Higher-order effects of initial deformation on the vibrations of crystal plates*

Xanthippi Markenscoff†

Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

A system of plate equations for the thickness-shear and flexural vibrations superposed on large initial deflection due to bending is derived; in the stress–strain relations the terms associated with the fourth-order elastic stiffness coefficients are retained. An explicit formula for the change in the fundamental cutoff thickness shear frequency is obtained and the effects of the terms associated with the fourth-order constrants appear to be significant for large gradients of the rotation angles.

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INTRODUCTION

The changes in the resonance frequencies of crystal plates subjected to initial stresses have been studied both experimentally and theoretically in the last twenty years.[1] The ratio of the frequency-change to the resonance frequency without initial stress, $(f-f_0)/f_0$, has an order of magnitude of 10^{-5} to 10^{-8} but accurate predictions of it have been obtained[1,2] by including in the stress-strain relation the nonlinear terms associated with the third-order elastic constants and taking into account the initial deformations. However, in the case of a plate subjected to large initial bending some discrepancies have been observed between experimental and both analytical and numerical results as described in Ref. 2. It was conjectured that including the fourth-order elastic constants in the stress-strain relations would accommodate these differences.

In Ref. 1 the basic plate equations of the theory of small deformations superposed on finite deformations are derived in Lagrangian formulation while in the stress-strain relations terms up to quadratic in the strain are kept. In the present paper the cubic terms in the strain, associated with the fourth-order elastic coefficients, have been included and plate equations are derived to accommodate the thickness-shear and flexural vibrating of rotated Y-cuts of quartz superposed on a state of large initial bending. The initial strain components are assumed small but large deflection gradients and rotation angles are allowed. An explicit formula is obtained for the change of the fundamental thickness-shear frequencies in terms of the initial deformation and second-, third-, and fourth-order constants.

The higher-order elastic constants can be measured by observations in phenomena in which the elastic nonlinearity is manifested. For a detailed discussion on this the reader is referred to Thurston.[3] Measurements of some of the fourth-order elastic constants have already been made. Based on the phenomena of intermodulation in thickness-shear and trapped energy resonators Tiersten[4] obtained an estimate of C_{6666} for AT-cut quartz.[4] Using shock-compression experiments Fowles[5] and Graham[6] have measured the longitudinal constants C_{1111} and C_{3333} for different cuts of quartz and sapphire. Their technique, however, is restricted to solids that can sustain large elastic compressions in uniaxial strain and to longitudinal elastic coefficients only. Some fourth-order constants for several cesium halides have been measured by ultrasonic techniques[7] and combinations of fourth-order constants of fused quartz have been determined in uniaxial tension experiments.[8] Also Thurston and Shapiro demonstrated that the growth of the third-harmonic in an initially sinusoidal finite-amplitude wave depends on both third- and fourth-order elastic constants.[9]

In this work a different method for the determination of the fourth-order constant is presented. The constants C_{6611}, C_{6633}, and C_{6655} can be determined to accommodate the differences between experiments and analytical predictions. As it appears from Fig. 6 and in particular Fig. 7—where the differences are more pronounced—of Ref. 2, a symmetric function in ψ with respect to $\psi = 180°$ is needed and the additional quadratic terms included in the formula for $\Delta f/f$ provide qualitatively this type of distribution. The order of magnitude of the fourth-order constants involved may be seen to be the same as that of the constants measured by the previously mentioned investigators.

I. STRESS–STRAIN RELATION

The notation followed here, shown in Table I, is the same as in Ref. 1, Table I—and the formulation of the problem is Lagrangian. At the present final state and the initial static state, the stress–strain relations are expressed, respectively, by

$$\bar{T}_{ij} = C_{ijkl}\bar{E}_{kl} + \tfrac{1}{2} C_{ijklmn}\bar{E}_{kl}\bar{E}_{mn} + \tfrac{1}{6} C_{ijklmnpq}\bar{E}_{kl}\bar{E}_{mn}\bar{E}_{pq} + \text{higher-order terms (h.o.t.) in } \bar{E}_{kl}\ , \tag{1}$$

$$T_{ij} = C_{ijkl}E_{kl} + \tfrac{1}{2} C_{ijklmn}E_{kl}E_{mn} + \tfrac{1}{6} C_{ijklmnpq}E_{kl}E_{mn}E_{pq} + \text{h.o.t. in } E_{kl}\ , \tag{2}$$

and their difference is defined as the incremental stress-strain relation

TABLE I. Notation used in Ref. 1.

