

# Nonlinear resonances in a class of multi-degree-of-freedom systems

S. Sridhar, A. H. Nayfeh, and D. T. Mook

*Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*  
(Received 27 December 1974)

An analysis is presented of the superharmonic, subharmonic, and combination resonances in a multi-degree-of-freedom system which has cubic nonlinearity and modal viscous damping and is subject to harmonic excitation. It is shown that, in the absence of internal resonances, the steady-state response contains only the modes which are directly excited. It is shown that, in the presence of internal resonances, modes other than those that are directly excited can appear in the response. The strong influence of internal resonances is exhibited in numerical examples involving hinged-clamped beams. It is shown that when a multimode solution exists the lowest mode can dominate the response, even when it is not directly excited.

Subject Classification: 40.30, 40.20.

## INTRODUCTION

Here, attention is focused on systems whose motions are governed by a set of second-order nonlinear ordinary differential equations with constant coefficients. The forcing terms are all assumed to have a single frequency which along with their amplitudes is a constant. It is known that the nonlinear forced response may involve different kinds of resonances. Denoting the excitation frequency by  $\lambda$  and the natural frequencies by  $\omega_i$ ,  $i = 1, 2, 3, \dots$ , one can classify these resonances as follows: (1) main or harmonic,  $\lambda \approx \omega_k$ ; (2) superharmonic,  $\lambda \approx \omega_k/n$ ; (3) subharmonic,  $\lambda \approx n\omega_k$ ; (4) combination,  $n\lambda \approx m_1\omega_1 + m_2\omega_2 + \dots + m_i\omega_i$ ; (5) rational,  $\lambda \approx (m/n)\omega_k$ ; and (6) internal,  $m_1\omega_1 + m_2\omega_2 + \dots + m_i\omega_i \approx 0$ ; where  $n$ ,  $m$ , and  $m_i$  are integers. In the following, a number of works are cited as examples of the extensive literature available on the study of nonlinear resonances. All the studies were concerned with the search for periodic solutions and their stability. In many cases, experimental and/or analog computer results were also presented.

Several books, such as Hayashi,<sup>1</sup> can be found in the field. An analysis leading to the classification of a class of dynamical systems with cubic nonlinearities was presented by Sethna.<sup>2</sup> However, the response analysis was restricted to a two-degree-of-freedom system. In a later paper,<sup>3</sup> he studied the superharmonic and subharmonic resonances in a two-degree-of-freedom system with quadratic nonlinearities. In both papers, Sethna paid particular attention to internal resonances. A thorough investigation of the different nonlinear resonances, including internal resonances, in a two-degree-of-freedom system can be found in the book on the problems of rotor dynamics by Tondl.<sup>4</sup> A six-degree-of-freedom system was considered by Efstathiades and Williams<sup>5</sup> in their study of vibration-isolating systems. Plotnikova<sup>6</sup> obtained the conditions for the stability of periodic solutions under main resonance for rather general two-degree-of-freedom systems.

Mettler<sup>7</sup> gives an excellent survey of the nonlinear vibration problems in mechanical systems including applications to elastic bodies subjected to gyroscopic and nonconservative follower forces. Combination and subharmonic resonances in systems having both quadratic

and cubic nonlinearities were studied by Yamamoto and Hayashi.<sup>8</sup> Much of their analysis was concerned with two-degree-of-freedom systems. However, they did present some more general results but did not include the effects of internal resonances. Szemplinska-Stupnicka, in a number of papers,<sup>9-12</sup> presented analyses of the various nonlinear resonances in multiple-degree-of-freedom systems. She also made a comparative study of the different approximation methods used in the analysis of nonlinear vibrations. An earlier work on such a comparative study is due to Newland.<sup>13</sup>

Most of the works cited above are concerned with discrete mechanical systems. However, as noted by Mettler, an analysis of the vibrations of elastic bodies leads to a set of nonlinear differential equations, the nonlinearities being essentially quadratic and/or cubic. Superharmonic resonances of different modes in straight beams were studied by Bennett and Easley<sup>14</sup> and Bennett.<sup>15</sup> They pointed out the inadequacy of a single-mode analysis to fully describe the response. Tseng and Dugundji<sup>16,17</sup> reported on the superharmonic, subharmonic and rational resonances in straight beams and superharmonic resonances in buckled beams.

The approximations that are invariably used in the analysis of weakly nonlinear systems can be broadly classified as (1) the perturbation method, usually the method developed by Krylov, Bogoliubov, and Mitropolsky which in the first approximation is known as the method of averaging, and (2) the Galerkin and the Ritz methods which originate from the variational principles of mechanics. In a noteworthy article, Rosenberg<sup>18</sup> gives a detailed account of the so-called geometrical methods which are more concerned with the qualitative and mathematical aspects of the nature of solutions of nonlinear systems. A significant feature of these methods is that their applicability is not restricted to weakly nonlinear systems. Many of the results in Ref. 18 are taken from previously published papers by Rosenberg.

Another method which is popular in the analysis of weakly nonlinear systems is the method of multiple scales. A detailed description of the method along with an exhaustive bibliography is given in the book by Nayfeh.<sup>19</sup> Recently, this method was applied to the anal-

ysis of ship motions by Nayfeh, Mook, and Marshall<sup>20</sup> and Mook, Marshall, and Nayfeh<sup>21</sup> and to the study of the large-amplitude vibrations of structural elements by Nayfeh, Mook, and Sridhar<sup>22</sup> and Nayfeh, Mook, and Lobitz.<sup>23</sup> The results of the last four papers exhibit the strong influence of internal resonances on the nonlinear resonant responses.

Although a large amount of literature is available on the subject of nonlinear resonances in weakly nonlinear multi-degree-of-freedom systems, this body of knowledge suffers from some deficiencies. Firstly, many of the studies are confined to two degrees of freedom. Even in studies of systems with more than two degrees of freedom, the analyses are restricted to the study of some specific resonance. Thus the available information is in some sense disjointed. Secondly, the phenomenological behavior of systems with internal resonances has not been explored in any depth. The works of Sethna, Tondl, and the papers mentioned in the previous paragraph are some efforts in this direction.

The present study is an effort to correct the above deficiencies in a class of nonlinear systems by presenting a unified method for the analysis of superharmonic, subharmonic, and combination resonances (these will be referred to as the external resonances) which takes internal resonances into account.

**I. METHOD OF SOLUTION**

In the present study, consideration is given to a system governed by a set of equations having the form

$$\frac{d^2 u_n}{dt^2} + \omega_n^2 u_n = \epsilon \left( -2c_n \frac{du_n}{dt} + \sum_{m,p,q} \Gamma_{nmpq} u_m u_p u_q \right) + P_n \cos \lambda t, \tag{1}$$

$n = 1, 2, \dots,$

where the  $\omega_n$  are the natural frequencies;  $\epsilon$  is a dimensionless parameter; the  $c_n$  are the modal damping coefficients; the  $\Gamma_{nmpq}$  are constant coefficients; the amplitudes of the excitation  $P_n$  are  $O(1)$ ; and the frequency of the excitation  $\lambda$  is not near any  $\omega_n$ . The derivative-expansion version of the method of multiple scales<sup>19</sup> is used to construct the first terms in the asymptotic expansions of the  $u_n$  which are uniformly valid for small  $\epsilon$  and all  $t$ .

Following the method of multiple scales, one introduces two time scales,

$$T_j = \epsilon^j t, \quad j = 0 \text{ and } 1, \tag{2}$$

and assumes expansions for the  $u_n$ ,

$$u_n(t; \epsilon) \sim u_{n0}(T_0, T_1) + \epsilon u_{n1}(T_0, T_1) + \dots, \quad n = 1, 2, \dots \tag{3}$$

Substituting Eqs. 2 and 3 into Eq. 1 and balancing powers of  $\epsilon$  yield

$$D_0^2 u_{n0} + \omega_n^2 u_{n0} = P_n \cos \lambda T_0, \tag{4}$$

and

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2D_0 D_1 u_{n0} - 2c_n D_0 u_{n0} + \sum_{m,p,q} \Gamma_{nmpq} u_{m0} u_{p0} u_{q0},$$

where

$$D_j = \partial / \partial T_j. \tag{5}$$

The solution of Eq. 4 can be written as

$$u_{n0} = A_n(T_1) \exp(i\omega_n T_0) + K_n \exp(i\lambda T_0) + cc, \tag{6}$$

where

$$K_n = \frac{1}{2} P_n (\omega_n^2 - \lambda^2)^{-1}$$

and  $cc$  represents the complex conjugate of the preceding terms. At this point, the  $A_n$  are unknown. They are determined by eliminating the secular terms at the next level of approximation.

Substituting Eq. 6 into Eq. 5 leads to

$$D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2i\omega_n (D_1 A_n + c_n A_n) \exp(i\omega_n T_0) - 2ic_n \lambda K_n \exp(i\lambda T_0) + \sum_{m,p,q} \Gamma_{nmpq} \left[ \sum_{j=1}^{27} B_j \exp(i\Lambda_j T_0) \right] + cc, \quad n = 1, 2, \dots, \tag{7}$$

where the  $\Lambda_j$  are linear combinations of the frequencies and the  $B_j$  are functions of the  $A_n$ ; they are listed in Appendix A. In order to obtain a uniformly valid expansion, the terms that produce secular terms in the  $u_{n1}$  must vanish. This so-called solvability condition yields the equations for the determination of the  $A_n$ .

