Second-order constitutive relations for transversely isotropic piezoelectric porous materials

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Based on the theory of invariants, polynomial constitutive relations for transversely isotropic piezoelectric porous materials are derived from the polynomial integrity bases for an energy density function depending on a symmetric second-order tensor and two vectors. They are assumed to be smooth functions of their arguments, are expanded about the values their arguments take in the reference configuration and all terms up to the quadratic terms in the gradients of the mechanical displacement, the electric potential, and the gradients of the volume fraction are kept. The second-order constitutive relations so obtained are then specialized to the case of infinitesimal deformations and weak electric fields, and also to the case of infinitesimal deformations and strong electric fields.

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INTRODUCTION

The effect of nonlinearity in the constitutive relations of piezoelectric ceramics has been of recent interest because of their use in smart structures. Nelson[1] has given, for all crystal classes, representations of quadratic piezoelectric constitutive relations generated by an energy density function of a symmetric second-order tensor and a vector.

Many piezoelectric materials are porous.[2–4] Here, based on the theory of invariants, nonlinear form invariant polynomial constitutive relations for transversely isotropic piezoelectric porous materials are derived. They are then reduced to second-order and linear constitutive relations, and constitutive relations for small deformations and strong electric fields.

I. EQUATIONS FOR A NONLINEAR PIEZOELECTRIC POROUS MATERIAL

Let the coordinates of a material particle with respect to a rectangular Cartesian coordinate system be X_K in the reference configuration, its spatial coordinates in the current configuration be x_k, then the balance laws for a nonlinear piezoelectric porous material are[5–6]

$$\begin{aligned}
&\rho_0=\rho\mathscr{J}, \quad \mathscr{J}=\det(x_{k,K}),\\
&[T_{KL}x_{k,L}+\mathscr{J}X_{K,l}\epsilon_0(E_kE_l-\tfrac{1}{2}E_mE_m\delta_{kl})]_{,K}+\rho_0 f_k\\
&\quad=\rho_0\delta_{kK}\ddot{U}_K,\\
&(\Pi_K+\mathscr{J}X_{K,k}\epsilon_0E_k)_{,K}=0,\\
&h_{K,K}+\rho_0(l+g)=\rho_0\overline{(k\dot{\psi})},
\end{aligned}\tag{1}$$

where ρ is the mass density, ρ_0 is the mass density of the porous material in the reference configuration, T_{KL} is the second Piola–Kirchhoff stress tensor, U_K is the mechanical displacement vector, δ_{kK} is the shifter, Π_K is the material electric polarization, $E_k=-\phi_{,k}$ is the electric field, ϕ is electric potential, ϵ_0 is the permittivity of the free space, δ_{kl} is the Kronecker delta, h_K is the equilibrated stress, g is the intrinsic equilibrated body force, l is the extrinsic equilibrated body force, k is the equilibrated inertia, and ψ is the volume fraction of voids or the porosity of the material. Throughout this paper, a repeated index implies summation over the range of the index, and a comma followed by K (i) implies partial differentiation with respect to X_K (x_i). A dot above a quantity signifies its material time derivative. Balance laws (1) are accompanied by constitutive relations

$$T_{KL}=\frac{\partial\Sigma}{\partial E_{KL}}, \quad \Pi_K=-\frac{\partial\Sigma}{\partial W_K}, \quad h_K=\frac{\partial\Sigma}{\partial V_K}, \tag{2}$$

where $\Sigma(E_{KL},W_K,V_K)$ is an energy density function that also depends on ψ, that dependence is not written explicitly. In Eq. (2) E_{KL} is the Green–Lagrange strain tensor, W_K is the electric field in material form, and V_K is the material gradient of ψ:

$$\begin{aligned}
&E_{KL}=\tfrac{1}{2}(U_{K,L}+U_{L,K}+U_{M,K}U_{M,L}),\\
&W_K=x_{k,K}E_k=-x_{k,K}\phi_{,k}=-\phi_{,K}; \quad V_K=x_{k,K}\psi_{,k}=\psi_{,K}.
\end{aligned}\tag{3}$$

II. FORM INVARIANT POLYNOMIAL CONSTITUTIVE RELATIONS

Let the material be invariant under rotations about a unit vector **a** and reflections about planes containing **a**. Then any scalar polynomial function of a symmetric tensor **E** and two vectors **W** and **V** must be a polynomial function of the following invariants called the polynomial integrity bases:[7]

$$\begin{aligned}
&I_1=\mathbf{a}\cdot\mathbf{E}\cdot\mathbf{a}, \quad I_2=\operatorname{tr}\mathbf{E}, \quad I_3=\mathbf{a}\cdot\mathbf{W}, \quad I_4=\mathbf{a}\cdot\mathbf{V},\\
&II_1=\mathbf{a}\cdot\mathbf{E}^2\cdot\mathbf{a}, \quad II_2=\operatorname{tr}\mathbf{E}^2, \quad II_3=\mathbf{W}\cdot\mathbf{W},\\
&II_4=\mathbf{a}\cdot\mathbf{E}\cdot\mathbf{W}+\mathbf{W}\cdot\mathbf{E}\cdot\mathbf{a}, \quad II_5=\mathbf{V}\cdot\mathbf{V},\\
&II_6=\mathbf{a}\cdot\mathbf{E}\cdot\mathbf{V}+\mathbf{V}\cdot\mathbf{E}\cdot\mathbf{a}, \quad II_7=\mathbf{W}\cdot\mathbf{V},
\end{aligned}\tag{4}$$

