A NOVEL GRID ANALOGY FOR TRANSVERSELY LOADED ORTHOTROPIC PLATES

by

Ramakrishna Ganesan Iyer

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APPROVED:

Joseph R. Lerski, Chairman

Siegfried M. Holzer

Geza Ifju

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(ABSTRACT)

The objective of the study was to develop a modeling technique to idealize thin, orthotropic plates into an equivalent grid, suitable for the matrix-displacement method of analysis. The formulation is an alternative to the classical plate theory and finite element method. The grid model is unique compared to other grid formulations in its applicability to plates exhibiting material orthotropy in addition to isotropy and shape orthotropy. Further, the grid model has less members compared to the previous grid formulations.

A function was developed to establish the grid member cross-sectional size for various boundary conditions and material properties. The maximum error between predicted and published theoretical and experimental deflections was 3% for plates in isotropic and advanced composite materials. The maximum error between predicted and experimental deflections for wood based composite plates was 20%.
Acknowledgements

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I am also grateful to my other committee members Dr. Geza Ifju and Dr. Siegfried M. Holzer for their valuable suggestions and appreciative comments, which provided me with the zeal to evolve the novel grid formulation.

I would be failing in my duty if I do not place on record the help rendered by for the conduct of the experiments.

Lastly, I wish to express my heartfelt gratitude to my fellow students for their help rendered throughout the project.
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LIST OF SYMBOLS

\( A_{y} \) grid member cross-sectional dimension based on flexure
\( A_{d} \) membrane stiffness matrix
\( A_{t} \) grid member cross-sectional dimension based on torsion
\( A_{xx} \) grid member cross-sectional dimension for isotropic plate simply supported on all edges
\( A_{sc} \) grid member cross-sectional dimension for isotropic plate simply supported on four corners
\( A' \) square cross-sectional dimension of grid internal member
\( B_{ij} \) coupling stiffness matrix
\( B_{xy}, B_{yx} \) torsional rigidity of grid member
\( B' \) square cross-sectional dimension of grid boundary member
\( D_{ij} \) bending stiffness matrix
\( D_{xg}, D_{yg} \) flexural stiffness of grid
\( D_{xp}, D_{yp} \) flexural stiffness of plate
\( E, E_{L}, E_{T} \) modulus of elasticity
\( E^{a} \) modified modulus of elasticity of grid members
\( E_{iga}, E_{iga} \) global panel moduli of elasticity
$E_u, E_h$ local panel moduli of elasticity
$E_l$ stiffness
$G, G_{cl}$ modulus of rigidity
$2H_g$ torsional stiffness of grid
$2H_p$ torsional stiffness of Plate
$J$ index number
$L, T$ principal material directions
$M_x, M_y, M_{xy}$ moment resultants
$N_x, N_y, N_{xy}$ in-plane stress resultants
$P$ applied Load
$P_x$ in-plane compressive force
$Q$ transformed stiffness coefficients
$Q, Q_{ij}$ stiffness coefficients
$Q_x, Q_y$ shear force resultants
$U$ strain energy
$V$ total potential energy
$a, b$ lengths of the sides of rectangular plate, rectangular grid
$a^*$ non-dimensionalized grid cross-sectional size
$b_x, b_y, c$ spacing of grid members
$dx, dy, dz$ dimensions of differential element
$h$ depth of grid member
$k$ torsion coefficient for rectangular sections
$l$ span
$m$ arbitrary lamina in laminated plate
$n$ number of laminae in laminated plate
$q_0, q$ transverse loading
$t$ thickness of plate
$t_m, t_{(n-1)}$ distances of an arbitrary lamina from the mid-plane
$t_x, t_y$  width of grid member

$u, v, w$  translational displacements

$\bar{u}, \bar{v}, \bar{w}$  translational displacements of a point not on the mid-plane.

$x, y, z$  orthogonal cartesian coordinates

2ss, 2fr  plate simply supported on two opposite edges and free at the other two edges

4ss  plate simply supported on all four edges

4c  plate simply supported on four corners

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$  stress components

$\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$  strain components

$\theta_x, \theta_y, \theta_z$  rotational displacements

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$  stress components at a point not on the mid-plane

$\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$  strain components at a point not on the mid-plane

$\Omega$  potential energy due to external loads

$\nu, \nu_{LT}, \nu_{TL}$  Poisson ratios

$\alpha$  angle between L and x axes

$\kappa_x, \kappa_y$  bending curvatures

$\kappa_{xy}$  twisting curvature

$\Delta$  central deflection
INTRODUCTION

Plates are widely used in structural systems because they can economically cover large areas, as in flooring or sheathing applications. However, analyzing plates in structures is complex because the plate response is mathematically described by a bi-harmonic fourth order governing differential equation. Furthermore, exact solutions have been found only for a few boundary conditions and loadings on plates exhibiting isotropic properties. The complexity of analysis increases if the plates are constructed of orthotropic composite materials. In literature, few cases have been solved exactly for both isotropic and orthotropic plates.

Moreover, classical methods of analysis are often unsuitable for analyzing structural systems consisting of beams and columns in addition to plates. Approximate numerical methods have been developed to analyze such complex structures. The exact solutions have been used as benchmarks to verify the solutions obtained with numerical methods. Today, the most popular of the numerical methods are the matrix-displacement method and the finite element method.

The matrix-displacement and the finite element methods are suitable for the integrated analysis of structural systems.
The finite element method is more versatile than the matrix-displacement method, in that, it idealizes a planar continuum into a planar structure comprising smaller, finite, discrete, planar elements. For example, a plate is idealized and analyzed as a discrete assemblage consisting of smaller plate elements. In the idealization, each small element is connected to its neighboring elements at nodal points only.

The matrix-displacement method is a method capable of analyzing structures comprising one-dimensional elements such as bars and beams.

Despite the advantages of the finite element method, it is not suitable for many currently available micro-computers due to extensive memory requirements and greater central processing unit (CPU) time for computation. The matrix-displacement method, though lacking the versatility of the finite element method, is well suited for micro-computer implementation. The research discussed in this thesis is a component of a larger project, whose goal is to develop a design methodology for panel-deck pallets. Since the global project uses the matrix-displacement method for analysis, an approach that is compatible with the overall design method was required. Therefore the purpose of the research discussed in this thesis was to investigate the suitability of the matrix-displacement method for analyzing plates. If successful, the results will be used in a larger model that incorporates non-rigid joints and material variability to predict the structural performance of panel-deck pallets.

1.1 RESEARCH NEED

The advent of the finite element method reduced the interest of researchers in the area of matrix-displacement method applied to problems in planar elasticity. Using the matrix-displacement method to analyze a plate requires an analog model consisting of assemblage of one-dimensional elements, called a grid. Hrennikoff (1) idealized a plate into a grid con-
isting of square frames. Each square frame consisted of side and diagonal members. Yettram and Hussain (2) also idealized a plate as square frames consisting of side members and diagonals. Salonen (3) used a triangular frame model for the idealization and solved the plate problem by the Livesley elastic method. The findings of these early researchers are restricted to isotropic materials. However, applicability of the grid method of analysis to orthotropic materials has not been studied. The development of the grid method for orthotropic plate problems will be unique in that it will provide an alternative to the classical laminated plate analysis method (4). Further, structures consisting of orthotropic plates, such as wood composite based pallets and floor sheathings, can be analyzed by the matrix-displacement method on micro-computers.

1.2 OBJECTIVE

The objectives of this research were to develop an equivalent grid model, suitable for the matrix-displacement analysis of transversely loaded, orthotropic, thin plates, subjected to three different boundary conditions and to verify the model theoretically and experimentally.

To accomplish the objective, the study was conducted as follows: First, the mathematical expressions for the stiffnesses of a thin orthotropic plate and a grid comprising orthotropic members were formulated. By equating the plate and grid stiffnesses, the compatibility conditions for grid-plate equivalence were established.

The grid-plate analog model was validated for isotropic and orthotropic plates by comparison with classical solutions and published literature for several loads and boundary conditions. The sensitivity of the model to the input parameters was then studied by varying the material properties. Lastly, the model predictions were compared with experimental results for orthotropic wood based composite panels.

INTRODUCTION
Chapter 2 deals with the development of the theoretical model of the equivalent grid and the solution methodology. Chapter 3 discusses the experimental design and methods used to generate the required data for experimental verification of the theoretical model with wood composite based panels. The results of the grid model are compared in chapter 4 with published literature and the generated experimental data. Chapter 5 presents the conclusions and suggestions for further research.
1.3 OVERVIEW

This chapter contains the mathematical formulation and solution methodology to the problem of a grid idealization of thin plates subjected to out-of-plane loads. The formulation is a simple alternative to the classical thin plate theory for isotropic and orthotropic materials.

First, the basic assumptions in the theory of thin plates are enumerated and justified. The governing differential equations for a thin orthotropic plate, subjected to transverse loads, are derived systematically using the energy criterion. The equations are specialized for unidirectional and cross-ply laminates. From the equations, mathematical expressions for the flexural and torsional stiffnesses for isotropy and orthotropy are defined. Then, corresponding expressions are developed for an orthogonal grid comprising isotropic or orthotropic elements. The stiffness of the plate and grid are equated and algebraic equations in terms of the grid member sizes are developed.
The solution of the algebraic equations yields two bounds, based on flexural and torsional equivalence, for the dimensions of the internal grid members. The dimensions of the grid members are varied between these bounds to simulate the behavior of a plate under various boundary conditions. Loads are discretized based on a tributary area concept of load transfer on the equivalent grid. Displacements and stresses in the plate are obtained by analyzing the equivalent grid by the matrix-displacement method.

1.4 STIFFNESS FORMULATION OF A THIN PLATE

A plate is a 3-d continuum and is subjected to a three dimensional state of stress.

Consider a rectangular orthotropic plate of length a, width b, and thickness t, comprising n orthotropic laminae. The plate is subjected to edge loading and surface loading, normal to the plane of the plate. The reference axes are defined at the middle plane of the plate, using the rectangular cartesian system, shown in Figure 1. Translational displacements, along the three coordinate axes, are denoted by $u$, $v$, and $w$ respectively. Rotational displacements about the three axes are denoted by $\theta_x$, $\theta_y$, and $\theta_z$ respectively.

The strain energy $U$ of the plate is given by:

$$ U = \frac{1}{2} \int_{vol} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz} \right) \, dx \, dy \, dz $$

[2.1]

where

$\sigma_x$, $\sigma_y$, and $\sigma_z$ = normal stresses;

$\tau_{xy}$, $\tau_{xz}$, and $\tau_{yz}$ = shear stresses;
Figure 1. Orthotropic plate in rectangular cartesian coordinates
\[ \varepsilon_x, \varepsilon_y, \text{ and } \varepsilon_z = \text{normal strains;} \]
\[ \gamma_{xy}, \gamma_{xz}, \text{ and } \gamma_{yz} = \text{shear strains;} \]
\[ dx, dy, dz = \text{dimensions of a differential element (Figure 2)} \]

The subscripts in equation [2.1] refer to the directions and planes of the defined cartesian coordinate system. The exact solution of the above energy equation is complex. However, the problem can be simplified by making some rational assumptions based on the Kirchhoff’s postulates (4), which are discussed next:

1. The plate is thin i.e. the length \( a \), and width \( b \) are much greater than the thickness \( t \);
   Thin plates are often used in many applications. For example, the ratio of the length of the smallest side to panel thickness of a wood composite based pallet is at least 28 (5).

2. Each lamina obeys Hooke’s law.

3. The displacements \( u, v, \text{ and } w \) are small compared to the thickness \( t \) and the mathematical model is restricted to linear analysis. This assumption is justified, since designs are restricted to small deflections both from strength and serviceability considerations. Linear analysis is applicable in steel plates and concrete slabs, when the transverse deflection \( w \) is less than 0.2 times the thickness (6) and in wood based composites, when \( w \) is less than 0.5 times the thickness (7).

4. The transverse shear strains \( \varepsilon_{xy}, \varepsilon_{yz} \) are negligible. This implies that no relative slip occurs between the laminae of the orthotropic plate. Hence, normals to the plate remain normal and straight before and after deformation. This is justified in thin plates, since the contribution of transverse shear components \( \tau_{xz} \varepsilon_{xz} \) and \( \tau_{yz} \varepsilon_{yz} \) to the overall potential energy, is negligible compared with the contribution of the in-plane components \( \sigma_x \varepsilon_x, \sigma_y \varepsilon_y, \text{ and } \tau_{xy} \gamma_{xy} \).
Figure 2. Stresses on a differential element
5. The normals to the plate are inextensible. This assumption is justified by comparing the effects due to the transverse normal stress and in-plane stresses. The magnitude of the transverse normal stress, \( \sigma_z \), is very small compared to the in-plane normal stresses, \( \sigma_x \) and \( \sigma_y \). For example, the allowable in-plane stresses \( \sigma_x \) and \( \sigma_y \) in steel pressure vessels is 20000 psi, whereas the applied pressure \( \sigma_z \) is only 15-20 psi. Since the magnitude of the transverse normal stress is very low, the magnitude of transverse normal strain \( \varepsilon_z \) will also be small by Hooke's law. Thus the contribution to the potential energy by \( \sigma_z \varepsilon_z \) is negligible compared with the contribution of the in-plane components \( \sigma_x \varepsilon_x \), \( \sigma_y \varepsilon_y \), and \( \tau_{xy} \gamma_{xy} \).

6. There are no body forces.

The assumptions discussed above reduce the three-dimensional problem to an approximate 2-d formulation.

Internal forces and moments, in terms of resultants per unit distance, along the edges of an orthotropic plate element of size \( dx \) and \( dy \), (Figure 3), are related to the internal stresses by the following equations:

\[
N_x = \sum_{m=1}^{n} \int_{l_{m-1}}^{l_m} \overline{\sigma}_x \, dz \\
N_y = \sum_{m=1}^{n} \int_{l_{m-1}}^{l_m} \overline{\sigma}_y \, dz \\
N_{xy} = \sum_{m=1}^{n} \int_{l_{m-1}}^{l_m} \overline{\tau}_{xy} \, dz \\
N_{yx} = \sum_{m=1}^{n} \int_{l_{m-1}}^{l_m} \overline{\tau}_{yx} \, dz
\]
Figure 3. Stress resultants on differential element
\[ Q_x = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \tau_{xz} \, dz \quad Q_y = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \tau_{yz} \, dz \]
\[ M_x = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \sigma_x \, z \, dz \quad M_y = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \sigma_y \, z \, dz \]
\[ M_{xy} = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \tau_{xy} \, z \, dz \quad M_{yx} = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \tau_{yx} \, z \, dz \]

where

\( N_x, N_y, N_{xy}, N_{yx} \) = in-plane normal and shearing force intensities lb/in.;
\( Q_x, Q_y \) = transverse shearing forces lb/in.;
\( M_x, M_y \) = bending moments in.-lb/in.;
\( M_{xy}, M_{yx} \) = torsional moments in.-lb/in.;
\( \sigma_x, \sigma_y, \tau_{xy} \) etc. = stress intensities at any point on the thickness of the plate, other than the mid-plane, \( z = 0 \). (Figure 4);
\( n \) = the number of laminae in the plate;
\( t_m \) and \( t_{m-1} \) = the distances of the \( m^{th} \) lamina from the mid-plane.

