A DEFLECTION THEORY FOR ANISOTROPIC PLATES IN WHICH COUPLING BETWEEN LATERAL DEFLECTION AND IN-PLANE DISPLACEMENT IS PRESENT AND THE EFFECT OF COUPLING ON THE BUCKLING LOAD

by

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TABLE OF CONTENTS

I. LIST OF FIGURES

II. LIST OF SYMBOLS

a	plate length in x-direction, in.
b	plate length in y-direction, in.
h	thickness of plate, in.
a_n, b_n, c_n, d_n, e_n, f_n, g_n, h_n, c'_n, d'_n, h'_n	dimensionless quantities in roots of characteristic equations
k	dimensionless load parameter, $\frac{\bar{N}_x b^2}{\pi^2 D_1}$ (k is negative for compressive loads)
m_1, m_2, m_3, m_4	roots of characteristic equations
n	positive integers
q	lateral load intensity, pounds per square inch
u, v, w	components of displacement in the x-, y-, and z-directions respectively, in.

w_x, w_y	lateral deflection solutions due to the coupling terms C_{11} and C_{21}, respectively, as used in Appendix C
x, y, z	orthogonal coordinate system; z measured normal to the plane of the plate and x and y parallel to the axes of principal stiffness
C_{11}, C_{12}, C_{21}, C_{22}	coupling elastic constants associated with bending and stretching and defined by equations (1), in.
C_K	coupling elastic constant associated with twisting and shearing and defined by equations (1), in.
D_1, D_2	bending stiffness in the x- and y-directions, respectively, in-lb
D	bending stiffness when $D_1 = D_2 = D$, in-lb
D_{xy}	twisting stiffness relative to the x- and y-directions, in-lb
E_1, E_2	extensional stiffnesses in the x- and y-directions, respectively, lb/in.
F	force function defined by equation (11)
G_K	shearing stiffness in xy-plane, lb/in.
H	$H = \mu_y D_1 + D_{xy}$, in-lb
$K_{o_{11}}$	dimensionless coefficient for initial deflection (see ref. 7)
M_x, M_y	resultant internal bending-moment intensities acting on cross sections originally perpendicular to the x- and y-axes, respectively, lb

M_{xy}	resultant internal twisting-moment intensity acting in cross sections originally perpendicular to the x- and y-axes, lb
$\bar{M}_x$, $\bar{M}_y$, $\bar{M}_{xy}$	boundary values of the resultant bending and twisting moments
N_x, N_y	resultant internal normal-force intensities acting in planes I and II of cross sections originally perpendicular to the x- and y-axes, respectively, lb/in.
N_{xy}	resultant internal shear-force intensity acting in plane III of cross sections originally perpendicular to the x- and y-axes, lb/in.
$\bar{N}_x$, $\bar{N}_y$, $\bar{N}_{xy}$	boundary values of the resultant normal and shear forces, lb/in.
$\bar{Q}_x$, $\bar{Q}_y$	boundary values of the shear-force intensities acting in z-direction on planes originally perpendicular to the x- and y-axes, respectively, lb/in.
T	total external work done by the boundary forces, in-lb
U	total strain energy of the system, in-lb
V	total potential energy of the system, in-lb
X_n	x-wise components of the small deflection solution to differential equation of equilibrium

Y_n	y-wise components of the small deflection solution to differential equation of equilibrium
α_n	$\alpha_n = \frac{n\pi}{b}$
β_n	$\beta_n = \frac{n\pi}{a}$
ϵ_x, ϵ_y	normal strain of plane I in x-direction and of plane II in y-direction, respectively
γ_{xy}	shear strain of plane III with respect to the x- and y-directions
μ_x, μ_y	Poisson's ratios associated with bending in x- and y-directions, respectively, and defined by equations (1)
μ_1, μ_2	Poisson's ratios associated with extensions in the x- and y-directions, respectively, and defined by equations (1)
δ	total deflection of the midpoint of a plate with an initial deviation from flatness, in.
δ_o	initial deflection of the midpoint of a plate with an initial deviation from flatness, in.

III. INTRODUCTION

Recent aircraft wing designs have incorporated plates which are integrally stiffened only on one side. The effectiveness of integral stiffening is partly due to the fact that direct stresses in the skin are conducted into the integral stiffeners; thus the stiffeners are exploited in carrying stresses in their transverse as well as their longitudinal directions. The stress distribution through the thickness of the skin and stiffener of a plate integrally stiffened on one side is such that there is no plane of symmetry of stresses. This lack of symmetry introduces bending and twisting moments which produce curvature of the plate when direct stresses are applied in a plane of the plate. This effect is referred to as coupling.

General force-distortion equations (that is, equations relating forces, moments, strains, and curvatures) are presented in reference 1 for plates integrally stiffened on one side. Reference 1 includes the effect of coupling in the force-distortion equations and presents a method for calculating the associated elastic constants. Figure 1, taken from reference 1, illustrates several types of integrally stiffened panels.

The significance of coupling is most apparent in problems which would involve stability considerations were coupling not present. Because of this coupling, lateral deflection of the plate occurs at loads less than the buckling load of the equivalent uncoupled plate. The problem has therefore a completely new aspect which prohibits its being treated as a conventional stability problem.

Wing surfaces constructed of plates exhibiting the coupling effect will take on curvatures, thus the associated lateral deflections, when they are stressed in their plane. These lateral deflections may be large enough to affect adversely the wing's aerodynamic properties. Existing plate theory does not include the effect of coupling; therefore it cannot determine the magnitude of lateral deflections due to coupling.

A deflection theory is developed herein which takes coupling into account. This deflection theory is applicable to any isotropic or anisotropic body which behaves essentially as a plate. The effect of transverse shear deformation, important in corrugated-core sandwich plates which may exhibit coupling (ref. 2), is not included in the present theory.

Mathematically, a complete elastic theory may be specified by the following components:

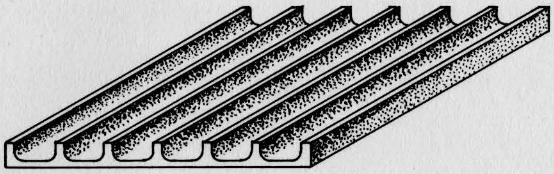
1. Force-distortion equations
2. Equations relating strains and displacements
3. The minimization of a potential energy expression for the system
4. Geometric boundary conditions

The force-distortion equations may be obtained from references 1 or 2 when considering integrally stiffened plates or corrugated-core sandwich plates, or they may be determined experimentally in other cases. The strain-displacement equations for plates in which coupling is present are the same as those used for conventional plates. Reference 4 presents such a set. The potential energy expression is derived in Appendix A and the geometric boundary conditions are determined by the particular problem considered.

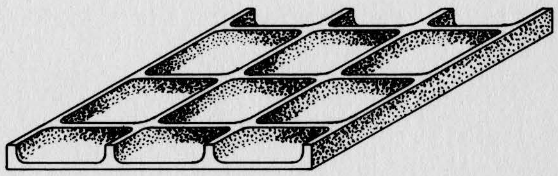
An elastic theory may also be constituted of the first two and fourth components as listed plus equilibrium and natural boundary conditions. The equilibrium equations and the so-called "natural" boundary equations are derived in Appendix B by means of the principle of minimum potential energy (ref. 5) in conjunction with the calculus of variations. The potential energy expression, equilibrium equations, and natural boundary equations are presented in the text under the section entitled "Results and Discussion."

The equilibrium equations can also be derived by consideration of the forces and moments acting on the faces of a differential element of the plate and application of the condition that the element must be in a state of static equilibrium. By use of the principle of minimum potential energy, however, the natural boundary conditions which must be satisfied are obtained in addition to the equations of equilibrium.

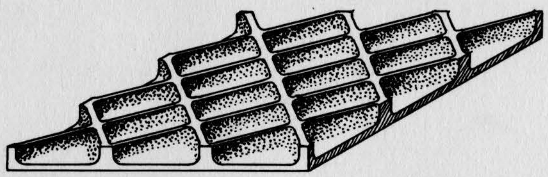
An exact small-deflection solution to the differential equation of equilibrium is presented in Appendix C for the case of an anisotropic plate exhibiting coupling. The plate is considered to be simply supported on all edges and is loaded by an evenly distributed in-plane compressive load. This solution illustrates that insofar as small-deflection theory is concerned, this problem is not the familiar stability problem but one of rapid growth of deflections with small increases in load. This rapid growth will occur at a load level which is lower than or equal to the uncoupled buckling load. Coupling is therefore shown to be a detrimental effect in this respect, depending on the magnitude of the coupling terms.



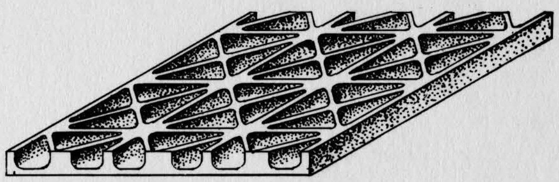
(a) Longitudinal or transverse



(b) Longitudinal and transverse



(c) Skewed



(d) Skewed plus longitudinal and transverse

Figure 1.– Examples of integrally stiffened plates.

IV. STATEMENT OF THE PROBLEM

The first purpose of the present paper is to provide an elastic theory from which problems involving coupling may be approached. Potential energy and equilibrium expressions will be derived for these are the components of the theory which are lacking. The potential energy expression may be used either in small- or large-deflection analysis. Equations of equilibrium are presented for both small- and large-deflection theory.

The second purpose of the present paper is to determine the effect of coupling on deflections and buckling of a simply supported anisotropic plate in compression. A small deflection analysis of this problem is made using the theory presented herein. No large deflection analysis is attempted; however, an estimation of the large deflection effect of coupling is made.

V. FORCE-DISTORTION EQUATIONS

References 1 and 2 present force-distortion equations for integrally stiffened plates and corrugated-core sandwich plates, respectively, which contain added terms due to coupling between distortions in the plane of the plate and lateral deflections of the plate. The existence and evaluation of these coupling terms will not be treated in this paper since ample discussion may be found on these subjects in references 1 and 2. The force-distortion equations of references 1 and 2 are given in the following form:

$$\left.\begin{aligned}
M_x &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{11}N_x + C_{12}N_y \\
M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{21}N_x + C_{22}N_y \\
M_{xy} &= D_{xy} \frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \\
\epsilon_x &= C_{11} \frac{\partial^2 w}{\partial x^2} + C_{21} \frac{\partial^2 w}{\partial y^2} + \frac{N_x}{E_1} - \frac{\mu_2}{E_2} N_y \\
\epsilon_y &= C_{12} \frac{\partial^2 w}{\partial y^2} + C_{22} \frac{\partial^2 w}{\partial y^2} - \frac{\mu_1}{E_1} N_x + \frac{N_y}{E_2} \\
\gamma_{xy} &= -2C_K \frac{\partial^2 w}{\partial x \partial y} + \frac{N_{xy}}{G_K}
\end{aligned}\right\} \quad (1)$$

(The transverse shear deflection terms of ref. 2 are not included in this set. That effect is not within the scope of this paper.) This set of equations assumes that constructions such as integrally stiffened or corrugated-core sandwich plates may be replaced by uniform thickness anisotropic plates in which $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, and $\frac{\partial^2 w}{\partial x \partial y}$ represent curvatures and twist respectively and N_x, N_y, and N_{xy} act in arbitrarily located plane I, II, and III, respectively. Figure 2, which is taken from reference 2, illustrates the forces and moments acting on an infinitesimal element of an equivalent uniform thickness anisotropic plate. The material properties of this idealized element may be considered to vary unsymmetrically about the midplane to permit coupling to exist even when the resultant forces are applied in the midplane. The location of the planes in which the forces act is arbitrary for the force-distortion equations in equation 1 for the sake of generality. These general expressions will be used in the ensuing derivations of the potential energy and equilibrium expressions.

