,$$

where

$$t = \omega T, \quad k = \omega L_0/c, \quad g = \cos \beta_m z,$$

$$\hat{k} = (k^2 - \beta_m^2)^{1/2}, \quad \beta_m = m\pi L_0/L_z,$$

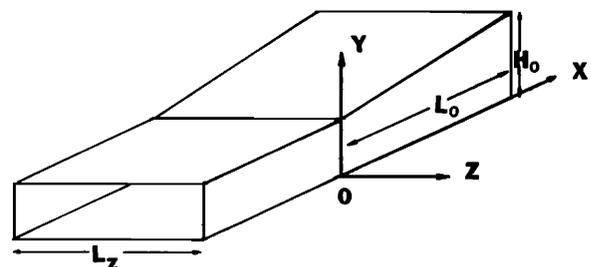


FIG. 1. Waveguide having a slope discontinuity at $x = 0$.

and

$$\epsilon = H_0/L_0.$$

Following the procedure outlined in Part I¹ the coefficients in asymptotic representation of the complex amplitudes of the pressure satisfy the homogeneous Webster horn equation to $O(\epsilon^2)$.

$$\begin{bmatrix} p_0^\pm \\ \epsilon p_1^\pm \end{bmatrix} = \text{Re} \left\{ \begin{bmatrix} \left[\exp\left(\int^x \theta(\tau, \hat{k}) d\tau\right) + R_0^\pm \exp\left(\int^x \theta^*(\tau, \hat{k}) d\tau\right) \right] \\ 0 \\ 0 \\ \left[\exp\left(\int^x \theta(\tau, \hat{k}) d\tau\right) + R_1^\pm \exp\left(\int^x \theta^*(\tau, \hat{k}) d\tau\right) \right] \end{bmatrix} \times \begin{bmatrix} A^\pm / \sqrt{h} \\ \epsilon A_1^\pm / \sqrt{h} \end{bmatrix} e^{-i\tau} \right\} + O(\epsilon^2), \quad (2)$$

where the superscript (\pm) denotes the side at which we are tabulating the pressure, and θ represents the spatially varying complex wavenumber. Substituting the aforementioned solution form into the Webster horn equation yields

$$\theta^2 + \theta' + \left[\hat{k}^2 - \left(\frac{1}{\sqrt{h}} \right) \left(\frac{d^2 \sqrt{h}}{dx^2} \right) \right] = 0, \quad (3)$$

the inhomogeneous Riccati equation. If we assume

$$|\theta^2| \gg |\theta'|, \quad (4)$$

the complex wavenumber can be approximated as

$$\theta \simeq i \left[\hat{k}^2 - (1/\sqrt{h})(\sqrt{h})'' \right]^{1/2}. \quad (5)$$

As a result of the assumption made in Eq. (4), we will limit $h(x)$ and \hat{k} the values which satisfy the inequality

$$|\hat{k}^2 - (1/\sqrt{h})(\sqrt{h})''|^{3/2} > |(1/\sqrt{h})(\sqrt{h})''|. \quad (6)$$

Hence, we will be working with frequencies significantly above the cutoff frequency. Of course, in the portion of the guide which has uniform height, θ is equal to wavenumber \hat{k} .

To evaluate the second-order pressure coefficient p_2^\pm , we must solve the inhomogeneous Webster equation. Using the aforementioned assumptions, we can show

$$p_2^\pm = (F + \{ [h(x)h'(x)]/6 \} p_0') + (y^2/2)(h'/h) p_0' + O(h'^2), \quad (7)$$

where F represents the homogeneous solution to the Webster horn equation.

$$F = \left(\frac{A_2^\pm}{\sqrt{h}} \right) \left[\exp\left(\int^x \theta\right) + R_2^\pm \exp\left(\int^x \theta^*\right) \right].$$

B. SOLUTION IN THE INCOMPRESSIBLE REGION BY CONFORMAL MAPPING

Following the nondimensionalization procedure outlined in Part I¹ the inner variables are

$$\hat{u} = u, \quad \hat{v} = v\epsilon, \quad \hat{w} = w,$$

and

$$\hat{p} = p.$$

$$\left[\frac{d^2}{dx^2} + 1/h \left(\frac{dh}{dx} \right) + \hat{k}^2 \right] (p_0 + \epsilon p_1) = 0. \quad (1)$$

Unfortunately this equation cannot be analytically solved in closed form for an arbitrary choice of $h(x)$. Therefore, only an approximate form for the pressure coefficients will be given here. Representing the pressure as the sum of two of dispersive waves yields

For the coordinates the nondimensionalization

$$\hat{x} = x/\epsilon, \quad \hat{y} = y,$$

and

$$\hat{z} = z,$$

where

$$t = \omega T, \quad k = \omega L_0/c, \quad \beta_m = m\pi L_0/L_2,$$

$$g(\hat{z}) = \cos \beta_m \hat{z},$$

and

$$\hat{k}^2 = k^2 - \beta_m^2.$$

The coefficients in the asymptotic representation of the complex pressure amplitude in the incompressible region satisfy the Laplace equation at ϵ and the Poisson equation at ϵ^2 .

