

First l^2 -Cohomology Groups

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(ABSTRACT)

We want to take a look at the first cohomology group $H^1(G, l^2(G))$, in particular when G is locally-finite. First, though, we discuss some results about the space $H^1(G, \mathbb{C}G)$ for G locally-finite, as well as the space $H^1(G, l^2(G))$ when G is finitely generated. We show that, although in the case when G is finitely generated the embedding of $\mathbb{C}G$ into $l^2(G)$ induces an embedding of the cohomology groups $H^1(G, \mathbb{C}G)$ into $H^1(G, l^2(G))$, when G is countably-infinite locally-finite, the induced homomorphism is not an embedding. However, even though the induced homomorphism is not an embedding, we still have that $H^1(G, l^2(G)) \neq 0$ when G is countably-infinite locally-finite. Finally, we give some sufficient conditions for $H^1(G, l^2(G))$ to be zero or non-zero.

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Chapter 1

Introduction

Let G be a locally-finite group. In the book [4] by Warren Dicks and M. J. Dunwoody, they show that $H^1(G, \mathbb{C}G)$ has infinite dimension over \mathbb{C} if G is countably infinite, and $H^1(G, \mathbb{C}G) = 0$ otherwise. In the paper [3] by D. F. Holt, Holt explores in detail why $H^1(G, \mathbb{C}G) = 0$ when G becomes uncountable. It is of interest whether or not such a result holds when we look at the first cohomology groups with coefficients in $l^2(G)$. In this paper we show that when G is countably-infinite locally-finite, $H^1(G, l^2(G)) \neq 0$, but that when $|G| > \aleph_1$, $H^1(G, l^2(G)) = 0$, however, it is still unclear whether or not the cohomology group vanishes for the general uncountable locally-finite group.

For a finitely generated group G , M. Bekka and A. Valette discuss how the embedding of $\mathbb{C}G$ into $l^2(G)$ induces an embedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, l^2(G))$ in [1], which yields many nice results. First, this gives some relation between results on the cohomology groups $H^1(G, \mathbb{C}G)$ and $H^1(G, l^2(G))$. It also gives information about the ends of groups (discussed in [4] and [5]) given information about the L^2 -cohomology of a group. Most especially, when a finitely-generated group has vanishing L^2 -cohomology group, then the group G has 1 end. We will look at this embedding in more detail later in this paper.

Bekka and Valette also discuss how the amenability of a group affects its first L^2 -cohomology. In particular, for a finitely generated group G , $H^1(G, l^2(G))$ is Hausdorff if and only if G is non-amenable. A slightly more general result to this affect is stated by Alain Guichardet in [10], and the result is revisited in this paper. In particular, we show that if G is non-amenable then $H^1(G, l^2(G))$ is Hausdorff. While the converse does hold when G is countable, we will show that the converse

can fail for an uncountable group.

Finally, this paper will look at a result of M. Bourdon, F. Martin, and A. Valette in [2], which states that given $p \in [1, \infty)$, and $N \subset H \subset G$ a chain of groups with G finitely-generated, and N infinite and normal in G , if $H^1(H, l^p(H)) = 0$ then $H^1(G, l^p(G)) = 0$. We will attempt to generalize this result to G not necessarily finitely-generated. We will then show that if G has an infinite center, in particular if G is abelian, then $H^1(G, l^2(G)) = 0$. We conclude with an example of an uncountable group with non-zero first L^2 -cohomology group.

Chapter 2

Definitions

We begin this paper with some basic definitions from group cohomology and amenability.

Definition. Let R be a ring with identity and G be a group. Define the group ring, denoted RG , to be the set of all finite sums

$$\sum_{g \in S} a_g g$$

where S is a finite subset of G and $a_g \in R$. This is a ring under componentwise addition, and multiplication given by $a_1 g_1 a_2 g_2 = (a_1 a_2)(g_1 g_2)$ extended to all of RG by distributive laws.

Definition. A G -module A will be an abelian group A together with a homomorphism $\phi: G \rightarrow \text{Aut } A$, so that G acts on A on the left by automorphisms.

Note here that since A is an abelian group, saying A is a G -module is the same as viewing A as $\mathbb{Z}G$ -module, where $\mathbb{Z}G$ is the group ring of G with coefficients in \mathbb{Z} . We now turn to defining the 1st cohomology group of G with coefficients in A .

Definition. Let G be a group and A be a G -module, then we define $C^0(G, A) = A$ and $C^n(G, A)$ to be the set of all maps from G^n to A . We will call these n -cochains of G with values in A . $C^n(G, A)$ is an additive abelian group under pointwise addition, and in the case of $n = 0$ under the group structure of A .

Definition. For $n \geq 0$, define the n th coboundary homomorphism from $C^n(G, A)$ to $C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \dots, g_{n+1}) &= g_1 \cdot f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned}$$

We will justify later, by defining the cohomology groups in a different way, that $d_n \circ d_{n-1} = 0$ for $n \geq 1$. For now, we will assume this and continue with attempting to define $H^1(G, A)$ using these cochain groups and the coboundary homomorphisms.

Definition. Let $Z^n(G, A) = \ker d_n$ for $n \geq 0$, $B_n(G, A) = \text{Im } d_{n-1}$ for $n \geq 1$, and $B_0(G, A) = 1$. Then, since $d_n \circ d_{n-1} = 0$ for $n \geq 1$, $B^n(G, A)$ is a subgroup of $Z^n(G, A)$, and we can define $H^n(G, A)$ to be the quotient group $Z^n(G, A)/B^n(G, A)$.

We will denote $Z^1(G, A)$ by Z_G , and by definition this will be the subgroup of maps in $C^1(G, A)$ satisfying $f(gh) = f(g) - g \cdot f(h)$ for all $g, h \in G$. These elements will be referred to these as 1-cocycles. Then let $B_G = B^1(G, A)$, which by definition is the subgroup of maps in $C^1(G, A)$ satisfying $f(g) = g \cdot a - a$ for some $a \in A$, and we will refer to these maps as 1-coboundaries. This will give us $H^1(G, A) = Z_G/B_G$.

We also take a quick look at $H^0(G, A)$. Since $B^0(G, A) = 0$, we have that $H^0(G, A) = Z^0(G, A)$, and note that the $\ker d_0 = \{a \in A : g \cdot a = a \text{ for all } g \in G\}$. Therefore, $H^0(G, A) = A^G$, where A^G is all elements in A fixed by G -action.

Now, to give some justification of this definition of $H^1(G, A)$, and to understand better why we have $d_n \circ d_{n-1} = 0$ for $n \geq 1$, we consider the standard resolution of \mathbb{Z} ,

$$\dots \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\text{aug}} \mathbb{Z} \rightarrow 0 \quad (2.1)$$

where $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} \mathbb{Z}G$ (where we have $n+1$ factors). When we make F_n a G -module under the G -action $g \cdot g_0 \otimes g_1 \otimes \dots \otimes g_n = gg_0 \otimes g_1 \otimes \dots \otimes g_n$, it can be shown that F_n becomes a free $\mathbb{Z}G$ -module with basis $1 \otimes g_1 \otimes \dots \otimes g_n$ with $g_i \in G$. Then, using the augmentation map $\text{aug}: \mathbb{Z}G \rightarrow \mathbb{Z}$ defined by $\text{aug}(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g$,

and the maps $d_n: F_{n+1} \rightarrow F_n$ for $n \geq 0$ defined by

$$\begin{aligned} d_n(1 \otimes g_1 \otimes \dots \otimes g_{n+1}) &= g_1 \cdot (1 \otimes g_2 \otimes \dots \otimes g_{n+1}) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (1 \otimes g_1 \otimes \dots \otimes g_{i-1} \otimes g_i g_{i+1} \otimes g_{i+2} \otimes \dots \otimes g_{n+1}) \\ &\quad + (-1)^n (1 \otimes g_1 \otimes \dots \otimes g_n), \end{aligned}$$

(1.1) becomes a free G -module resolution of \mathbb{Z} .

By applying $\mathbb{Z}G$ -module homomorphisms from (1.1) into A , we obtain the cochain complex

$$0 \rightarrow \text{Hom}_{\mathbb{Z}G}(F_0, A) \xrightarrow{d_0} \text{Hom}_{\mathbb{Z}G}(F_1, A) \xrightarrow{d_1} \dots$$

whose cohomology groups are $\text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, A)$ by definition. So, if we define $H^1(G, A)$ to be $\text{Ext}_{\mathbb{Z}G}^1(\mathbb{Z}, A)$, we see that the maps in $\ker d_1$ are the maps satisfying $f(gh) = f(h) + gf(h)$ for all $g, h \in G$, and the maps in the image of d_0 are maps satisfying $f(g) = g \cdot a - a$ for some $a \in A$. Since a homomorphism from F_n to A is determined by where the basis elements are sent, we can see why it makes sense to view the elements of $\text{Hom}_{\mathbb{Z}G}(F_n, A)$ as maps from G^n to A , giving some insight into the previous definition of $H^1(G, A)$.

Also, by defining $H^n(G, A) = \text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, A)$, we can get a long exact sequence in group cohomology.

Theorem 1. Let

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence of G -modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \xrightarrow{\delta_0} H^1(G, A) \rightarrow H^1(G, B) \rightarrow \dots \\ \dots \xrightarrow{\delta_{n-1}} H^n(G, A) \longrightarrow H^n(G, B) \longrightarrow \dots \end{aligned}$$

The final definition relating directly to the cohomology groups addresses what sort of group homomorphisms induce homomorphisms between corresponding cohomology groups.

Definition. Let A be a G -module and A' be a G' -module, then the group homomorphisms $\phi: G' \rightarrow G$ and $\psi: A \rightarrow A'$ are said to be compatible if $\psi(\phi(g')a) = g'\psi(a)$ for all $g' \in G'$ and $a \in A$.