Since \( \tau_{xy} \) equals \( \tau_{yx} \) from equilibrium consideration, \( N_{xy} \) equals \( N_{yx} \) and \( M_{xy} \) equals \( M_{yx} \).

In general, the force and moment intensities are functions of the coordinates \( x, y \).

The total potential energy, \( V \), of a plate under edge and surface loading is the sum of the strain energy, \( U \), and the potential energy, \( \Omega \), by the externally applied loads.

THEORETICAL FORMULATION AND SOLUTION METHODOLOGY
Figure 4. Stress intensity at any point through the thickness other than the mid-plane
\[ V = U + \Omega \]  \hspace{1cm} \text{[2.3]}  

A loaded plate is in equilibrium, if its potential energy \( V \) is stationary. \( V \) is stationary, when the integrand of equation [2.1] satisfies the Euler's equations of the calculus of variations (8).

The strain energy of a three-dimensional orthotropic medium imposing the Kirchoff's assumptions is given by:

\[ U = \frac{1}{2} \sum_{m=1}^{n} \int \int \int (\bar{\sigma}_x \bar{\varepsilon}_x + \bar{\sigma}_y \bar{\varepsilon}_y + \bar{\tau}_{xy} \bar{\gamma}_{xy}) \; dx \; dy \; dz \]  \hspace{1cm} \text{[2.4]}  

This expression will be reduced to two dimensions by invoking the constitutive relations for the material and strain displacement relations.

1.4.1 Constitutive Relations

For anisotropic, linearly elastic materials there are 21 independent constants in the Hooke's law relationship between stresses and strains (4). For a two-dimensional orthotropic lamina the number of elastic constants is reduced to five. The constitutive relation for a two-dimensional Hookean orthotropic lamina is given by:

\[
\begin{bmatrix}
\sigma_L \\
\sigma_T \\
\tau_{LT}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_L \\
\varepsilon_T \\
\gamma_{LT}
\end{bmatrix}
\]  \hspace{1cm} \text{[2.5]}  

where,
\[ Q_{11} = \frac{E_L}{1 - \nu_{LT}\nu_{TL}} \]

\[ Q_{12} = \frac{E_L\nu_{TL}}{1 - \nu_{LT}\nu_{TL}} = \frac{E_T\nu_{LT}}{1 - \nu_{LT}\nu_{TL}} \]

\[ Q_{22} = \frac{E_T}{1 - \nu_{LT}\nu_{TL}} \]

\[ Q_{66} = G_{LT} \]

L and T = the fiber direction and perpendicular to fiber direction respectively in a lamina coincident to the orthogonal coordinates x, y (Figure 5);

\[ Q_{11}, Q_{12}, Q_{22}, Q_{66} \] = stiffness coefficients;

\( E_L, E_T \) = modulii of elasticity along the L, and T directions;

\( \nu_{LT}, \nu_{TL} \) = poisson’s ratios in the L, and T directions;

\( G_{LT} \) = the modulus of rigidity in the L-T plane.

The expression for \( Q_{12} \) above reduces to the following reciprocal relation (4):

\[ \frac{E_L}{\nu_{LT}} = \frac{E_T}{\nu_{TL}} \tag{2.6} \]

Equation [2.6] enables reduction of the five elastic constants to four independent elastic constants.

The constitutive relation for the lamina along arbitrary orthogonal cartesian coordinates x, y is given by (4):

\[ [\bar{\sigma}_x] = [\bar{Q}] [\bar{\varepsilon}_x] \tag{2.7} \]

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Figure 5. Principal material directions in an orthotropic lamina.
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

where

\[
\bar{Q}_{11} = Q_{11} \cos^4 \alpha + 2(Q_{12} + 2Q_{66}) \sin^2 \alpha \cos^2 \alpha + Q_{22} \sin^4 \alpha
\]

\[
\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \alpha \cos^2 \alpha + Q_{12} (\sin^4 \alpha + \cos^4 \alpha)
\]

\[
\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin \alpha \cos^3 \alpha + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \alpha \cos \alpha
\]

\[
\bar{Q}_{22} = Q_{11} \sin^4 \alpha + 2(Q_{12} + 2Q_{66}) \sin^2 \alpha \cos^2 \alpha + Q_{22} \cos^4 \alpha
\]

\[
\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \alpha \cos \alpha + (Q_{12} - Q_{22} + 2Q_{66}) \sin \alpha \cos^3 \alpha
\]

\[
\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \alpha \cos^2 \alpha + Q_{66} (\sin^4 \alpha + \cos^4 \alpha)
\]

where, \( \alpha \) is the angle measured anticlockwise (Figure 6), between L direction and x axis.

The stress resultants defined in equation [2.2] are expressed in terms of strains by substituting the constitutive relations.

Thus,
Figure 6. Fibers of lamina at an angle to the cartesian axes
\[ N_x = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{11} \bar{\epsilon}_x + \bar{Q}_{12} \bar{\epsilon}_y + \bar{Q}_{16} \bar{\gamma}_{xy} \right] dz \]

\[ N_y = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{12} \bar{\epsilon}_x + \bar{Q}_{22} \bar{\epsilon}_y + \bar{Q}_{26} \bar{\gamma}_{xy} \right] dz \]

\[ N_{xy} = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{16} \bar{\epsilon}_x + \bar{Q}_{26} \bar{\epsilon}_y + \bar{Q}_{66} \bar{\gamma}_{xy} \right] dz \]

\[ M_x = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{11} \bar{\epsilon}_x + \bar{Q}_{12} \bar{\epsilon}_y + \bar{Q}_{16} \bar{\gamma}_{xy} \right] z dz \]

\[ M_y = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{12} \bar{\epsilon}_x + \bar{Q}_{22} \bar{\epsilon}_y + \bar{Q}_{26} \bar{\gamma}_{xy} \right] z dz \]

\[ M_{xy} = \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \left[ \bar{Q}_{16} \bar{\epsilon}_x + \bar{Q}_{26} \bar{\epsilon}_y + \bar{Q}_{66} \bar{\gamma}_{xy} \right] z dz \]

1.4.2 Strain Displacement Relations

The assumption of normals to the mid-plane remaining straight, normal, and inextensional during deformation, permits the displacement components at any point in the plate
(\(\overline{U}, \overline{V}, \) and \(\overline{W}\)) to be expressed in terms of the corresponding mid-plane quantities, \(u, v,\) and \(w\) by the relations:

\[
\begin{align*}
\overline{U} &= u + z \theta_z \\
\overline{V} &= v + z \theta_x \\
\overline{W} &= w
\end{align*}
\]  [2.10]

\[
\begin{align*}
\theta_x &= -\frac{\partial w}{\partial y} \\
\theta_y &= -\frac{\partial w}{\partial x}
\end{align*}
\]

The strains at any point on the plate for linear elasticity are expressed by:

\[
\begin{align*}
\overline{\varepsilon}_x &= \frac{\partial \overline{U}}{\partial x} \\
\overline{\varepsilon}_y &= \frac{\partial \overline{V}}{\partial y} \\
\overline{\gamma}_{xy} &= \frac{\partial \overline{U}}{\partial y} + \frac{\partial \overline{V}}{\partial x}
\end{align*}
\]  [2.11]

Corresponding middle surface strain displacement relations are:

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{align*}
\]  [2.12]

Substituting equation [2.10] for \(\overline{U}, \overline{V}, \overline{W}, \theta_x,\) and \(\theta_y\) in equation [2.11], gives:

\[
\begin{align*}
\overline{\varepsilon}_x &= \varepsilon_x + z \kappa_x \\
\overline{\varepsilon}_y &= \varepsilon_y + z \kappa_y \\
\overline{\gamma}_{xy} &= \gamma_{xy} + z \kappa_{xy}
\end{align*}
\]  [2.13]

where
\[ \kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y}. \]  

[2.14]

The quantities \( \kappa_x, \kappa_y \) are defined as the bending curvatures of the plate along the \( x, y \) axes and \( \kappa_{xy} \) as the twisting curvature of the plate.

Substituting equations [2.8] for stresses in equation [2.4] for strain energy gives:

\[
U = \frac{1}{2} \sum_{m=1}^{n} \int_0^x \int_0^y \left[ \bar{Q}_{11} \bar{\varepsilon}_x^2 + \bar{Q}_{12} \bar{\varepsilon}_y^2 + \bar{Q}_{66} \bar{\gamma}_{xy}^2 + 2 \bar{Q}_{12} \bar{\varepsilon}_x \bar{\varepsilon}_y + 2 \bar{Q}_{16} \bar{\varepsilon}_x \bar{\gamma}_{xy} + 2 \bar{Q}_{26} \bar{\varepsilon}_y \bar{\gamma}_{xy} \right] \, dx \, dy \, dz
\]

[2.15]

Substituting equations [2.13] for strains, in equation [2.15] for strain energy and integrating in the \( z \) direction, between limits \( t_m \) and \( t_{m-1} \) to reduce the problem from 3-d to 2-d, gives:

\[ U = \frac{1}{2} \int_0^x \int_0^y \left[ A_{11} \bar{\varepsilon}_x^2 + 2 A_{12} \bar{\varepsilon}_x \bar{\varepsilon}_y + A_{22} \bar{\varepsilon}_y^2 + 2 (A_{16} \bar{\varepsilon}_x + A_{26} \bar{\varepsilon}_y) (\bar{\gamma}_{xy}) + A_{66} (\bar{\gamma}_{xy})^2 - B_{11} \bar{\gamma}_x \bar{\gamma}_x - 2 B_{12} (\bar{\varepsilon}_x \bar{\gamma}_x + \bar{\varepsilon}_y \bar{\gamma}_y) - B_{22} \bar{\gamma}_y \bar{\gamma}_y - 2 B_{16} (\bar{\gamma}_x \bar{\gamma}_{xy} + 2 \bar{\varepsilon}_x \bar{\gamma}_{xy}) - 2 B_{26} (\bar{\gamma}_y \bar{\gamma}_{xy} + 2 \bar{\varepsilon}_y \bar{\gamma}_{xy}) + 4 B_{66} \bar{\gamma}_{xy} \bar{\gamma}_{xy} + D_{11} \bar{\kappa}_x^2 + 2 D_{12} \bar{\kappa}_x \bar{\kappa}_y + D_{22} \bar{\kappa}_y^2 + 4 (D_{16} \bar{\kappa}_x + D_{26} \bar{\kappa}_y) \bar{\kappa}_{xy} + 4 D_{66} \bar{\kappa}_{xy}^2 \right] \, dx \, dy
\]

[2.16]

where

\[
[A_{ij}] = \sum_{m=1}^{n} \bar{A}_{ij} (t_m - t_{m-1}), \quad \text{defined as Membrane stiffness matrix}
\]

\[
[B_{ij}] = \frac{1}{2} \sum_{m=1}^{n} \bar{B}_{ij} (t_m - t_{m-1}), \quad \text{defined as Coupling stiffness matrix}
\]

\[
[D_{ij}] = \frac{1}{3} \sum_{m=1}^{n} \bar{D}_{ij} (t_m - t_{m-1}), \quad \text{defined as Bending stiffness matrix}
\]
1.4.3 Potential Energy of External Loads

The potential energy of the external loads $\Omega$ is equal to the sum of the potential energies of the edge force $P_x$ and surface loads $q(x, y)$. $P_x$ is the in-plane compressive edge load, (in pounds) uniformly distributed along the edges as shown in Figure 7.

$$\Omega = \int_0^a \int_0^b \left( \frac{P_x}{b} \epsilon_x - q w \right) dx \, dy \quad [2.17]$$

Substituting equations [2.16 & 2.17] for $U$ and $\Omega$, in equation [2.3], the expression for total potential energy $V$, reduces to 2-dimensions.

1.4.4 Equations of Equilibrium

Applying Euler’s equations of calculus of variations (8) to equation [2.3] for the total potential energy, and substituting equation [2.9] for the strains in terms of the stress resultants, we obtain the equations of equilibrium:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad [2.18]$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - (N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}) = q(x, y)$$

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Figure 7. External loads on the plate
1.4.5 Governing Differential Equation of a Plate Subjected to Transverse Loads

For the case of small displacements in a plate subjected to transverse loads the terms involving $N_x$, $N_y$, and $N_y$ vanish from the equations of equilibrium.

The governing differential equation in terms of stress resultants is:

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = q(x, y) \quad [2.19]$$

The governing differential equation of an orthotropic plate subjected to transverse loads, in terms of displacements, is obtained by substituting equations [2.9] for moment resultants in terms of the strain displacement relations in equation [2.19].

Thus,

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4 D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 D_{26} \frac{\partial^4 w}{\partial x \partial y^3} +$$

$$D_{22} \frac{\partial^4 w}{\partial y^4} - B_{11} \frac{\partial^3 u}{\partial x^3} - 3 B_{16} \frac{\partial^3 u}{\partial x^2 \partial y} - (B_{12} + 2 B_{66}) \frac{\partial^3 u}{\partial x \partial y^2} -$$

$$B_{26} \frac{\partial^3 u}{\partial y^3} - B_{16} \frac{\partial^3 u}{\partial x^3} - (B_{12} + 2 B_{66}) \frac{\partial^3 v}{\partial x^2 \partial y} -$$

$$3 B_{26} \frac{\partial^3 v}{\partial x \partial y^2} - B_{22} \frac{\partial^3 v}{\partial y^3} = q \quad [2.20]$$

For a uni-directional or mid-plane, symmetric, cross-ply, orthotropic laminate, the terms $B_{11}$, $D_{16}$, and $D_{26}$ vanish and the governing equation reduces to:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q \quad [2.21]$$

The following plate stiffnesses are defined:

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• the flexural stiffness in the x direction, \( D_{xp} = D_{11} \) \[2.22.1\]
• the torsional stiffness, \( 2H_{p} = 2(D_{12} + 2D_{66}) \) \[2.22.2\]
• the flexural stiffness in y direction, \( D_{yp} = D_{12} \) \[2.22.3\]

The subscripts \( p, x, y \) refer to plate and reference axes respectively.