The solvability condition involves the first term as well as all other terms for which  $\Lambda_j \approx \omega_n$  on the right-hand side of Eq. 7. An investigation of Appendix A shows that  $\Lambda_1$  through  $\Lambda_7$  are linear combinations of the natural frequencies only and that it is always possible for  $\Lambda_1$  through  $\Lambda_6$  to be equal to  $\omega_n$ . For example,  $\Lambda_2 = \omega_n$ , when  $p = n$ , while  $\Lambda_4 = \omega_m + \omega_p - \omega_q = \omega_n$ , when  $m = n$  and  $p = q$ . Consequently, the first six terms in the sum in Eq. 7 enter the solvability condition. The first three terms are linear and the other three are cubic in the  $A_n$ . If any other combination of natural frequencies is approximately equal to  $\omega_n$  (i.e., the natural frequencies are commensurable; for example,  $\omega_2 \approx 3\omega_1$ ,  $\omega_3 \approx 2\omega_1 + \omega_2$ ,  $\omega_4 \approx \omega_1 + \omega_2 + \omega_3$ ), an internal resonance is said to exist. When  $\Lambda_j$  for  $j > 7$  is approximately equal to  $\omega_n$ , an external resonance is said to exist. The frequency combinations associated with external resonances always contain  $\lambda$ .

In general, when there are external and internal resonances, the solvability conditions have the form

$$-2i\omega_n (D_1 A_n + c_n A_n) + A_n \sum_j \gamma_{nj} A_j \bar{A}_j + 2H_{nn} A_n + R_n + N_n = 0, \quad n = 1, 2, \dots, \tag{8}$$

where  $R_n$  is due to internal resonances, if any;  $N_n$  is due to external resonances, if any; and

$$\begin{aligned} \gamma_{nj} &= 3\Gamma_{mmm}, & n = j, \\ &= 2(\Gamma_{nnjj} + \Gamma_{njjn} + \Gamma_{njjn}), & n \neq j, \end{aligned}$$

and

$$H_{nk} = \sum_{i,j} (\Gamma_{nki} + \Gamma_{nik} + \Gamma_{nijk}) K_i K_j.$$

When  $N_n \neq 0$ , the  $n$ th mode is said to be directly excited. The specific form of  $R_n$  and  $N_n$  depend on the types of internal and external resonances present in the system; various possibilities are considered in the next three sections.

**II. THE CASE OF NO RESONANCES**

In the absence of any resonance,  $R_n = N_n = 0$  for all  $n$ . Letting

$$A_n = \frac{1}{2} a_n(T_1) \exp[i\alpha_n(T_1)], \tag{9}$$

with real  $a_n$  and  $\alpha_n$  in Eq. 8 and separating the result into real and imaginary parts, yields

$$\omega_n a_n' + c_n \omega_n a_n = 0 \tag{10a}$$

and

$$\omega_n a_n \alpha_n' + \frac{1}{8} a_n \sum_j \gamma_{nj} a_j^2 + H_{nn} a_n = 0, \tag{10b}$$

where primes denote differentiation with respect to  $T_1$ . Equation 10a shows that  $a_n \rightarrow 0$  as  $T_1 \rightarrow \infty$ , and the steady-state amplitude  $\hat{a}_n = 0$ . Thus, the steady-state solution has the form

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + 0(\epsilon). \tag{11}$$

**III. THE CASE OF NO INTERNAL RESONANCE**

In this case  $R_n = 0$  for all modes and  $N_s = 0$  for all modes  $s$  which are not involved in any resonances with the excitation. Consequently, the amplitudes and phases of these  $s$  modes are governed by Eqs. 10 and hence the steady-state amplitude  $\hat{a}_s = 0$ . The specific form of  $N_n$  for the resonating modes depends on the type of resonance involved. In this section, the cases,  $3\lambda \approx \omega_k$ ,  $2\lambda \approx \omega_m \pm \omega_k$ , and  $\lambda \approx \omega_m \pm \omega_p \pm \omega_k$  are considered.

**A. The case of  $3\lambda \approx \omega_k$**

In this case, the only resonance is due to  $3\lambda$  being near  $\omega_k$  (superharmonic resonance). The detuning parameter  $\sigma$  is used to express the nearness of  $3\lambda$  to  $\omega_k$  as follows:

$$3\lambda = \omega_k + \epsilon\sigma.$$

Then,

$$N_k = F_k \exp(i\sigma T_1), \quad F_k = \sum_{m,p,q} \Gamma_{kmpq} K_m K_p K_q.$$

After separating Eq. 8 into real and imaginary parts, one obtains

$$-\omega_k(a_k' + c_k a_k) + F_k \sin \mu = 0 \tag{12a}$$

and

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + H_{kk} a_k + F_k \cos \mu = 0, \tag{12b}$$

where

$$\mu = \sigma T_1 - \alpha_k.$$

The steady-state solution corresponds to all  $a_n' = 0$  and  $\mu' = 0$ ; thus Eqs. 12 can be reduced to

$$-\omega_k c_k \hat{a}_k + F_k \sin \hat{\mu} = 0 \tag{13a}$$

and

$$\omega_k \hat{a}_k \sigma + \frac{1}{8} \gamma_{kk} \hat{a}_k^3 + H_{kk} \hat{a}_k + F_k \cos \hat{\mu} = 0. \tag{13b}$$

It is noted that  $\hat{a}_k = 0$  is not a solution of Eqs. 13, thus, solving for  $\hat{a}_k$  and  $\hat{\mu}$  and substituting the result into Eqs. 3, 6, and 9 can only yield a solution of the form

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + 0(\epsilon), \quad n \neq k \tag{14a}$$

and

$$u_k = P_k(\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_k \cos(3\lambda t - \hat{\mu}) + 0(\epsilon). \tag{14b}$$

The last term in Eq. 14b is essentially the  $k$ th mode of the linear, homogeneous solution; the difference between this term and the actual mode lies in the frequency, which the nonlinearity slightly adjusts so that

$$\omega_k + \epsilon \hat{\alpha}_k' = 3\lambda.$$

Because the frequencies  $\omega_k$  and  $\lambda$  are commensurable, this mode interacts with the excitation through the nonlinear terms in Eq. 1 and hence forms part of the steady-state solution in spite of the presence of damping.

**B. The case of  $2\lambda \approx \omega_m \pm \omega_k$  when  $m \neq k$**

In this case, the only resonance for which the details are presented is due to  $\lambda$  being near  $\omega_m + \omega_k$ . The results for  $2\lambda$  near  $\omega_m - \omega_k$  can be obtained from those predicted below by simply changing the sign of  $\omega_k$ .

The detuning parameter  $\sigma$  is used to express the nearness of  $2\lambda$  to  $\omega_m + \omega_k$  as follows:  $2\lambda = \omega_m + \omega_k + \epsilon\sigma$ . Then,  $N_n = 0$  for  $n \neq m$  and  $k$ , while

$$N_m = H_{mk} \bar{A}_k \exp(i\sigma T_1) \text{ and } N_k = H_{km} \bar{A}_m \exp(i\sigma T_1).$$

After separating Eq. 8 into real and imaginary parts, one obtains

$$-\omega_k(a_k' + c_k a_k) + \frac{1}{2} H_{km} a_m \sin \mu = 0, \tag{15a}$$

$$-\omega_m(a_m' + c_m a_m) + \frac{1}{2} H_{mk} a_k \sin \mu = 0, \tag{15b}$$

$$\omega_k a_k \alpha_k' + \frac{1}{8} a_k \sum_j \gamma_{kj} a_j^2 + H_{kk} a_k + \frac{1}{2} H_{km} a_m \cos \mu = 0, \tag{15c}$$

and

$$\omega_m a_m \alpha_m' + \frac{1}{8} a_m \sum_j \gamma_{mj} a_j^2 + H_{mm} a_m + \frac{1}{2} H_{mk} a_k \cos \mu = 0, \tag{15d}$$

where  $\mu = \sigma T_1 - \alpha_k - \alpha_m$ .

The steady-state solution corresponds to all  $a_n' = 0$  and  $\mu' = 0$ . In contrast with the superharmonic resonance, it is noted that  $\hat{a}_m = \hat{a}_k = 0$  is a steady-state solution of Eqs. 15. For a nontrivial solution, neither  $a_m$  or  $a_k$  is zero, and Eqs. 15 can be reduced to

$$-\omega_k c_k \hat{a}_k + \frac{1}{2} H_{km} \hat{a}_m \sin \hat{\mu} = 0, \tag{16a}$$

$$-\omega_m c_m \hat{a}_m + \frac{1}{2} H_{mk} \hat{a}_k \sin \hat{\mu} = 0, \tag{16b}$$

and

$$\begin{aligned} \sigma + \frac{1}{8} \left( \frac{\gamma_{kk}}{\omega_k} + \frac{\gamma_{mk}}{\omega_m} \right) \hat{a}_k^2 + \frac{1}{8} \left( \frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{km}}{\omega_k} \right) \hat{a}_m^2 + \frac{H_{kk}}{\omega_k} + \frac{H_{mm}}{\omega_m} \\ + \frac{1}{2} \left( \frac{H_{km} \hat{a}_m}{\omega_k \hat{a}_k} + \frac{H_{mk} \hat{a}_k}{\omega_m \hat{a}_m} \right) \cos \hat{\mu} = 0. \end{aligned} \tag{16c}$$

Solving Eqs. 16 yields  $\hat{a}_k$ ,  $\hat{a}_m$ , and  $\hat{\mu}$  which, when substituted into Eqs. 15c and 15d, give  $\alpha_k'$  and  $\alpha_m'$ . Hence, the steady-state solution has either the form given in Eq. 11 or the following form:

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + 0(\epsilon), \quad n \neq m \text{ and } k, \tag{17a}$$

$$u_m = P_m(\omega_m^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_m \cos[(\omega_m + \epsilon \hat{\alpha}_m')t + \tau_m] + 0(\epsilon), \tag{17b}$$

and

$$u_k = P_k (\omega_k^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_k \cos [(\omega_k + \epsilon \hat{\alpha}'_k) t + \tau_k] + O(\epsilon), \quad (17c)$$

where  $\tau_m$  and  $\tau_k$  are constants depending on the initial conditions. The last terms in Eqs. 17b and 17c appear as a result of the resonance in spite of the presence of damping. The nonlinearity adjusts the frequencies so that

$$\omega_m + \epsilon \hat{\alpha}'_m + \omega_k + \epsilon \hat{\alpha}'_k = \omega_m + \omega_k + \epsilon \sigma = 2\lambda.$$

If both solutions ( $\hat{a}_m = \hat{a}_k = 0$  and  $\hat{a}_m \neq 0$ ) are stable, then the initial conditions determine which solution represents the response.