Compatible homomorphisms induce group homomorphisms on the corresponding cohomology groups. Note that we can make A into a G' -module via ϕ , and the condition for ψ and ϕ to be compatible is precisely the condition that ψ is a G' -module homomorphism when A is made into a G' -module in this way.

To see that compatible maps induce group homomorphisms on the corresponding cohomology groups, first induce a homomorphism $\phi^n: (G')^n \rightarrow G^n$. This gives us a map from $C^n(G, A)$ to $C^n(G', A)$, where f maps to $f \circ \phi^n$. Then ψ induces a homomorphism from $C^n(G', A)$ to $C^n(G', A')$ where f is mapped to $\psi \circ f$. Let λ_n denote the resulting map from $C^n(G, A)$ to $C^n(G', A')$ that maps f to $\psi \circ f \circ \phi^n$.

If ψ and ϕ are compatible, then we can check that coboundaries are sent to coboundaries and cocycles to cocycles, and thus we get an induced group homomorphism on the cohomology $\lambda_n: H^n(G, A) \rightarrow H^n(G', A')$.

Before we move on to looking at specific cohomology groups, we make some definitions of amenability, which relates closely to the first cohomology group with coefficients in $l^2(G)$, especially in the case where G is finitely generated. We give two equivalent definitions for amenability.

Definition 1. A group G is said to be amenable if given $\epsilon > 0$ and finite $F \subset G$, there exists a finite subset U of G such that $|U \Delta g \cdot U|/|U| < \epsilon$ for all $g \in F$. If we can always find such a U we say G satisfies the Følner condition. (Here Δ denotes the symmetric difference of sets).

Definition 2. A group G is said to be amenable if given any finite subset F of G and $\epsilon > 0$, we can find some $f \in l^1(G)$ with $f \geq 0$, $\|f\|_1 = 1$, and $\|g \cdot f - f\|_1 < \epsilon$ for all $g \in F$. This is referred to as Reiter's Property.

Properties of Amenability.

1. If G is amenable, then every subgroup H of G is amenable.
2. Given normal subgroup N of G , G is amenable if and only if N and G/N are amenable.
3. G is amenable if and only if every finitely generated subgroup of G is amenable.

4. If G is abelian then G is amenable.

The proofs of these properties is omitted, but [8] looks at these and other properties in more detail. We will always be viewing our group G with the discrete topology, so G will certainly be locally compact.

Chapter 3

$H^1(G, \mathbb{C}G)$

While in this paper we are concerned mainly with the L^2 -cohomology of a group, much insight can be gained by first considering the cohomology groups with coefficients in $\mathbb{C}G$, since $\mathbb{C}G$ can be viewed as the subgroup of $l^2(G)$ with finite support. The following proposition addresses the cohomology group when the group G is finite.

Proposition 1. Let G be a finite group. Then $H^1(G, \mathbb{C}G) = 0$.

Proof. We first note that $\mathbb{C}G$ can be written as $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G$, and when G is finite, we have $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{C})$. Therefore, by Shapiro's Lemma we get the following equivalences,

$$H^1(G, \mathbb{C}G) \cong H^1(G, \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G) \cong H^1(G, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{C})) \cong H^1(1, \mathbb{C}) = 0.$$

■

To characterize the groups when G is not finite, we first take a look at defining the ends of the group. To look at the ends of a group, we want to consider almost invariant subsets of the group, so we make the following definitions.

Definition. We define the equivalence relation of almost equality on the group G by saying that if $A, B \subset G$, then $A \sim_a B \iff |A \Delta B| < \infty$.

Definition. We say that a subset A of G is almost (left) invariant if $gA \sim_a A$

for all $g \in G$. Note here that any subset of G that is almost equal to an almost invariant subset is also almost invariant.

The definition of the ends of G is well understood by considering the cohomology groups of G with coefficients in \mathbb{Z}_2G (where \mathbb{Z}_2 is the field with 2 elements), but we would like to relate this definition of ends to the cohomology groups with coefficients in $\mathbb{C}G$ as best we can. For now, we focus on \mathbb{Z}_2 and follow the process of Cohen in [5].

Definition. Let F be a field and G be a group. Then define $\overline{FG} := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, F)$.

Now note that $\overline{\mathbb{Z}_2G}$ can be viewed as the set of all subsets of G by looking at where the elements of G are sent under a particular homomorphism. Then \mathbb{Z}_2G is a G -submodule of $\overline{\mathbb{Z}_2G}$ and is identified with all finite subsets of G . If we let $\phi G = \overline{\mathbb{Z}_2G}/\mathbb{Z}_2G$, we see that a subset of G is almost invariant if and only if its image in ϕG is invariant. So, we would like to look at the invariant subsets of ϕG , or $H^0(G, \phi G)$.

Definition. The number of ends of a group G , denoted $e(G)$, is defined to be the $\dim H^0(G, \phi G)$ (as a \mathbb{Z}_2 vector space).

To study the dimension of this \mathbb{Z}_2 -vector space, we consider the short exact sequence

$$0 \rightarrow \mathbb{Z}_2G \rightarrow \overline{\mathbb{Z}_2G} \rightarrow \phi G \rightarrow 0$$

which gives rise to the exact sequence of \mathbb{Z}_2 -vector spaces

$$0 \rightarrow H^0(G, \mathbb{Z}_2G) \rightarrow H^0(G, \overline{\mathbb{Z}_2G}) \rightarrow H^0(G, \phi G) \rightarrow H^1(G, \mathbb{Z}_2G) \rightarrow H^1(G, \overline{\mathbb{Z}_2G}).$$

Now noting that by Shapiro's Lemma, $H^1(G, \overline{\mathbb{Z}_2G}) = H^1(\{e\}, \mathbb{Z}_2) = 0$, and that $H^0(G, \mathbb{Z}_2G) = 0$ and $H^0(G, \overline{\mathbb{Z}_2G}) = \mathbb{Z}_2$, we find that $\dim H^0(G, \phi G) = 1 + \dim H^1(G, \mathbb{Z}_2G)$, i.e. the number of ends of a group G is $1 + \dim H^1(G, \mathbb{Z}_2G)$.

So, we can equivalently define the number of ends of a group as follows.

Definition. Then number of ends of a group G is $1 + \dim H^1(G, \mathbb{Z}_2G)$.

To relate this to the cohomology groups with coefficients in $\mathbb{C}G$, note that $\overline{\mathbb{C}G}$ can be viewed as the set of maps $\{f: G \rightarrow \mathbb{C}\}$. Again, we can view $\mathbb{C}G$ as a G -submodule of $\overline{\mathbb{C}G}$, and it will correspond to the functions with compact (finite) support. Now, considering $\psi G = \overline{\mathbb{C}G}/\mathbb{C}G$, we want to use a basis for $H^0(G, \phi G)$ to define a linearly independent set of functions in $H^0(G, \psi G)$ (i.e. we take a basis of almost invariant sets, and make a linearly independent set of almost invariant functions.)

So let $\{E_i\}_{i \in I}$ be a basis for $H^0(G, \phi G)$. Let $\{\overline{E}_i\}_{i \in I}$ denote the corresponding elements of $l^\infty(G)$, defined to be 1 on the set E_i and 0 outside that set. To see the linear independence of the \overline{E}_i , assume that $\sum_{j \in J} a_j \overline{E}_j = 0$ in $H^0(G, \psi G)$, where $a_j \in \mathbb{C}$. Then we have that $g \cdot \sum_{j \in J} a_j \overline{E}_j = 0$ for all $g \in G$, and since $g \cdot \overline{E}_j = \overline{E}_j$ in $H^0(G, \psi G)$, we must have that $\bigcup_{j \in J} E_j = \emptyset$ in $H^0(G, \phi G)$, i.e. $\sum_{j \in J} b_j E_j = 0$ where $b_j = 0, 1$. Since the E_i are a basis in $H^0(G, \phi G)$, we see that all of the $b_j = 0$, and hence all of the $a_j = 0$.

In general, however, these functions will not span $\psi(G)$, at least in the case when G is locally finite. We give an example of a function which cannot be written as a sum of such elements in the results section of this paper. While we do believe $\dim_{\mathbb{Z}_2} H^1(G, \mathbb{Z}_2 G) = \dim_{\mathbb{C}} H^1(G, \mathbb{C}G)$, it is not in the natural way. However, since these elements of $\psi(G)$ are linearly independent, we do see clearly that $\dim_{\mathbb{Z}_2} H^1(G, \mathbb{Z}_2 G) \leq \dim_{\mathbb{C}} H^1(G, \mathbb{C}G)$, and in particular if $H^1(G, \mathbb{C}G) = 0$ then G has one end. Also, if $e(G) > 1$ then $H^1(G, \mathbb{C}G) \neq 0$. For the scope of this paper this alone will be useful.

The most general result on the cohomology groups of G with coefficients in $\mathbb{C}G$ can be found in [4], and it is proven using the theory of groups acting on graphs. While we will focus specifically on the case when G is locally finite, the following theorem gives a complete characterization of when the first cohomology group does or does not vanish.

Theorem 2.([4]) For any nonzero abelian group A , the following are equivalent:

- (a) $e(G) > 1$.
- (b) $H^1(G, AG) \neq 0$.
- (c) One of the following holds:
 - (i) $G = B *_C D$, where $B \neq C \neq D$ and C is finite;
 - (ii) $G = B *_C x$, where C is finite;

- (iii) G is countably-infinite locally-finite.
- (d) $e(G)$ is 2 or ∞ .

In this theorem, $B *_C D$ refers to the free product of B and D amalgamating C , and $B *_C x$ denotes the HNN extension of A by $x: C \rightarrow A$. AG denotes the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Z}G$. The theory of groups acting on graphs actually goes further to produce the following result on the number of ends of G .

Theorem 3. $e(G) = 2$ if and only if G has an infinite cyclic subgroup of finite index.