The above stiffnesses can be obtained in terms of the elastic, material properties and plate thickness by substituting the definition of bending stiffness \( D_{ij} \) from equation [2.16].

For a uni-directional laminate, the stiffnesses \( D_{ij} \) in terms of the material properties and plate thickness reduce to:

\[
D_{11} = \frac{E_{L} \frac{t^3}{12}}{12(1 - \nu_{LT} \nu_{TL})}
\]

\[
D_{12} = \frac{E_{L} \nu_{TL} \frac{t^3}{12}}{12(1 - \nu_{LT} \nu_{TL})} = \frac{E_{T} \nu_{LT} \frac{t^3}{12}}{12(1 - \nu_{LT} \nu_{TL})}
\]

\[
D_{66} = G_{LT} \frac{t^3}{12}
\]

[2.23]

Thus the plate stiffnesses are expressed by:

\[
D_{xp} = \frac{E_{L} \frac{t^3}{12}}{12(1 - \nu_{LT} \nu_{TL})}
\]

\[
2H_{p} = 2 \left[ \frac{\nu_{LT} E_{T}}{(1 - \nu_{LT} \nu_{TL})} + 2 G_{LT} \right] \left[ \frac{t^3}{12} \right]
\]

\[
D_{yp} = \frac{E_{T} \frac{t^3}{12}}{12(1 - \nu_{LT} \nu_{TL})}
\]

[2.24]

For isotropic materials, the plate stiffnesses are obtained by substituting \( E_{L} = E_{T} = E \), \( \nu_{LT} = \nu_{TL} = \nu \), and \( G_{LT} = G = \frac{E}{2(1 + \nu)} \). Thus,
\[ D_x = D_y = H = \frac{E t^3}{12(1 - \nu^2)} \]  

[2.25]

1.5 STIFFNESS FORMULATION OF AN ORTHOTROPIC GRID

1.5.1 General

Consider an orthogonal grid (Figure 8) of overall dimensions \( a \times b \) referred to rectangular cartesian coordinates \( x, y, z \). The grid members are orthotropic. Members of the grid parallel to the \( x \) axis are characterized by:

- spacing, \( b_x \)
- width, \( t_x \)
- depth, \( h \)
- Modulus of Elasticity, \( E_L \)
- Poisson Ratio, \( \nu_{LT} \)
- Modulus of Rigidity, \( G_{LT} \)

Members of the grid parallel to the \( y \) axis are characterized by:

- spacing, \( b_y \)
- width, \( t_y \)
- depth, \( h \)
- Modulus of Elasticity, \( E_T \)
- Poisson Ratio, \( \nu_{LT} \)
- Modulus of Rigidity, \( G_{LT} \)
Figure 8. Orthogonal grid
From Hooke’s law, the strains along the x and y axes are:

\[
\varepsilon_x = \frac{\sigma_x}{E_L} - \nu_{TL} \frac{\sigma_y}{E_T} \left[ \frac{t_y}{b_y} \right]
\]

\[
\varepsilon_y = \frac{\sigma_y}{E_T} - \nu_{LT} \frac{\sigma_x}{E_L} \left[ \frac{t_x}{b_x} \right]
\]  \[2.26\]

The connection between the grid members occupies only a finite area of the section as shown in Figure 8. Hence, the overall Poisson’s ratio effect in the above equations is modified by the ratios \(\frac{t_x}{b_x}\) and \(\frac{t_y}{b_y}\).

The expressions for stresses are:

\[
\sigma_x = E_L \left[ \varepsilon_x + \nu_{TL} \frac{\sigma_y}{E_T} \left( \frac{t_y}{b_y} \right) \right]
\]

\[
\sigma_y = E_T \left[ \varepsilon_y + \nu_{LT} \frac{\sigma_x}{E_L} \left( \frac{t_x}{b_x} \right) \right]
\]  \[2.27\]

Replacing \(\sigma_x\) and \(\sigma_y\) from the right hand side of equation [2.26], we get:

\[
\sigma_x = E_L \frac{\left[ \varepsilon_x + \nu_{TL} \varepsilon_y \left( \frac{t_y}{b_y} \right) \right]}{1 - \nu_{LT} \nu_{TL} \frac{t_x}{b_x} \frac{t_y}{b_y}}
\]

\[
\sigma_y = E_T \frac{\left[ \varepsilon_y + \nu_{LT} \varepsilon_x \left( \frac{t_x}{b_x} \right) \right]}{1 - \nu_{LT} \nu_{TL} \frac{t_x}{b_x} \frac{t_y}{b_y}}
\]  \[2.28\]
Replacing \( v_{LT} \) from equation [2.28] by substituting the reciprocal equation [2.6], the stresses in the grid are expressed as:

\[
\sigma_x = E_L^* \left[ \varepsilon_x + v_{LT} \varepsilon_y \left( \frac{t_y}{b_y} \right) \left( \frac{E_T}{E_L} \right) \right] \tag{2.29.1}
\]

\[
\sigma_y = E_T^* \left[ \varepsilon_y + v_{LT} \varepsilon_x \left( \frac{t_x}{b_x} \right) \right] \tag{2.29.2}
\]

where

\[
E_{i-L, T}^* = \frac{E_i}{1 - v_{LT}^2} \left( \frac{t_y}{E_L} \right) \left( \frac{t_x}{b_x} \right) \tag{2.30}
\]

From Kirchoff's beam theory (9),

\[
\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2} \tag{2.31}
\]

\[
\varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}
\]

Substituting equation [2.31] for strains, in equations [2.29.1 & 2.29.2] for stresses, we obtain:

\[
\sigma_x = -E_L^* \left[ \frac{\partial^2 w}{\partial x^2} + v_{LT} \frac{\partial^2 w}{\partial y^2} \left( \frac{t_y}{b_y} \right) \left( \frac{E_T}{E_L} \right) \right]
\]

\[
\sigma_y = -E_T^* \left[ \frac{\partial^2 w}{\partial y^2} + v_{LT} \frac{\partial^2 w}{\partial x^2} \left( \frac{t_x}{b_x} \right) \right] \tag{2.32}
\]
1.5.2 Grid Stress Resultants

The bending moments per unit width, obtained by summing the moments of the tensile and compressive forces about the neutral axis of the grid element, (Figure 9) are:

\[
M_x = -\frac{1}{b_x} \left[ E_L \frac{h}{2} \frac{\partial^2 w}{\partial x^2} \frac{1}{2} \frac{h}{3} h + \nu_{LT} E_L \frac{E_T}{E_L} \frac{t_y}{b_y} \frac{\partial^2 w}{\partial y^2} \frac{1}{2} \frac{1}{2} t_x \frac{h}{2} \frac{2}{3} h \right] \\
M_y = -\frac{1}{b_y} \left[ E_T \frac{h}{2} \frac{\partial^2 w}{\partial y^2} \frac{1}{2} \frac{2}{3} t_y \frac{h}{2} + \nu_{LT} E_T \frac{E_T}{E_L} \frac{t_x}{b_x} \frac{\partial^2 w}{\partial x^2} \frac{1}{2} \frac{1}{2} t_y \frac{h}{2} \frac{2}{3} h \right]
\]

Simplifying,

\[
M_x = - \left[ E_L h^3 \frac{t_x}{12 b_x} \frac{\partial^2 w}{\partial x^2} + \nu_{LT} E_L \frac{E_T}{E_L} \frac{t_x}{b_x} \frac{t_y}{b_y} \frac{h^3}{12} \frac{\partial^2 w}{\partial y^2} \right] \\
M_y = - \left[ E_T h^3 \frac{t_y}{12 b_y} \frac{\partial^2 w}{\partial y^2} + \nu_{LT} E_T \frac{t_x}{b_x} \frac{t_y}{b_y} \frac{h^3}{12} \frac{\partial^2 w}{\partial x^2} \right]
\]

[2.33]

Considering the twist of an individual grid member, torsional moments per unit width are:

\[
M_{xy} = B_{xy} \frac{\partial^2 w}{\partial x \partial y} M_{yx} = -B_{yx} \frac{\partial^2 w}{\partial x \partial y}
\]

[2.34]

where \( B_{wy} \) and \( B_{yx} \) are the torsional moments of the grid members parallel to the x and y directions respectively. For rectangular sections, the torsional rigidity of members along the x, and y directions are:

\[
B_{xy} = \frac{G_{LT} k t_x^3 h}{b_x} \quad \text{for} \quad t_x < h \\
B_{yx} = \frac{G_{LT} k t_y^3 h}{b_y} \quad \text{for} \quad t_y < h
\]

[2.35.1] [2.35.2]
Figure 9. Bending moment per unit width of grid
where $k$ is the torsion coefficient for a rectangular section and is dependent on the ratio of the sides of the grid member (10). A plot of $k$ versus depth to width ratio for rectangular sections is shown in Figure 10. The value of $k$ for square cross-sections from Figure 10 is 0.14. If the grid member width $t_x$ or $t_y$ is greater than $h$, the two are interchanged in equations [2.35.1 & 2.35.2].

For uni-directional laminated beams and for symmetric, cross-ply, orthotropic laminated beams $G_{lt}$ equals $G_{tt}$. Substituting the equations [2.33 & 2.34] for the stress resultants in the governing differential equation [2.19], and noting that $M_{xy}$ equals $M_{yx}$ for equilibrium, the grid stiffnesses are:

- flexural stiffness in the $x$ direction $D_{xx} = E_t h^3 \frac{t_x}{12 b_x}$ \[2.36.1\]
- flexural stiffness in the $y$ direction $D_{yy} = E_t h^3 \frac{t_y}{12 b_y}$ \[2.36.2\]
- torsional stiffness $2H_g = kh \left[ G_{lt} \frac{t_x^2}{b_x} + G_{lt} \frac{t_y^2}{b_y} \right] + \frac{\nu_{tt} t_x t_y h^3}{12 b_x b_y} \left[ \frac{E_t}{E_l} + \frac{1}{E_t} \right]$ \[2.37\]

The subscript 'g' refers to the grid.
Figure 10. Torsion coefficient for rectangular sections
1.6 SOLUTION METHODOLOGY

1.6.1 Equivalent Grid-Plate Relations

The deflection and stress responses of a structure depend on its stiffness. A plate and a grid will be equivalent, when their stiffnesses are identical. Thus for a plate to be idealized in to an equivalent grid, the plate stiffnesses are required to be transformed by assigning suitable cross-sectional dimensions to the grid members. Thus the grid - plate equivalence relations can be obtained by equating the stiffnesses [equations 2.24, 2.36, & 2.37] of the grid and the plate.

Thus the grid-plate relations are:

- Flexural Stiffness in x-direction

\[ D_x = \frac{E_L h^3 t_x}{12 b_x} = \frac{E_L t^3}{12(1 - \nu_{LT} \nu_{TL})} \tag{2.38.1} \]

- Flexural Stiffness in y-direction

\[ D_y = \frac{E_T h^3 t_y}{12 b_y} = \frac{E_T t^3}{12(1 - \nu_{LT} \nu_{TL})} \tag{2.38.2} \]

- Torsional Stiffness
By substituting \( t \), the plate thickness, for the grid variables \( b_x, b_y, t_x, t_y, \) and \( h \) in the left hand side of equations [2.38.1 to 2.38.3], the formulation reduces to the right hand side of the equations and hence is verified.

1.6.2 Establishing Grid Member Sizes

The three grid - plate equations [3.37.1 to 3.38.3] have five unknowns \( (t_x, t_y, h, b_x, b_y) \) and therefore have many solutions. The unknowns pertain to the grid and are hereafter collectively referred to as grid variables. Therefore, for each solution, at least two of the five grid variables must be assumed. The logical approach will be to assume values for the spacings \( b_x \) and \( b_y \) and solve the grid-plate equations for the remaining three unknowns. The resulting grid can then be structurally analyzed to determine the responses. Depending on the choice made for the spacings \( (b_x, b_y) \), some of the solutions for the remaining unknowns are likely to yield impractical values, such as \( t_x, t_y \) being greater than \( b_x, b_y \) respectively. To avoid these impractical solutions, further assumptions are required to satisfy geometric interdependence of the grid variables.
By requiring the cross-sections of the grid members to be square with equal dimensions 
\((t_x = t_y = h = A)\) and by assuming equal grid spacings along \(x\) and \(y\) directions 
\((s_x = s_y = c)\) the grid-plate equations reduce to two variables \(A\) and \(c\).

Thus equations [2.38.1 - 2.38.3] reduce to:

- **Flexural Stiffness**

\[
E_L A^4 + 12\nu_{LT} \frac{v_{TL}}{c} A^2 - 12 c D_{x,y} = 0 \quad [2.39]
\]

Note: The equation [2.38.2] for flexural stiffness \(D_y\) also reduces to the above equation [2.39] on substituting the reciprocal equation [2.6].

- **Torsional Stiffness**

\[
0.28 G_{LT} v_{LT} v_{TL} A^6 - E_T \nu_{LT} \frac{c}{6} A^5 - 0.28 G_{LT} c^2 A^4 - [2H_p] \nu_{LT} v_{TL} c A^2 + [2H_p] c^3 = 0 \quad [2.40]
\]

Assuming a finite value for \(c\), equation [2.39] can be solved exactly by factoring, and equation [2.40] can be solved numerically by the Newton-Raphson method (11). Solution of the equations yields lower and upper bounds, for the grid member size, \(A\), based on flexural and torsional equivalence referred to hereafter as \(A_f\) and \(A_t\), respectively.
1.7 GRID MEMBER SIZE FOR DIFFERENT BOUNDARY CONDITIONS

The deflection and stress responses of a plate under a given load will depend on the boundary conditions. The boundary condition triggers the plate stiffnesses \((2H_p, D_{np}, \text{ and } D_{pp})\) to interact in resisting the applied load. The extent of interaction of the stiffnesses varies with the boundary condition, causing a difference in the response of the plate. Analogously, since the equivalent grid should respond similarly, the stiffnesses of the grid members should be modeled such that the grid imitates the behavior of the plate under different boundary conditions. The range in the values of \(A_r\) and \(A_i\) for the grid members permit the matching of the grid response to different boundary conditions of the plate.

