Sound waves in two-dimensional ducts with sinusoidal walls

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(Received 7 March 1974; revised 21 April 1974)

The method of multiple scales is used to analyze the wave propagation in two-dimensional hard-walled ducts with sinusoidal walls. For traveling waves, resonance occurs whenever the wall wavenumber is equal to the difference of the wavenumbers of any two duct acoustic modes. The results show that neither of these resonating modes could occur without strongly generating the other.

Subject Classification: 20.45, 20.50.

INTRODUCTION

We consider acoustic waves propagating in an inviscid, nonconducting, perfect gas confined in a two-dimensional hard-walled duct having a nonuniform cross section. We assume the walls to have weak sinusoidal undulations. We make lengths, velocities, and time dimensionless using the average width of the duct d, the undisturbed speed of sound c, and the characteristic time d/c. In dimensionless quantities, the walls of the duct are located by

$$y=\epsilon\sin k_w x\ , \qquad \text{lower wall}\ , \tag{1a}$$

$$y=1+\epsilon\sin(k_w x+\theta)\ , \qquad \text{upper wall}\ , \tag{1b}$$

where ϵ is a small dimensionless parameter characterizing the weakness of the wall undulations, k_w is the wavenumber of the undulations, and θ is the phase difference between the undulations of the two walls.

Since the gas is assumed to be inviscid and nonconducting, its irrotational motion can be described by a dimensionless potential function $\tilde{\phi}(x, y, t)$. For linear motions and harmonic time variations of the form $\tilde{\phi}(x, y, t)=\phi(x,y)\exp(-i\omega t)$, ϕ satisfies

$$\nabla^2\phi+\omega^2\phi=0\ , \tag{2}$$

where ω is the dimensionless frequency of oscillation. For an inviscid fluid and a hard wall, the flow is tangential to the walls; that is,

$$\phi_y=\epsilon\phi_x k_w\cos k_w x\ , \qquad \text{at } y=\epsilon\sin k_w x\ , \tag{3}$$

$$\phi_y=\epsilon\phi_x k_w\cos(k_w x+\theta)\ , \qquad \text{at } y=1+\epsilon\sin(k_w x+\theta)\ . \tag{4}$$

Straightforward expansions of the form $\phi_0+\epsilon\phi_1$ were obtained for the solutions of this problem by Isakovitch[1] for the case of a waveguide with only one undulating wall, by Samuels[2] for the case of a waveguide with in-phase wall undulations (i.e., $\theta=0$), and by Salant[3] for the above general problem. Unfortunately, all the above expansions are not uniform because the correction $\epsilon\phi_1$ dominates the first term ϕ_0 for frequencies near what Isakovitch, Samuels, and Salant call the resonant frequencies. In this paper, we determine a uniform expansion by using the method of multiple scales.[4] Before determining this uniform expansion, we carry out a straightforward expansion in the next section to exhibit the nonuniformity.

I. A STRAIGHTFORWARD EXPANSION

In this section, we follow Salant and seek an expansion of the form

$$\phi(x,y)=\phi_0(x,y)+\epsilon\phi_1(x,y)+\cdots\ . \tag{5}$$

Substituting Eq. 5 into Eqs. 2–4, transferring the boundary conditions from $y=\epsilon\sin k_w x$ and $y=1+\epsilon\sin(k_w x+\theta)$ to $y=0$ and $y=1$ by developing ϕ_x and ϕ_y in Taylor series expansions, expanding for small ϵ, and equating coefficients of like powers of ϵ, we obtain

Order ϵ^0

$$\nabla^2\phi_0+\omega^2\phi_0=0\ , \tag{6}$$

$$\phi_{0y}=0\ , \quad \text{at } y=0\ , \tag{7}$$

$$\phi_{0y}=0\ , \quad \text{at } y=1\ ; \tag{8}$$

Order ϵ

$$\nabla^2\phi_1+\omega^2\phi_1=0\ , \tag{9}$$

$$\phi_{1y}=\phi_{0x}k_w\cos k_w x-\phi_{0yy}\sin k_w x\ , \quad \text{at } y=0\ , \tag{10}$$

$$\phi_{1y}=\phi_{0x}k_w\cos(k_w x+\theta)-\phi_{0yy}\sin(k_w x+\theta)\ , \quad \text{at } y=1\ . \tag{11}$$

The solution of Eqs. 6–8 is taken to be

$$\phi_0=A\cos n\pi y\exp(ik_n x)\ , \tag{12}$$

where

$$k_n^2=\omega^2-n^2\pi^2\ , \quad n=0,1,2,\cdots\ . \tag{13}$$

For traveling waves, $n\pi$ must be less than ω.

