STUDY OF FERROMAGNETIC SYSTEMS WITH MANY PHASE TRANSITIONS

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(ABSTRACT)

The change in the number of phase transitions for perturbations of finite range interactions is studied.

A Monte-Carlo simulation was performed for a translation invariant spin 1/2 ferromagnetic model in Z^2 with fundamental bonds

$A = \{(0,0);(0,1)\}$

$B = \{(0,0);(2,0)\}$

$C = \{(0,0);(0,1);(1,1);(1,0)\}$

The model exhibits one phase transition if the coupling constant J(A) is zero, but two phase transitions were found when J(A) is non zero and small enough.

The generalization of this situation is provided by a construction, due to J. Slawny, which through a sequence of progressively smaller perturbations yields models with an arbitrary minimum number of phase transitions. However, such construction requires the existence of interactions

with one fundamenta bond such that for all values of the coupling constants the Gibbs state is unique even when the interaction is perturbed by an arbitrary finite range perturbation of small enough norm. In this work it is proven that such property is exhibited by some translation invariant systems in Z^{ν} with finite state space at each point. The proof applies to models with real interactions and whose fundamental bonds are all multiple of a single bond which is of prime order and which is obtained as the product -in the group ring structure of the dual space- of one dimensional bonds whose non trivial projections at each lattice site are unique. The proof is based on the Dobrushin-Pecherski criterion concerning the uniqueness of Gibbs states under perturbations. Such criterion is restated so that only transition functions on sets of simple geometry are involved.

In addition, an algebraic characterization is presented for the set of Gibbs states for ferromagnetic systems for which the state space at each lattice site is a compact abelian group. This is a generalization of the theory originally introduced by Slawny for spin 1/2 ferromagnetic models and later extended by Pfister to ferromagnetic models for which the state space at each point is a finite product of tori and finite abelian groups.

A

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Chapter I

INTRODUCTION

The statistical mechanics of classical lattice systems is a well founded field from the mathematical standpoint. The basic framework was introduced in the late sixties ([26],[5]), and a very powerful machinery has been developed since then ([27],[17]). Probably the central phenomenon in this subject is the occurrence of phase transitions. These are characterized by sudden changes in observables like magnetization, density, etc; which are triggered by an infinitesimal change in some parameter of the system. Phase transitions have been defined and studied along two separated tracks: The first one relates the occurrence of phase transitions with the lack of anlyticity of the pressure ([26] chapter 5); the second one defines a phase transition as a change in the number of extremal points in the Choquet simplex of Gibbs states ([27] chapter 1). This work is devoted to the second point of view.

The basic motivation of the present work was to explore aspects related to the number of phase transitions that can be expected for ferromagnetic systems. The initial move was to undertake a Monte-Carlo calculation of a system where the "turning on" of a one extra coupling constant was expected

to have a noticeable effect in the number of phase transitions. It was found that the number of phase transitions increased from one to two for small non zero values of the perturbation. This fact motivated a theorem due to J. Slawny [30] that proves the existence of the extra phase transitions and moreover yields a procedure to construct for each natural n, a large family of ferromagnetic models with at least n phase transitions.

The plan of the dissertation is as follows. In chapter II a brief discussion of the general formalism is presented mainly to introduce the notation used in the work. The formalism of classical lattice statistical mechanics was developed in two equivalent ways: In Lanford-Ruelle formalism ([26],[20]), physical states are probability measures whose conditional probabilities on finite volumes coincide with the normalized Boltzman factors. On the other hand, in Dobrushin formalism states are defined by the marginal distributions of random fields. Both formalisms are used in this work. The variety of techniques available in the framework of these formalisms depends on the richness of the mathematical structure of the space of configurations. The notion of symmetry requires the action of certain sets on this space or on the spaces of functions defined on it. The most conspicuous of these actions is that of Z^{ν} which

incorporates the notion of translation invariance. Another direction in wich the configuration space can acquire additional structure is by endowing the state space at each point with a richer mathematical structure. In this work the models have a state space at each site that is a compact abelian group. In this situation one can use the techniques of the harmonic analysis which yields, for instance, very useful high and low themperature expansions ([12] part III). If in addition the state space at each point is finite, it can be endowed with a ring structure ([28],[29]), which yields a structure of module for the dual of the configuration space. This module is in fact a ring if the lattice is Z^{ν}. All these structures are presented in chapter II.

In chapter III we discuss an algebraic characterization of the set of invariant Gibbs states for ferromagnetic systems in which the state space at each site is a compact abelian group. By virtue of such characterization, the existence of phase transitions (involving a change in the number of quasiperiodic phases), corresponds to the change of size of a group of symmetries or, in terms of the dual space, of a group of observables. The characterization was introduced by Slawny [28] for spin 1/2 ferromagnetic models, and was later generalized by Pfister [25] to ferromagnetic

models for which the state space at each point is a product of tori and finite abelian groups. In the present work the theory is extended to ferromagnetic systems for which the state space at each point is an arbitrary compact abelian group. In fact, the only new result is lemma III.1 which reduces, under a suitable change of variables, to a basic result due to Ginibre ([10] example 4). The rest of chapter III is a reordering and transcription of the results introduced in references [23] and [25] to the more general setting.

The Monte-Carlo simulation is described in chapter IV. The theorem to prove the existence of the observed phase transitions and the construction of models with an arbitrary minimun number of phase transitions, are presented in chapter V. All the results of this last chapter are due to J. Slawny [30], and they are included as a motivation for the rather technical work undertaken in chapters VI and VII.

In these last two chapters we turn to the problem of obtaining a family of models that have only one Gibbs state even when perturbed by small enough finite range perturbations. Such family is needed for the construction of systems with many phase transitions described in chapter V. There are few criteria to prove the uniqueness of the Gibbs state, for instance Dobrushin criterion [6]; the use of

cluster expansions in combination with Kirkwood-Salsburg equations ([26] chapter 4,[8]); and the Gruber-Merlini equations [13]. These last two techniques however, are expansions around zero interaction and as such they are suited for the high temperature region, but not for perturbations of a non necessarily small interaction. Slawny [31] has obtained a general expression for a cluster expansion around a given (non zero) interaction that seems the natural tool to study perturbation. However its application for the kind of systems discussed in chapter VII requires some combinatoric estimation that could not be obtained. Instead, the technique adopted was the Dobrushin-Pecherski criterion [7] which is a generalization of the Dobrushin criterion. DP criterion had to be slightly restated for the present application in a way that does not involve arbitrary geometries, but only a particular family of simple parallelepipeds. There is no change in DP proof, only the remark that the sets involved in it are of this simple family. This restatement is discussed in detail in chapter VI, which is also intended to be a more pedagogical introduction to the beautiful but notationally involved proof presented in [7]. The criterion is applied in chapter VII to the so-called one bond systems. For these systems the high temperature expansion yields a sufficient condition for

DP criterion in terms of the relation between the number of cycles of a given length and this length. With this sufficient condition the uniqueness of the Gibbs state is proven for finite range perturbations of some one bond systems in Z^{ν} whose basic bond satisfies certain regularity requirements. These systems can therefore be used for the construction of chapter V. The results of chapter VII are new.

Chapter II

GENERAL FORMALISM OF CLASSICAL LATTICE SYSTEMS

2.1 LANFORD-RUELLE FORMALISM

Let L be a countable set called lattice and for each $i\epsilon L$, Ω_i a separable compact space with finite measure μ_i. The duple (Ω_i,μ_i) is called the (microscopic) state space at the point i. Generally the measure μ_i is a canonical measure and the microscopic state space is labelled only by Ω_i. The configuration space is the set $X=\underset{i\epsilon L}{X}\,\Omega_i$; its elements are called configurations. The configuration space is endowed with the product topology, therefore by Tychonov and the fact that L is countable X is a separable compact.

If Λ is a finite subset of L, the space of configurations on Λ is the set $X_\Lambda=\underset{i\epsilon\Lambda}{X}\,\Omega_i$. For $x\epsilon X$ let's denote x_Λ its restriction to Λ and pr_Λ the projection $pr_\Lambda(x)=x_\Lambda$. Denote also:

$$C=\{f:X\to R \text{ continuous}\}$$

$$C_\Lambda=\{f:X_\Lambda\to R \text{ continuous}\},\ \text{for } \Lambda\subset L$$

$$C_o=\cup\{C_\Lambda:\ \text{finite } \Lambda\subset L\}$$

C is a Banach space with the supremun norm and identifying $f\epsilon C_\Lambda$ with $f\circ pr_\Lambda\ \epsilon C$ one has that by Stone Weierstrass

$$\overline{C}_o = C \tag{II.1}$$

The elements of C are called observables. A local observable is an element of C_o, i.e. a function that depends on the microscopic states at only a finite number of points.

Another relevant space is

$$E(X)=\{\text{probability measures on } X\}$$

The elements of E(X) are called (macroscopic) states. The measured value of an observable f for a system in a state μ is given by $\mu(f)$. Physically, the only available operation to describe and characterize states is the measurement of observables. Accordingly, E(X) is endowed with the vague (or weak *) topology inherited from the dual of C. By Banach-Alaoglu E(X) is a compact set which is metrizable because X is separable. E(X) is also convex. Every state $\mu\in E(X)$ defines, by restriction, an state $\mu_\Lambda\in E(X_\Lambda)$ characterized by the property:

(II.2) $$\mu_\Lambda(f)=\mu(f\circ pr_\Lambda), \text{ for every } f\in C_\Lambda$$

To specify a physical model one also needs a Hamiltonian i.e. a particular interaction. An interaction is a family of functions $\phi=(\phi_M)_{\text{finite } M\subset L}$ where $\phi_M\in C_M$ and $\phi_\emptyset=0$. In general many of such ϕ_M are zero. One defines the set

$$B=\{A\subset L:\ \phi_A\neq 0\}$$

whose elements are called bonds. Defining a norm

$$\|\phi\| = \sup_{a\in L} \sum_{A: a\in A} \|\phi_A\|_\infty, \tag{II.3}$$

one has a Banach space

$$\mathcal{B} = \{\phi: \|\phi\| < \infty\}$$

called the space of locally bounded interactions. An interaction ϕ is of range k for some natural k if $|M|>k$ implies $\phi_M=0$. The subspace of range k interactions is denoted $\mathcal{B}_k$. If the lattice L is a subset of R^ν, an equivalent definition of range is used replacing "cardinal of M" by "diameter of M". This later definition will be assumed from chapter IV on. For each finite $\Lambda\subset L$ the Hamiltonian on Λ for free boundary condition is the function of C_Λ

$$H_\Lambda = \sum_{A\subset\Lambda} \phi_A. \tag{II.4}$$

On the other hand, if $y\in X_{L\setminus\Lambda}$, the Hamiltonian on Λ for the boundary condition y is the function of C_Λ defined by

$$H_\Lambda^y(x) = \sum_{A: A\cap\Lambda\neq\phi} \phi_A(x\vee y) \tag{II.5}$$

where $x\in X_\Lambda$ and

$$(x \vee y)_i = \begin{cases} x_i & \text{if } i \in \Lambda \\ y_i & \text{otherwise} \end{cases}$$

A <u>model</u> is a triple $(L, (\Omega_i)_{i \in L}, \phi)$.

The canonical measures μ_i define a conspicuous element of $E(X)$, namely the free or non-interacting state:

$$\nu = \bigotimes_{i \in L} [\mu_i / \mu_i(\Omega_i)]. \tag{II.6}$$

The Maxwell-Boltzman factors define the only physically meaningful states on X_Λ: for finite $\Lambda \subset L$, $f \in C_\Lambda$

$$\rho_\Lambda^y(f) = (1/Z_\Lambda^y) \int_{X_\Lambda} f(x) \exp(-H_\Lambda^y(x)) \, \nu_\Lambda(dx) \tag{II.7}$$

$$\rho_\Lambda(f) = (1/Z_\Lambda) \int_{X_\Lambda} f(x) \exp(-H_\Lambda(x)) \, \nu_\Lambda(dx), \tag{II.8}$$

where

$$Z_\Lambda^y = \int_{X_\Lambda} \exp(-H_\Lambda^y(x)) \, \nu_\Lambda(dx) \tag{II.9}$$

$$Z_\Lambda = \int_{X_\Lambda} \exp(-H_\Lambda(x)) \, \nu_\Lambda(dx). \tag{II.10}$$

The distinctive feature of these measures is the presence of the exponential density which factorizes whenever the Hamiltonian decomposes as a sum of two parts. This causes the Boltzman measures to factor into two succesive averages. Indeed, a straightforward calculation shows that if Λ, N are finite sets, $\Lambda \subset N \subset L$, and $f \in C_\Lambda$

$$\rho_N^y(f) = \int_{X_N} \rho_\Lambda^{z\vee y}(f)\ \rho_N^y(dx_\Lambda \otimes dz_{N\setminus\Lambda}) \tag{II.11}$$

This property of iterative averaging is the distinctive mark of the presence of the Maxwell-Boltzman factor and hence is the property selected to define physical states when the number of sites is infinite and no consistent definition of such a factor is available.

Definition II.1: μ is a Gibbs state for a model $(L,(\Omega_i)_{i\in L},\phi)$ if $\mu\in E(X)$ and for every finite $\Lambda\subset L$, $f\in C_\Lambda$

$$\mu(f) = \int_X \rho_\Lambda^z(f)\ \mu(dx_\Lambda \otimes dz_{L\setminus\Lambda}) \tag{II.12}$$

The definition can be stated in terms of standard concepts in probability theory. Given a probability space (X,Σ,μ), for every measurable set A and every σ-algebra $\Sigma'\subset\Sigma$ there exists a function $P_\mu(A|\Sigma')$: $X \to R$ which is Σ' measurable and satisfies

$$\int_{A'} P_\mu(A|\Sigma')(x)\ \mu(dx) = \mu(A\cap A') \tag{II.13}$$

for every $A\in\Sigma'$. The function $P_\mu(A|\Sigma')$ is termed the conditional probability of A given the σ-algebra Σ'. If X is separable compact the conditional probabilities can be chosen so that for each $x\in X$, $P_\mu(\cdot|\Sigma')(x)$ is a measure on X. If for $\Lambda\subset L$ one denotes $\Sigma_{L\setminus\Lambda}$ the σ-algebra of $X_{L\setminus\Lambda}$; then the definition II.1 is equivalent to

Definition II.1': μ is a Gibbs state for a model $(L,(\Omega_i)_{i\in L},\phi)$ if $\mu\in E(X)$ and for every finite $\Lambda\subset L$, and every measurable set $A\subset\Lambda$

$$P_\mu(A|\Sigma_{L\setminus\Lambda})(x) = \rho_\Lambda^x(A)$$

for μ-almost all $x\in X_{L\setminus\Lambda}$.

Sometimes It is of interest to introduce the inverse temperature β in the exponents of (II.7)-(II.10). This is equivalent to consider a family of interactions $\beta\phi$ where $\beta\geq 0$ and $\phi\in\mathcal{B}$.

2.2 DOBRUSHIN FORMALISM

For the case in which the state space is the same at each point, Dobrushin [5] has developed a formalism completely equivalent to the one discussed in the previous section. Such formalism presents classical lattice statistical mechanics as a particular application of random field theory. Let us recall some basic definitions of this theory [9]. Consider a probability space (W,Π,σ), where σ is a measure for the σ-algebra Π; a set L ;and a measura space (Ω,T), where T is a σ-algebra of subsets of Ω.

Definition II.2: A random field ξ with parameter space L and values in Ω is a map $\xi: L \times W \to \Omega$ such that for every $i \in L$, $\xi(i,\bullet): W \to \Omega$ is a random variable.

In some texts such object is called a random function and the label "field" is reserved for the case $L \subset R^{\nu}$ for some ν. If $\nu=1$ and L is interpreted as time, ξ is called a random or stochastic process.

According to the definition, a full description of a random field requires the knowledge of seven objects, namely W, Π, σ, L, Ω, T and ξ. In practice, this is more than the information that can be compiled from real-life measurements of real-life fields. Indeed, it is impossible to measure the field for all values of the parameter space L, only finite (yet as large as wanted) subsets of L are accesible in any experiment. That means that the relevant events are basically those belonging to a finite number or copies of T. Denote for $\Lambda \subset L$: $\Sigma = \underset{i \in L}{X}\, T$, $\Sigma_\Lambda = \underset{i \in \Lambda}{X}\, T$, $\xi_\Lambda = \underset{i \in \Lambda}{X}\, \xi_i$. Then

Definition II.3: The marginal distributions (or finite distributions) of the random field ξ are the measures σ_Λ^ξ defined on Σ_Λ by

$$\sigma_\Lambda^\xi(A) = \sigma(\xi_\Lambda^{-1}(A))$$

for any finite $\Lambda \subset L$ and $A \in \Sigma_\Lambda$.

The marginal distributions are the restrictions to Σ_Λ of the measure σ^ξ defined by

$$\sigma^\xi(A) = \sigma(\xi^{-1}(A)),\ A\in\Sigma. \tag{II.14}$$

The fact that the marginal distributions are the only objects that can be measured in practice motivates:

Definition II.4: Two random fields

$$\xi:\ L\times(W,\Pi,\sigma)\to(\Omega,T)$$
$$\eta:\ L\times(W',\Pi',\sigma')\to(\Omega,T)$$

are stochastically equivalent (in the wide sense) if for every finite $\Lambda\subset L$,

$$\sigma^\xi_\Lambda = (\sigma')^\eta_\Lambda.$$

Definition II.5: A physical random field (or random field in the wide sense) is a class of stochastically equivalent random fields. A physical random field with representant ξ will be denoted $[\xi]$.

In Dobrushin formalism the possible states of a macroscopic system formed by L copies of a measure space (Ω,T) are described by the physical random fields with parameter space

L and values on Ω. The equivalence between this formalism and Lanford-Ruelle's is contained in the following theorem:

Theorem II.6 (Kolmogorov): Let Ω be a complete separable space and $\mathcal{T}$ a σ-algebra of subsets of Ω that contains the Borelians. Consider a physical random field with parameter space L, values on $(\Omega,\mathcal{T})$ and marginal distributions $\{P_\Lambda\}_{\text{finite } \Lambda\subset L}$. Define $X=\Omega^L$, $\Sigma=\underset{i\in L}{X}\,\mathcal{T}$. Then there exists a unique probability measure μ on Σ satisfying $\mu_\Lambda=P_\Lambda$ for every finite $\Lambda\subset L$. Moreover, if η: $L\times X \to \Omega$ is defined by $\eta(i,x)=x_i$, then $\mu_\Lambda=\mu_\Lambda^\eta$ for every finite $\Lambda\subset L$.

The field defined by η, L, X, Σ, μ, Ω, $\mathcal{T}$ is called the canonical representation of the original physical random field. The equivalence of the Lanford-Ruelle and Dobrushin formalisms for a system with lattice L and state space at each point Ω (with the Borelian σ-algebra) is given by the bijective correspondence

Physical random field with marginal distributions $\{P_\Lambda\}$	$\longleftrightarrow$	$\mu\in E(X)$ with $\mu_\Lambda=P_\Lambda$

With this equivalence definition II.1' can be reworded as:

Definition II.7: A Gibbsian field for a model (L,Ω,ϕ) is a physical random field such that for every finite $\Lambda\subset L$, $A\in\Sigma_\Lambda$:

$$P_\mu(A|\Sigma_{L\setminus\Lambda})(x)=\rho_\Lambda^x(A)$$

for μ-almost all $x\in X_{L\setminus\Lambda}$. Here μ is defined in theorem II.6.

As discussed above, the distinctive role played by the measures ρ_Λ^x in the last definition can be traced back to the property (II.11) of iterative averaging. However, there may be families of measures satisfying (II.11) but not of the form (II.7), i.e. not related with any interaction. This observation led Dobrushin to propose a slight generalization of the notion of Gibbsian field. In what follows fix L and (Ω,T).

Definition II.8: A transition function for a finite set $\Lambda\subset L$ is a map

$$\begin{aligned} P_\Lambda:\ \Sigma_\Lambda \times X_{L\setminus\Lambda} &\to R \\ (A,y) &\to P_\Lambda(A|y) \end{aligned}$$

such that for each $y\in X_{L\setminus\Lambda}$ $P_\Lambda(\cdot|y)$ is a measure on X_Λ.

Definition II.9: A specification (or system of transition functions) is a set of transition functions $\{P_\Lambda\}_{\text{finite } \Lambda\subset L}$ satisfying the following condition of consistency:
If Λ and M are finite subsets of L, $\Lambda\subset M$, then for every $A\in\Sigma_\Lambda$, $y\in X_{L\setminus\Lambda}$:

$$(II.15)\qquad P_M(A|y) = \int_{X_M} P_\Lambda(A|z\vee y)\ P_M(dx_\Lambda \otimes dz_{M\setminus\Lambda}|y).$$

Definition II.10: A Physical random field $[\xi]$ is said to be consistent with the specification $\{P_\Lambda\}$ if, for every finite $\Lambda \subset L$, $A \in \Sigma_\Lambda$, $y \in X_{L\setminus\Lambda}$ and for every representant ξ of the physical field

$$P_{\sigma\xi}(A|\Sigma_{L\setminus\Lambda})(y) = P_\Lambda(A|y) \tag{II.16}$$

for σ-almost all $y \in X_{L\setminus\Lambda}$. (The LHS is the conditional probability (II.13) for the measure (II.14)).

