

A theory for the optimal control of the far-field acoustic pressure radiating from submerged structures^{a)}

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This paper is concerned with the problem of suppressing the far-field acoustic pressure radiating from a structure interacting with fluid. In particular, it develops a modern control theory for the optimal control of the vibration of a submerged structure in a way that the far-field acoustic radiation pressure is minimized.

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INTRODUCTION

Vibrating structures tend to radiate acoustic pressure throughout the surrounding medium. At the same time, the surrounding medium exerts pressure on the structure, thus affecting its dynamic response. The magnitude of this interaction depends on the nature of the surrounding medium. Indeed, for a structure vibrating in air the interaction effect is negligible, so that the response characteristics of the structure are essentially the same as those of a structure vibrating *in vacuo*. On the other hand, the response characteristics of a structure vibrating in a relatively dense fluid can differ substantially from the characteristics *in vacuo*.¹

The acoustic radiation pressure tends to be significant if the interface area between the vibrating structure and the fluid is large, such as in the case of a plate. The vibrating plate generates an acoustic pressure wave in the fluid. Conversely, the fluid pressure exerts a load on the plate, where the load distribution depends on the plate acceleration at a given point. As a result, the coefficient of the acceleration term in the plate differential equation of motion changes. This causes coupling of the *in vacuo* plate modes, so that the eigenvalues and eigenfunctions of a plate interacting with fluid differ from the *in vacuo* eigenvalues and eigenfunctions.¹

In many cases, the acoustic radiation pressure is undesirable, particularly the far-field sound radiation pressure,^{1,2} so that the interest lies in suppressing it; this can be done by suppressing the plate vibration. A common approach is to suppress the plate vibration by passive means, which amounts to adding damping materials to the plate. As it turns out, some modes of vibration contribute more to the acoustic radiation pressure than others, so that it is only natural to attempt to suppress only the plate modes that contribute in a significant way to the acoustic radiation pressure. However, there exists no procedure capable of tailoring the passive damping so as to suppress selected modes only.

Moreover, the extent of suppression is likely to be insufficient in many applications. In view of this, a more attractive alternative may be to suppress the acoustic radiation pressure by means of active control, and in particular by means of feedback control.³

Although the idea of active noise control has been around for three decades, most of the work on the subject is of a more recent vintage. This work is concerned overwhelmingly with noise radiated by structures vibrating in air. One approach to the active noise control is based on the principle of superposition of two sound fields generated by independent sources. The idea is to suppress the sound field generated by a primary source by means of a secondary source.⁴⁻⁸ This is an open-loop approach³ that is likely to experience difficulties when the primary source of noise is of a complex nature, such as in the case of several primary sources, or a distributed source. In such cases, a number of secondary sources may be necessary. Another approach to noise control is to control the vibration of the structure producing the noise.⁹⁻¹⁵ In particular, in Refs. 14 and 15 modern control is used to design controls for the suppression of the far-field sound pressure radiating from a vibrating plate. The control is implemented by discrete actuators placed so as to control the modes most responsible for the far-field acoustic radiation pressure.

The situation is markedly different when the structure radiating the acoustic pressure vibrates in a fluid instead of air. As pointed out above, the structure-fluid interaction tends to couple the *in vacuo* modes of the structure.¹⁶ In this case, these *in vacuo* modes can be used as admissible functions for the purpose of spatial discretization and truncation of the distributed-parameter problem. As in Refs. 14 and 15, Ref. 16 develops a relation between the modes of vibration of the structure and the far-field sound radiation pressure permitting a decision as to the modes to be controlled.

This paper develops a general modern control theory for the optimal control of the far-field acoustic pressure radiating from structures submerged in fluid. One difficulty in Refs. 14-16 is the selection of the modes to be controlled so as to minimize the far-field acoustic radiation pressure. Indeed, in Refs. 14-16 this selection is made by trial and error.

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This problem is overcome here by including in the performance measure a term placing a penalty on the far-field radiation pressure, in addition to a penalty on the state of the structure and a penalty on the control effort. In this manner, the choice of modes to be controlled is made automatically in the process of minimizing the performance measure.