Substituting for ϕ_0 into Eqs. 10 and 11, we have

$$\phi_{1y}=\tfrac{1}{2}iA(k_nk_w-n^2\pi^2)\exp[i(k_n+k_w)x] + \tfrac{1}{2}iA(k_nk_w+n^2\pi^2)\exp[i(k_n-k_w)x]\ , \quad \text{at } y=0\ , \tag{14}$$

$$\phi_{1y}=\tfrac{1}{2}iA(k_nk_w-n^2\pi^2)\cos n\pi\exp[i(k_n+k_w)x+i\theta] + \tfrac{1}{2}iA(k_nk_w+n^2\pi^2)\cos n\pi\exp[i(k_n-k_w)x-i\theta]\ , \quad \text{at } y=1\ . \tag{15}$$

We seek a particular solution to Eqs. 9, 14, and 15 of the form

$$\phi_1=\tfrac{1}{2}iA(k_nk_w-n^2\pi^2)\Phi_1(y)\exp[i(k_n+k_w)x] + \tfrac{1}{2}iA(k_nk_w+n^2\pi^2)\Phi_2(y)\exp[i(k_n-k_w)x]\ . \tag{16}$$

Substituting Eq. 16 into Eqs. 9, 14, and 15 and equating the coefficients of $\exp[i(k_n\pm k_w)x]$, we obtain

$$\Phi_1'' + \alpha_1^2\Phi_1 = 0 , \quad \alpha_1^2 = \omega^2 - (k_n + k_w)^2 , \tag{17a}$$

$$\Phi_1'(0) = 1 , \tag{17b}$$

$$\Phi_1'(1) = \cos n\pi \exp(i\theta) ; \tag{17c}$$

$$\Phi_2'' + \alpha_2^2\Phi_2 = 0 , \quad \alpha_2^2 = \omega^2 - (k_n - k_w)^2 , \tag{18a}$$

$$\Phi_2'(0) = 1 , \tag{18b}$$

$$\Phi_2'(1) = \cos n\pi \exp(-i\theta) . \tag{18c}$$

The solutions of Eqs. 17 and 18 are

$$\Phi_j = (\alpha_j \sin\alpha_j)^{-1}\{\sin\alpha_j \sin\alpha_j y + [\cos\alpha_j - \cos n\pi \times \exp(-i\theta\cos\pi j)]\cos\alpha_j y\} , \quad j = 1 \text{ and } 2 . \tag{19}$$

The present expansion is in agreement with that obtained by Salant.

The functions $\Phi_j \to \infty$, and hence, $\phi_1 \to \infty$ as $\alpha_j \to m\pi$, which corresponds to the resonant frequencies

$$\omega_r^2 = (k_n \pm k_w)^2 + m^2\pi^2 , \quad \text{with integer } m . \tag{20}$$

Hence, the above straightforward expansion is not valid when $\omega \to \omega_r$. An expansion valid near the resonant frequencies is obtained in the next section by using the method of multiple scales. Since $k_m^2 = \omega^2 - m^2\pi^2$ from Eq. 13, the resonant frequencies occur whenever

$$k_w = k_n \pm k_m . \tag{21}$$

Note that, for the special case $\theta = 0$, m takes on all odd values when n is even and m takes on all even values when n is odd. For the special case $\theta = \pi$, $m - n$ is even. For $\theta \neq 0$ and π, m takes on all integer values. The resonant case $k_w = k_n + k_m$ occurs only for standing waves, while the other case occurs for both standing and traveling waves. In this paper, we consider the traveling case only.

II. EXPANSIONS VALID NEAR RESONANT FREQUENCIES

In this section, we determine uniform expansions for the resonant case $k_w \approx k_n - k_m$. To this end, we seek asymptotic expansions to the solutions of Eqs. 2–4 of the form

$$\phi(x, y) = \phi_0(x_0, x_1, y) + \epsilon\phi_1(x_0, x_1, y) + \cdots , \tag{22}$$

where $x_0 = x$ is a fast scale characterizing the wavelengths of the acoustic waves and $x_1 = \epsilon x$ is a slow scale characterizing the amplitude and phase modulations due to the resonance. Substituting Eq. 22 into Eqs. 2–4 and equating coefficients of like powers of ϵ, we obtain