$$\hat{p}_{1,\hat{x}\hat{x}} + \hat{p}_{1,\hat{y}\hat{y}} = 0, \quad \hat{p}_{2,\hat{x}\hat{x}} + \hat{p}_{2,\hat{y}\hat{y}} = -\hat{k}^2 p_0. \quad (8)$$

The zeroth-order pressure coefficient is

$$\hat{p} = \text{constant} = p_0(0^+) = p_0(0^-),$$

using the zeroth-order matching condition. If we let

$$\hat{p}_2 = \hat{p}_2 - \hat{k}^2 \hat{p}_0(\hat{x}^2/2), \quad (9)$$

we can rewrite Eq. (8) as

$$\nabla^2(\epsilon \hat{p}_1 + \epsilon^2 \hat{p}_2) = 0, \quad (10)$$

a set of two homogeneous Laplace equations with boundary condition that the normal velocity is zero on the boundary. The solution of Eq. (10) will be obtained using the Schwartz Christoffel transformation.^{4,5}

The conformal transformation between the physical plane, the \hat{z} plane, and the upper half ω plane, can be made using the expression

$$\frac{d\hat{z}}{d\omega} = \left(\frac{h(0^-)}{\pi} \right) (1 - \omega)^{[\epsilon h'(0^+)/\pi]},$$

$$\hat{z} = \frac{h(0^-)}{\pi} \int_1^\omega [(1-s)^{\epsilon h'(0^+)/\pi}] / s ds, \quad (11)$$

where $j = \sqrt{-1}$, $\hat{z} = \hat{x} + j\hat{y}$, and $\epsilon h'(0^+)$ denotes the angle which the incline makes with the horizontal axis (see Fig. 2).

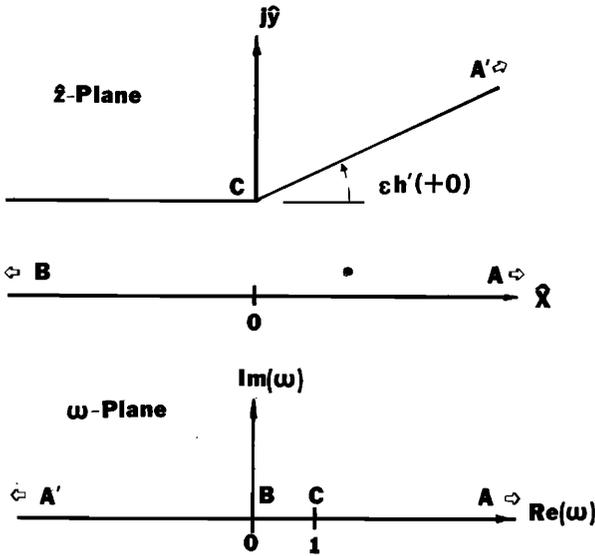


FIG. 2. Conformal transformation from \hat{z} plane to the w plane.

The perturbation parameter is in the exponent of Eq. (11) by virtue of the fact that we are working in an expanded horizontal scale \hat{x} in the incompressible region. Expanding Eq. (11) in Taylor series in about ϵ equal to zero yields

$$\frac{d\hat{z}}{d\omega} = \left(\frac{h(0^-)}{\pi\omega} \right) + \left(\frac{\epsilon h'}{\pi} \right) \ln(1-\omega) + \sum_{n=2}^{\infty} \left(\frac{(\epsilon h')^n}{\pi^n} \right) \ln^n \left(\frac{1-\omega}{n!} \right).$$

Since we are only matching the pressure coefficient to order ϵ^3 , it is only necessary to keep terms to $O(\epsilon^2)$. Truncating the aforementioned series at ϵ yields

$$\frac{d\hat{z}}{d\omega} = \left(\frac{h(0^-)}{\pi\omega} \right) \left[1 + \left(\frac{\epsilon h'}{\pi} \right) \ln(1-\omega) + O(\epsilon^2) \right]. \quad (12)$$

Integrating with respect to ω , we obtain two expressions, the first valid for $|\omega| > 1$, and the second valid for $|\omega| < 1$.

$$\hat{z} = \left(\frac{h(0^-)}{\pi} \right) \left\{ \ln \omega + \left(\frac{\epsilon h'}{\pi} \right) \left[\left(\frac{\pi^2}{6} \right) - \sum_{n=1}^{\infty} \left(\frac{\omega^n}{n^2} \right) \right] \right\},$$

for $|\omega| < 1$,

and

$$\hat{z} = \left(\frac{h(0^-)}{\pi} \right) \left\{ \ln \omega + \left(\frac{\epsilon h'}{\pi} \right) \left[\left(\frac{\ln^2(\omega)}{2} \right) + j\pi \ln \omega - \left(\frac{\pi^2}{6} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{(\omega^n n^2)} \right) \right] \right\},$$

for $|\omega| > 1$. (13)