	Total (at present state)	Initial (at initial state)	Incremental
Displacement	$\bar{U}_i$	U_i	$u_i = \bar{U}_i - U_i$
Kirchhoff–Piola stress	$\bar{T}_{ij}$	T_{ij}	$t_{ij} = \bar{T}_{ij} - T_{ij}$
Lagrangian strain	$\bar{E}_{ij}$	E_{ij}	$\eta_{ij} = \bar{E}_{ij} - E_{ij}$

$$t_{ij} = C_{ijkl}\eta_{kl} + C_{ijklmn}E_{kl}\eta_{mn} + \tfrac{1}{2}C_{ijlkmnpq}E_{kl}E_{mn}\eta_{pq} + \text{h.o.t.}\,, \tag{3}$$

where the products of the incremental strain have been omitted, the superposed motion being infinitesimal. Following Mindlin's[10] general procedure of series expansion in the thickness coordinate x_2, Eqs. (2) and (3) yield:

$$T_{ij}^{(e)} = \sum_f a_{(ef)}C_{ijkl}E_{kl}^{(f)} + \frac{1}{2}\sum_{f,g} a_{(efg)}C_{ikjlmn}E_{kl}^{(f)}E_{mn}^{(g)} + \frac{1}{6}\sum_{f,g,h} a_{(efgh)}C_{ijklmnpq}E_{kl}^{(f)}E_{mn}^{(g)}E_{pq}^{(h)}\,, \tag{4}$$

$$t_{ij}^{(e)} = \sum_f a_{(ef)}C_{ijkl}\eta_{kl}^{(f)} + \sum_{f,g} a_{(efg)}C_{ijklmn}E_{kl}^{(f)}\eta_{mn}^{(g)} + \frac{1}{2}\sum_{f,g,h} a_{(efgh)}C_{ijklmnpq}E_{kl}^{(f)}E_{mn}^{(g)}\eta_{pq}^{(h)}\,, \tag{5}$$

where

$$a_{(efgh\ldots)} = \begin{cases} 2b^{s+1}/s+1\,, & s \text{ even} \\ 0\,, & s \text{ odd} \end{cases}$$

$s = e + f + g + h + \cdots$, and $2b$ is the plate thickness.

Truncating the infinite series and retaining terms of orders 0 and 1 only, the stress-strain relations (4) and (5) reduce to

$$T_p^{(0)} = 2b(C_{pq} + \tfrac{1}{2}C_{pq}^{(0)} + \tfrac{1}{6}C_{pq}^{(00)})E_q^{(0)} + \tfrac{1}{3}b^3(C_{pq}^{(1)} + C_{pq}^{(10)})E_q^{(1)}\,, \tag{6}$$

$$T_p^{(1)} = \tfrac{2}{3}b^3(C_{pq} + C_{pq}^{(0)} + \tfrac{1}{2}C_{pq}^{(00)} + \tfrac{1}{10}b^2C_{pq}^{(11)})E_q^{(1)}\,, \tag{7}$$

$$t_p^{(0)} = 2b(C_{pq} + C_{pq}^{(0)} + \tfrac{1}{2}C_{pq}^{(00)} + \tfrac{1}{6}b^2C_{pq}^{(11)})\eta_q^{(0)} + \tfrac{2}{3}b^3(C_{pq}^{(1)} + C_{pq}^{(10)})\eta_q^{(1)}\,, \tag{8}$$

$$t_p^{(1)} = \tfrac{2}{3}b^3(C_{pq}^{(1)} + C_{pq}^{(10)})\eta_q^{(0)} + \tfrac{2}{3}b^3(C_{pq} + C_{pq}^{(0)} + \tfrac{1}{2}C_{pq}^{(00)} + \tfrac{3}{10}b^2C_{pq}^{(11)})\eta_q^{(1)}\,, \tag{9}$$

where the abbreviated Voigt notation is used[1] and

$$\begin{aligned} C_{pq}^{(\alpha)} &= C_{pq}E_q^{(\alpha)} \\ C_{pq}^{(\alpha\beta)} &= C_{pqrs}E_r^{(\alpha)}E_s^{(\beta)}\,, \quad \alpha, \beta = 0, 1\,. \end{aligned} \tag{10}$$