I. THE BOUNDARY-VALUE PROBLEM FOR A VIBRATING SUBMERGED STRUCTURE

We are concerned with the problem of controlling the sound pressure radiating from a vibrating elastic structure submerged in fluid. The vibration of the elastic structure is governed by the partial differential equation¹⁷

$$\mathcal{L}w(P,t) + m(P)\ddot{w}(P,t) = f(P,t), \quad P \in \Omega, \quad (1)$$

where $w(P,t)$ is the displacement at time t of a typical point P inside the domain Ω of the structure, \mathcal{L} a homogeneous self-adjoint differential operator of order $2p$, in which p is an integer, $m(P)$ the mass density, and $f(P,t)$ a distributed force. The solution $w(P,t)$ of Eq. (1) is subject to the boundary conditions

$$B_i w(P,t) = 0 \quad P \in \partial\Omega, \quad i = 1, 2, \dots, p, \quad (2)$$

in which B_i are homogeneous boundary differential operators of order ranging from zero to $2p - 1$ and $\partial\Omega$ is the boundary of Ω . The force density $f(P,t)$ arises from a variety of sources, so that it is natural to express it in the form

$$f(P,t) = f_c(P,t) + f_d(P,t) + f_p(P,t), \quad (3)$$

where $f_c(P,t)$ is a control force, $f_d(P,t)$ a persistent disturbing force, and $f_p(P,t)$ the pressure exerted by the fluid on the structure. For some of the control terminology, the reader is urged to consult Ref. 3.

II. THE BOUNDARY-VALUE PROBLEM FOR THE FLUID MOTION

The small-amplitude acoustic waves propagating through an ideal homogeneous compressible fluid must satisfy the wave equation¹

$$\nabla^2 p(P_f,t) = (1/c^2)\ddot{p}(P_f,t), \quad (4)$$

where $p(P_f,t)$ is the excess acoustic pressure (over the hydrostatic pressure) at a point P_f in the fluid, ∇^2 the three-dimensional Laplace operator, and $c = \sqrt{B/\rho}$ the sound velocity of the fluid, in which B is the fluid bulk modulus and ρ the fluid density. At the fluid-structure interface, the pressure must satisfy the boundary condition

$$-\frac{\partial p(P_f,t)}{\partial n} \Big|_{P_f=P} = \rho\ddot{w}(P,t), \quad (5)$$

where n is the normal to the structure. Of course, the force density exerted by the fluid on the structure is

$$f_p(P,t) = -p(P_f,t) \Big|_{P_f=P}. \quad (6)$$

III. THE PERFORMANCE MEASURE FOR OPTIMAL CONTROL

The object is to minimize the far-field acoustic radiation pressure $p(R,\theta,\phi,t)$, where R,θ,ϕ are spherical coordinates

(Fig. 1). This is to be achieved by the control force $f_c(P,t)$ acting on the structure. We are interested in optimal control, in the sense that it minimizes the performance measure

$$J = \frac{1}{2} \int_{\Omega} [h_1(P)w^2(P,t_f) + h_2(P)\dot{w}^2(P,t_f)] d\Omega + \frac{1}{2} \int_{t_i}^{t_f} \left(\int_{\Omega} [w(P,t)\mathcal{L}w(P,t) + m(P)\dot{w}^2(P,t) + a(P)f_c^2(P,t)] d\Omega + \int_{\Omega_f} b(R,\theta,\phi)p^2(R,\theta,\phi,t) d\Omega_f \right) dt, \quad (7)$$

where $h_1(P), h_2(P), a(P)$, and $b(R,\theta,\phi)$ are weighting functions and Ω_f is the region over which the far-field acoustic radiation pressure is to be minimized; t_i is the initial time and t_f is the final time. Various control objectives can be achieved by changing the weighting functions.

IV. RELATION BETWEEN THE ELASTIC VIBRATION AND THE FLUID PRESSURE FOR A RECTANGULAR PLATE

The problem of designing controls for the suppression of the far-field acoustic pressure radiating from an arbitrary vibrating structure is extremely difficult when the structure interacts with a fluid. The basic approach is to use Eqs. (4) and (5) to develop a relation between the pressure at any point in the fluid and the vibration of the structure. The complexity of the problem can be reduced to some extent by considering a structure in the form of a rectangular plate interfacing with the fluid. In the case of transient vibration of the plate, the relation between the pressure at a point in the fluid and the vibration of a point on the plate involves a convolution integral in the time domain, which makes the control problem intractable. Fortunately, the situation is markedly better in the steady-state case, such as the case in which the plate vibrates harmonically. In this case, the relation can be written in the general form

$$p(P_f,t) = \int_{\Omega} F_p(P_f,P')\ddot{w}(P',t)d\Omega(P'), \quad (8)$$

where $F_p(P_f,P')$ is an influence function representing the pressure at P_f due to a unit acceleration of point P' on the