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$$III_1 = \mathrm{tr}\,\mathbf{E}^3, \quad III_2 = \mathbf{W}\cdot\mathbf{E}\cdot\mathbf{W},$$

$$III_3 = \mathbf{a}\cdot\mathbf{E}^2\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{E}^2\cdot\mathbf{a}, \quad III_4 = \mathbf{V}\cdot\mathbf{E}\cdot\mathbf{V},$$

$$III_5 = \mathbf{a}\cdot\mathbf{E}^2\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{E}^2\cdot\mathbf{a}, \quad III_6 = \mathbf{W}\cdot\mathbf{E}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{E}\cdot\mathbf{W},$$

where $\mathbf{a}\cdot\mathbf{b}$ indicates the inner product between vectors $\mathbf{a}$ and $\mathbf{b}$, and $\mathrm{tr}\,\mathbf{E}$ equals the sum of the diagonal terms of $\mathbf{E}$. With (4), any scalar form invariant polynomial function of the symmetric tensor $\mathbf{E}$ and vectors $\mathbf{W}$ and $\mathbf{V}$ can be written as

$$\Sigma = \Sigma(I_1,\ldots,\ I_4, II_1,\ldots,II_7,\ III_1,\ldots,III_6), \tag{5}$$

where Σ is a general polynomial function of its arguments. From (2) and (5), we obtain the following general form for the polynomial constitutive relations for a nonlinear transversely isotropic material:

$$\mathbf{T} = \frac{\partial\Sigma}{\partial I_1}\mathbf{a}\otimes\mathbf{a} + \frac{\partial\Sigma}{\partial I_2}\mathbf{1} + \frac{\partial\Sigma}{\partial II_1}(\mathbf{a}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{a}) + 2\frac{\partial\Sigma}{\partial II_2}\mathbf{E} + \frac{\partial\Sigma}{\partial II_4}(\mathbf{a}\otimes\mathbf{W} + \mathbf{W}\otimes\mathbf{a}) + \frac{\partial\Sigma}{\partial II_6}(\mathbf{a}\otimes\mathbf{V} + \mathbf{V}\otimes\mathbf{a}) + 3\frac{\partial\Sigma}{\partial III_1}\mathbf{E}^2$$
$$+ \frac{\partial\Sigma}{\partial III_2}\mathbf{W}\otimes\mathbf{W} + \frac{\partial\Sigma}{\partial III_4}\mathbf{V}\otimes\mathbf{V} + \frac{\partial\Sigma}{\partial III_6}(\mathbf{W}\otimes\mathbf{V} + \mathbf{V}\otimes\mathbf{W}) + \frac{\partial\Sigma}{\partial III_3}(\mathbf{a}\otimes\mathbf{E}\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{E}\otimes\mathbf{a} + \mathbf{W}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{W})$$
$$+ \frac{\partial\Sigma}{\partial III_5}(\mathbf{a}\otimes\mathbf{E}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{E}\otimes\mathbf{a} + \mathbf{V}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{V}), \tag{6}$$

$$-\mathbf{\Pi} = \frac{\partial\Sigma}{\partial I_3}\mathbf{a} + 2\frac{\partial\Sigma}{\partial II_4}\mathbf{E}\cdot\mathbf{a} + 2\frac{\partial\Sigma}{\partial II_3}\mathbf{W} + \frac{\partial\Sigma}{\partial II_7}\mathbf{V} + 2\frac{\partial\Sigma}{\partial III_3}\mathbf{E}^2\cdot\mathbf{a} + 2\frac{\partial\Sigma}{\partial III_2}\mathbf{E}\cdot\mathbf{W} + 2\frac{\partial\Sigma}{\partial III_6}\mathbf{E}\cdot\mathbf{V}, \tag{7}$$

$$\mathbf{h} = \frac{\partial\Sigma}{\partial I_4}\mathbf{a} + 2\frac{\partial\Sigma}{\partial II_6}\mathbf{E}\cdot\mathbf{a} + \frac{\partial\Sigma}{\partial II_7}\mathbf{W} + 2\frac{\partial\Sigma}{\partial II_5}\mathbf{V} + 2\frac{\partial\Sigma}{\partial III_5}\mathbf{E}^2\cdot\mathbf{a} + 2\frac{\partial\Sigma}{\partial III_6}\mathbf{E}\cdot\mathbf{W} + 2\frac{\partial\Sigma}{\partial III_4}\mathbf{E}\cdot\mathbf{V}, \tag{8}$$