From these two expressions we can obtain the asymptotic behavior of the pressure as \hat{x} tends to $\pm \infty$. As ω tends to zero, \hat{x} tends to minus infinity. Hence,

$$\hat{z} = [h(0^-)/\pi] \ln \omega + (\epsilon h'/\pi)(\pi^2/6), \quad \text{as } \hat{x} \rightarrow -\infty. \quad (14)$$

As ω tends to infinity, \hat{x} tends to plus infinity. Hence,

$$\hat{z} = [h(0^-)/\pi] \{ \ln \omega + (\epsilon h'/\pi) [\ln^2(\omega)/2 + j\pi \ln \omega - (\pi^2/6)] \}, \quad \text{as } \hat{x} \rightarrow +\infty. \quad (15)$$

A point in the ω plane is related to the pressure as

$$\omega = \exp \left(\pi(\tilde{p}_1 + \epsilon \tilde{p}_2) / \{ i[q_0(0^-) + \epsilon q_1(0^-)] \} \right), \quad (16)$$

where

$$\tilde{p}_1 + \epsilon \tilde{p}_2 = \text{Re}(\tilde{p}_1 + \epsilon \tilde{p}_2).$$

Note that ω is a function of two complex variables. The real part of $\tilde{p}_1 + \epsilon \tilde{p}_2$ is the acoustic pressure. Care must be taken when taking the real part. The term $i\{[q_0(0^-) + \epsilon q_1(0^-)]\}$ is independent of this operation. Substituting Eq. (16) into Eqs. (14) and (15), keeping terms to order ϵ and taking the real part of the result yields

$$\lim_{\hat{x} \rightarrow \infty} \tilde{p}_1 + \epsilon \tilde{p}_2 = [i\hat{x}/h(0^-)] [q_0(0^-) + \epsilon q_1(0^-)] - [i\epsilon q_0(0^-)h'(0^+)/6 + O(\epsilon^2)]$$

and

$$\lim_{\hat{x} \rightarrow \infty} \tilde{p}_1 + \epsilon \tilde{p}_2 = [i\hat{x}/h(0^-)] [q_0(0^-) + \epsilon q_1(0^-)] - \epsilon i q_0(0^-) \left[\left(\frac{h'(0^+)}{h^2(0^-)} \right) \left(\frac{\hat{x}^2 - \hat{y}^2}{2} \right) + i\epsilon q_0(0^-) [h'(0^+)/6] \right]. \quad (17)$$

Using Eq. (9) and the matching condition outlined in Part I¹ yields the junction conditions

$$p_0(0^+) = p_0(0^-) = \hat{p}_0, \quad p_1(0^+) = p_1(0^-), \\ q_0(0^+) = q_0(0^-), \quad q_1(0^+) = q_1(0^-),$$

and

$$F(0^-) = -q_1(0^-) [h'(0^+)/6],$$

and

$$F(0^+) = 0.$$

The value for the nondimensional impedance and admittance in the two-port model shown in Fig. 3 is

$$Z = (\Delta p_0 + \epsilon \Delta p_1 + \epsilon^2 \Delta p_2) / (q_0 + \epsilon q_1) / \hat{k} \\ = -i\epsilon^2 \hat{k} h'(0^+) / 3$$

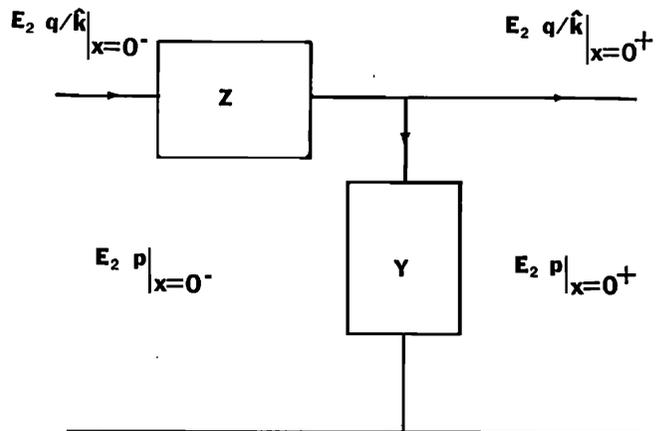


FIG. 3. Two-port representation of the slope discontinuity. Z is equal to $-i\epsilon^2 \hat{k} h'(0^+)/3$ and Y is equal to zero.

and $Y = 0$, where Δp_n denotes $p_n(0^-) - p_n(0^+)$ for $n = 0$ and 1, and

$$\Delta p_2 = -p_2(0^+, y) + y^2 \frac{h'(0^+)}{2h(0^-)} p_0'(0^-) + p_2(0^-).$$