Some of the coupling terms in these equations may be eliminated by choosing the plane in which the resultant forces act to be so located as to cause a moment, and thus a curvature counter to that given by the coupling effect. Thus if the plane in which N_x is acting is so located as to eliminate coupling between curvature in the x-direction and strains in the x-direction and if the plane in which N_y is acting is so located as to eliminate coupling between curvatures and strains in the y-direction, the force-distortion equations take on the form:

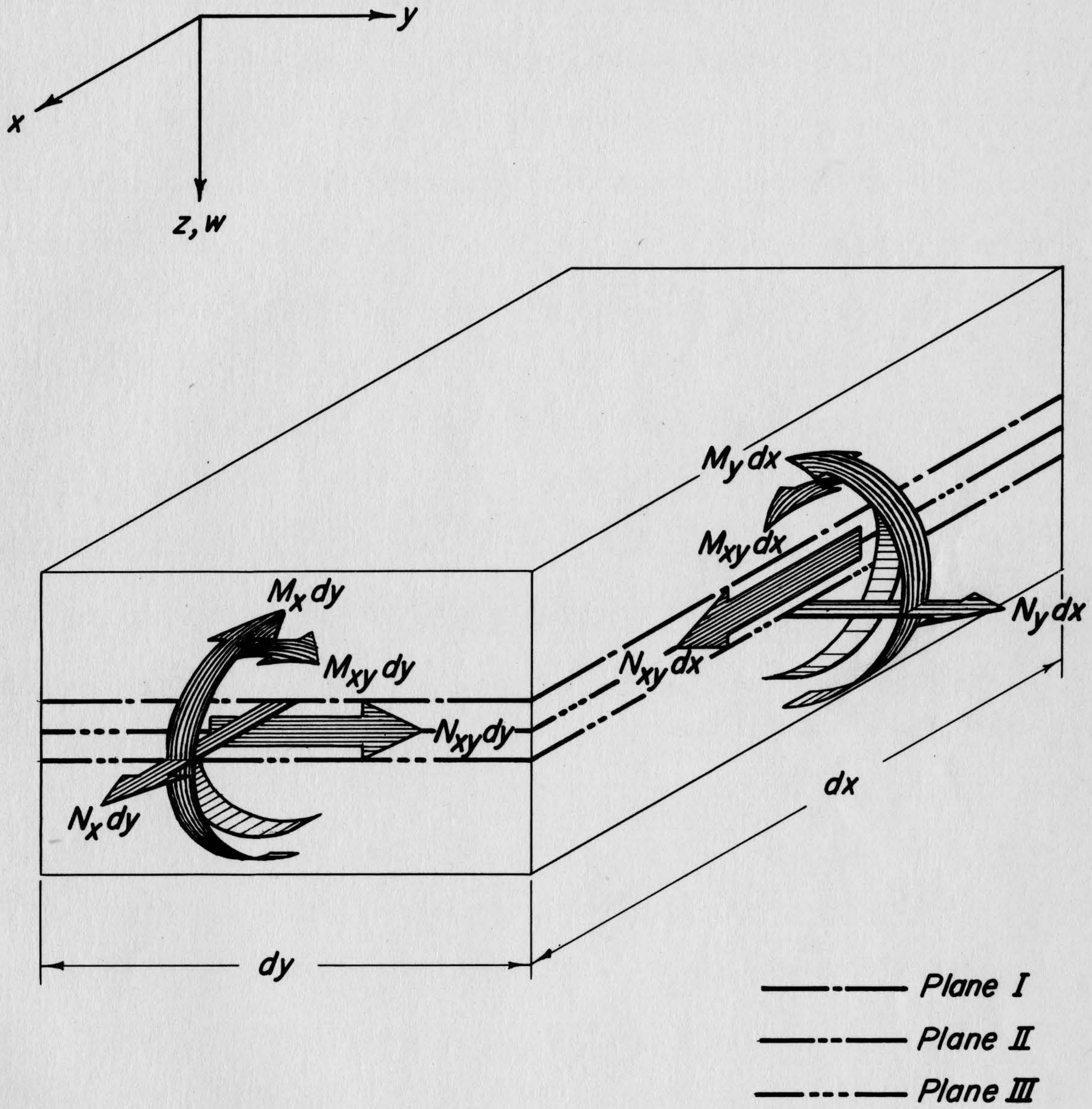


Figure 2.- Forces and moments acting on element.

$$\left.\begin{aligned}
M_x &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{12}N_y \\
M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{21}N_x \\
M_{xy} &= D_{xy}\frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \\
\epsilon_x &= C_{21}\frac{\partial^2 w}{\partial y^2} + \frac{N_x}{E_1} - \frac{\mu_2}{E_2}N_y \\
\epsilon_y &= C_{12}\frac{\partial^2 w}{\partial x^2}\frac{\mu_1}{E_1}N_x + \frac{N_y}{E_2} \\
\gamma_{xy} &= -2C_K\frac{\partial^2 w}{\partial x \partial y} + \frac{N_{xy}}{G_K}
\end{aligned}\right\} \qquad (2)$$

The forces N_x and N_y could be so placed as to eliminate coupling between M_x and N_y and M_y and N_x. Then, the force-distortion equations become:

$$\left.\begin{aligned}
M_x &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{11}N_x \\
M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{22}N_y \\
M_{xy} &= D_{xy}\frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \\
\epsilon_x &= C_{11}\frac{\partial^2 w}{\partial w^2} + \frac{N_x}{E_1} - \frac{\mu_2}{E_2} N_y \\
\epsilon_y &= C_{22}\frac{\partial^2 w}{\partial y^2} - \frac{\mu_1}{E_1} N_x + \frac{N_y}{E_2} \\
\gamma_{xy} &= -2C_K \frac{\partial^2 w}{\partial x \partial y} + \frac{N_{xy}}{G_K}
\end{aligned}\right\} \quad (3)$$

By choosing the proper plane in which N_{xy} is to act, uncoupling between N_{xy} and M_{xy} may also be accomplished. In general, three of the coupling terms given in equations (1) may be eliminated by proper choice of planes in which the forces N_x, N_y, and N_{xy} act. For an unsymmetric construction, no more than three of the terms may be eliminated. If the plate is of a symmetric type construction, all coupling may be eliminated by locating the forces N_x, N_y, and N_{xy} in the midplane of the plate.

Force-distortion equations as given in equations (1), (2), and (3) are of the form which would apply for any unsymmetrical type plate construction. Methods for computing the elastic constants of integrally stiffened plates are presented in references 1 and 6, and methods for computing the elastic constants of corrugated-core sandwich plates are presented in reference 2. The elastic constants for the above-mentioned constructions or any other unsymmetrical type construction may of course be determined experimentally.

VI. STRAIN-DISPLACEMENT EQUATIONS

The strain-displacement equations applicable to this theory in which coupling is considered are the same as those given in reference 4 for ordinary plate theory. They are

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned}\right\} \tag{4}$$

VII. RESULTS AND DISCUSSION

A. Potential Energy of the Plate

The derivation of the potential energy for a plate that deforms according to the force-distortion relations given in equations (1) is presented in Appendix A. The analysis of the external work given in Appendix A has been limited to cases in which the reactions do no work. Thus, only combinations of free, simply supported, or clamped edge conditions may be considered using this analysis. The analysis may be generalized by considering the energy of distorting springs capable of resisting displacements and rotations distributed along the edges of the plate.

The form of the potential energy expression presented below is obtained by integrating by parts the terms in equation (A4) which contain the boundary forces $\overline{N}_x$, $\overline{N}_y$, and $\overline{N}_{xy}$. Some of the resulting terms, those which correspond to the equations of equilibrium for forces in the plane of the plate, have been eliminated because the forces must satisfy equilibrium at all times. (These equations of equilibrium are derived later in the appendix.) This procedure is the same as that used in the appendix of reference 3. The strain-displacement equations are applied to eliminate u- and v-displacements from the energy expression. The resulting form of the energy expression is

$$\left.\begin{aligned}
V = \frac{1}{2}\int_0^a \int_0^b \Bigg\{ &- M_x \frac{\partial^2 w}{\partial x^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} - M_y \frac{\partial^2 w}{\partial y^2} + N_x\left[-\epsilon_x + \left(\frac{\partial w}{\partial x}\right)^2\right] + \\
&N_y\left[-\epsilon_y + \left(\frac{\partial w}{\partial x}\right)^2\right] + N_{xy}\left[-\gamma_{xy} + 2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]\Bigg\} dx\, dy\, - \\
&\int_0^b \left| - \bar{M}_x \frac{\partial w}{\partial y} + \bar{M}_{xy} \frac{\partial w}{\partial y} + \bar{Q}_{xw} \right|_0^a dy - \int_0^a \left| - \bar{M}_y \frac{\partial w}{\partial y} + \right. \\
&\left. \bar{M}_{xy} \frac{\partial w}{\partial x} + \bar{Q}_{yw} \right|_0^b dx - \int_0^a \int_0^b qw\, dx\, dy
\end{aligned}\right\} \quad (5)$$

In this form, the potential energy is a function of only the deflection and the in-plane forces when the substitution from equations (1) is made for the moments and strains.

The in-plane forces N_x, N_y, and N_{xy} in equation (5) are not exactly those which would prevail in the absence of coupling. The distribution of the in-plane forces within the boundaries of the plate is not only a function of the in-plane boundary forces but also a function of the lateral deflections of the plate as may be seen from equations (1). However, the change in the distribution of in-plane forces due to coupling will be small in most practical cases. This is because changes in horizontal load arise from changes in curvature of the plate, and the change of curvature will be very small in the present applications of small deflections theory. Thus, changes in the forces from one point to another in the plate will be assumed to be small. In most practical problems the distribution of in-plane forces within the boundaries of the plate may be assumed to be that which would prevail in the absence of coupling.

By observing the above simplifying assumption, the potential energy expression, equation (5), may be used in conjunction with the force-distortion equations in the conventional manner to solve small-deflection problems for plates in which coupling is present.

B. Equilibrium and Boundary Equations

The equilibrium and boundary equations for a plate in which coupling is present are derived in Appendix B. As would be expected, these equations are the same as those for an ordinary isotropic or anisotropic plate when they are expressed in terms of the forces. However, when the expressions for the moments as given in equations (1) are substituted into the equilibrium and boundary equations, the effect of coupling is exhibited.

1. Small-deflection equations.- For problems in which only small-deflections are considered, the equations of equilibrium for in-plane forces may be disregarded because in-plane extensions due to curvature are negligible. The remaining equilibrium equation and natural broundary equations are

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2(\mu_y D_1 + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - C_{11} \frac{\partial^2 N_x}{\partial x^2} - C_{12} \frac{\partial^2 N_y}{\partial x^2} +$$

$$2C_K \frac{\partial^2 N_{xy}}{\partial x \partial y} - C_{21} \frac{\partial^2 N_x}{\partial y^2} - C_{22} \frac{\partial^2 N_y}{\partial y^2} = q + N_x \frac{\partial^2 w}{\partial x^2} + \tag{6}$$

$$N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

Along $x = 0, a$

$$\bar{M}_x = - D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{11}N_x + C_{12}N_y \tag{7}$$

or

$$\frac{\partial w}{\partial x} = 0, a \tag{8}$$

$$\bar{M}_{xy} = D_{xy} \frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \tag{9}$$

or

$$\frac{\partial w}{\partial y} = 0 \tag{10}$$

$$\left.\begin{aligned} \bar{Q}_x = - D_1 \frac{\partial^3 w}{\partial x^3} - (D_1\mu_y + D_{xy}) \frac{\partial^3 w}{\partial x \partial y^2} + C_{11} \frac{\partial N_x}{\partial x} + C_{12} \frac{\partial N_y}{\partial y} - \\ C_K \frac{\partial N_{xy}}{\partial y} + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \end{aligned}\right\} \tag{11}$$

or

$$w = 0 \tag{12}$$

Along $y = 0, b$

$$\bar{M}_y = -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{21}N_x + C_{22}N_y \tag{13}$$

or

$$\frac{\partial w}{\partial y} = 0 \tag{14}$$

$$\bar{M}_{xy} = D_{xy} \frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \tag{15}$$

or

$$\frac{\partial w}{\partial x} = 0 \tag{16}$$

$$\left.\begin{aligned}\bar{Q}_y = &- D_2 \frac{\partial^3 w}{\partial y^3} - (D_2\mu_x + D_{xy}) \frac{\partial^3 w}{\partial x^2 \partial y} + C_{21} \frac{\partial N_x}{\partial y} + C_{22} \frac{\partial N_y}{\partial y} + \\ &C_K \frac{\partial N_{xy}}{\partial x} + N_x \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x}\end{aligned}\right\} \tag{17}$$

or

$$w = 0 \tag{18}$$

Equations (6) through (18) become those for an uncoupled anisotropic plate when the coupling terms are zero. The natural boundary equations are restricted to problems in which the edges are not elastically restrained.