### C. The case of $\lambda \approx \omega_m \pm \omega_p \pm \omega_k$

The case of  $\lambda$  being near  $\omega_m + \omega_p + \omega_k$  is considered first. The results for this case are then specialized to yield the results for  $\lambda$  being near  $3\omega_k$  (subharmonic resonance) and  $\lambda$  being near  $2\omega_k + \omega_m$ .

In this case, the detuning is introduced as follows:

$$\lambda = \omega_m + \omega_p + \omega_k + \epsilon \sigma.$$

Then,  $N_n = 0$ ,  $n \neq m, p$ , and  $k$ , while

$$N_m = H_{mpk} \bar{A}_p \bar{A}_k \exp(i\sigma T_1),$$

$$N_p = H_{pkm} \bar{A}_m \bar{A}_k \exp(i\sigma T_1),$$

and

$$N_k = H_{kmp} \bar{A}_m \bar{A}_p \exp(i\sigma T_1),$$

where

$$H_{mpk} = \sum_j (\Gamma_{mpkj} + \Gamma_{mkpj} + \Gamma_{mpjk} + \Gamma_{mkjp} + \Gamma_{mjpk} + \Gamma_{mjkp}) K_j.$$

Then, Eq. 8 yields

$$-\omega_k (a'_k + c_k a_k) + \frac{1}{4} H_{kmp} a_m a_p \sin \mu = 0, \quad (18a)$$

$$-\omega_p (a'_p + c_p a_p) + \frac{1}{4} H_{pkm} a_k a_m \sin \mu = 0, \quad (18b)$$

$$-\omega_m (a'_m + c_m a_m) + \frac{1}{4} H_{mpk} a_p a_k \sin \mu = 0, \quad (18c)$$

$$\omega_k a_k \alpha'_k + \frac{1}{8} a_k \sum_j \gamma_{kj} \alpha_j^2 + H_{kkk} a_k + \frac{1}{4} H_{kmp} a_m a_p \cos \mu = 0, \quad (18d)$$

$$\omega_p a_p \alpha'_p + \frac{1}{8} a_p \sum_j \gamma_{pj} \alpha_j^2 + H_{ppp} a_p + \frac{1}{4} H_{pkm} a_k a_m \cos \mu = 0, \quad (18e)$$

and

$$\omega_m a_m \alpha'_m + \frac{1}{8} a_m \sum_j \gamma_{mj} \alpha_j^2 + H_{mmm} a_m + \frac{1}{4} H_{mpk} a_p a_k \cos \mu = 0, \quad (18f)$$

where

$$\mu = \sigma T_1 - \alpha_m - \alpha_p - \alpha_k.$$

The steady-state response corresponds to all  $a'_n = 0$  and  $\mu' = 0$ . As in the previous case, a trivial solution is possible. For a nontrivial solution, Eqs. 18 can be reduced to

$$-\omega_k c_k \hat{a}_k + \frac{1}{4} H_{kmp} \hat{a}_m \hat{a}_p \sin \hat{\mu} = 0, \quad (19a)$$

$$-\omega_p c_p \hat{a}_p + \frac{1}{4} H_{pkm} \hat{a}_k \hat{a}_m \sin \hat{\mu} = 0, \quad (19b)$$

$$-\omega_m c_m \hat{a}_m + \frac{1}{4} H_{mpk} \hat{a}_p \hat{a}_k \sin \hat{\mu} = 0, \quad (19c)$$

and

$$\sigma + \frac{1}{8} \left( \frac{\gamma_{kk}}{\omega_k} + \frac{\gamma_{pp}}{\omega_p} + \frac{\gamma_{mm}}{\omega_m} \right) \hat{a}_k^2 + \frac{1}{8} \left( \frac{\gamma_{pp}}{\omega_p} + \frac{\gamma_{kk}}{\omega_k} + \frac{\gamma_{mm}}{\omega_m} \right) \hat{a}_p^2$$

$$+ \frac{1}{8} \left( \frac{\gamma_{mm}}{\omega_m} + \frac{\gamma_{pp}}{\omega_p} + \frac{\gamma_{kk}}{\omega_k} \right) \hat{a}_m^2 + \frac{H_{kk}}{\omega_k} + \frac{H_{pp}}{\omega_p} + \frac{H_{mm}}{\omega_m} + \frac{1}{4} \left( \frac{H_{kmp} \hat{a}_p \hat{a}_m}{\omega_k \hat{a}_k} + \frac{H_{pkm} \hat{a}_k \hat{a}_m}{\omega_p \hat{a}_p} + \frac{H_{mpk} \hat{a}_p \hat{a}_k}{\omega_m \hat{a}_m} \right) \cos \hat{\mu} = 0. \quad (19d)$$

The steady-state response for the subharmonic resonant case  $\lambda \approx 3\omega_k$  can be obtained from Eqs. 19 by first setting  $\gamma_{ij} = 0$  when  $i \neq j$  and then letting  $p = m = k$ . The result is

$$-\omega_k c_k \hat{a}_k + \frac{1}{4} F_k \hat{a}_k^2 \sin \hat{\mu} = 0 \quad (20a)$$

and

$$\omega_k \sigma + \frac{3}{8} \gamma_{kk} \hat{a}_k^2 + \frac{3}{4} F_k \hat{a}_k \cos \hat{\mu} + 3H_{kk} = 0, \quad (20b)$$

where

$$F_k = H_{kkk} \text{ and } \hat{\mu} = \sigma T_1 - 3\hat{\alpha}_k.$$

The steady-state response for the case  $\lambda \approx 2\omega_k + \omega_m$  can be obtained by letting  $\gamma_{ij} = 0$  when  $i \neq p$  and then setting  $p = k$  in Eqs. 19. The result is

$$-\omega_k c_k \hat{a}_k + \frac{1}{4} H_{kkm} \hat{a}_k \hat{a}_m \sin \hat{\mu} = 0, \quad (21a)$$

$$-\omega_m c_m \hat{a}_m + \frac{1}{4} H_{mkk} \hat{a}_k^2 \sin \hat{\mu} = 0, \quad (21b)$$

and

$$\sigma + \frac{1}{8} \left( \frac{2\gamma_{kk}}{\omega_k} + \frac{\gamma_{mk}}{\omega_m} \right) \hat{a}_k^2 + \frac{1}{8} \left( \frac{2\gamma_{km}}{\omega_k} + \frac{\gamma_{mm}}{\omega_m} \right) \hat{a}_m^2 + \frac{2H_{kk}}{\omega_k} + \frac{H_{mm}}{\omega_m} + \frac{1}{4} \left( \frac{2H_{kkm} \hat{a}_m}{\omega_k} + \frac{H_{mkk} \hat{a}_k^2}{\omega_m \hat{a}_m} \right) \cos \hat{\mu} = 0, \quad (21c)$$

where

$$\hat{\mu} = \sigma T_1 - 2\hat{\alpha}_k - \hat{\alpha}_m.$$

The cases  $\lambda \approx \omega_m + \omega_p - \omega_k$  and  $\lambda \approx \omega_m - 2\omega_k$  can be obtained from the above results by simply changing the sign of  $\omega_k$ . Changing the sign of  $\omega_m$  leads to the results for the cases  $\lambda \approx \omega_k + \omega_p - \omega_m$  and  $\lambda \approx 2\omega_k - \omega_m$ .

As in the cases considered previously, the modes which interact with the excitation can form part of the steady-state solution, or the steady-state solution can have the form given in Eq. 11.

In the next section, internal resonances are considered.

## IV. THE CASE OF INTERNAL RESONANCE

For any given frequency of the excitation  $\lambda$ , the modal content of the steady-state response depends on the internal resonances present in the system. In this paper, consideration is given to systems having an internal resonance which involves four modes:

$$\omega_a + \omega_b + \omega_c + \epsilon \sigma_1 = \omega_d.$$

Systems having internal resonances which involve three modes ( $\omega_b + 2\omega_c \approx \omega_d$ ) and two modes ( $3\omega_c \approx \omega_d$ ) are treated as special cases.

An investigation of Appendix A ( $\Lambda_4$  through  $\Lambda_7$ ) shows that the contribution to Eq. 8 due to the internal resonance is

$$R_a = Q_a \bar{A}_b \bar{A}_c A_d \exp(i\sigma_1 T_1), \quad (22a)$$

$$R_b = Q_b \bar{A}_c A_d \bar{A}_a \exp(i\sigma_1 T_1), \quad (22b)$$

$$R_c = Q_c A_d \bar{A}_a \bar{A}_b \exp(i\sigma_1 T_1), \tag{22c}$$

and

$$R_d = Q_d A_a A_b A_c \exp(-i\sigma_1 T_1), \tag{22d}$$

where the  $Q_n$  are constants involving the  $\Gamma_{nmbq}$ .