C. Composite solution

We will now determine the pressure in the guide. Using the matching results given in Sec. I B, the composite pres-

$$E_2 p = \text{Re} \left\{ [A^-(1 + \epsilon)(e^{ikx} + Re^{-ikx}) + i\epsilon^2 q_0(0^-)[h'(0^+)/6] C_1 e^{-ikx} + \epsilon^2 C_3 \right. \\ \left. + [\epsilon(\hat{p}_1 + \epsilon\hat{p}_2) - \epsilon[\hat{x}p_0^-(0^-) + \epsilon\hat{x}p_1^-(0^-)]] e^{-it} \right\} + O(\epsilon^3), \quad \text{for } x < 0,$$

$$E_2 p = \text{Re} \left\{ \left[A^+/\sqrt{h} \exp\left(\int_0^x \theta\right) (1 + \epsilon) + i\epsilon^2 q_0(0^-)[h'(0^+)/6] C_2 \exp\left(\int_0^x \theta\right) \sqrt{h} + \epsilon^2 C_3 \right. \right. \\ \left. \left. + \epsilon^2 [h'h/6 + y^2/2(h'/h)] p_0^{+'} + \epsilon(\hat{p}_1 + \epsilon\hat{p}_2) - \epsilon\hat{x}[p_0^-(0^+) + \epsilon p_1^-(0^+)] \right. \right. \\ \left. \left. - \epsilon^2 [h(0^+)h'(0^+)/6 - (\hat{x}^2/2 - \hat{y}^2/2)h'(0^+)/h'(0^+)/h(0^+)] p_0^{+'}(0^+) \right] e^{-it} \right\} + O(\epsilon^3) \quad \text{for } x > 0,$$

where

$$A^+ = \frac{A - 2i\hat{k}\sqrt{h}}{i\hat{k} + \theta - \frac{1}{2}h'/h} \Big|_{x=0^+},$$

$$R = \frac{i\hat{k} - \theta + \frac{1}{2}(h'/h)}{i\hat{k} + \theta - \frac{1}{2}(h'/h)} \Big|_{x=0^+},$$

$$C_1 = \frac{[\theta - \frac{1}{2}(h'/h)]}{i\hat{k} + \theta - \frac{1}{2}(h'/h)} \Big|_{x=0^+},$$

$$C_2 = \frac{i\hat{k}}{i\hat{k} + \theta - \frac{1}{2}(h'/h)} \Big|_{x=0^+},$$

$$C_3 = iq_0(0^-)[h'(0^+)/6] C_2,$$

$$p_0^-(0^-) = p_0^+(0^+) = (1 + R)A^-,$$

$$q_0(0^-) = h(0^-)\hat{k}(1 - R)A^-,$$

$$p_0^-(0^-) = p_0^+(0^+) = i\hat{k}(1 - R)A^-,$$

$$\theta = i[\hat{k}^2 - (\sqrt{h''}/\sqrt{h})]^{1/2},$$

and

$$p_0^+ = \left(\frac{A^+}{\sqrt{h}} \right) \left[\exp\left(\int^x \theta\right) \right].$$

II. ACOUSTIC WAVE PROPAGATION HAVING A RIGHT ANGLE BEND

The transmission of long wavelength acoustic waves in a waveguide with right angle bend will be investigated in this section. Morse and Ingard² determined the acoustic impedance of a bent duct under the assumption that the fluid in the vicinity of the junction behaved incompressibly. Under this assumption they determined that the impedance resulting from the sudden change in waveguide geometry can be modeled as a mass reactance. We will show that their impedance expression represents the limiting case of h^+h^- tending to zero. In actuality there are two impedance elements which parametrize the acoustic behavior in a bent waveguide. The first represents the mass reactance across the bend and the

sure solution can be tabulated to $O(\epsilon^3)$.

Consider the physical system shown in Fig. 1. We choose H_0 to equal the maximum height of the guide and L_0 is chosen such that the ratio of these two quantities equals the wall slope at x equal to 0^+ . Using the approximate solution for the Webster horn equation given in Sec. I A and the solution of Laplace equation given in Sec. I B and applying the matching conditions given in Sec. I B, we obtain

second represents the compliance of the fluid in the bend. It is shown that both of these elements appear at the same order of approximation. We will also obtain the composite solution for the acoustic pressure valid to $O(\epsilon^2)$. A plane acoustic pressure wave of unit amplitude with harmonic time dependence $e^{-i\omega T}$ is launched in the positive direction at $X = -\infty$. At $X = -H_0h^+$ a portion of the incident pressure wave is reflected and the remaining portion is transmitted to the branch extending to positive Y infinity (see Fig. 4). A reflected wave is initiated as a result of the junction impedance introduced by the sudden change in the waveguide's geometry.

Unlike the previous example the junction discontinuity introduced by the right angle bend is of finite thickness. Hence, compressibility of the fluid must be taken into account at the junction. We will show that to $O(\epsilon^2)$, the nondimensional acoustic impedance across the junction can be represented by a simple two port (see Fig. 5). The series ele-

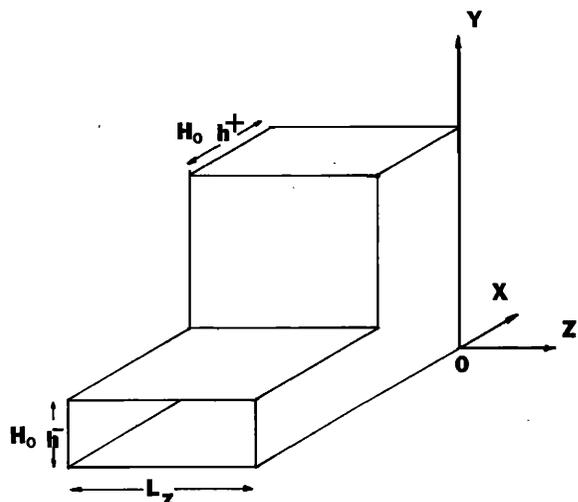


FIG. 4. Waveguide with a right angle bend.