The case when G is locally-finite is of particular interest in this paper, and we show a more direct proof of G countably-infinite locally finite implies $H^1(G, \mathbb{C}G) \neq 0$ in the last section of this paper. For now, though, we want to look carefully at the case when G is uncountable locally-finite, since in this case $H^1(G, \mathbb{C}G)$ vanishes. We look at why this happens as discussed by Holt in [3]. We start with a lemma regarding functions in $\overline{\mathbb{C}G}$.

Lemma 1. Let H and K be proper subgroups of a periodic group G , with $\langle H, K \rangle = G$. Let $f: G \rightarrow \mathbb{C}$ be constant on each coset $Hg \neq H$ and $Kg \neq K$. Then f is constant on $G - (H \cap K)$. Further, if f is also constant on K , f is constant on all of G .

Proof. Since G is locally finite, we know that given $h \in H - K$ and $k \in K - H$, $(kh^{-1})^n = 1$ and $(hk^{-1})^m = 1$ for some n and m . Therefore, we have a word of the form $hk^{-1}hk^{-1} \dots hk^{-1}h \in K$ and a word of the form $kh^{-1}kh^{-1} \dots kh^{-1}k \in H$. Without loss of generality assume $w_1 := hk^{-1}hk^{-1} \dots hk^{-1}h$ is the shortest word with one of these properties. Then $hk^{-1}hk^{-1} \dots hk^{-1}h \in K$, and so $hk^{-1}hk^{-1} \dots hk^{-1}h = k_1$ for some $k_1 \in K$. By rearranging terms, we see that

$$\begin{aligned} hk^{-1}hk^{-1} \dots hk^{-1}h &= k_1 \\ \Rightarrow hk_1^{-1}hk^{-1} \dots hk^{-1}h &= k. \end{aligned}$$

Since $h \notin K$ and f is constant on $Kg \neq K$, we see that $f(h) = f(k^{-1}h)$. Now, $k^{-1} \notin H$, so $k^{-1}h \notin H$, and so f constant on $Hg \neq H$ implies that $f(k^{-1}h) = f(hk^{-1}h)$. Note here that no terminal segment of w_1 can be in $H \cup K$, since this would contradict the fact that w_1 is the shortest word having one of the previously discussed properties. So, we use that f is constant on the cosets $Hg \neq H$ and

$Kg \neq K$ to get that

$$\begin{aligned} f(h) &= f(k^{-1}h) = f(hk^{-1}h) = \dots \\ &= f(hk_1^{-1}hk^{-1}\dots hk^{-1}hk^{-1}h) = f(k). \end{aligned}$$

Therefore, f is constant on $(H \cup K) - (H \cap K)$.

Now let $g \in G - (H \cap K)$. Since $G = \langle H, K \rangle$, we can write $g = g_1 \dots g_r$ where $g_i \in H \cup K$ for all i . We can also choose r to be minimal, so that $g_r \in H \cup K$, but $g_i \dots g_r \notin H \cup K$ for any $1 < i < r$. (We can do this by replacing g_i with $g_i \dots g_r$ if $g_i \dots g_r \in H \cup K$ for some i). Note here that by assuming r is minimal, we also have that $g_r \notin H \cap K$, since this would imply that $g_1 g_2 \in H \cup K$.

So, if $g_r \in H$, $g_{r-1} \notin H$, so $f(g_r) = f(g_{r-1}g_r)$. Then, $g_{r-1}g_r \notin H \cup K$, so $f(g_{r-1}g_r) = f(g_{r-2}g_{r-1}g_r)$. Continuing in this way, we see that $f(g_r) = f(g_{r-1}g_r) = \dots = f(g_1 g_2 \dots g_r) = f(g)$, and since $g_r \in (H \cup K) - (H \cap K)$, we have that f is constant on $G - (H \cap K)$.

To see the last part of the lemma, assume further that f is constant on K . Then, we know that f is constant on $G - (H \cap K)$, and since $K - H$ is non-empty, let $g_1 \in K - H$. For any $g_2 \in H \cap K$, $f(g_1) = f(g_2)$, and thus f is constant on all of G . ■

The lemma will be useful in proving an important theorem, however, before we look at the theorem, we take a closer look at what the elements of $H^1(G, A)$ look like when A is a G -submodule of $\overline{\mathbb{C}G}$. This will be helpful in trying to understand when an element is or is not zero in the quotient.

Proposition 2. Let A be a G -submodule of $\overline{\mathbb{C}G}$. Given an element $f \in Z_G$, f can be written in the form $f(g) = g \cdot \alpha - \alpha$ for some $\alpha \in \overline{\mathbb{C}G}$. Conversely, if $\alpha \in \overline{\mathbb{C}G}$ is such that $g \cdot \alpha - \alpha \in A$ for all $g \in G$, then $g \cdot \alpha - \alpha \in Z_G$.

Proof. To see the first part of the proposition, let $f \in H^1(G, A)$, and define

$\alpha(g) = -f(g)(g)$ for all $g \in G$. Then given any $g, h \in G$, we have that

$$\begin{aligned} (g \cdot \alpha - \alpha)(h) &= (g \cdot \alpha)(h) - \alpha(h) = \alpha(g^{-1}h) - \alpha(h) \\ &= -f(g^{-1}h)(g^{-1}h) + f(h)(h) \\ &= -f(g^{-1}h)(g^{-1}h) + f(gg^{-1}h)(h) \\ &= -f(g^{-1}h)(g^{-1}h) + (f(g) + g \cdot f(g^{-1}h))(h) \\ &= -f(g^{-1}h)(g^{-1}h) + f(g)(h) + f(g^{-1}h)(g^{-1}h) = f(g)(h), \end{aligned}$$

which shows that $f(g) = g \cdot \alpha - \alpha$ for all g .

For the converse, note that $f(g) = g \cdot \alpha - \alpha$ is a map from $G \rightarrow A$ by assumption, so all that we need to show is that $f(g) - f(gh) + g \cdot f(h) = 0$ for any $g, h \in G$. So given $g, h \in G$,

$$f(g) - f(gh) + g \cdot f(h) = g \cdot \alpha - \alpha - (gh \cdot \alpha - \alpha) + g \cdot (h \cdot \alpha - \alpha) = 0.$$

Therefore, $f \in Z_G$, and thus it represents an element in the quotient $H^1(G, A)$. ■

We now move to proving the following theorem:

Theorem 4. Let G be a locally finite group and $f \in H^1(G, \mathbb{C}G)$. If there exists an infinite, proper subgroup H of G such that $f(H) \subseteq \mathbb{C}H$, then f is of the form $f(g) = g \cdot \alpha - \alpha$ for some $\alpha \in \mathbb{C}G$. (i.e. $f \in B_G$)

Proof. We know that if we define $\alpha \in \overline{\mathbb{C}G}$ by $\alpha(g) = -f(g)(g)$, then we have $f(g) = g \cdot \alpha - \alpha$ for all $g \in G$ by the previous proposition. Now note that for any constant $\beta \in \overline{\mathbb{C}G}$, we have $f(g) = g \cdot (\alpha + \beta) - (\alpha + \beta)$ for all $g \in G$. Therefore, if we can show that α is constant outside some finite subset of G , then we can write $f(g) = g \cdot (\alpha - \beta) - (\alpha - \beta)$ where $\alpha - \beta \in \mathbb{C}G$, which is the desired result.

Let $k \in G - H$, and note that $K := \langle k \rangle$ is a finite subgroup of G . Since $f(k^i)$ has finite support for all i , for each i we have that $f(k^i)(k^i g) = 0$ for all but finitely many $g \in G$. Also note that if $f(k^i)(k^i g) = 0$ for all i we get that

$$\alpha(k^i g) = -f(k^i g)(k^i g) = -f(k^i)(k^i g) - k^i f(g)(k^i g) = -f(g)(g) = \alpha(g),$$

and thus α is constant on the coset Kg . This implies that there is a finite subset of G such that α is not constant on Kg . Let H_1 be the finite subgroup of $\langle H, k \rangle$

generated by the $g \in \langle H, k \rangle$ such that α is not constant on Kg . Then each $g \in H_1$ can be written as a word on H and K , and define H_0 to be the subgroup of H generated by the elements of H in each of these words. Since H_1 is finite and each word has finite length, we see that H_0 is a finite subgroup of H . Then, by defining $L = \langle H_0, k \rangle$, we see that for any $g \in \langle H, k \rangle - L$, α is constant on Kg .

We claim that α is constant outside of $H \cap L$, noting that L is finitely generated and therefore finite, and thus $H \cap L$ is a finite subgroup of G . Since L is a finite subgroup of $\langle H, k \rangle$, there exists some $h \in H - L$, and by definition we have $k \notin H$, so L and H are both proper subgroups of $\langle H, k \rangle$. Therefore, if we can show that α is constant on cosets $Hg \neq H$ and $Lg \neq L$, then we can use the lemma to deduce that α is constant on $\langle H, L \rangle - (H \cap L) = \langle H, k \rangle - (H \cap L)$.

To see that α is constant on $Hg \neq H$, we simply use that $f(H) \subseteq \overline{\mathbb{C}H}$ to see that $f(h)(hg) = 0$ for all $g \in G - H$. Therefore, given $g \in \langle H, k \rangle - H$ we have

$$\alpha(hg) = -f(hg)(hg) = -f(h)(hg) - h \cdot f(g)(hg) = -f(g)(g) = \alpha(g)$$

and thus α is constant on Hg .

To show that α is constant on $Lg \neq L$, first note that if $lg \in H$ for any $l \in L$, then $Lg = Llg$, and thus we can make the assumption that if $lg \in H$ for any $l \in L$ then $g \in H$. After making this assumption, fix a coset $Lg \neq L$. If $K = L$ then α is constant on $Kg \neq K$ as cosets in $\langle H, k \rangle$ by the definition of L , so we have that α is constant on Lg .