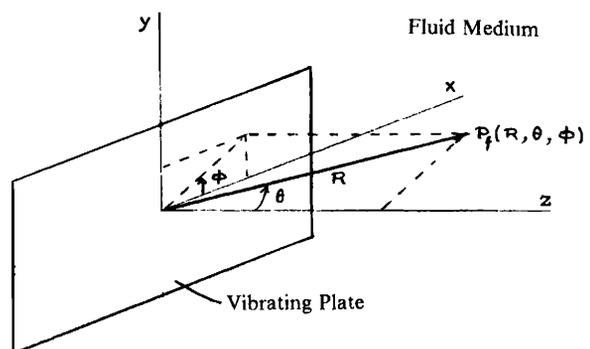


FIG. 1. A plate vibrating in a fluid medium.

plate. Then, from Eq. (6), the pressure loading at point P on the plate is simply

$$f_p(P,t) = -p(P_f,t)|_{P_f=P} \\ = - \int_{\Omega} F_p(P,P') \ddot{w}(P',t) d\Omega(P') \quad (9)$$

and the far-field acoustic radiation pressure has the expression

$$p(R,\theta,\phi,t) = \int_{\Omega} F_p(R,\theta,\phi,P') \ddot{w}(P',t) d\Omega(P'). \quad (10)$$

In the case in which the acoustic pressure radiates from a rectangular plate vibrating harmonically, the three-dimensional wave equation, Eq. (4), reduces to the Helmholtz equation

$$(\nabla^2 + k^2)p(P_f) = 0, \quad (11)$$

where $p(P_f)$ is the pressure amplitude and

$$k = \omega/c \quad (12)$$

is the acoustic wave number, in which ω is the frequency of the harmonic oscillation. Equation (11) is subject to the boundary condition

$$-\frac{\partial p(P_f)}{\partial z} \Big|_{P_f=P} = \rho \ddot{w}(P), \quad (13)$$

where $\ddot{w}(P)$ is the acceleration amplitude.

The problem of determining the influence function $F_p(P_f,P')$ can be treated conveniently by means of a double Fourier transformation. For any function $f(x,y)$, the Fourier transform pair is defined as

$$\tilde{f}(\gamma_x,\gamma_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i(\gamma_x x + \gamma_y y)} dx dy, \quad (14a)$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\gamma_x,\gamma_y) e^{i(\gamma_x x + \gamma_y y)} d\gamma_x d\gamma_y. \quad (14b)$$

Then, letting x,y,z be a set of Cartesian coordinates attached to the plate (Fig. 1), where x and y are in the plane of the plate and coincident with the symmetry axes and z is normal to the plate, and Fourier transforming both sides of Eq. (13), we can write

$$-\frac{\partial \tilde{p}(\gamma_x,\gamma_y)}{\partial z} \Big|_{z=0} = \rho \tilde{\ddot{w}}(\gamma_x,\gamma_y). \quad (15)$$

Solving Eq. (11) in conjunction with Eq. (13), the acoustic radiation pressure can be shown to have the form of the inverse Fourier transform¹

$$p(x,y,z) = \frac{i\rho}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{\ddot{w}}(\gamma_x,\gamma_y) \exp[i(\gamma_x x + \gamma_y y) + i(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2} z]}{(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2}} d\gamma_x d\gamma_y, \quad (16)$$

so that the amplitude of the pressure exerted by the fluid on the structure is

$$f_p(x,y) \\ = -p(x,y,0) \\ = -\frac{i\rho}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{\ddot{w}}(\gamma_x,\gamma_y) \exp[i(\gamma_x x + \gamma_y y)]}{(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2}} \\ \times d\gamma_x d\gamma_y. \quad (17)$$

Another relation of interest is that between the elastic vibration and the far-field sound radiation pressure. Note that far field is defined as a region in the fluid in which the relationships between pressure, fluid velocity, and sound intensity approach those for a plane wave.^{1,2} For rectangular planar radiators, the far-field sound radiation pressure is given by Rayleigh's formula¹

$$p(R,\theta,\phi) = \frac{\rho e^{ikR}}{2\pi R} \int_{-b}^b \int_{-a}^a \ddot{w}(x,y) \\ \times \exp[-ik \sin \theta (x \cos \phi + y \sin \phi)] dx dy, \quad (18)$$

where R,θ,ϕ are spherical coordinates.

V. THE OPEN-LOOP, *IN VACUO* EIGENVALUE PROBLEM

As pointed out in Sec. III, our interest lies in suppressing the far-field acoustic radiation pressure. The state of the art

does not permit a solution of the problem in terms of partial differential equations, so that spatial discretization is a virtual necessity. Moreover, the far-field acoustic pressure depends on the modes of vibration of the structure, so that it appears desirable to reformulate the problem in terms of modes. To this end, we consider the open-loop, *in vacuo* modes of vibration of the structure, obtained by setting $f(P,t) \equiv 0$ in Eq. (1), and using the method of separation of variables to eliminate the time dependence.¹⁷ Although the modes thus obtained are not the modes for the problem at hand, they do represent a complete set of admissible functions¹⁷ that can be used to reduce the partial differential equation to a set of ordinary differential equations.