Every specification has at least one physical random field consistent with it, but in general there is more than one.

Definition II.11: For a model (L,Ω,ϕ)

i) The functions P_Λ with $P_\Lambda(A|y)=\rho_\Lambda^y(A)$ are called the Gibbsian transition functions of the model

ii) The family of such transition functions is the Gibbsian specification.

Hence definition II.7 can be rephrased by saying that a Gibbsian field for a model is a physical random field consistent with the Gibbsian specification for the model.

Dobrushin formalism will be referred in connection with Dobrushin criteria for uniqueness of the Gibbs state under perturbation (chapter VI). Except there, the Lanford-Ruelle formalism will be used throughout.

2.3 PURE PHASES. INVARIANT STATES

The usual situation in Statistical Mechanics is to have fixed lattice and state spaces at each point and to analyze the effect of changing the interaction. Hence the set of Gibbs states for a given model $(L,(\Omega_i)_{i\in L},\phi)$ is denoted by $\Delta(\phi)$. The basic properties of $\Delta(\phi)$ are summarized in the following theorem. Let's first recall that if E is a convex subset of a real vector space, a point $x\in E$ is an extremal point if $x=\lambda x_1+(1-\lambda)x_2$, $0<\lambda<1 \Rightarrow x_1=x_2=x$. The set of extremal points of E is denoted $\mathcal{E}(E)$

Theorem II.12: [27] $\Delta(\phi)$ is a non empty, convex and compact subset of E(X). Moreover $\Delta(\phi)$ is a Choquet simplex, that is for each $\rho\in\Delta(\phi)$ there exists a unique measure δ_ρ concentrated on $\mathcal{E}(\Delta(\phi))$ such that

$$\rho(f) = \int_{\Delta(\phi)} f(\xi)\ \delta_\rho(d\xi),$$

every $f\in C$, or equivalently

$$\rho = \int_{\Delta(\phi)} \xi\ \delta_\rho(d\xi) \qquad \text{(II.17)}$$

in weak sense.

Definition II.13: The pure phases of a model are the elements of the set $\mathcal{E}(\Delta(\phi))$.

When under a change in the interaction the number of pure phases changes, the system undergoes a phase transition.

To discuss translation invariance and other symmetries the notion of action of a group on a set is useful.

Definition II.14: A group G is said to act on a set S if there is a function

$$G \times S \to S$$
$$(g,s) \to gs$$

such that

$$1s=s, \quad (gg')s=g(g's)$$

Note that for each $g \in G$ the map $g: s \to gs$ is inversible. Hence G is said to act by automorphisms, in particular if S is a topological space and the maps g are continuous, G is said to act by homeomorphisms. The action of G on S induces an action of G on any space of functions on S defined by

$$(gF)(s) = F(g^{-1}s) \tag{II.18}$$

Proposition II.15: If the group G acts on S:

i) The relation

$$s \equiv s' \iff \text{there exists } g \in G \text{ s.t. } s=gs', \quad (s,s' \in S)$$

is an equivalence relation on S. The equivalence classes are called orbits.

ii) For each $s\in S$ the set $G_s=\{g\in G: gs=s\}$ is a subgroup of G called the isotropy group of S.

iii) The orbit of $s\in S$ is in one-one and onto correspondence with G/G_s.

For a set $A\subset S$ and $g\in G$ one denotes $gA=\{gs: s\in A\}$ and one says that A is G-invariant if gA=A. In particular an element $s\in S$ is G-invariant if $G_s=G$ or equivalently if its orbit has only one element, namely s itself. More generally, if the orbit of s is finite (i.e. G/G_s is a finite set), s is said G-periodic.

In many situations the group Z^ν has a natural action on the lattice L. Typically $L\subset R^\nu$, hence the action of Z^ν on L is termed translation and it is denoted additively: $(a,i)\mapsto i+a$, $a\in L$, $i\in Z^\nu$. If the spaces Ω_i are chosen so that $\Omega_{i+a}=\Omega_i$ for every $a\in Z^\nu$, then the action of Z^ν on the lattice induces an action on X and on turn on C and E(X) by successive application of (II.18). This action is denoted $(\tau_a)_{a\in Z^\nu}$ in order not to mix multiplicative and additive notations. For example, if $x\in X$, $\tau_a x$ is the translated configuration

$$(\tau_a x)_i=x_{i-a}$$

The action of Z^ν on an interaction is defined as

(II.19) $$(\tau_a\phi)_A(x) = \phi_{A-a}(\tau_a x)$$

The set of $\phi\epsilon\,\mathcal{B}$ that are translation invariant is denoted $\mathcal{B}_I$. Likewise the set of translation invariant states and the set of translation invariant Gibbs states are denoted $E_I(X)$, $\Delta_I(\phi)$ respectively. If Z^ν acts on the lattice L, the latter is partitioned into a number of Z^ν orbits. A <u>fundamental lattice</u> L_o is defined picking one representant of each orbit. The choice of L_o is not unique, but it is supossed fixed before hand in each problem. In an analogous fashion, to define a translation invariant interaction it is enough to pick one bond of each Z^ν orbit of B. A set of representants of the Z^ν orbits of B is called a <u>fundamental family of bonds</u> and denoted B_o. Such set is certainly non unique and it is minimal with respect to the property

(II.20) $A\epsilon B$ => there exists $A'\epsilon B_o$, $a\epsilon Z^\nu$ such that $A=\tau_a A'$

The properties of the set of translation invariant Gibbs states are summarized in the following theorem.

<u>Theorem II.16</u>: Suppose $\phi\epsilon\,\mathcal{B}_I$

i) $E_I(X)$ and $\Delta_I(\phi)$ are non empty, convex and compact subsets of E(X). Moreover both are Choquet simplexes,

ii) $\Delta_I(\phi)$ is a face of $E_I(X)$, i.e. $\mathcal{E}(\Delta_I(\phi))\subset\mathcal{E}(E_I(X))$.

The elements of $\mathcal{E}(\Delta_I(\phi))$ are called ergodic states because they are indeed ergodic measures. Let us remark that $\Delta_I(\phi)$ is not a face of $\Delta(\phi)$, hence in general the extremal points of $\Delta_I(\phi)$ (ergodic states) admit non trivial decompositions into extremal points of $\Delta(\phi)$ (pure phases). Such situation is known in physics as spontaneous symetry breakdown.

2.4 MODELS FOR WHICH THE STATE SPACE AT EACH POINT HAS AN ABELIAN GROUP STRUCTURE

If the spaces Ω_i, $i\in L$ are compact abelian groups, the measures μ_i are taken to be the corresponding Haar measures. The configuration space X also results a compact abelian group (product group) and the free measure ν is its normalized Haar measure. For such models the powerful techniques of the harmonic analysis are available. In what follows multiplicative notation will be used both for X and its dual group $\hat{X}$.

If $\hat{\Omega}_i$ is the dual group of Ω_i, one has that $\hat{X} = \bigoplus_{i\in L} \hat{\Omega}_i$. Every character ξ on X is of the form

$$\xi(x) = \prod_{j\in A} \xi_j(x_j), \quad x\in X \tag{II.21}$$

where each $\xi_j \in \hat{\Omega}_j$ and $A \subset L$ is the finite set

$$\{j\in L: \xi_j \text{ is not trivial}\}$$

called the support of ξ which will be denoted $\underline{\xi}$ in the sequel. Moreover, each ξ_j is of the form $\xi_j=\exp(i\Phi_j)$ where Φ_j (phase) takes values on the interval $[0,2\pi)$. If this set is endowed with the toroidal topological group structure, each Φ_j becomes a continuous homomorphism. As the σ-algebra generated by the open sets of the torus coincide with the σ-algebra of R restricted to $[0,2\pi)$, the map Φ_j with image on $[0,2\pi)$ is measurable. In (II.21)

$$\xi(x) = \exp[i \sum_{j\in\xi} \Phi_j(x_j)]. \tag{II.22}$$

Given an interaction ϕ, each function ϕ_A is uniquely defined by its Fourier coefficients, hence there is no loss of generality in defining the interactions as just the set of (the negative of the) Fourier coefficients.

Definition II.17: An interaction for a model with Ω_i compact abelian for each $i\in L$, is a family of complex numbers $J=(J(\xi))_{\xi\in\hat{X}}$ such that

$$J(\xi^{-1}) = \bar{J}(\xi) \tag{II.23}$$

This last condition ensures that the Hamiltonians

$$H_\Lambda = - \sum_{\xi\in\hat{X}_\Lambda} J(\xi)\ \xi \tag{II.24}$$

are real valued functions on X_Λ. In fact with (II.23)

$$H_\Lambda = - \sum_{\xi\in\hat{X}_\Lambda} \mathrm{Re}[J(\xi)\ \xi], \tag{II.25}$$

and in general

$$(II.26) \qquad H_\Lambda^y(x) = - \sum_{\xi:\underline{\xi}\cap\Lambda\neq\phi} \mathrm{Re}[J(\xi)\ \xi(xvy)]$$

The set of bonds is $B=\{\xi\in\hat{X}:\ J(\xi)\neq 0\}$ and for finite $\Lambda\subset L$ we will denote $B_\Lambda=B\cap X_\Lambda$. Now $\mathcal{B}$ is defined as the space of interactions J such that

$$(II.27) \qquad \|J\| = \sup_{a\in L} \sum_{\xi:a\in\underline{\xi}} |J(\xi)|$$

is finite. In particular this implies that B is a countable family. An interaction is of finite range if each point of L belongs to the support of only a finite number of bonds (for example if the fundamental family of bonds B_o is finite). The coefficient $J(\xi)$ is called the <u>coupling constant</u> for the bond ξ. For the high and low temperature expansions it is convenient to group the bonds that are powers of the same character. This is accomplished [12] by introducing a family $\tilde{B}$ of characters on X such that each $\xi\in B$ is of the form $\xi=\chi^l$ for some $l\in Z$, $\chi\in\tilde{B}$. We will allow a little less freedom in the definition of B than in Ref [12] in order to make this B also useful for Slawny's theory of translation invariant Gibbs states (chapter III).

<u>Definition II.18</u>: For a given set of bonds B, the family $\tilde{B}$ is chosen so to satisfy the following properties:

i) For every $\xi\in B$ there exist $\chi\in\widetilde{B}$, $l\in Z$ so that $\xi=\chi^l$. If ord(χ)=order of $\chi < \infty$, then $l\in Z_{ord(\chi)}$.

ii) For each $\chi\in B$, there are two bonds $\xi_1=\chi^{l_1}$, $\xi_2=\chi^{l_2}$ with $|l_1|$, $|l_2|$ coprime.

iii) $\widetilde{B}$ is minimal with respect to properties i) and ii).

It is immediate that a set satisfying i) and iii) can be found. To make it satisfy ii) one has to consider for each χ the set $K = \{|l|: \chi^l\in B\}$ and k = g.c.d.(K). If $k=1$ then ii) is satisfied, if $k\neq 1$ replace χ by $\chi'=\chi^k$. If Ω is isomorphic to Z_2 (spin ½) $\widetilde{B}=B$, but in general $\widetilde{B}$ need not be a subset of B. Condition i) is the original condition stated in ref (12), condition iii) is just convenient for simplicity, and condition ii) is required so to have proposition II.19 which establishes that the subgroups S, $\overline{B}$ defined in relation with high and low temperature expansions are the same as those used in Slawny's theory of invariant Gibbs states. We will denote $\widetilde{B}_\Lambda=\widetilde{B}\cap X_\Lambda$, and if $\xi=\chi^l$ $J(\xi)$ will be written as $J(\chi,l)$. Hence the Hamiltonians are

$$(II.28)\quad H_\Lambda = \sum_{\chi\in\widetilde{B}_\Lambda}[-\sum_{l\in Z} J(\chi,l)\,\chi^l] = \sum_{\chi\in\widetilde{B}_\Lambda} H^{(\chi)}$$

with the convention $Z_\infty=Z$.

The following group homomorphisms are of relevance [29]:

$$\gamma: X \to \prod_{\chi\in\widetilde{B}} (\text{Image } \chi)$$

$$x \longmapsto (\chi(x))_{\chi\in\widetilde{B}}$$

(II.29) $$\varepsilon: \bigoplus_{\chi\in\widehat{B}} Z_{\text{ord}(\chi)} \to \hat{X}$$

$$\alpha \longmapsto \prod_{\chi\in\widetilde{B}} \chi^{\alpha(\chi)}$$

If the interaction is finite range, the map ε can be extended to

(II.30) $$\tilde{\varepsilon}: \prod_{\chi\in\widetilde{B}} Z_{\text{ord}(\chi)} \to \prod_{i\in L} \hat{\Omega}_i$$

The image space in (II.30) is in fact the dual of the dense subgroup

$$X_f = \bigoplus_{i\in L} \Omega_i$$

The map (II.29) defines the following subgroups:

$$S = \text{Ker } \gamma \quad \text{(symmetry group)}$$

(II.31) $$K = \text{Ker } \varepsilon \quad \text{(group of cycles)}$$

$$\overline{B} = \text{Im } \varepsilon$$

When $\tilde{\varepsilon}$ can be defined, K and B can be extended to

(II.32) $$K_{inf} = \text{Ker } \tilde{\varepsilon}$$

$$\overline{B}_{inf} = \text{Im } \tilde{\varepsilon}$$

For the groups in the domain of ε or $\tilde{\varepsilon}$ additive notation will be used. When the configuration space X is restricted to X_Λ a subindex Λ will be used for the spaces (II.31), (II.32).

Proposition II.19:

i) $S=\{y\in X: \xi(y)=1 \text{ for every } \xi\in\bar{B}\}$

ii) $\bar{B}=\{\eta\in X: \text{there exist } \alpha\in Z^B \text{ such that } \eta=\prod_{\xi\in B}\xi^{\alpha(\xi)}\}$

iii) $S=\bar{B}^\perp$, $S^\perp=\bar{B}$

Proof: To proof i) and ii) one has to prove that the subgroup generated by $\tilde{B}$ is exactly equal to $\bar{B}$. Such set contains $\bar{B}$ because $\tilde{B}$ generates $\bar{B}$. The opposite contention follows from requirement ii) of definition II.18. Indeed, for every $\chi\in B$ there exist ξ_1, $\xi_2\in B$, $\xi_1=\chi^l$, $\xi_2=\chi^k$ with $k,l>0$ and $(l,k)=1$ (Both k,l can be chosen positive because B is closed under inversion). Hence if a,b are natural numbers such that $ak+bl=1$, one has that $\chi=\xi_1^a\xi_2^b$. This proves that $\tilde{B}$ is generated by B.

The proof that $S=\bar{B}^\perp$ is immediate using for example the RHS in i) and ii). Taking anihilators to both sides one obtains the other identity in iii) because $\bar{B}$ is a closed subgroup ($\hat{X}$ is discrete).∎

Finally, for future uses, let us recall the high temperature expansion for the systems of this section. This is obtained by expanding $\exp[-H^{(\chi)}]$ in Fourier series for a fixed finite $\Lambda \subset L$ and all $\chi \in B_\Lambda$, and then using these expansions to compute

$$Z_\Lambda = \int \prod_{\chi \in \tilde{B}_\Lambda} \{\exp[-H^{(\chi)}]\}\ \nu_\Lambda(d\chi).$$

The result is ([12] p.263)

$$(II.33)\quad Z_\Lambda = \prod_{\chi \in \tilde{B}_\Lambda} f(\chi,0) \sum_{\alpha \in K_\Lambda} \prod_{\chi \in \tilde{B}_\Lambda} t(\chi,\alpha(\chi))$$

with

$$(II.34)\quad f(\chi,l) = \int_{X_\Lambda} \chi^l(x)\ \exp[-H^{(\chi)}(x)]\ \nu_\Lambda(d\chi)$$

$$(II.35)\quad t(\chi,l) = f(\chi,l)/f(\chi,0)$$

These coefficients have the following important property:

Proposition II.20: If the interaction is real $|t(\chi,l)|<1$ for every $\chi \in \tilde{B}$, $l \in Z_{ord(\chi)}$.

Proof: If in the RHS of (II.34) the exponential is expanded in power series and the series integrated termwise, by the orthogonality of the characters one obtains that $f(\chi,l)$ is a sum of powers of coupling constants. Hence, if the coupling constants are real so are the $f(\chi,l)$. Therefore, for every χ, l:

$$f(\chi,l) = \int \mathrm{Re}[\chi^l(x)]\ \exp[-H_\Lambda^{(\chi)}(x)]\ \nu_\Lambda(d\chi)$$

and as the exponential is strictly positive and $-1 \le \mathrm{Re}(\chi^1) \le 1$:

$$(II.36) \quad -\int \exp[-H_\Lambda^{(\chi)}]\, \nu_\Lambda(dx) \le \int \mathrm{Re}[\chi^1(x)] \exp[-H_\Lambda^{(\chi)}(x)]\, \nu_\Lambda(dx) \le \int \exp[-H_\Lambda^{(\chi)}(x)]\, \nu_\Lambda(dx)$$

Moreover, for every $\xi \in X$ the set $\{x: \mathrm{Re}\,\xi(x)<1\}$ has non zero measure otherwise it would contradict the fact that $\int \xi\, \nu(dx)=0$. This implies that the rightmost inequality in (II.36) is sharp. By an analogous argument the leftmost inequality is also sharp. Hence, for every $\chi, 1$:

$$|f(\chi,1)| < f(\chi,0). \blacksquare$$

2.5 SYSTEMS WITH FINITE STATE SPACE AT EACH POINT

An n-component lattice system is a system for which $|\Omega_i|=n$ for each $i \in L$. For such system the state space at each point can be endowed with a finite abelian group structure in many different ways. For an n-component system we will identify each Ω_i with G_n the group of n-roots of the unit. This is a compact group with the discrete topology and its Haar measure is (any multiple of) the counting measure. Moreover, both Ω and $\hat{\Omega}$ have a ring structure with the ring product shown in table 1, which becomes a field structure if n is prime ([15],[29]). As the ring sum is already denoted as a product, the ring product will be denoted by a dot "•".

TABLE 1

Ring product for a finite group and its dual

Group:	$G_n=\{w_k=\exp(i2\pi k/n),\ k=o,1,\ldots,n-1\}$
Product:	$\exp(i2\pi k/n)\bullet\exp(i2\pi m/n)=\exp(i2\pi km/n)$
Group:	$G_n=\{s^k,\ k=0,1,\ldots,n-1;\ s^k(w)=w^k\}$
Product:	$s^k\bullet s^m=s^{km}(\text{mod } n)$

For the ring G_n the neutral element for the ring sum is $w_0=1$, while that for the ring product is w_1. One has:

(II.37) $\qquad 1\bullet w_k=1, \quad$ for every $w_k\epsilon G_n$

(II.38) $\qquad (w_1)^k=w_k, \qquad k=0,1,\ldots,n-1.$

Moreover

(II.39) $\qquad (w_s\bullet w_t)^k=(w_s)^k\bullet w_t=w_s\bullet(w_t)^k$

For the ring $\hat{G}_n$ the neutral element for the sum is $s^0=\hat{1}$, while that for the ring product is s^1. One has:

(II.40) $\qquad \hat{1}\bullet\hat{w} = \hat{1}, \quad$ for every $\hat{w}\epsilon\hat{\Omega}$

The configuration space X for such systems becomes a G_n-module with action

(II.41) $\qquad (w\bullet x)_i=w\bullet x_i, \quad i\epsilon L,\ w\epsilon G_n,\ x\epsilon X.$

In particular the dense subgroup

(II.42) $\qquad X_f=\bigoplus_{i\epsilon L}\Omega_i$

is a free G_n-module with base $\{e_a;\ a\epsilon L\}$ where

(II.43) $\qquad (e_a)_i=\begin{cases} w_1 & \text{if } i=a \\ 1 & \text{otherwise} \end{cases}$

Every $x \in X_f$ can be written as $x = \prod_{a \in L} (x_a \bullet e_a)$ with almost all x_a equal to 1. The characters in X are linear for the action (II.40), i.e.

$$\xi(w \bullet x) = w \bullet \xi(x) \tag{II.44}$$

where the action on the RHS is that of G_n.

We will consider n-components systems with $L = Z^\nu$. For them the dual of the configuration space can be endowed of a ring structure, namely the group ring structure ([16] p 117):

$$\hat{X} = \hat{\Omega}[Z^\nu]$$

With our multiplicative notation the ring product of $\hat{X}$ is defined by

$$(\xi \bullet \eta)_i = \prod_{j \in Z^\nu} \xi_j \bullet \eta_{i-j} \tag{II.45}$$

with "$\bullet$" given in the second row of table 2.1.

If one defines $\hat{e}_a \in \hat{X}$ by:

$$(\hat{e}_a)_i = \begin{cases} s^1 & \text{if } i=a \\ \hat{1} & \text{otherwise} \end{cases} \tag{II.46}$$

then it is straightforward to check that

$$\tau_a \xi = \hat{e}_a \bullet \xi \tag{II.47}$$

$$\xi^k = (\hat{e}_0)^k \bullet \xi \tag{II.48}$$

for $a \in Z^\nu$, $k \in Z$, $\xi \in \hat{X}$.