According to Timoshenko (12), "the deflections of a square, isotropic plate simply supported on two opposite edges and free at the other two (2ss,2fr plate), and subjected to a uniformly distributed load, differ but little from the deflections and moments of a plate bent in to a cylindrical surface."

The cylindrical configuration implies beam action in the plate. Extending the same rationale to a 2ss,2fr orthotropic plate, whose \(L\) direction is parallel to the span, the deflection will be primarily controlled by the flexural stiffness \(D_{np}\). A tertiary, if not negligible role will be played by the flexural stiffness \(D_{pp}\) and torsional stiffness \(2H_p\). Instead, if the \(T\) direction of the...
same plate is parallel to the span the deflection will be primarily controlled by the flexural stiffness $D_{\sigma}$.

Accordingly, the member size $A^*$ of the grid equivalent to a 2ss 2fr orthotropic plate should be based purely on the equivalence of flexural stiffness [2.39].

On the other hand, if the same orthotropic plate is simply supported on its four edges (4 ss plate), the response will be different from the 2ss 2fr plate. The two extra supports of the 4ss plate trigger all the stiffnesses $2H_{\alpha}, D_{\sigma},$ and $D_{\rho}$. It can therefore be deduced that the member size $A^*$ for a grid equivalent to a 4ss plate should therefore be greater than $A_\gamma$. Thus the required member size $A^*$ of the grid will depend on the boundary condition of the plate problem being considered. The tributary area (Figure 11), which a typical internal member (size $A^*$) along the $x$ or $y$ axes of the equalized grid represents, is equal to $b_x \times a$ and $b_y \times b$ respectively.

1.8 BOUNDARY MEMBERS

The tributary areas for the boundary members are equal to half of the tributary areas of the internal members (Figure 11). If the boundary members are also assumed to be of square cross-section their dimensions are calculated as:

Dimensions of the internal grid member $= A^* \times A^*$
Figure 11. Tributary area of internal and boundary members of grid
Moment of inertia of the internal grid member = \( \frac{A^4}{12} \)

Moment of inertia of boundary member required = \( \frac{A^4}{24} \)

Therefore the boundary member size, \( B' = \left( \frac{A^4}{2} \right)^{\frac{1}{4}} \)

1.9 LOAD REPRESENTATION

The following plate load cases are considered for representation on the equivalent grid:

1. uniformly distributed load (UDL);
2. line load applied at the center of the plate (LL);
3. concentrated load at the center of the plate (Conc.Ld).

Loads on the equivalent grid can be applied both on the members or the joints between the grid members. The concept of load representation is explained with the help of an illustrative example. Consider a square plate of dimensions \( a \times a \), (Figure 12 a) subjected to an UDL of \( q \) psi. The plate is unsmearred to yield an equivalent grid also of dimensions \( a \times a \) (Figure 12 b). The grid comprises six members, three along \( x \) axis and three along \( y \) axis. The spacing of the members along both the \( x \) and \( y \) axes is \( c \). Thus the grid is divided in to four square sub panels each of dimensions \( c \times c \).
Uniformly distributed loads on the surface of the plate are represented on the grid members by a triangular discretization scheme (Figure 12 c). The discretization is carried out by dividing each sub-panel into triangles by joining its diagonals. Thus a typical sub-panel ABCD is divided into four triangles along the lines (AC and BD) of diagonal symmetry. The UDL of the plate is represented on the grid by hatched lines. The uniform load acting on each triangular zone is condensed in a member load. For example, the load from triangle DNC is transferred to its base DC and load from triangle DNA transferred to its base AD. Thus all internal members receive twice the load received by the boundary members. Therefore load transferred to the internal member, DC, equals the loads transferred from the triangles DMC and DNC. Thus, load representation on internal member DC equals $\frac{1}{2} (c) q$, and, on boundary member DA equals $\frac{1}{4} (c) q$.

Line loads applied at the center of the plate, are represented on the equivalent grid as distributed loads on the members. A concentrated load at the center of the plate is represented on the grid at its central joint.

1.10 SOLUTION METHOD

After the member dimension $A^r$ is determined and the loads are represented using the rationale discussed in the previous section, the grid is analyzed by the matrix-displacement method (13). For a three-dimensional analysis every node of a member has six degrees of
Figure 12. Load Representation: (a) square plate under udl (b) equivalent grid (c) triangular discretization for udl
freedom (13) comprising three translations and three rotations. Depending on the type of
problem being analyzed, only some degrees of freedom, of the six will be present. Others will
either be absent or be arrested.

The rationale for assigning the degrees of freedom to any member end of the equivalent grid
is based on the loading and boundary condition of the original plate. Since the scope of the
problem is restricted to plates under transverse loads only and no in-plane loads, the in-plane
degrees of freedom are absent. The active degrees of freedom at any node are translation
w along the axis z, and rotations \( \theta_x \) and \( \theta_y \) about the x, and y axes respectively. The
modeling of the nodes of the equivalent grid for these three degrees of freedom at the
boundaries will depend on the plate boundary conditions. This is explained with an example.

Consider a plate simply supported on its four edges (Figure 13). A point (\( x, y = 0 \)) on the
boundary along the x axis (not including corner points) can neither translate in the vertical
direction (\( w = 0 \)) nor rotate about the y-axis (\( \theta_y = 0 \)). The only active degree of freedom at
this node will be \( \theta_x \). Similarly any point (\( x = 0, y \)) other than the corners, on the boundary
along the y axis will have only one active degree of freedom, \( \theta_y \), which is rotation about the
x axis. Furthermore all the degrees of freedom at the the corners of the plate will be ar-
rested. Therefore the problem is simplified by eliminating unnecessary degrees of freedom in
the model. This is important for microcomputer applications.

The grid solutions were obtained using SPACEPAL (14), a three dimensional space frame
computer program based on the matrix-displacement analysis method (13).
Figure 13. Plate simply supported on all edges
Since SPACEPAL is a 3-D analysis program six possible displacements can occur at any joint.

An array called JCODE is used to define the directions (relative to the global coordinate system) in which each joint is constrained or free to move. The notation and sequence adopted for specifying the nodal degrees of freedom in SPACEPAL are given next.

1. Notation: A symbol 0 denotes that the node can deform and 1 denotes that the degree of freedom is either arrested (by a support) or absent.

2. Sequence: The degrees of freedom for any node of the grid are specified in the following sequence:

- translation $u$ along the x direction;
- translation $v$ along the y direction;
- translation $w$ along the z direction;
- rotation $\theta_x$ about the x direction;
- rotation $\theta_y$ about the x direction;
- rotation $\theta_z$ about the x direction.

The scope of the present work is restricted to the following boundary conditions:

1. 2ss, 2fr plate;
2. 4ss plate;
3. 4cor plate.
The JCODE used to model the various nodes of the equivalent grid for different boundary conditions covered under the scope of the work are given next. The nodes of the grid equivalent to a 2ss, 2fr plate are specified as follows:

1. internal nodes: 110001
2. all boundary nodes including corner nodes: 111011 (span of the plate is parallel to the y axis)
3. all boundary nodes including corner nodes: 111101 (span of the plate is parallel to the x axis)

The nodes of the grid equivalent to a 4 ss plate are specified as follows:

1. internal nodes: 110001;
2. boundary nodes parallel to x axis: 111011;
3. boundary nodes parallel to y axis: 111101;
4. corner node: 111111.

The nodes of the grid, equivalent to a plate simply supported on its four corners are specified as follows:

1. corner node: 111001
2. all other nodes: 110001
1.11 RESPONSE OF THE EQUIVALENT GRID

The structural analysis of the equivalent grid gives generalized (13) displacements and forces at the joints in the global frame of reference and the member end forces in the local frames of reference. For example, the grid solution for a 4ss plate under UDL will comprise deflections and rotations of the joints, shear forces and bending moments of the various members of the grid. If the forces are needed in the form of stress resultants (bending moments or shear forces), then the forces determined by the grid solution must be smeared over the respective distributary regions. For example, the bending moment, $M_x$, found by the grid solution for an internal member along the $x$ axis has to be divided by the spacing $b_x$ to find the bending moment per inch $M'_x$.

The model developed was validated with classical and experimental solutions, details of which are presented in chapter 4. The next chapter discusses the experimental data collected for the comparison.
EXPERIMENTAL METHODS

The previous chapter described the development of a theoretical model for a grid that simulates the behavior a homogenous, orthotropic plate subjected to transverse loads. This chapter discusses the experiments designed to generate data for verifying the theoretical model.

The tests were conducted on wood based composite panels, including plywood and oriented strand board. First, experiments were conducted to determine the moduli of elasticity of the sample panels. Then, each panel was supported under its four corners and subjected to concentrated and line loads. The deflection response of the panels were measured and are used in the next chapter to verify the theoretical model. The experimental design of the tests and their procedures are discussed in the following sections.
Wood based panels exhibit orthotropic stiffness and strength properties due to the inherent orthotropic characteristics of their 'wood' components, and also due to their construction. For example, plywood is an 'all veneer panel' consisting of an odd or even number of cross-laminated layers glued together and oriented strand board (OSB) is a panel 'of compressed strand-like particles arranged in layers (usually three to five) oriented at right angles to one another.' (15). The orthogonal and oriented construction imparts orthotropic strength and stiffness properties to these panels respectively.

Wood based composite panels are heterogeneous and exhibit variable orthotropic properties. The variability is caused by the morphological characteristics of the veneers and the strands. The heterogeneity is due to manufacturing defects such as voids, imperfect glue lines, and variations in thickness. Visually perceivable voids in one or more plies are common in plywood. Consequently, with wood based panels, variable elastic properties are evident, even within an individual panel. Classical structural analysis methods, however, assume material homogeneity and invariant properties. To conduct simplified structural analysis of wood composite based structures homogenous properties must be assumed. The American Plywood Association (APA) has addressed this need from a statistical perspective and has provided minimum property values for plywood and OSB (16, 17) for use by designers.
An individual panel of a grade, selected at random, may therefore have properties, which are significantly different from those specified by APA. Such a panel used in a structure will respond consistent with its material properties and not with the specified minimum properties. Verification of an analysis model for such a panel will therefore require the actual material properties as an input parameter. Further these properties, which will be variable, must be averaged to satisfy the requirement of material homogeneity. Therefore, to validate the theoretical model presented in Chapter 3, the elastic properties of sample panels must be evaluated nondestructively. Then structure response tests on the same panels need be conducted and compared with the predicted response.

A homogenous, orthotropic lamina is characterized by four material properties (4). The properties of interest are the moduli of elasticity in the directions \( L \) and \( T \), the poisson's ratio \( \nu_{LT} \), and the modulus of rigidity \( G_{LT} \).

Lee (18) reports the dominant influence of \( \frac{E_L}{E_T} \) in comparison with other elastic properties for an orthotropic panel subjected to transverse loads. Therefore, I decided to experimentally determine only the properties \( E_L \) and \( E_T \), and assume other required properties for the sample panels, based on published data.
1.12.1 Materials

Plywood is available in different thicknesses in a variety of grades ranging from structural grade (A-C) to construction grade (C-D) (16). These grades differ in stiffness and strength. OSB is classified by various thicknesses only. The above panels are commercially available in standard sizes, 4 feet wide by 8 feet length.

One number each of the following panels were used for the experiments.

- Plywood \( \frac{3}{4}'' \) thick, A-C Grade, Southern Pine, 5 plies;
- Plywood \( \frac{1}{2}'' \) thick, A-C Grade, Southern Pine, 5 plies;
- Plywood \( \frac{1}{2}'' \) thick, C-D Grade, Southern Pine, 4 plies;
- OSB \( \frac{7}{16}'' \) thick.

The nomenclature for plywood grades A-C etc. follow APA’s classification for plywood (16).

Panel thicknesses given above, are nominal values. The materials represent a spectrum of commercially available wood composite based panels. These panels were selected, because they have a wide range of stiffness properties (16).

Two samples, each of dimensions 4 feet \( \times \) 4 feet, were cut from each panel and tested. The thickness was measured at several locations on each panel, with a vernier caliper, to an accuracy of 0.001". The actual average thickness of each sample was thus determined.
The following are defined consistent with Chapter 2 for the development of the theoretical model:

- for plywood, \( L \) and \( T \) refer to the direction parallel and perpendicular to the grain of the face ply respectively; and

- for OSB, \( L \) and \( T \) refer to the direction parallel and perpendicular to the direction, most strands are oriented.

The directions \( L \) and \( T \) are also collectively referred to as principal material directions (PMD) by some authors (4).

The following tests were conducted:

1. non-destructive determination of the moduli of elasticity of sample panels;
2. structure response on the same panels.

### 1.12.2 Test Set-Up and Instrumentation

All tests were conducted on a steel reaction frame of plan dimensions 71" x 65" and height 78". The load is applied through a hydraulic cylinder mounted on the top of the frame. Two steel I beams of size, 5" x 5" and length 8 feet were used as the reaction supports for the panels. The I-beams are rigidly clamped to the frame with heavy duty C-clamps. The load
was applied with a hand operated hydraulic cylinder connected to a 500 lb. BLH load cell. A universal head was used to transfer the load from the hydraulic cylinder to the panel. The universal head enabled vertical application of the load to the panel, thus eliminating any misalignment errors from the hydraulic cylinder assembly. The load cell was calibrated using a Tinius Olsen universal testing machine. The applied load was measured through a Vishay signal conditioner and a digital multimeter. The Vishay conditioner is a low noise, self-excited bridge amplifier circuit, with variable gain, which converts the change in resistance of the load cell to a proportional voltage. The load cell resistance changes with the load applied by the hydraulic cylinder. The conditioner also isolates the load cell from effects of electrical interference and noise from other equipment in its vicinity. Care was taken to operate the hydraulic cylinder gradually to avoid dynamic influences on the tests. Panel deflections were measured at several locations at discrete load levels. The details of the tests are presented next.

1.13 NON-DESTRUCTIVE DETERMINATION OF THE MODULI OF ELASTICITY

The moduli of elasticity, $E_L$ and $E_T$, of the panels were determined by conducting flexure tests under a line-load. The flexure tests were conducted with two opposite edges of the panel simply supported and the other two edges free. The simply supported boundary conditions
were simulated at the panel edges by the rounded flange ends of the steel beams. The
modulus of elasticity, $E_L$, was obtained by testing the panel with its $L$ direction aligned parallel to the span, and the modulus of elasticity, $E_T$, was obtained by testing the panel with its $T$ direction aligned parallel to the span.

