THE ELASTIC EFFECT OF SPANDRELS AND COLUMNS

ON THE MOMENTS IN SLABS DUE TO VERTICAL LOADS

by

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PREFACE

This thesis will be concerned with the comparison of results obtained from the mathematical and experimental solution of flat plates supported by beams and columns. The experimental solution is taken from a thesis by John W. Flemer,[1] also for the degree of Master of Science in Architectural Engineering at Virginia Polytechnic Institute. Mathematical solutions of the problem with both pinned and fixed edges will be worked out and the comparison made.

The method of difference equations as developed by Dr. H. Marcus,[3] translated into English and explained by Joseph A. Wise,[4] and the superposition principal as described by D. L. Holl[5] will be used in these solutions of the problem.

It is hoped that this thesis will be of use in the investigation of the problem of flat plates. This thesis is not meant to be an end in itself, but only one of the many means toward the goal of a more exact solution of flat plates to be available for the use of the designing engineer.

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INTRODUCTION

The solution for moments, stresses, and deflections in flat plates has been of interest to the engineer for many years. The equation developed by Lagrange forms the basis for most of the attempts at the exact solution, but has not as yet been solved for the general case. Many solutions by approximations have been worked out but most of them are either very involved and require the use of higher mathematics, or are not accurate enough for the use of the engineer.

Dr. H. Marcus[3] gave us one solution by approximations that uses difference equations. His solution was based upon the analogy of a membrane and the plate. Two theorems that make the analogy possible are: (a) The deflection of a membrane loaded with loads proportional to those on a given plate may be considered as the sum of the principal moments of the actual plate. (b) A second membrane may be loaded with elastic weights proportional to these moment sums and, subjected to appropriate boundary conditions, the deflection of the latter membrane will be proportional to the deflection of the plate under the given loading system.

In a paper by Joseph A. Wise,[4] Mr. Wise presents the solution of Dr. Marcus using the elastic web. Mr. Wise solves an example problem of a square flat plate carrying a uniform load, and pinned on all edges.

A paper presented by Mr. D. L. Holl[5] shows the solution of flat plates by the method of the difference equations for flat plates with various edge conditions. Mr. Holl uses the method of superposition of deflections to solve for plates with edge conditions other than pinned. In his paper Mr. Holl works several examples of square plates supporting a center point load, and having edge conditions variously fixed, pinned, or free.

This thesis will attempt to use the methods that are mentioned above, and apply them to the specific flat plate that was used in the experiment. The article by Mr. Wise forms the basic reference for the work done in this thesis. The article by Mr. Holl was used as reference for the solution of the plate with fixed edges.

The method of difference equations with the use of a sixteen mesh elastic web has an accuracy that varies less than two and a half percent from the more exacting solutions of Nadai and Estanave. As the mesh size is decreased the accuracy increases, and a very accurate solution may be obtained without having the work become too involved to make the solution practical for the engineer.

In this thesis a thirty six mesh web is used so that a comparison of these results with the experimental results may be made. In the experiment concentrated loads were applied at various points on the plate, and the effects of all of them were combined to give the effect of a concentrated load. In the analytical solution a uniform load, equivalent to the concentrated load is used.

THE THEORY OF THE

DIFFERENCE METHOD

FOR THE

SOLUTION OF FLAT PLATES

A DISCUSSION OF THE METHOD OF DIFFERENCE EQUATIONS FOR THE SOLUTION OF FLAT PLATES

The method of using difference equations for the solution of flat plates is comparatively new; Dr. Marcus[3] first published the method in nineteen twenty five. Since then it has been translated into English, and expanded upon in many articles by mathematicians, and engineers. Among the men who have presented the work in English with sample problems of different types are : Stephen Timoshenko,[6] Joseph A. Wise,[4] and D. L. Holl.[5] The works of these men was found to be the most complete on the subject that was available. None of these articles alone would form a complete discussion of the subject, but from the three of them a considerable understanding of the subject may be obtained.

The mathematics of the problem become complicated only in the process of proving the analogies and deriving the difference equations. The mathematics involved in the solution of the problem are very simple in comparison with the mathematics in some of the other solutions of the flat plate. The simple nature of the mathematics involved prompted Mr. Wise[4] to remark in his article; "The deflections of the web are determined by means of difference equations that are easily set up, and the sole mathematical device necessary for actual computation is the solution of simultaneous linear equations."

The difference method is apparently quite accurate enough for most practical work that is done in engineering today. All of the authors that were mentioned remarked upon the accuracy of the method. Mr. Timoshenko[6] in his book, speaking of the sample problem in the book in which he used a mesh width of one quarter of the length, said that error of the calculated maximum deflection as compared with the exact deflection was less than one percent, and the maximum bending moment at the center of the plate was about four and one half percent less than the exact value. By taking twice the number of subdivisions of the length of plate he found that the error in the bending moment was less than one percent.

The method of difference equations may be applied to almost any type of flat plate, under any type of loading. Mr. Wise[4] in his article said, "This paper will present the application of the method for the case of square and rectangular slabs freely supported at the four edges, although the method can be generally applied to almost any case of shape, supporting conditions, or loading."

The ease with which the problem may be handled, the accuracy of the method, and the possibility of its use on almost any type of plate problem justifies further work being done on this subject. It is hoped that this thesis will be a step toward the further development of the method, for use in the field of plate design.

AN EXPLANATION OF THE THEORY OF THE DIFFERENCE EQUATION METHOD

The theory of the difference method as applied to plates depends upon the analogy between the plate and two membranes. The first membrane is loaded in the same manner as the plate, and the deflection of the membrane is proven analogous to the moment sum in the plate. The second membrane is loaded with the elastic, moment-sum, loads and the deflection in this membrane is proven proportional to the deflection in the plate due to the original load.

It would probably be impossible to load a membrane with a varying load, such as the moment-sum load, so to facilitate the working of the problem the membranes are replaced by elastic webs. The elastic web is conceived as a network of perfectly elastic wires covering the same area as the plate. The loading on the plate is converted into equivalent concentrated loads at the intersections of the wires. The end conditions, representing different conditions of fixity or freedom acquired from the supports, are determined from the theory of the action of elastic webs.

The deflections of the web under the various loading conditions give the moments, stresses, and deflections of the plate. The deflections of the web are obtained directly from the solution of the simultaneous linear difference equations, written for the web.

THE DERIVATION OF THE THEORY OF THE DIFFERENCE EQUATION METHOD FOR THE SOLUTION OF FLAT PLATES

A list of the characters and symbols that will be used in the derivation of the theory will be given here for easy reference to aid in the understanding of the theory.

E = the modulus of elasticity of the material.

h = the thickness of the plate.

H = the horizontal component of the stress in the wire.

$\bar{M}$ = the moment-sum, a measure of the bending of the plate.

N = a constant factor, equal to $\frac{Eh^3}{12(1-\nu)}$

p = the uniform load on the plate.

P = the concentrated load at any point x, y.

R = the tension in the wires.

S = the horizontal component of the membrane surface stresses.

w = the deflection of the membrane and web at any point, due to the original loading.

z = the deflection of the web at any point , due to the elastic (w) loading.

ζ = the deflection of the plate at any point, due to original loading.

κ = the ratio of λ_x to λ_y.

λ = the mesh width.

ν = Poisson's Ratio

ω = the angle between the wire and the horizontal.

The basic differential equation for flat plates as derived by Lagrange is;

$$\frac{\partial^4 \zeta}{\partial x^4} + 2\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \frac{\partial^4 \zeta}{\partial y^4} = \frac{P}{N} \text{ ------------------ } (1)$$

This equation is derived upon the principal assumptions, (a) that the material is homogeneous, isotropic, and perfectly elastic, (b) that the plate is medium thick (not so thin to act as a membrane, nor so thick that the distribution of stresses at the ends appreciably influences the results), (c) that the center of the plate deflects only up or down, and does not rotate, (d) that all flexure is within the elastic limit of the material.

Equation (1) can be resolved into simpler form by the use of the operator symbol

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Then,

$$\frac{\partial^4 \zeta}{\partial x^4} + 2\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \frac{\partial^4 \zeta}{\partial y^4} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \zeta = \nabla^2 \nabla^2 \zeta = \nabla^4 \zeta = \frac{P}{N} \text{ ---- } (2)$$

If we define a new function $\overline{M}$ as;

$$\overline{M} = -N\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) \text{ ------------------ } (3)$$

the equation of the elastic surface becomes

$$\nabla^2 \overline{M} = -P \text{ ------------------------ } (4)$$

The function $\overline{M}$ is a measure of the bending of the plate.

In Figure 1 a portion of the plate is represented, dx wide,

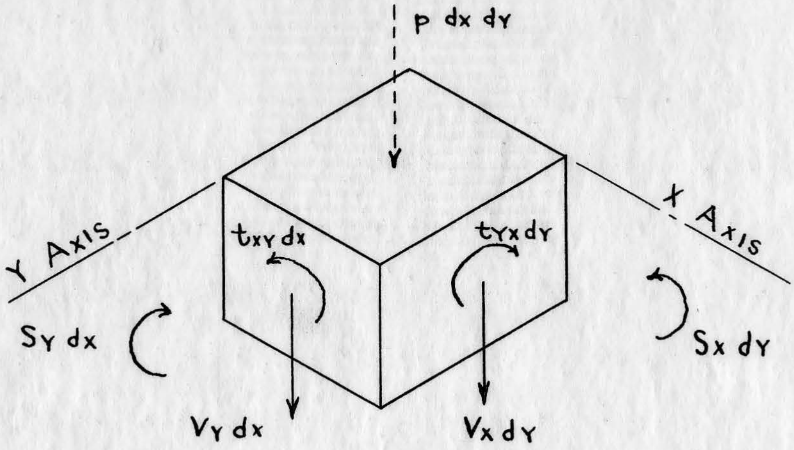


Figure 1 - Stresses on the Element of Plate

dy long, and h thick, the notations S_x and S_Y represent bending moments normal to the faces of the prism, V_x and V_Y represent vertical shearing stresses, and t_{XY} and t_{YX} represent torsional stresses on the faces of the prism. All of these are in terms of one unit of width of element. It can be shown that,[3]

$$S_x = -N\left(\frac{\partial^2 \zeta}{\partial x^2} + \nu \frac{\partial^2 \zeta}{\partial Y^2}\right) \text{ - - - - - - - - - - - - - - - - (5)}$$

$$S_Y = -N\left(\frac{\partial^2 \zeta}{\partial Y^2} + \nu \frac{\partial^2 \zeta}{\partial x^2}\right) \text{ - - - - - - - - - - - - - - - - (6)}$$

$$t_{xY} = -N(1-\nu)\frac{\partial^2 \zeta}{\partial x \partial Y} \text{ - - - - - - - - - - - - - - - - - (7)}$$

$$V_x = -N\frac{\partial}{\partial x}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial Y^2}\right) \text{ - - - - - - - - - - - - - - - (8)}$$

$$V_Y = -N\frac{\partial}{\partial Y}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial Y^2}\right) \text{ - - - - - - - - - - - - - - - (9)}$$

$$S_X + S_Y = -N(1-\nu)\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) = (1+\nu)\bar{M}$$