where $\mathbf{1}$ is the identity tensor and $\mathbf{u}\otimes\mathbf{v}$ denotes tensor product between tensors $\mathbf{u}$ and $\mathbf{v}$. In order to derive a second-order theory, we assume that Σ is a smooth function of its arguments and write its Taylor series expansion about the values the arguments take in the reference configuration and only keep all terms up to degree three in $\mathbf{E}$, $\mathbf{W}$, and $\mathbf{V}$:

$$\Sigma = \alpha_1 I_1 + \alpha_2 I_2 + \beta I_3 + \gamma I_4 + c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2 + \epsilon_1 I_3^2 + \epsilon_2 II_3 + \kappa_1 I_4^2 + \kappa_2 II_5 + e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 II_4$$
$$+ f_1 I_1 I_4 + f_2 I_2 I_4 + f_3 II_6 + g_1 I_3 I_4 + g_2 II_7 + \lambda_1 I_1^3 + \lambda_2 I_2^3 + \lambda_3 I_1^2 I_2 + \lambda_4 I_2^2 I_1 + \lambda_5 II_1 I_1 + \lambda_6 II_1 I_2 + \lambda_7 II_2 I_1 + \lambda_8 II_2 I_2$$
$$+ \lambda_9 III_1 + \mu_1 I_3^3 + \mu_2 II_3 I_3 + \xi_1 I_4^3 + \xi_2 II_5 I_4 + \nu_1 I_1^2 I_3 + \nu_2 I_3^2 I_1 + \nu_3 I_2^2 I_3 + \nu_4 I_3^2 I_2 + \nu_5 II_1 I_3 + \nu_6 II_2 I_3 + \nu_7 II_3 I_1 + \nu_8 II_3 I_2$$
$$+ \nu_9 II_4 I_1 + \nu_{10} II_4 I_2 + \nu_{11} II_4 I_3 + \nu_{12} III_2 + \nu_{13} III_3 + \nu_{14} I_1 I_2 I_3 + \eta_1 I_1^2 I_4 + \eta_2 I_4^2 I_1 + \eta_3 I_2^2 I_4 + \eta_4 I_4^2 I_2 + \eta_5 II_1 I_4 + \eta_6 II_2 I_4$$
$$+ \eta_7 II_5 I_1 + \eta_8 II_5 I_2 + \eta_9 II_6 I_1 + \eta_{10} II_6 I_2 + \eta_{11} II_6 I_4 + \eta_{12} III_4 + \eta_{13} III_5 + \eta_{14} I_1 I_2 I_4 + \zeta_1 I_3^2 I_4 + \zeta_2 I_4^2 I_3 + \zeta_3 II_3 I_4$$
$$+ \zeta_4 II_5 I_3 + \zeta_5 II_7 I_3 + \zeta_6 II_7 I_4 + \delta_1 I_1 I_3 I_4 + \delta_2 I_2 I_3 I_4 + \delta_3 II_4 I_4 + \delta_4 II_6 I_3 + \delta_5 II_7 I_1 + \delta_6 II_7 I_2 + \delta_7 III_6, \tag{9}$$

where α_1, α_2, β, and γ will be shown to represent initial fields. Here, $c_1 - c_5$, ϵ_1, ϵ_2, κ_1, κ_2, $e_1 - e_3$, $f_1 - f_3$, g_1, and g_2 are 17 constants for the quadratic terms in Σ or for the linear constitutive relations; $\lambda_1 - \lambda_9$ represent cubic terms containing $\mathbf{E}$ alone, μ_1 and μ_2 cubic terms containing $\mathbf{W}$ alone, ξ_1 and ξ_2 cubic terms containing $\mathbf{V}$ alone, $\nu_1 - \nu_{14}$ cubic terms containing $\mathbf{E}$ and $\mathbf{W}$, $\eta_1 - \eta_{14}$ cubic terms containing $\mathbf{E}$ and $\mathbf{V}$, $\zeta_1 - \zeta_6$ cubic terms containing $\mathbf{W}$ and $\mathbf{V}$, and $\delta_1 - \delta_7$ cubic terms containing $\mathbf{E}$, $\mathbf{W}$, and $\mathbf{V}$. There are 54 constants in all for cubic terms in Σ or the quadratic constitutive relations. Substitution of (9) into (6)–(8) gives