Substituting Eqs. 9 and 22 into Eq. 8, separating the real and imaginary parts, and setting  $a'_n = 0$  for all  $n$  lead to the following equations governing the amplitudes and the phases of the steady-state solution:

$$-\omega_n c_n \hat{a}_n + \frac{1}{8} \hat{R}_n \sin \hat{\beta} + \hat{N}_n^{(1)} = 0, \tag{23a}$$

$$\omega_n \hat{a}_n \hat{a}'_n + \frac{1}{8} \hat{\alpha}_n \sum_j \gamma_{nj} \hat{a}_j^2 + H_{nn} \hat{a}_n + \frac{1}{8} \hat{R}_n \cos \hat{\beta} + \hat{N}_n^{(2)} = 0, \tag{23b}$$

for  $n = a, b,$  and  $c$ ;

$$\omega_d c_d \hat{a}_d + \frac{1}{8} \hat{R}_d \sin \hat{\beta} - N_d^{(1)} = 0, \tag{24a}$$

$$\omega_d \hat{a}_d \hat{a}'_d + \frac{1}{8} \hat{\alpha}_d \sum_j \gamma_{dj} \hat{a}_j^2 + H_{dd} \hat{a}_d + \frac{1}{8} \hat{R}_d \cos \hat{\beta} + \hat{N}_d^{(2)} = 0; \tag{24b}$$

$$\omega_n c_n \hat{a}_n + \hat{N}_n^{(1)} = 0, \tag{25a}$$

$$\omega_n \hat{a}_n \hat{a}'_n + \frac{1}{8} \hat{\alpha}_n \sum_j \gamma_{nj} \hat{a}_j^2 + H_{nn} \hat{a}_n + \hat{N}_n^{(2)} = 0, \tag{25b}$$

for  $n \neq a, b, c,$  and  $d$ ; and

$$\hat{\beta}' = \sigma_1 - \hat{\alpha}'_a - \hat{\alpha}'_b - \hat{\alpha}'_c + \hat{\alpha}'_d = 0, \tag{26}$$

where

$$\hat{R}_a = Q_a \hat{a}_b \hat{a}_c \hat{a}_d, \quad \hat{R}_b = Q_b \hat{a}_c \hat{a}_d \hat{a}_a,$$

$$\hat{R}_c = Q_c \hat{a}_d \hat{a}_a \hat{a}_b, \quad \hat{R}_d = Q_d \hat{a}_a \hat{a}_b \hat{a}_c.$$

Because the  $N_n^{(1),(2)}$  appear as a result of external resonances, they can be functions of the  $a_n$  and the  $\alpha_n$  (see Sec. III); hence, the precise form of this set of simultaneous equations depends on the external resonance involved. Several possibilities are considered next.

### A. The case of no external resonance

In this case, all  $N_n = 0$ ; thus, Eq. 25a leads directly to

$$\hat{a}_n = 0$$

for  $n \neq a, b, c,$  and  $d$ . Equations 23a and 24a lead to

$$-\omega_a c_a \hat{a}_a + \frac{1}{8} Q_a \hat{a}_b \hat{a}_c \hat{a}_d \sin \hat{\beta} = 0, \tag{27a}$$

$$-\omega_b c_b \hat{a}_b + \frac{1}{8} Q_b \hat{a}_c \hat{a}_d \hat{a}_a \sin \hat{\beta} = 0, \tag{27b}$$

$$-\omega_c c_c \hat{a}_c + \frac{1}{8} Q_c \hat{a}_d \hat{a}_a \hat{a}_b \sin \hat{\beta} = 0, \tag{27c}$$

and

$$\omega_d c_d \hat{a}_d + \frac{1}{8} Q_d \hat{a}_a \hat{a}_b \hat{a}_c \sin \hat{\beta} = 0. \tag{27d}$$

For structural elements, nontrivial solutions cannot exist in the absence of external excitations and the presence of damping (i.e., the system cannot be self excited). This condition demands that the signs of  $Q_a, Q_b, Q_c,$  and  $Q_d$  be the same. This can be seen by supposing such a solution does exist and then combining Eq. 27 to obtain

$$\left(\frac{a_a}{a_d}\right)^2 = -\frac{\omega_d c_d Q_d}{\omega_a c_a Q_a}, \tag{28a}$$

$$\left(\frac{a_b}{a_d}\right)^2 = -\frac{\omega_d c_d Q_d}{\omega_b c_b Q_b}, \tag{28b}$$

and

$$\left(\frac{a_c}{a_d}\right)^2 = -\frac{\omega_d c_d Q_d}{\omega_c c_c Q_c}. \tag{28c}$$

Clearly, the supposition of a nontrivial solution is inconsistent with Eqs. 28 if the signs of  $Q_a, Q_b, Q_c,$  and  $Q_d$  are the same.

### B. The case of an external resonance

In this section, several possibilities are considered.

If none of the modes involved in the internal resonances are directly excited, then it follows immediately from Eqs. 28 that

$$\hat{a}_a = \hat{a}_b = \hat{a}_c = \hat{a}_d = 0. \tag{29}$$

Hence, the internal resonance has no influence on the solution, which would be obtained as outlined in Sec. III.

If two (one) of the lower modes involved in the internal resonance are (is) directly excited, then it follows from Eqs. 28 that the amplitudes of the remaining, unexcited lower mode(s) and the highest mode  $\hat{a}_d$  are zero. Therefore,  $\hat{R}_n = 0$  for  $n = a, b, c,$  and  $d$ . Again the internal resonance has no influence on the solution, which would be obtained as outlined in Sec. III.

If all three of the lower modes are directly excited, then, depending on the type of external resonance, the amplitudes of the lower modes may be either zero or nonzero. When  $\hat{a}_a, \hat{a}_b,$  and  $\hat{a}_c$  are not zero, it follows from Eq. 27d that  $\hat{a}_d$  cannot be zero.

If the  $d$ th mode is the only mode involved in the internal resonance to be directly excited, then, depending on the type of external resonance,  $\hat{a}_d$  may be either zero or nonzero. Hence, it follows from Eqs. 27 and 28 that there are three possibilities:

$$(1) \hat{a}_d = 0, \text{ and thus } \hat{a}_a = \hat{a}_b = \hat{a}_c = 0.$$

$$(2) \hat{a}_d \neq 0, \text{ and } \hat{a}_a = \hat{a}_b = \hat{a}_c = 0.$$

$$(3) \hat{a}_a, \hat{a}_b, \hat{a}_c, \hat{a}_d \text{ are nonzero.}$$

From the last two subcases considered, it follows that the terms appearing in Eqs. 23 and 24 as a result of the internal resonance can be responsible for a transfer of energy from a directly excited mode to a mode which is not directly excited. In the next section, the response of a hinged-clamped beam to a harmonic excitation is presented as a numerical example.

### V. THE BEAM AS AN EXAMPLE

Large-amplitude vibrations of beams supported in such a way as to restrict longitudinal movement at the ends are accompanied by stretching of the neutral plane. One must account for this stretching by using nonlinear strain-displacement relationships, and consequently, the equations governing the lateral vibrations are nonlinear. In the present example, a hinged-clamped beam is considered. Modal viscous damping is included, and the excitation is taken to be harmonic.

The nondimensional form of the governing equation<sup>22</sup> is

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \epsilon \left\{ -2c \frac{\partial w}{\partial t} + \nu \left[ \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx \right] \frac{\partial^2 w}{\partial x^2} \right\} + p(x, t), \tag{29}$$

where  $l$  is the nondimensional length of the beam,  $\nu$  is a constant which depends on the type of axial restraint [for a rigid restraint  $\nu = 1/(2l)$ ],  $\epsilon$  is a small nondimensional parameter defined as

$$\epsilon = r^2/L^2,$$

$r$  is the radius of gyration of the cross-section area, and  $L$  is a characteristic length.

It is noted that  $p(x, t)$  is  $O(1)$ , which is in contrast with the example considered in Ref. 22, where  $p = O(\epsilon)$ . In the following,  $p$  is assumed to vary sinusoidally with time; that is,

$$p(x, t) = P(x) \cos \lambda t.$$

The deflection  $w$  is expanded in terms of the linear, free-oscillation modes  $\phi_m$  (taken to be orthonormal) as follows:

$$w(x, t) = \sum_m \phi_m(x) u_m(t). \tag{30}$$

Substituting Eq. 30 into Eq. 29 and using the orthonormality of the  $\phi_m$  yields

$$\frac{d^2 u_n}{dt^2} + \omega_n^2 u_n = \epsilon \left( -2c_n \frac{du_n}{dt} + \sum_{m,p,q} \Gamma_{nmpq} u_m u_p u_q \right) + P_n \cos \lambda t, \tag{31}$$

for  $n = 1, 2, \dots$ ,

where

$$\Gamma_{nmpq} = \nu \left[ \int_0^l \phi_n \frac{d^2 \phi_q}{dx^2} \right] \left[ \int_0^l \frac{d\phi_m}{dx} \frac{d\phi_p}{dx} dx \right].$$

The five lowest natural frequencies for  $l = 2$  are

$$\begin{aligned} \omega_1 &= 3.8545, & \omega_2 &= 12.491, & \omega_3 &= 26.062, \\ \omega_4 &= 44.568, & \omega_5 &= 68.007. \end{aligned}$$

It is noted that  $\omega_2$  and  $\omega_1$  are nearly in the ratio of 3 to 1. Thus, there is a two-mode, internal resonance. The nearness of  $\omega_2$  to  $3\omega_1$  is expressed quantitatively by the detuning parameter  $\sigma_1$  as follows:

$$\omega_2 = 3\omega_1 + \epsilon \sigma_1; \quad \epsilon \sigma_1 = 0.9275.$$

Numerical results are presented for superharmonic, subharmonic, and combination resonances. Some typical values of the coefficients are given in Appendix B.