Following the procedure described above, we recall that for a rectangular plate $P = x,y$ and assume a solution of Eq. (1) with $f \equiv 0$ in the form

$$w(x,y,t) = W(x,y)F(t). \quad (19)$$

Then, separating variables, we conclude that $F(t)$ is harmonic, so that $\ddot{F}(t) = -\omega^2 F(t)$, and that $W(x,y)$ must satisfy the differential equation

$$\mathcal{L}W(x,y) = \lambda m(x,y)W(x,y), \quad \lambda = \omega^2, \\ -a < x < a, \quad -b < y < b. \quad (20)$$

Moreover, inserting Eq. (19) into Eqs. (2) and eliminating $F(t)$, we obtain the boundary conditions

$$B_i W(x,y) = 0, \quad -a < x < a, \quad y = \pm b$$

$$\text{or } -b < y < b, \quad x = \pm a, \quad i = 1, 2. \quad (21)$$

We refer to the problem described by Eqs. (20) and (21) as the open-loop, *in vacuo* eigenvalue problem, or simply as the open-loop eigenvalue problem.

Because the operator \mathcal{L} is self-adjoint, the solution of the eigenvalue problem consists of a denumerably infinite set of real eigenvalues λ_r and eigenfunctions $W_r(x, y)$ ($r = 1, 2, \dots$). For a positive definite operator \mathcal{L} , all $\lambda_r = \omega_r^2$ are positive, where ω_r are the natural frequencies of the plate. If \mathcal{L} is only positive semidefinite, then the eigenvalue problem admits rigid-body modes with zero natural frequencies. Moreover, the eigenfunctions are orthogonal with respect to m and \mathcal{L} and can be normalized so as to satisfy

$$\int_{-b}^b \int_{-a}^a m(x, y) W_r(x, y) W_s(x, y) dx dy = \delta_{rs}, \quad r, s = 1, 2, \dots, \quad (22a)$$

$$\int_{-b}^b \int_{-a}^a W_s(x, y) \mathcal{L} W_r(x, y) dx dy = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots, \quad (22b)$$

where δ_{rs} is the Kronecker delta. Note that, because \mathcal{L} is self-adjoint, the subscripts r and s can be interchanged in Eqs. (22b) without affecting the results.

VI. PROBLEM DISCRETIZATION

In Sec. V, we made the point that the design of controls for suppression of the far-field acoustic radiation pressure based on partial differential equations is not within the state of the art, so that the only viable alternative is spatial discretization. This amounts to representing the solution as a linear combination of the open-loop modes of vibration. Hence, let us assume that the plate displacement can be expressed in the form of the series

$$w(x, y, t) = \sum_{r=1}^{\infty} W_r(x, y) q_r(t), \quad (23)$$

where $W_r(x, y)$ are the open-loop modes introduced in Sec. V and $q_r(t)$ are time-dependent generalized coordinates, referred to as “modal” coordinates. Inserting Eq. (23) into Eq. (1) (with P replaced by x, y), multiplying through by $W_s(x, y)$, integrating over the plate, and considering Eq. (3), as well as Eqs. (22), we obtain the “modal” equations

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = f_{cr}(t) + f_{dr}(t) + f_{pr}(t), \quad r = 1, 2, \dots, \quad (24)$$

in which

$$f_{jr}(t) = \int_{-b}^b \int_{-a}^a W_r(x, y) f_j(x, y, t) dx dy, \quad j = c, d, p; \quad r = 1, 2, \dots \quad (25)$$

represent “modal” forces. It must be stressed here that, although they have the appearance of an independent set, Eqs. (24) are coupled through the terms $f_{cr}(t)$ and $f_{pr}(t)$, which depend in general on all the modal coordinates.

At this point, we wish to derive an explicit expression for the modal forces due to fluid pressure. To this end, we recall Eq. (9) and write

$$\begin{aligned} f_p(x, y, t) &= - \int_{-b}^b \int_{-a}^a F_p(x, y, x', y') \ddot{w}(x', y', t) dx' dy' \\ &= - \sum_{s=1}^{\infty} \int_{-b}^b \int_{-a}^a F_p(x, y, x', y') \\ &\quad \times W_s(x', y') dx' dy' \ddot{q}_s(t). \end{aligned} \quad (26)$$