For most practical problems there will be no gradient of the internal in-plane forces and the coupling terms in equation (6) will be zero. When lateral deflections of the plate occur, there may be some redistribution of the internal in-plane forces due to coupling, but for the small-deflections case this redistribution may be neglected for reasons stated previously. With the elimination of the coupling terms from the equation of equilibrium by the above considerations, only the natural boundary equations will impose the effect of coupling upon the problem.

In the case of a plate clamped on all edges, there will be no effect of coupling because the natural boundary equations which contain the coupling terms do not apply. For a plate simply supported on all edges, equations (7) and (13) are the only boundary equations which will impose the effect of coupling upon the problem. The presence of the coupling terms in the boundary equations produces the same effect as that of moments distributed along the edges of the plate. For example; if equation (7) is applicable in a given problem and $\overline{M}_x = 0$, the coupling terms of equation (7) may be thought of as external moments acting on the edges $x = 0, a$ of magnitudes $- C_{11}N_x$ and $- C_{21}N_y$. Thus, for a simply supported plate in compression, the problem becomes that of an uncoupled simply supported plate in compression acted upon by the evenly distributed moments $- C_{11}\overline{N}_x$ along $x = 0, a$ and $- C_{21}\overline{N}_x$ along $y = 0, b$.

The differential equations of equilibrium subject to the boundary conditions discussed may be solved in many ways depending upon the distribution of the in-plane forces. The solution for the case of a simply supported anisotropic plate in compression is presented in Appendix C.

It may be seen from the above discussion that many problems which would require a stability analysis for the uncoupled case become problems of deflection in the presence of coupling.

2. Large-deflection equations.- For large-deflection problems where extensional stresses become significant the equilibrium equations for all three directions must be considered simultaneously with the force-distortion and the strain-displacement equations. Combining these three sets of equations in a manner similar to that given in reference 4 results in the following two equations:

$$\left.\begin{aligned}&\frac{1}{E_2}\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G} - 2\frac{\mu_1}{E_1}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_1}\frac{\partial^4 F}{\partial y^4} + C_{12}\frac{\partial^4 w}{\partial x^4} + \\ &\qquad (C_{11} + C_{22})\frac{\partial^4 w}{\partial x^2 \partial y^2} + C_{21}\frac{\partial^4 w}{\partial y^4} = \\ &\qquad \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\end{aligned}\right\} \quad (19)$$

$$\left.\begin{aligned} & D_1 \frac{\partial^4 w}{\partial x^4} + 2(\mu_x D_1 + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - C_{12} \frac{\partial^4 F}{\partial x^4} - \\ & \qquad (C_{11} + C_{22} + 2C_K) \frac{\partial^4 F}{\partial x^2 \partial y^2} - C_{21} \frac{\partial^4 F}{\partial y^4} = \\ & \qquad q + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad (20)$$

where

$$\left.\begin{aligned} N_x &= \frac{\partial^2 F}{\partial y^2} \\ N_y &= \frac{\partial^2 F}{\partial x^2} \\ N_{xy} &= - \frac{\partial^2 F}{\partial x \partial y} \end{aligned}\right\} \quad (21)$$

Equations (19) and (20) together with the boundary conditions determine w and F. The loads may then be determined by equations (21). The general solution to the differential equations given in (19) and (20) is unknown.

C. Exact Small-Deflection Solution for a Simply Supported Anisotropic Plate in Compression

The exact small-deflection solution to equation (6) for a simply supported anisotropic plate is presentediin Appendix C. The solution was obtained by separating the problem into two parts: one part for coupling between x-wise forces and curvatures in the x-direction; the other part for coupling between x-wise forces and curvature in the y-direction. When coupling in only one direction is present, the methods of separation of variables is applied in solving the differential equation of equilibrium. The deflections obtained from the solution of these two problems are superimposed to obtain the solution for the general case in which coupling exists in both directions.

As shown in Appendix C, there are several different solutions to the differential equations depending on whether the roots of the characteristic equations are real, imaginary, or complex. This is determined by the load and plate properties.

For a given set of plate properties, load-deflection curves may be computed from the equations presented in Appendix C. From such curves, a load level may be identified at which the deflections grow rapidly with small increase of load. This load will be below or equal to the uncoupled buckling load.

Computations have been made using the solutions presented in Appendix C for the deflection of a point in the plate for cases in which the flexural stiffnesses of the plates D_1 and D_2 are equal. The aspect ratio is varied from 1 to 3 and the value of H/D is varied from 0.125 to 4. (Equal flexural stiffness and values of H/D would be found in integrally stiffened plates, such as shown in part (b) and the 45^o case of part (c) of fig. 1.) The computations are presented in figures 3 through 7 as plots of w_x/C_{11} and w_y/C_{21} against a load parameter k. For given values of C_{11} and C_{21}, the total deflection $w(w = w_x + w_y)$ at the point of maximum deflection may be computed from the information provided in figures 3

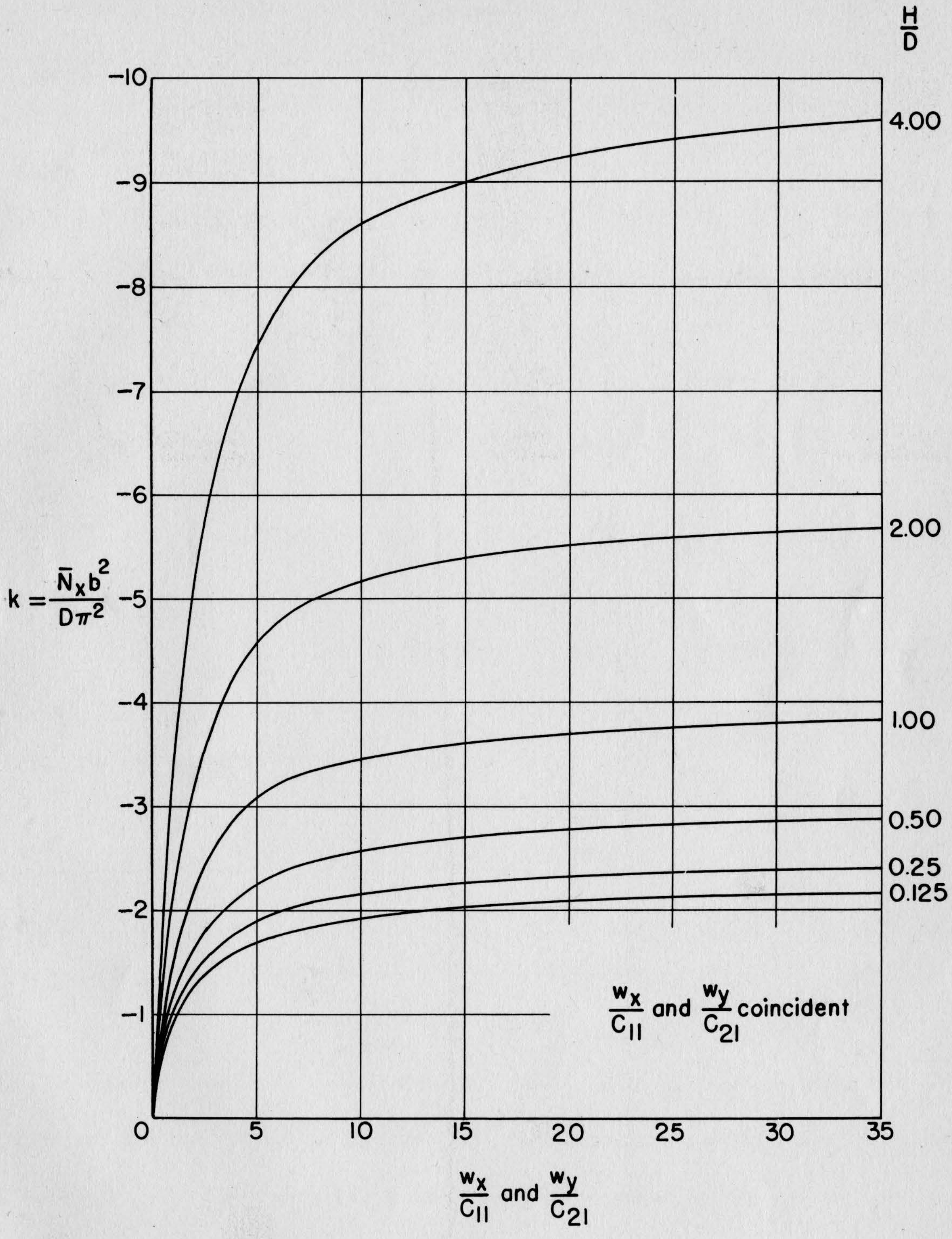


Figure 3.- Dimensionless load-deflection curves for plate in which $D_1 = D_2 = D$ and $a/b = 1$. Deflections computed at $x = a/2$, $y = b/2$.

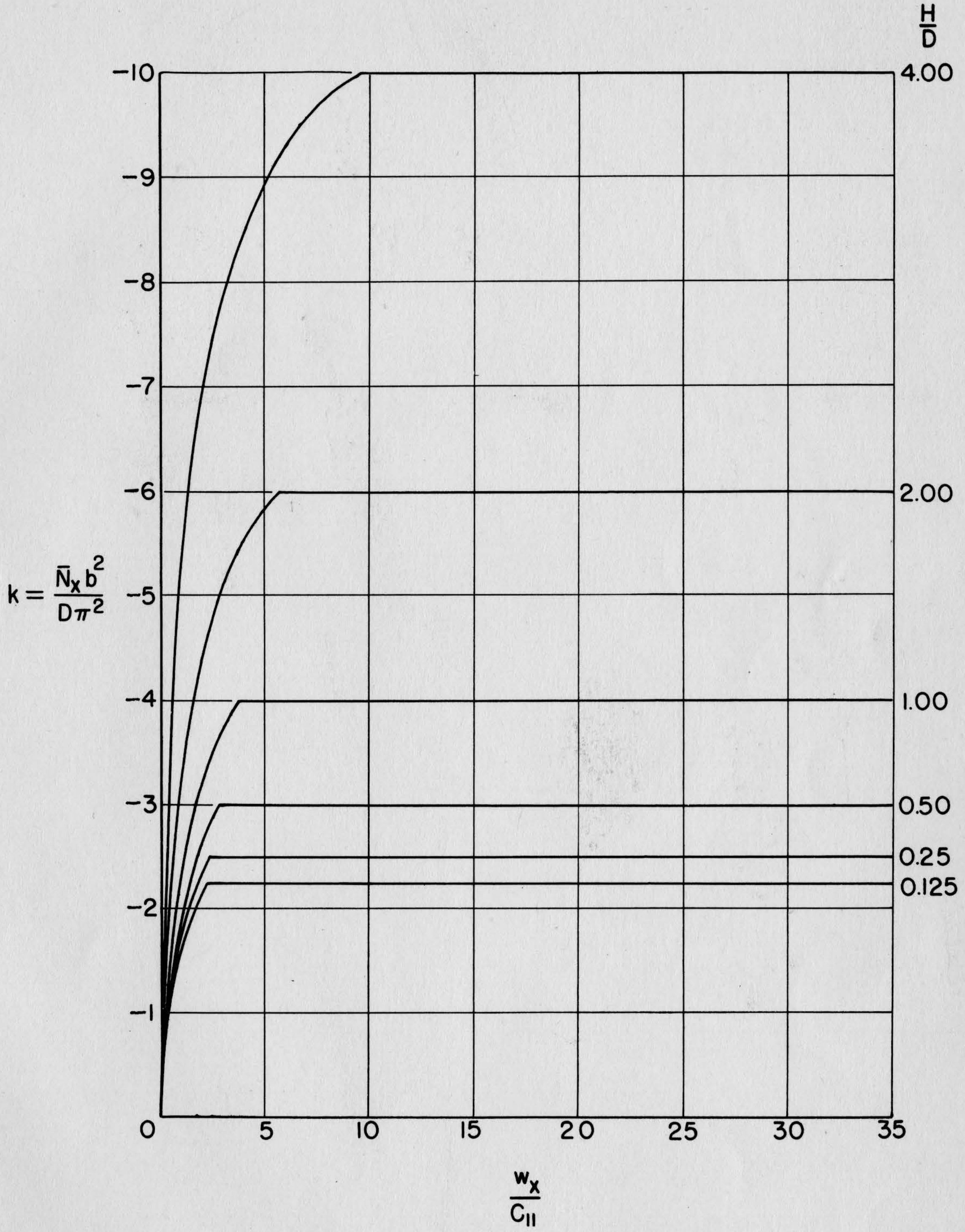


Figure 4.- Dimensionless load-deflection curves for plate in which $D_1 = D_2 = D$ and $a/b = 2$. Deflection computed at $x = a/4$, $y = b/2$. (Deflection due to C_{11} only.)