**A. The case of  $3\lambda \approx \omega_1$**

In this case, a second detuning parameter is defined as follows:

$$3\lambda = \omega_1 + \epsilon \sigma_2.$$

Thus,

$$N_1 = F_1 \exp(i\sigma_2 T_1), \quad F_1 = \sum_{m,p,q} \Gamma_{1mpq} K_m K_p K_q$$

and

$$N_n = 0, \quad \text{for } n > 1.$$

Equations 23–26 can be reduced to

$$-\omega_1 c_1 \hat{a}_1 + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \sin \hat{\beta} + F_1 \sin \hat{\mu} = 0, \tag{32a}$$

$$-\omega_2 c_2 \hat{a}_2 - \frac{1}{8} Q_2 \hat{a}_1^3 \sin \hat{\beta} = 0, \tag{32b}$$

$$\begin{aligned} \omega_2(\sigma_1 - 3\sigma_2) \hat{a}_2 - \frac{1}{8}(\gamma_{22} \hat{a}_2^3 + \gamma_{21} \hat{a}_2 \hat{a}_1^2) \\ - H_{22} \hat{a}_2 - \frac{1}{8} Q_2 \hat{a}_1^3 \cos \hat{\beta} = 0, \end{aligned} \tag{32c}$$

$$\begin{aligned} \omega_1 \sigma_2 \hat{a}_1 + \frac{1}{8}(\gamma_{11} \hat{a}_1^3 + \gamma_{12} \hat{a}_1 \hat{a}_2^2) + H_{11} \hat{a}_1 \\ + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \cos \hat{\beta} + F_1 \cos \hat{\mu} = 0, \end{aligned} \tag{32d}$$

and

$$\hat{a}_n = 0, \quad \text{for } n > 2,$$

where

$$\hat{\beta} = \sigma_1 T_1 - 3\hat{\alpha}_1 + \hat{\alpha}_2 \tag{32e}$$

and

$$\hat{\mu} = \sigma_2 T_1 - \hat{\alpha}_1. \tag{32f}$$

Because  $F_1$  is independent of  $\hat{a}_1$ , it follows from Eqs. 32 that neither  $\hat{a}_1$  nor  $\hat{a}_2$  can be zero. This is in agreement with the comments of Sec. IV; here all the lower modes are excited. The steady-state solution has the form

$$u_1 = P_1(\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_1 \cos(3\lambda t - \hat{\mu}) + O(\epsilon), \tag{33a}$$

$$u_2 = P_2(\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_2 \cos(9\lambda t - 3\hat{\mu} + \hat{\beta}) + O(\epsilon), \tag{33b}$$

and, for  $n > 2$ ,

$$u_n = P_n(\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + O(\epsilon). \tag{33c}$$

It is noted that the nonlinearity adjusts the frequencies of the second and the first modes such that they are precisely in the ratio of three to one and the frequency of the first mode is precisely three times that of the excitation.

For some arbitrary values of the excitation amplitude and the damping coefficients (here, for simplicity,  $c_1 = c_2 = c$ ) Eqs. 32 were solved by using a Newton-Raphson procedure. In Fig. 1,  $\hat{a}_1$  and  $\hat{a}_2$  are plotted as functions of the detuning parameter  $\epsilon \sigma_2$ . For the sake of clarity, only the stable portions of the complete solution are shown in this figure as well as in all those that follow. (The manner in which the stability was studied is discussed briefly in the next section.) It is noted that  $\hat{a}_2$  is always smaller than  $\hat{a}_1$ .

**B. The case of  $\lambda \approx 3\omega_2$**

In this case, the second detuning parameter is defined as follows:

$$\lambda = 3\omega_2 + \epsilon \sigma_2.$$

Thus,

$$N_2 = F_2 \bar{A}_2^2 \exp(i\sigma_2 T_1), \quad F_2 = H_{222},$$

and

$$N_n = 0, \quad \text{for } n = 1, 3, 4, \dots$$

Equations 23–26 can be reduced to

$$-\omega_1 c_1 \hat{a}_1 + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \sin \hat{\beta} = 0, \tag{34a}$$

$$-\omega_2 c_2 \hat{a}_2 - \frac{1}{8} Q_2 \hat{a}_1^3 \sin \hat{\beta} + \frac{1}{4} F_2 \hat{a}_2^2 \sin \hat{\mu} = 0, \tag{34b}$$

$$\omega_1(\sigma_1 + \frac{1}{3}\sigma_2) \hat{a}_1 + \frac{1}{8}(\gamma_{11} \hat{a}_1^3 + \gamma_{12} \hat{a}_1 \hat{a}_2^2) + 3H_{11} \hat{a}_1 + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \cos \hat{\beta} = 0, \tag{34c}$$

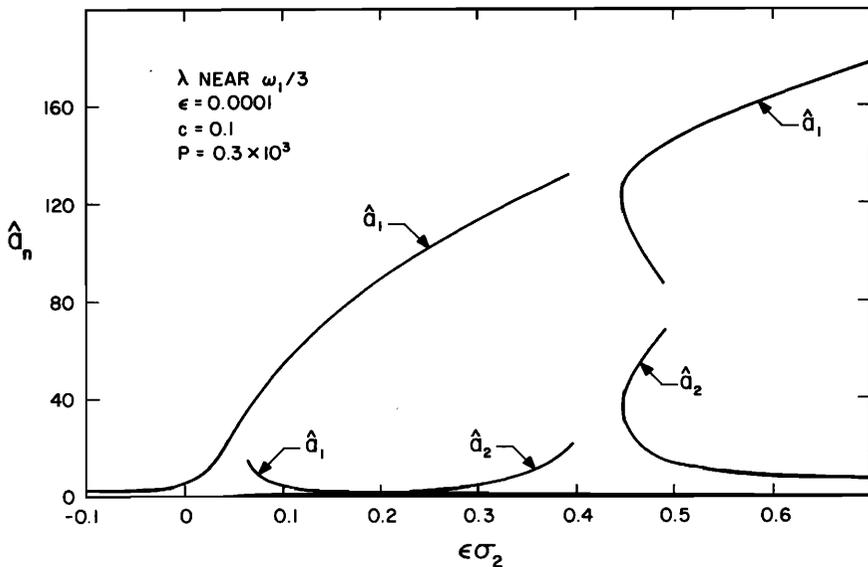


FIG. 1. Variations of the steady-state amplitudes of the first and second modes with the frequency of the excitation ( $\lambda - \omega_1/3 = \epsilon\sigma_2$ ).

$$\omega_2 \sigma_2 \hat{a}_2 + \frac{3}{8} (\gamma_{22} \hat{a}_2^3 + \gamma_{21} \hat{a}_2 \hat{a}_1^2) + 3H_{22} \hat{a}_2 + \frac{3}{8} Q_2 \hat{a}_1^3 \cos \hat{\beta} + \frac{3}{4} F_2 \hat{a}_2^2 \cos \hat{\mu} = 0, \quad (34d)$$

and

$$\hat{a}_n = 0, \text{ for } n > 2,$$

where

$$\hat{\beta} = \sigma_1 T_1 - 3\hat{\alpha}_1 + \hat{\alpha}_2 \quad (34e)$$

and

$$\hat{\mu} = \sigma_2 T_1 - 3\hat{\alpha}_2. \quad (34f)$$

Equations 34 reveal that there are three possible solutions. This is in agreement with comments of Sec. IV.

When  $\hat{a}_1$  and  $\hat{a}_2$  are zero, the steady-state solution is given by

$$u_n = P_n (\omega_n^2 - \lambda^2)^{-1} \cos \lambda t + 0(\epsilon) \quad (35)$$

for all  $n$ .

When  $\hat{a}_1 = 0$  and  $\hat{a}_2 \neq 0$ , the steady-state solution has the form

$$u_2 = P_2 (\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_2 \cos [\frac{1}{3}(\lambda t - \hat{\mu})] + 0(\epsilon), \quad (36)$$

and the remaining  $u_n$  are given by Eq. 35. It is noted that the nonlinearity adjusts the frequency of the second mode such that it is precisely one third of that of the excitation.

When  $\hat{a}_1$  and  $\hat{a}_2$  differ from zero, the steady-state solution has the form

$$u_1 = P_1 (\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_1 \cos [\frac{1}{3}(\lambda t - \hat{\mu}) - \frac{1}{3}\hat{\beta}] + 0(\epsilon), \quad (37a)$$

$$u_2 = P_2 (\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_2 \cos [\frac{1}{3}(\lambda t - \hat{\mu})] + 0(\epsilon), \quad (37b)$$

and the remaining  $u_n$  are given by Eq. 35. It is noted that the nonlinearity adjusts the frequencies of the second and the first modes such that they are precisely in the three to one ratio and the frequency of the second mode is precisely one-third of that of the excitation.

The first case (Eq. 35) is of little interest and no results are presented.

For the second case (Eq. 36),  $\hat{a}_2$  is plotted as a function of the detuning  $\epsilon\sigma_2$  in Fig. 2(a). As one might expect, this result resembles the solution of the Duffing equation for subharmonic resonance.

For the third case (Eqs. 37),  $\hat{a}_1$  and  $\hat{a}_2$  are plotted as functions of detuning  $\epsilon\sigma_2$  in Fig. 2(b). The values of the amplitude of the excitation and the damping coefficients are the same in both figures. It is noted that, when  $\hat{a}_1$  is not zero, it is greater than  $\hat{a}_2$  over a considerable range of the detuning.