Now assume that there exists $g \in L - K$. Note that α is constant on Lg if and only if $\alpha(lg) = \alpha(g)$ for all $l \in L$, if and only if $\alpha g^{-1}(l) = \alpha g^{-1}(1)$ for all $l \in L$, if and only if αg^{-1} is constant on L . Since we have assumed that there exists $g \in (H \cap L) - K$, and certainly $k \notin H \cap L$, both $H \cap L$ and K are proper subgroups of $L = \langle H \cap L, K \rangle$. Therefore, to show that αg^{-1} is constant on L , we consider the cosets Kl and $(H \cap L)l$ in L , and we aim to show that αg^{-1} is constant on all cosets Kl and on all cosets $(H \cap L)l \neq (H \cap L)$, and then use the second part of the lemma to conclude that αg^{-1} is constant on $\langle K, H \cap L \rangle = L$.

First, we look at the cosets Kl . Since $g \in \langle H, k \rangle - L$, we have that $lg \notin L$ for any $l \in L$. Therefore, by the definition of L we have that α is constant on Klg for all $l \in L$. This gives us that

$$\alpha g^{-1}(kl) = \alpha(klg) = \alpha(lg) = \alpha g^{-1}(l),$$

for all $k \in K$ and $l \in L$, and thus αg^{-1} is constant on Kl for all $l \in L$.

Now we consider the cosets $(H \cap L)l \neq (H \cap L)$. To show that αg^{-1} is constant on these cosets, we use the assumption that if $lg \in H$ for any $l \in L$, then $g \in H$.

If $lg \in H$ for some $l \in L$, then by our assumption we have that $g \in H$, which implies that $l \in H$. Therefore, $(H \cap L)l = (H \cap L)$. If $lg \notin H$ for any $l \in L$, then α is constant on Hlg for all $l \in L$, and hence on $(H \cap L)lg$ for all $l \in L$. Therefore, for any $x \in H \cap L$ and $l \in L$, we have that

$$\alpha g^{-1}(xl) = \alpha(xlg) = \alpha(lg) = \alpha g^{-1}(l),$$

which implies that αg^{-1} is constant on $(H \cap L)l$.

So, αg^{-1} is constant on Kl for all $l \in L$ and on $(H \cap L)l \neq H \cap L$, and we can now apply the second part of the lemma to conclude that αg^{-1} is constant on L , or, equivalently α is constant on Lg . This gives us that α is constant on Lg for all $g \in \langle H, k \rangle - L$ and Hg for all $g \in \langle H, k \rangle - H$, and since H and L are both proper subgroups of $\langle H, L \rangle$, we apply the first part of the lemma to deduce that α is constant on $\langle H, L \rangle - (H \cap L)$.

Finally, let $k' \in G - H$ such that $k \neq k'$. By the same proof we see that α is constant on $\langle H, L' \rangle - (H \cap L')$, so $\alpha(h) = \alpha(k')$ for all but finitely many $h \in H$. We also have that $\alpha(h) = \alpha(k)$ for all but finitely many $h \in H$. Therefore, since H is countable, there exists $h \in H$ such that $\alpha(k') = \alpha(h) = \alpha(k)$, and thus α is constant on $G - (H \cap L)$, and this concludes the proof. ■

Finally, we arrive at the desired result:

Theorem 5. Let G be an uncountable locally-finite group. Then $H^1(G, \mathbb{C}G) = 0$.

Proof. Let $f \in H^1(G, \mathbb{C}G)$ and let H_1 be any countable subgroup of G . Then the support of $f(h)$ is finite for all $h \in H_1$, so we can find a countable subgroup H_2 such that $H_1 \subseteq H_2$ and $f(H_1) \subseteq \mathbb{C}H_2$. By induction, we can create a chain of countable subgroups of G such that $f(H_i) \subseteq \mathbb{C}H_{i+1}$ for all i and $H_1 \subseteq H_2 \subseteq H_3 \dots$. Letting $H = \bigcup_{i=1}^{\infty} H_i$ we see that H is a countable subgroup, and thus a proper

subset of G , with $f(H) \subseteq \mathbb{C}H$. Therefore, $f \in B_G$ by the previous theorem which implies that $H^1(G, \mathbb{C}G) = 0$. ■

Corollary 1. Uncountable locally finite groups have one end.

Proof. If G is an uncountable locally-finite group, $H^1(G, \mathbb{C}G) = 0$ which implies that $e(G) = 1$. ■

We now make a quick note of some interest regarding locally-finite groups. When G is uncountable, $H^1(G, \mathbb{C}G) = 0$, but when G is countable, $H^1(G, \mathbb{C}G) \neq 0$, and, in fact, $\dim_{\mathbb{C}} H^1(G, \mathbb{C}G) = \infty$. This gives rise to the natural question of whether or not something similar happens when dealing with coefficients in $l^2(G)$. We will see in the results sections that while we don't have an embedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, l^2(G))$ when G is countably-infinite locally-finite, we can still show that $H^1(G, l^2(G)) \neq 0$ in that case. Of interest in future work finding whether or not $H^1(G, l^2(G))$ vanishes when G becomes uncountable, as it does with coefficients in $\mathbb{C}G$.

Chapter 4

$H^1(G, l^2(G))$

We now turn to the L^2 -cohomology of a group G . First note that if G is a finite group, then $l^2(G) = \mathbb{C}G$, and so we again have that $H^1(G, l^2(G)) = 0$. However, if we let G be an infinite group, $l^2(G)$ is the closure of $\mathbb{C}G$ in the l^2 -norm, and much changes with regards to the cohomology. We start with trying to determine when the group $H^1(G, l^2(G))$ is even a Hausdorff space (i.e. when is B_G closed in Z_G). The following lemma will give some insight on the issue.

Definition. For this section, let $l^2(G)^G$ denote the space of functions $\{f: G \rightarrow l^2(G)\}$.

Lemma 2. Let G be an infinite group. Then there is a continuous map $f: l^2(G) \rightarrow B_G$ given by $f(e) = g \cdot e - e$, and this map has a continuous inverse if and only if G is non-amenable.

Proof. To say that the map is continuous (where the topology on $l^2(G)^G$ is pointwise convergence) is the same as saying any net e_λ such that $e_\lambda \rightarrow 0$ implies that $g \cdot e_\lambda - e_\lambda \rightarrow 0$. Since $\|e_\lambda\|_2 \rightarrow 0$, we see that for any $g \in G$, $\|g \cdot e_\lambda - e_\lambda\|_2 \leq \|g \cdot e_\lambda\|_2 + \|e_\lambda\|_2 \rightarrow 0$, and thus the map is continuous.

Now assume that G is amenable. Let $\{F_\lambda\}_{\lambda \in L}$, be the compact (finite) sets of G , where L is a directed set by inclusion. Let $\epsilon_\lambda = \frac{1}{|F_\lambda|}$ for $\lambda \in L$, and note that this is a net contained in \mathbb{R}^+ with $\lim_{\lambda \in L} \epsilon_\lambda = 0$. Then, by the Følner condition, for each $\lambda \in L$, find U_λ such that $|U_\lambda \Delta g \cdot U_\lambda| / |U_\lambda| < \epsilon_\lambda$ for all $g \in F_\lambda$. Then we can

let

$$e_\lambda = \frac{\sum_{u \in U_\lambda} u}{\|\sum_{u \in U_\lambda} u\|_2},$$

noting that $\|e_\lambda\|_2 = 1$. Then, given any $g \in G$, we have that $g \in F_{\lambda'}$ for some λ' , and we get that for $\lambda > \lambda'$ (where $>$ is the operation in L), we have

$$\begin{aligned} \|g \cdot e_\lambda - e_\lambda\|_2 &= \frac{\|\sum_{u \in U_\lambda \Delta g \cdot U_\lambda} u\|_2}{\|\sum_{u \in U_\lambda} u\|_2} \\ &= \sqrt{\frac{|U_\lambda \Delta g \cdot U_\lambda|}{|U_\lambda|}} < \sqrt{\epsilon_\lambda} \rightarrow 0. \end{aligned}$$

Therefore, we have a net converging in B_G , but not converging in $l^2(G)$, and thus the inverse map is not continuous.

Now assume that the inverse map is not continuous and, for contradiction, that G is non-amenable. Then we must have a countable subgroup $H = h_1, h_2, h_3, \dots$ of G with H non-amenable (since if every finitely generated subgroup of G is amenable, then G is amenable). Since the inverse map is not continuous, there exists a net $\{e_\lambda\}_{\lambda \in L} \subset l^2(G)$ with $\lim_{\lambda \in L} g \cdot e_\lambda - e_\lambda = 0$ for all $g \in G$, but $\lim_{\lambda \in L} e_\lambda \neq 0$. Fix $\epsilon > 0$ and $F \subset H$ where F is finite. We aim to show that there exists $b \in l^1(G)$ with $\|b\|_1 = 1$ and $\|g \cdot b - b\| < \epsilon$ for all $g \in F$ (i.e. that H has the Reiter Property).