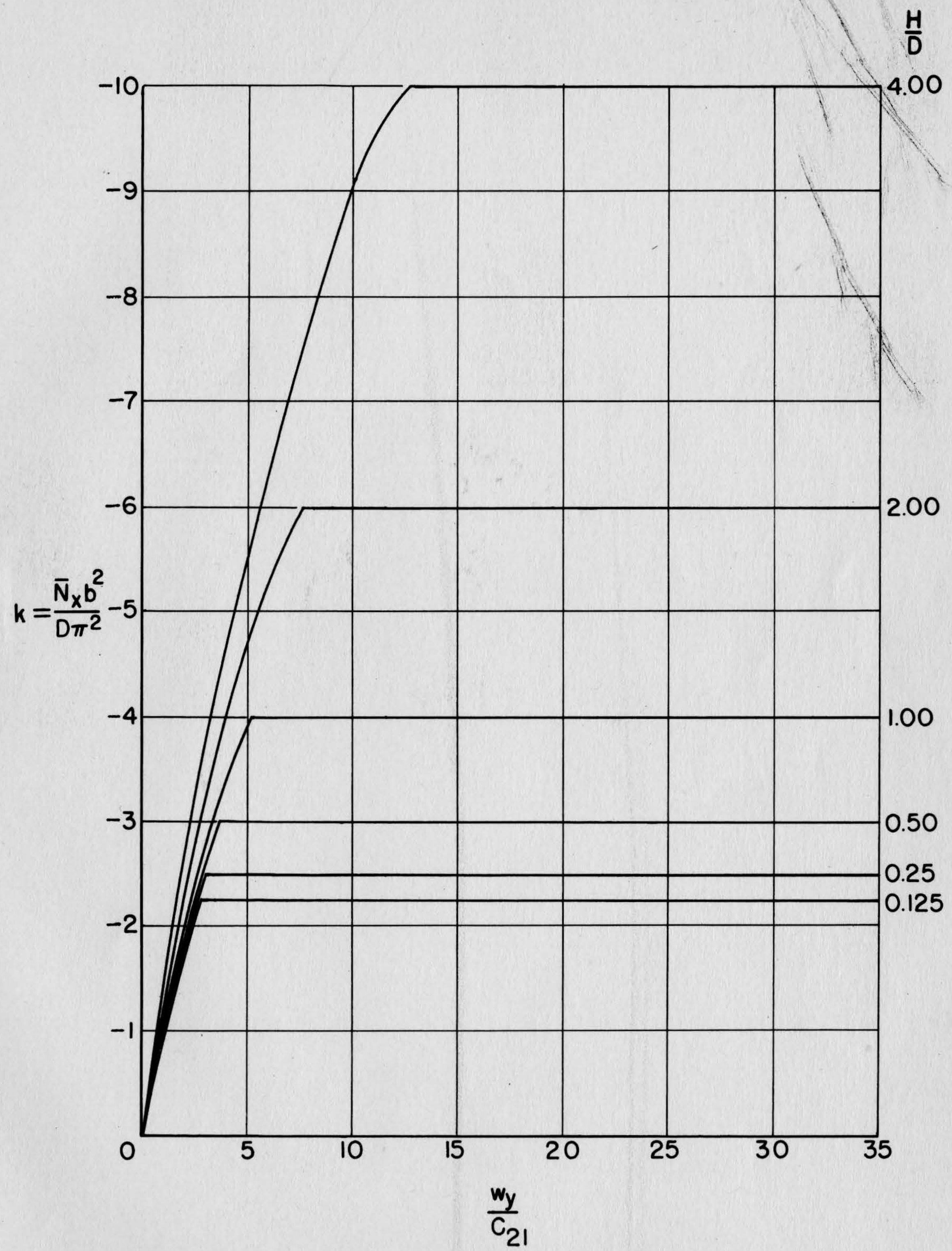


Figure 5.– Dimensionless load–deflection curves for plate in which $D_1 = D_2 = D$ and $a/b = 2$. Deflection computed at $x = a/4$, $y = b/2$. (Deflection due to C_{21} only.)

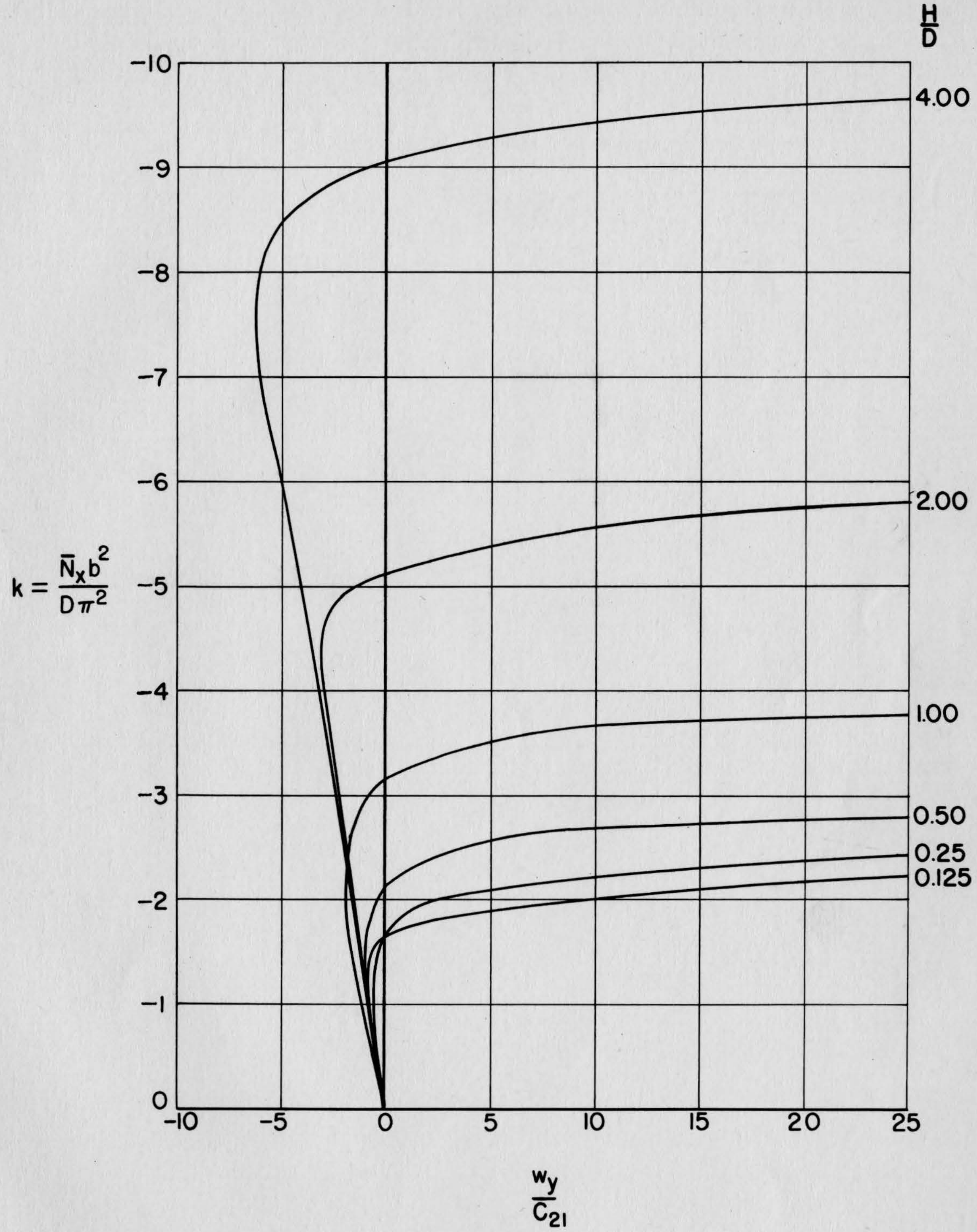


Figure 6.– Dimensionless load-deflection curves for plate in which $D_1 = D_2$ and $a/b = 3$. Deflections computed at $x = a/2$, $y = b/2$. (Deflection due to C_{21} only.)

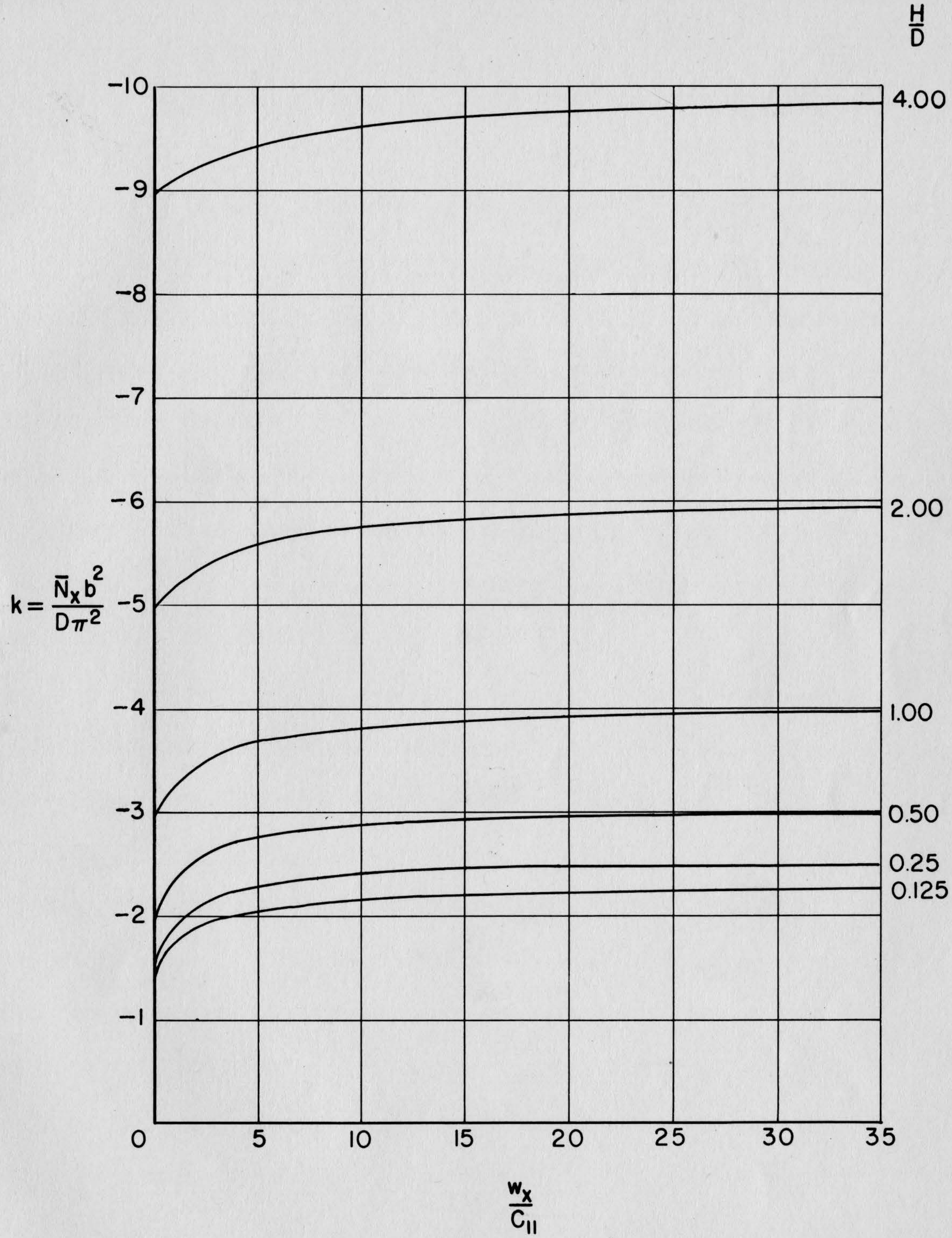


Figure 7.- Dimensionless load-deflection curves for plate in which $D_1 = D_2$ and $a/b = 3$. Deflections computed at $x = a/2$, $y = b/2$. (Deflection due to C_{11} only.)

through 7. From a plot of the total deflection against the load parameter k, the load level at which the deflections begin to grow rapidly may be identified. This load level is similar to the familar buckling load in that in each case a rapid growth of deflection occurs.