If we now consider a thin plate or a membrane, and if it is subjected to fiber stresses parallel to the surface only, it is described by the differential equation[7]

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{S} \quad \text{-----------------(10)}$$

where w is the deflection of the membrane at any point x,y; P is the intensity of loading, and S is the horizontal component of the surface stresses. From the conditions of static equilibrium, S is constant for the entire membrane. If S is made equal to unity, then comparing equation (10) with equation (4) (restated for comparison)

$$\nabla^2 \bar{M} = \frac{\partial^2 \bar{M}}{\partial x^2} + \frac{\partial^2 \bar{M}}{\partial y^2} = -P \quad \text{-----------------(4)}$$

we can state the following law: A membrane loaded with the same loading, P, and having a value of S = 1, forms a moment diagram for the elastic plate. Note that the "moment" in the above statement is really the moment-sum function $\bar{M}$, as defined before. It can also be shown that the following law is true: A membrane loaded with the elastic loads $P = \frac{\bar{M}}{N}$ and having a value of S = 1, forms a deflection diagram of the elastic plate.

The sum of the second derivative of equation (3) with respect to x, and ν times the second derivative with respect to y gives,

$$\frac{\partial^2 \bar{M}}{\partial x^2} + \nu \frac{\partial^2 \bar{M}}{\partial y^2} = -N\left[\frac{\partial^4 \zeta}{\partial x^4} + (1+\nu)\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \nu\frac{\partial^4 \zeta}{\partial y^4}\right]$$

Also, the sum of the second derivative of equation (3) with respect to y, and ν times the second derivative with respect to x gives,

$$\frac{\partial^2 \bar{M}}{\partial y^2} + \nu \frac{\partial^2 \bar{M}}{\partial x^2} = -N\left[\frac{\partial^4 \zeta}{\partial y^4} + (1+\nu)\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \nu \frac{\partial^4 \zeta}{\partial x^4}\right]$$

The sum of the second derivatives with respect to x and y of equations (5) and (6) gives,

$$\nabla^2 S_x = \frac{\partial^2 S_x}{\partial x^2} + \frac{\partial^2 S_x}{\partial y^2} = -N\left[\frac{\partial^4 \zeta}{\partial x^4} + (1+\nu)\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \nu \frac{\partial^4 \zeta}{\partial y^4}\right]$$

$$\nabla^2 S_y = \frac{\partial^2 S_y}{\partial x^2} + \frac{\partial^2 S_y}{\partial y^2} = -N\left[\frac{\partial^4 \zeta}{\partial y^4} + (1+\nu)\frac{\partial^4 \zeta}{\partial x^2 \partial y^2} + \nu \frac{\partial^4 \zeta}{\partial x^4}\right]$$

And if we define

$$\left.\begin{aligned} -\Pi_x &= \frac{\partial^2 \bar{M}}{\partial x^2} + \nu \frac{\partial^2 \bar{M}}{\partial y^2} \\ -\Pi_y &= \frac{\partial^2 \bar{M}}{\partial y^2} + \nu \frac{\partial^2 \bar{M}}{\partial x^2} \\ -\Pi_{xy} &= (1+\nu)\frac{\partial^2 \bar{M}}{\partial x \partial y} \end{aligned}\right\} \text{-------------------} (11)$$

and combine them with the previous equations we get

$$\left.\begin{aligned} \nabla^2 S_x &= -\Pi_x \\ \nabla^2 S_y &= -\Pi_y \\ \nabla^2 t_{xy} &= -\Pi_{xy} \end{aligned}\right\} \text{-------------------} (12)$$

From these last equations we can derive the law: The membrane carrying loads defined by equations (11) and having $S = 1$, forms a moment diagram for the S_x, S_Y, and t_{XY} moments of the elastic plate.

The shearing forces V_x and V_Y can be obtained from a simple transformation of equations (8) and (9)

$$V_x = -N \frac{\partial}{\partial x} \nabla^2 \zeta = \frac{\partial \bar{M}}{\partial x}$$

$$V_Y = -N \frac{\partial}{\partial Y} \nabla^2 \zeta = \frac{\partial \bar{M}}{\partial Y}$$

From the preceding results, it can be seen that the complete solution of the plate is dependent upon the two differential equations

$$\nabla^2 \bar{M} = -P \quad ; \quad \nabla^2 \zeta = -\frac{\bar{M}}{N}$$

In plates that are freely supported at the edges on unyielding supports, ζ and $\bar{M}$ are zero at these edges, and therefore, the first differential equation suffices to determine $\bar{M}$ for the plate, such plates are called "Statically Determinate". If a plate has its edges restrained both equations are necessary for its solution, and it is called "Statically Indeterminate". In the case of the statically determinate plates, the $\bar{M}$ diagram obtained from the membrane is a true moment diagram, but this is not true of the statically indeterminate plates.

The solution of the plate has been made dependent upon the solution of the membrane, and now the membrane will be replaced by the elastic web.

The elastic web is shown in Figure (2), and is composed of two sets of perfectly elastic wires crossing at right angles. The mesh width is uniform in each direction, λ_x in the direction of the x axis, and λ_y in the direction of the y axis. The nodes or intersection points of the web are designated as shown in the figure. The loading of the plate is replaced by equivalent concentrated loads at the nodal points. The separate wires have direct tensile stresses R_i, R_ℓ, R_m and R_n respectively for the wires running from k to i, l, m, and n. The angles ω_i, and ω_ℓ designate the angles that wires R_i and R_ℓ make with the x axis, and ω_m and ω_n designate the angles wires R_m and R_n make with the y axis.

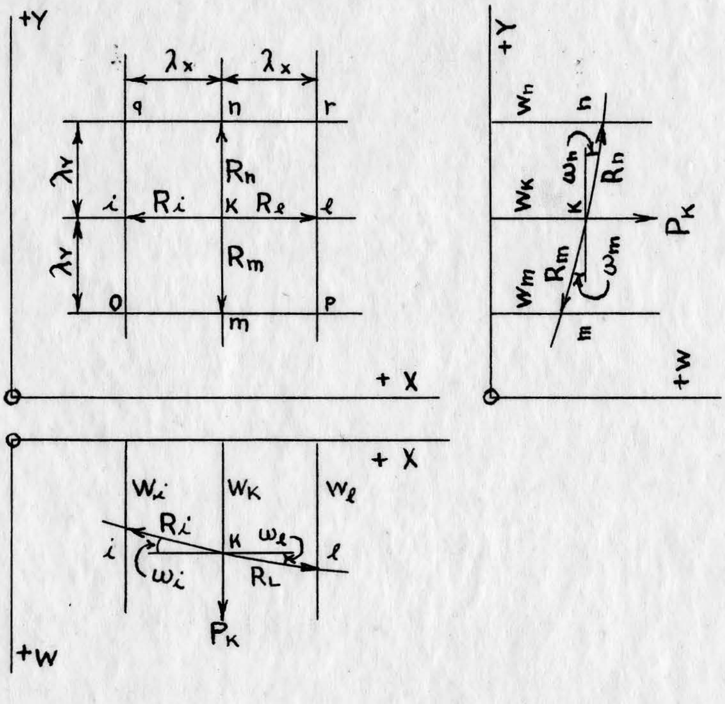


Figure 2. The Elastic Web with Rectangular Meshes

From conditions of static equilibrium at k in the three directions, the following equations are obtained:

$$R_i \cos \omega_i - R_l \cos \omega_l = 0$$

$$R_m \cos \omega_m - R_n \cos \omega_n = 0$$

$$P_K + (R_l \sin \omega_l - R_i \sin \omega_i) + (R_n \sin \omega_n - R_m \sin \omega_m) = 0$$

Call the horizontal components of the stresses in the wires running in the x and y directions H_X and H_Y respectively. Then the following relations are true:

$$R_i \cos \omega_i = R_l \cos \omega_l = H_X$$

$$R_m \cos \omega_m = R_n \cos \omega_n = H_Y$$

Substituting in the previous equation we get,

$$H_X(\tan \omega_i - \tan \omega_l) + H_Y(\tan \omega_m - \tan \omega_n) = P_K$$

The difference operator is defined as $(\Delta wk)x = w_l - w_K$, or the second difference of w_K is $(\Delta^2 w_K)x = - P.\lambda^2 = (w_K - w_i) - (w_l - w_K)$

By this definition we can get;

$$\lambda_X \tan \omega_l = w_l - w_i = (\Delta w_K)_X$$

$$\lambda_Y \tan \omega_n = w_n - w_m = (\Delta w_K)_Y$$

$$\lambda_X(\tan \omega_l - \tan \omega_i) = (w_l - w_K) - (w_K - w_i) = (\Delta^2 w_K)_X$$

$$\lambda_Y(\tan \omega_n - \tan \omega_m) = (w_n - w_K) - (w_K - w_m) = (\Delta^2 w_K)_Y$$

Then substituting in the previous equation,

$$\frac{H_X}{\lambda_X}(\Delta^2 w_K)_X + \frac{H_Y}{\lambda_Y}(\Delta^2 w_K)_Y = - P_K \qquad \text{——————} (10)$$