Two known properties of the group ring are:

Theorem II.21: [21] R ring with identity, G group

i) R[G] is Noetherian <=> R is Noetherian and G finitely generated,

ii) If R is a field, $R[Z^\nu]$ is a unique factorization domain.

As a consequence

Corollary II.22: For an n-component lattice system in Z^ν

i) $\hat{X}$ is a Noetherian ring,

ii) If n is prime $\hat{X}$ is a unique factorization domain.

Chapter III

STATES FOR SYSTEMS IN WHICH THE STATE SPACE AT EACH POINT IS A COMPACT ABELIAN GROUP

3.1 THE BASIC LEMMA

In this chapter we place ourselves in the framework of section 2.4. The lattice L and the spaces $(\Omega_i)_{i\in L}$ are considered fixed throughout the chapter; only the interactions will vary. The following notation will be used: If J and J' are two interactions, $J \geq |J'|$ means $J(\xi) \geq |J'(\xi)|$ for every character ξ. In particular an interaction is ferromagnetic if $J \geq 0$. To simplify the notation the normalized Haar measure $\nu(dx)$ will be denoted plainly as "dx". Likewise $\nu_\Lambda(dx) = dx_\Lambda$. Slawny [28] introduced a useful description of translation invariant, or more generally periodic and quasiperiodic states for ferromagnetic 2-component systems at almost all temperatures. The description was extended by Pfister [25] to ferromagnetic systems for which the state space at each point is a product of one dimensional tori and finite abelian groups. Here the theory will be generalized to all ferromagnetic systems in the setting of section II.

The theory is based on correlation inequalities that exist for ferromagnetic systems. Ginibre [10] reduced the task of

proving such inequalities to showing that suitable functions D on X×X satisfy:

(III.1) $\int_{X\times X} D(x,y)\, dx\, dy \geq 0$

The following lemma -slight generalization of example 4 in [10]- provides all the functions D needed for the theory.

Lemma III.1: Pick a natural n and consider the set $M_n = \bigoplus_{a\in L} R^n$. Each element t of M_n is of the form $\vec{t}=(t_a)_{a\in L}$ with $t_a=(t_a^1, t_a^2, \ldots, t_a^n)$. Let F be a real valued function defined on M_n which is bounded, measurable and such that for every $a\in L$ and every integer k there exists an $\varepsilon=\varepsilon(k,a)$ with $\varepsilon^2=\varepsilon$, $\varepsilon(-k)=\varepsilon(k)$, so that if $p=(2\pi,2\pi,\ldots,2\pi)$,

(III.2) $F(t_a + 2kp) = \varepsilon\, F(t_a)$.

Then, if $\xi^1, \xi^2, \ldots, \xi^n$ are characters on X, $\xi^j = \exp(i\Sigma\Phi_a^j)$, and one defines the map $\vec{\theta}$ from X to M_n as $\theta_a=(\Phi_a^1, \ldots, \Phi_a^n)$; one has that:

(III.3) $\int_{X\times X} F(\vec{\theta}(y)+\vec{\theta}(x))\, F(\vec{\theta}(y)-\vec{\theta}(x))\, dy\, dx \geq 0$

Proof: As the map θ_a with values on $[0,2\pi)^n$ is measurable, the change in variables

(III.4) $t = \theta_a(x_a)$,

(the index "a" is dropped from the LHS in order to simplify the notation), defines a measure dt on $[0,2\pi)^n$. Denote $Q=\theta_a(\Omega_a)$.

Claim: there is a closed subgroup H of R^n such that $p\in H$ and: i) $Q=H\cap[0,2\pi)^n$, and ii) dt is a (normalization of the) Haar measure of H.

Indeed, if the set $[0,2\pi)^n$ is endowed with the topological group structure of the n-dimensional torus, the map θ_a is a continuous homomorphism onto a closed subgroup Γ_a of T^n. Such subgroup is of the form $H/2\pi Z^n$ where H is a closed subgroup of R^n that contains p ([3] section VII.4). All such groups H are isomorphic to $R^p\times Z^q$ for some p, q with $p+q\leq n$. Therefore the Γ_a is isomorphic to $T^p\times F$ were T is the torus and F the product of q finite subgroups of T. The measure dt is invariant under Γ_a-translations because θ_a is a homomorphism and dx_a is invariant under Ω_a-translations. Hence, by uniqueness of the Haar measure, dt is the Haar measure of Γ_a; which on the set $Q=H\cap[0,2\pi)^n$ coincides with the restriction of a (normalization of the) Haar measure of H. This proves the claim.

Thus, under the change of variables (III.4) the part of the integral in (III.3) corresponding to the measure $dx_a dy_a$ becomes

$$\int_{Q\times Q} F(t'+t)\,F(t'-t)\,dt'\,dt \qquad \text{(III.5)}$$

where dt is the Haar measure on a group $H \supset Q$.
At this point the proof provided by Ginibre ([10], example 4) applies but considering n dimensions at a time instead of only one. For the sake of completeness, the proof will be repeated here. By hypothesis the integrand in (III.5) is invariant under the changes

(III.6) $\quad t' \to t' \pm p$

(III.7) $\quad t \to t \pm p$

As the measures are Haar measures, they are also invariant under such changes. By the invariance under the transformation of the bottom of line (III.6), the integral over t' in (III.5) can be replaced by 1/2 the integral over $(Q-p) \cup Q$. An analogous change on the range of integration over t can be made by the invariance under the bottom transformation of line (III.7). Hence (III.5) becomes

(III.8) $$(1/4) \int_{[(Q-p)\cup Q]^2} F(t'+t)\, F(t'-t)\, dt'\, dt$$

Let's now agree to denote for $t, s \in R^n$; $t \geq s$, if $t^i \geq s^i$ for $1 \leq i \leq n$, and define the set

$$D(\geq,\geq,+,>p)=\{(t,t') \in [(Q-p)\cup Q]^2 : t' \geq 0, t \geq 0, t'+t > p\}$$

where the first two inequality signs in the argument of D refer to the first two inequalities inside the curly

bracket, and the "+" sign and the last inequality sign refer to the analogous signs in the last condition on the RHS. In an analogous fashion one defines

$$D(<,\geq,-,>-p)=\{(t,t')\epsilon\{(Q-p)\cup Q]^2 : t'<0, t\geq 0, t'-t>-p\}$$

etc. (This domains correspond to the triangular domains in figure 1 of reference [10]). Using the invariance of the integrand and the measure under the simultaneous application of the transformations of the bottom of lines (III.6) and (III.7), one gets that in (III.8) the integral over $D(\geq,\geq,+,>p)$ is the same as the integral over $D(<,<,+,>-p)$ (triangles labelled "3" in fig. 1 of ref. [10]). In the same fashion, combining one transformation of (III.6) with one of (III.7) one proves that the following pairs of domains contribute the same to the integral (8): (the labels of the corresponding triangles in figure 1 of ref. [10] are given in brackets)

$D(\geq,\geq,+,\leq p)$ and $D(<,<,+,\leq -p)$ (label 1)

$D(\geq,<,-,>p)$ and $D(<,\geq,-,>-p)$ (label 2)

$D(\geq,<,-,\leq p)$ and $D(<,\geq,-,\leq -p)$ (label 4)

Hence the integral in (III.8) becomes

$$(III.9) \qquad (1/8) \int_{D\times D} F(t'+t)\, F(t'-t)\, dt'\, dt$$

where

$$D = D(\geq,\geq,+,\leq p) \cup D(<,<,+,>-p) \cup D(<,\geq,-,>-p) \cup D(\geq,<,-,\leq p)$$

$$= \{(t,t')\in[(Q-p)\cup Q]^2 : -p<t'+t\leq p;\ -p<t'-t\leq p\}$$

Finally, one considers the change of variables

$$\alpha = t' + t, \qquad \beta = t' - t$$

This change is an automorphism in H×H, hence the normalized Haar measure remains unchanged:

$$dt'\ dt = d\alpha\ d\beta$$

and morever D factorizes into two identical domains in α and β; $D=D(\alpha)\times D(\beta)$, with

$$D(\alpha)=\{\alpha=t'+t:\ t',t\in[(Q-p)\cup Q]^2\ -p<\alpha\leq p\}$$

Therefore, the partial integration over the variables at the point $a\in L$ factorizes into integrals over the same domain with the same measure.

Repeating the above procedure for all $a\in L$ such that $0_a\neq 0$ (for the other a's the integral is trivial), one obtains that the integral (III.3) is proportional to

$$[\ \int F(\alpha)\ d\alpha\ ]^2 \geq 0 \qquad \blacksquare$$

Remarks: i) The same proof holds if M_n is extended to $M= \underset{a\epsilon L}{X} R^{(N)}$ and instead of considering a finite set of characters one considers any set with the property that for each $a\epsilon L$ only a finite number of characters of the set has $\xi_a \neq 1$. Such generality is not needed in the sequel.

ii) If F is periodic in each coordinate, i.e. there exists a positive integer m such that for each $a\epsilon L$ $F(t_a+2mp)=F(t_a)$; and if also the Fourier series of F is absolutely summable (for instance if F is a finite linear combination of positive definite functions), then the positivity of (III.5) can be proved using Fourier expansions [18]. Indeed in such case F is an integrable function defined on $\underset{a\epsilon L}{\oplus} T(m)$, with $T(m)=R^n/2m\pi Z^n$, and by the 2π-invariance of the integrand and the measure the integral (III.5) becomes

$$(1/m^2) \int_{\Gamma\times\Gamma} F(t'+t)\, F(t'-t)\, dt'\, dt$$

where Γ is some closed subgroup of T(m), both measures are equal to the normalized Haar measure on Γ and "+" and "-" are the operations on T(m). If the Fourier series of F,

$$F(t) = \Sigma\, a(\gamma)\, \gamma(t),$$

is replaced in the integral and termwise integrated, one ends up with integrals of the sort

$$\iint \gamma_a(t'+t)\, \gamma'_a(t'-t)\, dt'\, dt = \int (\gamma_a\gamma'_a)(t')dt' \int (\gamma_a\bar{\gamma}'_a)(t)dt$$

Due to the orthonormality of the characters, such integral is zero unless $\gamma_a=\gamma'_a$ and $(\gamma_a)^2=1$ in which case it is 1. Integrating in the same fashion for all $a\in L$ one finally gets that the integral (III.5) is equal to $\Sigma(a_\gamma)^2$ where the sum ranges over all characters γ such that $\gamma^2=1$. As F is real valued these coefficients $a(\gamma)$ are real, hence the integral is non-negative.

Corollary III.2: If $\mathcal{D}$ is the set of functions of the form

(III.10) $\quad \Sigma a_i \; F_i(\vec{\theta}(y)+\vec{\theta}(x)) \; F_i(\vec{\theta}(y)-\vec{\theta}(x))$

with $a_i > 0$, F_i as in the lemma, and the sum converging (almost) uniformly in x,y; then (III.1) holds for every D in $\mathcal{D}$. Note that $\mathcal{D}$ is closed under multiplication.

We will denote $\mathcal{D}_\Lambda$ as the set of functions in $\mathcal{D}$ which depend only on configurations on X_Λ:

$$\mathcal{D}_\Lambda=\{D\in\mathcal{D}:\ D(x,y)=D(x_\Lambda,y_\Lambda)\}$$

Using trigonometric identities as in references [23] and [25] one can prove:

Corollary III.3: The following functions are in $\mathcal{D}$

i) $\mathrm{Re}[\xi(x)]+\mathrm{Re}[e^{i\alpha}\,\xi(y)]$, ξ character on X, $\alpha\in R$

ii) $\exp[-H_\Lambda(x)-H'_\Lambda(y)]$, if $J > |J'|$, Λ finite subset of L

iii) $\exp[\lambda \mathrm{Re}(\xi(x))\ \mathrm{Re}(\xi(y))]$, ξ character on X, $\lambda \in R$

Proof: i) [23] Using that
$\cos(a)+\cos(b)=2\cos[\frac{1}{2}(a+b)]\cos[\frac{1}{2}(b-a)]$ one can write

$$\mathrm{Re}[\xi(x)]+\mathrm{Re}[e^{i\alpha}\ \xi(y)]=2\cos\{\tfrac{1}{2}[\Sigma(\Phi_a(x)+\Phi_a(y))+\alpha]\}$$
$$\times \cos\{\tfrac{1}{2}[\Sigma(\Phi_a(y)-\Phi_a(x))+\alpha]\}$$

Each factor is a function F as in the lemma.

ii) [23] If $J'(\xi)=|J'(\xi)|\ \exp(i\alpha_\xi)$

$$-H_\Lambda(x)-H'_\Lambda(y)=\Sigma\{\tfrac{1}{2}[J(\xi)+|J'(\xi)|][\mathrm{Re}\xi(x)+\mathrm{Re}(\exp(i\alpha_\xi)\xi(y))]$$
$$+\tfrac{1}{2}[J(\xi)-|J'(\xi)|][\mathrm{Re}\xi(x)-\mathrm{Re}(\exp(i\alpha_\xi)\xi(y))]\}$$

where the sum is over all characters on X_Λ. By i) and the hypothesis this is a sum as in (III.7). The exponential of functions of $\mathcal{D}$ is in $\mathcal{D}$ because the series of the exponential has positive coefficients.

iii) [25] Using the identity
$\cos(a)\cos(b)=\frac{1}{2}[\cos(a+b)+\cos(a-b)]$ one has:

$$\exp[\lambda \mathrm{Re}(\xi(x))\ \mathrm{Re}(\xi(y))]=\exp\{\lambda/2\ \cos[\Sigma(\Phi_a(y)+\Phi_a(x))]\}$$
$$\times \exp\{\lambda/2\ \cos[\Sigma(\Phi_a(y)-\Phi_a(x))]\}$$

Each function is an F as in the lemma.∎

From ii) above and corollary 1:

Corollary III.4: If $J>|J'|$ and Λ a finite subset of L, then for every $D\epsilon \mathcal{D}_\Lambda$

$$\int D(x,y)\ \rho_\Lambda(dx)\ \rho'_\Lambda(dy) \geq 0 \qquad \text{(III.11)}$$

3.2 CORRELATION INEQUALITIES

This section is included for the sake of completeness. It includes the basic properties of the state ρ^+ needed for the generalization of Slawny's theory discussed in next section. The results are all well known, except perhaps for two crucial observations (theorems III.6 i) and III.7 ii) below) that were first discussed in [23]. As a preliminary result let's state the observation made at the begining of the proof of proposition II.20. By Fourier expansion of the exponent and termwise integration, or by changing variables in the integral, one can prove:

Proposition III.5: If J is a real interaction, then for all finite $\Lambda \subset L$, and all characters ξ on X_Λ, $\rho_\Lambda(\xi)$ is a real number.

Theorem III.6: (Correlation inequalities)

i) (Generalized first Griffiths inequality)

$$J \geq |J'| \Rightarrow \rho_\Lambda(\mathrm{Re}(\xi)) \geq \rho'_\Lambda(\mathrm{Re}(e^{i\alpha}\xi))$$

for every finite $\Lambda \subset L$, $\xi \epsilon \hat{X}_\Lambda$, and $\alpha \epsilon R$.

ii) (First Griffiths inequality)

$$J \geq 0 \Rightarrow \rho_\Lambda(f) \geq 0$$

for every finite $\Lambda \subset L$, and every positive definite integrable function f on X_Λ.

iii) (Second Griffiths inequality)

$$J \geq 0 \Rightarrow \rho_\Lambda(fg) \geq \rho_\Lambda(f)\rho_\Lambda(g)$$

for every finite $\Lambda \subset L$, and every positive definite square integrable real valued functions f,g on X_Λ.

Proof: i)[23]

$$\rho_\Lambda(\mathrm{Re}(\xi)) - \rho'_\Lambda(\mathrm{Re}(e^{i\alpha}\xi)) = \int [\mathrm{Re}(\xi(x)) - \mathrm{Re}(e^{i\alpha}\xi(y))] \times [\mathrm{Re}(\xi(x)) - \mathrm{Re}(e^{i\alpha}\xi(y))]\, \rho_\Lambda(dx)\, \rho'_\Lambda(dy)$$

Each factor of the integrand is in $\mathcal{D}$ (Corollary III.3 i), hence the integral is non-negative by corollary III.4.

ii) From i) and proposition III.5, $\rho_\Lambda(\xi) \geq 0$ for every $\xi \in X_\Lambda$. Then use that every positive definite integrable function has a Fourier series with positive coefficients that converges absolutely and hence it can be termwise integrated.

ii) [10] If $\xi, \eta \in \hat{X}_\Lambda$:

$$\rho_\Lambda(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta)) - \rho_\Lambda(\mathrm{Re}(\xi))\ \rho_\Lambda(\mathrm{Re}(\eta)) = \tfrac{1}{2}\int [\mathrm{Re}(\xi(x))-\mathrm{Re}(\xi(y))]$$

$$\times\ [\mathrm{Re}(\eta(x))-\mathrm{Re}(\eta(y))]\ \rho_\Lambda(dx)\ \rho_\Lambda(dy)$$

Hence by corollaries III.2, III.3 i), and III.4:

$$\rho_\Lambda(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta)) \geq \rho_\Lambda(\mathrm{Re}(\xi))\ \rho_\Lambda(\mathrm{Re}(\eta)) \tag{III.13}$$

If f,g are square integrable on X_Λ; f, g, and fg are integrable. Positive definite functions have absolutely summable Fourier series, hence termwise integration is allowed. If the functions are real valued, their Fourier series are in terms of the real part of the characters; and if they are positive definite, the coefficients of the series are positive. Then termwise integration together with (III.13) yield the proposed inequality.∎

For every finite $\Lambda \subset L$ and every boundary condition y, the Gibbsian measure ρ_Λ^y can be written as a Gibbsian measure with free boundary conditions but for a different interaction. Indeed, as every character ξ such that $\underline{\xi} \cap \Lambda \neq \phi$ can be written as $\xi = \xi'\xi''$ with $\underline{\xi}' \subset \Lambda$ and $\underline{\xi}'' \subset L\setminus\Lambda$, one has that

$$\rho_\Lambda^y(J) = \rho_\Lambda(J^{y,\Lambda}) \tag{III.14}$$

with

$$J^{y,\Lambda}(\xi) = \sum_{\eta \in X_{L\setminus\Lambda}} J(\xi\eta)\ \eta(y) \tag{III.15}$$

If J is a ferromagnetic interaction, in general $J^{y,\Lambda}$ is no longer ferromagnetic except if $y_a=1$ for every $a\in L$. Denoting ρ_Λ^+ the measure for such boundary condition one has that

$$\rho_\Lambda^+(J) = \rho_\Lambda(J^{+,\Lambda}) \tag{III.16}$$

with

$$J^{+,\Lambda}(\xi) = \sum_{\eta\in X_{L\setminus\Lambda}} J(\xi\eta) \tag{III.17}$$

Obviously

$$J^{+,\Lambda} \geq |J^{y,\Lambda}| \tag{III.18}$$

for every $y\in X_{L\setminus\Lambda}$; therefore by theorem III.6 i)

$$\rho_\Lambda^+(\mathrm{Re}(\xi)) \geq \rho_\Lambda^y(\mathrm{Re}(e^{i\alpha}\xi)) \tag{III.19}$$

for every $\xi\in\hat{X}_\Lambda$, $\alpha\in R$, $y\in X_{L\setminus\Lambda}$. Thus, if N,Λ are finite sets, $\Lambda\subset N\subset L$, and $\xi\in\hat{X}_\Lambda$, one has from (II.11) and (III.19):

$$\rho_\Lambda^+(\mathrm{Re}(\xi))=(1/Z_N^+)\int_{X_N}\rho_\Lambda^{z\vee y}(\mathrm{Re}(\xi))\,\rho_N^+(dx_\Lambda\times dz_{N\setminus\Lambda})$$

$$\leq (1/Z_N^+)\int_{X_N}\rho_\Lambda^+(\mathrm{Re}(\xi))\,\rho_N^+(dx_\Lambda\times dz_{N\setminus\Lambda}) = \rho_\Lambda^+(\mathrm{Re}(\xi))$$

Hence, by proposition III.5:

$$\rho_N^+(\xi) \leq \rho_\Lambda^+(\xi) \tag{III.20}$$

for every $\xi\in\hat{X}_\Lambda$. This means that for ferromagnetic systems, for each character ξ the net $\{\rho_\Lambda^+(\xi)$: Λ finite with $\underline{\xi}\subset\Lambda\subset L\}$ is

convergent because it is decreasing and bounded below (by zero due to the first Griffiths inequality). The state defined by such limits is a Gibbs state denoted as ρ^+. With this state ρ^+ we can prove the "infinite lattice version" of corollary III.4:

Proposition III.7: If $J \geq |J'|$, then for each $\mu \in \Delta(J')$:

$$\int D(x,y)\ \rho^+(dx)\ \mu(dy) \geq 0$$

where either

i) $D \in \mathcal{D}_M$ for some finite $M \subset L$, or

ii) $D \in \mathcal{D}$ is continuous.