$$\mathbf{T} = (\alpha_1 + 2c_1 I_1 + c_3 I_2 + e_1 I_3 + f_1 I_4 + 3\lambda_1 I_1^2 + 2\lambda_3 I_1 I_2 + \lambda_4 I_2^2 + \lambda_5 II_1 + \lambda_7 II_2 + 2\nu_1 I_1 I_3 + \nu_2 I_3^2 + \nu_7 II_3 + \nu_9 II_4 + \nu_{14} I_2 I_3$$
$$+ 2\eta_1 I_1 I_4 + \eta_2 I_4^2 + \eta_7 II_5 + \eta_9 II_6 + \eta_{14} I_2 I_4 + \delta_1 I_3 I_4 + \delta_5 II_7)\mathbf{a}\otimes\mathbf{a} + (\alpha_2 + 2c_2 I_2 + c_3 I_1 + e_2 I_3 + f_2 I_4 + 3\lambda_2 I_2^2 + \lambda_3 I_1^2$$
$$+ 2\lambda_4 I_1 I_2 + \lambda_6 II_1 + \lambda_8 II_2 + 2\nu_3 I_2 I_3 + \nu_4 I_3^2 + \nu_8 II_3 + \nu_{10} II_4 + \nu_{14} I_1 I_3 + 2\eta_3 I_2 I_4 + \eta_4 I_4^2 + \eta_8 II_5 + \eta_{10} II_6 + \eta_{14} I_1 I_4 + \delta_2 I_3 I_4$$
$$+ \delta_6 II_7)\mathbf{1} + (c_4 + \lambda_5 I_1 + \lambda_6 I_2 + \nu_5 I_3 + \eta_5 I_4)(\mathbf{a}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{a}) + 2(c_5 + \lambda_7 I_1 + \lambda_8 I_2 + \nu_6 I_3 + \eta_6 I_4)\mathbf{E} + (e_3 + \nu_9 I_1 + \nu_{10} I_2$$
$$+ \nu_{11} I_3 + \delta_3 I_4)(\mathbf{a}\otimes\mathbf{W} + \mathbf{W}\otimes\mathbf{a}) + (f_3 + \eta_9 I_1 + \eta_{10} I_2 + \eta_{11} I_4 + \delta_4 I_3)(\mathbf{a}\otimes\mathbf{V} + \mathbf{V}\otimes\mathbf{a}) + 3\lambda_9\mathbf{E}^2 + \nu_{12}\mathbf{W}\otimes\mathbf{W} + \eta_{12}\mathbf{V}\otimes\mathbf{V} + \nu_{13}(\mathbf{a}$$
$$\otimes\mathbf{E}\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{E}\otimes\mathbf{a} + \mathbf{W}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{W}) + \eta_{13}(\mathbf{a}\otimes\mathbf{E}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{E}\otimes\mathbf{a} + \mathbf{V}\otimes\mathbf{E}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{E}\otimes\mathbf{V}) + \delta_7(\mathbf{W}\otimes\mathbf{V} + \mathbf{V}\otimes\mathbf{W}), \tag{10}$$

$$-\mathbf{\Pi} = (\beta + 2\epsilon_1 I_3 + e_1 I_1 + e_2 I_2 + g_1 I_4 + 3\mu_1 I_3^2 + \mu_2 II_3 + \nu_1 I_1^2 + 2\nu_2 I_3 I_1 + \nu_3 I_2^2 + 2\nu_4 I_3 I_2 + \nu_5 II_1 + \nu_6 II_2 + \nu_{11} II_4 + \nu_{14} I_1 I_2$$
$$+ 2\zeta_1 I_3 I_4 + \zeta_2 I_4^2 + \zeta_4 II_5 + \zeta_5 II_7 + \delta_1 I_1 I_4 + \delta_2 I_2 I_4 + \delta_4 II_6)\mathbf{a} + 2(e_3 + \nu_9 I_1 + \nu_{10} I_2 + \nu_{11} I_3 + \delta_3 I_4)\mathbf{E}\cdot\mathbf{a}$$
$$+ 2(\epsilon_2 + \mu_2 I_3 + \nu_7 I_1 + \nu_8 I_2 + \zeta_3 I_4)\mathbf{W} + (g_2 + \zeta_5 I_3 + \zeta_6 I_4 + \delta_5 I_1 + \delta_6 I_2)\mathbf{V} + 2\nu_{13}\mathbf{E}^2\cdot\mathbf{a} + 2\nu_{12}\mathbf{E}\cdot\mathbf{W} + 2\delta_7\mathbf{E}\cdot\mathbf{V}, \tag{11}$$

$$\mathbf{h}=(\gamma+2\kappa_1 I_4+f_1 I_1+f_2 I_2+g_1 I_3+3\xi_1 I_4^2+\xi_2 II_5+\eta_1 I_1^2+2\eta_2 I_4 I_1+\eta_3 I_2^2+2\eta_4 I_4 I_2+\eta_5 II_1+\eta_6 II_2+\eta_{11} II_6+\eta_{14} I_1 I_2 +2\zeta_2 I_3 I_4+\zeta_1 I_3^2+\zeta_3 II_3+\zeta_6 II_7+\delta_1 I_1 I_3+\delta_2 I_2 I_3+\delta_3 II_4)\mathbf{a}+2(f_3+\eta_9 I_1+\eta_{10} I_2+\eta_{11} I_4+\delta_4 I_3)\mathbf{E}\cdot\mathbf{a} +(g_2+\zeta_5 I_3+\zeta_6 I_4+\delta_5 I_1+\delta_6 I_2)\mathbf{W}+2(\kappa_2+\xi_2 I_4+\eta_7 I_1+\eta_8 I_2+\zeta_4 I_3)\mathbf{V}+2\eta_{13}\mathbf{E}^2\cdot\mathbf{a}+2\delta_7\mathbf{E}\cdot\mathbf{W}+2\eta_{12}\mathbf{E}\cdot\mathbf{V}. \tag{12}$$