We now relate this to the membrane by noting that from the definition for S given on page 10.

$$H_Y = \lambda_Y S_X , \quad H_Y = \lambda_X S_Y$$

Also, $P_K = p_K \lambda_X \lambda_Y$

And consequently,

$$\frac{S_x}{\lambda_x^2}(\Delta^2 w_K)_x + \frac{S_Y}{\lambda_Y^2}(\Delta^2 w_K)_Y = -p_K \text{ - - - - - - - - - - - - - - - - } (11)$$

If $H_x = H_Y = H$, or $S_x = S_Y = S$

$$\frac{1}{\lambda_x}(\Delta^2 w_K)_x + \frac{1}{\lambda_Y}(\Delta^2 w_K)_Y = -\frac{P_K}{H} \text{ - - - - - - - - - - - - - - - - } (12)$$

$$\frac{(\Delta^2 w_K)_x}{\lambda_x^2} + \frac{(\Delta^2 w_K)_Y}{\lambda_Y^2} = -\frac{p_K}{S} \text{ - - - - - - - - - - - - - - - - - } (13)$$

If $\lambda_x = \lambda_Y = \lambda$ we obtain in the simple formula,

$$(\Delta^2 w_K)_x + (\Delta^2 w_K)_Y = -P_K\frac{\lambda}{H} = -p_K\frac{\lambda^2}{S}$$

This can also be written,

$$4w_K - (w_i + w_\ell + w_m + w_n) = \frac{\lambda P_K}{H} = \frac{\lambda^2 p_K}{S} \text{ - - - - - - - - - - - - } (14)$$

This is the basic equation used in the solution of square plates, freely supported at the edges.

It may be noted that if the mesh widths are decreased indefinitely equation (13) in the limit becomes the basic equation of the membrane,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{S}$$

For rectangular plates, whose ratio of longer side to the shorter side is

$$\frac{\lambda_x}{\lambda_Y} = K$$

equation (13) becomes

$$(\Delta^2 w_K)_x + K^2(\Delta^2 w_K)_Y = -p_K\frac{\lambda_x^2}{S}$$

or in the expanded form

$$2w_K(1+K^2)-(w_i+w_\ell)-K^2(w_m+w_n)=p_K\frac{\lambda_x^2}{S}=K^2p_K\frac{\lambda_Y^2}{S}$$

This is the basic equation for freely supported rectangular plates.

To determine the deflections the same equations are used, only the load p_K becomes the "elastic load" $\frac{\bar{M}}{N}$. The value of S can be made unity, and then the values of w will numerically equal the moment-sum at the corresponding points. The values of ζ will be numerically equal to the values of $\frac{Z}{N}$ at the corresponding points. Note that in applying equations (14) and (15) in finding deflections, w is replaced by z. Having the values of $\bar{M}$ and ζ we can find the stress-moments, torsions and shears directly from them.

Loading our web now with the elastic weight $p_K = w_K$ and taking S equal to unity, using equation (13),

$$\frac{Z_i-2Z_K+Z_\ell}{\lambda_x^2}+\frac{Z_m-2Z_K+Z_n}{\lambda_Y^2}=\frac{1}{\lambda_x^2}(\Delta^2Z_K)_x+\frac{1}{\lambda_Y^2}(\Delta^2Z_K)_Y=-w_K=-\bar{M}_K \quad \text{--(16)}$$

We now evaluate the various partial derivatives necessary for determining the stress-moments, torsions and shears.

$$\left.\begin{aligned}
\left(\frac{\partial Z}{\partial x_K}\right)&=\frac{Z_\ell-Z_i}{2\lambda_x}=\frac{(\Delta Z_K)_x}{\lambda_x}\\
\left(\frac{\partial Z}{\partial Y_K}\right)&=\frac{Z_n-Z_m}{2\lambda_Y}=\frac{(\Delta Z_K)_Y}{\lambda_Y}\\
\left(\frac{\partial^2 Z}{\partial x_K^2}\right)&=\frac{Z_i-2Z_K+Z_\ell}{\lambda_x^2}=\frac{1}{\lambda_x^2}(\Delta^2Z_K)_x\\
\left(\frac{\partial^2 Z}{\partial Y_K^2}\right)&=\frac{Z_m-2Z_K+Z_n}{\lambda_Y^2}=\frac{1}{\lambda_Y^2}(\Delta^2Z_K)_Y\\
\left(\frac{\partial^2 Z}{\partial x\,\partial Y_K}\right)&=\frac{(Z_o+Z_R)-(Z_p+Z_q)}{4\lambda_x\lambda_Y}=\frac{(\Delta^2Z_K)_{xY}}{\lambda_x\lambda_Y}
\end{aligned}\right\} \text{----------(17)}$$

Substituting these values in equations (5), we get,

$$S_x = -\left(\frac{\partial^2 Z}{\partial x^2} + \nu\frac{\partial^2 Z}{\partial y^2}\right) = \frac{2Z_k - Z_i - Z_\ell}{\lambda^2_x} + \nu\,\frac{2Z_k - Z_m - Z_n}{\lambda^2_Y}$$

$$S_Y = -\left(\frac{\partial^2 Z}{\partial Y^2} + \nu\frac{\partial^2 Z}{\partial x^2}\right) = \frac{2Z_k - Z_m - Z_n}{\lambda^2_Y} + \nu\,\frac{2Z_k - Z_i - Z_\ell}{\lambda^2_x}$$

$$t_{xY} = -(1-\nu)\frac{\partial^2 Z}{\partial x\,\partial Y} = (1-\nu)\left[\frac{(Z_p + Z_q) - (Z_o + Z_r)}{4\lambda_x\lambda_Y}\right]$$

$$v_x = \frac{\partial w}{\partial x} = \frac{1}{2\lambda_x}(W_\ell - W_i)$$

$$v_Y = \frac{\partial w}{\partial Y} = \frac{1}{2\lambda_Y}(w_n - w_m)$$

Substituting the values of ς and $\bar{M}$ for z and w, we get,

$$\left.\begin{aligned}
\frac{\lambda^2_x}{N} S_x &= (2\varsigma_K - \varsigma_i - \varsigma_\ell) + \nu K(2\varsigma_K - \varsigma_m - \varsigma_n) \\
\frac{\lambda^2_x}{N} S_Y &= K(2\varsigma_K - \varsigma_m - \varsigma_n) + \nu(2\varsigma_K - \varsigma_i - \varsigma_\ell) \\
\frac{\lambda^2_x}{N} t_{xY} &= \frac{1-\nu}{4} K\left[(\varsigma_p + \varsigma_q) - (\varsigma_o + \varsigma_r)\right] \\
2\lambda_x \lambda_x &= \bar{M}_\ell - \bar{M}_i \\
2\lambda_x \lambda_Y &= K(\bar{M}_n - \bar{M}_m)
\end{aligned}\right\} \text{-----------} (18)$$

The value of ν that is used in the above formulas influences the resulting values to a considerable extent. For concrete, at the usual values of working stresses, ν is approximately 0.20. However, as the stresses approach the ultimate, ν decreases for concrete in tension[8] and as the most unfavorable case would be ν equal to zero, that value may be considered as the proper one to be used in determining stresses, etc., in plates approaching a condition of rupture. In determining the value of working stress to be used in the plate, if ν is taken as zero, no notice need be taken of the fact that the stresses in the plate increase toward the ultimate strength of the materials, the moments do not change proportionately to the loads, because the stresses will have been calculated upon the basis of the most unfavorable condition of the plate. If a value of $\nu = 0.20$ is used, the working stress may be taken less conservatively to allow for the less rapid increase of stresses that occur as the loads increase.

THE SOLUTION OF FLAT PLATES

BY THE

THEORY OF THE DIFFERENCE METHOD

A list of the characters and symbols that will be used in working the problems will be given here for easy reference to aid in the understanding of the problem.

E = the modulus of elasticity of the material.

f = the fiber stresses in the plate at the points indicated.

h = the thickness of the plate.

L = the length of a side of the plate.

N = a constant factor, equal to $\frac{Eh^3}{12(1-\nu)}$

p = the uniform load on the plate.

S_x = the bending moment in the plate, in the x direction.

S_Y = the bending moment in the plate, in the y direction.

w = the deflection of the web due to the original load.

z = the deflection of the web due to the elastic load.

z_o = z values taken from the problem of the pinned plate, and used in the problem of the fixed plate.

ζ = the deflection of the plate.

ζ_o = the deflection taken from the problem of the pinned plate, and used in the problem of the fixed plate.

ζ' = the superimposed deflections.

λ = the length of the web.

ν = Poisson's Ratio.

A SQUARE PLATE SIMPLY SUPPORTED ON ALL EDGES AND SUPPORTING A UNIFORM LOAD

$L = 14$ in.

$\lambda = \frac{L}{6} = 2.3333$ in.

$\lambda^2 = 5.4444$ sq. in.

$p = 420$ lb./sq. in.

Poisson's Ratio = .3

$N = 343{,}407$ in. lb.