In the section entitled Estimate of large-deflection effects it will be found that the limit of accuracy of this particular small-deflection solution for computing deflections is approximately two-tenths of the thickness of the plate. For some modern wing constructions which incorporate very thick skins, deflections of two-tenths of the thickness may be significant. Small-deflection theory will be of practical value in computing the magnitude of prebuckling deflections when coupling is present in these plates.

1. Illustrative example.- Figure 8 illustrates load-deflection curves obtained from the data of figures 3 through 7 by choosing $C_{11} = 0.0500h$ and $C_{21} = 0.0166h$. These load-deflection curves apply to the case $H/D = 1$, and $a/b = 1,2,$ and 3. (This represents a fairly thick ribbing for most integrally stiffened plates.)

When $a/b = 1$, the mode of deflection due to coupling (ie. one-half wave length in each direction) is identical to that in which the uncoupled plate will buckle upon reaching the buckling load. Thus, there is no transition from one mode to another, and the curve is smooth. An effective buckling load of $k \cong -3.2$ is estimated for this case. When $a/b = 2$, the abrupt divergence of deflection at $-k = 4$ is due to the incompatibility between the mode of deflection due to the coupling action and that into which the plate will buckle

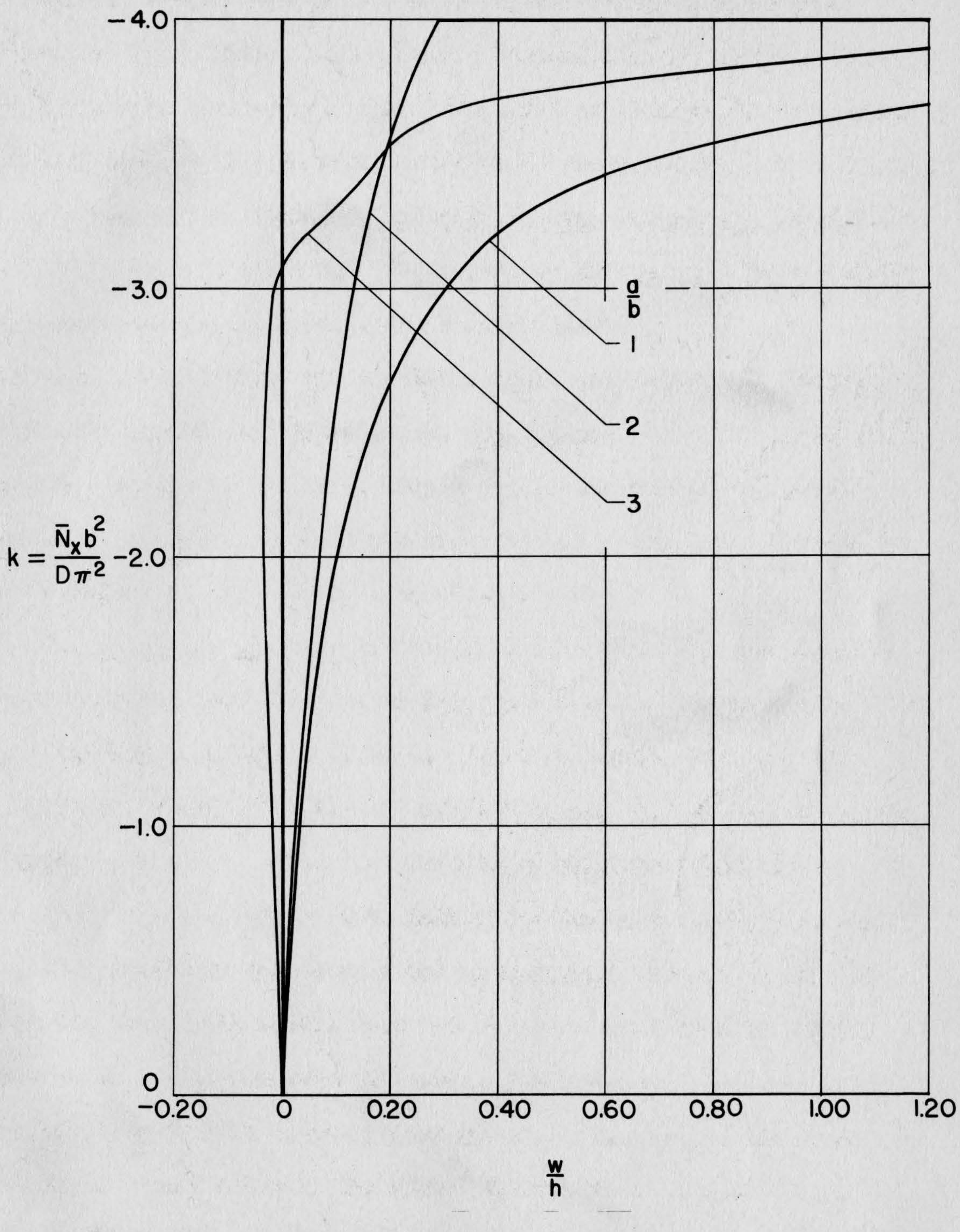


Figure 8.- Dimensionless load-deflection curves for plate in which $D_1 = D_2 = D$, $C_{11} = 0.0500$, and $C_{21} = 0.0166$. Deflections computed at $x = \frac{a}{2}$ and $y = \frac{b}{2}$ for $\frac{a}{b} = 1$ and 3. Deflections computed at $x = \frac{a}{4}$ and $y = \frac{b}{2}$ for $\frac{a}{b} = 2$.

upon reaching the uncoupled buckling load. The abrupt divergence at $k = -4$ is then a yielding of the former mode to the latter. When $a/b = 3$, there is some degree of compatibility between the mode of deflection due to the coupling action and the uncoupled buckling mode. The yielding to the uncoupled buckling mode is, therefore, less abrupt than the $a/b = 2$ case, but it is more abrupt than the $a/b = 1$ case. An effective buckling load of $k \cong -3.6$ is estimated for this case.

As the aspect ratio is increased above $a/b = 3$ the mode of deflection due to coupling action becomes more incompatible with the buckling mode. The effect of coupling on the uncoupled buckling load will accordingly become negligible.

For cases in which the bending stiffnesses differ in the orthogonal directions, no such general statements can be made because the buckles in the uncoupled buckling mode are not necessarily square. These cases must be treated individually.

2. Estimation of large-deflection effects.- It was found that the deflections due to coupling as indicated by small-deflection theory are similar to those which are indicated by the small-deflection solution to the problem of a plate with a small initial deviation from flatness subject to the same boundary conditions. A large-deflection analysis of the compressive buckling of a square plate simply supported on all edges with small initial deviations from flatness is presented in reference 7.

In both the problem of initial deviations from flatness and the problem of coupling a divergence of deflections is incurred by the presence of deflections prior to the uncoupled flat plate buckling load. Also, in each case the post-buckling state approaches that of the uncoupled flat plate. (In the post-buckling state, the effect of coupling will become negligible because as the load increases above the load level at which the deflections grow rapidly, the rate of deflection growth decreases; therefore, the effect of coupling decreases.) Thus, it may be assumed that the transition from the stage in which small-deflection theory is applicable for the case of coupling to large-deflection theory for the uncoupled flat plate is the same as that for a similar plate with an initial deviation from flatness. That is, when small-deflection theory for each case indicates coincident load-deflection curves, or nearly coincident in the range in which the deflections grow rapidly, the transition to large-deflection theory for the uncoupled flat plate is the same. This point is illustrated in figure 9.

Figure 9 illustrates a few cases in which the equivalence between the initial deviations from flatness and the coupling terms have been established. There are an infinity of combinations of the two coupling terms for which this equivalence could have been established.

Reference 7 presents information only for square isotropic plates. The results may be applied as an approximation to the deflections of isotropic plates of greater aspect ratios than those presented in figures 3 through 7. In the case of large aspect ratios, the buckles in the post-buckling stage will tend toward their natural square shape

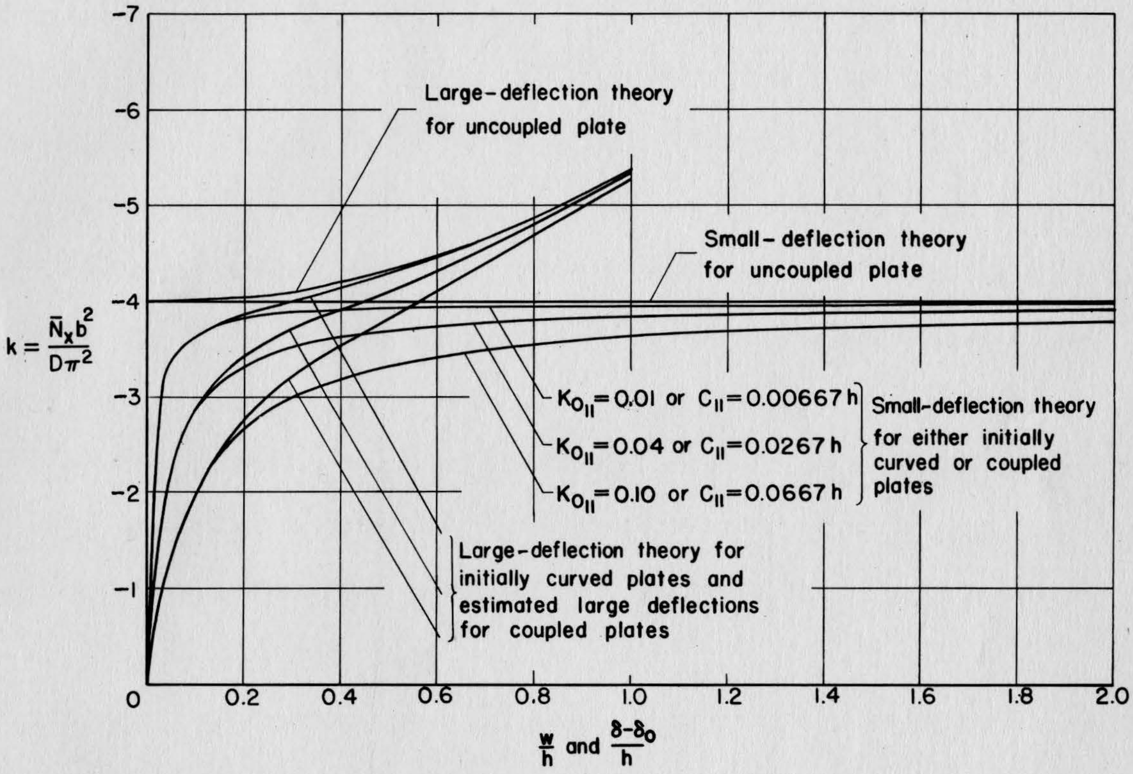


Figure 9.- Estimated dimensionless load-deflection curves for the transition from small-deflection theory for plates exhibiting coupling to large-deflection theory for flat uncoupled plates. $D_1 = D_2 = D = H$, $a/b = 1$.

and the information of figure 9 will therefore furnish a good approximation to the deflections to be expected.

VIII. CONCLUSIONS

The elastic theory presented herein forms a basis from which problems involving coupling may be treated. The significance of coupling is most apparent in problems which would involve stability considerations in the absence of coupling. The presence of deflections due to coupling prior to reaching the uncoupled buckling load forces the problem to be treated as one of deflections rather than stability. Coupling has a general detrimental effect upon this type of problem in that it lowers the load at which deflections grow rapidly (that is, buckling in the uncoupled case).

The effect of coupling on the buckling of plates of equal bending stiffnesses in their two orthogonal directions becomes negligible as the aspect ratio of the plate becomes large. Some lateral deflection prior to buckling will occur, however, even for large aspect ratios.

The general anisotropic plate is considered in the small-deflections solution to the differential equation of equilibrium. Computations were made for cases in which the bending stiffnesses were equal. The equations and methods are applicable to cases in which the bending stiffnesses are not equal.

IX. ACKNOWLEDGMENTS

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XII. APPENDIX A

Derivation of Potential Energy Expression

The derivations made in this section shall be made only for rectangular plates the edges of which are $x = 0,a$ and $y = 0,b$. However, the methods are general and may be applied to a plate of any shape. The following methods are similar to those used in reference 3.

Strain energy: The strain energy produced by the moments M_x, M_y, and M_{xy} and the horizontal loads N_x, N_y, and N_{xy} can be obtained by considering the work done by these forces in distorting a differential element such as shown in figure 2.