The tests were conducted on local regions of the panel and the global panel. Three local regions, along the $L$ and $T$ directions were marked on each panel. The scheme of local region partitioning is shown in Figure 14. The local region tests were conducted to assess the variability in the values of the moduli of elasticity in every panel. These tests were also conducted to compare the global and average properties of the individual panels. I felt, that this approach would provide me with a rational basis for calculating the equivalent homogenous property values for individual panels.

The following symbols are defined:

1. global moduli of elasticity are denoted by $E_{LG}$ and $E_{TG}$.

2. local region moduli are denoted by $E_{l,i}$ and $E_{n,i}$, where $i = 1, 2, 3$ representing the regions defined in Figure 14.

The alignment of the panel to obtain the moduli of elasticity $E_{LG}$, $E_{TG}$, $E_{L}$ and $E_{T}$ followed the alignment defined earlier for $E_L$ and $E_T$. 

EXPERIMENTAL METHODS 54
Figure 14. Scheme of local region partitioning for property determination tests
The line load was simulated by placing a stiff wooden loading bar of length 48" and dimensions 2" x 4" on the panel. The loading bar was placed on the panels coincident with the span center.

The span for the global property test was 44", to provide a bearing distance of 2" on each support. The span for the local region tests was 21". Deflections were measured with dial gauges to an accuracy of 0.001" over a range of 1.0". For the global tests the deflections were measured at the center of the panel and the centers of the two free edges. For the local tests the deflections were measured at the center of the local region and the centers of the free edges of the local regions. Since the same panels were to be used for the structure tests load magnitudes were kept low to prevent failure in the panels. The levels were selected based on span rating values (16) specified on the panels by the manufacturers. The panels were checked for uniform contact both with the supporting steel beams and loading bar. This was ascertained by visual examination for passage of light at the junction of the panel and support steel beams/loading bar. All samples, except the \( \frac{1}{2} \)" thick A-C grade and C-D grade plywood panels were in uniform contact. It was necessary to apply an initial load of 25 lbs. for the A-C grade (\( \frac{1}{2} \)" thick) panels and about 50 to 75 lbs. for the C-D grade panels. The initial load was applied with the hydraulic cylinder. After ensuring the uniform contact, the horizontal alignment of the panel on the support beams was checked with a 48" long spirit level. The \( \frac{3}{4} \)" thick A-C grade panels were level along and perpendicular to the span. The remaining panels were level along the span but were not level perpendicular to the span. I concluded that the set-up was perfectly level but the panels other than the \( \frac{3}{4} \)" thick A-C grade plywood panels were deflecting unevenly, confirming the variability in stiffness properties within the panels.
The panels deflected by 0.25" under the pre-load. This configuration was considered to be the initial reference state for these panels.

The load was increased in steps of 25 lbs. for the A-C grade plywood panels and 5-10 lbs. for the C-D grade plywood and OSB panels and deflections were measured at each load step. The deflections were measured immediately (less than one minute) after applying the step load to avoid any time dependent effects.

The global deflection of C-D grade plywood and OSB were typically 0.8", even at only 25 lbs. of load. To prevent permanent damage to these panels higher loads were not applied. A-C grade plywood panels were loaded up to 300 lbs.

The residual deflection (set) of all the panels was checked with the spirit level after unloading. Only the OSB samples showed permanent set. The residual deflections in these panels were greater than the initial deflection by 0.05". The panels, however, did not show any visual signs of damage. Superfesky (7) also observed the same phenomenon and the difference in deflections was attributed to time dependent effects. This is justifiable since OSB has a large volume fraction of glue. Glues are polymeric substances which are known to exhibit viscoelastic behavior.

The panels did not bend symmetrically. In the case of the C-D grade plywood and OSB the deflections at the free edges were greater than the center deflection. This demonstrates the variability in the moduli of elasticity within the panels. The deflection for the $E_T$ tests were greater than the $E_L$ tests indicating orthotropic behavior.
In general, the conduct of the test was relatively easier with A-C grade plywoods than the C-D grade plywood or OSB. Because of the large deflections, even with small load increments, few data points could be obtained for the global tests of C-D grade plywood and OSB. The response of these panels under local region tests was gradual compared to the global tests.

The testing arrangement for \( \frac{3}{4} \) A-C plywood is shown in Figure 15.

### 1.13.1 Data Reduction

The objective of these experiments was to measure the stiffness of each panel. This requires a symmetrical deflection response of the panel. Since the panels did not respond symmetrically both in the global and local tests, due to the variability of properties, I averaged the three deflections measured in each experiment and used that average deflection to compute the average global/local panel stiffness. This simplification reduces the problem to the bending of a simply supported beam subjected to a central load. The beam spans for calculation of the global stiffness and local stiffness are 44 and 21 inches respectively. The width of the beam for both global and local stiffness calculations is 48 inches. The load-deflection response was linear for all the panels. The data is presented in the form of load-deflection plot in Figure 16 for one sample only of 0.75\( '' \) A-C grade plywood. The slopes of the load-deflection responses for all the samples are reported in Table 1.
Figure 15. Property determination test: Plywood 0.75", A-C grade
Figure 16. Load-deflection response for Plywood 0.75", A-C grade: (a) $E_L$ test  
(b) $E_T$ test
Table 1: Slopes of load-deflection curves for sample panels

<table>
<thead>
<tr>
<th>Panel</th>
<th>Sample</th>
<th>Global Test</th>
<th>Local Test Region</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
<tr>
<td></td>
<td></td>
<td>L  T</td>
<td>L  T</td>
<td>L  T</td>
</tr>
<tr>
<td>PWAC0.75&quot;</td>
<td>1</td>
<td>108 978</td>
<td>9412 9412</td>
<td>9024  7374 9703 8054</td>
</tr>
<tr>
<td>PWAC0.75&quot;</td>
<td>2</td>
<td>921 918</td>
<td>5628 10091</td>
<td>8151  8345 7666</td>
</tr>
<tr>
<td>PWAC0.5&quot;</td>
<td>1</td>
<td>409 284</td>
<td>3853 1179</td>
<td>3824  1409 3824 1955</td>
</tr>
<tr>
<td>PWAC0.5&quot;</td>
<td>2</td>
<td>384 211</td>
<td>3651 1754</td>
<td>3623  1725 3651 1869</td>
</tr>
<tr>
<td>PWCD</td>
<td>1</td>
<td>299 54</td>
<td>2812 501</td>
<td>2831  501 2639 501</td>
</tr>
<tr>
<td>PWCD</td>
<td>2</td>
<td>310 59</td>
<td>2985 462</td>
<td>2331  482 2196 539</td>
</tr>
<tr>
<td>OSB</td>
<td>1</td>
<td>209 84</td>
<td>1618 713</td>
<td>1579  693 1540 693</td>
</tr>
<tr>
<td>OSB</td>
<td>2</td>
<td>170 89</td>
<td>1926 693</td>
<td>1502  636 1540 597</td>
</tr>
</tbody>
</table>

Note: PW = Plywood
1.13.2 Calculation of the Modulii of Elasticity

The moduli of elasticity of the panels were calculated as follows:

First, the panel flexural stiffnesses are calculated using the following formula:

\[
[EJ]_J = \frac{Pl^3}{48\Delta}
\]

where \( EJ \) = stiffness (Global or Local) (lbs-in²)

\( J = L, T \) directions defined earlier

\( P = \) Applied load, (lbs.)

\( l = \) span, 44" for global tests and 21" for local tests

\( \Delta = \) average central deflection (in.)

The quantity \( \frac{P}{\Delta} \) was obtained from the global and local tests conducted.

Next the panel stiffnesses per unit width are calculated using the following relation:

\[
D_J = \frac{[EJ]_J}{a}
\]

where \( a \) is the length of the side of the panel = 48". It can be seen that the above quantities can be obtained from the load deflection curves. Using equation (2.24) for the flexural stiffnesses \( D_L \) and \( D_T \), the moduli of elasticity of the panels are computed from the following equation:
The values for $v_{LT}$ was assumed as 0.3 for all the panels based on Bodig and Jayne (19). The average of the actual thicknesses measured was used for the value of $t$. The global and local moduli of elasticity are reported in Table 2.
Table 2: Global and local moduli of elasticity

| Panel (type & sample no.) | \( E_l \) psi \( \times 10^6 \) Local | | | Global | \( (\text{Local} \ Global)\% \) | \( E_T \) psi \( \times 10^6 \) Local | | | Global | \( (\text{Local} \ Global)\% \) | \( E_{LG} \) |
|---------------------------|----------------------|----------------------|---------------------|---------------------|----------------------|----------------------|---------------------|---------------------|---------------------|---------------------|
| PWAC3/4*(1)              | 0.97 0.93 1.00       | 0.97 1.024 -5.27     | 0.97 0.76 0.83 0.85 | 0.927 8.30 1.105  |
|                           | .58 0.81 0.86 0.75   | 0.873 -14.09 1.04    | 0.84 0.79 0.89      | 0.870 2.30 1.003    |
| PWAC1/2*(1)              | 1.34 1.33 1.33 1.33  | 1.309 1.60 0.41 0.49 | 0.68 0.53 0.908     | -41.63 1.762       |
| PWAC1/2*(2)              | 1.27 1.27 1.27 1.27  | 1.230 3.25 0.61 0.60 | 0.65 0.62 0.675     | -8.15 1.822        |
| PWCD(1)                  | 1.46 1.47 1.37 1.43  | 1.430 0.00 0.26      | 0.26 0.26 0.26      | 0.260 0.00 5.500    |
| PWCD(2)                  | 1.55 1.21 1.14 1.30  | 1.480 -12.16 0.24    | 0.25 0.28 0.28      | 0.280 1.07 5.290    |
| OSB(2)                   | 0.84 0.82 0.80 0.82  | 0.996 -17.67 0.37    | 0.36 0.36 0.36      | 0.402 -10.45 2.480  |
| OSB(2)                   | 1.00 0.78 0.80 0.86  | 0.810 6.20 0.36      | 0.33 0.31 0.33      | 0.426 -22.53 1.901  |

Note: PW = Plywood
1.14 STRUCTURE RESPONSE TESTS

The same panels previously tested were used for the structure response tests. The panel was supported under its four corners in the same test set-up used for the property tests. The corner supports in the test arrangement were simulated by welding steel blocks of length two inches by width one inch by thickness one inch on the steel support beams. The span of the panel in both directions were 46 inches. The instrumentation and conduct of the experiment was the same as in the property tests.

The panels were tested under the following loads:

1. centrally applied concentrated load;
2. line load along the L direction;
3. line load along the T direction.

Line loads were simulated with the loading bar arrangement of the property tests. The concentrated load was simulated by placing the spherical seat directly on the panel center. Deflections were measured at the panel center. The test arrangement with a $\frac{3}{4}$" thick A-C grade plywood is shown in Figure 17. The panels did not show a symmetric deflection response in these tests. The problems faced in conducting the structure tests were similar to those experienced in the property tests. The load deflection response for a sample of 0.75" A-C grade plywood is shown in Figure 18. The deflection response for all the panels are compared with the results of the theoretical model in chapter 4.
Figure 17. Four corner panel test for Plywood 0.75", A-C grade
Figure 18. Load-deflection response of Plywood 0.75\textdegree, A-C grade: four corner test: (a) line load parallel to grain (b) line load perpendicular to grain
Chapter 3 described the experiments designed to collect data on wood based orthotropic panels. This chapter presents the solutions obtained from the theoretical grid model for isotropic and orthotropic plates under various boundary conditions and loads. The results are presented in two parts. Part 1 presents comparisons of the grid model with classical, numerical and experimental results reported in the literature for isotropic and orthotropic plate bending problems. Part 2 presents comparisons between the grid model solutions and experimental data discussed in chapter 3. The procedure for proportioning the grid member between the bounds $A_r$ and $A_e$ to simulate the response of a plate under different boundary conditions is also discussed. The exact value varied with the ratio of $\frac{E}{G_{LT}}$. The case studies also established the member size $A^*$ of a grid equivalent to a plate (isotropic and orthotropic) supported on four corners (4 cor).
1.15 PART 1: CASE STUDIES REPORTED IN LITERATURE

Case studies are presented first for isotropic plates and then for orthotropic plates. The variables $b_x$ and $b_y$ were selected to equal $\frac{1}{6}$ times the side of the plate for all the cases. This spacing is consistent with Szilard (6), who states: “the central deflection of a square, isotropic plate is obtained to an accuracy up to 2% with such a mesh”. Details of formulating the member dimensions $A_x$ and $A_y$ are presented as an illustration for the first case only. For the remaining cases, only the member dimensions $A_x$ and $A_y$ are given, since the formulation procedure is identical. Details of plate dimensions, boundary conditions, material properties and loads considered for the various case studies are presented with the numerical results in the following sections.

1.15.1 Case 1: Simply Supported, Square, Isotropic Plate

The specifications adopted for this case are given below.

1. Material
   - Type- Isotropic
   - Properties
     - Modulus of Elasticity $E = 30 \times 10^6$ psi.
     - Poisson’s ratio $\nu = 0.3$
1. Modulus of Rigidity \( G = 11538462 \) psi.

The magnitude of the modulus of rigidity was calculated from the relation

\[
E = 2G (1 + \nu).
\]

2. Plate geometry, \((a \times b \times t) = 48\) inches \(\times 48\) inches \(\times 0.5\) inches

3. Boundary Conditions = 4 ss

4. Loads

- a concentrated load of magnitude 100 lbs. applied at the center of the plate (Conc.Ld);
- line load (LL) of magnitude 1 lb./in. applied centrally;
- uniformly distributed load (UDL) of magnitude 1 psi;

5. Grid Member Spacing, \( b_x = b_y = 8\) inches

6. Grid Member Cross-Sectional Shape = Square

**Formulation**

By substituting the above properties in equations [2.24] the plate stiffnesses are obtained:

- Flexural Stiffness \( D_{xx} = D_{yy} = D = 343406.59 \) in.lb.
- Torsional Stiffness \( 2H_p = 686813.18 \) in.lb.