II. INCREMENTAL MOTION SUPERPOSED ON INITIAL BENDING

In this paper the initial stress–strain–displacement fields for a plate subjected to initial bending are taken the same as in Ref. 2, where a detailed analysis is presented. The initial strains $E_i^{(0)}(i=1, 6)$ are assumed small but the initial rotation angles of the plate elements $U_1^{(1)}$, $U_3^{(1)}$, as well as the deflection gradients $U_{2,1}^{(0)}$, $U_{2,3}^{(0)}$ are considered large and terms up to quadratic in these quantities and their derivatives are retained. Therefore under these assumptions terms in $C_{pq}^{(01)}$, $C_{pq}^{(00)}$ are discarded.

The stress equations of motion for vibrations superposed on initial bending are presented in Ref. 2 [Eqs. (23)]. Substituting the stress in terms of the strain according to (6)–(9) and using the strain–displacement relations (27) and (28) of Ref. 2, the displacement equations of motion in the thickness-shear and flexural modes are obtained:

$$\begin{aligned} &P_1u_{1,11}^{(1)} + P_2u_{2,1}^{(0)} + P_3u_1^{(1)} = \rho\ddot{u}_1^{(1)}\,, \\ &P_4u_{2,11}^{(0)} + P_5u_{1,1}^{(1)} = \rho\ddot{u}_2^{(0)}\,, \end{aligned} \tag{11}$$

where

$$P_1 = C_{11} + C_{11}^{(0)} + 2C_{11}E_1^{(0)} + \tfrac{3}{10}b^2C_{11}^{(11)}\,,$$

$$\begin{aligned} P_2 = {}&-\frac{3k^2}{b^2}C_{66} + kC_{16,1}^{(1)} + kC_{56,3}^{(1)} + k(C_{66,1}^{(1)} + C_{46,3}^{(1)})U_1^{(1)} - (C_{11,1}^{(1)} + C_{15,3}^{(1)})U_1^{(1)} - (C_{15,1}^{(1)} + C_{55,3}^{(1)})U_3^{(1)} - C_{11}^{(1)}U_{1,1}^{(1)} \\ &- C_{55}^{(1)}U_{3,3}^{(1)} - C_{15}^{(1)}(U_{3,1}^{(1)} + U_{1,3}^{(1)}) + C_{11}U_{2,11}^{(1)} + C_{55}U_{2,33}^{(1)} - \frac{3k^2}{b^2}[C_{66}^{(0)} + C_{66}U_{1,1}^{(0)} + C_{66}U_2^{(1)} + C_{56}U_3^{(1)} + C_{12}(U_1^{(0)})^2] - \tfrac{1}{2}k^2C_{66}^{(11)}\,, \end{aligned}$$

$$\begin{aligned} P_3 = {}&-\frac{3k^2}{b^2}C_{66} + k(C_{16,1}^{(1)} + C_{56,3}^{(1)}) + k(C_{66,1}^{(1)} + C_{46,3}^{(1)})U_1^{(1)} + C_{11}(U_1^{(1)}U_{1,1}^{(1)})_{,1} + C_{13}(U_3^{(1)}U_{1,3}^{(1)})_{,1} + \frac{3}{2b^3}(T_{6,1}^{(1)} + T_{4,3}^{(1)}) \\ &- \frac{3k^2}{b^2}[C_{66}^{(0)} + 2C_{66}E_1^{(0)}] - \tfrac{1}{2}k^2C_{66}^{(11)}\,, \end{aligned}$$