Order ϵ^0

$$\frac{\partial^2\phi_0}{\partial x_0^2} + \frac{\partial^2\phi_0}{\partial y^2} + \omega^2\phi_0 = 0 , \tag{23}$$

$$\phi_{0y} = 0 , \quad \text{at } y = 0 , \tag{24}$$

$$\phi_{0y} = 0 , \quad \text{at } y = 1 ; \tag{25}$$

Order ϵ

$$\frac{\partial^2\phi_1}{\partial x_0^2} + \frac{\partial^2\phi_1}{\partial y^2} + \omega^2\phi_1 = -2\frac{\partial^2\phi_0}{\partial x_0\partial x_1} , \tag{26}$$

$$\frac{\partial\phi_1}{\partial y} = k_w \frac{\partial\phi_0}{\partial x_0}\cos k_w x_0 - \frac{\partial^2\phi_0}{\partial y^2}\sin k_w x_0 , \quad \text{at } y = 0 , \tag{27}$$

$$\frac{\partial\phi_1}{\partial y} = k_w \frac{\partial\phi_0}{\partial x_0}\cos(k_w x_0 + \theta) - \frac{\partial^2\phi_0}{\partial y^2}\sin(k_w x_0 + \theta) , \quad \text{at } y = 1 . \tag{28}$$

The solution of Eqs. 23–25 is taken to contain the two interacting modes; that is,

$$\phi_0 = A_m(x_1)\cos m\pi y \exp(ik_m x_0) + A_n(x_1)\cos n\pi y \exp(ik_n x_0) , \tag{29}$$

where $A_m(x_1)$ and $A_n(x_1)$ are still undetermined at this level of approximation; they are determined at the next level of approximation. Substituting for ϕ_0 from Eq. 29 into Eqs. 26–28, we obtain

$$\frac{\partial^2\phi_1}{\partial x_0^2} + \frac{\partial^2\phi_1}{\partial y^2} + \omega^2\phi_1 = -2ik_m A_m' \cos m\pi y \exp(ik_m x_0) - 2ik_n A_n' \cos n\pi y \exp(ik_n x_0) , \tag{30}$$

$$\phi_{1y} = \tfrac{1}{2}i \sum_{j=m,n} A_j(k_j k_w - j^2\pi^2)\exp[i(k_j + k_w)x_0] + \tfrac{1}{2}i \sum_{j=m,n} A_j(k_j k_w + j^2\pi^2)\exp[i(k_j - k_w)x_0] , \quad \text{at } y = 0 , \tag{31}$$

$$\phi_{1y} = \tfrac{1}{2}i \sum_{j=m,n} A_j(k_j k_w - j^2\pi^2)\cos j\pi \exp[i(k_j + k_w)x_0 + i\theta] + \tfrac{1}{2}i \sum_{j=m,n} A_j(k_j k_w + j^2\pi^2)\cos j\pi \exp[i(k_j - k_w)x_0 - i\theta] , \quad \text{at } y = 1 , \tag{32}$$

where primes denote differentiation with respect to x_1.

Equations 30–32 will have a finite solution, and hence the resulting expansion is uniform if, and only if, a so-called solvability condition is satisfied. To determine this solvability condition and carry out the expansion further, we express the nearness of k_w to $k_n - k_m$ by introducing the detuning parameter σ according to

$$k_w = k_n - k_m + \epsilon\sigma , \quad \sigma = 0(1) , \tag{33}$$

and express $(k_n - k_w)x_0$ and $(k_m + k_w)x_0$ as

$$(k_n - k_w)x_0 = k_m x_0 - \sigma x_1 , \tag{34a}$$

$$(k_m + k_w)x_0 = k_n x_0 + \sigma x_1 . \tag{34b}$$

To determine the solvability condition for Eqs. 30–32, we seek a particular solution of the form

$$\phi_1 = i\Phi_m(y)\exp(ik_m x_0) + i\Phi_n(y)\exp(ik_n x_0) . \tag{35}$$

Substituting Eq. 35 into Eqs. 30–32, using Eqs. 34, and equating the coefficients of $\exp(ik_m x_0)$ and $\exp(ik_n x_0)$ on both sides, we obtain

$$\Phi_m'' + m^2\pi^2\Phi_m = -2k_m A_m' \cos m\pi y , \tag{36}$$

$$\Phi_m'(0) = \tfrac{1}{2}A_n(k_n k_w + n^2\pi^2)\exp(-i\sigma x_1) , \tag{37}$$

$$\Phi_m'(1) = \tfrac{1}{2}A_n(k_n k_w + n^2\pi^2)\cos n\pi \exp[-i(\sigma x_1 + \theta)] , \tag{38}$$

$$\Phi_n'' + n^2\pi^2\Phi_n = -2k_n A_n' \cos n\pi y , \tag{39}$$

$$\Phi_n'(0) = \tfrac{1}{2}A_m(k_m k_w - m^2\pi^2)\exp(i\sigma x_1) , \tag{40}$$

$$\Phi_n'(1) = \tfrac{1}{2}A_m(k_m k_w - m^2\pi^2)\cos m\pi \exp[i(\sigma x_1 + \theta)] . \tag{41}$$