Since F is finite $F \subset \{h_1, h_2, \dots, h_n\}$ for some n . Since $e_\lambda \not\rightarrow 0$, there exists $\delta > 0$ such that for every $\lambda \in L$, there exists $x > \lambda$ such that $\|e_x\|_2 > \delta$. Let $X = \{x \in L : \|e_x\|_2 > \delta\} \subset L$, so that $\{e_x\}_{x \in X}$ is a subnet of e_λ . Since

$$\begin{aligned} \|g \cdot e_\lambda - e_\lambda\|_2 \rightarrow 0 &\Rightarrow \|g \cdot e_x - e_x\|_2 \rightarrow 0 \\ &\Rightarrow \|g \cdot \frac{e_x}{\|e_x\|_2} - \frac{e_x}{\|e_x\|_2}\|_2 \rightarrow 0 \end{aligned}$$

for all $g \in G$. So, we can find x_1 such that for $x \geq x_1$, $\|h_1 \cdot \frac{e_x}{\|e_x\|_2} - \frac{e_x}{\|e_x\|_2}\|_2 < \frac{\epsilon}{2}$. Then, we can find $x_2 > x_1$ such that for $x \geq x_2$, we have $\|h_2 \cdot \frac{e_x}{\|e_x\|_2} - \frac{e_x}{\|e_x\|_2}\|_2 < \frac{\epsilon}{2}$. Continuing in this way we arrive at x_n such that $\|h_i \cdot \frac{e_{x_n}}{\|e_{x_n}\|_2} - \frac{e_{x_n}}{\|e_{x_n}\|_2}\|_2 < \frac{\epsilon}{2}$ for all h_i , $1 \leq i \leq n$. Letting $f = \frac{e_{x_n}}{\|e_{x_n}\|_2}$, we see that $\|f\|_2 = 1$, $\|g \cdot f - f\| < \frac{\epsilon}{2}$ for all $g \in F$.

To finish, let $b = |f|^2$ and note that $b \in l^1(G)$ with $\|b\|_1 = 1$, $b \geq 0$, and given any

$g \in F$,

$$\begin{aligned} \|h \cdot b - b\|_1 &= \sum_{g \in G} ||f(h^{-1}g)|^2 - |f(g)|^2| \\ &\leq \sum_{g \in G} |f(h^{-1}g)^2 - f(g)^2| \\ &\leq \sum_{g \in G} |f(h^{-1}g) - f(g)|(|f(h^{-1}g)| + |f(g)|). \end{aligned}$$

then, Hölder's inequality gives us,

$$\begin{aligned} \sum_{g \in G} |f(h^{-1}g) - f(g)|(|f(h^{-1}g)| + |f(g)|) &\leq \|f(h^{-1}g) - f(g)\|_2 \|(|f(h^{-1}g)| + |f(g)|)\|_2 \\ &\leq \|f(h^{-1}g) - f(g)\|_2 (\|f(h^{-1}g)\|_2 + \|f(g)\|_2) \\ &= 2\|f(h^{-1}g) - f(g)\|_2 < \epsilon \end{aligned}$$

since $\|f\|_2 = 1$. Therefore, H has the Reiter property and is amenable, which is a contradiction. \blacksquare

We claim that this Lemma completely determines when a group will be Hausdorff in the case when G is countable. When G is countable, the topology of pointwise convergence on the space $l^2(G)^G$ (the set of all $f: G \rightarrow l^2(G)$) is given by the countable family of seminorms, $\|f(g_i)\|_2$, and thus $l^2(G)^G$ is a Frechet space. So, since a closed subspace of a Frechet space is a Frechet space, and any bijective, continuous map from a Frechet space to a Frechet space has a continuous inverse, we know that if G is amenable then $H^1(G, l^2(G))$ cannot be Hausdorff. When looking at the case when G is non-amenable, we actually don't need G to be countable, as seen in the following Theorem.

Theorem 6. Let G be a nonamenable group. Then $H^1(G, l^2(G))$ is Hausdorff.

Proof. The goal is to prove that B_G is a closed subspace of Z_G , and thus the quotient is Hausdorff. If G is nonamenable, then the map $l^2(G) \rightarrow B_G$ has a continuous inverse. When this is the case we note that the topology on $l^2(G)$, which is a Banach space, makes B_G into a complete metric space. So we aim to show that given any $x \in \overline{B}_G$, there exists a Cauchy net in B_G converging to x , and thus $x \in B_G$.

To see this first note that $l^2(G)$ is a locally convex space, since it's topology is given by the seminorms $\{p_g\}_{g \in G}$, where for $e \in l^2(G)^G$, $p_g(e) = \|e(g)\|_2$. So if

we let $\phi = \{F_\alpha\}_{\alpha \in A}$ be the set of all finite subsets of G directed by inclusion, we can get a directed family of seminorms, $\{p_{F_\alpha}\}_{\alpha \in A}$ where $p_{F_\alpha} = \sum_{g \in F_\alpha} p_g$, and this directed family defines the same topology (namely, the topology of pointwise convergence) on G .

Since $x \in \overline{B_G}$, any neighborhood of x in $l^2(G)^G$ will intersect with B_G nontrivially. Therefore, for any $\alpha \in A$, we can find x_α such that $p_{F_\alpha}(x_\alpha - x) < \epsilon_\alpha$, where we let $\epsilon_\alpha = \frac{1}{|F_\alpha|}$. Since the entourages on the uniform space $l^2(G)^G$ are given by the directed family of seminorms from the previous paragraph, in order to show that $\{x_\alpha\}$ is Cauchy, given any seminorm p_{F_β} and $\epsilon > 0$, we can find γ such that for $\lambda, \mu > \gamma$, $p_{F_\beta}(f_\lambda - f_\mu) < \epsilon$. Let γ be such that $F_\beta \subset F_\gamma$ and $\epsilon_\gamma < \epsilon/2$. Then we have

$$\begin{aligned} p_{F_\beta}(f_\lambda - f_\mu) &\leq p_{F_\gamma}(f_\lambda - f_\mu) \\ &\leq p_{F_\gamma}(f_\mu - f) + p_{F_\gamma}(f_\lambda - f) \\ &\leq p_{F_\mu}(f_\mu - f) + p_{F_\lambda}(f_\lambda - f) \\ &< \epsilon_\mu + \epsilon_\lambda \leq 2\epsilon_\gamma < \epsilon. \end{aligned}$$

Therefore, x_α is a Cauchy net, and thus, using the fact that B_G is a complete metric space, it converges to a unique element $x \in B_G$. This implies that B_G is closed in $l^2(G)^G$, and thus in Z_G , and we must have that $H^1(G, l^2(G))$ is Hausdorff. ■

In the case when G is countable we can see this using sequences instead of nets, since in this case $l^2(G)^G$ is a metric space, but the result holds even in the more general case. This gives the following corollary.

Corollary 2. Let G be a countable group. Then $H^1(G, l^2(G))$ is Hausdorff if and only if G is non-amenable.

Proof. This follows directly from the Theorem and the discussion immediately preceding it. ■

Take careful note, here, that we are not sure whether this corollary holds when G is uncountable. While the Theorem shows that G non-amenable certainly still implies that $H^1(G, l^2(G))$ is Hausdorff, if G is amenable all we know is that B_G is not a Frechet space. However, since $l^2(G)^G$ is no longer a Frechet space when G is uncountable, this does not necessarily imply that B_G is not closed in Z_G .

Also, this corollary already gives us an instance of the cohomology of a group with coefficients in $\mathbb{C}G$ differing from the cohomology with coefficients in $l^2(G)$. Namely, if G is a countable, amenable, group which is not a free product with amalgamation, HNN extension, or locally finite, then $H^1(G, \mathbb{C}G) = 0$ but $H^1(G, l^2(G))$ is not even Hausdorff. To see an example of such a group, consider $G = \mathbb{Z} \times \mathbb{Z}$. This is abelian and therefore amenable, and also is torsion-free. Any non-trivial, torsion-free free product contains $F(x, y)$ (the free group on two generators) as a subgroup, and is therefore non-amenable, so G is certainly not a free product, and the torsion free version of an HNN extension is just the infinite cyclic group, so G is not an HNN extension. Finally, G is certainly not locally finite since it has $\langle (1, 0) \rangle = \mathbb{Z}$ which is certainly not finite, therefore, $H^1(G, \mathbb{C}G) = 0$ but $H^1(G, l^2(G)) \neq 0$.

The following proposition helps us extract further information about $H^1(G, l^2(G))$ from our knowledge of $H^1(G, \mathbb{C}G)$ in the finitely generated case.

Proposition 3. Let G be a finitely generated group. Then the G -module embedding of $\mathbb{C}G \hookrightarrow l^2(G)$ induces an embedding of groups $H^1(G, \mathbb{C}G) \hookrightarrow H^1(G, l^2(G))$.

Letting ψ be the identity and ϕ be the natural embedding of $\mathbb{C}G$ into $l^2(G)$, we see that the two homomorphisms are compatible (as in the definition given in the first section of this paper), and therefore we have a group homomorphism $\theta: H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^2(G))$. To see that θ is injective, let $b \in \ker \theta$. Then there is an $\alpha \in l^2(G)$ such that $b(g) = g \cdot \alpha - \alpha$. We claim that $\alpha \in \mathbb{C}G$, i.e. α has finite support.

To see this, let $\alpha = \sum_{g \in G} a_g g$, and note that since $b \in H^1(G, \mathbb{C}G)$, $b(g) \in \mathbb{C}G$ for all $g \in G$. Therefore, if we define $\phi(h) = \{g \in G : a_{h^{-1}g} - a_g \neq 0\}$, we see that $|\phi(h)| < \infty$ for all $h \in G$ (this is precisely the condition that $b(h) \in \mathbb{C}G$ for all $h \in G$). Let S be a generating set for G , and assume it is closed under inverses. Let $F(G) = \bigcup_{s \in S} \phi(s)$, and note that this is still a finite set. Let X be the Cayley graph of G , with vertex set G and edge set $\{(g, sg), s \in S\}$. Since $F(G)$ is finite, we know that $X - F(G)$ has finitely many connected components.

Let q and r be in an infinite component of $X - F(G)$. Then $r = s_1 s_2 \dots s_n q$ for some $s_i \in S$. Since this path is completely outside of $F(G)$, we know that $a_{s_{i-1} s_i \dots s_n q} = a_{s_i \dots s_n q}$ for all i . Therefore, $a_q = a_{s_1 s_2 \dots s_n q} = a_r$, and this holds for any r and q in an infinite component of $X - F(G)$. This implies that α is constant on all infinite components of $X - F(G)$, and since $\alpha \in l^2(G)$, it must be equivalently zero on all infinite components of $X - F(G)$. But since there are only finitely

many finite components of $X - F(G)$, we have that $\{g \in G : a_g \neq 0\}$ is a finite set. Therefore, $\alpha \in \mathbb{C}G$, which tells us that $b(g) = g \cdot \alpha - \alpha$ where $\alpha \in \mathbb{C}G$, and thus b is a 1-cocycle and we have that θ is injective. ■

This injection certainly gives us more information on when $H^1(G, l^2(G)) \neq 0$, and we can also extract some information on the number of ends of a group given its L^2 cohomology, as seen in the next two corollaries.