### C. The case of $2\lambda \approx \omega_2 + \omega_3$

In this case, the second detuning parameter is defined as follows:

$$2\lambda = \omega_2 + \omega_3 + \epsilon\sigma_2.$$

Thus,

$$N_2 = H_{23} \bar{A}_3 \exp(i\sigma_2 T_1), \quad N_3 = H_{32} \bar{A}_2 \exp(i\sigma_2 T_1),$$

and

$$N_n = 0 \text{ for } n = 1, 4, 5, \dots$$

Equations 23–26 can be reduced to

$$-\omega_1 c_1 \hat{a}_1 + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \sin \hat{\beta} = 0, \quad (38a)$$

$$-\omega_2 c_2 \hat{a}_2 - \frac{1}{8} Q_2 \hat{a}_1^3 \sin \hat{\beta} + \frac{1}{2} H_{23} \hat{a}_3 \sin \hat{\mu} = 0, \quad (38b)$$

$$-\omega_3 c_3 \hat{a}_3 + \frac{1}{2} H_{32} \hat{a}_2 \sin \hat{\mu} = 0, \quad (38c)$$

and, for a nontrivial solution,

$$\sigma_1 + \frac{1}{8} \left( \frac{3\gamma_{11}}{\omega_1} - \frac{\gamma_{21}}{\omega_2} \right) \hat{a}_1^2 + \frac{1}{8} \left( \frac{3\gamma_{12}}{\omega_1} - \frac{\gamma_{22}}{\omega_2} \right) \hat{a}_2^2 + \frac{1}{8} \left( \frac{3\gamma_{13}}{\omega_1} - \frac{\gamma_{23}}{\omega_2} \right) \hat{a}_3^2 + \frac{3H_{11}}{\omega_1} - \frac{H_{22}}{\omega_2} + \frac{1}{8} \left( \frac{3Q_1}{\omega_1} \hat{a}_1 \hat{a}_2 - \frac{Q_2 \hat{a}_1^3}{\omega_2 \hat{a}_2} \right) \cos \hat{\beta} - \frac{1}{2} \frac{H_{23} \hat{a}_3}{\omega_2 \hat{a}_2} \cos \hat{\mu} = 0 \quad (38d)$$

and

$$\sigma_2 + \frac{1}{8} \left( \frac{\gamma_{21}}{\omega_2} + \frac{\gamma_{31}}{\omega_3} \right) \hat{a}_1^2 + \frac{1}{8} \left( \frac{\gamma_{22}}{\omega_2} + \frac{\gamma_{32}}{\omega_3} \right) \hat{a}_2^2 + \frac{1}{8} \left( \frac{\gamma_{23}}{\omega_2} + \frac{\gamma_{33}}{\omega_3} \right) \hat{a}_3^2 + \frac{H_{22}}{\omega_2} + \frac{H_{33}}{\omega_3} + \frac{1}{8} \frac{Q_2 \hat{a}_1^3}{\omega_2 \hat{a}_2} \cos \hat{\beta} + \frac{1}{2} \left( \frac{H_{23} \hat{a}_3}{\omega_2 \hat{a}_2} + \frac{H_{32} \hat{a}_2}{\omega_3 \hat{a}_3} \right) \cos \hat{\mu} = 0, \quad (38e)$$

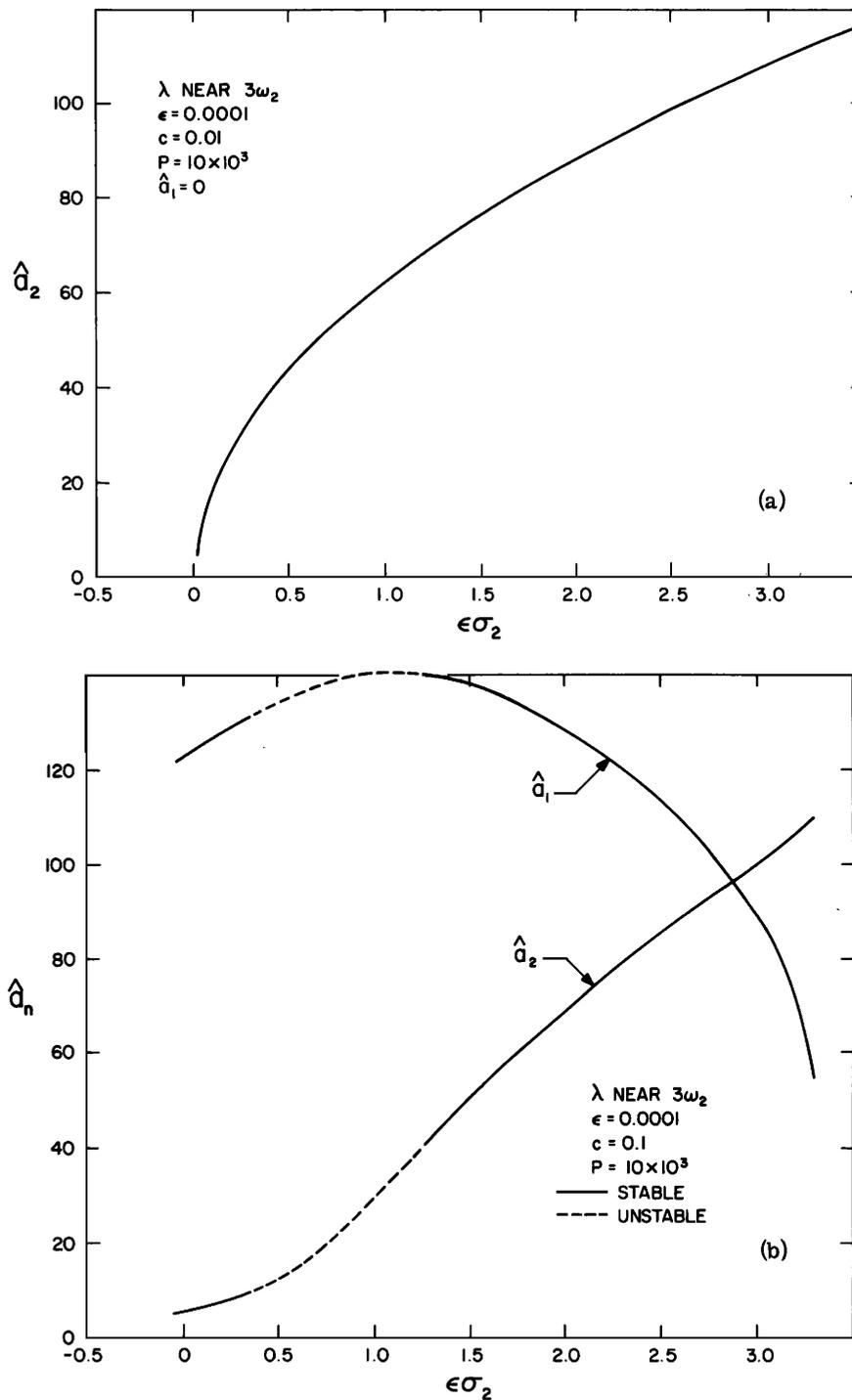


FIG. 2. (a) Variations of the steady-state amplitude of the second mode with the frequency of the excitation ( $\lambda - 3\omega_2 = \epsilon\sigma_2$ ) when the steady-state amplitude of the first mode is zero. (b) Variations of the steady-state amplitudes of the first and second modes with the frequency of the excitation ( $\lambda - 3\omega_2 = \epsilon\sigma_2$ ).

where

$$\hat{\beta} = \sigma_1 T_1 - 3\hat{a}_1 + \hat{a}_2 \tag{38f}$$

and

$$\hat{\mu} = \sigma_2 T_1 - \hat{a}_2 - \hat{a}_3. \tag{38g}$$

In this case also there are three possible solutions.

When  $\hat{a}_1$ ,  $\hat{a}_2$ , and  $\hat{a}_3$  are zero, the steady-state solution is given by Eq. 35 for all  $n$ .

When  $a_1 = 0$  and  $\hat{a}_2$  and  $\hat{a}_3$  differ from zero, the steady-state solution is obtained by solving Eqs. 38b, 38c, and 38e for  $\hat{a}_2$ ,  $\hat{a}_3$  and  $\hat{\mu}$ , after setting  $\hat{a}_1 = 0$ , and then obtain-

ing  $\hat{\alpha}'_2$  and  $\hat{\alpha}'_3$  from Eqs. 24b and 24c, which become

$$\omega_2 \hat{a}_2 \hat{\alpha}'_2 + \frac{1}{8} \hat{a}_2 (\gamma_{22} \hat{a}_2^2 + \gamma_{23} \hat{a}_3^2) + H_{22} \hat{a}_2 + \frac{1}{8} Q_2 \hat{a}_1^3 \cos \hat{\beta} + \frac{1}{2} H_{23} \hat{a}_3 \cos \hat{\mu} = 0 \tag{39a}$$

and

$$\omega_3 \hat{a}_3 \hat{\alpha}'_3 + \frac{1}{8} \hat{a}_3 (\gamma_{32} \hat{a}_2^2 + \gamma_{33} \hat{a}_3^2) + H_{33} \hat{a}_3 + \frac{1}{2} H_{32} \hat{a}_2 \cos \hat{\mu} = 0. \tag{39b}$$

Then, the solution is given by

$$u_2 = P_2 (\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_2 \cos [(\omega_2 + \epsilon \hat{\alpha}'_2) t + \tau_2], \tag{40a}$$

$$u_3 = P_3 (\omega_3^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_3 \cos [(\omega_3 + \epsilon \hat{\alpha}'_3) t + \tau_3], \tag{40b}$$

and the remaining  $u_n$  are given by Eq. 35. It is noted

that the nonlinearity adjusts the frequencies of the second and the third mode such that the resonant frequency combination is satisfied exactly; that is,

$$\omega_2 + \epsilon \hat{\alpha}'_2 + \omega_3 + \epsilon \hat{\alpha}'_3 = \omega_2 + \omega_3 + \epsilon \sigma_2 = 2\lambda .$$