Proof: For case i), from (III.18) and corollary III.4

$$\int_{X_\Lambda \times X_\Lambda} D(x,y)\ \rho^+_\Lambda(dx)\ \rho^z_\Lambda(dy) \geq 0$$

for every finite Λ with $M \subset \Lambda \subset L$. Integrating both sides with respect to $\mu(dz)$ and taking the limit $\Lambda \to \infty$ one obtains the thesis.

For case ii), consider the net

$$\{D_\Lambda(x,y)=D(x_\Lambda \vee 1, y_\Lambda \vee 1):\ \Lambda \subset L \text{ finite}\}$$

$D_\Lambda \to D$ pointwise and $\|D_\Lambda\|_\infty \leq \|D\|_\infty$. By case i) and dominated convergence, D satisfies the thesis.∎

Other properties of ρ^+ needed in the sequel are:

Theorem III.8: Consider a ferromagnetic interaction J:

i) If $J \geq |J'|$, then for every $\mu \in \Delta(J')$, $\xi \in \hat{X}$, $\alpha \in R$

$$|\mu(Re(e^{i\alpha}\xi))| \leq \rho^+(Re(\xi))$$

ii) If $\mu(Re(\xi)) = \rho^+(Re(\xi))$ for some $\mu \in \Delta(J)$, $\xi \in \hat{X}$, then $\mu(Im(\xi)) = 0$.

iii) $\rho^+ \in \mathcal{C}(\Delta(J))$.

iv) If $J \in \mathcal{B}_I$; ρ^+ is translation invariant.

v) $\rho^+(f) \geq 0$ for every integrable positive definite function on X.

vi) $\rho^+(\xi) > 0$ for every $\xi \in B$.

Proof: i) As in theorem III.6 i), using proposition III.7.

ii)[23] For any $\alpha \in R$, $\xi \in \hat{X}$

$$\mu(Re(e^{\pm i\alpha}\xi)) = \cos(\alpha)\, \mu(Re(\xi)) \mp \sin(\alpha)\, \mu(Im(\alpha))$$

From i) and the hypothesis this implies that

$$\cos(\alpha)\, \rho^+(Re(\xi)) \mp \sin(\alpha)\, \mu(Im(\xi)) \leq \rho^+(Re(\xi))$$

for any $\alpha \in R$; i.e., if α is positive and small:

$$\pm\mu(Im(\xi)) \leq \rho^+(Re(\xi))[(1-\cos(\alpha))/\sin(\alpha)] \underset{\alpha \to 0}{\to} 0$$

iii) Suppose there exist $\rho_1; \rho_2 \in \Delta(J)$, $0<\lambda<1$, such that

$$\rho^+(Re(\xi)) = \lambda\rho_1(Re(\xi)) + (1-\lambda)\rho_2(Re(\xi))$$

As λ and $1-\lambda$ are positive and from i) both $\rho_1(\mathrm{Re}(\xi))$ and $\rho_2(\mathrm{Re}(\xi))$ are smaller or equal than $\rho^+(\mathrm{Re}(\xi))$, the previous formula forces

$$\rho_1(\mathrm{Re}(\xi)) = \rho_2(\mathrm{Re}(\xi)) = \rho^+(\mathrm{Re}(\xi))$$

then by ii) $\rho_1(\xi)=\rho_2(\xi)=\rho^+(\xi)$ for every character ξ. Hence, $\rho_1=\rho_2=\rho^+$.

iv) If Z^ν acts on L, then for $\xi\epsilon\hat{X}$, $a\epsilon Z^\nu$, $\tau_a\xi$ is also a character. If $J\epsilon\mathfrak{B}_I$, then $\tau_a\mu\epsilon\Delta(J)$ if $\mu\epsilon\Delta(j)$ and $a\epsilon Z^\nu$. Using repeatedly i) and the fact that from ii) $\rho^+(\eta)\epsilon R$ for every character η, one gets that for all characters ξ, $a\epsilon Z^\nu$:

$$\rho^+(\tau_{-a}\xi) = \tau_a\rho^+(\xi) \leq \rho^+(\xi) = \tau_{-a}\rho^+(\tau_{-a}\xi) \leq \rho^+(\tau_{-a}\xi)$$

Hence $\tau_a\rho^+(\xi)=\rho^+(\xi)$ for every character ξ and $a\epsilon Z^\nu$; which implies the translation invariance of ρ^+.

v) From (III.16) and the first Griffiths inequality one gets in the limit $\Lambda \to \infty$:

$$\rho^+(\xi) \geq 0$$

for every character ξ. For integrable positive definite functions the result follows from a termwise integration of the Fourier series.

vi) If $\Lambda\supset\underline{\xi}$, $J^{+,\Lambda} \geq J'$ with

$$J'(\eta)=\begin{cases} J(\xi) & \text{if } \xi=\eta \\ 0 & \text{otherwise} \end{cases}$$

Then from theorem III.6 i):

$$\rho_\Lambda^+(\xi) \geq \rho'_{\underline{\xi}}(\xi) = \int_{\underline{\xi}} \xi(y)\, \exp[J(\xi)\, \mathrm{Re}(\xi(y))]\, dy \,/\, Z_{\underline{\xi}}$$

Expanding the exponential in power series and termwise integrating one gets a series of terms which are non negative because $J(\xi)>0$ and the characters are orthonormal. Neglecting all but the one corresponding to first order in $J(\xi)$ one gets

$$\rho_\Lambda^+(\xi) \geq J(\xi) \int_{\underline{\xi}} \xi(y)\, \mathrm{Re}(\xi(y))\, dy \,/\, Z_{\underline{\xi}} = J(\xi)\,/(2Z_{\underline{\xi}}) > 0.\blacksquare$$

3.3 TRANSLATION INVARIANT GIBBS STATES FOR FERROMAGNETIC SYSTEMS

In this section the inverse temperature β will be introduced. This is accomplished by fixing a lattice L, spaces $(\Omega_i)_{i\in L}$ and an interaction $J\in\mathcal{B}$; and considering the family of models $(L,(\Omega_i)_{i\in L},\beta J)$. In this section we will consider $L\subset R^\nu$ such that Z^ν acts on L and L_o is finite. The interaction J will be assumed ferromagnetic and translation invariant.

The groups S and $\bar{B}$ defined in (II.31) are β-independent, on the other hand the set

(III.21) $$B^+(\beta)=\{\xi\epsilon\hat{X}:\ \rho^+_{(\beta)}(\xi)>0\}$$

is certainly β-dependent. The crucial result for Slawny's theory is that B^+ is a group. For spin 1/2 models this fact follows almost immediately from the correlation inequalities [28], but the proof is more involved in the general case. Such proof is due to Pfister [25] and it is based in the first result of the following lemma:

Lemma III.9: [25] Consider characters ξ, η

i) $\rho^+(\xi)=0$, $\rho^+(\eta)>0 \Rightarrow \rho^+(\xi\eta)=0$

ii) If $\mu\epsilon\Delta(J)$, then:

$$\rho^+(Re(\xi))=\mu(Re(\xi)),\ \rho^+(Re(\eta))=\mu(Re(\eta))>0 \Rightarrow \rho^+(Re(\xi\eta))=\mu(Re(\xi\eta));$$

$$\rho^+(Re(\xi\eta^{-1}))=\mu(Re(\xi\eta^{-1}));$$

(Note that from proposition III.5 and theorem III.8 ii), the prefix "Re" can be omitted in all the statements of the lemma).

Proof: i) By corollary III.3 iii) and proposition III.7:

$$\int[Re(\eta(x))+Re(\eta(y))]\exp[-\lambda Re(\xi(x))Re(\xi(y))]\rho^+(dx)\rho^+(dy) > 0$$

for any $\lambda>0$. Expanding the exponential one gets:

$$0 \leq \int[\mathrm{Re}(\eta(x))+\mathrm{Re}(\eta(y))][1-\lambda\mathrm{Re}(\xi(x))\mathrm{Re}(\xi(y))]\rho^+(dx)\rho^+(dy) + o(\lambda^2)$$

$$= 2\rho^+(\mathrm{Re}(\eta))-2\lambda\rho^+(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))\rho^+(\mathrm{Re}(\xi)) + o(\lambda^2)$$

Using that $\rho^+(\mathrm{Re}(\xi))=0$ and dividing by λ one obtains

$$0 \geq \rho^+(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))\rho^+(\mathrm{Re}(\xi)) + o(\lambda^2)/\lambda$$

for every $\lambda>0$. The first summand is non-negative by part v) of theorem III.8, hence taking the limit $\lambda \to 0$:

$$\rho^+(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))\rho^+(\mathrm{Re}(\xi)) = 0$$

As by hypothesis $\rho^+(\mathrm{Re}(\xi))>0$:

$$\rho^+(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta)) = 0$$

This, together with the identity

(III.23) $$\cos(a)\cos(b) = \tfrac{1}{2}[\cos(b+a)+\cos(b-a)]$$

implies that

$$\rho^+(\mathrm{Re}(\xi\eta))+\rho^+(\mathrm{Re}(\xi\eta^{-1})) = 0$$

As both summands are non-negative (theorem III.8 v), they both must be zero. From proposition III.5, this implies the thesis.

ii) By corollary III.3 iii) and proposition III.7:

$$0 \leq \int[\mathrm{Re}(\xi(x))-\mathrm{Re}(\xi(y))]\exp[\pm\lambda \mathrm{Re}(\eta(x))\mathrm{Re}(\eta(y))]\rho^{+}(dx)\mu(dy)$$

$$= \int[\mathrm{Re}(\eta(x))-\mathrm{Re}(\eta(y))][1\pm\lambda \mathrm{Re}(\xi(x))\mathrm{Re}(\xi(y))]\rho^{+}(dx)\mu(dy) + O(\lambda^2)$$

Considering $\lambda>0$ and using that ρ^{+} and μ coincide on $\mathrm{Re}(\xi)$ and $\mathrm{Re}(\eta)$, one gets:

$$|\rho^{+}(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))-\mu(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))|\ \rho^{+}(\mathrm{Re}(\eta)) + o(\lambda^2)/\lambda \leq 0$$

As $\rho^{+}(\mathrm{Re}(\eta))>0$, one has in the limit $\lambda \to 0$:

$$\rho^{+}(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta)) = \mu(\mathrm{Re}(\xi)\ \mathrm{Re}(\eta))$$

By (III.23) this implies

$$\rho^{+}(\mathrm{Re}(\xi\eta))+\rho^{+}(\mathrm{Re}(\xi\eta^{-1})) = \mu(\mathrm{Re}(\xi\eta))+\mu(\mathrm{Re}(\xi\eta^{-1}))$$

As each summand of the LHS is larger than the corresponding summand of the RHS (theorem III.8 i), there must be equality summandwise.∎

Theorem III.10:

i) For every β, $B^{+}(\beta)$ is a subgroup of $\hat{X}$ and $B^{+}(\beta)\supset\bar{B}$.

ii) $\beta_1\geq\beta_2 \Rightarrow B^{+}(\beta_1)\supset B^{+}(\beta_2)$.

Proof: i)Consider ξ, η in B^{+}. We have to prove that $\rho^{+}(\xi^{-1}\eta)>0$. But if it were not, by lemma III.9 i)

$$\rho^{+}(\eta) = \rho^{+}(\xi(\xi^{-1}\eta)) = 0$$

against the fact that $\eta \epsilon B^+$. By part vi) of theorem III.8 B^+ contains the generators of $\bar{B}$.

ii) If $\beta_1 \geq \beta_2$ then $\beta_1 J \geq \beta_2 J$. Then the thesis follows from part i) of theorem III.7.∎

In analogy to the relation

(III.26) $$S = \bar{B}^{\perp}$$

one defines

(III.27) $$S^+(\beta) = [B^+(\beta)]^{\perp}$$

As $B^+ \supset \bar{B}$, the subgroup S^+ is contained in S:

(III.28) $$S^+(\beta) = \{x \epsilon S: \xi(x) = 1 \text{ for every } \xi \epsilon B^+\}$$

As ρ^+ is translation invariant, so are $\bar{B}$, B^+ and hence its anihilators S, S^+. Moreover the group S acts on X through the group product:

$$S \times X \to X$$
$$(y,x) \to yx;$$

with

$$(yx)_a = y_a x_a; \qquad a \epsilon L$$

which induces an action of S on $\hat{X}$ and E(X) by successive application of the prescription (II.18). The action on $\hat{X}$ is

$$(y\xi)(x) = \xi(y^{-1}x); \quad y\in S,\ \xi\in\hat{X},\ x\in X$$

or, as characters are group homomorphisms,

(III.29) $$y\xi = \xi(y^{-1})\ \xi$$

For the action on E(X), let's denote

$$y\mu = \mu_y$$

From (II.18) and (II.29) one gets

(III.30) $$\mu_y(\xi) = \xi(y)\ \mu(\xi); \qquad \xi\in\hat{X},\ y\in S,\ \mu\in E(X)$$

The elements of S define symmetries in the sense that their action leaves the interaction invariant. Indeed, from (III.29) and proposition II.19 i), $y\xi=\xi$ for every $y\in S$, $\xi\in B$. The interpretation of the subgroup of symmetries S^+ is contained in the following proposition.

Proposition III.11:

i) Let $y\in S$, then

$$\rho_y^+ = \rho^+ \iff y\in S^+$$

ii) $\mu_y = \mu$ for every $y\in S^+$, $\mu\in\Delta(J)$

Proof: i) Immediate from (III.30) and (III.28)

ii) If $y \in S^+$, $\mu \in \Delta(J)$, by (III.30) and (III.28):

$$\mu_y(\xi) = \mu(\xi) \quad \text{for each } \xi \in B^+$$

If $\xi \in B^+$, from theorem III.7 i)

$$\delta(\mathrm{Re}(\xi)) = \rho^+(\xi) = 0, \quad \text{for every } \delta \in \Delta(J)$$

Then, from part ii) of the same theorem $\delta(\xi)=\rho^+(\xi)=0$. In particular

$$\mu(\xi) = \mu_y(\xi) = 0; \qquad \xi \in B^+. \blacksquare$$

Mathematically, i) means that S^+ is the isotropy group of ρ^+ under the action of S. Physically it means that S^+ is the set of symmetries of S that are not broken by ρ^+. In fact, by ii) the symmetries of S^+ are never broken; but Gibbs states other than ρ^+ may preserve more symmetries than those. With respect to S, the state ρ^+ has minimal symmetry. We now state without proof an important result that is in fact valid for more general symmetries than those of S [4].

Theorem III.12: If $y \in S$

i) $\mu \in \Delta(J) \Rightarrow \mu_y \in \Delta(J)$

ii) $\mu \in \mathcal{E}(\Delta(J)) \Rightarrow \mu_y \in \mathcal{E}(\Delta(J))$

Combining this theorem with proposition 11 ii) we conclude that the action

$$(III.31) \qquad S/S^+ \times \Delta(J) \to \Delta(J)$$
$$([y],\mu) \to \mu_y$$

is well defined and it is by homeomorphisms. Moreover it leaves $\mathcal{E}(\Delta(J))$ invariant, hence it is also an action on the set of pure phases. Slawny [28] used the S/S^+-orbit of ρ^+ to establish a convenient description of the set

$$(III.32) \qquad \Delta^+(\beta)=\{\mu\in\Delta(\beta J):\ \mu(Re(\xi))=\rho^+(Re(\xi)),\ \xi\in\bar{B}\}$$

Theorem III.13: [28] For each $\mu\in\Delta^+$ there exists a unique probability measure δ on S/S^+ such that

$$(III.33) \qquad \mu = \int_{S/S^+} \rho^+_{[y]}\ \delta(d[y])$$

Proof: a) Existence.

As $\Delta(J)$ is a Choquet simplex, there exists a probability measure λ supported on $\mathcal{E}(\Delta(J))$ such that

$$(III.34) \qquad \mu = \int_{\mathcal{E}(\Delta(J))} \omega\ \lambda(d\omega)$$

Pick $\xi\in B$, by hypothesis

$$\mu(Re(\xi)) = \rho^+(Re(\xi)),$$

therefore, by the positivity of the measure λ and theorem III.8 i):

$$\omega(\mathrm{Re}(\xi)) = \rho^{+}(\mathrm{Re}(\xi)) \tag{III.35}$$

for λ-almost all ω. As B is countable, the set

$$\{\omega\in\Delta(J):\ \omega(\mathrm{Re}(\xi))\neq\rho^{+}(\mathrm{Re}(\xi)) \text{ for some } \xi\in B\}$$

has λ-measure 0. Therefore:

$$\mathrm{supp}(\lambda) \subset \Delta^{+}\cap \mathcal{E}(\Delta(J))$$

Claim: If $\omega\in[\Delta^{+}\cap \mathcal{E}(\Delta(J))]$, then there exists an $y\in S$ such that $\omega=\rho^{+}_{y}$.

Indeed, consider

$$\bar{\omega} = \int_S \omega_y\, dy, \qquad \bar{\rho}^{+} = \int_S \rho^{+}_{y}\, dy \tag{III.36}$$

where dy is the normalized Haar measure on S. For a character ξ one has, from (III.30):

$$\bar{\omega}(\xi) = \omega(\xi) \int_S \xi(y)\, dy$$

As the restriction of ξ defines a character on S, by orthonormality of the characters of a compact abelian group the integral on the right is zero unless $\xi\in S^{\perp}=\bar{B}$. Hence

$$\bar{\omega}(\xi)=\begin{cases}\omega(\xi)=\rho^{+}(\xi) & \text{if } \xi\epsilon\bar{B}\\ 0 & \text{otherwise}\end{cases}$$

Hence, $\bar{\omega}=\rho^{+}$ if $\omega\epsilon\Delta^{+}$; in particular

(III.37) $$\bar{\omega} = \bar{\rho^{+}}$$

The map

(III.38) $$S \to \Delta(J)$$
$$y \to \omega_y$$

is continuous, hence translating the measure dy on S via such map one gets a measure λ_ω on $\Delta(J)$. As $\omega\epsilon\,\mathcal{E}(\Delta(J))$, so do all ω_y (theorem III.12 ii), hence λ_ω is a probability measure supported on $\mathcal{E}(\Delta(J))$. Changing variables, (III.36) yields:

(III.39) $$\bar{\omega} = \int_{\mathcal{E}(\Delta(J))} \varepsilon\ \lambda_\omega(d\varepsilon); \qquad \bar{\rho^{+}} = \int_{\mathcal{E}(\Delta(J))} \varepsilon\ \lambda_{\rho^{+}}(d\varepsilon)$$

By (III.37) and the uniqueness of the Choquet decomposition

$$\lambda_\omega = \lambda_{\rho^{+}}$$

in particular

$$\mathrm{supp}(\lambda_\omega) \cap \mathrm{supp}(\lambda_{\rho^{+}}) \neq \phi$$

this in turn implies that

$$\{\omega_y : y \in S\} \cap \{\rho^+_y : y \in S\} \neq \phi$$

Hence, there exist x, y in S such that

$$\omega_x = \rho^+_y \Rightarrow \omega = \rho^+_{yx^{-1}}$$

which proves the claim.

The measure δ is obtained translating the measure λ on (III.34) to the orbit of ρ^+ via the action (III.31).

b) Uniqueness.
Each measure δ satisfying (III.33) can be translated to $\Delta(J)$ via the action (III.31). Under such translation (III.33) yields a decomposition of μ in extremal points of $\Delta(J)$. By uniqueness of such decomposition the measure δ must be unique.∎

As the action of Z^ν on S leaves S^+ invariant, it can be defined on the quotient S/S^+ and hence on the measures of S/S^+.

Proposition III.14: The map

(III.40) $$E(S/S^+) \to \Delta^+$$
$$\delta \to \mu_\delta = \int \rho^+_{[y]}\, \delta(d[y])$$

is a bijection, preserves convex combinations and satisfies

$$\tau_a \delta = \tau_a \mu_\delta$$

for every $a \in Z^\nu$. Hence the correspondence preserves extremality and the properties: translation invariance, ergodicity, periodicity, and quasiperiodicity.

Proof: The bijectiveness was proven in the previous theorem, the remaining statements are a direct consequence of the uniqueness part of the theorem.∎

This proposition is of use only if we can prove that Δ^+ contains enough interesting Gibbs states. This is proved in the next theorem. Let's first recall that the pressure for a finite region $\Lambda \subset L$ and boundary condition y is:

$$P_\Lambda^y = (1/|\Lambda|) \ln Z_\Lambda^y \qquad \text{(III.41)}$$

It is well known that:

Proposition III.15 [17]:

i) For each finite $\Lambda \subset L$, $y \in X$, the function $J \to P_\Lambda^y(J)$ is convex,

ii) For $J \in \mathcal{B}_I$

$$P(J) = \lim_{\Lambda \to \infty} P_\Lambda^y(J) \qquad \text{(III.42)}$$

exists, it is independent of y, and the convergence is uniform in y. The function P(J) is called the pressure.

iii) The function $\mathcal{B}_I \ni J \to P(J) \in R$ is convex and continuous.

If a family of interactions $(\beta J)_{\beta\geq 0}$ is considered then the function $P(\beta)=P(\beta J)$ is also convex and continuous.