Corollary 3. Let G be a finitely generated group and suppose G satisfies one of the following:

1. $G = B *_C D$ where $B \neq C \neq D$ and C is finite;
2. $G = B *_C x$ where C is finite;

then $H^1(G, l^2(G)) \neq 0$.

Corollary 4. Let G be a finitely generated group. Then if $H^1(G, l^2(G)) = 0$, G is non-amenable and $e(G) = 1$.

Proof. We know that if G satisfies (1) or (2), $H^1(G, \mathbb{C}G) \neq 0$ (from the last section), so Corollary 1 follows from the embedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, l^2(G))$.

Corollary 4 also follows from the embedding, since $H^1(G, l^2(G)) = 0 \Rightarrow H^1(G, \mathbb{C}G) = 0 \Rightarrow e(G) = 1$. ■

Chapter 5

Results

Our first result is on the topic of the embedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, l^2(G))$. In particular, the following proposition shows that if G is countably-infinite locally-finite, the map induced from the natural embedding of $\mathbb{C}G$ into $l^2(G)$ is no longer an embedding itself. To show this, we construct a function which is non-zero in $H^1(G, \mathbb{C}G)$ but whose image is zero in $H^1(G, l^2(G))$. Because the induced homomorphism is no longer an embedding, even though $H^1(G, \mathbb{C}G) \neq 0$, $H^1(G, l^2(G))$ could vanish for G countably-infinite locally-finite. This turns out not to be the case, however, as we show later. Before looking at the construction of this function, we prove a useful lemma about the uniqueness of an element of $H^1(G, A)$, in particular when $A = \mathbb{C}G$ or $A = l^2(G)$.

Lemma 3. Let A be a G -submodule of $\overline{\mathbb{C}G}$ and $f \in H^1(G, A)$. If $f(g) = g \cdot \alpha - \alpha = g \cdot \beta - \beta$ for all g and $\alpha, \beta \in \overline{\mathbb{C}G}$, then $\alpha = \beta + C$ where C is a constant function in $\overline{\mathbb{C}G}$.

Proof.

$$g \cdot \alpha - \alpha = g \cdot \beta - \beta \Rightarrow \alpha(g^{-1}h) - \alpha(h) = \beta(g^{-1}h) - \beta(h)$$

for all $g, h \in G$. So, given any $g_1, g_2 \in G$, and letting $g^{-1} = g_1 g_2^{-1}$ and $h = g_2$, we see that we must have

$$\alpha(g_1) - \beta(g_1) = \alpha(g_2) - \beta(g_2).$$

As this is true for any $g_1, g_2 \in G$, we see that $\alpha = \beta + C$ for some constant $C \in \overline{\mathbb{C}G}$.

■

We now look at constructing a non-zero element of $H^1(G, \mathbb{C}G)$ whose image is zero in $H^1(G, l^2(G))$ (under the group homomorphism induced by the embedding of $\mathbb{C}G$ into $l^2(G)$).

Proposition 4. Let G be countably-infinite locally-finite. Then the group homomorphism $\phi: H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^2(G))$ induced by the embedding $\theta: \mathbb{C}G \hookrightarrow l^2(G)$ is not an embedding.

Since G is countable, we can write $G = \{g_1, g_2, g_3, \dots\}$. Then, letting $G_i = \langle g_1, \dots, g_i \rangle$, we can write $G = \cup_{i=1}^{\infty} G_i$, and note that G_i is finite for all i . We can also assume that $G_i \neq G_{i+1}$ for any i (by throwing out any duplicates). Let $e_i = \sum_{g \in G_i} 1 \cdot g$, an element of $\mathbb{C}G$. Then define

$$\alpha = \sum_{i=1}^{\infty} 2^{-i} \frac{e_i}{|G_i \setminus G_{i-1}|}$$

where we let $G_0 = \emptyset$. Note that if this is a summable function, then it is square summable (i.e. $\alpha \in l^2(G)$). To see that it is summable, if we write $\alpha = \sum_{g \in G} a_g g$, then we can write $\sum_{g \in G} |a_g|$ as

$$\begin{aligned} \sum_{i=1}^{|G_i|} \frac{\sum_{j=1}^{\infty} 2^{-j}}{|G_i|} + \sum_{i=|G_1|+1}^{|G_2|} \frac{\sum_{j=2}^{\infty} 2^{-j}}{|G_2| - |G_1|} + \dots \\ = 1 + \frac{1}{2} + \frac{1}{4} + \dots = 2, \end{aligned}$$

and thus $\alpha \in l^2(G)$. We aim to show that there exists a non-zero element of $H^1(G, \mathbb{C}G)$ which maps to zero in $H^1(G, l^2(G))$. So let us examine the map $f(g) = g \cdot \alpha - \alpha$. Given any $g \in G$, then there exists n such that $g \in G_i$ for $i \geq n$. We claim that the support of $f(g)$ is contained in G_n . To see this, note that for any $h \in G \setminus G_n$, $h \in G_{i+1} \setminus G_i$ for some $i \geq n$, so $hG_i = g^{-1}hG_i$ since $g^{-1} \in G_n \subseteq G_i$. Therefore, we have that $\alpha(g^{-1}h) = \alpha(h) \Rightarrow (g \cdot \alpha - \alpha)(h) = 0$, and, thus, $f(g)$ has finite support. So for any $g \in G$, $f(g) \in \mathbb{C}G$, and since

$$f(g) - f(gh) + gf(h) = g\alpha - \alpha - (gh\alpha - \alpha) + g(h\alpha - \alpha) = 0,$$

we must have that $f(g) \in Z_G$, and thus is an element in the quotient $H^1(G, \mathbb{C}G)$. By the lemma, if $f(g) = g \cdot \beta - \beta$ for some $\beta \in \overline{\mathbb{C}G}$, then $\beta = \alpha + C$ for some constant C in $\overline{\mathbb{C}G}$. But since α is only constant on finite sets, there cannot exist

a constant C in $\mathbb{C}G$ such that $\alpha + C \in \mathbb{C}G$, so $f \notin B_G$.

Note here that the map $\phi: H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^2(G))$ is best understood when we view $f \in H^1(G, A)$ as a map $f: G \rightarrow A$. Then we see that $\phi(f)(g) = \theta(f(g))$. So, $g \cdot \alpha - \alpha$ is a non-zero element of $H^1(G, \mathbb{C}G)$ whose image is zero in $H^1(G, l^2(G))$, since $\alpha \in l^2(G)$. Therefore, for a countably infinite, locally finite group G , the map $H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^2(G))$ is not an embedding. ■

Corollary 5. Let G be a countably-infinite locally-finite group. Then $H^1(G, \mathbb{C}G) \neq 0$.

Proof. The function constructed in the previous proposition gives a non-zero element of $H^1(G, \mathbb{C}G)$. ■

Note here that this corollary can also be seen using the techniques of groups acting on graphs, as seen in [4], and the improved version of this corollary is given in the section on $H^1(G, \mathbb{C}G)$.

A similar construction yields a non-zero element of $H^1(G, l^2(G))$, which tells us that although the induced map on the cohomology groups is not an embedding, we still have that $H^1(G, l^2(G)) \neq 0$ for G countably infinite locally-finite.

Proposition 5. Let G be countably-infinite locally-finite. Then $H^1(G, l^2(G)) \neq 0$.

Proof. Given $g \in G$, we know that $g \in G_i \setminus G_{i-1}$ for some i , where the G_i are as in the first proposition. Now define $\alpha \in \overline{\mathbb{C}G}$ by $\alpha(g) = \frac{1}{\sqrt{i}}$. Note that $\alpha \notin l^2(G)$ since $\sum_{g \in G} |\alpha_g|^2 \geq \sum_{i=1}^{\infty} \frac{1}{i} = \infty$. We see by the argument in the previous proposition that $g \cdot \alpha - \alpha$ is a non-zero element of $H^1(G, \mathbb{C}G)$, so we consider the image of this element in $H^1(G, l^2(G))$. Assume for contradiction that $g \cdot \alpha - \alpha = g \cdot \beta - \beta$ for some $\beta \in l^2(G)$. Then by the Lemma we know that $\beta = \alpha + C$ and thus $\|\beta\|_2 = \|\alpha + C\|_2 = \infty$, which implies $\beta \notin l^2(G)$ which is a contradiction. Therefore, the image of $g \cdot \alpha - \alpha$ in $H^1(G, l^2(G))$ is non-zero, and thus $H^1(G, l^2(G)) \neq 0$. ■

Before moving on to our final results, we note that the non-zero element of $H^1(G, \mathbb{C}G)$ constructed for these propositions shows that a basis for the almost invariant subsets of G does not span the space of almost invariant functions in $H^1(G, \mathbb{C}G)$, at least in the case when G is countably-infinite locally-finite. These functions are almost invariant (since each $g \in G$ only changes the coefficients of finitely many g , as discussed in the first proposition), but the functions achieve infinitely many values in \mathbb{C} . While the function is constant only on finite sets, there are infinitely many of these, and therefore the support of the function (namely, all of G) cannot be written as the sum of finitely many constant functions whose support is a basis set of the almost invariant subsets of G .

The next result gives us a characterization of some groups that have vanishing first L^2 -cohomology groups.

Proposition 6. Let G be a group with an infinite, countable, normal subgroup N . If $H^1(N, l^2(N)) = 0$, then $H^1(G, l^2(G)) = 0$.