$$P_4 = k^2C_{66} + k^2C_{66}^{(0)} + 2k^2C_{66}U_2^{(1)} + C_{56}(U_{2,3}^{(0)} - U_3^{(1)}) + \tfrac{1}{3}b^3k^2C_{66}^{(11)}\,,$$

$$P_5 = k^2C_{66} + k^2C_{66}(U_{1,1}^{(0)} + U_1^{(1)})^2 + U_2^{(1)}) + kC_{56}U_{2,3}^{(0)} - \frac{1}{2b}U_1^{(1)}(T_{1,1}^{(1)} + T_{5,3}^{(1)} + T_6^{(0)}) + \tfrac{1}{3}b^3k^2C_{66}^{(11)}\,.$$

The displacement equations of motion accounting for the coupling with the $u_3^{(1)}$ mode as well and variations of the incremental motion in the $-X_3$ direction have been derived, but being very lengthy are not reported here. They will be published in a subsequent Technical Report.

In the previous studies[1,2] of plate vibrations it was shown that differences between the changes in the fundamental thickness-shear cutoff frequency at zero wave number and the changes in the thickness-shear frequencies are very small. A simplified formula is obtained in a similar fashion as in Ref. 2 for the variation of the fundamental cutoff frequency of rotated Y cuts of quartz:

$$\frac{\Delta f}{f_0} = E_1^{(0)} + \frac{C_{66}^{(0)}}{2C_{66}} - \frac{b^2}{\sqrt{3}\pi C_{66}} \Big(C_{16,1}^{(1)} + C_{56,3}^{(1)} + (C_{66,1}^{(1)} + C_{46,3}^{(1)}) U_3^{(1)} + \frac{1}{k} C_{11} (U_1^{(1)} U_{1,1}^{(1)})_{,1} + \frac{1}{k} C_{13} (U_3^{(1)} U_{1,3}^{(1)})_{,1} + \frac{3}{2b^3 k} (T_{6,1}^{(1)} + T_{4,3}^{(1)}) - \tfrac{1}{2} k C_{66}^{(11)} \Big) + \mathrm{h.o.t.} \quad , \tag{12}$$

where

$$C_{66}^{(0)} = C_{661} E_1^{(0)} + C_{662} E_2^{(0)} + C_{663} E_{30}^{(0)} + C_{664} E_4^{(0)} \quad , \tag{13}$$

$$C_{16,1}^{(1)} = C_{165} E_{5,1}^{(1)} \; , \tag{14}$$

$$C_{56,3}^{(1)} = C_{561} E_{1,3}^{(1)} + C_{563} E_{3,3}^{(1)} \quad ,$$

$$C_{66}^{(11)} = C_{6611} (E_1^{(1)})^2 + C_{6633} (E_3^{(1)})^2 + C_{6655} (E_5^{(1)})^2 \quad , \tag{15}$$

$k^2 = \frac{1}{12}\pi^2$, k being a correction factor.[1]

As it may be seen from Eq. (12), the frequency change depends on the longitudinal strain $E_1^{(0)}$ (i.e., the stretching of the middle plane) directly, as well as through the third-order elastic constants [Eq. (13)], on shear strain [Eq. (14)] and on the square of the longitudinal strain due to bending [Eq. (15)]. For $E_i^{(1)} \simeq E_{i,j}^{(1)} \simeq 0.1$ (in the case of the plate of Ref. 2 this would result in strain at the outer fibers of $\simeq 0.1\%$), the contribution of the terms appearing in Eq. (15) is of the same order as those in Eq. (14), provided that the associated fourth-order constants are bigger than the third-order ones by a factor of 10. This is to be expected given the order of magnitude found for the fourth-order constants by the previously mentioned investigators. An estimate for the fourth-order constant involved is not made here, however, because more data are needed and further experiments required.

The phenomenon of small amplitude vibrations or wave-motion superposed as a state of initial bending, could be used to determine the fourth-order elastic coefficients. A shortcoming of this method is that the higher-order terms related to the fifth- and higher-order constants do not vanish in formula (12). A thorough study of wave motion superposed on large bending is required in order to obtain all the fourth-order elastic coefficients and this constitutes a topic for further research.

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† Present address: Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139.

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