Inserting Eq. (26) into Eqs. (25) with $j = p$, we obtain

$$\begin{aligned} f_{pr}(t) &= \int_{-b}^b \int_{-a}^a W_r(x, y) f_p(x, y, t) dx dy \\ &= \int_{-b}^b \int_{-a}^a W_r(x, y) \left(- \sum_{s=1}^{\infty} \int_{-b}^b \int_{-a}^a F_b(x, y, x', y') \right. \\ &\quad \left. \times W_s(x', y') dx' dy' \right) dx dy \ddot{q}_s(t) \\ &= - \sum_{s=1}^{\infty} c_{rs} \ddot{q}_s(t), \quad r = 1, 2, \dots, \end{aligned} \quad (27)$$

where

$$c_{rs} = \int_{-b}^b \int_{-a}^a W_r(x, y) \left(\int_{-b}^b \int_{-a}^a F_p(x, y, x', y') \times W_s(x', y') dx' dy' \right) dx dy, \quad r, s = 1, 2, \dots \quad (28)$$

Next, we write Eqs. (24) in the matrix form

$$\ddot{\mathbf{q}}(t) + \Lambda \mathbf{q}(t) = \mathbf{f}_c(t) + \mathbf{f}_d(t) + \mathbf{f}_p(t) \quad (29)$$

in which $\Lambda = \text{diag}[\omega_1^2, \omega_2^2, \dots]$; the notation for the various vectors is obvious. In particular, using Eqs. (27) the modal pressure vector can be written as

$$\mathbf{f}_p(t) = -C \ddot{\mathbf{q}}(t), \quad (30)$$

where $C = [c_{rs}]$ is the matrix with the entries given by Eqs. (28). In view of Eq. (30), Eq. (29) can be rewritten in the form

$$(I + C) \ddot{\mathbf{q}}(t) + \Lambda \mathbf{q}(t) = \mathbf{f}_c(t) + \mathbf{f}_d(t). \quad (31)$$

VII. TRUNCATED EQUATIONS FOR CONTROL

Equation (31) represents an infinite set of ordinary differential equations. The design of feedback control for an infinite-dimensional system is not feasible, nor is it necessary. Indeed, in practice only a finite number of modes are excited, so that we propose to truncate the system. To this end, we write the modal displacement vector in the form

$$\mathbf{q}(t) = [\mathbf{q}_C^T(t) \quad \mathbf{q}_R^T(t)]^T, \quad (32)$$

where $\mathbf{q}_C(t)$ is an n -dimensional vector of controlled modal displacements and $\mathbf{q}_R(t)$ is an infinite-dimensional vector of uncontrolled, or residual modal displacements. Moreover, we assume that the coupling between the controlled and residual modes tends to disappear as the number of controlled modes increases, so that we can write

$$C = \text{block-diag}[C_C \quad C_R], \quad (33)$$

where the notation is obvious. This assumption tends to be true in the case of harmonic excitation of the submerged plate.¹ Then, if we introduce the notation

$$\Lambda = \text{block-diag}[\Lambda_C \quad \Lambda_R] \quad (34)$$

Eq. (31) can be separated into an equation for the controlled modes and another equation for the residual modes, or

$$[I + C_C] \ddot{\mathbf{q}}_C(t) + \Lambda_C \mathbf{q}_C(t) = \mathbf{f}_{cC}(t) + \mathbf{f}_{dC}(t), \quad (35a)$$

$$[I + C_R] \ddot{\mathbf{q}}_R(t) + \Lambda_R \mathbf{q}_R(t) = \mathbf{f}_{cR}(t) + \mathbf{f}_{dR}(t). \quad (35b)$$

We propose to carry out the control by means of point actuators. Point actuators can be treated as distributed by writing

$$f_c(x, y, t) = \sum_{k=1}^N F_{ck}(t) \delta(x - x_k, y - y_k), \quad (36)$$

where N is the number of actuators, F_{ck} are force amplitudes, and $\delta(x - x_k, y - y_k)$ are spatial Dirac delta functions.³ Inserting Eq. (36) into the corresponding ones in Eqs. (25) and separating the controlled and the residual modal forces, we obtain

$$\mathbf{f}_{cC}(t) = B'_{cC} \mathbf{F}_c(t), \quad (37a)$$

$$\mathbf{f}_{cR}(t) = B'_{cR} \mathbf{F}_c(t), \quad (37b)$$

where

$$B'_{cC} = [B'_{crk}] = [W_r(x_k, y_k)], \quad (38a)$$

$$r = 1, 2, \dots, n; \quad k = 1, 2, \dots, N,$$

$$B'_{cR} = [B'_{crk}] = [W_r(x_k, y_k)], \quad (38b)$$

$$r = n + 1, n + 2, \dots; \quad k = 1, 2, \dots, N.$$

Introducing Eqs. (37) into Eqs. (35), we have

$$(I + C_C) \ddot{\mathbf{q}}_C(t) + \Lambda_C \mathbf{q}_C(t) = B'_{cC} \mathbf{F}_c(t) + \mathbf{f}_{dC}(t), \quad (39a)$$

$$(I + C_R) \ddot{\mathbf{q}}_R(t) + \Lambda_R \mathbf{q}_R(t) = B'_{cR} \mathbf{F}_c(t) + \mathbf{f}_{dR}(t). \quad (39b)$$