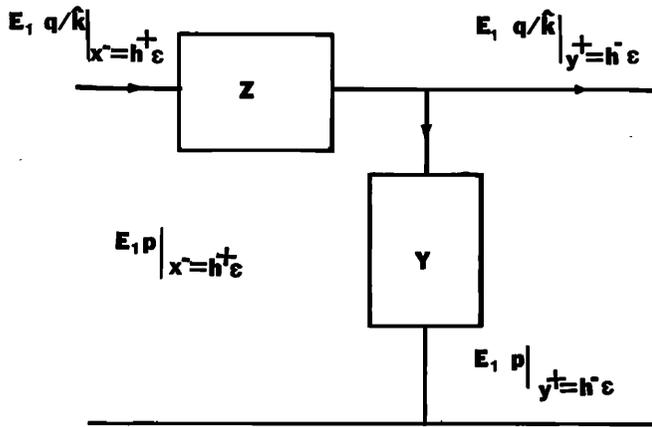


FIG. 5. Two-port representation of a right angle bend. Z is equal to $-iek\Delta\phi$ and Y is equal to $-iekh^-h^+$.

ment represents the increase in acoustic mass resulting from the presence of the bend. The shunt element represents the effect of the fluid's compressibility in the bend.

As X approaches $-H_0h^+$ and Y approaches $+H_0h^-$ from the wave regions, the vertical velocity V is $O(U)$. Therefore, the solution obtained for the wave region has to be singular in this limit. Hence, when $X = O(H_0)$ and $Y = O(H_0)$, U is $O(V)$.

The bent geometry will be split into three domains (see Fig. 6). Two of the domains will be termed the wave regions and the remaining domain is termed the incompressible region. L_0 and H_0 are equal to the typical width of the tube and the acoustic wavelength of the disturbance, respectively. We also assume that $h^+ = O(h^-)$. The perturbation parameter ϵ is the ratio of H_0 and L_0 . The nondimensionalization of the variables in each of the regions is as follows: For $X < -H_0h^-$, $V = O(\epsilon U)$,

$$x^- = X/L_0, \quad y^- = Y/H_0, \quad z^- = Z/L_0, \\ u^- = U/U_0, \quad v^- = V/(U_0\epsilon),$$

and

$$w^- = w/U_0.$$

For $Y > H_0h^-$, $U = O(\epsilon V)$,

$$x^+ = X/H_0, \quad y^+ = Y/L_0, \quad z^+ = Z/L_0, \\ u^+ = U/(U_0\epsilon), \quad v^+ = V/U_0,$$

and

$$w^+ = W/U_0.$$

B. Solution in the incompressible region by conformal mapping

Following the procedure outlined in Part I¹ the perturbation coefficient for complex amplitude of the pressure in the incompressible region satisfied the Laplace equation at ϵ .

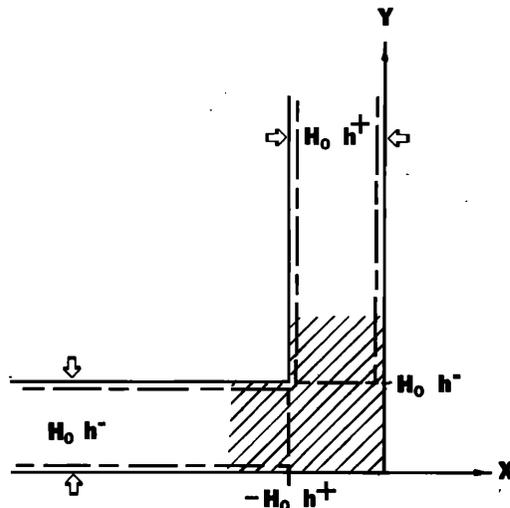


FIG. 6. Schematic diagram of a waveguide with a right angle bend. Dashed line denotes wave region; shaded region represents incompressible region.