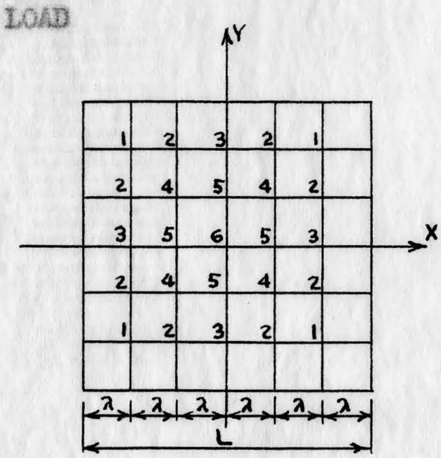


Figure 3, The Plate and Web

Applying equation (14) to each point 1 through 6 respectively we get; Equation (14) is $4w_K - (w_i + w_\ell + w_m + w_n) = \frac{p\lambda^2}{S}$

Point (1) $4w_1 - 2w_2 = p\lambda^2$

" (2) $-w_1 + 4w_2 - w_3 - w_4 = p\lambda^2$

" (3) $-2w_2 + 4w_3 - w_5 = p\lambda^2$

" (4) $-2w_2 + 4w_4 - 2w_5 = p\lambda^2$

" (5) $-w_3 - 2w_4 + 4w_5 - w_6 = p\lambda^2$

" (6) $-4w_5 + 4w_6 = p\lambda^2$

These equations solved simultaneously give;

$w_1 = 0.951922\, p\lambda^2 = 0.026442\, pL^2$

$w_2 = 1.403845\, p\lambda^2 = 0.038996\, pL^2$

$w_3 = 1.538460\, p\lambda^2 = 0.042735\, pL^2$

$w_4 = 2.124999\, p\lambda^2 = 0.059028\, pL^2$

$w_5 = 2.346153\, p\lambda^2 = 0.065171\, pL^2$

$w_6 = 2.596153\, p\lambda^2 = 0.072115\, pL^2$

Loading the plate with these Elastic Loads, and again applying equation (14) to each point 1 through 6 respectively, we get;

Point (1) $4z_1 - 2z_2 = w_1\lambda^2 = 0.0007345\ pL^4$

" (2) $-z_1 + 4z_2 - z_3 - z_4 = w_2\lambda^2 = 0.0010832\ pL^4$

" (3) $-2z_2 + 4z_3 - z_5 = w_3\lambda^2 = 0.0011871\ pL^4$

" (4) $-2z_2 + 4z_4 - 2z_5 = w_4\lambda^2 = 0.0016397\ pL^4$

" (5) $-z_3 - 2z_4 + 4z_5 - z_6 = w_5\lambda^2 = 0.0018103\ pL^4$

" (6) $-4z_5 + 4z_6 = w_6\lambda^2 = 0.0020032\ pL^4$

These equations solved simultaneously give;

$z_1 = 0.001109\ pL^4 = 17,893$

$z_2 = 0.001853\ pL^4 = 29,898$

$z_3 = 0.002110\ pL^4 = 34,044$

$z_4 = 0.003110\ pL^4 = 50,179$

$z_5 = 0.003547\ pL^4 = 57,230$

$z_6 = 0.004048\ pL^4 = 65,313$

From page 16 we can see that the deflection equals $\frac{Z}{N}$, and is;

$\zeta_1 = 0.052$ in.

$\zeta_2 = 0.087$ in.

$\zeta_3 = 0.099$ in.

$\zeta_4 = 0.146$ in.

$\zeta_5 = 0.167$ in.

$\zeta_6 = 0.190$ in.

The axial moments as arrived at from equations (18) are;

Equations (18) are,

$$\frac{\lambda_x^2}{N} S_x = (2\zeta_k - \zeta_i - \zeta_l) + \nu K(2\zeta_k - \zeta_m - \zeta_n)$$

$$\frac{\lambda_x^2}{N} S_Y = K(2\zeta_k - \zeta_m - \zeta_n) + \nu(2\zeta_k - \zeta_i - \zeta_l)$$

$Sx_1 = Sy_1 = 1{,}081 + 324 = 1{,}405$ in. lb.

$Sx_2 = 1{,}443 + 533 = 1{,}976$ in. lb.

$Sy_2 = 1{,}776 + 433 = 2{,}199$ in. lb.

$Sx_3 = 1{,}523 + 598 = 2{,}121$ in. lb.

$Sy_3 = 1{,}994 + 457 = 2{,}451$ in. lb.

$Sx_4 = Sy_4 = 2{,}430 + 729 = 3{,}159$ in. lb.

$Sx_5 = 2{,}590 + 832 = 3{,}422$ in. lb.

$Sy_5 = 2{,}774 + 777 = 3{,}551$ in. lb.

$Sx_6 = Sy_6 = 2{,}969 + 891 = 3{,}860$ in. lb.

From the equation $f = \frac{Mc}{I}$ we may get the surface stresses which are;

$fx_1 = fy_1 = 33{,}720$ psi

$fx_2 = 47{,}352$ psi

$fy_2 = 52{,}776$ psi

$fx_3 = 50{,}904$ psi

$fy_3 = 58{,}824$ psi

$fx_4 = fy_4 = 75{,}816$ psi

$fx_5 = 82{,}128$ psi

$fy_5 = 85{,}224$ psi

$fx_6 = fy_6 = 92{,}640$ psi

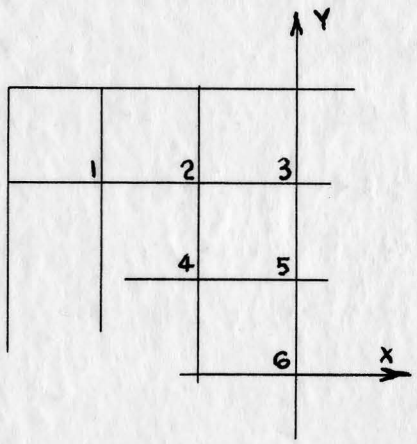


Figure 4

The Points at which these Stresses apply

A SQUARE FLAT PLATE SUPPORTING A UNIFORM LOAD, AND HAVING FIXED EDGES

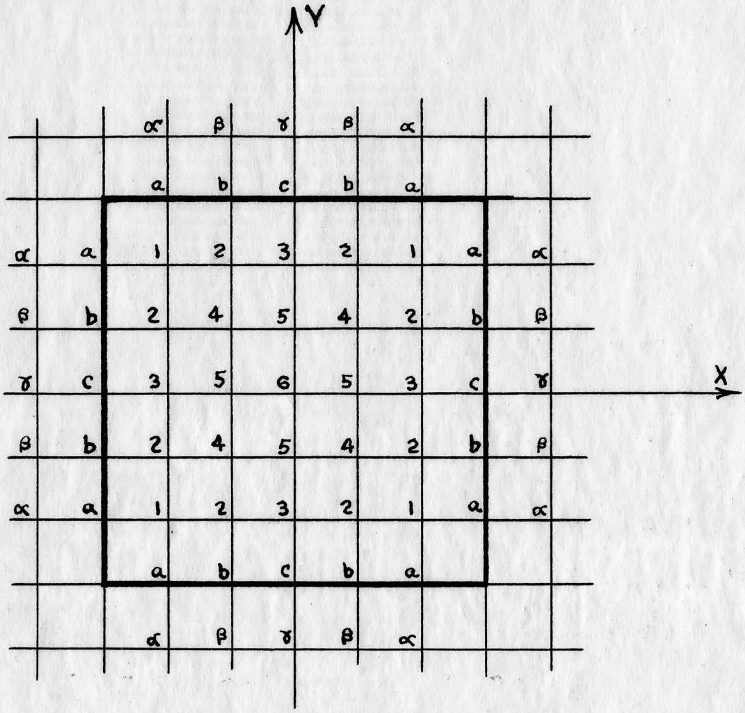


Figure 5 The Plate and Web

In the solution of this problem the deflection of the plate subjected only to moments along the edges, of sufficient magnitude to prevent the edges from rotating when subjected to a uniform load, is superimposed upon the deflection of the plate caused by the uniform load. The solution will be similar to the previous solution, except that the first elastic web for the correction will be loaded with zero loads and the results in terms of the boundary moments carried through the following work. The boundary points a, b, and c will also be included, and values found for them.

The first step will be to apply equation (14) to each point 1 through 6 respectively. We get;

Point (1) $4w_1 - 2w_2 \qquad - 2w_a = 0$

" (2) $-w_1 + 4w_2 - w_3 - w_4 \qquad - w_b = 0$

" (3) $-2w_2 + 4w_3 - \quad - w_5 - w_c = 0$

" (4) $-2w_2 + 4w_4 - 2w_5 = 0$

" (5) $-w_3 - 2w_4 + 4w_5 \qquad - w_6 = 0$

" (6) $-4w_5 + 4w_6 = 0$

These equations solved simultaneously give;

$$w_1 = 0.63462\ w_a + 0.26923\ w_b + 0.09616\ w_c$$

$$w_2 = 0.26923\ w_a + 0.53846\ w_b + 0.19231\ w_c$$

$$w_3 = 0.19231\ w_a + 0.38462\ w_b + 0.42308\ w_c$$

$$w_4 = 0.25000\ w_a + 0.50000\ w_b + 0.25000\ w_c$$

$$w_5 = 0.23077\ w_a + 0.46154\ w_b + 0.30769\ w_c$$

$$w_6 = 0.23077\ w_a + 0.46154\ w_b + 0.30769\ w_c$$

Loading the plate with these elastic loads, and applying equation (14) to each point 1 through 6 respectively we get;

Point (1) $4z_1 - 2z_2 = 0.63462w_a + 0.26923w_b + 0.09616w_c$

" (2) $-z_1 + 4z_2 - z_3 - z_4 = 0.26923w_a + 0.53846w_b + 0.19231w_c$

" (3) $-2z_2 + 4z_3 - z_5 = 0.19231w_a + 0.38462w_b + 0.42308w_c$

" (4) $-2z_2 + 4z_4 - 2z_5 = 0.25000w_a + 0.50000w_b + 0.25009w_c$

" (5) $-z_3 - 2z_4 + 4z_5 - z_6 = 0.23077w_a + 0.46154w_b + 0.30769w_c$

" (6) $-4z_5 + 4z_6 = 0.23077w_a + 0.46154w_b + 0.30769w_c$

These equations solved simultaneously give;

$$\left.\begin{aligned}
z_1 &= 0.35688\,w_a + 0.39645\,w_b + 0.19859\,w_c \\
z_2 &= 0.39645\,w_a + 0.65828\,w_b + 0.34911\,w_c \\
z_3 &= 0.39719\,w_a + 0.69822\,w_b + 0.44305\,w_c \\
z_4 &= 0.56250\,w_a + 1.00000\,w_b + 0.56250\,w_c \\
z_5 &= 0.60355\,w_a + 1.09172\,w_b + 0.65089\,w_c \\
z_6 &= 0.66124\,w_a + 1.20710\,w_b + 0.72781\,w_c
\end{aligned}\right\} \text{------} (19)$$

Apply equation (14) to points a, b, and c on the boundary, we get;