The grid-plate algebraic equations are obtained by substituting material properties and the values for grid spacing in equations [2.38], which are:

\[
(1.92 \times 10^9)A_f^4 + (2060439.5)A_f^2 - 2.1098 \times 10^9 = 0 \quad [4.1.1]
\]
\[ A_t^6 - (41.26984) A_t^5 - (711.11112) A_t^4 - (1.7006802) A_t^2 + 1209.3726 = 0 \quad [4.1.2] \]

Solving equation \([4.1.1]\) by factoring and equation \([4.1.2]\) by the Newton-Raphson first order method (11), we get:

- \( A_t = 1.0235849 \text{ inches} \) \quad \([4.2.1]\)

- \( A_t = 1.124041941 \text{ inches} \) \quad \([4.2.2]\)

The grid was analyzed with SPACEPAL using a number of values for the member dimension between the limits \( A_t \) and \( A_r \) for the loads detailed earlier. The solution obtained with the member dimension \( A_{4ss} = \frac{1}{2} \cdot (A_r + A_t) \) gave excellent results for deflection and bending moment in comparison with the classical solutions (12) and are reported in Tables 3 and 4.

RESULTS AND DISCUSSION
Table 3 Central deflections of simply supported square isotropic plate

<table>
<thead>
<tr>
<th>Load</th>
<th>Deflection inches</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Classical Solution</td>
<td>Grid Solution</td>
</tr>
<tr>
<td>Conc</td>
<td>0.0077827</td>
<td>0.0076723</td>
</tr>
<tr>
<td>Line</td>
<td>0.0021706</td>
<td>0.0021921</td>
</tr>
<tr>
<td>UDL</td>
<td>0.0627599</td>
<td>0.0625487</td>
</tr>
</tbody>
</table>
Table 4 Maximum forces in a simply supported square isotropic plate

<table>
<thead>
<tr>
<th>Load</th>
<th>Bending Moment lb.in./in.</th>
<th>Shear Force lb./in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Classical Solution</td>
<td>Grid Solution</td>
</tr>
<tr>
<td>UDL</td>
<td>110.36</td>
<td>103.35</td>
</tr>
</tbody>
</table>
1.15.2 Case 2: Central Deflections of a Simply Supported, Isotropic Plate

The specifications for this case were identical to Case 1 except \( \nu \) was 0.25. Accordingly the value of \( G \) was calculated as 12000000 psi, from the relation \( E = 2G(1 + \nu) \). The plate was analyzed under concentrated load and line load. The magnitude of the applied loads was the same as Case 1. Results of the grid analysis method compared with classical solutions (12) are reported in Table 5.
Table 5 Central deflections of simply supported square isotropic plate

<table>
<thead>
<tr>
<th>Load</th>
<th>Deflection inches</th>
<th></th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Classical Solution</td>
<td>Grid Solution</td>
<td></td>
</tr>
<tr>
<td>Conc</td>
<td>0.0080179</td>
<td>0.0079757</td>
<td>-0.53</td>
</tr>
<tr>
<td>Line</td>
<td>0.0022361</td>
<td>0.0022441</td>
<td>0.36</td>
</tr>
</tbody>
</table>
1.15.3 Case 3: Sensitivity of Modulus of Elasticity of Isotropic Materials

To study the sensitivity of the solution by the grid method to the modulus of elasticity the next case study was carried out with the following specifications:

Specifications

1. Material
   - Type: Isotropic
   - Properties
     - Modulus of Elasticity $E$ was varied from $30 \times 10^3$ to $1 \times 10^6$ psi.
     - Poisson's ratio $v = 0.3$
     - Modulus of Rigidity $G$ was calculated for each value of $E$ from the relation
       \[ E = 2G(1 + v). \]

2. Plate geometry, $(a \times b \times t) = 46$ inches x $46$ inches x $0.5$ inches

3. Boundary Conditions = 4 ss

4. Grid Member Spacing $b_x = b_y = 7.6666667$ inches

5. Grid Member Cross-Sectional Shape = Square

The member size $A_{4ss}$ used in the analysis with the grid model was the average of the member sizes $A_x$ and $A_y$ consistent with case studies 1 and 2. It should be noted that the member sizes $A_x$ and $A_y$ and therefore $A_{4ss}$ for all the cases were the same.
The grid member cross-sectional sizes for all the values of $E$ were:

- Grid Member Cross-Sectional Size $A_x = 1.012623979''$
- Grid Member Cross-Sectional Size $A_y = 1.111600032''$
- Grid Member Cross-Sectional Size $A_{4ss} = 1.082112''$

The central deflections obtained with the parametric studies are compared with the classical solutions (12) for UDL and concentrated loads and are presented in Tables 6 and 7 respectively.
Table 6  Sensitivity studies for different values of $E$ on square isotropic plate UDL

<table>
<thead>
<tr>
<th>$E$ $10^6$ psi</th>
<th>Classical Solution</th>
<th>Grid Solution</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.0552372</td>
<td>0.0551212</td>
<td>-0.21</td>
</tr>
<tr>
<td>25</td>
<td>0.0657430</td>
<td>0.0658811</td>
<td>-0.21</td>
</tr>
<tr>
<td>20</td>
<td>0.0828558</td>
<td>0.0826818</td>
<td>-0.21</td>
</tr>
<tr>
<td>15</td>
<td>0.1104745</td>
<td>0.1102425</td>
<td>-0.21</td>
</tr>
<tr>
<td>10</td>
<td>0.1657117</td>
<td>0.1653637</td>
<td>-0.21</td>
</tr>
<tr>
<td>5</td>
<td>0.3314235</td>
<td>0.3307274</td>
<td>-0.21</td>
</tr>
<tr>
<td>1</td>
<td>1.6571173</td>
<td>1.6536367</td>
<td>-0.21</td>
</tr>
</tbody>
</table>
Table 7. Sensitivity studies for different values of $E$ on square isotropic plate
Concentrated Load

<table>
<thead>
<tr>
<th>$E \times 10^4$ psi</th>
<th>Deflection inches</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Classical Solution</td>
<td>Grid Solution</td>
<td>Error %</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.0071476</td>
<td>0.0071546</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.0085458</td>
<td>0.0085541</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.0107215</td>
<td>0.0107318</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.0142953</td>
<td>0.0143091</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0214430</td>
<td>0.0214637</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0428860</td>
<td>0.0429274</td>
<td>-0.097</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.2144303</td>
<td>0.2146388</td>
<td>-0.097</td>
<td></td>
</tr>
</tbody>
</table>
The error between the classical and grid solutions was found to be consistent for both udl and concentrated load for all values of $E$. This was due to the constant ratio of the grid and plate stiffnesses. The ratio between the flexural stiffness of the grid and plate was 1.211 and torsional stiffnesses was 0.831 for all values of $E$. The grid stiffnesses were calculated by substituting the value of $A_{4ss}$ in equations [2.36 & 2.37]. Plate stiffnesses were calculated from equations [2.24].

1.15.4 Case 4: Isotropic Plate, Simply Supported on Two edges and Free at the other Two Edges

The next case presented is a 2 ss 2 fr square, isotropic plate subjected to a UDL of 1 psi. The specifications adopted were identical to those for case 1. The member size used for the grid model was $A_r$ due because a 2ss 2 fr plate will be more controlled by the flexural stiffness and not the torsional stiffness. The results obtained for the case study compared with classical results (12) are presented in Table 8.
<table>
<thead>
<tr>
<th>Load</th>
<th>Deflection inches</th>
<th>Classical Solution</th>
<th>Grid Solution</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>UDL</td>
<td></td>
<td>0.19894</td>
<td>0.1995987</td>
<td>-1.35</td>
</tr>
</tbody>
</table>
1.15.5 Case 5: Square Isotropic Plate Supported on Four Corners

The next case study was on a square isotropic plate simply supported at its four corners and subjected to a concentrated load at the center. The specifications adopted were from the experimental and theoretical work carried out by Yettram and Hussain (2) which are reproduced below.

1. Material
   - Type- Isotropic
   - Properties
     - Modulus of Elasticity $E = 10 \times 10^6$ psi.
     - Poisson's ratio $= 0.3$
     - Modulus of Rigidity $G = 3846153.846$ psi.

The magnitude of the modulus of rigidity satisfies the relation $E = 2G(1 + \nu)$.

2. Plate geometry, $(a \times b \times t) = 24$ inches $\times 24$ inches $\times 0.2496$ inches

3. Boundary Conditions $= 4cor$

4. Grid Member Spacing $b_x = b_y = 4.0$ inches

5. Grid Member Cross-Sectional Shape $= Square$

RESULTS AND DISCUSSION
The grid member cross-sectional size computed from flexural and torsional considerations are given below.

- \( A_r = 0.5111265210 \) inches
- \( A_t = 0.5613566286 \) inches
- \( A_{4ss} = 0.5362415 \) inches

The boundary condition of a four corner simply supported plate falls in between the cases of a 2ss 2fr plate and a 4 ss plate, implying that the required member size is between \( A_r \) and \( A_{4ss} \). Trials were conducted with the dimensions of the grid member adjusted between \( A_r \) and \( A_{4ss} \). The grid member cross-sectional size size of \( A_r + \frac{1}{3} (A_{4ss} - A_r) \) gave excellent results when compared with Yettram and Hussain's experimental results. The value of the member dimensions calculated on the above rationale was \( A_{4cor} = 0.5153123 \) inches and \( B_{4cor} = 0.4333242 \) inches. The results of the grid analysis are compared with Yettram and Hussain (2) in Table 9. The comparison is made for a load value of 1000 lbs. only. Since the grid method is a linear displacement model comparisons for other load values can easily be made.
Table 9 Central deflections of a four corner simply supported square isotropic plate

<table>
<thead>
<tr>
<th>Load</th>
<th>Deflection inches</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental Solution</td>
<td>Grid Solution</td>
</tr>
<tr>
<td>Conc</td>
<td>1.486</td>
<td>1.5085612</td>
</tr>
</tbody>
</table>
1.15.6 Case 6: Central Deflection of a Simply Supported Rectangular Isotropic Plate

The case consisted of a rectangular isotropic plate of aspect ratio $\frac{a}{b}$ equal to 1.2. The specifications for this case are similar to those in Case study 1 except that the dimensions of the plate were 48'' × 40''. The member dimensions $A_t$, $A_x$, and $A_{4ss}$ are the same as those of case 1, since the stiffness formulations are identical. The results obtained for the plate are compared with classical solutions (12) in Table 10.
Table 10 Central deflections of a simply supported rectangular isotropic plate (48 inches x 40 inches)

<table>
<thead>
<tr>
<th>Load</th>
<th>Classical Solution</th>
<th>Grid Solution</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conc</td>
<td>0.0630389</td>
<td>0.0631268</td>
<td>0.14</td>
</tr>
<tr>
<td>Line</td>
<td>0.0148908</td>
<td>0.0149708</td>
<td>0.54</td>
</tr>
<tr>
<td>UDL</td>
<td>0.0420446</td>
<td>0.0405629</td>
<td>-3.52</td>
</tr>
</tbody>
</table>
1.15.7 Case 7: Parametric Studies with Isotropic Material Properties

To develop a rational procedure for modeling orthotropic materials, where $E_L$, $E_T$, $G_{LT}$, and $\nu_{LT}$ are independent of each other, I decided to study the response of the model by maintaining $E_L = E_T = E$ and vary the value of $G$. The specifications for the parametric studies were the same as those for case study 1 except the plate size analyzed was 46 inches x 46 inches. The value of $\frac{E}{G}$ was varied in the range of 2.6 to 600. For these computations the value of $E$ was frozen at $30 \times 10^6$ psi. The value of $\nu$ was maintained constant at 0.3.

Since few closed form solutions are available for orthotropic plate bending problems I decided to use the series solution approach presented by Lee (18) to evaluate benchmark solutions. Accordingly, a computer program was developed and values of the central deflection were obtained by considering 400 terms of the Fourier series. The parametric studies were conducted on a 4ss plate. The member size input for each formulation was A4ss similar to the approach adopted for the previous case studies. The ratios of the plate stiffnesses and the errors in deflections are presented in Table 11.
Table 1: Parametric studies for different values of $\frac{E}{G_{LT}}$
on square plate

<table>
<thead>
<tr>
<th>$\frac{E}{G_{LT}}$</th>
<th>$\frac{D_{grid}}{D_{plate}}$</th>
<th>$\frac{2H_{grid}}{2H_{plate}}$</th>
<th>Error - udl/conc. Id.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6</td>
<td>1.211</td>
<td>0.831</td>
<td>-0.21</td>
</tr>
<tr>
<td>3.0</td>
<td>1.233</td>
<td>0.817</td>
<td>-1.64</td>
</tr>
<tr>
<td>6.0</td>
<td>1.377</td>
<td>0.738</td>
<td>-10.20</td>
</tr>
<tr>
<td>12.0</td>
<td>1.583</td>
<td>0.646</td>
<td>-20.75</td>
</tr>
<tr>
<td>24.0</td>
<td>1.823</td>
<td>0.562</td>
<td>-30.75</td>
</tr>
<tr>
<td>30.0</td>
<td>2.356</td>
<td>0.686</td>
<td>-46.13</td>
</tr>
<tr>
<td>300.0</td>
<td>2.356</td>
<td>0.421</td>
<td>-45.73</td>
</tr>
<tr>
<td>600.0</td>
<td>2.396</td>
<td>0.412</td>
<td>-46.60</td>
</tr>
</tbody>
</table>
From Table 11 it is clear that the accuracy of the solution is related to the ratio of $\frac{E}{G}$.