In the incompressible region the nondimensionalization is for $X > -H_0h^+$, $Y < H_0h^-$, and $U = O(V)$,

$$\hat{x} = X/H_0 = x^-/\epsilon = x^+, \quad \hat{y} = Y/H_0 = y^- = y^+/\epsilon, \\ \hat{v} = V/U_0 = v^- \epsilon = v^+, \quad \hat{u} = U/U_0 = u^- = u^+ \epsilon,$$

and

$$\hat{w} = w^+ = w^-.$$

Following the same nondimensionalization scheme as given in Sec. I A

$$t = \omega T, \quad k = \omega L_0/c, \quad \beta_m = m\pi L_0/L_z, \\ \hat{k} = (k^2 - \beta_m^2)^{1/2}, \quad g = \cos \beta_m z,$$

and

$$\epsilon = H_0/L_0.$$

A. Solution in the wave region

In the wave regions the coefficients of the asymptotic representation of pressure satisfy the homogeneous Helmholtz equation to $O(\epsilon^2)$.

$$\left(\frac{d^2}{dx^{-2}} + \hat{k}^2 \right) (p_0^- + \epsilon p_1^-) = 0 \quad \text{for } x^- < -\epsilon h^+, \\ \left(\frac{d^2}{dy^{+2}} + \hat{k}^2 \right) (p_0^+ + \epsilon p_1^+) = 0 \quad \text{for } y^+ > \epsilon h^-. \quad (18)$$

The solution of these equations can be represented as the sum of two simple plane waves.

$$\begin{bmatrix} p_0^- \\ \epsilon p_1^- \end{bmatrix} = \text{Re} \left\{ \begin{bmatrix} e^{ikx^-} + R_0 e^{-ikx^-} & 0 \\ 0 & e^{-ikx^-} + R_1 e^{-ikx^-} \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} e^{-it} \right\} + O(\epsilon^2), \quad \text{for } x^- < -h^+ \epsilon. \quad (19)$$

$$\begin{bmatrix} p_0^+ \\ \epsilon p_1^+ \end{bmatrix} = \text{Re} \left\{ \begin{bmatrix} e^{+iky^+} & 0 \\ 0 & e^{+iky^+} \end{bmatrix} \begin{bmatrix} T_0 \\ \epsilon T_1 \end{bmatrix} e^{-it} \right\} + O(\epsilon^2), \quad \text{for } y^+ > h^- \epsilon. \quad (20)$$

$$\hat{p}_{1xx} + \hat{p}_{1yy} = 0. \quad (21)$$

The zeroth-order pressure coefficient is

$$\hat{p}_0 = \text{constant} = \lim_{\epsilon \rightarrow 0} p_0^+(y^+ = h^- \epsilon) \\ = \lim_{\epsilon \rightarrow 0} p_0^-(x^- = -h^+ \epsilon),$$

which is obtained by using the zeroth-order matching condition. The solution of Eq. (21) will be obtained using the Schwartz Christoffel transformation.

The conformal transformation between the physical plane, the \hat{z} plane, and the lower half ω plane, can be made using the relation

$$\frac{d\hat{z}}{d\omega} = - \left(\frac{h^+}{\pi(1-\omega)} \right) \left(\frac{\beta-\omega}{\omega} \right)^{1/2}, \quad (22)$$

where

$$\gamma = h^-/h^+, \quad \beta = 1 + \gamma^2, \quad j = \sqrt{-1},$$

and

$$\hat{z} = \hat{x} + j\hat{y}.$$

See Fig. 7. Integrating with respect to ω yields

$$\hat{z} = \left\{ (h^+/\pi) \cosh^{-1}[(\beta - 2\omega)/\beta] + (h^-/\pi) \times \cosh^{-1}[\beta(\omega - 1) - 2\omega]/[\beta(1 - \omega)] \right\}, \quad (23)$$

where the nondimensional complex potential is

$$\Phi = (1/\pi) \log(\omega - 1) = -j\chi + \hat{\phi}, \quad (24)$$

where $\hat{p}_1 = iq_0 \operatorname{Re} \Phi$, and q_0 is the volume velocity at $x^- = -\epsilon h^+$.

Using Eqs. (23) and (24) the pressure in the incompressible region can be tabulated to $O(\epsilon^2)$.

$$E_1 \hat{p} = \hat{p}_0 + \epsilon iq_0 \operatorname{Re} \Phi[\omega(\hat{x}, \hat{y})] + O(\epsilon^2), \quad (25)$$

where Φ is an implicit function of \hat{x} and \hat{y} .

Now we must determine the asymptotic behavior of the velocity potential as \hat{x} approaches $-\infty$ and \hat{y} approaches ∞ . From these results we can determine the first-order pressure drop and asymptotic values needed to evaluate the composite solution.