Point (a) $-z_\alpha + 4z_a - z_b - z_1 = w_a$

" (b) $-z_\beta - z_a + 4z_b - z_c - z_2 = w_b$

" (c) $-z_\gamma - 2z_b + 4z_c - z_3 = w_c$

But as the deflection on the boundary is equal to zero, z_a, z_b, and z_c must be equal to zero, and we have left;

Point (a) $-z_\alpha - z_1 = w_a$

" (b) $-z_\beta - z_2 = w_b$

" (c) $-z_\gamma - z_3 = w_c$

Substituting in the values of z_1, z_2, and z_3 from equation (19) we get

$$\left.\begin{aligned}
&\text{Point (a)} \quad z_\alpha = -1.35688\,w_a - 0.39645\,w_b - 0.19859\,w_c \\
&\text{" (b)} \quad z_\beta = -0.39645\,w_a - 1.65828\,w_b - 0.34911\,w_c \\
&\text{" (c)} \quad z_\gamma = -0.39719\,w_a - 0.69822\,w_b - 1.44305\,w_c
\end{aligned}\right\} \text{---} (20)$$

From the solution for the pinned plate we see that the z values at points 1, 2, and 3 are;

Point (1) $z_{01} = 17,893$ or $3,286\,\lambda^2$

" (2) $z_{02} = 29,898$ or $5,491\,\lambda^2$

" (3) $z_{03} = 34,044$ or $6,253\,\lambda^2$

In working with afixed condition at the boundary we must assume that the plate is continuous over the rigid support, and then that the deflections are symmetrical about the support. If we assume this to be true we may say that the deflections $z_{o\alpha}$, $z_{o\beta}$, and $z_{o\gamma}$ are equal to $-z_{o1}$, $-z_{o2}$, and $-z_{o3}$ respectively.

We may now say that

$$z_{o1} = (z_\alpha - z_1) \div 2 = 3,286\ \lambda^2$$

$$z_{o2} = (z_\beta - z_2) \div 2 = 5,491\ \lambda^2$$

$$z_{o3} = (z_\gamma - z_3) \div 2 = 6,253\ \lambda^2$$

Then from equations (19), (20), and (21) we have all of the conditions necessary to find the moment-sum at the boundary.

Point (a) $0.85688\, w_a + 0.39645\, w_b + 0.19859\, w_c = -3,286\ \lambda^2$

" (b) $0.39645\, w_a + 1.15828\, w_b + 0.34911\, w_c = -5,491\ \lambda^2$

" (c) $0.39719\, w_a + 0.69822\, w_b + 0.94305\, w_c = -6,253\ \lambda^2$

These equations solved simultaneously give;

$$w_a = -1,546.22\ \lambda^2$$

$$w_b = -3,101.29\ \lambda^2$$

$$w_c = -3,683.30\ \lambda^2$$

If these values are now substituted into equations (19) we will get the superimposed z values which are;

$$z_1 = -2{,}512.79\,\lambda^2 = -13{,}681$$

$$z_2 = -3{,}940.29\,\lambda^2 = -21{,}452$$

$$z_3 = -4{,}442.41\,\lambda^2 = -24{,}186$$

$$z_4 = -6{,}042.90\,\lambda^2 = -32{,}900$$

$$z_5 = -6{,}716.38\,\lambda^2 = -36{,}567$$

$$z_6 = -7{,}446.73\,\lambda^2 = -40{,}543$$

From these z values we may get the superimposed deflections

$\zeta_1' = -0.0398$ in.

$\zeta_2' = -0.0625$ in.

$\zeta_3' = -0.0704$ in.

$\zeta_4' = -0.0958$ in.

$\zeta_5' = -0.1065$ in.

$\zeta_6' = -0.1181$ in.

Now if we superimpose these deflections upon the deflections for the plate with the pinned edges we will have the deflection for the plate with fixed edges.

From Pinned Plate	Superimposed Deflection	Final Deflection
$\zeta_{01} = 0.052$ in.	$\zeta_1' = -0.0398$ in.	$\zeta_1 = 0.0122$ in.
$\zeta_{02} = 0.087$ in.	$\zeta_2' = -0.0625$ in.	$\zeta_2 = 0.0245$ in.
$\zeta_{03} = 0.099$ in.	$\zeta_3' = -0.0704$ in.	$\zeta_3 = 0.0286$ in.
$\zeta_{04} = 0.146$ in.	$\zeta_4' = -0.0958$ in.	$\zeta_4 = 0.0502$ in.
$\zeta_{05} = 0.167$ in.	$\zeta_5' = -0.1065$ in.	$\zeta_5 = 0.0605$ in.
$\zeta_{06} = 0.190$ in.	$\zeta_6' = -0.1181$ in.	$\zeta_6 = 0.0719$ in.

From these deflections we may find the moments and stresses by the same method as in the previous problem. The moments may be found by equations (18), and are;

$Sx_a = -\ 462$ in. lb.

$Sy_a = -1,539$ in. lb.

$Sx_b = -\ 927$ in. lb.

$Sy_b = -3,091$ in. lb.

$Sx_c = -1,082$ in. lb.

$Sy_c = -3,608$ in. lb.

$Sx_1 = Sy_1 = -\ 82$ in. lb.

$Sx_2 = +\ 494$ in. lb.

$Sy_2 = +\ 79$ in. lb.

$Sx_3 = +\ 455$ in. lb.

$Sy_3 = -\ 53$ in. lb.

$Sx_4 = Sy_4 = +\ 1,262$ in. lb.

$Sx_5 = +\ 1,687$ in. lb.

$Sy_5 = +\ 1,683$ in. lb.

$Sx_6 = Sy_6 = +\ 1,869$ in. lb.

From these moments we may find the stresses in the plate by using $f = \frac{Mc}{I}$ which are;

$f_{x_a} = -11{,}088$ psi

$f_{y_a} = -36{,}936$ psi

$f_{x_b} = -22{,}248$ psi

$f_{y_b} = -74{,}184$ psi

$f_{x_c} = -25{,}968$ psi

$f_{y_c} = -86{,}592$ psi

$f_{x_1} = f_{y_1} = -1{,}968$ psi

$f_{x_2} = +11{,}856$ psi

$f_{y_2} = +1{,}896$ psi

$f_{x_3} = +10{,}920$ psi

$f_{y_3} = -1{,}272$ psi

$f_{x_4} = f_{y_4} = +30{,}288$ psi

$f_{x_5} = +40{,}488$ psi

$f_{y_5} = +40{,}392$ psi

$f_{x_6} = f_{y_6} = +44{,}856$ psi

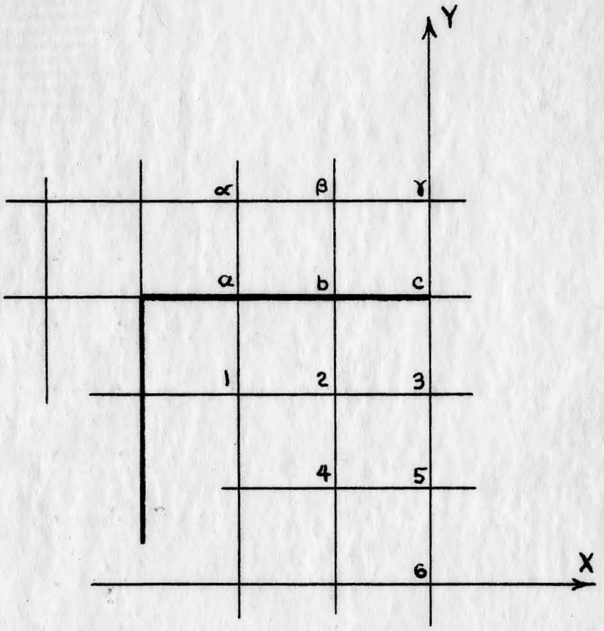


Figure 6 To Illustrate at Which Points These Values Are Valid

A COMPARISON OF THE THEORETICAL

AND

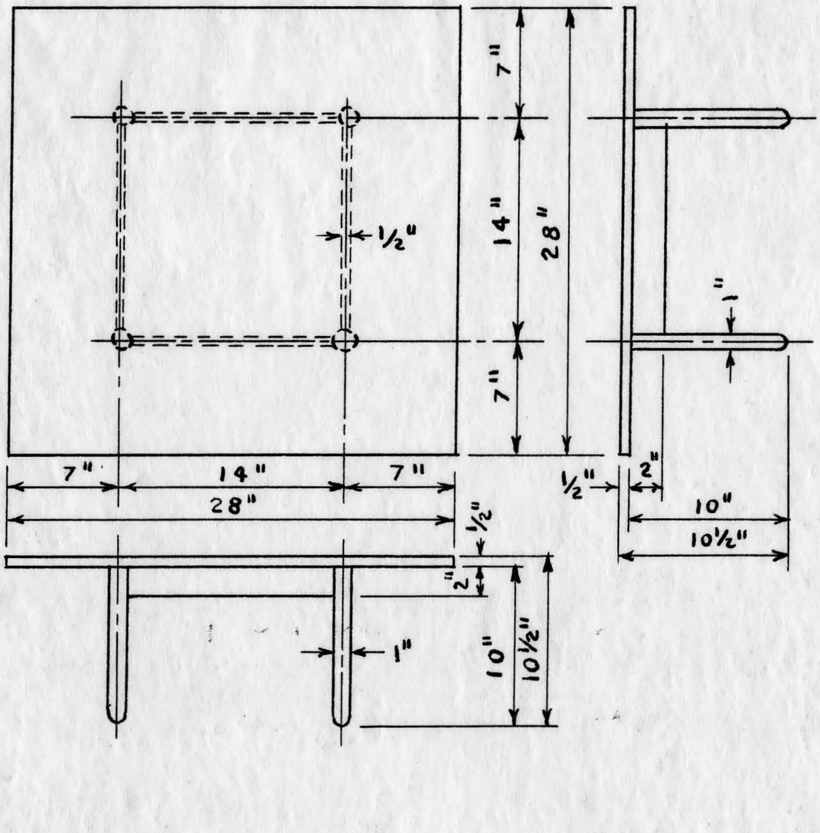
EXPERIMENTAL SOLUTION OF FLAT PLATES

THE EXPERIMENTAL STRESSES IN A FLAT PLATE

Taken from the thesis "THE EFFECT OF FLOOR SLABS WITH SPANDRELS ON THE MOMENT IN COLUMNS DUE TO VERTICAL LOADS" by John W. Flemer[1]

The plate that was used in the experiment was a steel plate twenty eight inches square, and one half inch thick. This plate was welded to four steel columns, one inch in diameter and ten inches long, placed fourteen inches on centers. Between the columns beams, two inches by one half inch, were welded to the plate and columns. The columns had a pinned condition at their base. The model was constructed in this manner to simulate a bay in an ordinary building frame.