The general solution of Eq. 36 is

$$\Phi_m = c_1 \cos m\pi y + c_2 \sin m\pi y - (k_m/m\pi) y A_m' \sin m\pi y \ . \tag{42}$$

Substituting Eq. 42 into Eqs. 37 and 38, we have

$$m\pi c_2 = \tfrac{1}{2} A_n (k_n k_w + n^2\pi^2) \exp(-i\sigma x_1) \ , \tag{43}$$

$$(m\pi c_2 - k_m A_m') \cos m\pi = \tfrac{1}{2} A_n (k_n k_w + n^2\pi^2) \times \cos n\pi \exp[-i(\sigma x_1 + \theta)] \ . \tag{44}$$

Solving Eq. 43 for c_2 and substituting the result into Eq. 44, we obtain the following solvability condition for Eqs. 36–38:

$$A_m' = \tfrac{1}{2} k_m^{-1} (k_n k_w + n^2\pi^2) [1 - (-1)^{m+n} \exp(-i\theta)] A_n \exp(-i\sigma x_1) \ . \tag{45}$$

Similarly, the solvability condition for Eqs. 39–41 is

$$A_n' = \tfrac{1}{2} k_n^{-1} (k_m k_w - m^2\pi^2) [1 - (-1)^{m+n} \exp(i\theta)] A_m \exp(i\sigma x_1) \ . \tag{46}$$

We seek a solution to Eqs. 45 and 46 of the form

$$A_m = a_m \exp(s x_1) \ , \quad A_n = a_n \exp[(s + i\sigma) x_1] \ , \tag{47}$$

where a_j and s are constants. Substituting this assumed solution into Eqs. 45 and 46 and eliminating the a's, we get

$$s(s + i\sigma) = \Omega \ , \tag{48a}$$

where

$$\Omega = \tfrac{1}{2} (k_m k_n)^{-1} (k_m k_w - m^2\pi^2)(k_n k_w + n^2\pi^2) [1 - (-1)^{m+n} \cos\theta] \ . \tag{48b}$$

The solution of Eq. 48a is

$$s = \tfrac{1}{2} i [-\sigma \pm (\sigma^2 - 4\Omega)^{1/2}] \ . \tag{49}$$

Since $k_w = k_n - k_m + \epsilon\sigma$ from Eq. 33,

$$k_m k_w - m^2\pi^2 = k_n k_m - k_m^2 + \epsilon\sigma k_m - m^2\pi^2 = k_n k_m - \omega^2 + \epsilon\sigma k_m < 0 \ ,$$

because $k_n < \omega$ and $k_m < \omega$. Hence s is pure imaginary and A_m and A_n are bounded as a consequence. Therefore, ϕ_0 and ϕ_1 are bounded according to Eqs. 29, 42, and 43, and the response is not very large in contrast with the straightforward expansion obtained in the previous section. However, Eqs. 29, 45, and 46 show that the nth mode cannot exist without the mth resonant mode. Consequently, sinusoidal wall undulations can be used to generate the mth mode from the nth mode if $k_m \approx k_n - k_w$.

III. CONCLUDING REMARKS

A straightforward perturbation solution of the form $\phi = \phi_0 + \epsilon\phi_1$ is obtained for the acoustic wave propagation in a hard-walled two-dimensional duct whose walls have weak sinusoidal undulations of the order of the small dimensionless parameter ϵ. The results show that if the mth mode corresponding to the frequency ω with the wavenumber k_m passes through the duct, the wall undulations will generate two weak waves with the wavenumbers $k_w + k_m$ and $k_w - k_m$, where k_w is the wavenumber of the wall undulations. If $k_w \mp k_m \approx k_n$ where k_n is the wavenumber of the nth mode, the straightforward expansion breaks down because $\epsilon\phi_1$ is not small compared with ϕ_0. For traveling waves, only the resonant case $k_w \approx k_n - k_m$ occurs. An expansion valid when $k_w \approx k_m - k_n$ is then obtained. The results show that the mth mode cannot exist without strongly generating the nth mode; however, both modes travel unattenuated through the duct.

ACKNOWLEDGMENTS

The author is grateful to Mr. O. R. Asfar for helpful comments and discussions and to Dr. J. E. Kaiser for reading the manuscript. This work was supported by the NASA Langley Research Center under Grant NGR-47-004-109.

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