VIII. OPTIMAL FEEDBACK CONTROL

At this point, we turn our attention to the question of designing controls so as to suppress the far-field acoustic radiation pressure. As pointed out in Sec. III, the object is to suppress the sound pressure in an optimal fashion. To generate the necessary control forces, we must cast Eq. (39a) in state form. To this end, we introduce the controlled state vector

$$\mathbf{x}(t) = [\mathbf{q}_C^T(t) \quad \dot{\mathbf{q}}_C^T(t)]^T. \quad (40)$$

Then, adjoining the identity $\dot{\mathbf{q}}_C(t) \equiv \dot{\mathbf{q}}_C(t)$, Eq. (39a) can be written in the standard state form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{F}_c(t) + D\mathbf{f}_{dC}(t), \quad (41)$$

where

$$A = \begin{bmatrix} 0 & I \\ -(I + C_C)^{-1} \Lambda_C & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ (I + C_C)^{-1} B'_{cC} \end{bmatrix}, \quad (42)$$

$$D = \begin{bmatrix} 0 \\ (I + C_C)^{-1} \end{bmatrix}.$$

Before optimal controls can be determined, it is necessary to discretize the performance measure, Eq. (7). To this end, we refer to Secs. VI and VII retain only the effect of the controlled modes, and rewrite Eq. (23) in the truncated form

$$w(P, t) \cong \sum_{i=1}^n W_i(P) q_i(t) = \mathbf{W}_C^T(P) \mathbf{q}_C(t), \quad (43)$$

in which $\mathbf{W}_C(P) = [W_1(P) W_2(P) \cdots W_n(P)]^T$ is a vector of open-loop eigenfunctions corresponding to the controlled modes, where in the case at hand $P = x, y$. Hence, retaining the effect of the controlled modes only, the first integral in Eq. (7) can be approximated by

$$\int_{\Omega} [h_1(P) w^2(P, t_f) + h_2(P) \dot{w}^2(P, t_f)] d\Omega \cong \mathbf{x}^T(t_f) H \mathbf{x}(t_f), \quad (44)$$

where

$$H = \text{block-diag} \left(\int_{\Omega} h_1(P) \mathbf{W}_C(P) \mathbf{W}_C^T(P) d\Omega \right. \\ \left. \int_{\Omega} h_2(P) \mathbf{W}_C(P) \mathbf{W}_C^T(P) d\Omega \right). \quad (45)$$

Moreover,

$$\int_{\Omega} [w(P, t) \mathcal{L} w(P, t) + m(P) \dot{w}^2(P, t)] d\Omega \cong \mathbf{x}^T(t) Q \mathbf{x}(t), \quad (46)$$

where

$$Q^* = \begin{bmatrix} \Lambda_C & 0 \\ 0 & I \end{bmatrix} \quad (47)$$

and, replacing x_k, y_k by P_k in Eq. (36), we can write

$$\int_{\Omega} a(P) f_c^2(P, t) d\Omega = \mathbf{F}_c^T(t) R^* \mathbf{F}_c(t), \quad (48)$$

where

$$R^* = \text{diag}[a(P_k)]. \quad (49)$$

Finally, for the far-field acoustic radiation term in Eq. (7), we refer to Eq. (10) and write

$$p(R, \theta, \phi, t) = \mathbf{P}^T(R, \theta, \phi) \dot{\mathbf{q}}_C(t), \quad (50)$$

where

$$\mathbf{P}(R, \theta, \phi) \cong \int_{\Omega} F_p(R, \theta, \phi, P') \mathbf{W}_C(P') d\Omega(P'). \quad (51)$$