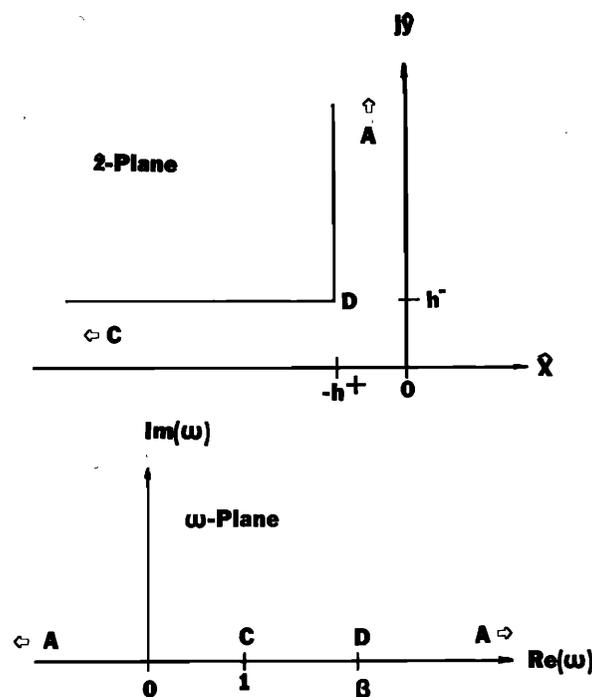


FIG. 7. Conformal transformation of a waveguide with a right angle bend to the lower half w plane, where $\beta = 1 + \gamma^2$ and $\gamma = h^+/h^-$.

Let us first determine asymptotic behavior of the velocity potential in the limit as \hat{x} goes to $-\infty$. In Fig. 7, we see that as \hat{x} goes to $-\infty$, ω approaches one. Since the domain of investigation is of infinite extent we need not apply Lagrange's expansion theorem. Using Eqs. (23) and (24) and taking the limit as ω tends to 1, the asymptotic behavior of the velocity potential is

$$\lim_{\hat{x} \rightarrow -\infty} \operatorname{Re} \Phi = (1/\pi) [\log(4\gamma^2/(1 + \gamma^2)) + (1/\gamma\pi) \times \cos^{-1}[(\gamma^2 - 1)/(\gamma^2 + 1)] + (\hat{x}/h^-)] \quad (26)$$

for $\gamma > 1$,

and

$$\lim_{\hat{x} \rightarrow -\infty} \operatorname{Re} \Phi = (1/\pi) \{ \log[(4\gamma^2)/(1 + \gamma^2)] - (1/\gamma\pi) \times \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)] + (1/\gamma) + (\hat{x}/h^-) \} \quad \text{for } \gamma < 1. \quad (27)$$

Following a similar process the asymptotic behavior for the velocity potential in the limit as \hat{y} goes to infinity can be obtained. As \hat{y} approaches infinity, ω approaches infinity. Using Eqs. (23) and (27) and taking the limit as ω tends to infinity yields

$$\lim_{\hat{y} \rightarrow \infty} \operatorname{Re} \Phi = ((1/\pi) \{ \log[4/(1 + \gamma^2)] + \gamma \cos^{-1}[(\gamma^2 - 1)/(\gamma^2 + 1)] - (\hat{y}/h + \gamma\pi) \}) \quad \text{for } \gamma > 1, \quad (28)$$

and

$$\lim_{\hat{y} \rightarrow \infty} \operatorname{Re} \Phi = -((1/\pi) \log\{4/(1 + \gamma^2) - \gamma \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)] - (\hat{y}/h^+) \}) \quad \text{for } \gamma < 1. \quad (29)$$

The potential drop across the bend is

$$\Delta \hat{\phi} = \lim_{\hat{y} \rightarrow \infty} \{ \operatorname{Re} \Phi - [(\hat{y} - h^-)/h^+] \} - \lim_{\hat{x} \rightarrow -\infty} \{ \operatorname{Re} \Phi - [(\hat{x} + h^+)/h^-] \} = (1/\pi) 2 \log[(1 + \gamma^2)/4\gamma] + [2\gamma + (1/\gamma)] - [\gamma + (1/\gamma)](1/\pi) \cos^{-1}[(\gamma^2 - 1)/(\gamma^2 + 1)] \quad \text{for } \gamma > 1, \quad (30)$$

and

$$\Delta \hat{\phi} = (2/\pi) \log[(1 + \gamma^2)/4\gamma] + \gamma + [\gamma + (1/\gamma)] \times (1/\pi) \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)], \quad \text{for } \gamma < 1. \quad (31)$$

Since the guide is of infinite extent, evanescent waves generated by the bend do not influence on the matching process. The junction conditions at the bend are

$$\begin{aligned} p_0^-(x^- = -\epsilon h^+) &= p_0^+(y^+ = \epsilon h^-) = \hat{p}_0, \\ -p_1^+(y^+ = \epsilon h^-) + p_1^-(x^- = -\epsilon h^+) &= -i\Delta \hat{\phi} q_0^-(x^- = -\epsilon h^+), \\ q_0^-(x^- = -\epsilon h^+) &= q_0^+(y^+ = +\epsilon h^-), \end{aligned} \quad (32a)$$

and

$$\begin{aligned} -q_1^+(y^+ = +\epsilon h^-) + q_1^-(x^- = -\epsilon h^+) \\ = -i\hat{k}^2 p_0^-(x^- = -\epsilon h^+)h^+h^-, \end{aligned} \quad (32b)$$

in the limit as ϵ goes to zero.

Using Eqs. (32a) and (32b) we can determine the values for the nondimensional impedance and admittance in the model shown in Fig. 5. From Eq. (32a) we see that

$$Z = -i\epsilon\hat{k}\Delta\hat{\phi},$$

and from (32b)

$$Y = -i\epsilon\hat{k}h^+h^-.$$

C. Composite solution

Using the solutions obtained for the wave and incompressible regions, the composite solution for the pressure will be obtained. In this section we will state the composite solutions for the zeroth- and first-order approximation of the pressure.