The work done by the moment $M_x\,dy$ in rotating through the distortion $-\frac{\partial^2 w}{\partial x^2}\,dx$ with respect to the opposite face of the element is

$$-\frac{1}{2} M_x \frac{\partial^2 w}{\partial x^2}\,dx\,dy \qquad \text{(a)}$$

Similarly, the work done by the moment $M_y\,d_x$ is

$$-\frac{1}{2} M_y \frac{\partial^2 w}{\partial y^2}\,dx\,dy \qquad \text{(b)}$$

The work done by the twisting moment $M_{xy}\,dy$ in twisting through the distortion $\frac{\partial^2 w}{\partial x \partial y}\,dx$ with respect to the opposite face of the element is

$$\frac{1}{2} M_{xy} \frac{\partial^2 w}{\partial x \partial y} \qquad \text{(c)}$$

and the work done by the twisting moment $M_{xy}\ dx$ on the adjacent face of the element is likewise

$$\frac{1}{2} M_{xy} \frac{\partial^2 w}{\partial x \partial y} \tag{d}$$

The work done by the horizontal direct load $N_x\ dy$ in moving through the distortion $\epsilon_x\ dy$ with respect to the opposite face of the element is

$$\frac{1}{2} N_x\ \epsilon_x\ dx\ dy \tag{e}$$

Similarly, the work done by the horizontal direct load $N_y\ dx$ is

$$\frac{1}{2} N_y\ \epsilon_y\ dx\ dy \tag{f}$$

The work done by the horizontal shear loads $Nxy\ dy$ and $N_{xy}\ dy$ in causing the distortion γ_{xy} is

$$\frac{1}{2} N_{xy} \gamma_{xy}\ dx\ dy \tag{g}$$

Assumptions have been made in the above derivations that the shearing forces on adjacent faces are equal and that the twisting moments on adjacent faces are equal. These are assumptions on the equilibrium of the element and will therefore be reflected in the derivation of the equilibrium equations in which the strain energy expression will be used. It is necessary that this assumption be made because the assumptions are inherent in the force-distortion relations which are being considered. The correctness of these assumptions is substantiated in reference 6 for integrally stiffened plates.

Integration of each of the components of strain energy, (a) through (g), over the area of the rectangular plate gives the total strain energy of the plate, thus

$$U = \frac{1}{2}\int_0^a\int_0^b\left(-M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x\partial y} + N_x\varepsilon_x + N_y\varepsilon_y + N_{xy}\gamma_{xy}\right)dx\,dy \tag{A1}$$

The coupling effect is characterized by lateral deflections of points in the plate resulting from the application of horizontal loads, therefore, the strains in the plane of the plate must be expressed in terms of lateral as well as horizontal displacements. The relations for the strain ε_x, ε_y, and γ_{xy} in terms of the displacements u, v, and w are given in equation (4). Substitution of equation (4) into equation (A1) permits the strain energy to be expressed in terms of only the loads and displacements.

$$U = \frac{1}{2}\iint\left(-M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial x^2} + 2M_{xy}\frac{\partial^2 w}{\partial x\partial y} + N_x\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + N_y\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + N_{xy}\left[\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right)dx\,dy \tag{A2}$$

External work: The boundary conditions considered in the following do not include those of the type such that the reactions do work. That is, elastic supports are not included. The work done by the horizontal boundary forces $\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$ is

$$\int_0^b \left| \bar{N}_x u + \bar{N}_{xy} v \right|_0^a dy + \int_0^a \left| \bar{N}_y v + N_{xy} u \right|_0^b dx \qquad \text{(a)}$$

The work done by the boundary moments $\bar{M}_x$, $\bar{M}_y$, and $\bar{M}_{xy}$ is

$$\int_0^b \left| - \bar{M}_x \frac{\partial w}{\partial x} + \bar{M}_{xy} \frac{\partial w}{\partial y} \right|_0^a dy + \int_0^a \left| - \bar{M}_y \frac{\partial w}{\partial y} + \bar{M}_{xy} \frac{\partial w}{\partial x} \right|_0^b dx \qquad \text{(b)}$$

The work done by the boundary transverse shear forces $\bar{Q}_x$ and $\bar{Q}_y$ is

$$\int_0^b \left| \bar{Q}_x w \right|_0^a dy + \int_0^b \left| \bar{Q}_y w \right|_0^a dx \qquad \text{(c)}$$

The work done by a lateral loading $q(x,y)$ over the plate is

$$\int_0^a \int_0^b qw \, dx \, dy \qquad \text{(d)}$$

The total external work T is then the sum of the expressions (a) through (d).

$$T = \int_0^b \left| \overline{N}_x u + \overline{N}_{xy} v - \overline{M}_x \frac{\partial w}{\partial x} + \overline{M}_{xy} \frac{\partial w}{\partial y} + \overline{Q}_x w \right|_0^a dy + \int_0^a \left| \overline{N}_y v + \right.$$

$$\left. \overline{N}_{xy} u - \overline{M}_y \frac{\partial w}{\partial y} + \overline{M}_{xy} \frac{\partial w}{\partial x} + \overline{Q}_y w \right|_0^b dx + \int_0^a \int_0^b qw \, dx \, dy \tag{A3}$$

The total potential energy of the system is given by subtracting the external work from the strain energy, thus

$$V = U - T = \frac{1}{2} \int_0^a \int_0^b \left\{ - M_x \frac{\partial^2 w}{\partial x^2} - M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_x \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] + \right.$$

$$\left. N_y \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] + N_{xy} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \right\} dx \, dy - \int_0^b \left| \overline{N}_x u + \overline{N}_{xy} v - \right.$$

$$\left. \overline{M}_x \frac{\partial w}{\partial x} + \overline{M}_{xy} \frac{\partial w}{\partial y} + \overline{Q}_x w \right|_0^a dy - \int_0^a \left| \overline{N}_y v + \overline{N}_{xy} u - \overline{M}_y \frac{\partial w}{\partial y} + \right.$$

$$\left. \overline{Q}_y w \right|_0^b dx - \int_0^a \int_0^b qw \, dx \, dy$$

$$\tag{A4}$$

XIII. APPENDIX B

Derivation of Equilibrium and Boundary Equations

It is known from the principle of minimum potential energy that for an elastic system under the action of external loads to be in static equilibrium the displacements which occur in the system are those for which the potential energy of the system is a minimum. The potential energy expression derived in Appendix A shall therefore be minimized in this section by the methods of the calculus of variations (ref. 8). The equations thus obtained by minimizing the potential energy with respect to the displacements u, v, and w will be the equations of equilibrium in the x-, y-, and z-directions, respectively, and associated natural boundary equations.

In order that the potential energy expression, equation (A5), may be expressed in terms of displacements only, the following set of force displacement equations, rather than the conventional set presented in the text, shall be used wherever necessary. The b_{ij} are elastic constants related to those given in equations (1).

The relationships between the elastic constants of this form and the forms presented in the text are easily derived; however, they will not be required in the following treatment and therefore are not given.

$$
\left.
\begin{aligned}
M_x &= -b_{11}\,\frac{\partial^2 w}{\partial x^2} - b_{12}\,\frac{\partial^2 w}{\partial y^2} + b_{14}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + b_{15}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{dw}{dy}\right)^2\right] \\
M_y &= -b_{12}\,\frac{\partial^2 w}{\partial y^2} - b_{22}\,\frac{\partial^2 w}{\partial y^2} + b_{24}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + b_{25}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{dw}{dy}\right)^2\right] \\
M_{xy} &= b_{33}\,\frac{\partial^2 w}{\partial x \partial y} + b_{36}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \\
N_x &= -b_{14}\,\frac{\partial^2 w}{\partial x^2} - b_{24}\,\frac{\partial^2 w}{\partial y^2} + b_{44}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + b_{45}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] \\
N_y &= -b_{15}\,\frac{\partial^2 w}{\partial x^2} - b_{25}\,\frac{\partial^2 w}{\partial y^2} + b_{45}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + b_{55}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] \\
N_{xy} &= 2b_{36}\,\frac{\partial^2 w}{\partial x \partial y} + b_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]
\end{aligned}
\right\} \quad \text{(B1)}
$$

The first variation of the potential energy (equation (A5)) with respect to u-displacements is

$$
\begin{aligned}
\delta_u V = \frac{1}{2}\int_0^a \int_0^b \Bigg\{\Bigg[& -b_{14}\,\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta u}{\partial x} - b_{24}\,\frac{\partial^2 w}{\partial y}\frac{\partial \delta u}{\partial x} + 2b_{36}\,\frac{\partial^2 w}{\partial x \partial y}\frac{\partial \delta u}{\partial y} + \\
& N_x\,\frac{\partial \delta u}{\partial x} + b_{44}\,\frac{\partial \delta u}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + b_{45}\,\frac{\partial \delta u}{\partial y}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \\
& N_{xy}\,\frac{\partial \delta u}{\partial y} + b_{66}\,\frac{\partial \delta u}{\partial y}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\Bigg\} dx\,dy - \int_0^b \left|\overline{N}_x \delta u\right|_0^a dy \qquad \text{(B2)} \\
& \int_0^a \left|\overline{N}_{xy} \delta u\right|_0^b dx
\end{aligned}
$$

Simplifying equation (B2) by substitution from equation (B1) yields

$$\delta_u V = \int_0^a \int_0^b \left(N_x \frac{\partial \delta u}{\partial x} + N_{xy} \frac{\partial \delta u}{\partial y} \right) dx dy - \int_0^b \left| \bar{N}_x \delta u \right|_0^a dy - \int_0^a \left| \bar{N}_{xy} \delta u \right|_0^b dx \quad \text{(B3)}$$

The first term of the above expression can be integrated by parts, thus only δu remains rather than derivatives of δu.

$$\delta_u V = -\int_0^a \int_0^b \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u dx dy - \int_0^b \left| \left(\bar{N}_x - N_x \right) \delta u \right|_0^a dy -$$

$$\int_0^a \left| \left(\bar{N}_{xy} - N_{xy} \right) \delta u \right|_0^b dx \quad \text{(B4)}$$

The minimizing condition on the potential energy is that the value of $\delta_u V$ must be zero for any value of δu. Imposing this condition results in the following differential equation of equilibrium and natural boundary equations:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad \text{(B5)}$$

$$\begin{aligned} \bar{N}_x = N_x \quad &\text{or} \quad \delta u = 0 \quad \text{along} \quad x = 0,\ a \\ \bar{N}_{xy} = N_{xy} \quad &\text{or} \quad \delta u = 0 \quad \text{along} \quad y = 0,\ b \end{aligned} \quad \text{(B6)}$$

Similarly, the first variation of equation (A5) with respect to v-displacements and the minimization of the result leads to the following differential equation of equilibrium and boundary equations.