Then, letting

$$\bar{P} = \int_{\Omega_f} b(R, \theta, \phi) \mathbf{P}(R, \theta, \phi) \mathbf{P}^T(R, \theta, \phi) d\Omega_f, \quad (52)$$

we can write

$$\int_{\Omega_f} b(R, \theta, \phi) p^2(R, \theta, \phi, t) dW_f \cong \dot{\mathbf{x}}^T(t) P^* \dot{\mathbf{x}}(t), \quad (53)$$

where

$$P^* = \begin{bmatrix} 0 & 0 \\ 0 & \bar{P} \end{bmatrix}. \quad (54)$$

Note that evaluation of the matrix \bar{P} , Eq. (52), represents a very difficult problem, and in all likelihood the integration will have to be carried out numerically. Inserting Eqs. (44), (46), (48), and (53) into Eq. (7), we obtain the discretized performance measure

$$J = \frac{1}{2} \mathbf{x}^T(t_f) H \mathbf{x}(t_f) \\ + \frac{1}{2} \int_{t_i}^{t_f} (\mathbf{x}^T Q^* \mathbf{x} + \mathbf{F}_c^T R^* \mathbf{F}_c + \dot{\mathbf{x}}^T P^* \dot{\mathbf{x}}) dt. \quad (55)$$

The performance measure given by Eq. (55) contains the time derivative of the state vector and the optimal control formalism is in terms of the state vector and not its time derivative. To resolve this slight inconvenience, we insert the state equations, Eq. (41), into Eq. (55) and obtain the performance measure

$$J = \frac{1}{2} \mathbf{x}^T(t_f) H \mathbf{x}(t_f) + \frac{1}{2} \int_{t_i}^{t_f} (\mathbf{x}^T Q \mathbf{x} + \mathbf{F}_c^T R \mathbf{F}_c + \mathbf{f}_{dc}^T S \mathbf{f}_{dc} + \mathbf{x}^T T \mathbf{F}_c + \mathbf{x}^T U \mathbf{f}_{dc} + \mathbf{F}_c^T V \mathbf{f}_{dc}) dt, \quad (56)$$

where

$$Q = Q^* + A^T P^* A, \quad R = R^* + B^T P^* B, \quad S = D^T P^* D, \\ T = 2A^T P^* B, \quad U = 2A^T P^* D, \quad V = 2B^T P^* D. \quad (57)$$

To derive the optimal control law, we introduce the Hamiltonian³

$$\mathcal{H} = \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \frac{1}{2} \mathbf{F}_c^T R \mathbf{F}_c + \frac{1}{2} \mathbf{f}_{dc}^T S \mathbf{f}_{dc} + \mathbf{x}^T T \mathbf{F}_c + \mathbf{x}^T U \mathbf{f}_{dc} \\ + \mathbf{F}_c^T V \mathbf{f}_{dc} + \mathbf{p}^T (A \mathbf{x} + B \mathbf{F}_c + D \mathbf{f}_{dc}), \quad (58)$$

where \mathbf{p} is the costate vector. Then, we have the relations³

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = A \mathbf{x} + B \mathbf{F}_c + D \mathbf{f}_{dc} \quad (59a)$$

$$\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = -(Q \mathbf{x} + T \mathbf{F}_c + U \mathbf{f}_{dc} + A^T \mathbf{p}) \quad (59b)$$

$$\frac{\partial \mathcal{H}}{\partial \mathbf{F}_c} = R \mathbf{F}_c + T^T \mathbf{x} + V \mathbf{f}_{dc} + B^T \mathbf{p} = \mathbf{0}. \quad (59c)$$

From Eq. (59c), we obtain the control law

$$\mathbf{F}_c = -R^{-1} (T^T \mathbf{x} + V \mathbf{f}_{dc} + B^T \mathbf{p}). \quad (60)$$

Equation (60) contains the costate vector \mathbf{p} , which can be eliminated by assuming that

$$\mathbf{p} = K \mathbf{x} + \mathbf{v}, \quad (61)$$

where K is a matrix and \mathbf{v} a vector, both still to be determined. Then, combining Eqs. (59a), (59b), (60), and (61), we obtain

$$\dot{\mathbf{p}} = \dot{K} \mathbf{x} + K \dot{\mathbf{x}} + \dot{\mathbf{v}} \\ = (\dot{K} + KA - KBR^{-1}T^T - KBR^{-1}B^TK) \mathbf{x} \\ + \dot{K}(D - BR^{-1}V) \mathbf{f}_{dc} - KBR^{-1}B^T \mathbf{v} + \dot{\mathbf{v}} \\ = -(Q - TR^{-1}T^T + TR^{-1}K + A^TK) \mathbf{x} \\ + (TR^{-1}V - U) \mathbf{f}_{dc} + (TR^{-1}B^T - A^T) \mathbf{v}. \quad (62)$$

To satisfy Eq. (62), we choose K and \mathbf{v} so that

$$\dot{K} = -(Q - TR^{-1}T^T) - (A^T - TR^{-1}B^T)K \\ - K(A - BR^{-1}T^T) \\ + KBR^{-1}B^TK = 0, \quad K(t^f) = H \quad (63)$$

and

$$\dot{\mathbf{v}} = (TR^{-1}B^T - A^T + KBR^{-1}B^T) \mathbf{v} \\ + [TR^{-1}V - U - K(D - BR^{-1}V)] \mathbf{f}_{dc}, \\ \mathbf{v}(t_f) = \mathbf{0}. \quad (64)$$