Equations (10)–(12) are polynomial representations of $\mathbf{T}$, $\mathbf{\Pi}$, and $\mathbf{h}$ of degree two in components of $\mathbf{E}$, $\mathbf{W}$, and $\mathbf{V}$. It can be seen that terms in (10) involving α_1 and α_2 do not depend upon $\mathbf{E}$, $\mathbf{W}$, or $\mathbf{V}$ and hence represent the initial stress. Terms involving β in (11) and γ in (12) are similar. The derivations in this section are for the constitutive relations generated by an energy density function of a symmetric tensor and two vectors in general. When applied to the case of a piezoelectric porous material, all of the material parameters should be considered as functions of ψ, the porosity of the material.

Passman and Batra[6] assumed that $\mathbf{T}$, $\mathbf{\Pi}$, and $\mathbf{h}$ also depend upon $\dot{\psi}$, the rate of change of the porosity. If we adopt this assumption, then the material parameters will also depend upon $\dot{\psi}$. Henceforth, we disregard the dependence of $\mathbf{T}$, $\mathbf{\Pi}$, and $\mathbf{h}$ upon $\dot{\psi}$.

III. SECOND-ORDER CONSTITUTIVE RELATIONS

By second-order constitutive relations we mean relations that contain all quadratic terms of the mechanical displacement gradient, electric potential gradient, and the gradient of the volume fraction of voids. Equations (10)–(12) contain some higher-order terms in this sense. To get second-order constitutive relations, we make the following decompositions:

$$\mathbf{E}=\mathbf{E}^{(1)}+\mathbf{E}^{(2)},\quad E_{KL}^{(1)}=\tfrac{1}{2}(U_{K,L}+U_{L,K}),$$
$$E_{KL}^{(2)}=\tfrac{1}{2}U_{M,K}U_{M,L}, \tag{13}$$
$$\mathbf{W}=\mathbf{W}^{(1)},\quad W_K^{(1)}=-\phi_{,K},\quad \mathbf{V}=\mathbf{V}^{(1)},\quad V_K^{(1)}=\psi_{,K},$$

and expansions

$$\begin{aligned}
&I_1=I_1^{(1)}+I_1^{(2)},\quad I_1^{(1)}=\mathbf{a}\cdot\mathbf{E}^{(1)}\cdot\mathbf{a},\quad I_1^{(2)}=\mathbf{a}\cdot\mathbf{E}^{(2)}\cdot\mathbf{a},\\
&I_2=I_2^{(1)}+I_2^{(2)},\quad I_2^{(1)}=\mathrm{tr}\,\mathbf{E}^{(1)},\quad I_2^{(2)}=\mathrm{tr}\,\mathbf{E}^{(2)},\\
&I_3=I_3^{(1)},\quad I_3^{(1)}=\mathbf{W}^{(1)}\cdot\mathbf{a},\quad I_4=I_4^{(1)},\quad I_4^{(1)}=\mathbf{V}^{(1)}\cdot\mathbf{a},\\
&II_1=II_1^{(2)}+\cdots,\quad II_1^{(2)}=\mathbf{a}\cdot(\mathbf{E}^{(1)})^2\cdot\mathbf{a},\\
&II_2^{(2)}=\mathrm{tr}(\mathbf{E}^{(1)})^2,\quad II_3^{(2)}=\mathbf{W}^{(1)}\cdot\mathbf{W}^{(1)},\\
&II_4^{(2)}=\mathbf{a}\cdot\mathbf{E}^{(1)}\cdot\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\cdot\mathbf{E}^{(1)}\cdot\mathbf{a},\quad II_5^{(2)}=\mathbf{V}^{(1)}\cdot\mathbf{V}^{(1)},\\
&II_6^{(2)}=\mathbf{a}\cdot\mathbf{E}^{(1)}\cdot\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\cdot\mathbf{E}^{(1)}\cdot\mathbf{a},\quad II_7^{(2)}=\mathbf{W}^{(1)}\cdot\mathbf{V}^{(1)},
\end{aligned} \tag{14}$$

where a superscript enclosed in parentheses indicates the order of the quantity. We have written $\mathbf{W}$ and $\mathbf{V}$ as $\mathbf{W}^{(1)}$ and $\mathbf{V}^{(1)}$ to make formally superscripts of different terms homogeneous. Substituting (13) and (14) into (10)–(12), and keeping terms up to second order, we obtain the following second-order representations for $\mathbf{T}$, $\mathbf{\Pi}$, and $\mathbf{h}$:

$$\begin{aligned}
&\mathbf{T}=\alpha_1\mathbf{a}\otimes\mathbf{a}+\alpha_2\mathbf{1}+\mathbf{T}^{(1)}+\mathbf{T}^{(2)},\\
&\mathbf{\Pi}=-\beta\mathbf{a}+\mathbf{\Pi}^{(1)}+\mathbf{\Pi}^{(2)},\\
&\mathbf{h}=\gamma\mathbf{a}+\mathbf{h}^{(1)}+\mathbf{h}^{(2)},
\end{aligned} \tag{15}$$

where

$$\mathbf{T}^{(1)}=(2c_1 I_1^{(1)}+c_3 I_2^{(1)}+e_1 I_3^{(1)}+f_1 I_4^{(1)})\mathbf{a}\otimes\mathbf{a}+(2c_2 I_2^{(1)}+c_3 I_1^{(1)}+e_2 I_3^{(1)}+f_2 I_4^{(1)})\mathbf{1}+c_4(\mathbf{a}\otimes\mathbf{E}^{(1)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(1)}\otimes\mathbf{a})+2c_5\mathbf{E}^{(1)} +e_3(\mathbf{a}\otimes\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\otimes\mathbf{a})+f_3(\mathbf{a}\otimes\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\otimes\mathbf{a}), \tag{16}$$

$$\begin{aligned}
\mathbf{T}^{(2)}=&[2c_1 I_1^{(2)}+c_3 I_2^{(2)}+3\lambda_1(I_1^{(1)})^2+2\lambda_3 I_1^{(1)}I_2^{(1)}+\lambda_4(I_2^{(1)})^2+\lambda_5 II_1^{(2)}+\lambda_7 II_2^{(2)}+2\nu_1 I_1^{(1)}I_3^{(1)}+\nu_2(I_3^{(1)})^2+\nu_7 II_3^{(2)}+\nu_9 II_4^{(2)}\\
&+\nu_{14}I_2^{(1)}I_3^{(1)}+2\eta_1 I_1^{(1)}I_4^{(1)}+\eta_2(I_4^{(1)})^2+\eta_7 II_5^{(2)}+\eta_9 II_6^{(2)}+\eta_{14}I_2^{(1)}I_4^{(1)}+\delta_1 I_3^{(1)}I_4^{(1)}+\delta_5 II_7^{(2)}]\mathbf{a}\otimes\mathbf{a}+[2c_2 I_2^{(2)}+c_3 I_1^{(2)}\\
&+3\lambda_2(I_2^{(1)})^2+\lambda_3(I_1^{(1)})^2+2\lambda_4 I_1^{(1)}I_2^{(1)}+\lambda_6 II_1^{(2)}+\lambda_8 II_2^{(2)}+2\nu_3 I_2^{(1)}I_3^{(1)}+\nu_4(I_3^{(1)})^2+\nu_8 II_3^{(2)}+\nu_{10}II_4^{(2)}+\nu_{14}I_1^{(1)}I_3^{(1)}\\
&+2\eta_3 I_2^{(1)}I_4^{(1)}+\eta_4(I_4^{(1)})^2+\eta_8 II_5^{(2)}+\eta_{10}II_6^{(2)}+\eta_{14}I_1^{(1)}I_4^{(1)}+\delta_2 I_3^{(1)}I_4^{(1)}+\delta_6 II_7^{(2)}]\mathbf{1}+(\lambda_5 I_1^{(1)}+\lambda_6 I_2^{(1)}+\nu_5 I_3^{(1)}+\eta_5 I_4^{(1)})\\
&\times(\mathbf{a}\otimes\mathbf{E}^{(1)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(1)}\otimes\mathbf{a})+c_4(\mathbf{a}\otimes\mathbf{E}^{(2)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(2)}\otimes\mathbf{a})+2(\lambda_7 I_1^{(1)}+\lambda_8 I_2^{(1)}+\nu_6 I_3^{(1)}+\eta_6 I_4^{(1)})\mathbf{E}^{(1)}+2c_5\mathbf{E}^{(2)}\\
&+(\nu_9 I_1^{(1)}+\nu_{10}I_2^{(1)}+\nu_{11}I_3^{(1)}+\delta_3 I_4^{(1)})(\mathbf{a}\otimes\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\otimes\mathbf{a})+(\eta_9 I_1^{(1)}+\eta_{10}I_2^{(1)}+\eta_{11}I_4^{(1)}+\delta_4 I_3^{(1)})(\mathbf{a}\otimes\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\otimes\mathbf{a})\\
&+3\lambda_9(\mathbf{E}^{(1)})^2+\nu_{12}\mathbf{W}^{(1)}\otimes\mathbf{W}^{(1)}+\eta_{12}\mathbf{V}^{(1)}\otimes\mathbf{V}^{(1)}+\nu_{13}(\mathbf{a}\otimes\mathbf{E}^{(1)}\cdot\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\cdot\mathbf{E}^{(1)}\otimes\mathbf{a}+\mathbf{W}^{(1)}\otimes\mathbf{E}^{(1)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(1)}\otimes\mathbf{W}^{(1)})\\
&+\eta_{13}(\mathbf{a}\otimes\mathbf{E}^{(1)}\cdot\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\cdot\mathbf{E}^{(1)}\otimes\mathbf{a}+\mathbf{V}^{(1)}\otimes\mathbf{E}^{(1)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(1)}\otimes\mathbf{V}^{(1)})+\delta_7(\mathbf{W}^{(1)}\otimes\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\otimes\mathbf{W}^{(1)}),
\end{aligned} \tag{17}$$