Lemma III.16 [22]: If $P(\beta)$ is differentiable at β_o

$$(III.43) \quad (\partial P/\partial\beta)(\beta_o) = (1/|L_o|) \sum_{\xi\in B} J(\xi)\mu(\xi)$$

for every $\mu\in\Delta_I(\beta_o J)$.

Proof: For a positive integer n, denote J^n the finite range truncation

$$J^n(\xi)=\begin{cases} J(\xi) & \text{if } \operatorname{diam}(\xi)\leq n \\ 0 & \text{otherwise} \end{cases}$$

and for positive integers j, k, with $j\leq k$, denote C_j the cube in Z^ν of side 2j centered at 0, $\delta_k C_j$ its internal k-boundary, and Λ_j the set formed by L_o copies of C_j:

$$C_j=\{a\in Z^\nu:\ |a_i|\leq j,\ i=1,\ldots,\nu\}$$

$$\delta_k C_j=\{a\in Z^\nu:\ j-k\leq|a_i|\leq j,\ i=1,\ldots,\nu\}$$

$$\Lambda_j=\bigcup_{a\in L}\tau_a C_j$$

$$\delta_k\Lambda_j=\bigcup_{a\in L}\delta_k(\tau_a C_j)$$

For finite $\Lambda\subset L$ one easily checks that

$$(III.44) \quad (\partial P_\Lambda^y/\partial\beta)(\beta_o) = (1/|\Lambda|)\,\rho^y_{(\beta_o)}(H_\Lambda^y)$$

From now on the argument β_o will be omitted in most of the expressions. For the finite range interaction J^n only the points at a distance less than n from the boundary "feel" the boundary condition. In particular

$$(III.45) \qquad H^{\gamma}_{\Lambda_j}(J^n) = \Sigma\, J^n(\xi)\,\xi + \Sigma\, (J^n)^{\gamma,\Lambda}(\xi)\,\xi$$

where the second sum runs over all $\xi\epsilon B$ such that $\xi\cap\delta_n\Lambda_j\neq\phi$, and the first sum over the remaining $\xi\epsilon B_{\Lambda_j}$. Now denote H_j the Hamiltonian obtained summing all the bonds with support intersecting Λ_j:

$$H_j = \sum_{a\epsilon L} \sum_{\xi\epsilon B: a\epsilon\xi} \sum_{b\epsilon C} J(\xi)\,\tau_b\xi$$

One has that

$$(III.46) \qquad \|H^{\gamma}_{\Lambda_j}(J^n) - H_j\|_\infty \leq \|J-J^n\|\,|\Lambda_j\backslash\delta_n\Lambda_j| + 2\|J\|\,|\delta_n\Lambda_j|$$

From (III.44) and (III.46):

$$(III.47) \qquad |\mu[(\partial P^{(\bullet)}_{\Lambda_j}/\partial\beta) - (1/|\Lambda_j|\rho^{(\bullet)}_{\Lambda_j}(H_j)]| \leq$$

$$\leq 2\|J-J^n\|\,|\Lambda_j\backslash\delta_n\Lambda_j|/|\Lambda_j| + 2\|J\|\,|\delta_n\Lambda_j|/|\Lambda_j|$$

By translation invariance

$$(III.48) \qquad \mu[\rho^{(\bullet)}_{\Lambda}(H_j)] = \mu(H_j) = |C_j| \sum_{\xi\epsilon B} J(\xi)\,\mu(\xi)$$

Moreover the functions

$$f^{\gamma}_j(\beta) = P^{\gamma}_{\Lambda_j}(\beta)$$

are differentiable, convex, and

$$f_j^y(\beta) \underset{j}{\to} P(\beta)$$

In such situation, by a property of the convex functions, if $(\partial P/\partial\beta)(\beta_o)$ exists:

$$(\partial f_j^y/\partial\beta)(\beta_o) \underset{j}{\to} (\partial P/\partial\beta)(\beta_o)$$

pointwise in y. Also, from (III.44)

$$|(\partial f_j^y/\partial\beta)(\beta_o)| \leq \|J\|$$

hence, by dominated convergence:

$$\lim_j \mu(\partial P_{\lambda_j}^{(\bullet)}/\partial\beta) = \mu[(\partial P/\partial\beta)(\beta_o)] = (\partial P/\partial\beta)(\beta_o) \tag{III.49}$$

Taking limit over j in (III.47) and using (III.48) and (III.49), one obtains

$$|(\partial P/\partial\beta)(\beta_o) - (1/|L_o|) \sum_{\xi\in B} J(\xi)\mu(\xi)| \leq 2\|J-J^n\|$$

The limit $n \to \infty$ yields the thesis.∎

Theorem III.17: If $P(\beta)$ is differentiable at β_o $\Delta^+(\beta_o J)$ contains all Gibbs states that are invariant, periodic or quasiperiodic.

Proof ([15] appendix B, and reference cited therein) :

i) Case μ translation invariant.

By (III.43) and the fact that ρ^+ is translation invariant:

$$\sum_{\xi\epsilon B} J(\xi)\ \mu(Re(\xi)) = \sum_{\xi\epsilon B} J(\xi)\ \rho^{+}(Re(\xi))$$

As $J(\xi)>0$ and $\mu(Re(\xi))\leq\rho^{+}(Re(\xi))$, one obtains

$$\mu(Re(\xi)) = \rho^{+}(Re(\xi))$$

for every $\xi\epsilon B_{o}$. By translation invariance, the same equality extends to every $\xi\epsilon B$. From part ii) of lemma (III.9) the equality holds on B.

ii) Case μ periodic.

Let G_{o} be the subgroup of Z^{ν} that leaves μ invariant, and $G=Z^{\nu}/G_{o}$. Define

$$\mu = (1/|G|) \sum_{[a]\epsilon G} \tau_{[a]}\mu$$

This Gibbs state is translation invariant, hence from i), $\mu(\xi)=\rho^{+}(\xi)$ for every $\xi\epsilon B$. From this and the maximality of ρ^{+} one gets as in i) that

$$\rho^{+}(\xi) = \tau_{[a]}\mu(\xi)$$

for every $[a]\epsilon G$, $\xi\epsilon B$.

iii) The proof for the case μ quasiperiodic can be consulted in [15] app. B.∎

Combining this theorem with proposition III.14 one obtains:

Proposition III.18: If $P(\beta)$ is differentiable at β_o, the map (III.40) defines a one-one correspondence between translation invariant (resp. periodic, quasiperiodic) Gibbs states at β_o and the translation invariant (periodic, quasiperiodic) probability measures on $S/S^+(\beta_o)$. Such correspondence preserves extremality.

It well known that $P(\beta)$ is analytic at sufficiently high and sufficiently low β, and, moreover, as $P(\beta)$ is convex, it is differentiable for almost all values of β. If at an inverse temperature β_o where $P(\beta_o)$ is differentiable, there is a phase transition which changes the number of quasiperiodic pure phases, then by Slawny's theory the subspace $B^+(\beta)$ changes at β_o. In fact, from theorem III.10 ii) $B^+(\beta_o+\varepsilon) \supset B^+(\beta_o-\varepsilon)$ for ε small enough. The order parameters for the phase transition at β_o are the elements of $B^+(\beta_o+\varepsilon) \setminus B^+(\beta_o-\varepsilon)$.

Chapter IV

MONTE-CARLO SIMULATION OF A SPIN 1/2 MODEL

4.1 THE MODEL

For models with $\Omega=G_2$ there is only one non-trivial character in Ω. Hence, there is a one-one correspondence

(IV.1) $$X \to \{\text{finite subsets of } L\}$$
$$\xi \to \underline{\xi}$$

This correspondence is in fact a group isomorphism if the image is endowed with the symmetric difference as the group operation. The character with support on a set $A \subset L$ will be denoted in the sequel s_A:

$$s_A(x) = \prod_{a \in A} x_a; \qquad x \in X$$

Accordingly, for an interaction J we will denote $J(s_A)$ as J(A) and the bonds will be considered to be the elements of the set

$$B(J)=\{\text{finite } A \subset L: J(A) \neq 0\}$$

Moreover, if $L=Z^\nu$ the field structure of X (section 2.5) corresponds, via (IV.1), to the field structure of the set of finite subsets of Z^ν defined by the symmetric difference as the "sum" and the product

(IV.2) $$A \bullet D = \sum_{a \in A} \tau_a D$$

Here "Σ" is the symmetric difference. For this field the "zero" is the empty set and the "one" is the one point set $\{0 \in Z^\nu\}$.

The model chosen for the simulation is a translation invariant model in Z^ν with B_o formed by:

$A=\{(0,0);\ (0,1)\}$

$B=\{(0,0);\ (2,0)\}$

$C=\{(0,0);\ (0,1);\ (1,1);\ (1,0)\}$

The interaction defines a Hamiltonian

$$H_\Lambda = -J(A)[\Sigma\ s_{\tau_a A}] - J(B)[\Sigma\ s_{\tau_a B}] - J(C)[\Sigma\ s_{\tau_a C}]$$

where the sums range over $a \in Z^2$ such that $\tau_a A$, $\tau_a B$, $\tau_a C$ are in Λ, respectively. This model has the property of being soluble whenever any of J(A), J(B) or J(C) is zero. Indeed:

i) If J(C)=0, the lattice splits into two non-interacting sublattices, and the model of each sublattice is the well known two dimensional Ising model with zero magnetic field. Such model exhibits only one phase transition, which is of second order and occurs across the critical curve

(IV.3) $$sh(2J(A))\ sh(2J(B)) = 1$$

The onset of the phase transition is pointed by an abrupt change in the magnetization $\rho^{+}(s_{(0,0)})$. This order parameter is larger than zero for values of J(A), J(B) above the curve and it drops to zero at the curve. Therefore, for the Ising model , $S^{+}=\{1\}$ above (IV.3) and $S^{+}=\{-1,1\}=S$ below it. Thus, from the discussion of chapter III there are two translation invariant pure phases above (IV.3) and only one below it. In fact, there are no other pure phases because it has been proved that for the two dimensional Ising model there are no non-translation-invariant states. For our original model with J(C)=0, each Gibbs state is the tensor product of one Gibbs state for the Ising model in each of the sublattices. Hence, for coupling constants above (IV.3) there are four pure phases and only one below such curve. The phase transition is detected by the appearance of spontaneous magnetization, i.e. by the behavior of the order parameter $s_{(0,0)}$.

ii) If J(A)=0 the remaining fundamental bonds have a non-trivial common divisor for the product (IV.2). Indeed,

(IV.4) $B=D\bullet D$

$C=A\bullet D$

with

(IV.5) $D=\{(0,0);\ (1,0)\}$

By the Holsztynski-Slawny theory [15], the model reduces to an Ising model in the sense that if ρ'^{+} is the "+" state for an Ising model:

(IV.6) $$\rho^{+}(s_{D\bullet N}) = \rho'^{+}(s_N) ,$$

for every finite $N \subset Z^2$; and

(IV.7) $$\rho^{+}(s_M) = 0$$

if M is not divisible by D. Moreover the model has the same number of ergodic states as the Ising model. From (IV.6) and the known properties of the Ising model, one concludes that there are two ergodic states above the curve

(IV.8) $$sh(2J(B))\ sh(2J(C)) = 1$$

and only one below it. The phase transition is signaled by the behavior of $\rho'^{+}(s_{(0,0)})=\rho^{+}(s_D)$, i.e. the relevant order parameter is the "pseudomagnetization" s_D.

iii) If J(B)=0 one has:

$$A = e_o \bullet A$$

$$C = D \bullet A$$

where $e_o=\{(0,0)\}$. Hence by the Holsztynski- Slawny theory the model reduces to a one dimensional Ising model with magnetic field which does not exhibit any phase transition.

Therefore, there is only one ergodic phase for any value of J(A), J(C). By formulas analogous to (IV.6)-(IV.7) one concludes that in the case J(B)=0 both $\rho^+(s_{(0,0)})$ and $\rho^+(s_D)$ are zero because neither e_o nor D are divisible by A.

In conclusion,in the space of parameters (J(A),J(B),J(C)); the phase diagram for the ergodic phases on the boundary of the first octant has the features summarized in figure 1 To study the (ergodic) phase diagram inside the first octant (ferromagnetic region), a Monte-Carlo simulation was performed.

4.2 THE MONTE-CARLO TECHNIQUE

4.2.1 Advantages of the technique

The study of the phase diagram of our model involves the calculation of $\rho^+(h)$ for suitable order parameters h. A numerical evaluation therefore requires a double process: first, the numerical integration with measures ρ^+_Λ for finite $\Lambda \subset Z^2$; and second, the estimation of the limit $\Lambda \to \infty$. As to the first process, one has to evaluate integrals of the form

$$(IV.9) \qquad \langle h \rangle = \int h(x)\, f(x)\, dx$$

where dx is the counting measure in $G_2^{|\Lambda|}$ and

$$(IV.10) \qquad f(x) = \exp[-H^+_\Lambda(x)] \,/\, Z^+_\Lambda$$

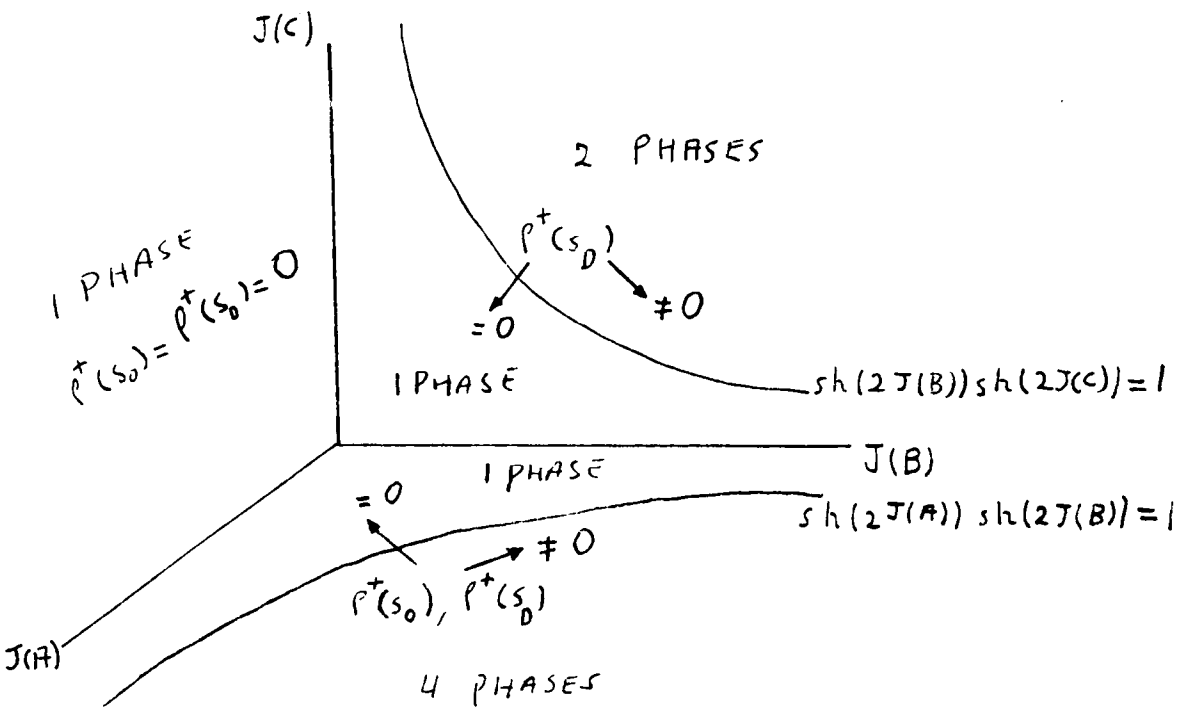

Figure 1: Number of ergodic phases and expectation values of order parameters in the boundary of the ferromagnetic region of the model

Two problems appear in evaluating (IV.9):

i) The number of dimensions of integration is very big (equals $|\Lambda|$),

ii) The factor Z_Λ^+ in the probability density f is imposible to calculate in practice.

The process of numerical integration consists in approximating $\langle h \rangle$ by a sum over N sample points:

$$\bar{h} = (1/N) \sum_{i=1}^{N} h(x_i)\, f(x_i) \qquad \text{(IV.11)}$$

It can be proved that for classical methods of numerical integration the difference between $\bar{h}$ and $\langle h \rangle$ grows, for fixed N, exponentially fast with the number of dimensions. In Monte-Carlo techniques the points are chosen randomly and the central limit theorem implies that

$$|\bar{h} - \langle h \rangle| \simeq \sigma/\sqrt{N} \qquad \text{(IV.12)}$$

where σ is the standard deviation. Therefore, for fixed N, the error in a Monte-Carlo integration is approximately independent of the number of dimensions. This makes such technique specially suitable for calculations in Statistical Mechanics. Moreover, it can be combined with the so-called importance sampling. This consists in replacing (IV.10) by

$$\bar{h} = (1/N) \sum_{i=1}^{N} h(x_i) \qquad \text{(IV.13)}$$

but with the points chosen with probability density f. Importance sampling improves the convergence specially when f is strongly peaked -as it is in (IV.10)- because it ensures that the more contributing points will be picked.

Nevertheless, the second problem discussed above remains, namely that the weight function f can not be evaluated. A technique has been developed to approximate such f through a Markov chain.

4.2.2 Markov chains

Let us review briefly the basic concepts of the theory of Markov chains [14]. Consider a system that has a countable set of possible states $\{S_i\}_{i\geq 1}$ and it can change of state only at discrete times. Denote x(i) the state of the system at time i. The conditional probability

$$P(x(t)=S_j \mid x(t_1)=S_{i_1},\ldots,x(t_n)=S_{i_n})$$

is the probability that the system be in the state S_j at time t given that it was in S_{i_1} at time t_1, etc. The system is a Markov chain if this conditional probability only depends on the immediately previous state:

$$(IV.14)\quad P(x(n)=S_j \mid x(1)=S_{i_1},\ldots,x(n-1)=S_{i_{n-1}}) = P(x(n)=S_j \mid x(n-1)=S_{i_{n-1}})$$

An stationary Markov process is a Markov process in which the probailities (IV.14) do not depend on t i.e.:

(IV.15) $$P(x(n)=S_j \mid x(n-1)=S_i) = P_{ij}$$

independently of n. These numbers define a matrix P of components P_{ij}. An stationary Markov process is:

stochastic if $P_{ii} \geq 0$ and $\sum_j P_{ij}=1$,

irreducible if for every i, j there exists n such that $(P^n)_{ij} \neq 0$,

periodic if it returns after multiples of a certain period d; i.e. $(P^n)_{ij} \neq 0$ if and only if n is a multiple of d.

The following theorem is the relevant result for our purposes:

Theorem IV.1: Consider a probability density $(f_i)_{i \geq 1}$. If a Markov process is characterized by a matrix P which is stochastic, aperiodic, irreducible, and such that

(IV.16) $$\sum_i f_i P_{ij} = f_j$$

Then

$$(P^n)_{ij} \underset{n}{\to} f_j.$$

In our case $f_i=f(x(i))$ for f given in (IV.10). Note that then the cumbersome factor Z_Λ^+ dissapears in (IV.16); i.e. the matrix P_{ij} may be defined with no reference to partition functions at all. The Monte-Carlo technique for the statistics of a classical lattice system is then as follows: A Markov process is set in the condition of theorem IV.1. The system is left to evolve following this process for a large enough number n of steps and the configuration obtained is used to evaluate (IV.13). Then, starting from this configuration, the process is repeated so as to obtain another configuration and hence another term in (IV.13); etc. If n is large enough, all the configurations are obtained approximately with probability density f by theorem IV.1, hence the sum (IV.13) converges to <h> (importance sample).

4.2.3 Expression for the matrix P

The probabilities P_{ij} are in general calculated in two steps [14]:

i) Define an stochastic, aperiodic, irreducible matrix p_{ij}. This is interpreted as the probability that the state j be selected as a candidate for a new state of a system currently in the state i.

ii) Define a set of numbers $0<W_{ij}\leq 1$ such that

(IV.17) $$f_i W_{ij} = f_j W_{ji}$$

W_{ij} is interpreted as the probability that, having chosen j as the candidate for a new state, the system actually change into this state j.

The probability P_{ij} is defined combining both probabilities. If $i \neq j$, the probability of a transition $i \to j$ is the product of choosing j as candidate to new state times the probability of making the transition after the choice:

(IV.18) $$P_{ij} = p_{ij} W_{ij}, \qquad i \neq j$$

On the other hand, the probability for a system to stay in the current state i has two contributions: on the one side the probability of choosing the state i as new state; and on the other hand, the probability of choosing any other j as candidate to new state but failing to make the transition:

(IV.19) $$P_{ii} = p_{ii} + \sum_{j \neq i} p_{ij}(1 - W_{ij})$$

The matrix P_{ij} is positive because p_{ij}; W_{ij} are numbers between 0 and 1, and it is aperiodic and irreducible because so is p_{ij} and because W_{ij} is strictly positive. Finally, the stochasticity and condition (IV.16) can be checked straightforwardly from (IV.18) and (IV.19).