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad \text{(B7)}$$

$$\left.\begin{aligned} \bar{N}_y = N_y \quad \text{or} \quad \delta v = 0 \quad \text{along} \quad y = 0, b \\ \bar{N}_{xy} = N_{xy} \quad \text{or} \quad \delta v = 0 \quad \text{along} \quad x = 0, a \end{aligned}\right\} \qquad \text{(B8)}$$

The first variation of the potential energy with respect to w-displacement is

$$\left.\begin{aligned}
\delta_w V = {} & \frac{1}{2}\int_0^a \int_0^b \Bigg\{ - M_x \frac{\partial^2 \delta w}{\partial x^2} - \frac{\partial^2 w}{\partial x^2}\Bigg[- b_{11}\frac{\partial^2 \delta w}{\partial x^2} - b_{12}\frac{\partial^2 \delta w}{\partial y^2} + b_{14}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \\
& b_{15}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Bigg] - M_y \frac{\partial^2 \delta w}{\partial y^2} - \frac{\partial^2 w}{\partial y^2}\Bigg[- b_{12}\frac{\partial^2 \delta w}{\partial x^2} - b_{22}\frac{\partial^2 \delta w}{\partial y^2} + b_{24}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \\
& b_{25}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Bigg] + 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\Bigg[b_{33}\frac{\partial^2 \delta w}{\partial x \partial y} + b_{36}\Big(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \\
& \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\Big)\Bigg] + N_x \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \Bigg[\frac{\partial u}{\partial x} + \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2\Bigg]\Bigg[- b_{14}\frac{\partial^2 \delta w}{\partial x^2} - b_{24}\frac{\partial^2 \delta w}{\partial y^2} + \\
& b_{44}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + b_{45}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Bigg] + N_y \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + \Bigg[\frac{\partial v}{\partial y} + \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2\Bigg]\Bigg[- b_{15}\frac{\partial^2 \delta w}{\partial x^2} - \\
& b_{25}\frac{\partial^2 \delta w}{\partial y^2} + b_{45}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + b_{55}\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\Bigg] + N_{xy}\Big(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\Big) + \\
& \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big)\Bigg[2b_{36}\frac{\partial^2 \delta w}{\partial x \partial y} + b_{66}\Big(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\Big)\Bigg]\Bigg\} dxdy - \int_0^b \Bigg| - \\
& \bar{M}_x \frac{\partial \delta w}{\partial x} + \bar{M}_{xy}\frac{\partial \delta w}{\partial y} + \bar{Q}_x \delta w \Bigg|_0^a dy - \int_0^a \Bigg| - \bar{M}_y \frac{\partial \delta w}{\partial y} + M_{xy}\frac{\partial \delta w}{\partial x} + \\
& \bar{Q}_y \delta w \Bigg|_0^b dx - \int_0^a \int_0^b q\,\delta w\,dxdy
\end{aligned}\right\}$$

(B9)

The above expression can be simplified by substituting the forces given by equation (B1) wherever possible. The expression for $\delta_w V$ then becomes

$$\delta_w V = \left\{ \begin{aligned} &\int_0^a \int_0^b \Big[-M_x \frac{\partial^2 \delta w}{\partial x^2} - M_y \frac{\partial^2 \delta w}{\partial y^2} + 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} + N_x \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + N_y \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \\ &\quad N_{xy}\left(\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x}\right)\Big] dxdy - \int_0^b \left| - \bar{M}_x \frac{\partial \delta w}{\partial x} + \bar{M}_{xy} \frac{\partial \delta w}{\partial y} + \bar{Q}_x \delta w \right|_0^a dy - \\ &\quad \int_0^a \left| - \bar{M}_y \frac{\partial \delta w}{\partial y} + \bar{M}_{xy} \frac{\partial \delta w}{\partial x} + \bar{Q}_y \delta w \right|_0^b dx - \int_0^a \int_0^b q\, \delta w dxdy \end{aligned} \right\} \tag{B10}$$

With the integration by parts of the first term of the above expression so as to eliminate derivatives of δw and the rearrangement of the resulting terms the expression for $\delta_w V$ becomes

$$
\left.
\begin{aligned}
\delta_w V = \int_0^a \int_0^b \Bigg[& -\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 M_y}{\partial y^2} - q - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - \\
& 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\Bigg] \delta w\,dx\,dy + \int_0^b \left| (\overline{M}_x - M_x)\,\delta \frac{\partial w}{\partial y} \right|_0^a dy + \int_0^a \left| (\overline{M}_y - M_y)\,\delta \frac{\partial w}{\partial y} \right|_0^b dx - \\
& \int_0^b \left| (\overline{M}_{xy} - M_{xy})\,\delta \frac{\partial w}{\partial y} \right|_0^a dy - \int_0^a \left| (\overline{M}_{xy} - M_{xy})\,\delta \frac{\partial w}{\partial x} \right|_0^b dx - \int_0^b \Bigg| \Bigg(\overline{Q}_x - \\
& \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - N_x \frac{\partial w}{\partial x} - N_{xy} \frac{\partial w}{\partial y}\Bigg) \delta w \Bigg|_0^a dy - \int_0^a \Bigg| \Bigg(\overline{Q}_y - \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - \\
& N_y \frac{\partial w}{\partial y} - N_{xy} \frac{\partial w}{\partial x}\Bigg) \delta w \Bigg|_0^b dx
\end{aligned}
\right\}
\qquad \text{(B11)}
$$

Imposing the minimizing condition that $\delta_w V = 0$ for any value of δw results in the following equation of equilibrium and natural boundary equations:

$$
\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) \qquad \text{(B12)}
$$

Along $x = 0, a$

$$
\left.
\begin{aligned}
& \frac{\partial w}{\partial x} = 0 \quad \text{or} \quad \overline{M}_x = M_x \\
& \frac{\partial w}{\partial y} = 0 \quad \text{or} \quad \overline{M}_y = M_y \\
& w = 0 \quad \text{or} \quad \overline{Q}_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y}
\end{aligned}
\right\}
\qquad \text{(B13)}
$$

Along $y = 0, b$

$$\left.\begin{aligned} &\frac{\partial w}{\partial y} = 0 \quad \text{or} \quad \bar{M}_y = M_y \\ &\frac{\partial w}{\partial x} = 0 \quad \text{or} \quad \bar{M}_{xy} = M_{xy} \\ &w = 0 \quad \text{or} \quad \bar{Q}_y = \frac{\partial M_y}{\partial x} - \frac{\partial M_{xy}}{\partial x} + N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \end{aligned}\right\} \text{(B14)}$$

The derived equilibrium and boundary equations when expressed in terms of the forces are the same as those for a conventional isotropic or anisotropic flat plate. When the expressions for the moments given in equations (1) is substituted in equation (B12), the equation of equilibrium for displacements in the z-direction exhibits the effect of coupling as

$$\left.\begin{aligned} &D_1 \frac{\partial^4 w}{\partial x^4} + 2(\mu_y D_1 + D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} - C_{11} \frac{\partial^2 N_x}{\partial x^2} - C_{12} \frac{\partial^2 N_y}{\partial x^2} + \\ &\qquad 2C_K \frac{\partial^2 N_{xy}}{\partial x \partial y} - C_{21} \frac{\partial^2 N_x}{\partial y^2} - C_{22} \frac{\partial^2 N_y}{\partial y^2} = q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + \\ &\qquad 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \text{(B15)}$$

The boundary equations for displacements in the z-direction upon substitution from equations (1) for the moments become along $x = 0, a$

$$\left.\begin{aligned}
&\frac{\partial w}{\partial x} = 0 \quad \text{or} \quad \overline{M}_x = -D_1 \frac{\partial^2 w}{\partial x^2} - D_1\mu_y \frac{\partial^2 w}{\partial y^2} + C_{11}N_x + C_{12}N_y \\
&\frac{\partial w}{\partial y} = 0 \quad \text{or} \quad \overline{M}_{xy} = D_{xy} \frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \\
&w = 0 \quad \text{or} \quad \overline{Q}_x = -D_1 \frac{\partial^3 w}{\partial x^3} - (\mu_y M_y + D_{xy}) \frac{\partial^3 w}{\partial x \partial y^2} + C_{11} \frac{\partial N_x}{\partial x} + C_{12} \frac{\partial N_y}{\partial x} - \\
&\qquad C_K \frac{\partial N_{xy}}{\partial y} + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y}
\end{aligned}\right\} \tag{B16}$$

Along $y = 0, b$

$$\begin{aligned}
&\frac{\partial w}{\partial y} = 0 \quad \text{or} \quad \overline{M}_y = -\mu_x D_2 \frac{\partial^2 w}{\partial x^2} - D_2 \frac{\partial^2 w}{\partial y^2} + C_{21}N_x + C_{22}N_y \\
&\frac{\partial w}{\partial x} = 0 \quad \text{or} \quad \overline{M}_{xy} = D_{xy} \frac{\partial^2 w}{\partial x \partial y} + C_K N_{xy} \\
&w = 0 \quad \text{or} \quad \overline{Q}_y = -D_2 \frac{\partial^3 w}{\partial y^3} - (\mu_x D_2 + D_{xy}) \frac{\partial^3 w}{\partial x^2 \partial y} + C_{21} \frac{\partial N_x}{\partial y} + \\
&\qquad C_{22} \frac{\partial N_y}{\partial y} + C_K \frac{\partial N_{xy}}{\partial x} + N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x}
\end{aligned} \tag{B17}$$

XIV. APPENDIX C

Exact Small-Deflection Solution for a Simply Supported Anisotropic Plate in Compression

The small-deflection solution to the differential equation of equilibrium, equation (6), will be found in this section for the case of a simply supported anisotropic plate acted upon by a uniformly distributed load $\bar{N}_x$. The plate is shown in figure 10.

The internal in-plane force N_x is considered to be evenly distributed over the width of the plate according to the discussion presented in the text. The differential equation of equilibrium becomes

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} = \bar{N}_x \frac{\partial^2 w}{\partial x^2} \tag{C1}$$

where $H = \mu_y D_1 + D_{xy}$ and $\bar{N}_x$ is tension. Of the natural boundary conditions given, only equations (7) and (13) will be different from those for an uncoupled plate. These two equations become

$$\left.\begin{aligned} 0 &= -D_1\left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) + C_{11}\bar{N}_x \quad \text{along } x = 0, a \\ 0 &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2}\right) + C_{21}\bar{N}_x \quad \text{along } y = 0, b \end{aligned}\right\} \tag{C2}$$

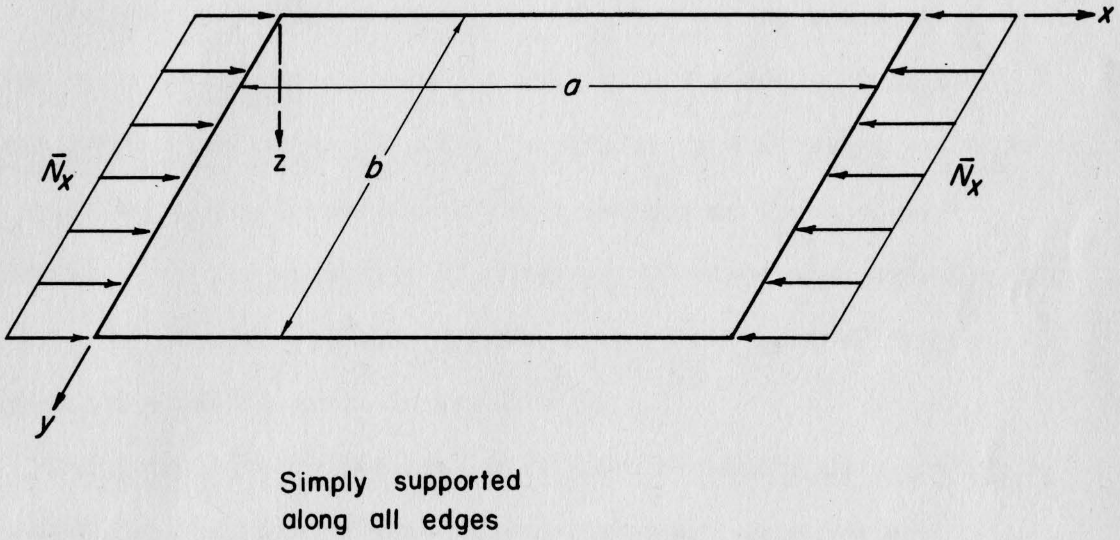

Figure 10.– Plate considered.

This natural boundary condition may be effected by considering the equivalent uncoupled plate to be acted upon by moments

$$\left.\begin{aligned} \overline{M}_x &= -C_{11}\overline{N}_x \quad \text{along} \quad x = 0,\, a \\ M_y &= -C_{21}\overline{N}_x \quad \text{along} \quad y = 0,\, b \end{aligned}\right\} \qquad \text{(C3)}$$

The problem shall be separated into two parts; the first is that which would result by considering the force-distortion equations as those given in equation (3), and the second is that which would result by considering the force-distortion equations as those given by equation (2). The solutions of these two problems when superimposed will be the general solution for the case of the general force-distortion equation given in equation (1).