Equation (63) is a nonlinear equation known as the matrix differential Riccati equation. The equation can be inte-

grated backward in time to obtain $K(t)$. Instead of solving a nonlinear matrix differential equation, it is possible to transform the problem into a linear one of twice the order.³ Moreover, if the system is controllable, $H = 0$ and A, B, Q, R , and T are constant, the Riccati matrix $K(t)$ tends to a constant matrix as t_f increases. Hence, if the control time t_f is reasonably large, the matrix differential Riccati equation, Eq. (63), reduces to a matrix algebraic Riccati equation, which can be solved by Potter's algorithm.³

Before we can complete the determination of the control law, we must obtain the costate vector \mathbf{p} , which according to Eq. (61) involves the vector \mathbf{v} , in addition to the matrix K . To produce the vector \mathbf{v} , we must solve Eq. (64). In the case in which the Riccati matrix depends on time, $K = K(t)$, Eq. (64) represents a time-varying system, and its solution can be obtained in discrete time.³ If K is constant, then Eq. (64) represents a time-invariant system and its solution has the form³

$$\mathbf{v}(t) = \int_{t_f}^t \Phi_{\mathbf{v}}(t - \tau) [TR^{-1}V - U - K(D - BR^{-1}V)] \\ \times \mathbf{f}_{dc}(\tau) d\tau, \quad (65)$$

where

$$\Phi_{\mathbf{v}}(t) = \exp(TR^{-1}B^T - A^T + KBR^{-1}B^T)t \quad (66)$$

is the transition matrix for system (64).

Finally, inserting Eq. (61) into Eq. (60), we obtain the control law

$$\mathbf{F}_c = -R^{-1} [(T^T + B^TK) \mathbf{x} + V \mathbf{f}_{dc} + B^T \mathbf{v}], \quad (67)$$

where \mathbf{v} is a function of \mathbf{f}_{dc} , as indicated by Eq. (65). Note that the control law given by Eq. (67) includes both feedback control of transient disturbances and control of the persistent disturbances.

IX. CLOSED-LOOP EQUATION AND SYSTEM RESPONSE

The closed-loop equation describing the motion of the plate interacting with the fluid is obtained by inserting Eq. (67) into Eq. (41). The result can be written in the form

$$\dot{\mathbf{x}}(t) = A_c \mathbf{x}(t) + (D - BR^{-1}V) \mathbf{f}_{dc}(t) - BR^{-1}B^T \mathbf{v}(t), \quad (68)$$

where

$$A_c = A - BG \quad (69)$$

is the closed-loop coefficient matrix, in which

$$G = R^{-1}(T^T + B^TK) \quad (70)$$

is the control gain matrix. Moreover, $\mathbf{v}(t)$ is the solution of Eq. (64).

Equation (68) is similar to Eq. (64), except that here the initial condition is not zero. Hence, once again we must distinguish between the case in which K depends on time and that in which K is constant. If K depends on time, the system is time-varying, and the solution can be obtained in discrete time. If K is constant, then A_c is constant, and a solution of Eq. (68) can be obtained in the closed form³

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau) [(D - BR^{-1}V) \times \mathbf{f}_{dc}(\tau) - BR^{-1}B^T \mathbf{v}(\tau)] d\tau, \quad (71)$$

where $\mathbf{x}(0)$ is the initial state and

$$\Phi(t) = e^{At} \quad (72)$$

is the state transition matrix.

X. SUMMARY AND CONCLUSIONS

A problem of current interest consists of the suppression of the far-field acoustic pressure radiated by a structure vibrating in a fluid. The interaction between the structure and fluid results in coupling of the *in vacuo* vibration modes of the structure, altering the eigenvalues and eigenfunctions of the structure. In suppressing the far-field acoustic radiation pressure, active control offers many advantages over passive control. In particular, it permits tailoring of the feedback controls so as to concentrate the control on the vibration modes contributing the most to the far-field acoustic radiation pressure.

This paper develops a general modern control theory for the optimal control of the far-field acoustic pressure radiating from a structure submerged in fluid. The theory considers simultaneously the boundary-value problem for the structure, the interaction between structure and fluid, and the control design. The control is designed in an optimal fashion by including in the performance measure a penalty on the far-field acoustic radiation pressure, in addition to the customary penalty on the state and on the control effort. The distributed-parameter problem is discretized in space by expanding the displacement of the structure in a series of admissible functions in the form of the *in vacuo* modes of the structure.

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