Figure 7 The Model



The plate between the columns was then divided into six sections in each direction, or into thirty six small squares in all. At the intersection of the squares SR - 4 strain gages, and strain rosettes were attached. The location of the gages and rosettes are shown in Figure 8 . At points "c" a SR - 4 strain gage was placed on the bottom side of the beam. At points "a", "b", "c", and "d" strain rosettes were attached to the top of the plate. At points "1", "2", "3", "4", "5", and "6" strain rosettes were attached to the bottom of the plate. All of the strains were obtained from these gages, and converted into stresses to give the stress curves which are Figures 14 and 15 .

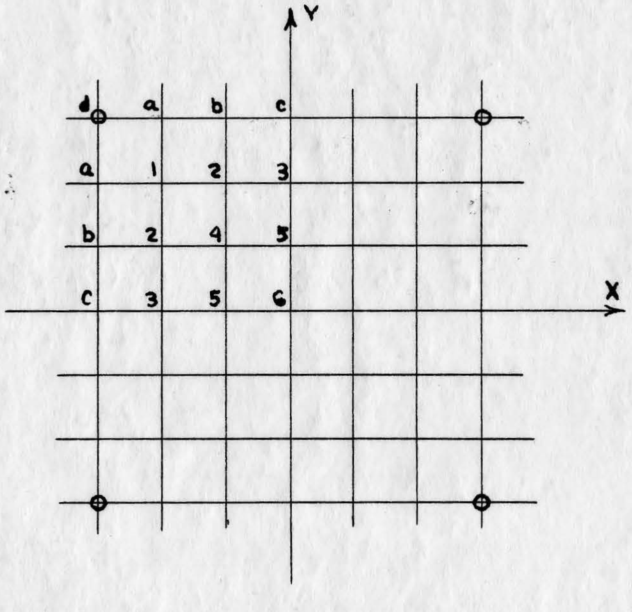


Figure 8 The Measuring Grid

For the loading, the plate was redivided between the columns into four sections in each direction, or into sixteen squares in all. The loading was done with a bridge affair that enabled it to be loaded with identical loads at two points at once. This type of gang loading was done throughout with the exception of the center point load which was by itself. The loading was done in this manner, loaded in corresponding opposite points on the plate, to prevent the plate from lifting one of the columns off the base when it was loaded in one quadrant, and thus producing undesirable stresses in the plate.

The loading grid, and the points of application are shown in Figure 9. Loads of five thousand five hundred pounds were placed at points 1 and 21, 2 and 20, 3 and 19, 4 and 18, 5 and 17, 6 and 16, 7 and 15, 8 and 14, 9 and 13, and 10 and 12 simultaneously, and a single load of five thousand five hundred pounds was placed at point 11. From each of these groups of loads the strain was recorded, and for the effect of a uniformly distributed load of five thousand five hundred pounds at each of these points the algebraic sum of the strains was used. From these strains the stresses were obtained, and the results are shown in Figures 14 and 15.

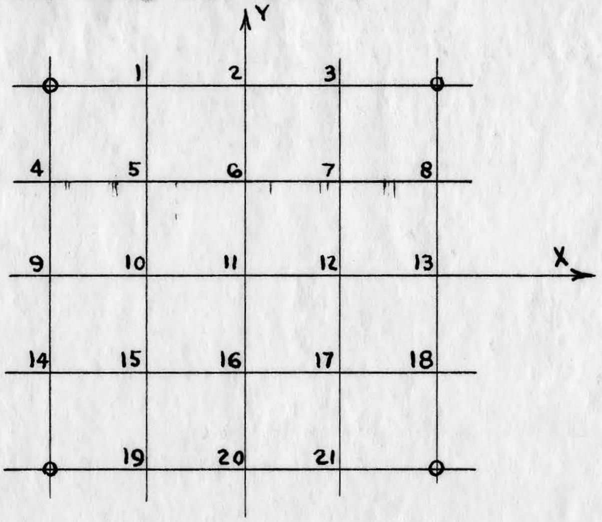


Figure 9 The Loading Grid

For the use in this thesis only the stresses in the x and y directions for the uniform load were used.