The increased error of the grid model with increasing values of $\frac{E}{G_{LT}}$ as seen in Table 11 indicates that the grid member size must be modified between the values of $A_{4ss}$ towards $A_r$. The member size was adjusted to obtain convergent solutions with those computed using the series solution. The central deflections obtained with the series and grid solutions for a square plate with varying $\frac{E}{G_{LT}}$ ratios, subjected to UDL and concentrated load are compared in Table 12.
Table 12  Sensitivity of Grid Member size to varying $\frac{E}{G_{LT}}$ ratios on a simply supported square plate

| $\frac{E}{G_{LT}}$ | $a'$ |  |  |  |  |  |  |  |
|-------------------|------|---|---|---|---|---|---|
|                   |      | UDL | Conc. Load | Deflection inches |      |  |  |  |
|                   |      | Series | Grid | Error % | Series | Grid | Error % |      |
| 2.6               | 1.000 | 0.0552372 | 0.0551212 | -0.21 | 0.00714768 | 0.0071546 | 0.097 |
| 6.0               | 0.667 | 0.0691770 | 0.0689062 | -0.39 | 0.00879236 | 0.0087662 | -0.290 |
| 12.0              | 0.500 | 0.0765149 | 0.0757790 | -0.96 | 0.00965468 | 0.0095608 | -0.970 |
| 24.0              | 0.333 | 0.0807934 | 0.0828332 | 2.50  | 0.01015587 | 0.0104060 | 2.500 |
| 30.0              | 0.275 | 0.0817066 | 0.0801065 | -1.95 | 0.01026270 | 0.0100546 | -2.500 |
| 300.0             | 0.275 | 0.0851703 | 0.0841127 | -1.24 | 0.01066751 | 0.0105239 | -1.350 |
| 600.0             | 0.275 | 0.0853713 | 0.0838951 | -1.70 | 0.01089098 | 0.0104948 | -1.830 |

Note: The member size $a'$ is shown on a scale of 0 to 1.
The grid member cross-sectional size $a^*$ yielding minimum error in central deflection is represented in Table 12 on a scale of range 0 and 1 representing the limits $A_r$ and $\frac{1}{2} (A_r + A_l)$ respectively. It can be seen that by systematically varying the member size of the grid, solutions with errors less than 3.0% can be obtained with the grid model. Therefore, it is hypothesized that a function based on the ratio $\frac{E}{G}$ can be developed to rationally establish the grid member cross-section size for a 4ss plate and thus to calibrate the model for various plate materials. The variation of the grid member cross-sectional size $a^*$ with $\frac{E}{G}$ is shown in Figure 19 a. A linear regression between $a^*$ and $\frac{E}{G}$ (Figure 19 b) yielded the equation $a^* = 1.234 + (-0.662) \log \left( \frac{E}{G} \right)$ with $r^2 = 0.984$. Thus, once the the grid member cross-sectional sizes $A_r$ and $\frac{1}{2} (A_r + A_l)$ are established, the regression equation can be used to determine the exact member cross-sectional size for any $\frac{E}{G}$ ratio and solve the resulting grid.

1.15.8 Case 8: Simply Supported Square Orthotropic Panels

This case consisted of orthotropic plates whose properties $E_L, E_T, G_{LT}$ and $\nu_{LT}$ are independent. The validation of the grid model was carried out using properties in the range typically found in wood composite based panels. The model was verified with data reported by Lee (18). The rationale for the member sizes was based on the results of the parametric studies obtained in Case study 7. The grid model was evaluated for a panel of dimensions 46 inches x 46 inches. All other parameters were the same as defined by Lee.
Figure 19. Variation of non-dimensionalized grid cross-sectional size $a^*$ versus $E/G_{LT}$, (a) actual data (b) linear regression of transformed data

RESULTS AND DISCUSSION
The central deflections obtained with the grid solutions for the plate subjected to UDL (1 psi.) and concentrated load (100 lbs. at the center) are compared in Table 13.
Table 13  Parametric Studies on Square Simply Supported Orthotropic Plates with varying $\frac{E}{G_{LT}}$ and $\frac{E_{L}}{E_{T}}$ ratios

<table>
<thead>
<tr>
<th>$\frac{E_{L}}{G_{LT}}$</th>
<th>$\frac{E_{L}}{E_{T}}$</th>
<th>$n_{u_{LT}}$</th>
<th>$n_{u_{IL}}$</th>
<th>t in.</th>
<th>a'</th>
<th>Deflection inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>UDL</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Conc. Load</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>series sol.</td>
</tr>
<tr>
<td>11.5</td>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>0.86</td>
<td>0.59</td>
<td>0.3903005</td>
</tr>
<tr>
<td>16.0</td>
<td>2</td>
<td>0.409</td>
<td>0.205</td>
<td>0.86</td>
<td>0.54</td>
<td>0.3848647</td>
</tr>
<tr>
<td>17.0</td>
<td>3</td>
<td>0.380</td>
<td>0.127</td>
<td>0.60</td>
<td>0.44</td>
<td>1.3048628</td>
</tr>
<tr>
<td>18.0</td>
<td>4</td>
<td>0.368</td>
<td>0.092</td>
<td>0.60</td>
<td>0.40</td>
<td>1.3706326</td>
</tr>
<tr>
<td>19.0</td>
<td>5</td>
<td>0.456</td>
<td>0.081</td>
<td>0.48</td>
<td>0.37</td>
<td>2.6102321</td>
</tr>
<tr>
<td>19.0</td>
<td>10</td>
<td>0.473</td>
<td>0.047</td>
<td>0.60</td>
<td>0.33</td>
<td>1.5230338</td>
</tr>
<tr>
<td>21.0</td>
<td>15</td>
<td>0.310</td>
<td>0.021</td>
<td>0.35</td>
<td>0.28</td>
<td>7.5355093</td>
</tr>
<tr>
<td>22.0</td>
<td>20</td>
<td>0.400</td>
<td>0.020</td>
<td>0.30</td>
<td>0.28</td>
<td>11.54947</td>
</tr>
</tbody>
</table>

Note: The member size a' is shown on a scale of 0 to 1.
The grid member cross-sectional size $a^*$ yielding minimum error in central deflection is re-presented in Table 13 on a scale of range 0 and 1 representing the limits $A_r$ and $\frac{1}{2} (A_r + A_s)$ respectively. The variation of the grid member cross-sectional size $a^*$ with $\frac{E}{G}$ is shown in Figure 20 a. A linear regression between $a^*$ and $\frac{E}{G}$ (Figure 20 b) yielded the equation $a^* = 1.910 + (-1.2079) \log \left( \frac{E}{G} \right)$, with $r^2 = 0.847$. Thus, once the grid member cross-sectional sizes $A_r$ and $\frac{1}{2} (A_r + A_s)$ are established, the regression equation can be used to determine the exact member cross-sectional size for any $\frac{E_L}{G_{LT}}$ ratio and solve the resulting grid.

### 1.15.9 Case 9: 2ss,2fr Square Orthotropic Plate

The next validation of the grid model was conducted using properties of advanced composite materials. The properties used in this study were from Pagano (20). Grid models were formulated with the above properties and verified against the published results (20). The specifications for the case study were:

1. **Material**
   - **Type:** Orthotropic
   - **Properties**
     - Modulus of Elasticity $E_L = 25 \times 10^6$ psi.
     - Modulus of Elasticity $E_T = 10^6$ psi.
     - Poisson’s ratio $\nu_{LT} = 0.25$
Figure 20. Variation of non-dimensionalized grid cross-sectional size $a$ versus $E_L / G_{LT}$ for orthotropic materials: (a) actual data (b) linear regression of transformed data

RESULTS AND DISCUSSION
Modulus of Rigidity $G_{LT} = 0.5 \times 10^6$ psi.

2. Plate geometry, $(a \times b \times t) = 48$ inches $\times$ 48 inches $\times$ 0.5 inches

3. Boundary Conditions $= 2$ss $2$fr

4. Loading $= \text{udl}$ 1 psi.

5. Grid Member Spacing $b_x$ and $b_y = 8$ inches

6. Grid Member Cross-Sectional Shape = Square

The grid member cross-sectional size used for this study was $A$, due to the boundary condition. $A$, was calculated as 1.0006162 inches. The deflection of the grid at the center of the plate was 0.2581604 inches. Pagano (20) presents the deflection for the case in a normalized form, which is reproduced below.

$$\bar{W} = \frac{50 E_T w}{b q_0 S^3}$$

where

- $\bar{W}$ = non-dimensionalized central deflection of plate
- $w$ = central deflection inches
- $b$ = 0.5 $l$
- $l$ = span inches
- $q_0$ = load intensity psi.
- $S = \frac{l}{t}$
- $t$ = thickness of plate inches

RESULTS AND DISCUSSION
The exact normalized solution for the case is 0.608 inches (20). Accordingly the normalized deflections were computed for the grid solution as $\bar{w} = 0.6079035$ inches corresponding to an error of only $-0.013\%$.

1.15.10 Case 10: 2ss 2fr Orthotropic Plate

The next validation of the grid model was conducted on a case solved using the Levy’s solution method by Reddy et. al, (21). The grid model was formulated with the following specifications:

1. Material
   - Type- Orthotropic
   - Properties
     - Modulus of Elasticity $E_L = 20.83 \times 10^6$ psi.
     - Modulus of Elasticity $E_T = 10.94 \times 10^6$ psi.
     - Poisson’s ratio $\nu_{LT} = 0.44$
     - Modulus of Rigidity $G_{LT} = 6.1 \times 10^6$ psi.

2. Plate geometry, $(a \times b \times t) = 48$ inches $\times$ 48 inches $\times$ 0.5 inches

3. Boundary Conditions = 2 ss 2 fr

4. Loading = udl 1 psi.

5. Grid Member Spacing $b_x$ and $b_y = 8$ inches

6. Grid Member Cross-Sectional Shape = Square

RESULTS AND DISCUSSION
The grid member cross-sectional size was $A_r$, corresponding to a $2ss \ 2fr$ plate. $A_r$ was calculated as 1.0264336 inches. The deflection of the plate at the center was 0.2843938 inches. Reddy et. al. (21) present deflections for the case in a normalized form, which is reproduced below.

$$\bar{w} = \frac{E_t \ t^3 \ w}{q_0 \ a^4} \times 100$$

where

- $\bar{w}$ = normalized central deflection of plate
- $w$ = central deflection, inches
- $t$ = thickness of plate, inches
- $q_o$ = load intensity, psi.
- $a$ = size of the square plate inches

The exact normalized solution for the case is 7.4667 (21). The normalized deflections computed by the grid method was $\bar{w} = 7.3262631$ thus giving an error of $-1.88\%$. 
1.16 SUMMARY

The results of the case studies 1 to 10 show:

1. for a simply supported, square, isotropic plate (satisfying the relationship

\[ E = 2G(1 + \nu) \]

the grid member square cross-sectional dimension, that best

simulates the plate's response is equal to the average of \( A_r \) and \( A_t \).

2. for a simply supported, square, orthotropic plate, the grid member's square cross-

sectional dimension to simulate the plate's response is based on the \( \frac{E_L}{G_{LT}} \) ratio. The

actual size is less than the value obtained for the isotropic case.

The analogy of a smaller grid member cross-sectional dimension for orthotropic materials

compared to isotropic materials should be valid for other boundary conditions, such as

a four corner supported plate.

3. The 4c case studies presented are restricted to isotropic plates. The grid member cross-

sectional square dimension simulating the plate's response is

\[ A_r + \frac{1}{3} \left( \frac{A_r + A_t}{2} \right) - A_r \]

It is hypothesized that the grid member cross-

sectional dimension for a four corner supported orthotropic plate should be less than

\[ A_r + \frac{1}{3} \left( \frac{A_r + A_t}{2} \right) - A_r \].

RESULTS AND DISCUSSION
1.17 PART 2: COMPARISON OF GRID MODEL RESULTS WITH EXPERIMENTAL DATA

In this section, the deflections predicted by the grid model are compared to the deflections measured in the structure response tests on wood based composite panels. The panels were supported on their four corners and tested under a central concentrated load, a line-load applied parallel to the L direction, and a line-load applied parallel to the T direction. The material properties $E_L$ and $E_T$ for the grid model were those experimentally determined in the property determination tests. The results were presented in Table 2, chapter 3. The property $G_{LT}$ for the plywood panels was assumed as 90000 psi. based on the recommendations of APA (16). Biblis (22) presents properties of OSB from three commercial mills and it is observed that $G_{LT}$ is approximately equal to one-third the value of $E_L$. Accordingly, $G_{LT}$ for OSB was assumed as one-third the value of $E_L$. For all the panels, the grid spacing was maintained as 7.3333333 inches. The member cross-sectional sizes $A_y$, $A_t$ were computed using the procedure presented in Case 1. The grid member cross-sectional dimension was selected as $A_y + \frac{1}{3} \left\{ \left( \frac{A_y + A_t}{2} \right) - A_y \right\}$. The central deflections of the panels corresponding to three to four load levels are compared Tables 14 to 21. The error in the central deflection between the theoretical model and experimental data is maximum 20%.
Table 14. Central deflections of four corner supported panel:

Plywood 0.75", A-C Grade, Sample 1

<table>
<thead>
<tr>
<th>Type</th>
<th>Magnitude lbs.</th>
<th>Load</th>
<th>Deflection inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Experiment</td>
</tr>
<tr>
<td>LLPA</td>
<td>57.0</td>
<td>0.123</td>
<td>0.1102</td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>0.160</td>
<td>0.1450</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>0.210</td>
<td>0.1934</td>
</tr>
<tr>
<td></td>
<td>125.0</td>
<td>0.259</td>
<td>0.2417</td>
</tr>
<tr>
<td>LLPE</td>
<td>50.5</td>
<td>0.104</td>
<td>0.0960</td>
</tr>
<tr>
<td></td>
<td>81.5</td>
<td>0.165</td>
<td>0.1545</td>
</tr>
<tr>
<td></td>
<td>105.0</td>
<td>0.209</td>
<td>0.1991</td>
</tr>
<tr>
<td></td>
<td>125.0</td>
<td>0.269</td>
<td>0.2370</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>51.0</td>
<td>0.130</td>
<td>0.1160</td>
</tr>
<tr>
<td></td>
<td>76.5</td>
<td>0.194</td>
<td>0.1736</td>
</tr>
<tr>
<td></td>
<td>100.5</td>
<td>0.253</td>
<td>0.2280</td>
</tr>
<tr>
<td></td>
<td>125.0</td>
<td>0.313</td>
<td>0.2836</td>
</tr>
</tbody>
</table>

Note:

LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
Table 15. Central deflections of four corner supported panel:

Plywood 0.75", A-G Grade, Sample 2

<table>
<thead>
<tr>
<th>Type</th>
<th>Load</th>
<th>Deflection inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Magnitude lbs.</td>
<td>Experiment</td>
</tr>
<tr>
<td>LLPA</td>
<td>150.0</td>
<td>0.321</td>
</tr>
<tr>
<td></td>
<td>175.5</td>
<td>0.372</td>
</tr>
<tr>
<td>LLPE</td>
<td>152.0</td>
<td>0.287</td>
</tr>
<tr>
<td></td>
<td>175.0</td>
<td>0.330</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>125.0</td>
<td>0.358</td>
</tr>
<tr>
<td></td>
<td>150.0</td>
<td>0.419</td>
</tr>
</tbody>
</table>

Note:

LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
### Table 16. Central deflections of four corner supported panel:

**Plywood 0.50”, A-C Grade, Sample 1**

<table>
<thead>
<tr>
<th>Type</th>
<th>Load</th>
<th>Deflection inches</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Magnitude lbs.</td>
<td>Experiment</td>
<td>Grid</td>
<td>% Error</td>
</tr>
<tr>
<td>LLPA</td>
<td>50.0</td>
<td>0.255</td>
<td>0.296</td>
<td>16.07</td>
</tr>
<tr>
<td></td>
<td>75.5</td>
<td>0.386</td>
<td>0.447</td>
<td>15.80</td>
</tr>
<tr>
<td></td>
<td>80.0</td>
<td>0.422</td>
<td>0.474</td>
<td>12.32</td>
</tr>
<tr>
<td>LLPE</td>
<td>51.0</td>
<td>0.250</td>
<td>0.281</td>
<td>12.40</td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>0.377</td>
<td>0.413</td>
<td>9.54</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>0.507</td>
<td>0.551</td>
<td>8.68</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>50.0</td>
<td>0.325</td>
<td>0.338</td>
<td>6.00</td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>0.472</td>
<td>0.508</td>
<td>7.63</td>
</tr>
<tr>
<td></td>
<td>101.0</td>
<td>0.616</td>
<td>0.677</td>
<td>9.90</td>
</tr>
</tbody>
</table>

**Note:**

LLPA: Line load parallel to grain centrally applied

LLPE: Line load perpendicular to grain centrally applied

Conc.Ld: Concentrated load at centre
Table 17. Central deflections of four corner supported panel:
Fplywood 0.50", A-C Grade, Sample 2

<table>
<thead>
<tr>
<th>Type</th>
<th>Load Magnitude lbs.</th>
<th>Deflection inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experiment</td>
<td>Grid</td>
</tr>
<tr>
<td>LLPA</td>
<td>75.5</td>
<td>0.536</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>0.721</td>
</tr>
<tr>
<td></td>
<td>125.0</td>
<td>0.905</td>
</tr>
<tr>
<td>LLPE</td>
<td>100.0</td>
<td>0.574</td>
</tr>
<tr>
<td></td>
<td>126.5</td>
<td>0.729</td>
</tr>
<tr>
<td></td>
<td>150.0</td>
<td>0.775</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>76.5</td>
<td>0.618</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>0.793</td>
</tr>
<tr>
<td></td>
<td>125.0</td>
<td>0.968</td>
</tr>
</tbody>
</table>

Note:
LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
Table 18. Central deflections of four corner supported panel:

Plywood 0.50", C-D Grade, Sample 1

<table>
<thead>
<tr>
<th>Type</th>
<th>Load Magnitude lbs.</th>
<th>Deflection inches</th>
<th>Experiment</th>
<th>Grid</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLPA</td>
<td>25.0</td>
<td>0.479</td>
<td>0.440</td>
<td>-8.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50.0</td>
<td>0.777</td>
<td>0.880</td>
<td>13.24</td>
<td></td>
</tr>
<tr>
<td>LLPE</td>
<td>50.5</td>
<td>0.569</td>
<td>0.661</td>
<td>16.17</td>
<td></td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>0.860</td>
<td>0.981</td>
<td>14.07</td>
<td></td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>1.145</td>
<td>1.309</td>
<td>12.51</td>
<td></td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>25.0</td>
<td>0.420</td>
<td>0.465</td>
<td>10.71</td>
<td></td>
</tr>
<tr>
<td></td>
<td>50.0</td>
<td>0.856</td>
<td>0.931</td>
<td>8.76</td>
<td></td>
</tr>
</tbody>
</table>

Note:

LLPA: Line load parallel to grain centrally applied

LLPE: Line load perpendicular to grain centrally applied

Conc.Ld: Concentrated load at centre
Table 19. Central deflections of four corner supported panel:
Plywood 0.50", C-D Grade, Sample 2

<table>
<thead>
<tr>
<th>Type</th>
<th>Load Magnitude lbs.</th>
<th>Deflection inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experiment</td>
<td>Grid</td>
</tr>
<tr>
<td>LLPA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25.0</td>
<td>0.377</td>
<td>0.399</td>
</tr>
<tr>
<td>50.0</td>
<td>0.775</td>
<td>0.799</td>
</tr>
<tr>
<td>55.0</td>
<td>0.857</td>
<td>0.878</td>
</tr>
<tr>
<td>LLPE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>75.0</td>
<td>0.772</td>
<td>0.897</td>
</tr>
<tr>
<td>100.0</td>
<td>1.052</td>
<td>1.195</td>
</tr>
<tr>
<td>127.5</td>
<td>0.344</td>
<td>1.524</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50.5</td>
<td>0.781</td>
<td>0.855</td>
</tr>
<tr>
<td>55.0</td>
<td>0.862</td>
<td>0.931</td>
</tr>
<tr>
<td>60.0</td>
<td>0.940</td>
<td>1.015</td>
</tr>
</tbody>
</table>

Note:
LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
Table 20. Central deflections of four corner supported panel:
OSB 0.4375", Sample 1

<table>
<thead>
<tr>
<th>Type</th>
<th>Magnitude lbs.</th>
<th>Experiment</th>
<th>Grid</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLPA</td>
<td>50.0</td>
<td>0.850</td>
<td>0.7000</td>
<td>-17.60</td>
</tr>
<tr>
<td></td>
<td>75.0</td>
<td>1.302</td>
<td>1.0500</td>
<td>-19.35</td>
</tr>
<tr>
<td></td>
<td>100.0</td>
<td>1.731</td>
<td>1.4000</td>
<td>-19.12</td>
</tr>
<tr>
<td>LLPE</td>
<td>25.0</td>
<td>0.239</td>
<td>0.2120</td>
<td>-11.30</td>
</tr>
<tr>
<td></td>
<td>50.0</td>
<td>0.449</td>
<td>0.4230</td>
<td>-6.42</td>
</tr>
<tr>
<td></td>
<td>75.5</td>
<td>0.668</td>
<td>0.6390</td>
<td>-4.40</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>25.0</td>
<td>0.442</td>
<td>0.6990</td>
<td>-18.25</td>
</tr>
<tr>
<td></td>
<td>50.5</td>
<td>0.832</td>
<td>1.0099</td>
<td>-21.38</td>
</tr>
</tbody>
</table>

Note:
LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
Table 21. Central deflections of four corner supported panel:
OSB 0.4375" Sample 2

<table>
<thead>
<tr>
<th>Type</th>
<th>Magnitude lbs.</th>
<th>Experiment</th>
<th>Grid</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>LLPA</td>
<td>35.0</td>
<td>0.569</td>
<td>0.5200</td>
<td>-8.61</td>
</tr>
<tr>
<td></td>
<td>40.0</td>
<td>0.651</td>
<td>0.5960</td>
<td>-8.76</td>
</tr>
<tr>
<td></td>
<td>45.0</td>
<td>0.737</td>
<td>0.6680</td>
<td>-9.36</td>
</tr>
<tr>
<td>LLPE</td>
<td>5.0</td>
<td>0.044</td>
<td>0.0472</td>
<td>7.27</td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>0.085</td>
<td>0.0944</td>
<td>11.06</td>
</tr>
<tr>
<td>Conc.Ld</td>
<td>30.0</td>
<td>0.502</td>
<td>0.5750</td>
<td>16.50</td>
</tr>
<tr>
<td></td>
<td>35.0</td>
<td>0.590</td>
<td>0.6710</td>
<td>13.73</td>
</tr>
<tr>
<td></td>
<td>40.0</td>
<td>0.677</td>
<td>0.7670</td>
<td>13.29</td>
</tr>
</tbody>
</table>

Note:
LLPA: Line load parallel to grain centrally applied
LLPE: Line load perpendicular to grain centrally applied
Conc.Ld: Concentrated load at centre
The grid model was compared with eighteen cases involving classical, numerical, and experimental results.

The deflections obtained from the grid model presented in Part 1 for isotropic and orthotropic plates compare extremely well to deflections predicted by classical methods. The average error for deflections in all the case studies was 1.5%. The modeling technique was established for the 2ss, 2fr, 4ss, and 4c boundary conditions. The comparison for forces could be made only for case 1, since benchmark results are not available in literature. The error in force prediction for the case was 3.09%.

Therefore, it seems that the grid analog model can be used to model the response of a plate subjected to transverse loads.

The grid member cross-sectional size $A_{4ss}$, equal to the average of $A_x$ and $A_y$ for isotropic materials, (rigorously satisfying the relation $E = 2G[1 + \nu]$) yields consistently accurate predictions for deflections. The results obtained for various values of $\nu$ also are satisfactory. The grid member cross-sectional dimension of $A_x$ adequately models the 2ss,2fr boundary condition for both isotropic and orthotropic materials also yields excellent solutions. The member cross-section of $A_x + \frac{1}{3}(A_y - A_x)$ for the 4c isotropic case compared well with experimental results obtained by Yettram and Hussain (2). The solution compares better than
the theoretical solution obtained by Yettram and Hussain for the problem. The case studies show that the model is valid for square and rectangular plates in isotropic materials.

The parametric studies for a 4ss orthotropic plate in Cases 7 and 8 showed that the cross-sectional size of the grid member varies with $\frac{E}{G}$. Functions were developed for the member cross-sectional size based on the $\frac{E}{G}$ ratio. Using these functions and the bounds for the grid member cross-sectional sizes, $(A_r, \frac{1}{2} (A_r + A_t))$ the required grid member cross-sectional size for a given orthotropic material can be easily computed. $A_r$ and $A_t$ are obtained by solving two algebraic equations. This function approach greatly simplifies the analysis of transversely loaded, orthotropic thin plates.

The model yields excellent results for a range of orthotropic material properties selected in cases 8 to 10 for both the 4ss and 2ss 2fr boundary conditions and is thus applicable to both wood based composites and man-made composites.

However, as shown in Part 2, the results of the grid model do not compare so well (relative to validations of Part 1) with the experimental measurements of specific wood composite based panels. The errors in deflections between the grid model and experimental data are 12% for the 0.75" thick A-C grade plywood panels, 16% for the 0.5" thick A-C grade plywood and C-D grade plywood panels and 21% for the OSB panels. There are several reasons for the discrepancy. The error due to membrane action in the plate affects both the property determination tests and the structure tests. However the results of both the tests show a linear
load-deflection response. If membrane action was significant, the load-deflection response would be non-linear. Therefore membrane effects are minimal.

Friction at the supports can also affect test results. This effect was minimized in the tests by using smooth supports. Friction would also produce errors in the determination of material properties.

The experimental method used for the determination of the moduli of elasticity can also contribute to the error. ASTM (22) specifies a pure moment applied flexure test to determine the panel moduli of elasticity. Since tests in this study had a minimum span to thickness ratio of 28, shear deflection effects should be negligible. Therefore, the departure from the ASTM test probably contributed minimal errors to the property estimation.

However, the use of bending tests to determine material properties is questionable, since bending produces a variable stress distribution in the specimen (4). Tension tests would be a better way to measure the material properties. However, these tests are destructive. Therefore the material properties measured in this study may not be representative of the actual material properties. It is reasoned that the tests have yielded lower properties than the actual state of the material.

The properties $G_L$ and $v_L$ were based on values reported in literature rather than experimental values. This assumption may also have contributed to the errors.
Lastly, the property variability in the panels was not explicitly accounted for in the analysis. This is particularly important, since average global properties were assigned to the grid members, instead of localized properties.
SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

1.19 SUMMARY and CONCLUSIONS

A novel and simple technique was developed to model an orthotropic plate into an equivalent orthotropic grid. The method is simple, in that, the formulation only involves solving two algebraic equations, to establish an upper and lower bound for the grid member cross-sectional size. A rationale for modeling the effect of boundary conditions and material properties on the deflections of the plate was established by varying the grid member's cross-sectional size. Case studies for isotropic and orthotropic materials with invariant properties established the validity of the grid method.
The model was verified for plates simply supported on four edges (4ss), simply supported on four corners (4c), and simply supported along two edges – free on other two edges (2ss 2fr). Other boundary conditions, such as a plate clamped on all four edges or a plate supported on semi-rigid joints could be considered in another study. The range between $A_r$ and $A_i$ for the member size will permit incorporation of other boundary conditions.

The grid member cross-sectional size equal to $A_r$ yields excellent results for 2ss 2fr isotropic and 2ss 2fr orthotropic plates. The case of 4c square isotropic plate under a central concentrated load, a difficult case for mathematical modeling (2), has also been successfully modeled by the grid method. The grid cross-sectional size of $\frac{1}{2} (A_r + A_i)$ yields excellent results for 4ss isotropic square and rectangular plates. For orthotropic plates the member cross-sectional size must be reduced with increasing $\frac{E_L}{G_{LT}}$ ratios. The maximum error between predicted and published theoretical and experimental deflections was 3%. Linear functions presented for establishing the grid member cross-sectional size for orthotropic materials make the formulation simple.

For materials exhibiting variable orthotropy, the analytical predictions differed from the experimental results by 20%. This is attributed to the incorrect evaluation of material properties of the panels and not to the model itself. Excellent validations obtained with published literature show that the model is able to analyze transversely loaded isotropic and orthotropic plates.
The formulation developed is an alternative to the formulations by the classical plate theory, and the finite element method. The grid model is unique compared to other previous grid formulations (1,2,3) in its applicability to plates exhibiting material orthotropy in addition to isotropy and shape orthotropy. Further, the present grid has less members compared to the previous grid formulations (1,2,3). Although a crude mesh was adopted for the discretization of isotropic and orthotropic plates, the results compare well with classical solutions. The model requires very few degrees of freedom and is suitable for integration with any commercial PC based matrix-displacement software.

1.20 RECOMMENDATIONS

The following recommendations are made for future work.

1. The problem of finding the member sizes for various boundary conditions will be best solved by an optimization approach. Once the member cross-sectional sizes $A_x$ and $A_y$ are evaluated by the procedure outlined in chapter 3, the objective function for the optimization problem will be to minimize errors in deflections for various boundary conditions. The constraints to the problem will be to ensure that the member dimension $t_x$ and $t_y$ are not greater than the spacings, $b_x$ and $b_y$ respectively of the grid members. The assumptions of the member cross-section to be square and equally spaced along both the $x$ and $y$ directions also need not be made if the optimization approach is adopted.
2. The method should be extended to cover boundary conditions and loads not considered in this study. To validate the model for boundary conditions and loads for which benchmarks are not available, experimental work must be conducted.

3. The present study is restricted to the transverse bending problem only. There is a need to study the grid model's response to plane stress problems.

4. Further, the function approach must be adopted for all the boundary conditions for orthotropic materials. This can culminate in useful recommendations for analysis.

5. There is a need to accurately evaluate material properties on specific wood composite based panels.
References


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