$$\mathbf{\Pi}^{(1)}=-(2\epsilon_1 I_3^{(1)}+e_1 I_1^{(1)}+e_2 I_2^{(1)}+g_1 I_4^{(1)})\mathbf{a}-2e_3\mathbf{E}^{(1)}\cdot\mathbf{a}-2\epsilon_2\mathbf{W}^{(1)}-g_2\mathbf{V}^{(1)}, \tag{18}$$

$$\begin{aligned}
\mathbf{\Pi}^{(2)}=&-[e_1 I_1^{(2)}+e_2 I_2^{(2)}+3\mu_1(I_3^{(1)})^2+\mu_2 II_3^{(2)}+\nu_1(I_1^{(1)})^2+2\nu_2 I_3^{(1)}I_1^{(1)}+\nu_3(I_2^{(1)})^2+2\nu_4 I_3^{(1)}I_2^{(1)}+\nu_5 II_1^{(2)}+\nu_6 II_2^{(2)}\\
&+\nu_{11}II_4^{(2)}+\nu_{14}I_1^{(1)}I_2^{(1)}+2\zeta_1 I_3^{(1)}I_4^{(1)}+\zeta_2(I_4^{(1)})^2+\zeta_4 II_5^{(2)}+\zeta_5 II_7^{(2)}+\delta_1 I_1^{(1)}I_4^{(1)}+\delta_2 I_2^{(1)}I_4^{(1)}+\delta_4 II_6^{(2)}]\mathbf{a}
\end{aligned}$$

$$-2(\nu_9 I_1^{(1)}+\nu_{10} I_2^{(1)}+\nu_{11} I_3^{(1)}+\delta_3 I_4^{(1)})\mathbf{E}^{(1)}\cdot\mathbf{a}-2e_3\mathbf{E}^{(2)}\cdot\mathbf{a}-2(\mu_2 I_3^{(1)}+\nu_7 I_1^{(1)}+\nu_8 I_2^{(1)}+\zeta_3 I_4^{(1)})\mathbf{W}^{(1)}$$
$$-(\zeta_5 I_3^{(1)}+\zeta_6 I_4^{(1)}+\delta_5 I_1^{(1)}+\delta_6 I_2^{(1)})\mathbf{V}^{(1)}-2\nu_{13}(\mathbf{E}^{(1)})^2\cdot\mathbf{a}-2\nu_{12}\mathbf{E}^{(1)}\cdot\mathbf{W}^{(1)}-2\delta_7\mathbf{E}^{(1)}\cdot\mathbf{V}^{(1)}, \tag{19}$$
$$\mathbf{h}^{(1)}=(2\kappa_1 I_4^{(1)}+f_1 I_1^{(1)}+f_2 I_2^{(1)}+g_1 I_3^{(1)})\mathbf{a}+2f_3\mathbf{E}^{(1)}\cdot\mathbf{a}+g_2\mathbf{W}^{(1)}+2\kappa_2\mathbf{V}^{(1)}, \tag{20}$$
$$\mathbf{h}^{(2)}=[f_1 I_1^{(2)}+f_2 I_2^{(2)}+3\xi_1(I_4^{(1)})^2+\xi_2 II_5^{(2)}+\eta_1(I_1^{(1)})^2+2\eta_2 I_4^{(1)}I_1^{(1)}+\eta_3(I_2^{(1)})^2+2\eta_4 I_4^{(1)}I_2^{(1)}+\eta_5 II_1^{(2)}+\eta_6 II_2^{(2)}+\eta_{11} II_6^{(2)}$$
$$+\eta_{14} I_1^{(1)}I_2^{(1)}+2\zeta_2 I_3^{(1)}I_4^{(1)}+\zeta_1(I_3^{(1)})^2+\zeta_3 II_3^{(2)}+\zeta_6 II_7^{(2)}+\delta_1 I_1^{(1)}I_3^{(1)}+\delta_2 I_2^{(1)}I_3^{(1)}+\delta_3 II_4^{(2)}]\mathbf{a}$$
$$+2(\eta_9 I_1^{(1)}+\eta_{10} I_2^{(1)}+\eta_{11} I_4^{(1)}+\delta_4 I_3^{(1)})\mathbf{E}^{(1)}\cdot\mathbf{a}+2f_3\mathbf{E}^{(2)}\cdot\mathbf{a}+(\zeta_5 I_3^{(1)}+\zeta_6 I_4^{(1)}+\delta_5 I_1^{(1)}+\delta_6 I_2^{(1)})\mathbf{W}^{(1)}$$
$$+2(\xi_2 I_4^{(1)}+\eta_7 I_1^{(1)}+\eta_8 I_2^{(1)}+\zeta_4 I_3^{(1)})\mathbf{V}^{(1)}+2\eta_{13}(\mathbf{E}^{(1)})^2\cdot\mathbf{a}+2\delta_7\mathbf{E}^{(1)}\cdot\mathbf{W}^{(1)}+2\eta_{12}\mathbf{E}^{(1)}\cdot\mathbf{V}^{(1)}. \tag{21}$$