The zeroth-order composite solution of the pressure is

$$E_{0,0}p^+(x^+, y^+) = \text{Re} [(T_0 e^{+iky^+})e^{-ix^+}] + O(\epsilon),$$

for $y^+ > h^-$ and $x^+ > -\epsilon h^+$,

$$E_{0,0}p^-(x^-, y^-) = \text{Re} [(e^{+ikx^-} + R_0 e^{-kx^-})e^{-iy^-}] + O(\epsilon)$$

for $y^- < h^-$ and $x^- < -\epsilon h^+$,

and

$$E_{0,0}p = \text{Re} [(1 + R_0)e^{-ix^+}] + O(\epsilon),$$

for $y^+ < \epsilon h^-$ and $x^- > -\epsilon h^+$,

where

$$T_0 = 2/(1 + \gamma), \quad R_0 = (\gamma - 1)/(\gamma + 1),$$

and

$$\gamma = h^-/h^+.$$

The first-order composite expansion of the pressure is

$$E_{1,1}p^+(x^+, y^+) = \text{Re} \{ [(T_0 + \epsilon T_1)e^{+iky^+} + \epsilon ikq_0[\phi(\hat{x}, \hat{y}) - (\hat{y} - h^-)/h^+] + C]e^{-ix^+} \} + O(\epsilon^2),$$

for $y^+ > h^-$ and $x^+ > -\epsilon h^+$,

$$E_{1,1}p^-(x^-, y^-) = \text{Re} \{ [e^{+ikx^-} + (R_0 + \epsilon R_1)e^{-ikx^-} + \epsilon ikq_0[\phi(\hat{x}, \hat{y}) - (\hat{x} - h^+)/h^-] + C]e^{-iy^-} \},$$

for $y^- < h^-$ and $x^- < -\epsilon h^+$,

$$E_{1,1}p = \text{Re} \{ [\epsilon ikq_0\phi(\hat{x}, \hat{y}) + C + p_0]e^{-ix^+} \} + O(\epsilon^2),$$

for $y^+ < \epsilon h^-$ and $x^+ > -\epsilon h^+$,

where

$$\begin{aligned} R_1 &= R_0[-i\hat{k}(h^+ + h^-) \\ &\quad + i[\hat{k}/(\gamma + 1)](-q_0\Delta\hat{\phi} + p_0h^+), \\ T_1 &= T_0[-i\hat{k}(h^+ + h^-)] + [i\hat{k}/(\gamma + 1)](q_0\Delta\hat{\phi} + p_0h^+), \\ q_0 &= \hat{k}h^-(1 - R_0), \quad p_0 = (1 + R_0), \\ C &= [i\hat{k}q_0\Delta\hat{\phi}\gamma/(1 + \gamma)] + i\hat{k}q_0\hat{\phi}^-, \end{aligned}$$

$$\hat{\phi}^+ = -\{(1/\pi)\log[4/(1 + \gamma^2)] + (\gamma/\pi)\} \\ \times \cos^{-1}[(\gamma^2 - 1)/(\gamma^2 + 1)] - 2\gamma, \quad \text{for } \gamma > 1,$$

$$\hat{\phi}^+ = -\{(1/\pi)\log[4/(1 + \gamma^2)] - (\gamma/\pi)\} \\ \times \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)] - \gamma, \quad \text{for } \gamma < 1,$$

$$\hat{\phi}^- = \{(1/\pi)\log[4\gamma^2/(1 + \gamma^2)] + [1/(\gamma\pi)]\} \\ \times \cos^{-1}[(\gamma^2 - 1)/(\gamma^2 + 1)] - (1/\gamma), \quad \text{for } \gamma > 1,$$

$$\hat{\phi}^- = (1/\pi)\log[4\gamma^2/(1 + \gamma^2)] - [1/(\gamma\pi)] \\ \times \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)], \quad \text{for } \gamma < 1,$$

$$\Delta\hat{\phi} = (2/\pi)\log[(1 + \gamma^2)/4\gamma] + [2\gamma + (1/\gamma)] \\ - [\gamma + (1/\gamma)](1/\pi)\cos^{-1}[(\gamma^2 - 1)/(\gamma^2 - 1)]$$

for $\gamma > 1$,

and

$$\Delta\hat{\phi} = (2/\pi)\log[1 + \gamma^2/4\gamma] + \gamma + [(1/\gamma) + \gamma](1/\pi) \\ \times \cos^{-1}[(1 - \gamma^2)/(\gamma^2 + 1)], \quad \text{for } \gamma < 1.$$

III. CONCLUSION

Acoustic wave propagation in a waveguide having discontinuity in wall slope and a waveguide having a right angle bend has been examined using the theoretical development given in Part I.¹ The junction impedances for each problem as well as the composite solutions for the pressure have been presented.

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