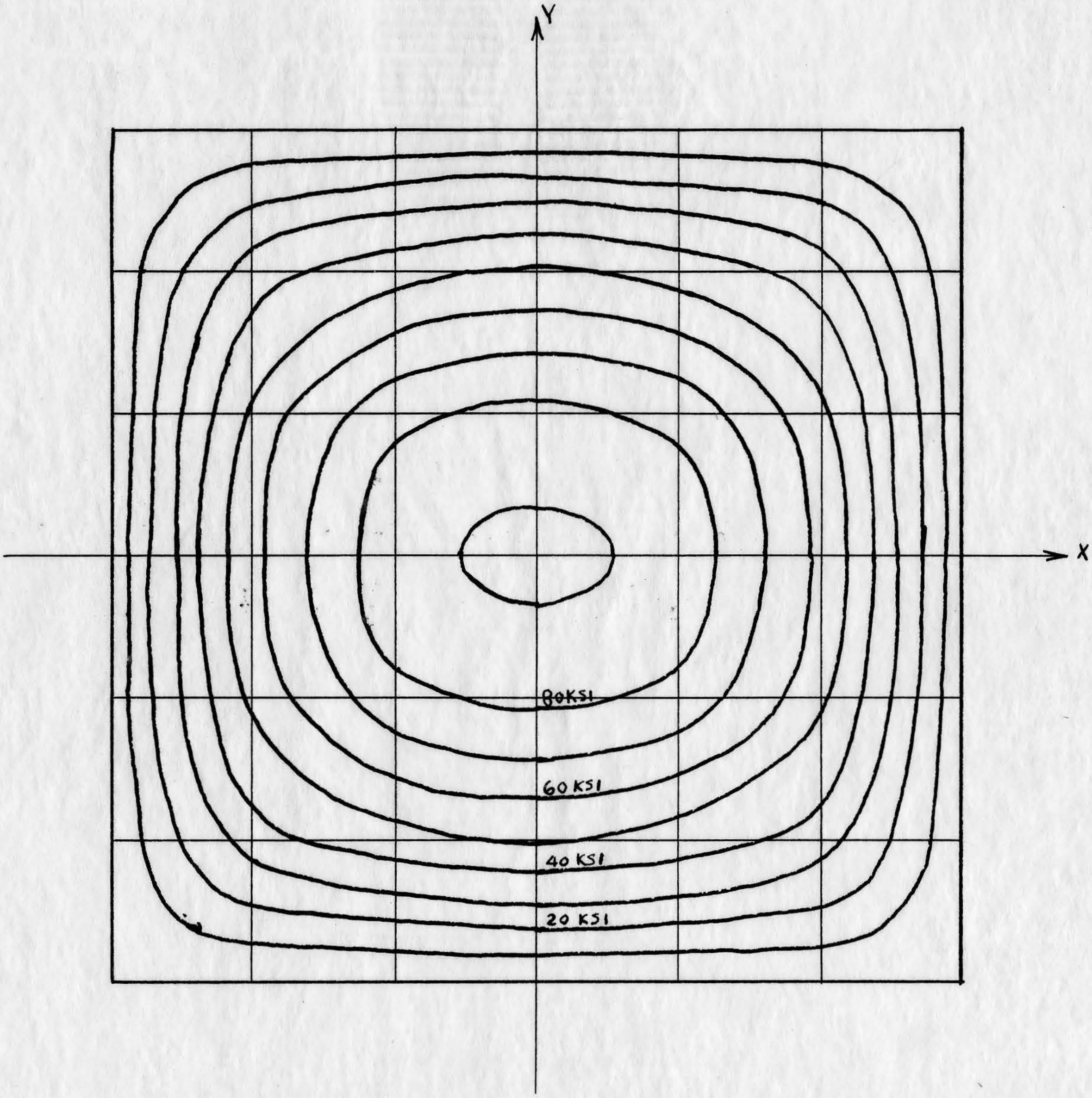
STRESS CURVES

FOR

THEORETICAL AND EXPERIMENTAL SOLUTIONS

OF THE

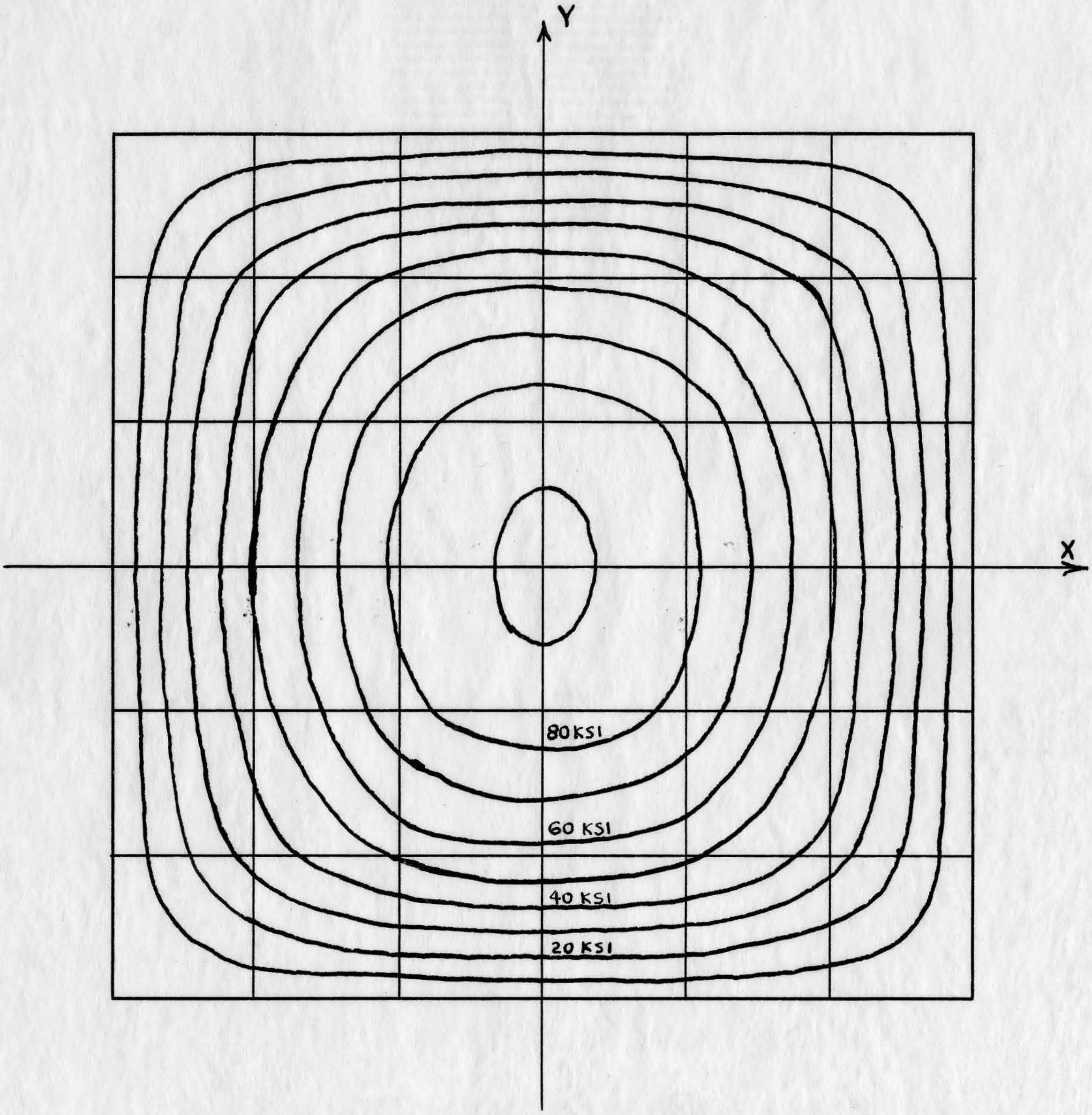
FLAT PLATE



STRESS CURVES

For Stresses in the X Direction for Pinned Plate

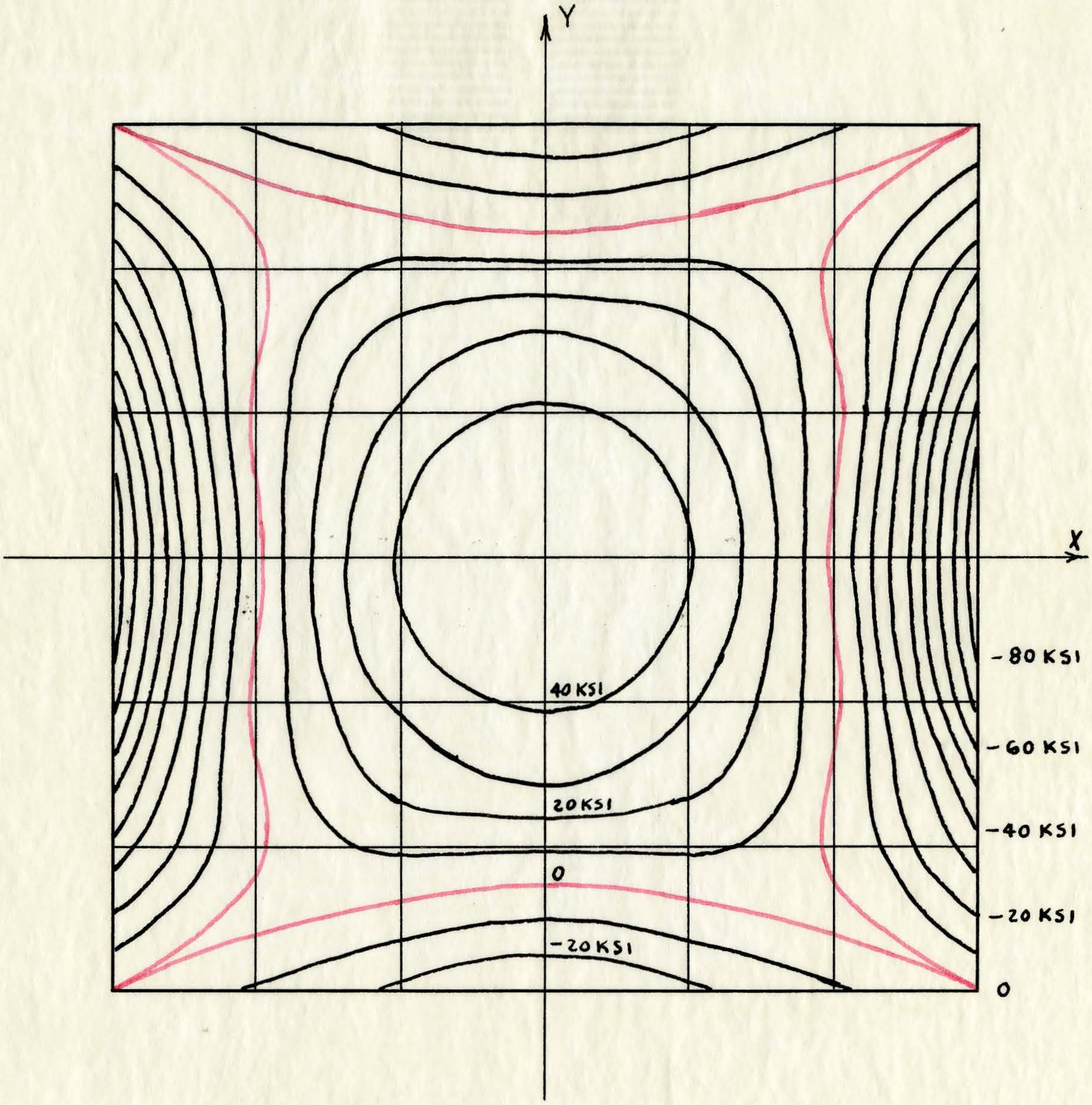
Figure 10



STRESS CURVES

For Stresses in the Y Direction for Pinned Plate

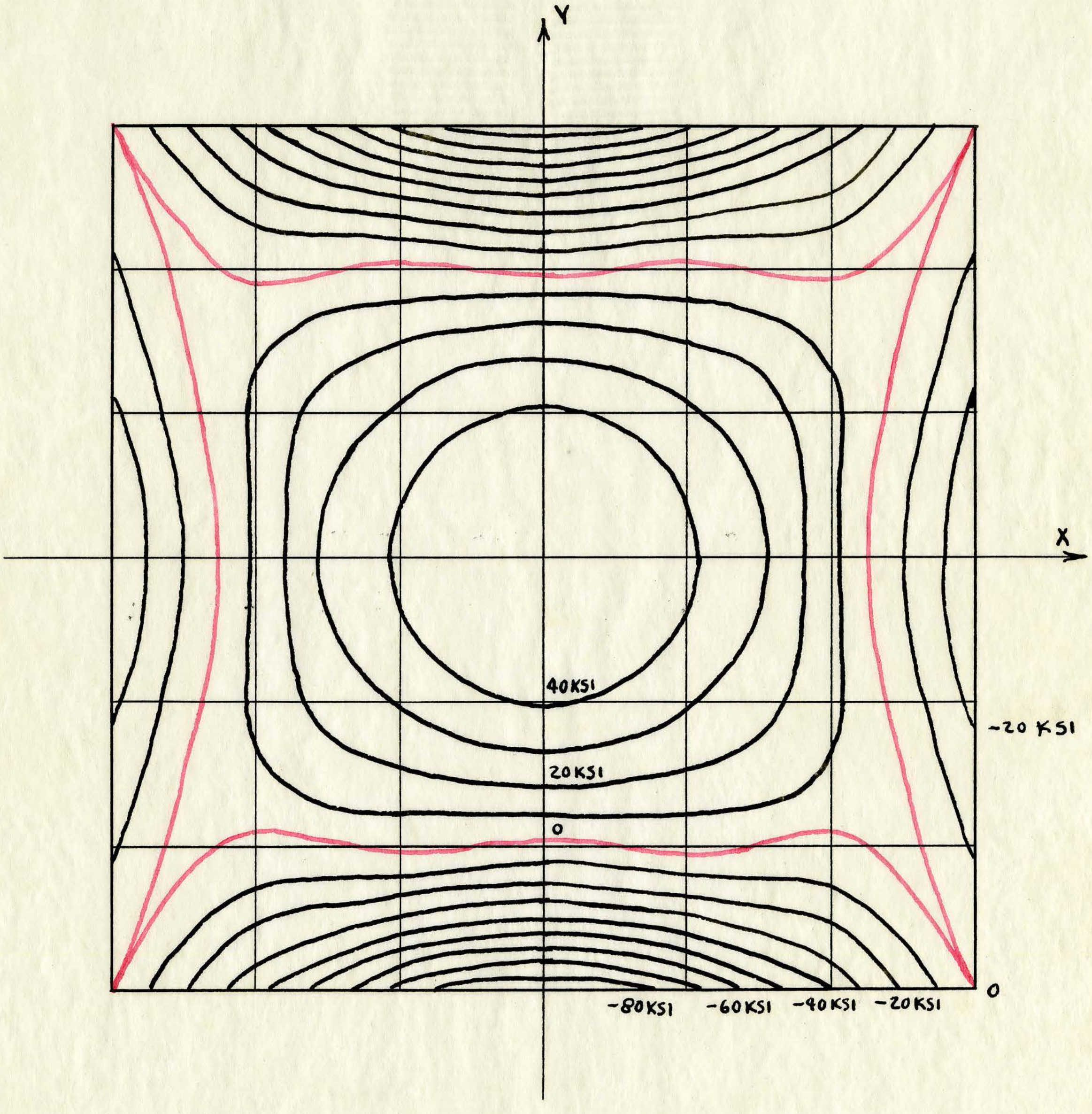
Figure 11



STRESS CURVES

For Stresses in the X Direction for Fixed Plate

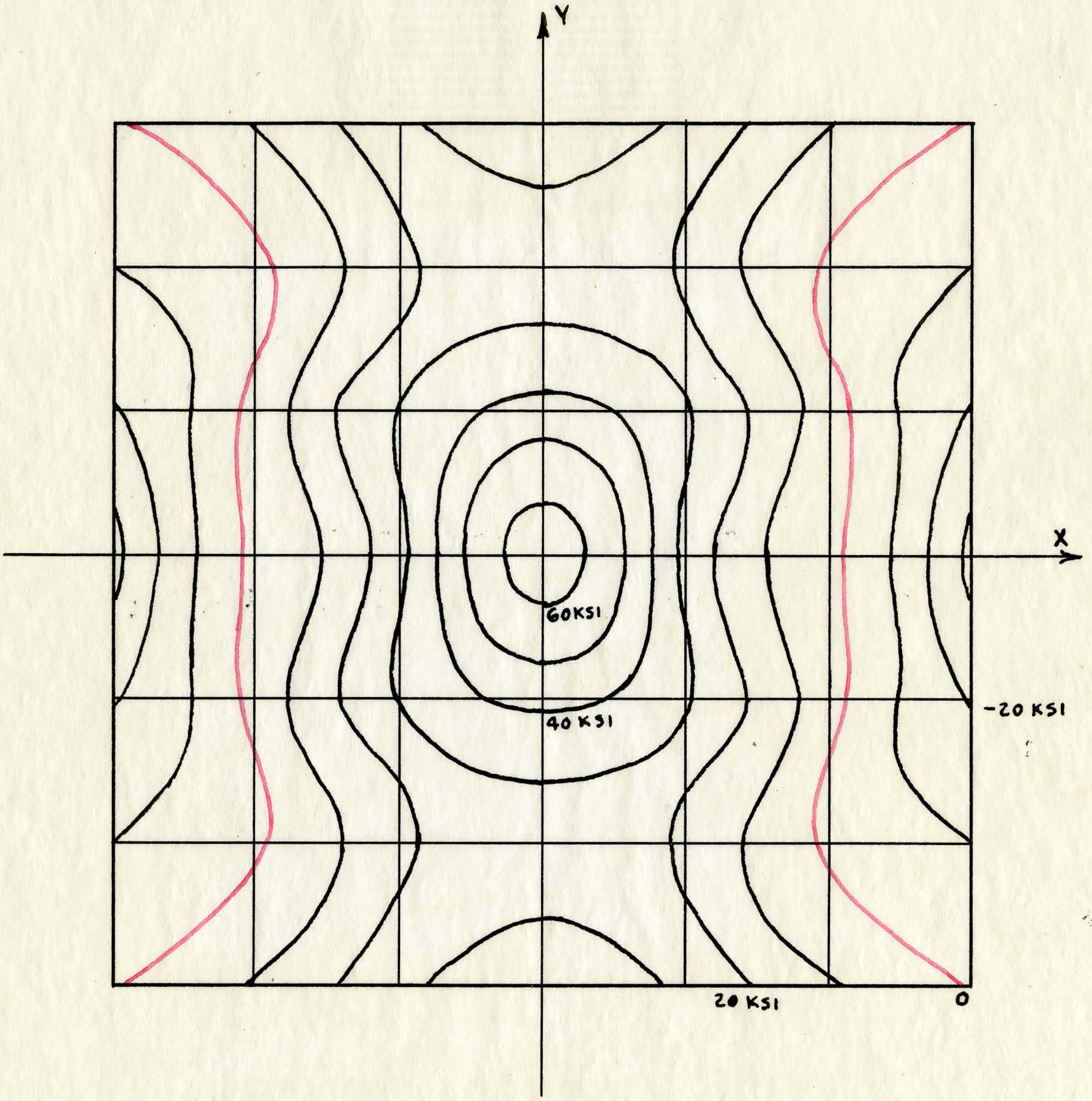
Figure 12



STRESS CURVES

For Stresses in the Y Direction for Fixed Plate

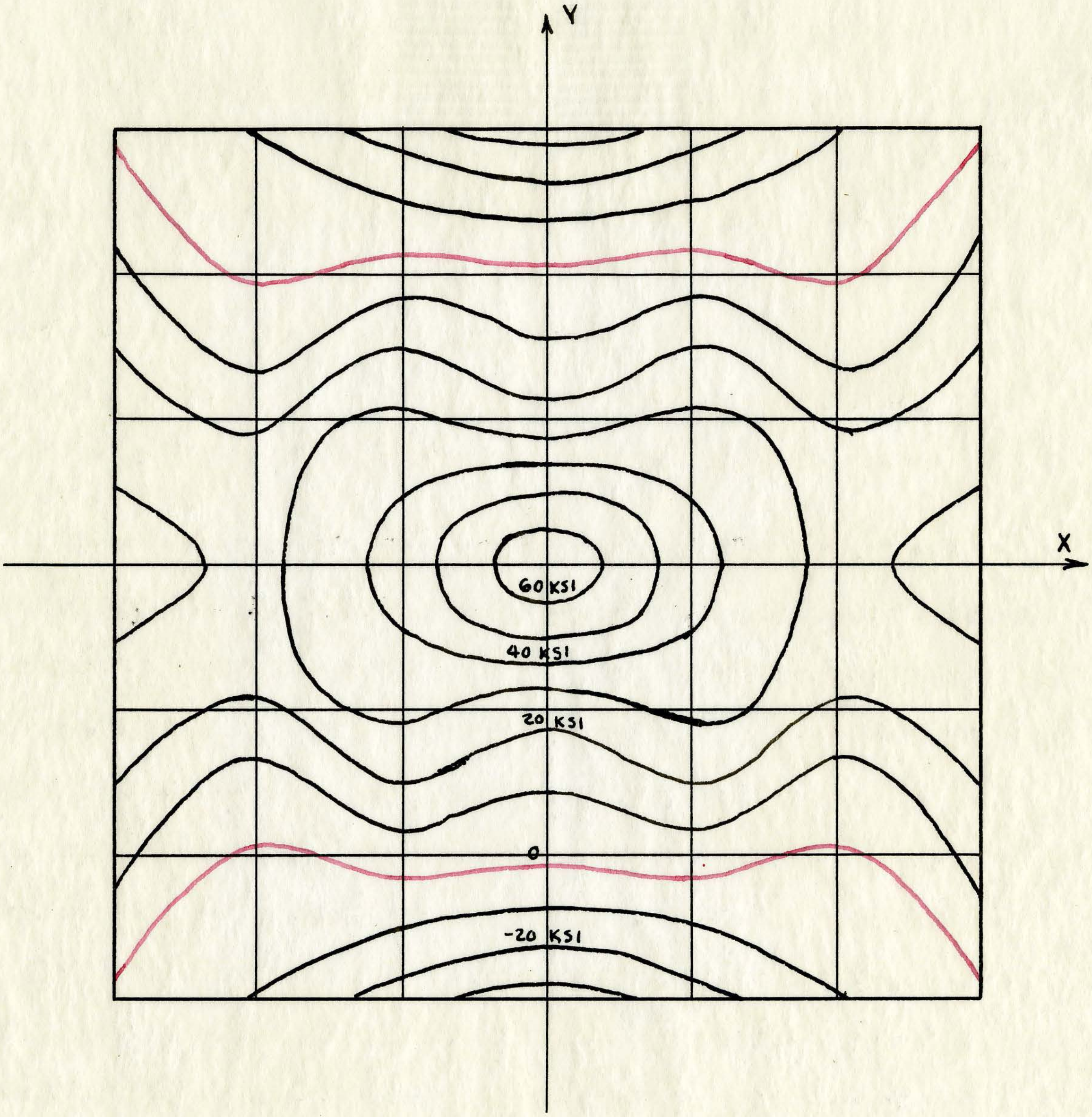
Figure 13



STRESS CURVES

For Stresses in the X Direction for Experimental Plate

Figure 14



STRESS CURVES

For Stresses in the Y Direction for Experimental Plate

Figure 15

CONCLUSION

A COMPARISON OF THE THEORETICAL AND EXPERIMENTAL RESULTS

At the end of the theoretical solutions and the experimental analysis of the flat plate the stress curves are drawn. The stresses were calculated for the x and y directions in order that a quick and qualitative comparison might be made. The stress curves were drawn as carefully as possible, but inaccuracies occurred because of the large web mesh size, drawings, and the large variations in the moments. Therefore we do not have exact results, but the ones we have are accurate enough to compare and draw conclusions from.