Proof. To see the proof of the proposition, consider the following exact sequence from group cohomology

$$0 \rightarrow H^1(G/N, A^N) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(N, A|_N)^{G/N} \quad (5.1)$$

where A is any G -module, and $H^1(N, A|_N)^{G/N}$ denotes the elements of $H^1(N, A|_N)$ fixed by G/N action. Applying this exact sequence to $A = l^2(G)$, and noting that since N is infinite, $l^2(G)^N = 0$, we see that the restriction map is injective. So we aim to show that $H^1(N, l^2(G)|_N) = 0$, which will then give us that $H^1(G, l^2(G)) = 0$.

Claim: Let G be a group with an infinite, countable, normal subgroup N . Then if $H^1(N, l^2(N)) = 0$, we must have $H^1(N, l^2(G)|_N) = 0$.

Proof of Claim. Let $l^2(G) = \bigoplus_{t \in T} l^2(tN)$ where T is a subset of G ranging over all cosets of N in G . Then let $b \in Z^1(N, l^2(G)|_N)$. Denumerate N , and we see that $b(n_i) \in l^2(G)|_N$, and thus has countable support. Therefore, for each i , there exists a countable subset of T where $b(n_i)$ is non-zero, and since N is countable, we have $\{t_j\}_1^\infty \subset T$ such that $(b(n_i))(x) = 0$ for all $x \notin t_j N$ for some j and all i .

Therefore, we can write $b(n) = (t_1 b_1(n), t_2 b_2(n), \dots)$ where $b_i(n) \in l^2(N)$, and thus there exists $f_i \in l^2(N)$ such that $b_i(n) = n f_i - f_i$. Then let $F_M = (t_1 f_1, t_2 f_2, \dots, t_M f_M, 0, 0, \dots) \in l^2(G)|_N$, and define $B_M(n) = n F_M - F_M$.

We see that this converges pointwise to b and $B_M \in B^1(N, l^2(G)|_N)$ for all M . This gives us that $H^1(N, l^2(G)|_N) = \overline{B^1(N, l^2(G)|_N)}$. But since we already have N non-amenable, which implies that $H^1(N, l^2(G)|_N)$ is Hausdorff, $B^1(N, l^2(G)|_N)$ is closed and thus $H^1(N, l^2(G)|_N) = 0$. ■

In particular, this gives us information on the L^2 -cohomology of some uncountable groups. However, we should note here that since such a group must have an infinite normal subgroup N with $H^1(N, l^2(N)) = 0$, such a group must necessarily be non-amenable, since any countable group whose first L^2 -cohomology is Hausdorff is non-amenable, and if G has a non-amenable subgroup, then G is non-amenable. So we are still unsure whether we can have $H^1(G, l^2(G)) = 0$ when G is amenable. While this is certainly not possible in the case when G is countable, we would like to know whether or not this could happen when G is uncountable.

A specific case where this seems plausible is the case when G is locally-finite, since we have that $H^1(G, \mathbb{C}G) \neq 0$ when G is countably infinite, but when G is uncountable locally-finite, $H^1(G, \mathbb{C}G) = 0$. If a similar result happens when we consider coefficients in $l^2(G)$, then certainly the space $H^1(G, l^2(G))$ is Hausdorff. Then note that since a group is amenable if and only if every finitely-generated subgroup is amenable, we have that any locally finite group is amenable (since all finitely-generated subgroups are finite). This would certainly be an interesting conclusion for future work.

While at this time we cannot say for sure whether the first L^2 -cohomology group vanishes for the general uncountable locally-finite group G , we can show that if $|G| > \aleph_1$ then $H^1(G, l^2(G)) = 0$. To do this we use the techniques of Holt's proof in [3]. We will first prove a Lemma, then look at the proof of the Theorem, which is very similar to the proof that $H^1(G, \mathbb{C}G) = 0$ for G uncountable locally-finite that we have already given. In fact, it also makes use of Lemma 1 that was stated and proved in Chapter 3.

Lemma 4. Let G be a group of cardinality \mathfrak{c} . Then let \mathfrak{d} be a cardinality such that $\aleph_0 \leq \mathfrak{d} < \mathfrak{c}$. Then given any $f \in Z_G$ we can find a subgroup H of cardinality \mathfrak{d} such that $f(H) \subset \overline{\mathbb{C}H}$.

Proof. Given $f \in Z_G$, let H_1 be a subgroup of cardinality \mathfrak{d} . Then $f(h)$ has countable support for all $h \in H_1$. Let H_2 be the subgroup generated by H_1 and

the supports of $f(h)$ for all $h \in H_1$, and note that $|H_2| = \mathfrak{d}$ and that $f(H_1) \subseteq \overline{\mathbb{C}H_2}$. Continuing in this way, we arrive at a chain of subgroups $H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$ such that $f(H_i) \subseteq H_{i+1}$ for all i . Letting $H = \bigcup H_i$, we see that $|H| = \mathfrak{d}$ and that $f(H) \subseteq \overline{\mathbb{C}H}$. ■

Theorem 7. Let G be an uncountable periodic group such that $|G| > \aleph_1$. Then $H^1(G, l^2(G)) = 0$.

Proof. Given $f \in Z_G$, let H be a proper subgroup of G with $|H| = \aleph_1$ such that $f(H) \subseteq \overline{\mathbb{C}H}$, noting that since $|G| > \aleph_1$, H is a proper subgroup of G . Let $\alpha : G \rightarrow \mathbb{C}$ be defined as $\alpha(g) = -f(g)(g)$, and by a previous lemma we have that $f(g) = g \cdot \alpha - \alpha$ for all $g \in G$. We first aim to show that α is constant outside some countable set.

To do this, let $k \in G - H$, and note that $K := \langle k \rangle$ is a finite subgroup of G . Since $f(k^i)$ has countable support for all i , we have that $f(k^i)(k^i g) = 0$ for all but countably many $g \in G$. Also note that if $f(k^i)(k^i g) = 0$ for all i we get that

$$\alpha(k^i g) = -f(k^i g)(k^i g) = -f(k^i)(k^i g) - k^i f(g)(k^i g) = -f(g)(g) = \alpha(g),$$

and thus α is constant on the coset Kg . This implies that there is a countable subset of G such that α is not constant on Kg . Let H_1 be the countable subgroup of $\langle H, k \rangle$ generated by the $g \in \langle H, k \rangle$ such that α is not constant on Kg . Then each $g \in H_1$ can be written as a word on H and K , and define H_0 to be the subgroup of H generated by the elements of H in each of these words. Since H_1 is countable and each word has finite length, we see that H_0 is a countable subgroup of H . Then, by defining $L = \langle H_0, k \rangle$, we see that for any $g \in \langle H, k \rangle - L$, α is constant on Kg .

We claim that α is constant outside of $H \cap L$, noting that L is countably generated and therefore countable, and thus $H \cap L$ is a countable subgroup of G . Since L is a countable subgroup of $\langle H, k \rangle$, there exists some $h \in H - L$, and by definition we have $k \notin H$, so L and H are both proper subgroups of $\langle H, k \rangle$. Therefore, if we can show that α is constant on cosets $Hg \neq H$ and $Lg \neq L$, then we can use lemma 1 to deduce that α is constant on $\langle H, L \rangle - (H \cap L) = \langle H, k \rangle - (H \cap L)$.

To see that α is constant on $Hg \neq H$, we simply use that $f(H) \subseteq \overline{\mathbb{C}H}$ to see that

$f(h)(hg) = 0$ for all $g \in G - H$. Therefore, given $g \in \langle H, k \rangle - H$ we have

$$\alpha(hg) = -f(hg)(hg) = -f(h)(hg) - h \cdot f(g)(hg) = -f(g)(g) = \alpha(g)$$

and thus α is constant on Hg .

To show that α is constant on $Lg \neq L$, first note that if $lg \in H$ for any $l \in L$, then $Lg = Llg$, and thus we can make the assumption that if $lg \in H$ for any $l \in L$ then $g \in H$. After making this assumption, fix a coset $Lg \neq L$. If $K = L$ then α is constant on $Kg \neq K$ as cosets in $\langle H, k \rangle$ by the definition of L , so the result is clear. So we can assume that there exists $g \in L - K$. Then note that α is constant on Lg if and only if $\alpha(lg) = \alpha(g)$ for all $l \in L$, if and only if $\alpha g^{-1}(l) = \alpha g^{-1}(1)$ for all $l \in L$, if and only if αg^{-1} is constant on L . Since we have assumed that there exists $g \in (H \cap L) - K$, and certainly $k \notin H \cap L$, both $H \cap L$ and K are proper subgroups of L . Therefore, to show that αg^{-1} is constant on L , we consider the cosets Kl and $(H \cap L)l$ in L , and we aim to show that αg^{-1} is constant on all cosets Kl and on all cosets $(H \cap L)l \neq (H \cap L)$, and then use the second part of the lemma to conclude that αg^{-1} is constant on $\langle K, H \cap L \rangle = L$.

First, we look at the cosets Kl . Since $g \in \langle H, k \rangle - L$, we have that $lg \notin L$ for any $l \in L$. Therefore, by the definition of L we have that α is constant on Klg for all $l \in L$. This gives us that

$$\alpha g^{-1}(kl) = \alpha(klg) = \alpha(lg) = \alpha g^{-1}(l),$$

for all $k \in K$ and $l \in L$, and thus αg^{-1} is constant on Kl for all $l \in L$.

Now we consider the cosets $(H \cap L)l \neq (H \cap L)$. To show that αg^{-1} is constant on these cosets, we use the assumption that if $lg \in H$ for any $l \in L$, then $g \in H$.