There are three standard choices for p_{ij}; all of them verify that

(IV.20) $p_{ij}=0$ if x(i) and x(j) differ in more than one site

Therefore, to choose a candidate to a new configuration is equivalent to choose a spin in the lattice that is a candidate to be flipped. This spin is chosen in one of the following ways:

i) At random: all spins have equal probability, i.e. $p_{ij}=1/|\Lambda|$ for every states x(i), x(j) differing at most in one site. In this case p_{ij} is symmetric and P_{ij} satisfies a sufficient condition for (IV.16), namely the detailed balance condition:

(IV.21) $$f_i P_{ij} = f_j P_{ji}$$

ii) Orderly: the sites are ordered in some fixed way, usually in lexicographic order, and the spins are tested sequentially.

iii) Random shuffle: The spins are tested orderly, but the order is changed each sweep of the lattice. The new order is chosen at random, hence this is a combination of the previous two methods.

The choices for W_{ij} are basically two: Denoting $U_i=H(x(i))$;

i) The original transition probability introduced by Metropolis et al. [24]:

$$W_{ij}=\begin{cases}\exp[-(U_j-U_i)] & \text{if } U_j>U_i \\ \\ i & \text{otherwise}\end{cases} \tag{IV.22}$$

ii) The "thermal bath" transition probabilities in which the remaining spins are considered a thermal bath for the spins to be flipped:

$$W_{ij} = \exp(-U_j)/[\exp(-U_j) + \exp(-U_i)] \tag{IV.23}$$

4.2.4 Relaxation and boundary effects

The word "simulation" is usually associated to this kind of technique. This is because one can think that the system is evolving driven by transition probabilities P_{ij} towards an state with probability distribution $(f_i)_{i\geq 1}$. In this regard, the process is a simulation of some mathematical dynamics that is not related with the true statistical mechanics evolution. The unit of "time" for this mathematical evolution is the Monte-Carlo step per spin (MCS)

$$1 \text{ MCS} = |\Lambda| \text{ spins tested}$$

As discussed above, there is a double source of statistical error for the process: On the one hand, that due to the finiteness of the sample of points for the numerical integration, i.e. with the fact that N in (IV.13) is finite. This error decreases if the number of sample configurations is increased. On the other hand, there is the error associated with the difference between $(P^n)_{ij}$ in theorem IV.1 and the true f_j. This is controled by taking n large enough. This value of n, measured in MCS is called relaxation time. This relaxation time is generally taken longer for the initial configuration, because one starts from a "seed" configuration that is very far from a typical one for the distribution f_i. For successive configurations, however, one assumes that $(P^n)_{ij}$ is already close to f_j, hence the relaxation time is taken smaller. If n_o is the initial relaxation time (in MCS), and μ the relaxation time between successive configurations, the Monte-Carlo simulation yields, in place of (IV.13):

$$\bar{h} = (1/N) \sum_{i=1}^{N} h(x(n_o+i\mu)) \qquad \text{(IV.24)}$$

Finally, one has to consider the thermodynamic limit $\Lambda \to \infty$. One expects that if Λ is large enough ρ_Λ^+ and ρ^+ differ little. Hence one may just accept the calculation made for a large Λ as the limit value.For a check one may repeat the

calculation for a larger Λ to see that there is no significative difference. It turns out that it is more convenient in this regard to use periodic boundary conditions instead of "+" ones; i.e. to consider

$$\rho^{per} = \lim_{\Lambda \to \infty} \rho_{\Lambda}^{per}$$

where ρ_{Λ}^{per} is the Gibbsian measure obtained for periodic boundary conditions on Λ. For a fixed Λ a better approximation to the thermodynamic limit is obtained for periodic boundary conditions in which the system has already no boundaries. The difference between ρ_{Λ}^{per} and ρ^{per} ("boundary effects") is that the former supresses configurations with periodicity of period strictly larger than the size of th "fundamental cell" Λ. To check that the contribution of such configurations is negligible a series of calculations with increasing lattice size is advised as well. For a more detailed study of the boundary effects and other limitations of the technique, references [1] and [2] can be consulted.

4.3 DESCRIPTION OF THE COMPUTER CODE

4.3.1 Introduction

Three computer codes were used to study the system described in section 6.1:

i) A code to evaluate the order parameters as a function of β for fixed coupling constants J(A), J(B), and J(C) (code SPIN).

ii) A code to evaluate the order parameters as a function of J(A) for fixed β, J(B), and J(C) (code VARA).

iii) A code to plot the variation of the relevant magnitudes as a function of β or J(A), so to reveal the existence of phase transitions (code SUMMARY).

Codes SPIN and VARA are built with the same basic program but run inside different loops. Only SPIN will be described in the sequel. The code SUMMARY is just a convenient tool and is of no interest here.

The code SPIN was developed by Dr. R. B. Jones, and it was subsequently reformed by the author to include periodical boundary conditions, ordered spin testing, and more detailed output including the printing of some intermediate configurations and the graphing of the relevant magnitudes as a function of the number of MCS. This last capability was

added to estimate the relaxation time that was needed in each case. A brief discussion of the characteristics of the code SPIN follows; the code is listed in appendix A.

4.3.2 General organization and INPUT data

The lattice is taken to be squared, of side LATS≤100. Only even values of LATS are accepted because the configurations of the symmetry group S have period 2 in the horizontal direction. An odd horizontal size for the lattice would tend to obstaculize the emergence of such configurations which may be very contributing. This would slow down the convergence. The configurations are stored in a logical matrix SPIN of size (LATS+2)×(LATS+4). An spin in the lattice is "up" (x_a=+1), if the corresponding position in SPIN is "true". The fact that the matrix SPIN has dimensions larger than the lattice is a remanent of the original code where "+" boundary conditions were used. In the final version with periodic boundary conditions the first and last rows and the first two and last two columns of the matrix SPIN are ignored. The initial configuration is defined with a fraction FRAC of rows with spins chosen at random and the rest set "up". This allows to compare the effect of the initial configuration when the latter changes from totally ordered (FRAC=0) to totally randomized (FRAC=LATS).

The inverse temperature is stored in a variable BETA. In the input and output β is measured in units of the critical inverse temperature for the two dimensional Ising model obtained for J(A)=0, J(B)=J(C)=1. Thus, the inverse temperature provided by the user and the one printed by the program correspond to a variable

$$BBTC = BETA/BETC$$

where

(IV.25) $$BETC = 1/2\ \ln(1+\sqrt{2})$$

is the solution of

$$sh^2(2\ BETC) = 1.$$

Values of order parameters are calculated for values of BETA in an interval ST≤BBTC≤SP with BBTC incremented an amount CT, which can be positive or negative. This allows to study histeresis effects. For each BETA there are CHK1 initial relaxation steps. For subsequent configurations, the relaxation time is μ=1 MCS (or $(LATS)^2$ trials). The starting configuration for a new value of BETA is always the last configuration of the previous BETA. Therefore, if CT is small the starting configuration is not far from a typical configuration of the new BETA; and the initial relaxation

time need not to be large. The only exception is the first value of BETA (ST*BETC if CT≥0; SP*BETC otherwise), for which an aditional relaxation time ADIC is introduced. Therefore the values of n_o, μ in (IV.24) are, in units of MCS:

$$\mu = 1$$

$$n_o = \begin{cases} \text{CHK1+ADIC} & \text{for the first value of } \beta \\ \\ \text{CHK1} & \text{for subsequent values of } \beta \end{cases}$$

The Markov process is defined with the transition probabilities P_{ij} given in (IV.18) and (IV.19). The matrix W_{ij} is defined as in (IV.22) and, for the purposes of comparing speed of convergence, the three types of p_{ij} discussed above are included in the program. However, most of the calculations were done with the less time consuming ordered trial. This is the option in effect in the listing of appendix A.

After the n_o relaxation steps have transcurred, the program starts the statistics of the configurations. This includes the accumulation of the values of $h(x(n_o+i\mu))$ for all the functions h of interest, and the relevant squares to compute the standard deviations. The number N of sample configurations in (IV.24) corresponds to the variable MCSS

in the program. However, the statistics of the MCSS configurations is broken down into sets of IFRQ configurations (Figure 2). Each of these sets is called a "block". Each block can be thought as a different experiment, hence looking to the successive block averages and standard deviations one can appreciate the repetitiveness of the simulation and detect problems like insufficient relaxation time. The error in h is estimated by the standard deviation among block averages.

The input deck for the code is summarized in table 2 The printed output of the program includes:

i) Block averages and standard deviations for the energy, magnetization and pseudomagnetization per spin.

ii) A final table with the overall averages and standard deviations of the energy and the two order parameters as a function of β. The table also shows the specific heat (as defined below).

iii) Intermediate configurations found at multiples of a certain number ICONFC of Monte-Carlo steps. This number ICONFC is provided by the user.

In addition the code provides a graphic output with the values of the energy and the order parameters as a function of the number of Monte-Carlo steps.

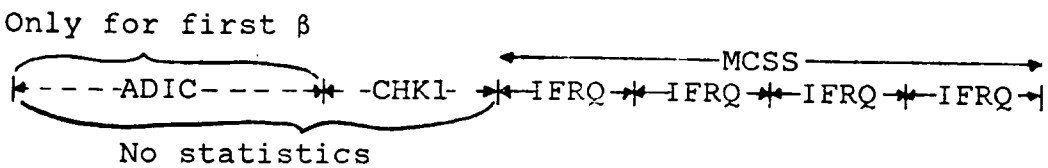

Figure 2: Scheme of the iteration procedure of the code SPIN. The horizontal line correponds to number of iterations ("time") measured in MCS. The variables used in the program for the length of each interval are written on top of them.

TABLE 2

Input deck for the program SPIN

	VARIABLE	FORMAT	MEANING
Card 1	ITIT	20A4	Tittle (at most 80 characters)
Card 2	ST	F5.2	Smallest value of BBTC
	SP	F5.2	Largest value of BBTC
	CT	F5.2	Increment in BBTC values (If CT<0 the first value of BBTC is SP)
	FRAC	F5.2	Fraction of the starting configuration chosen at random
Card 3	MCSS	I5	Number of effective MCS
	LATS	I5	Size of the square lattice (even)
	ICONFC	I5	Configuration printed each ICONFC Monte-Carlo steps
	ADIC	I5	Number of additional relaxation steps for the first BBTC
	CHK1	I5	Number of initial relaxation steps for subsequent values of BBTC
	IFRQ	I5	Size of the blocks
Card 4	AX	E15.4	Value of J(A)
	BX	E15.4	Value of J(B)
	CX	E15.4	Value of J(C)

4.3.3 Magnitudes calculated by the program

The adoption of periodic boundary conditions has an unpleasant side effect, namely that the magnetization $\rho^{per}(s_{(0,0)})$ is zero by the symmetry of the system under flipping. The standard solution to this problem is to use instead $\rho^{per}(|s_{(0,0)}|)$, where $s_{(0,0)}$ is the average of the magnetization at different points of the lattice. An analogous policy was adopted regarding the pseudomagnetization S_D. Therefore, the functions h integrated by the program are (in the nomenclature used in the code):

i) The energy per spin

$$\text{ENERGY} = H_{\Lambda}^{per} \; / \; |\Lambda| \tag{IV.26}$$

ii) The absolute value of the magnetization per spin (order parameter 1)

$$\text{ORD1} = \left| \sum_{i=1}^{|\Lambda|} s_i \right| \; / \; |\Lambda| \tag{IV.27}$$

iii) The absolute value of the pseudomagnetization S_D per spin (order parameter 2)

$$\text{ORD2} = \left| \sum_{i=1}^{|\Lambda|} s_{\tau_i D} \right| \; / \; |\Lambda| \tag{IV.28}$$

The energy per spin is important to detect first order phase transitions. With regard to the order parameters, the

question arises as to what extent ORD1 and ORD2 are as valid as the original $\rho^{+}(s_{(0,0)})$ and $\rho^{+}(s_{D})$ to point out phase transitions. In fact, by a slight generalization of a proof due to Griffiths [11] one obtains that

(IV.29) $$\rho^{per}(\text{ORD1}) \leq \rho^{+}(s_{(0,0)})$$

(IV.30) $$\rho^{per}(\text{ORD2}) \leq \rho^{+}(s_{D})$$

It is not known whether the opposite inequalities are true. Hence the parameters (IV.27), (IV.28) are -in Griffiths' words- "weaker" than the original parameters with "+" boundary conditions.

The program also computes the "specific heat"

(IV.31) $$c_{V} = \text{BBTC}^{2}(\Delta U)^{2}$$

$$\approx [k/(|\Lambda|^{2}\text{BETC}^{2})]\ [\partial(\rho^{per}(\text{ENERGY}))/\partial T]$$

where k is the Boltzman constant, T the temperature and ΔU the standard deviation of the block averages of ENERGY. Moreover, the code SUMMARY also plots the "susceptibilities"

(IV.32) $$\chi_{o} = \text{BBTC}\ (\Delta s_{(0,0)})^{2}$$

(IV.33) $$\chi_{D} = \text{BBTC}\ (\Delta s_{D})^{2}$$

Here $\Delta s_{(0,0)}$; Δs_D are the standard deviations for ORD1 and ORD2 respectively.

4.4 OUTCOME OF THE SIMULATION

4.4.1 Experimental arrangements

The purpose of the simulation was two-fold. On the one hand, to learn the basic aspects of the technique and to explore its limitations; and, on the other hand, to observe features of the phase diagram that suggest properies that could be later proved rigorously. It was not intended as a careful computer experiment mainly because of lack of sufficient funds. The simulation was focussed in studying how the phase diagram changes when $J(A)=J(C)=\beta$ and $J(A)$ increases. The goal was to observe what happens with the number of phases when the system with fundamental bonds $\{B,C\}$ is perturbed with an interaction $J(A)$. To this end, the lines

$$\beta \longmapsto (\beta J(A), \beta, \beta)$$

were analyzed with the program SPIN; and the lines

$$J(A) \longmapsto (\beta J(A), \beta, \beta)$$

with the program VARA. The lattice site was fixed in 16 and no intent was made to check the results for larger lattice size.

To see the extent of the influence of the initial configuration some runs were made for J(A)=0 and BBTC=2, with initial configurations with different degrees of randomization. At this low temperature the system is almost all the time with most of its spin aligned, and the time it takes for a randomized configuration to evolve into an ordered one gives an idea of the relaxation time that must be allowed in the simulation. An starting configuration with 30% of its spins chosen at random, needed about 30 MCS to reach a configuration with minimun energy, and it stood in it from there on (except for some minor fluctuations). An initial configuration with 50% randomly chosen spins demanded about 105 MCS for such stabilization, while for 75% of initial randomization 110 MCS were needed. On the basis of these figures it was estimated that an initial relaxation of 600 MCS was enough to make irrelevant the choice of the initial configuration. In all the remaining runs this starting configuration was taken to be the one with all spins up. There are two reasons for this choice: first, it is the cheapest configuration to generate in terms of computer time; and second, -as it was stated in a previous publication for which we do not have the reference- it is easier to obtain disorder from order than viceversa.

Some study as also made in relation with the different choices for p_{ij}. The three choices discussed above were compared for J(A)=0, BETA=BETC (the well known Ising model at critical temperature). In terms of computer time the most expensive is the random choice; the random shuffle is abour 1% cheaper, and the ordered trial costs 9% less. When using this last method a certain tendency was observed for the system to flip all the spins in a kind of oscillatory fashion. This phenomenon was observed previously by other authors [19] and it is explained in the following fashion: Once an spin has been flipped the probability for the contiguous spin to be flipped increases because the "energy barrier" (U_j-U_i in (IV.22)) is smaller. This collective flipping does not affect the average value of the order parameters because of the absolute value used in (IV.27)-(IV.28); but it produces larger fluctuations, i.e. it affects the accuracy of the calculation. No noticeable difference in fluctuations was observed between the random and random shuffle methods. Despite the larger fluctuations, the ordered method was adopted for all calculations due to its lower cost and to the fact that extreme accuracy was not an objective of the work.

In conclusion, the simulations were made with LATS=16, initial configuration with all spins up and ordered trial of

spins. The values of the parameters of figure 2 were typically ADIC=500, CHK1=100, IFRQ=50, and MCSS=500. The increment in BBTC (or in J(A) for the code VARA) was set to CT=0.05 for preliminary runs and CT=0.02 for a posterior fine tune in the position of the phase transitions.

4.4.2 Results observed

No first order phase transition was observed, and two lines of second order phase transitions were found in the plane (J(A),T). The upper line coresponds to the drop to zero of the pseudomagnetization ORD2 accompanied by the divergence of χ_D and c_V. The bottom line corresponds to the disappearance of the magnetization ORD1 and at the same time the divergence of χ_o. In figure 3 a graphic output of the code SPIN is shown as an example. It corresponds to J(A)=0.5, BBTC=0.76. Large fluctuations are observed in the magnetization, and it seems to decay suddenly to zero in the middle of the experiment. The interpretation is that the system is probably undergoing the correspondent phase transition. The fluctuations in the pseudomagnetization and the energy are also very big, which suggests that the second phase transition is very close of it may be taking place simultaneously with the first one.

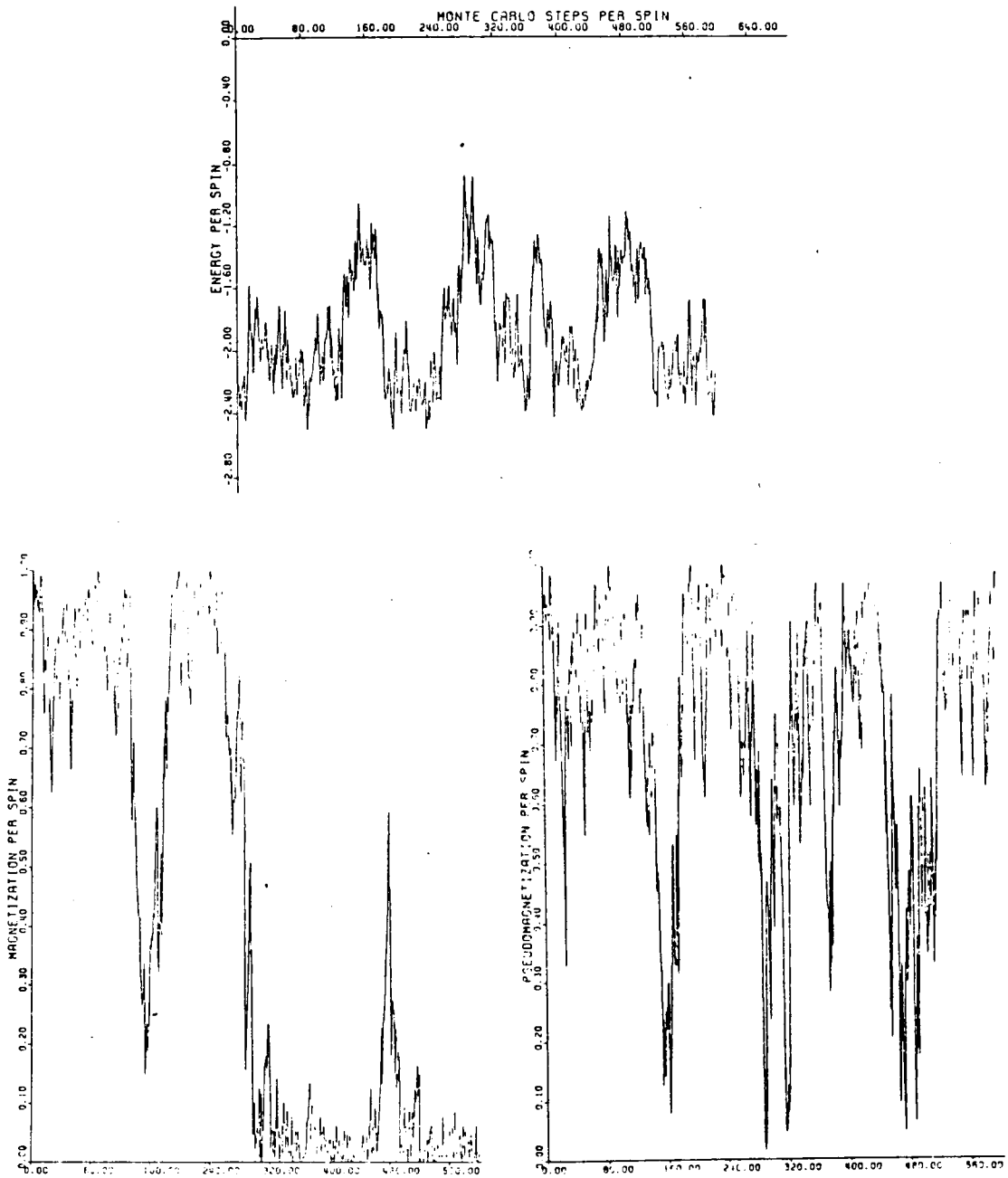

Figure 3: Evolution of the energy and order parameters in the Monte-Carlo simulation for BBTC=0.76, J(A)=0.5 (Code SPIN)

The results observed are summarized in figure 4, in which the y-axis corresponds to

$$T/T_c = 1/BBTC$$

The error bars come from an estimation of the uncertainty in the position of the maxima of the susceptibilities and of the points where the order parameters drop to zero. The vertical bars correspond to calculations made varying β (program SPIN), while horizontal ones to computations made varying J(A) (program VARA). For J(A)≥0.5 both phase transitions seem to happen almost simultaneously. For low values of T the program is no longer of practical use because there is a very high "energy barrier" for the flipping of one spin in a sea of aligned spins. Indeed, a short calculation shows that for T/T_c=0.565 the probability (IV.22) for turning one spin down in a configuration with all spins up is

$$\exp[-16\ BETC/0.565] \approx 1/256{,}000$$

Hence, for LATS=16 ($|\Lambda|$=256) one such transition will happen on the average every 1000 MCS. Hence if the system is "trapped" in an ordered configuration, its relaxation time is infinite for all practical purposes.