When  $\hat{a}_1$ ,  $\hat{a}_2$ , and  $\hat{a}_3$  are not zero, the steady-state solution is obtained by solving Eqs. 38 for  $\hat{a}_1$ ,  $\hat{a}_2$ ,  $\hat{a}_3$ ,  $\hat{\beta}$ , and  $\hat{\mu}$ , and then obtaining  $\hat{\alpha}'_1$ ,  $\hat{\alpha}'_2$ , and  $\hat{\alpha}'_3$  from

$$\omega_1 \hat{a}_1 \hat{\alpha}'_1 + \frac{1}{8} \hat{a}_1 (\gamma_{11} \hat{a}_1^2 + \gamma_{12} \hat{a}_2^2 + \gamma_{13} \hat{a}_3^2) + H_{11} \hat{a}_1 + \frac{1}{8} Q_1 \hat{a}_1^2 \hat{a}_2 \cos \hat{\beta} = 0 , \tag{41a}$$

$$\omega_2 \hat{a}_2 \hat{\alpha}'_2 + \frac{1}{8} \hat{a}_2 (\gamma_{21} \hat{a}_1^2 + \gamma_{22} \hat{a}_2^2 + \gamma_{23} \hat{a}_3^2) + H_{22} \hat{a}_2 + \frac{1}{8} Q_2 \hat{a}_1^3 \cos \hat{\beta} + \frac{1}{2} H_{23} \hat{a}_3 \cos \hat{\mu} = 0 , \tag{41b}$$

and

$$\omega_3 \hat{a}_3 \hat{\alpha}'_3 + \frac{1}{8} \hat{a}_3 (\gamma_{31} \hat{a}_1^3 + \gamma_{32} \hat{a}_2^2 + \gamma_{33} \hat{a}_3^2) + H_{33} \hat{a}_3 + \frac{1}{2} H_{32} \hat{a}_2 \cos \hat{\mu} = 0 . \tag{41c}$$

Then, the solution is given by

$$u_1 = P_1 (\omega_1^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_1 \cos [(\omega_1 + \epsilon \hat{\alpha}'_1) t + \tau_1] + 0(\epsilon) , \tag{42a}$$

$$u_2 = P_2 (\omega_2^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_2 \cos [(\omega_2 + \epsilon \hat{\alpha}'_2) t + \tau_2] + 0(\epsilon) , \tag{42b}$$

$$u_3 = P_3 (\omega_3^2 - \lambda^2)^{-1} \cos \lambda t + \hat{a}_3 \cos [(\omega_3 + \epsilon \hat{\alpha}'_3) t + \tau_3] + 0(\epsilon) , \tag{42c}$$

and the remaining  $u_n$  are given by Eq. 35. It follows from Eqs. 38f, 38g, and 42 that the nonlinearity adjusts the frequencies such that the frequencies of the first and second modes are precisely in the ratio of one to three and the sum of the frequencies of the second and

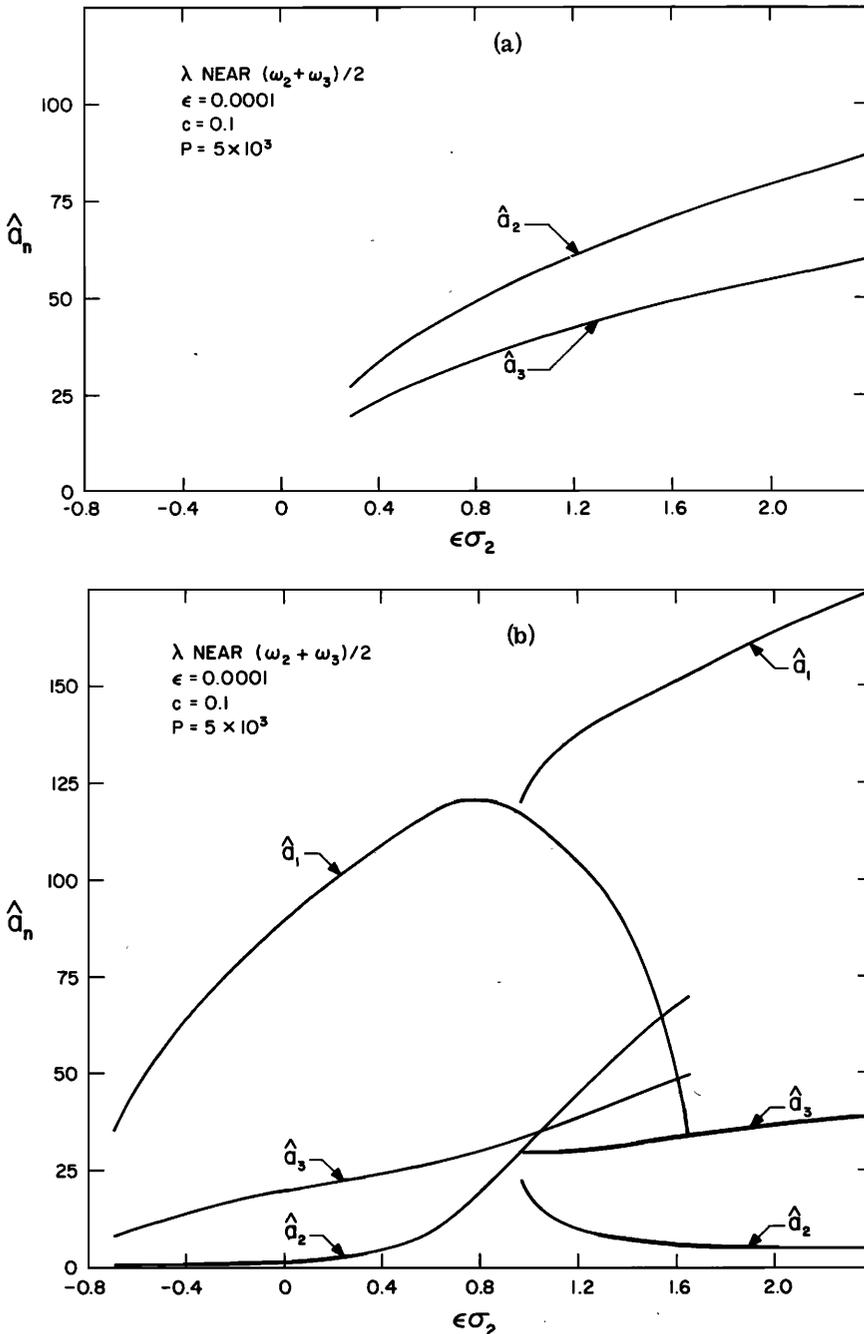


FIG. 3. (a) Variations of the amplitudes of the second and third modes with the frequency of the excitation [ $\lambda - (\omega_2 + \omega_3)/2 = \epsilon \sigma_2$ ] when the steady-state amplitude of the first mode is zero. (b) Variations of the amplitudes of the first, second, and third modes with the frequency of the excitation [ $\lambda - (\omega_2 + \omega_3)/2 = \epsilon \sigma_2$ ].

third modes is precisely  $2\lambda$ ; that is,

$$3(\omega_1 + \epsilon \hat{\alpha}'_1) = 3\omega_1 + \epsilon \sigma_1 + \epsilon \hat{\alpha}'_2 = \omega_2 + \epsilon \alpha'_2$$

and

$$\omega_2 + \epsilon \hat{\alpha}'_2 + \omega_3 + \epsilon \hat{\alpha}'_3 = \omega_2 + \omega_3 + \epsilon \sigma_2 = 2\lambda .$$

The first subcase (Eq. 35) is of little interest, and the results are not presented.

For the second subcase (Eqs. 40),  $\hat{a}_2$  and  $\hat{a}_3$  are plotted as functions of the detuning  $\epsilon \sigma_2$  in Fig. 3(a).

For the third subcase (Eqs. 42),  $\hat{a}_1$ ,  $\hat{a}_2$ , and  $\hat{a}_3$  are plotted as functions of the detuning  $\epsilon \sigma_2$  in Fig. 3(b). It is noted that  $\hat{a}_1$  is greater than  $\hat{a}_2$  and  $\hat{a}_3$  over a wide range of the frequency of the excitation.

## VI. STABILITY

The stability of the various branches was determined by adding an infinitesimal disturbance to the steady-state solution. From Eqs. 32, 34, and 38, one can obtain a system of linear, homogeneous equations, having constant coefficients, which govern the disturbance. Consequently, the disturbance will be of the form  $\exp(MT_1)$ , where  $M$  is an eigenvalue of the coefficient matrix. If the real parts of all the eigenvalues are negative, the branch is said to be stable; otherwise it is said to be unstable. More details can be found in Ref. 22.

## VII. CONCLUDING REMARKS

A method is presented for analyzing superharmonic, subharmonic, and combination resonances in a multi-degree-of-freedom system which has a cubic nonlinearity and modal viscous damping and is subject to harmonic excitations. The method of multiple scales is an elegant and effective method for studying resonances and systematically obtaining approximate solutions.

The present results reveal the following features of the steady-state response.

- (1) In the absence of internal resonances, only the directly excited modes can appear in the steady-state response.
- (2) The directly excited modes may not appear in the cases of subharmonic and combination resonances. But in the case of a superharmonic resonance, the directly excited mode always appears in the steady-state response.
- (3) In the presence of an internal resonance, it is possible for modes other than those that are directly excited to appear in the response.
- (4) If the highest mode in the internal resonance is directly excited, then either all or none of the lower modes are drawn into the response.
- (5) If all of the lower modes in the internal resonance are directly excited, then the highest mode is always drawn into the response.
- (6) If not more than two of the lower modes in a four-mode internal resonance (not more than one of the lower

modes in a three-mode internal resonance) are directly excited, then none of the other modes in the internal resonance appear in the response.