If $lg \in H$ for some $l \in L$, then by our assumption we have that $g \in H$, which implies that $l \in H$. Therefore, $(H \cap L)l = (H \cap L)$. If $lg \notin H$ for any $l \in L$, then α is constant on Hlg for all $l \in L$, and hence on $(H \cap L)lg$ for all $l \in L$. Therefore, for any $x \in H \cap L$ and $l \in L$, we have that

$$\alpha g^{-1}(xl) = \alpha(xlg) = \alpha(lg) = \alpha g^{-1}(l),$$

which implies that αg^{-1} is constant on $(H \cap L)l$.

So, αg^{-1} is constant on Kl for all $l \in L$ and on $(H \cap L)l \neq H \cap L$, and we can now apply the second part of the lemma to conclude that αg^{-1} is constant on L ,

or, equivalently α is constant on Lg . This gives us that α is constant on Lg for all $g \in \langle H, k \rangle - L$ and Hg for all $g \in \langle H, k \rangle - H$, and since H and L are both proper subgroups of $\langle H, L \rangle$, we apply the first part of the lemma to deduce that α is constant on $\langle H, L \rangle - (H \cap L)$.

Finally, let $k' \in G - H$ such that $k \neq k'$. By the same proof we see that α is constant on $\langle H, L' \rangle - (H \cap L')$, so $\alpha(h) = \alpha(k')$ for all but countably many $h \in H$. We also have that $\alpha(h) = \alpha(k)$ for all but countably many $h \in H$. Therefore, since $|H| > \aleph_0$, there exists $h \in H$ such that $\alpha(k') = \alpha(h) = \alpha(k)$, and thus α is constant on $G - (H \cap L)$.

So, we can find a constant $\beta \in \overline{\mathbb{C}G}$ such that $\alpha - \beta$ has countable support. Let $\gamma = \alpha - \beta$. Assume for contradiction that $\gamma \notin l^2(G)$. Since $H \cap L$ is a proper subset of G , we can find $g \in G - (H \cap L)$. Then $\|g \cdot \gamma - \gamma\|_2^2 = \|g \cdot \gamma\|_2^2 + \|\gamma\|_2^2$ since the support of γ is in $H \cap L$. Since $g \cdot \gamma - \gamma \in l^2(G)$ for all $g \in G$, we must have $\|\gamma\|_2^2 < \infty$, and thus $f \in B_G$ which gives us that $H^1(G, l^2(G)) = 0$. ■

A fundamental idea in this proof is realizing that we can find an infinite, proper subgroup H of G such that $f(H) \subseteq \overline{\mathbb{C}H}$, and that α will be constant on the cosets $Hg \neq H$. For any uncountable group, we can actually show that this constant is the same for all $Hg \neq H$ so long as G has an infinite center. This leads us to the following Theorem which has interesting consequences.

Definition. Let $Z(G)$ denote the center of G . By the center of G we mean $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

Theorem 8. Let G be an uncountable group with $|Z(G)| = \infty$. Then $H^1(G, l^2(G)) = 0$.

Proof. Let H_1 be a countable subgroup of G such that $|H_1 \cap Z(G)| = \infty$ and let $f : G \rightarrow l^2(G)$ be a 1-cocycle. Since $f(g)$ has countable support for all g , we can find a countable subgroup $H_2 \supset H_1$ such that $f(H_1) \subset l^2(H_2)$. Continuing inductively, we arrive at a chain of countable subgroups $H_1 \subset H_2 \subset H_3 \subset \dots$. Let $H = \bigcup_1^\infty H_i$. Then we have $f(H) \subset l^2(H)$ and H is countable, thus a proper subgroup of G .

We can write $f(g) = g \cdot \alpha - \alpha$ where $\alpha(g) = -f(g)(g)$ as previously shown. Note that since $f(H) \subset l^2(H)$, we have that

$$\begin{aligned}\alpha(hg) &= -f(hg)(hg) \\ &= -f(h)(hg) - h \cdot f(g)(hg) \\ &= -f(g)(g) = \alpha(g)\end{aligned}$$

for any $h \in H$ and $g \in G - H$. Therefore, we see that α is constant on cosets $Hg \neq H$.

Let $\alpha(Hg_1) = c_1$ and $\alpha(Hg_2) = c_2$ for some cosets $Hg_1 \neq Hg_2$ where $g_1, g_2 \in G - H$. Since $h \in Z(G)$ for infinitely many $h \in H$, we get that

$$\begin{aligned}(g_1^{-1}g_2\alpha - \alpha)(hg_2) &= \alpha(g_2^{-1}g_1hg_2) - \alpha(hg_2) \\ &= \alpha(hg_2g_2^{-1}g_1) - \alpha(hg_2) \\ &= \alpha(hg_1) - \alpha(hg_2) = c_1 - c_2\end{aligned}$$

for infinitely many $h \in H$. Since $g \cdot \alpha - \alpha \in l^2(G)$ for any $g \in G$, we must have that $c_1 = c_2$. Therefore, α is constant on $G - H$. By adding or subtracting by a constant in $\mathbb{C}G$, we can assume that $f(g) = g \cdot \alpha - \alpha$ where α has countable support.

Finally, assume for contradiction that $\alpha \notin l^2(G)$. Since H is a proper subset of G , we can find $g \in G - H$. Then $\|g \cdot \alpha - \alpha\|_2^2 = \|g \cdot \alpha\|_2^2 + \|\alpha\|_2^2$ since the support of α is in H . Since $g \cdot \alpha - \alpha \in l^2(G)$ for all $g \in G$, we must have $\|\alpha\|_2^2 < \infty$, and thus $f \in B_G$ which gives us that $H^1(G, l^2(G)) = 0$. ■

We now make an interesting note regarding whether or not $H^1(G, l^2(G))$ is Hausdorff when G is amenable. Since abelian groups are amenable and $H^1(G, l^2(G)) = 0$ when G is abelian, we have that G amenable does not imply that $H^1(G, l^2(G))$ is non-Hausdorff when G is uncountable, which is surprising. Since locally-finite groups are amenable (since all finitely-generated subgroups are finite, and thus amenable), Theorem 7 gives us another example of an amenable group with Hausdorff first L^2 -cohomology. This raises the question of whether or not there exists an uncountable group with non-Hausdorff first L^2 -cohomology, since at this time we do not know of an example of such a group. This is an interesting topic to consider in the future.

Finally, up to this point, we have not seen an example of an uncountable group with non-zero first L^2 -cohomology. When we deal with coefficients in $\mathbb{C}G$, if G is

a free group, then $H^1(G, \mathbb{C}G) \neq 0$, so we would like to see if a similar result holds when dealing with coefficients in $l^2(G)$. In fact, such a result does hold.

Proposition 7. Let G be a free group. Then $H^1(G, l^2(G)) \neq 0$.

Proof. We aim to construct a non-zero element of $H^1(G, l^2(G))$. To do this, we use the techniques from a paper by R. Fox [12]. First, let x be a generator in G , and define $\alpha(w)$ for some word $w \in G$ by

$$\alpha(w) = \begin{cases} 1 & : \text{the word } w \text{ ends in } x \\ 0 & : \text{otherwise.} \end{cases}$$

Now consider $(w \cdot \alpha - \alpha)(u) = \alpha(w^{-1}u) - \alpha(u)$ for some $w, u \in G$. We note that in order for this difference to be non-zero we must have either (i) the word u ends in x but $w^{-1}u$ does not, or (ii) $w^{-1}u$ ends in x but u does not. We see that in order for either of these two conditions to occur, u must be an initial sequence of the word w . Therefore, $(w \cdot \alpha - \alpha)(u) = 0$ for all but finitely many u . In other words, $w \cdot \alpha - \alpha \in \mathbb{C}G \subset l^2(G)$ for all $w \in G$. By a previous proposition, this implies that $g \cdot \alpha - \alpha \in H^1(G, l^2(G))$.

By another previous lemma, we have that if $f(g) = g \cdot \alpha - \alpha = g \cdot \beta - \beta$ for some $f \in H^1(G, l^2(G))$, we must have $\alpha - \beta = C$ for some constant C in $\overline{\mathbb{C}G}$. Since $\alpha(x^i) = 1$ for all $i > 0$, and $\alpha(x^j) = 0$ for all $j \leq 0$, there does not exist a constant $C \in \overline{\mathbb{C}G}$ such that $\alpha + C \in l^2(G)$. Therefore, $g \cdot \alpha - \alpha \notin B_G$, and thus the image of $g \cdot \alpha - \alpha$ is non-zero in $H^1(G, l^2(G))$. ■

Note. This proof also demonstrates that $H^1(G, \mathbb{C}G) \neq 0$ when G is a free group, a fact already stated but not proven.

In the future, as well as topics stated earlier, we would like to arrive at a more complete characterization of when $H^1(G, l^2(G)) \neq 0$, both when G is countable or uncountable. In order to explore this topic, we would like to look more thoroughly at L^2 -invariants and L^2 -Betti numbers, both of which Lück discusses in [6], as well as in several other papers. A more thorough look into the theory of groups acting on graphs, which led to the general characterization of the $H^1(G, \mathbb{C}G)$ groups given in this paper, could also lead to advances along this line.

Chapter 6

Symbols and Notation

RG	group ring	(3)
$C^0(G, A)$	n -cochains	(3)
d_n	n th coboundary homomorphism	(4)
$Z^n(G, A)$	n -cocycles	(4)
$B^n(G, A)$	n -coboundaries	(4)
$H^n(G, A)$	n th cohomology group with coefficients in A	(4)
Z_G	1-cocycles	(4)
B_G	1-coboundaries	(4)
A^G	G -fixed elements of A	(4)
\approx	almost equality	(8)
\mathbb{Z}_2	2 element field	(9)
\overline{FG}	$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, F)$	(9)
$e(G)$	ends of G	(9)
$_ *_{\mathcal{C}} _$	free product with amalgamation	(11)
$_ *_{\mathcal{C}} x$	HNN extension	(11)
$l^2(G)^G$	space of functions from G to $l^2(G)$	(16)

Chapter 7

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