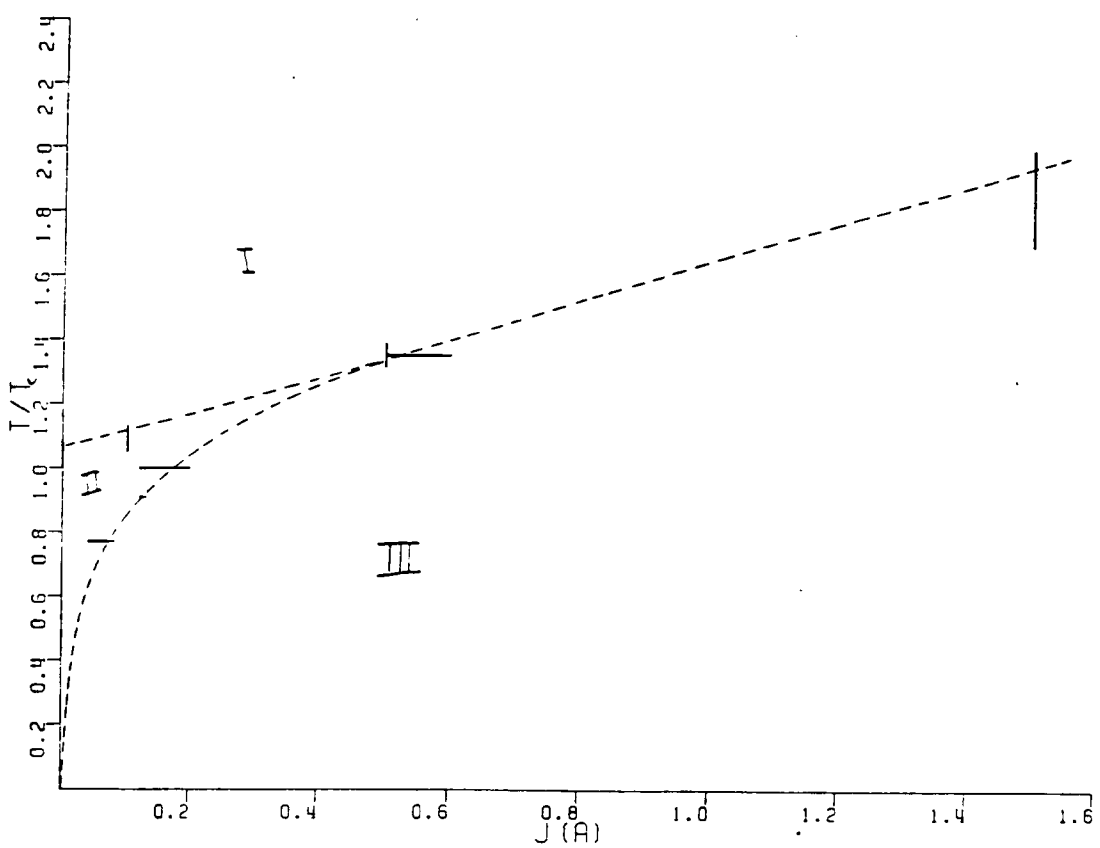

Figure 4: Phase diagram observed. In zone III both order parameters are different from zero; in zone II only the pseudomagnetization is non-zero; in zone I both order parameters are zero

If we assume equality in (IV.29)-(IV.30), some features of figure 4 can be explained from well known results. For instance, the fact that the critical curves have positive slope at all points follows from the generalized first Griffiths inequality. Indeed, the values of $\rho^+(s_U)$ increase with J(A) for any character s_U; hence if the expectation value is non-zero for a certain J(A), it is also non-zero for all J(A) larger than this. The lower critical curve was completed so to start from (J(A),T)=(0,0), because for $J(A)\neq 0$ the system has the "separation property" (theorem 3.8 of reference [15]); and hence by theorem 2.7 of [15] $\rho^+(s_{(0,0)})>0$ whenever $J(A)\neq 0$. What was not known a priori was whether a region II was present; i.e. whether there was a region with two different phase transitions or the zone with zero magnetization was confined to the axis J(A)=0. The simulation showed that the perturbation J(A) incremented the number of phase transitions from one to two. This result motivated a more general construction due to J. Slawny [30] that produces for a given n, a large family of models with at least n phase transitions. Such construction is presented in the next chapter.

Chapter V

SYSTEMS WITH MANY PHASE TRANSITIONS

5.1 PRELIMINARIES

A procedure to construct ferromagnetic systems with many phase transitions will be presented in this chapter. The construction is due to J. Slawny [30]. The models will be supposed ferromagnetic, translation invariant, and with $L=Z^{\nu}$. In addition, only spin 1/2 will be considered in theorems V.1 and V.2 below and in the main result -theorem V.6. However, these results generalize to p-component systems, with p prime. As in chapter IV, X will be identified with the finite subsets of Z^{ν}, with field structure defined by (IV.3). Before establishing the construction we need some preliminary results also due to Slawny.

We start with the formal statement of the property already used in (IV.6), (IV.7). Given an interaction $J\geq 0$ with set of fundamental bonds B_o, and a finite set $D\subset Z^{\nu}$; define a new interaction J' with fundamental bonds

$$B'_o = \{D\bullet C:\ C\in B_o\}$$

and coupling constants

$$J'(A)=\begin{cases} J(C) \text{ if } A=D\bullet C \\ \\ 0 \quad \text{otherwise} \end{cases}$$

Denote also (D) as the ideal

$$(D) = \{D\bullet C: C\subset Z^{\nu}\}$$

Theorem V.1 [15]: If ρ^{+} and ρ'^{+} are the Gibbs states with "+" boundary conditions for the interaction J and J' respectively, then:

$$\rho'^{+}(s_{D\bullet C}) = \rho^{+}(s_C) \quad \text{for any finite } C\subset Z^{\nu}$$

$$\rho'^{+}(s_A) = 0 \quad \text{if } A \notin (D).$$

The second result needed is a sufficient condition for S^{+} to be trivial at low temperature.

Theorem V.2 [15]: Consider an interaction J. If the set of fundamental bonds has g.c.d.(B_o)=1, then $\rho^{+}_{(\beta J)}(s_0)\neq 0$ at low temperatures.

Combining theorems V.1 and V.2 one has:

Corollary V.3: Consider an interaction J. If the set of fundamental bonds has g.c.d.(B_o)=D, then $\rho^{+}_{(\beta J)}(s_D)\neq 0$ at low temperatures.

In addition we need the following well known result which holds for any ferromagnetic system in the setting of chapter III.

Proposition V.4: If there is only one Gibbs state, then $B^+=\tilde{B}$.

Proof: By theorem III.10 i) $B^+\supset\tilde{B}$. To prove the opposite contention suppose $\xi\notin\tilde{B}$. As $\tilde{B}^\perp=S$, there exists $y\in S$ such that $\xi(y)\neq 1$. Suppose $\xi\in B^+$, then

$$\rho_y^+(\xi) = \xi(y)\rho^+(\xi) \neq \rho^+(\xi).$$

This is a contradiction because ρ_y^+ is also a Gibbs state (theorem III.12) and the Gibbs state was supposed to be unique.∎

5.2 THE BASIC RESULT

The construction of systems with many phase transitions is based in the following theorem that holds for n-component systems of arbitrary finite n.

Theorem V.5 [30]: Let J, K be two ferromagnetic finite range interactions such that $\overline{B(J)}\subset\overline{B(K)}$. Then, there exists a function $f: R_+\to R_+$ smooth and strictly increasing such that for every ferromagnetic interaction $I\in\mathcal{B}$ and every character ξ:

$$\rho^{+}_{(\beta J+I)}(\xi) \leq \rho^{+}_{(f(\beta)K+I)}(\xi).$$

f depends on K and B(J), but it is independent of the values of the coupling constants $J(\eta)$.

A first application of this result is the proof that there is indeed a region in the phase diagram of figure 4 with two phase transitions. Indeed, the interaction for the model of chapter IV can be written as $J=J'+I$ where I si the interaction defined on the bonds that are translated of A. By (IV.4), if one defines a translation invariant interaction K with $B_o(K)=\{D\}$; one has that $B(J')\subset B(K)$. Therefore, by the previous theorem:

$$\rho^{+}_{(\beta J'+\beta I)}(s_{(0,0)}) \leq \rho^{+}_{(f(\beta)K+\beta I)}(s_{(0,0)}) \tag{V.1}$$

But the model in the RHS is a two dimensional Ising model with coupling constants $f(\beta)K(D)$ for the horizontal bond and $\beta J(A)$ for the vertical one. Hence, for each β one has that the RHS of (V.1) is zero for J(A) such that

$$\mathrm{sh}[2f(\beta)K(D)]\ \mathrm{sh}[2\beta J(A)] < 1 \tag{V.2}$$

Hence, there is a $J_o(\beta)>0$ such that

$$\rho^{+}_{(\beta J)}(s_{(0,0)})=0 \quad \text{for } 0 \leq J(A) < J_o(\beta)$$

This example can be generalized. Indeed, in (V.2) we are using the fact that for a two dimensional Ising model with coupling constants J(B), J(A), and no magnetic field, the magnetization is zero for J(A) small enough. In fact, this is a consequence of the uniqueness of the Gibbs state for such model for small J(A) and proposition V.4. This suggests a generalization that leads to the construction presented in the next section

5.3 THE CONSTRUCTION

For $D \subset Z^{\nu}$, B_i a set of subsets of Z^{ν}, denote

$$D \bullet B_i = \{D \bullet C: C \in B_i\}$$

Theorem V.6 [30]: Consider a family of ferromagnetic interactions $J_1,\ldots,J_n$ with respective sets of fundamental bonds $B_1,\ldots,B_n$ such that the sets

$$D_k = \text{g.c.d.}(B_k,\ldots,B_n) \qquad 1 \leq k \leq n,$$

satisfy:

D1) For each $1 \leq k \leq n-1$, D_k does not belong to the ideal generated by $D_{k+1}, B_k, \ldots, B_1$; and D_n does not belong to the ideal generated by $B_n,\ldots,B_1$.

D2) For every $1 \leq k \leq n$, and every $r>0$ one has the following: For every interaction K_k of fundamental bond D_k, there

exists an $\varepsilon=\varepsilon(K_k,r)$ such that for every interaction I with range smaller or equal than r and $\|I\|\leq\varepsilon$, the interaction K_k+I has only one Gibbs state.

Then, there exist $\varepsilon_1,\ldots\varepsilon_n > 0$ and $0<T_1<\ldots<T_n$ such that if $J=\varepsilon_1 J_1+\ldots+\varepsilon_n J_n$:

$$\rho^{+}_{(\beta J)}(s_{D_k}) \begin{cases} = 0 & \text{for } T>T_k \\ \neq 0 & \text{for } T<T_k \end{cases}$$

Therefore J exhibits at least n phase transitions.

Proof: We will choose $\varepsilon_n,\ldots,\varepsilon_1$ and temperatures $\tau_n,\ldots,\tau_1$ inductively in the following fashion: (β_i will denote the inverse temperature for τ_i)
Pick $\varepsilon_n=1$ and τ_n a temperature so that

$$\rho^{+}_{(\beta_n J_n)}(s_{D_n}) \neq 0$$

Such temperature exists by corollary V.3. Suppose $\varepsilon_n,\ldots,\varepsilon_i$; $\tau_n,\ldots,\tau_i$ have already been chosen. Fix $T=\tau_i/2$ and define an interaction K_i with fundamental bond D_i and coupling constants $K(D_i)=1$. By theorem V.5 there exists $f_i(\beta)$ such that for all $\eta_1,\ldots,\eta_{i-1}$:

$$\text{(V.3)}\quad \rho^{+}_{(\beta(\eta_1 J_1+\ldots+\eta_{i-1}J_{i-1}+\varepsilon_i J_i+\ldots+\varepsilon_n J_n))}(s_{D_{i-1}}) \leq$$

$$\leq \rho^{+}_{(\beta(\eta_1 J_1+\ldots+\eta_{i-1}J_{i-1})+f_i(\beta)K_i)}(s_{D_{i-1}})$$

By hypothesis D2 there exists ε_{i-1} such that $\varepsilon_{i-1}\leq\varepsilon_i$ and if $|\eta_1|\ldots|\eta_{i-1}|\leq\varepsilon_{i-1}$, the interaction of the RHS of (V.3) has only one Gibbs state. Then, by hypothesis D1 and proposition V.4, the RHS of (V.3) is zero. In conclusion, we chose ε_{i-1} so that

$$(V.4)\quad |\eta_1|\ldots|\eta_{i-1}|\leq\varepsilon_{i-1};\ T=\tau_i/2 \Rightarrow$$

$$\Rightarrow \rho^{+}_{(\beta(\eta_1 J_1+\ldots+\eta_{i-1}J_{i-1}+\varepsilon_i J_i+\ldots+\varepsilon_n J_n))}(s_{D_{i-1}}) = 0$$

Choose τ_{i-1} such that

$$(V.5)\quad \rho^{+}_{(\beta_{i-1}(\varepsilon_{i-1}J_{i-1}+\ldots+\varepsilon_n J_n))}(s_{D_{i-1}}) \neq 0$$

Such temperature exists by corollary V.3, moreover; from (V.4):

$$(V.6)\qquad \tau_{i-1} < \tau_i/2$$

As $J_1,\ldots,J_{i-2}$ are ferromagnetic interactions, from (V.5) and the generalized first Griffiths inequality one has that

$$(V.7)\quad \rho^{+}_{(\beta_{i-1}(\eta_1 J_1+\ldots+\eta_{i-2}J_{i-2}+\varepsilon_{i-1}J_{i-1}+\ldots+\varepsilon_n J_n))}(s_{D_{i-1}}) \neq 0$$

for arbitrary $\eta_1,\ldots,\eta_{i-2}\geq 0$.

Having chosen $\varepsilon_1,\ldots,\varepsilon_n$ and $\tau_1,\ldots,\tau_n$; define

$$T_i = \sup\{T:\ \rho^{+}_{(\beta J)}(s_{D_i})\neq 0\}$$

These suprema are finite for $1\leq i\leq n-1$ because, from (V.4)

(V.8) $\qquad T_{i-1} \leq \tau_i/2 \qquad 2 \leq i \leq n$

T_n is also finite because as D_n is not in the ideal generated by $B_1,\ldots,B_n$;

$$\rho^{+}_{(\beta J)}(s_{D_n}) = 0$$

for T high enough. This is a consequence of proposition V.4 and the fact that there is only one Gibbs state at high temperature.

Finally, from (V.7):

$$\tau_i < T_i$$

thus by (V.8)

$$T_{i-1} < T_i .\blacksquare$$

To be able to apply the construction, one must obtain sets (characters) D_i satisfying the property D2. This means to find interactions with only one fundamental bond such that for all values of the coupling constant the Gibbs state remains unique even when the interaction is perturbed by arbitrary finite range interactions of small enough norm. The rest of the work is devoted to find a family of such interactions.

Chapter VI

UNIQUENESS OF GIBBS STATES UNDER PERTURBATION

6.1 DOBRUSHIN-PECHERSKI CRITERION

In this chapter the notation and framework of section 2.2 will be used. In addition the state space at each point Ω will be supposed finite and $L=Z^\nu$. The following notation will also be used: If $s=(s_1,\ldots,s_\nu)\in Z^\nu$, $|s|=\max|s_i|$. For $U\subset Z^\nu$, $\operatorname{diam}(U)=\sup\{|s-u|: s,u\in U\}$ and the boundary of U will be denoted ∂U. More generally, if k is a positive integer the k-(external) boundary of U is the set

$$\partial_k U=\{s\in Z^\nu\setminus U: \min\{|s-u|: u\in U\}\le k\}$$

with the convention $\partial_0=\phi$. If U, V are subsets of Z^ν, $\operatorname{dist}(U,V)=\min\{|u-v|: u\in U, v\in V\}$. The discrete metric on Ω^{Z^ν} will be denoted χ; $\chi(x,y)=1-\delta_{xy}$ and for finite $V\subset Z^\nu$ $\chi_V=\chi\circ(pr_V\times pr_V)$.

The results of this chapter are restricted to k-dependent physical fields for some natural k. These are fields for which the conditional probabilities for the measure μ of theorem II.6 satisfy

$$P_\mu(\cdot|\Sigma_{L\setminus\Lambda})=P_\mu(\cdot|\Sigma_{\partial_k\Lambda}) \tag{VI.1}$$

μ-almost everywhere for any finite $\Lambda \subset L$. For example an interaction of range k yields a Gibbsian k-dependent field. In anlogous way a k-dependent specification is an specification P satisfying

$$(VI.2) \qquad P_U(\cdot|x)=P_U(\cdot|x_{\partial_k \Lambda})$$

for every finite $U \subset L$, $x \in X_{L\setminus U}$.

For a transition function P_U a set $V \subset U$ and a configuration x one has the measure on V defined by

$$(VI.3) \qquad P_{V,U}(A|x)=P_U(A \times X_{U\setminus V}|x), \quad A \in \Sigma_V.$$

For a Gibbsian transition function this corresponds to the measure obtained by fixing a boundary condition $x \in X_{L\setminus U}$ outside U and integrating out the configurations on $U\setminus V$.

The space of probability measures on finite configuration spaces will be endowed with the variation distance:

$$R(\mu,\sigma)=\sup_{A \in \Sigma} |\mu(A)-\sigma(A)|, \quad \mu,\sigma \in E(X_\Lambda)$$

An important property of this distance is that

$$(VI.4) \qquad R(\mu,\sigma)=\inf_{\vartheta} \int_{X_\Lambda} \chi(x,y)\ \vartheta(dx,dy)$$

where the infimum is taken over all $\vartheta \in E(X_\Lambda \times X_\Lambda)$ such that for every $A \in \Sigma_\Lambda$:

$$\vartheta(X_\Lambda \times A)=\sigma(A), \qquad \vartheta(A \times X_\Lambda)=\mu(A)$$

Dobrushin and Pecherski [7] established a rather involved criterion for the uniqueness of Gibbs state under perturbation. Some definitions are needed to state it.

Definition VI.1: A function $\psi: Z^\nu \to R_+$ is in the class F_{pow} if

$$\lim_{N\to\infty} N^{\nu^2-\nu} \sum_{|t|>N} \psi(t) = 0 \tag{VI.5}$$

Definition VI.2: A k-dependent specification P satisfies condition $A(\psi,\varepsilon,r)$ where $\psi: Z^\nu \to R_+$, $\varepsilon>0$, $r\le\infty$; if for any finite $V\subset U\subset Z^\nu$ with $\mathrm{diam}(U)\le r$, and any $y,z\in X_{L\setminus U}$

$$R(P_{V,U}(\cdot|y),P_{V,U}(\cdot|z)) \le \sum_{v\in V,\, t\in \partial_k U} [\psi(v-t)+\varepsilon]\, \chi_t(y_t,z_t) \tag{VI.6}$$

The results of Dobrushin and Pecherski are the following:

Theorem VI.3 (Dobrushin-Pecherski): For every function ψ in the class F_{pow} there exist $\varepsilon_o=\varepsilon_o(\psi)$, $r_o=r_o(\psi)$, such that if an specification P satisfies condition $A(\psi,\varepsilon,r)$ for some $\varepsilon\le\varepsilon_o$, $r\ge r_o$, then there is a unique physical random field consistent with this specification.

Corollary VI.4: Consider a range k potential ϕ such that its Gibbsian specification satisfies condition $A(\psi,0,\infty)$ for $\psi\in F_{pow}$. Then there is an open neighborhood of ϕ in $\mathcal{B}_k$ where there is only one Gibbs state.