IV. LINEAR CONSTITUTIVE RELATIONS

The linear or first-order constitutive relations (16), (18), and (20), when terms are rearranged according to the order of dependence on $\mathbf{E}^{(1)}$, $\mathbf{W}^{(1)}$, and $\mathbf{V}^{(1)}$, are

$$\mathbf{T}^{(1)}=(2c_1 I_1^{(1)}+c_3 I_2^{(1)})\mathbf{a}\otimes\mathbf{a}+(2c_2 I_2^{(1)}+c_3 I_1^{(1)})\mathbf{1}+c_4(\mathbf{a}\otimes\mathbf{E}^{(1)}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{E}^{(1)}\otimes\mathbf{a})+2c_5\mathbf{E}^{(1)}+e_1 I_3^{(1)}\mathbf{a}\otimes\mathbf{a}+e_2 I_3^{(1)}\mathbf{1}$$
$$+e_3(\mathbf{a}\otimes\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\otimes\mathbf{a})+f_1 I_4^{(1)}\mathbf{a}\otimes\mathbf{a}+f_2 I_4^{(1)}\mathbf{1}+f_3(\mathbf{a}\otimes\mathbf{V}^{(1)}+\mathbf{V}^{(1)}\otimes\mathbf{a}),$$
$$\mathbf{\Pi}^{(1)}=-(e_1 I_1^{(1)}+e_2 I_2^{(1)})\mathbf{a}-2e_3\mathbf{E}^{(1)}\cdot\mathbf{a}-2\epsilon_1 I_3^{(1)}\mathbf{a}-2\epsilon_2\mathbf{W}^{(1)}-g_1 I_4^{(1)}\mathbf{a}-g_2\mathbf{V}^{(1)}, \tag{22}$$
$$\mathbf{h}^{(1)}=(f_1 I_1^{(1)}+f_2 I_2^{(1)})\mathbf{a}+2f_3\mathbf{E}^{(1)}\cdot\mathbf{a}+g_1 I_3^{(1)}\mathbf{a}+g_2\mathbf{W}^{(1)}+2\kappa_1 I_4^{(1)}\mathbf{a}+2\kappa_2\mathbf{V}^{(1)}.$$

These can be written a matrix form.[8,9]

V. SMALL DEFORMATIONS AND STRONG ELECTRIC FIELDS

In this case, (15), (16), (18), and (20) remain the same. Equations (17), (19), and (21) reduce to

$$\mathbf{T}^{(2)}=[\nu_2(I_3^{(1)})^2+\nu_7 II_3^{(2)}]\mathbf{a}\otimes\mathbf{a}+[\nu_4(I_3^{(1)})^2+\nu_8 II_3^{(2)}]\mathbf{1}+\nu_{11} I_3^{(1)}(\mathbf{a}\otimes\mathbf{W}^{(1)}+\mathbf{W}^{(1)}\otimes\mathbf{a})+\nu_{12}\mathbf{W}^{(1)}\otimes\mathbf{W}^{(1)}, \tag{23}$$
$$\mathbf{\Pi}^{(2)}=-[3\mu_1(I_3^{(1)})^2+\mu_2 II_3^{(2)}]\mathbf{a}-2\mu_2 I_3^{(1)}\mathbf{W}^{(1)}, \tag{24}$$
$$\mathbf{h}^{(2)}=[\zeta_1(I_3^{(1)})^2+\zeta_3 II_3^{(2)}]\mathbf{a}+\zeta_5 I_3^{(1)}\mathbf{W}^{(1)}, \tag{25}$$

which can also be written in a matrix form.[8,9]

VI. CONCLUSIONS

Constitutive relations (22) imply that for a linear theory, contributions to stresses, electric polarization, and the equilibrated stresses from the mechanical strains, the electric field, and the porosity gradients are additive. However, such is not the case in the nonlinear theory as should be evident from the second-order constitutive relations (15)–(21). When the changes in porosity are also infinitesimal, then the above-stated constitutive relations can be suitably modified. In them, the material parameters will depend upon the initial value of the porosity.

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