The first part of the problem is then to obtain the solution to equation (C1) subject to the following boundary conditions:

$$\left.\begin{aligned} & w\Big|_{x=0} = w\Big|_{x=a} = w\Big|_{y=0} = w\Big|_{x=b} = 0 \\ & \frac{\partial^2 w}{\partial x^2}\bigg|_{x=0} = \frac{\partial^2 w}{\partial x^2}\bigg|_{x=a} = \frac{C_{11}\overline{N}_x}{D_1} \\ & \frac{\partial^2 w}{\partial y^2}\bigg|_{y=0} = \frac{\partial^2 w}{\partial y^2}\bigg|_{y=b} = 0 \end{aligned}\right\} \qquad \text{(C4)}$$

Assume the deflection function

$$w_x = \sum_{n=1,3,5}^{\infty} X_n \sin \frac{n\pi y}{b} \tag{C5}$$

which automatically satisfies y-wise boundary conditions. By substituting the deflection function into equation (C1) the following equation is obtained:

$$X_n'''' - \alpha_n^2\left(2\frac{H}{D_1} + \frac{\bar{N}_x}{\alpha_n^2 D_1}\right)X_n'' + \frac{D_2}{D_1} X_n \alpha_n^4 = 0 \quad \left(\alpha_n = \frac{n\pi}{b}\right) \tag{C6}$$

The general solution to equation (C6) is

$$X_n = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} \tag{C7}$$

where the m's are the roots of the characteristic equation and are given by

$$m = \pm\alpha_n \sqrt{\frac{H}{D_1} + \frac{k}{2n^2} \pm \sqrt{\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 - \frac{D_2}{D_1}}} \tag{C8}$$

and

$$k = \frac{\bar{N}_x b^2}{\pi^2 D_1} \quad \text{(k will be negative for a compressive load.)} \tag{C9}$$

The roots of the characteristic equations may be of four types:

1. Roots all complex
2. Roots all imaginary
3. Roots all real
4. Roots real and imaginary

For the roots to be complex the condition is

$$\frac{D_2}{D_1} > \left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 \tag{C10}$$

Then

$$\left.\begin{aligned} m_1 &= \alpha_n(a_n + ib_n) \\ m_2 &= \alpha_n(a_n - ib_n) \\ m_3 &= -\alpha_n(a_n + ib_n) \\ m_4 &= -\alpha_n(a_n - ib_n) \end{aligned}\right\} \tag{C11}$$

where

$$a_n = \sqrt{\frac{\dfrac{D_2}{D_1} + \dfrac{H}{D_1} + \dfrac{k}{2n^2}}{2}}$$

and

$$b_n = \sqrt{\frac{\dfrac{D_2}{D_1} - \dfrac{H}{D_1} - \dfrac{k}{2n^2}}{2}} \tag{C12}$$

By substituting the values of the characteristics roots given in equation (C11) into equation (C7), the function X_n may be expressed in the form

$$X_n = \cosh \alpha_n a_n x(A \sin \alpha_n b_n x + B \cos \alpha_n b_n x) + \sinh \alpha_n a_n x(C \sin \alpha_n b_n x + D \cos \alpha_n b_n x) \tag{C13}$$

By substituting this form of X_n into the solution of the differential equation of equilibrium and applying the boundary conditions given in (C4) (a Fourier expansion for the x-wise curvature is used) X_n is determined in the following form

$$X_n = \frac{2C_{11}k}{\pi n^3 a_n b_n} \times \left[\frac{\sin \alpha_n b_n a \sinh \alpha_n a_n x \cos \alpha_n b_n x - \sinh \alpha_n a_n a \cosh \alpha_n a_n x \sin \alpha_n b_n x}{\cos \alpha_n b_n a + \cosh \alpha_n a_n a} + \sinh \alpha_n a_n x \sin \alpha_n b_n x\right] \tag{C14}$$

For cases in which

$$\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 \geq \frac{D_2}{D_1}$$

three possibilities remain; all roots real, all roots imaginary, or 2 roots real and 2 roots imaginary. Only the case for which all roots are real will be considered, for if the roots are imaginary, the relationships between hyperboic and trigonometric function may be substituted into the solution for real roots, thus changing that solution to the case for which all or part of the roots are imaginary.

Therefore, considering the roots to be real, the function X_n is of the form

$$X_n = A \cosh \alpha_n c_n x + B \sinh \alpha_n c_n x + C \cosh \alpha_n d_n x + D \sinh \alpha_n d_n x \tag{C15}$$

where

$$\begin{aligned} c_n &= \sqrt{\frac{H}{D_1} + \frac{k}{2n^2} + \sqrt{\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 - \frac{D_2}{D_1}}} \\ d_n &= \sqrt{\frac{H}{D_1} + \frac{k}{2n^2} - \sqrt{\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 - \frac{D_2}{D_1}}} \end{aligned} \tag{C16}$$

By substituting the form of X_n given by equation (C15) into the solution of the differential equation of equilibrium and applying the boundary conditions, equation (C4), X_n is determined in the following form:

$$X_n = \frac{4C_{11}k}{\pi n^3}\left(\frac{1}{c_n^2 - d_n^2}\right)\left[\left(\frac{1 - \cosh \alpha_n c_n a}{\sinh \alpha_n c_n a}\right)\sinh \alpha_n c_n x + \cosh \alpha_n c_n x - \right.$$

$$\left.\left(\frac{1 - \cosh \alpha_n d_n a}{\sinh \alpha_n d_n a}\right)\sinh \alpha_n d_n x - \cosh \alpha_n d_n x\right] \tag{C17}$$

If either or both of the parameters c_n and d_n are imaginary the function X_n may be found by substituting appropriatly the following relationships into equation (C17):

$$\left.\begin{aligned} \cosh f\,(c_n \text{ or } d_n) &= \cos f\,(c_n' \text{ or } d_n') \\ \sinh f\,(c_n \text{ or } d_n) &= i \sin f\,(c_n' \text{ or } d_n') \end{aligned}\right\} \quad (C18)$$

where

$$\left.\begin{aligned} C'_n &= \sqrt{-C_n^2} = \sqrt{-\frac{H}{D_1} - \frac{k}{2n^2} - \sqrt{\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 - \frac{D_2}{D_1}}} \\ d'_n &= \sqrt{-d_n^2} = \sqrt{-\frac{H}{D_1} - \frac{k}{2n^2} - \sqrt{\left(\frac{H}{D_1} + \frac{k}{2n^2}\right)^2 - \frac{D_2}{D_1}}} \end{aligned}\right\} \qquad (C19)$$

Thus, the deflection function is composed of a summation of X_n's where X_n may be of several types of solutions for given plate properties and values of n.

The second part of this problem, in which the force-distortion equations of equation (2) are assumed to govern, is solved in a similar manner to the first part. Equation (C1) is subjected to the following geometrical boundary conditions.

$$\left.\begin{aligned} & w\Big|_{x=0} = w\Big|_{x=a} = w\Big|_{y=0} = w\Big|_{y=b} = 0 \\ & \frac{\partial^2 w}{\partial x^2}\bigg|_{x=0} = \frac{\partial^2 w}{\partial x^2}\bigg|_{x=a} = 0 \\ & \frac{\partial^2 w}{\partial y^2}\bigg|_{y=0} = \frac{\partial^2 w}{\partial y^2}\bigg|_{y=b} = \frac{C_{21}\bar{N}_x}{D_2} \end{aligned}\right\} \qquad (C20)$$

The following deflection function will be used since it satisfies the x-wise boundary conditions automatically

$$w_y = \sum_{n=1,3,5}^{\infty} Y_n \sin\frac{n\pi x}{a} \tag{C21}$$

By substituting this deflection function into the differential equation of equilibrium, equation (C1), the following equation is obtained:

$$Y_n'''' - Y_n''\, 2\,\frac{H}{D_2}\,\beta_n^{\,2} + \frac{D_1}{D_2}\,\beta_n^{\,4}\left(1 + \frac{\bar{N}_x}{D_1\beta_n^{\,2}}\right)Y_n = 0 \tag{C22}$$

$$\left(\beta_n = \frac{n\pi}{a}\right)$$

The general solution to equation (C22) is

$$Y_n = C_1 e^{m_1 y} + C_2 e^{m_2 y} + C_3 e^{m_3 y} + C_4 e^{m_4 y} \tag{C23}$$

where the m's are the roots of the characteristic equation

$$m = \pm\beta_n\sqrt{\frac{D_1}{D_2}\left(\frac{H}{D_1} \pm \sqrt{\left(\frac{H}{D_1}\right)^2 - \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]}\right)} \tag{C24}$$

The roots of the characteristic equation may be of three types:

1. All roots complex
2. All roots real
3. Two roots real and two roots imaginary

When

$$\left(\frac{H}{D_1}\right)^2 < \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right] \tag{C25}$$

all the roots are complex. Then

$$\left.\begin{aligned} m_1 &= \beta_n(e_n + if_n) \\ m_2 &= \beta_n(e_n - if_n) \\ m_3 &= -\beta_n(e_n + if_n) \\ m_4 &= -\beta_n(e_n - if_n) \end{aligned}\right\} \tag{C26}$$

where

$$\left.\begin{aligned} e_n &= \sqrt{\frac{1}{2}\left(1 + \frac{H/D_1}{\sqrt{\frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]}}\right)} \\ f_n &= \sqrt{\frac{1}{2}\left(1 - \frac{H/D_1}{\sqrt{\frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]}}\right)} \end{aligned}\right\} \tag{C27}$$

By a process similar to that used in the first part of this problem when the roots were complex, the function Y_n is determined here as

$$Y_n = 2\,\frac{D_1}{D_2}\left(\frac{a}{b}\right)^2 \frac{C_{21}k}{\pi n^3 e_n f_n} \times$$

$$\left[\frac{\sin \beta_n f_n b \sinh \beta_n e_n y \cos \beta_n f_n y - \sinh \beta_n e_n b \cosh \beta_n e_n y \sin \beta_n f_n y}{\cos \beta_n f_n b + \cosh \beta_n e_n b} + \right.$$

$$\left. \sinh \beta_n e_n y \sin \beta_n f_n y \right] \tag{C28}$$

It is seen by inspecting equation (C24) that when

$$\left(\frac{H}{D_1}\right)^2 \geq \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right] \geq 0 \tag{C29}$$

all the roots of the characteristic equation are real. The form of the function Y_n is then the same as equation (C15) and is determined similarly for the boundary conditions given in equation (C20) to be

$$Y_n = \frac{D_1}{D_2}\left(\frac{a}{b}\right)^2 \frac{4C_{21}k}{\pi n^3}\left(\frac{1}{g_n^{\,2} - h_n^{\,2}}\right)\left[\left(\frac{1 - \cosh \beta_n g_n b}{\sinh \beta_n g_n b}\right) \sinh \beta_n g_n y + \cosh \beta_n g_n y - \right.$$

$$\left. \left(\frac{1 - \cosh \beta_n h_n b}{\sinh \beta_n h_n b}\right) \sinh \beta_n h_n y - \cosh \beta_n h_n y\right] \tag{C30}$$

where

$$\left.\begin{aligned} g_n &= \sqrt{\frac{D_1}{D_2}\left(\frac{H}{D_1} + \sqrt{\left(\frac{H}{D_1}\right)^2 - \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]}\right)} \\ h_n &= \sqrt{\frac{D_1}{D_2}\left(\frac{H}{D_1} - \sqrt{\left(\frac{H}{D_1}\right)^2 - \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]}\right)} \end{aligned}\right\} \tag{C31}$$

The remaining possibility for the roots given in equation (C24) is

$$\left(\frac{H}{D_1}\right)^2 \geq \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right] < 0 \tag{C32}$$

in which case two of the roots will be real and two will be imaginary. The form of the part of the solution containing the real roots will be the same as that given in the first two terms of equation (C30), but the part containing the imaginary roots must be determined by substituting relationships similar to those given in equation (C18) into the last two terms of equation (C30). The solution for the condition given in equation (C32) is

$$Y_n = \frac{D_1}{D_2}\left(\frac{a}{b}\right)^2 \frac{4C_{21}k}{\pi_m{}^3}\left(\frac{1}{g_n^2 + h_n'^2}\right)\left[\left(\frac{1 - \cosh \beta_n g_n b}{\sinh \beta_n g_n b}\right)\sinh \beta_n g_n y + \cosh \beta_n g_n y - \right.$$

$$\left.\left(\frac{1 - \cos\beta_n h'_n b}{\sin \beta_n h'_n b}\right)\sin \beta_n h'_n y - \cos \beta_n h'_n y\right] \tag{C33}$$

where g_n is given in equation (C32) and

$$h'_n = \sqrt{-h_n^2} = \sqrt{\left(\frac{D_1}{D_2}\sqrt{\left(\frac{H}{D_1}\right)^2 - \frac{D_2}{D_1}\left[1 + \left(\frac{a}{b}\right)^2 \frac{k}{n^2}\right]} - \frac{H}{D_1}\right)} \qquad \text{(C34)}$$

For the general case in which there is coupling between both moments and the load $\bar{N}_x$ (both C_{11} and C_{21} have values), the deflection function is

$$w = \sum_{n=1,3,5}^{\infty} X_n \sin \frac{n\pi y}{b} + \sum_{n=1,3,5}^{\infty} Y_n \sin \frac{n\pi y}{a} \qquad \text{(C35)}$$

where the X_n's and the Y_n's correspond to the functions derived herein for given plate properties and value of n.