We note that in the case of a two-mode internal resonance there is only one lower mode, and exciting this mode always draws the other mode into the response.

The numerical examples illustrate the possibility of the amplitude of the mode which is not directly excited being much larger than the amplitudes of the modes which are directly excited. This illustrates the importance of taking internal resonances into account in a nonlinear analysis.

## ACKNOWLEDGMENT

This work was supported by the Structures Division of the Langley Research Center, NASA, under Contract NAS1-10646-29.

## APPENDIX A

TABLE A-I. Coefficients  $B_j$  and frequency combinations  $\Lambda_j$  in Eqs. 7.

$j$	$B_j$	$\Lambda_j$	$j$	$B_j$	$\Lambda_j$
1	$2A_m K_p K_q$	$\omega_m$	15	$A_q K_m K_p$	$-2\lambda + \omega_q$
2	$2A_p K_q K_m$	$\omega_p$	16	$A_m A_p K_q$	$\lambda + \omega_m + \omega_p$
3	$2A_q K_m K_p$	$\omega_q$	17	$A_m A_p K_q$	$-\lambda + \omega_m + \omega_p$
4	$A_m A_p A_q$	$\omega_m + \omega_p - \omega_q$	18	$A_p A_q K_m$	$\lambda + \omega_p + \omega_q$
5	$A_m A_p A_q$	$\omega_m - \omega_p + \omega_q$	19	$A_p A_q K_m$	$-\lambda + \omega_p + \omega_q$
6	$A_m A_p A_q$	$-\omega_m + \omega_p + \omega_q$	20	$A_q A_m K_p$	$\lambda + \omega_q + \omega_m$
7	$A_m A_p A_q$	$\omega_m + \omega_p + \omega_q$	21	$A_q A_m K_p$	$-\lambda + \omega_m - \omega_p$
8	$3K_m K_p K_q$	$\lambda$	22	$A_m A_p K_q$	$\lambda + \omega_m - \omega_p$
9	$K_m K_p K_q$	$3\lambda$	23	$A_m A_p K_q$	$-\lambda + \omega_m - \omega_p$
10	$A_m K_p K_q$	$2\lambda + \omega_m$	24	$A_p A_q K_m$	$\lambda + \omega_p - \omega_q$
11	$A_m K_p K_q$	$-2\lambda + \omega_m$	25	$A_p A_q K_m$	$-\lambda + \omega_p - \omega_q$
12	$A_p K_q K_m$	$2\lambda + \omega_p$	26	$A_q A_m K_p$	$\lambda + \omega_q - \omega_m$
13	$A_p K_q K_m$	$-2\lambda + \omega_p$	27	$A_q A_m K_p$	$-\lambda + \omega_q - \omega_m$
14	$A_q K_m K_p$	$2\lambda + \omega_q$			

## APPENDIX B

Values of some of the constants in the numerical examples:

$$\epsilon = 0.0001 ,$$

$$Q_1 = 2\Gamma_{1121} + \Gamma_{1112} = -2.3108 ,$$

$$Q_2 = \Gamma_{2111} = -0.77027 ,$$

$$\gamma_{11} = 3\Gamma_{1111} = -6.213 ,$$

$$\gamma_{22} = 3\Gamma_{2222} = -86.26 ,$$

$$\gamma_{33} = 3\Gamma_{3333} = -414.5 ,$$

$$\gamma_{12} = 2(2\Gamma_{1122} + \Gamma_{1221}) = \gamma_{21} = -16.58 ,$$

$$\gamma_{13} = 2(2\Gamma_{1133} + \Gamma_{1331}) = \gamma_{31} = -34.73 ,$$

and

$$\gamma_{23} = 2(2\Gamma_{2233} + \Gamma_{2332}) = \gamma_{32} = -129.9 .$$

For the graphs, the spatial variation of the loading function was taken to be constant; i. e.,

$$P_n = \int_0^1 P(x)\phi_n(x)dx = P \int_0^1 \phi_n(x)dx .$$

Typical values of the other constants:

(1)  $\lambda \approx \omega_1/3$ —For  $P = 0.3 \times 10^3$  and  $\epsilon\sigma_2 = 0.1$ ,  
 $F_1 = -9459$ ,  $H_{11} = -1710$ , and  $H_{22} = -2264$ .

(2)  $\lambda \approx 3\omega_2$ —For  $P = 10 \times 10^3$  and  $\epsilon\sigma_2 = 2$ ,  $F_2 = 41.37$ ,  
 $H_{11} = -899.2$ , and  $H_{22} = -2713$ .

(3)  $\lambda \approx (\omega_2 + \omega_3)/2$ —For  $P = 5 \times 10^3$  and  $\epsilon\sigma_2 = 1.5$ ,  
 $H_{11} = -819.1$ ,  $H_{22} = -1914$ ,  $H_{33} = -9921$ , and  $H_{23} = H_{32} = -1054$ .

<sup>1</sup>C. Hayashi, *Nonlinear Oscillations in Physical Systems* (McGraw-Hill, New York, 1964).

<sup>2</sup>P. R. Sethna, "Coupling in certain classes of weakly nonlinear vibrating systems," in *Nonlinear Differential Equations and Nonlinear Mechanics*, J. P. LaSalle and S. Lefshetz, Eds. (Academic, New York, 1963), pp. 58–79.

<sup>3</sup>P. R. Sethna, "Vibrations of dynamical systems with quadratic nonlinearities," *J. Appl. Mech.* 32, 576–582 (1964).

<sup>4</sup>A. Tondl, *Some Problems in Rotor Dynamics* (Chapman and Hall, London, 1965).

<sup>5</sup>G. J. Efstathiades and C. J. H. Williams, "Vibration isolation using nonlinear springs," *Int. J. Mech. Sci.* 9, 27 (1967).

<sup>6</sup>G. V. Plotnikova, "On the stability of periodic solutions of non-self-contained quasi-linear systems with two degrees of freedom," *Prikl. Mat. Mekh.* 29, 1084–1091 (English transl.: 1273–1281) (1965).

<sup>7</sup>E. Mettler, "Stability and vibration problems of mechanical systems under harmonic excitation," in *Dynamic Stability of Structures*, G. Hermann, Ed. (Pergamon, New York, 1967), pp. 169–188.

<sup>8</sup>T. Yamamoto and S. Hayashi, "Summed and differential harmonic oscillations in nonlinear vibratory systems," *Nagoya U. Faculty Eng. Mem.* 18, 85–150 (1966).

<sup>9</sup>W. Szemplinska-Stupnicka, "On the phenomenon of the combination type resonance in non-linear two-degree-of-freedom systems," *Int. J. Non-Linear Mech.* 4, 335–359 (1969).

<sup>10</sup>W. Szemplinska-Stupnicka, "On the asymptotic, averaging and Ritz method in the theory of steady-state vibrations of

nonlinear systems with many degrees of freedom," *Arch. Mech.* 22, 161–181 (1970).

<sup>11</sup>W. Szemplinska-Stupnicka, "On the averaging and W. Ritz methods in the theory of nonlinear resonances in vibrating systems with multiple degrees of freedom," *Arch. Mech.* 24, 67–88 (1972).

<sup>12</sup>W. Szemplinska-Stupnicka, "On the stability limit of non-linear resonances in multiple-degree-of-freedom vibrating systems," *Arch. Mech.* 25, 501–511 (1973).

<sup>13</sup>D. E. Newland, "On the method of Galerkin, Ritz and Krylov-Bogolyubov in the theory of nonlinear oscillations," *Inst. J. Mech. Sci.* 2, 159–172 (1965).

<sup>14</sup>J. A. Bennett and J. G. Eisley, "A multiple degree-of-freedom approach to nonlinear beam variations," *AIAA J.* 8, 734–739 (1970).

<sup>15</sup>J. A. Bennett, "Ultraharmonic motion of a viscously damped nonlinear beam," *AIAA J.* 11, 710–715 (1973).

<sup>16</sup>W.-Y. Tseng and J. Dugundji, "Nonlinear vibrations of a beam under harmonic excitation," *J. Appl. Mech.* 32, 292–297 (1970).

<sup>17</sup>W.-Y. Tseng and J. Dugundji, "Nonlinear vibrations of a buckled beam under harmonic excitation," *J. Appl. Mech.* 38, 467–476 (1971).

<sup>18</sup>R. M. Rosenberg, "On nonlinear vibrations of systems with many degrees of freedom," *Adv. Appl. Mech.* 9, 155–242 (1966).

<sup>19</sup>A. H. Nayfeh, *Perturbation Methods* (Wiley-Interscience, New York, 1973), Chap. 6.

<sup>20</sup>A. H. Nayfeh, D. T. Mook, and L. R. Marshall, "Nonlinear coupling of pitch and roll modes in ship motions," *J. Hydro-naut.* 7, 145–152 (1973).

<sup>21</sup>D. T. Mook, L. R. Marshall, and A. H. Nayfeh, "Subharmonic and superharmonic resonances in the pitch and roll modes of ship motions," *J. Hydronaut.* 8, 32–40 (1974).

<sup>22</sup>A. H. Nayfeh, D. T. Mook, and S. Srinadar, "Nonlinear analysis of the forced response of structural elements," *J. Acoust. Soc. Am.* 55, 281–291 (1974).

<sup>23</sup>A. H. Nayfeh, D. T. Mook, and D. W. Lobitz, "A numerical-perturbation method for the nonlinear analysis of structural vibrations," *AIAA J.* 12, 1222–1228 (1974).