From the curves for the experimental analysis, Figures 14 and 15, we can see that the spandrel offers a considerable amount of torsional restraint to the slab. This restraint would be increased if the spandrels were continued, or other bays were added. The spandrels will have a tendency to increase the positive bending moment at the corners due to the torsion in them, but at the same time the ones framing perpendicular to these would decrease the moment at the corners by resisting the tendency to deflect. The columns would offer further resistance to the rotation at the corners, but this would be slight in comparison with the resistance the spandrels offer.

The experimental solution as expected was similar in some ways to both of the theoretical solutions, and was between them. The maximum positive stresses at the center of the plate may be seen to vary from 92.6 ksi for the pinned plate, to 65.4 ksi for the model, and to 44.8 ksi for the fixed plate. The maximum negative stresses on the spandrels vary from 0 ksi for the pinned plate, to - 35.7 ksi for the model, and to -86.6 ksi for the fixed plate. As can be seen the experimental results fall about halfway between the other two cases.

The resistance that the spandrel offers to the plate varies from a zero value at the columns to a maximum value at the center. From the work just completed in this thesis it seems possible that in a building frame consisting of beams and slabs that a certain portion of the design moment could be eliminated by the addition of a negative moment.

If the slab were designed as a simply supported slab it seems that the bending moment in the center could be reduced by about one third, and this value of one third of the maximum positive moment, placed at the edge of the slab in the middle as a negative bending moment.

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