Proof: Let $\psi \epsilon \mathcal{B}_k$, $V \subset U$, $\mathrm{diam}(U) \leq r_o(\psi)$. If P and Q are the respective specifications of ϕ, ψ, a straightforward calculation shows that for any $z \epsilon X_{L \setminus U}$:

$$R(P_{V,U}(\cdot|z), Q_{V,U}(\cdot z)) \leq \exp[c(r_o+k)^{\nu}\|\phi-\psi\|] - 1.$$

where c is a positive constant that depends only on ν.
Consider $\delta \geq 0$ satisfying $\exp(c(r_o+k)^{\nu}\delta)-1=\varepsilon_o/2$
and denote $O_\delta = \{\psi \epsilon \mathcal{B}_k : \|\psi-\phi\| < \delta\}$.
Then for $\psi \epsilon O_\delta$, $y, z \epsilon X_{L \setminus U}$; $y \neq z$:

$$R(Q_{V,U}(\cdot|z), Q_{V,U}(\cdot|y)) \leq R(Q_{V,U}(\cdot|z), P_{V,U}(\cdot|z)$$

$$+ R(P_{V,U}(\cdot|z), P_{(V,U}(\cdot|y)) + R(P_{V,U}(\cdot|y), Q_{V,U}(\cdot|y))$$

$$\leq \varepsilon_o/2 + \Sigma \psi(v-t)\chi(y_t, z_t) + \varepsilon_o/2$$

$$\leq \Sigma[\psi(v-t)+\varepsilon_o]\chi(y_t, z_t)$$

Hence there is a unique physical random field for the Gibbsian specification corresponding to each $\psi \epsilon O_\delta$. ∎

6.2 CRITERION IN TERMS OF PARTITION FUNCTIONS

Consider a Gibbsian specification. If $V \subset U$ are finite sets in Z^{ν} and f is a bounded and measurable function on X_V:

$$P_{V,U}(f|y) = (1/Z_U^y) \int_{X_V} dx\ f(x)\ \exp(-H_V(x))\, Z^{x \vee y}_{(U \setminus V) \cup \partial V}$$

where $x \vee y \in X_{V \cup (Z^\nu \setminus U)}$ coincides with x on V and with y on $Z^\nu \setminus U$. Therefore:

$$P_{V,U}(f|y) - P_{V,U}(f|z) = \int dx\, f(x) \exp(-H_V(x)) \left[Z^{x \vee y}_{(U \setminus V) \cup \delta V} / Z^y_U - Z^{x \vee z}_{(U \setminus V) \cup \delta V} / Z^z_U \right].$$

Hence, as

$$R(P_{V,U}(\cdot|y), P_{V,U}(\cdot|z)) \leq \sup_{f: \|f\|=1} |P_{V,U}(f|y) - P_{V,U}(f|z)|$$

one has that

$$(VI.7) \quad R(P_{V,U}(\cdot|y), P_{V,U}(\cdot|z)) \leq Z_V \| Z^{x \vee y}_{(U \setminus V) \cup \delta V} / Z^y_U - Z^{x \vee z}_{(U \setminus V) \cup \delta V} / Z^z_U \|_\infty .$$

This shows that a possible way to verify a condition $A(\psi, 0, \infty)$ is to play with the combinations of partition functions on the RHS of (VI.7). In this regard DP criterion would require the estimation of such partition functions for sets U, V of arbitrary geometry which is rather involved even for simple potentials. However, a closer look to DP proof reveals that no such generality is used in it. In fact the pairs of sets V, U involved in the proof have the following helpful characteristic: the sets V, and $(U \setminus V) \cup \delta V$ are formed by at most a fixed number $b(\nu)$ of parallelepipeds whose non zero dimensions are all proportional to the

parameter N of (VI.4). With this observation, in the condition (VI.5) one may restrict suitably the geometry of V, U and still theorem VI.3 is valid. This will be made in the sequel. For the sake of completeness and clarity, the part of the proof of DP theorem where such geometric restrictions play a role, will be repeated.

6.3 BASIC IDEAS OF THE PROOF OF DP THEOREM

The uniqueness of the field is proved in the following fashion: If $[\xi]$ and $[\eta]$ are two physical random fields consistent with the specification P, then for every cube V of an absorbing sequence of cubes and every $\varepsilon>0$, two representants $\xi=\xi(V,\varepsilon)\in[\xi]$; $\eta=\eta(V,\varepsilon)\in[\eta]$ are found such that

$$R(\sigma_V^{\xi},\sigma_V^{\eta}) \leq \varepsilon \tag{VI.8}$$

This implies that the marginal distributions for $[\xi]$ and $[\eta]$ are equal and hence $[\xi]=[\eta]$. Moreover, from (VI.4)

$$R(\sigma_V^{\xi},\sigma_V^{\eta}) \leq \int_{X_V\times X_V} \chi(x,y)\ \sigma^{\xi}(dx)\sigma^{\eta}(dy)$$

$$\leq \sum_{v\in V} (\sigma^{\xi}\times\sigma^{\eta})(\{(x,y)\in X_V\times X_V:\ x_v\neq y_v\})$$

Therefore

$$R(\sigma_V^{\xi},\sigma_V^{\eta}) \leq \sum_{v\in V} (\sigma^{\xi}\times\sigma^{\eta})(\chi_v)$$

From this expression one sees that (VI.8) is proven if one can prove that for every $\varepsilon>0$ there exist representants ξ, η such that

$$(VI.9) \qquad (\sigma^{\xi}\times\sigma^{\eta})(\chi_V) \leq \varepsilon, \quad \text{for every } V\in VII.$$

In fact in this expression ξ and η need not be representants of the physical field but what DP call a mutual realization.

Definition VI.5: Consider two physical random fields $[\xi]$, $[\eta]$ and a set $W\subset Z^{\nu}$. Two fields ξ_W, η_W defined on X_W are a mutual realization of the fields $[\xi]$, $[\eta]$ on W if for every $V\subset W$

$$P_V^{\xi_W}=P_V^{\xi}; \qquad P_V^{\eta_W}=P_V^{\eta}$$

If $W=Z^{\nu}$ then the fields of the realization are indeed representants of the physical fields.

The representants satisfying (VI.9) will be obtained by an iterative construction of mutual realizations for which the LHS of (VI.9) is progresively smaller. This process is based in the following reconstruction lemma whose proof is omitted.

Lemma VI.6 (Lemma 5.1 of [7]):
Consider $S\subset T\subset Z$ finite subsets with $\mathrm{diam}(T)\leq r$; $[\xi]$, $[\eta]$ physical random fields consistent with an specification P

satisfying (VI.6) for U=T, V=S and $\psi \epsilon F_{pow}$. If there exists a mutual realization $(\xi'_{Z^\nu \setminus T}, \eta'_{Z^\nu \setminus T})$ on $Z^\nu \setminus T$ such that

$$(\sigma^{\xi'} \times \sigma^{\eta'})(\chi_t) \leq c_t, \quad t \epsilon Z^\nu \setminus T, \; c_t \geq 0,$$

then for any $\delta > 0$ there exists a mutual realization $(\xi''_{(Z^\nu \setminus T) \cup S}, \eta''_{(Z^\nu \setminus T) \cup S})$ on $(Z^\nu \setminus T) \cup S$ such that

$$(VI.10) \quad (\sigma^{\xi''} \times \sigma^{\eta''})(\chi_S) \leq \sum_{s \epsilon S, t \epsilon \partial_k T} [\psi(t-s)+\varepsilon] c_t + \delta$$

and

$$(VI.10') \quad (\sigma^{\xi''} \times \sigma^{\eta''})(\chi_t) \leq c_t \quad \text{for every } t \epsilon Z^\nu \setminus T.$$

The pair $(\xi''_{(Z^\nu \setminus T) \cup S}, \eta''_{(Z^\nu \setminus T) \cup S})$ obtained in this lemma will be called the (T,S)- reconstruction of $[\xi]$, $[\eta]$. One notices that the (T,S) reconstruction does not alter the bound (VI.8) outside T, but it improves it inside S due to the damping properties of ψ.

The bounds c_t found in the proof of DP theorem take only a finite number of values. Thus, the following lemma is useful:

Lemma VI.7 (Lemma 5.2 of [7]):

If besides the conditions of lemma VI.6 one has that $\partial_k T = \bigcup_{i=1}^{n} T_i$ for some disjoints sets T_i, then if one denotes

$s_i=\inf\{|u-v|: v\in S, u\in T_i\}$, $\Xi(s)=\sum_{|t|\geq s}\sigma\ \psi(t)$, one has that the (T,S)-reconstruction satisfies:

$$(VI.11)\quad (\sigma^{\xi''}\times\sigma^{\eta''})(\chi_S)\leq \sum_{i=1}^{n}(\max_{t\in T} c_t)|T_i|\Xi(s_i)$$

$$+\varepsilon|S||\partial_k T|(\max_{t\in\partial_k T} c_t)+\mu\delta\ .$$

The reconstruction process for achieving the bound (VI.8) is made in the following way: First, Z^ν is partitioned into disjoints cubes of side N for a natural N large enough. This cubes are clasified into a finite number of families (colors) such that the reconstruction process in one of the cubes does not involve any other cube of the same family. The reconstruction is performed one family at a time, and the larger the number of families -i.e. the larger the number of stages in the process-, the worse the bound. Hence it is of interest to keep the number of colors at a minimun. The formalization of these ideas is as follows:

Definition VI.8:

i) If N is a natural number, an N-partition of Z^ν is a set of disjoints cubes of side N $\{V_i\}_{i\geq 1}$ whose union is Z^ν.

ii) For a given d>0, two cubes V_i, V_j of an N-partition are d-adjacent if $\partial_d V_i\cap\partial_d V_j\neq\phi$.

iii) An N-partition can be d-colored in n colors if there exists a function $V_i\longmapsto f(V_i)\in\{1,\ldots,n\}$ such that: V_i, V_j d-adjacent => $f(V_i)\neq f(V_j)$.

Lemma VI.9 (Lemma 5.3 of [7]):

If

(VI.12) $(N-1)/2^{\nu-1} > d$

there exists an N-partition of Z^ν that can be d-colored in $\nu+1$ colors.

This is the best possible result, no less than $\nu+1$ colors can be used because for any partition of Z^ν in cubes, each vertix has at least $\nu+1$ cubes around it. In what follows the cubes of a partition will be denoted with two indexes: a superscript indicating the color and a subindex labelling the cubes of that color.

6.4 RESTATEMENT OF DP RESULTS

To state DP-theorem in a way more useful for our purposes some extra notation is needed. Let's denote

$D(N,\nu,r_1,r_2,s_1,s_2)$=set of parallelepipeds in Z^n whose sides are of size $r_1N+s_1 \le L \le r_2N+s_2$.

$a(\nu)$=number of N-cubes d-adjacents to any cube of the partition of lemma VI.9, $b(\nu)=2\nu a(\nu)$,

$E(N,k,\nu,r_1,r_2,s_1,s_2,b,c)$=set formed by $V \subset Z^\nu$ that are the union of at most b parallelepipeds (V_i,W_j), with $V_i \in D(N,\nu,r_1,r_2,s_1,s_2)$; $W_j \in D(N,\nu-1,r_1,r_2,s_1,s_2)$ satisfying the following:

i) There is no ν-1-dimensional parallelepiped facing a ν-1 dimensional "face" of an adjacent parallelepiped.

ii) The set $\cup_i(V_i \cup \partial_k V_i)$ can be decomposed in at most c parallelepipeds of $D(N,\nu,r_1,r_2,s_1,s_2)$.

$Y(N,k,\nu)=\{(U,V)$: $V \subset U \subset Z^\nu$ and V, $(U\setminus V)\cup\partial V$ are sets in $E(N,k,\nu)\}$.

Figure 5 shows an example related with the second condition in the definition of E. The set of part a) of the figure is in E, while this in part b) is not because no matter how the set is decomposed into parallelepipeds one of the parallelepipeds has a side of size independent of N.

Finally, let's say that a k-dependent specification P satisfies condition $A_Y(\psi,\varepsilon,r)$ if it satisfies definition VI.2 for pairs $(U,V)\in Y$.

With this notation what DP have in fact proven in [7] is a somehow more general version of theorem VI.3 in which condition $A(\psi,\varepsilon,r)$ can be replaced by the slightly more relaxed condition $A_Y(\psi,\varepsilon,r)$. The proof is in two steps (Theorem VI.10 and Corollary VI.11 below).

Theorem VI.10: For each function $\psi\in F_{pow}$ and $k>0$ there exist $N_o(\psi)$, $\varepsilon_o(\psi)$, $r_o(\psi)$ and constants $A(N_o,\varepsilon_o)<1$, $A_{\nu+1}(N_o,\varepsilon_o)$

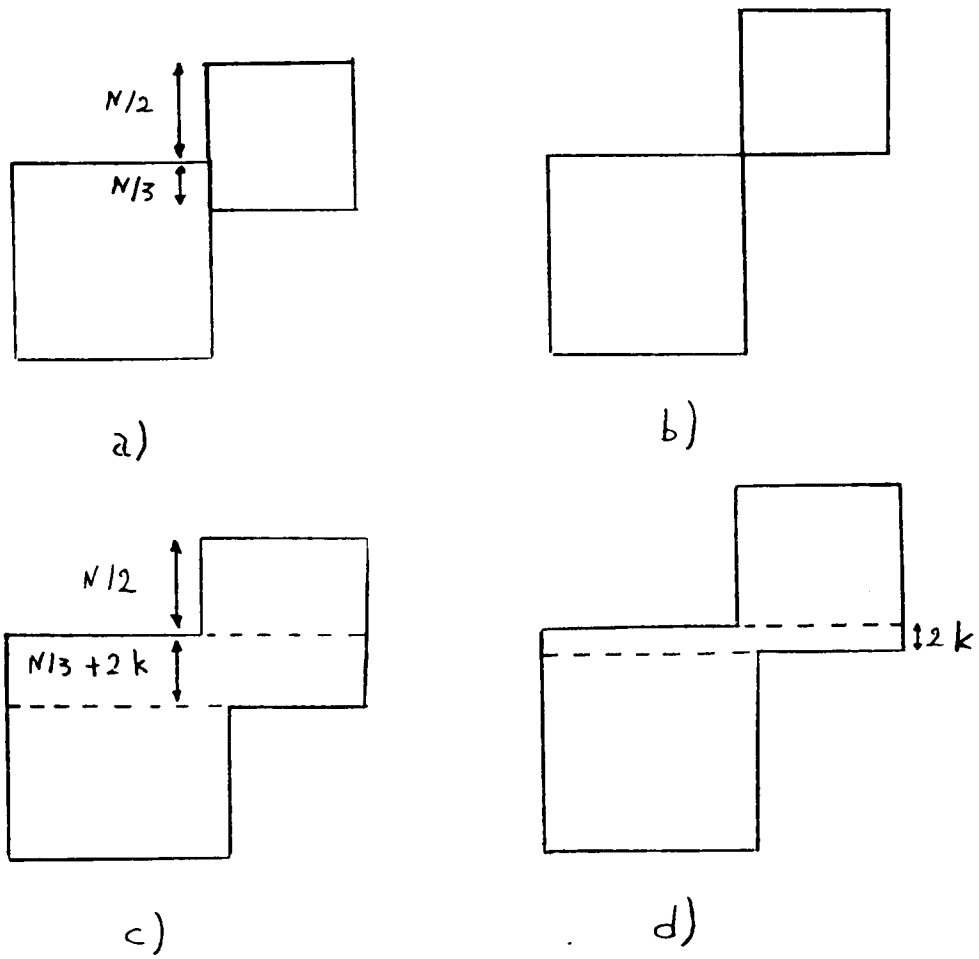

Figure 5: a) Example of a set in E; b) example of a set not in E. Parts C) and d) show respective decompositions of these examples into parallelepipeds.

such that the following holds: Consider $N \geq N_o$, an N-partition $\{V_i^j\}_{i\geq 1}^{1\leq j\leq \nu+1}$ that is d-colored in $\nu+1$ colors with $d=(N-1)/2^\nu$; and a k-dependent specification P satisfying condition $A_\gamma(\psi,\varepsilon,r)$ for some $\varepsilon\leq\varepsilon_o$, $r\geq r_o$, $N\geq N_o$. Suppose that $[\xi]$ and $[\eta]$ are physical random fields consistent with the specification P and that there exist respective representants ξ, η such that

$$(\sigma^\xi\times\sigma^\eta)(\chi_t)\leq C_o<\infty, \quad t\in Z^\nu\setminus(\cup_i V_i^1).$$

then for every $\delta>0$ there exist representants ξ', η' such that

$$(VI.13)\quad (\sigma^{\xi'}\times\sigma^{\eta'})(\chi_t)\leq \begin{cases} C_o A A_{\nu+1}{}^{+\delta}, & t\in\cup_i V_i^1 \\ C_o A^{2+\delta} & \text{otherwise} \end{cases}$$

Proof: First of all choose N so that

$$(VI.14)\qquad d=(N-1)/2^\nu>k$$

and r such that

$$(VI.15)\qquad N+2d\leq r.$$

Denote

$$(VI.16)\qquad M=d-k>0.$$

M will be the width of a zone around each N-cube from which a so-called "protection zone" (to be precised later) will be carved. Note that the bound (VI.10) for sets S inside a

cube or its protection zone does not involve points outside a strip of width d around the cube. As from (VI.14) $2d<N$, a reconstruction on a cube and/or its protection zone does not change bounds in points of non adjacent cubes or their protection zones. Condition (VI.15) ensures that the reconstruction lemma VI.6 can be applied to sets inside a cube or its protection zone. The geometrical arrangement is presented in figure 6 for the case $\nu=2$.

The reconstruction process proceeds in $\nu+1$ stages, each of them dealing with cubes of one color and their protection zones. Moreover each stage has in turn two steps, the first dealing with cubes themselves and the second with their protection zones.

First stage of the reconstruction process

For each of the cubes V_j^1 of the first color define the respective protection zone as

$$U(V_j^1)=(\text{cube of side } N+2M \text{ concentric with } V_j^1)\setminus V_j^1$$

i) First step (Figure 7 a)

Apply a (T,S)-reconstruction to $(\xi_{Z^\nu\setminus T}, \eta_{Z^\nu\setminus T})$ where $T=V_1^1 \cup U(V_1^1)$, $S=V_1^1$. Note that $(S,T)\in Y$; in particular $(T\setminus S)\cup \partial S$ can be decomposed in the following way (figure 7

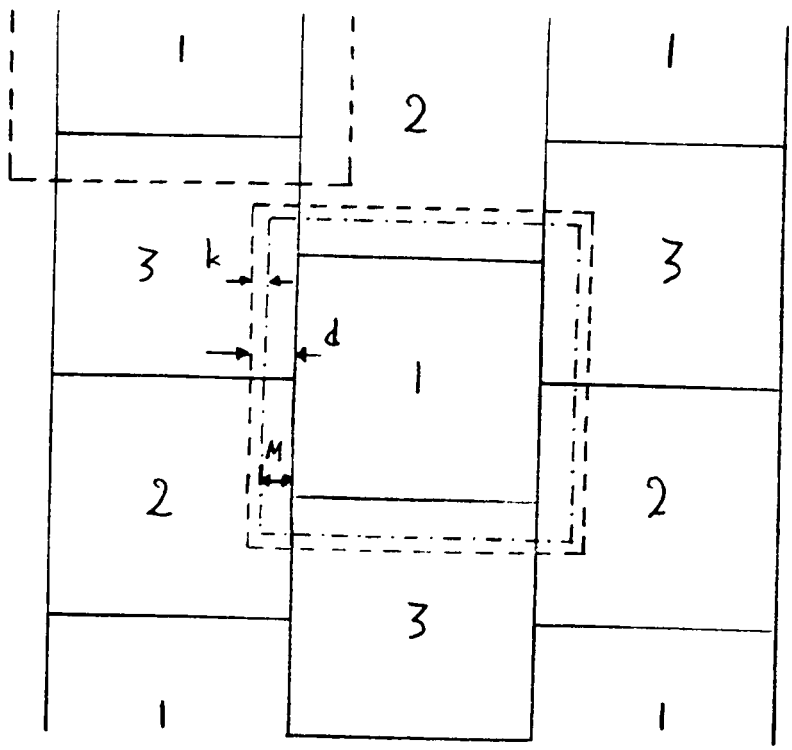

Figure 6: Geometrical arrangement of the different zones around cubes. Cubes of the same color have the zones involved in the reconstruction disjoint

c): If $V_1,\ldots,V_{a(\nu)}$ are the cubes adjacent to V_1^1, define for each $i=1,\ldots,a(\nu)$

V_i=parallelepiped obtained by moving the face of V_i facing V_1^1 so to become in contact with the adjacent face of V_1^1.

Then:

$$(VI.19)\qquad U(V_1^1)\cup\partial V_1^1 = \bigcup_{i=1}^{a(\nu)} \{V_i\cap[U(V_1^1)\cup\partial V_1^1]\}.$$

Each of the parallelepipeds $V_i\cap U(V_1^1)$ has one face of side M and the others sides of size at most N and at least the dimensions of the part of the face of V_1^1 facing V_i. This dimension has to be larger than d or otherwise more than $\nu+1$ colors would be needed for the partition. Hence the size of the sides of the $a(\nu)$ parallelepipeds is between $M=N/2^{\nu}-(k+1/2^{\nu})$ and N. The conditions i)-ii) of the definition of E are obviously satisfied; hence $U(V_1^1)\cup\partial V_1^1\in E$ for suitable parameters r_1,r_2,s_1,s_2,b,c.

(VI.19) is not the more economical way of decomposing $U(V_1^1)\cup\partial V_1^1$ into sets of $D(N,k,\nu)$, but it is general in the sense that the same decomposition can be used for all the stages of the reconstruction process, even when the protection zones become geometrically more complicated. In fact, protection zones for larger colors will be obtained "carving out" some of the sets of the decomposition (VI.19).

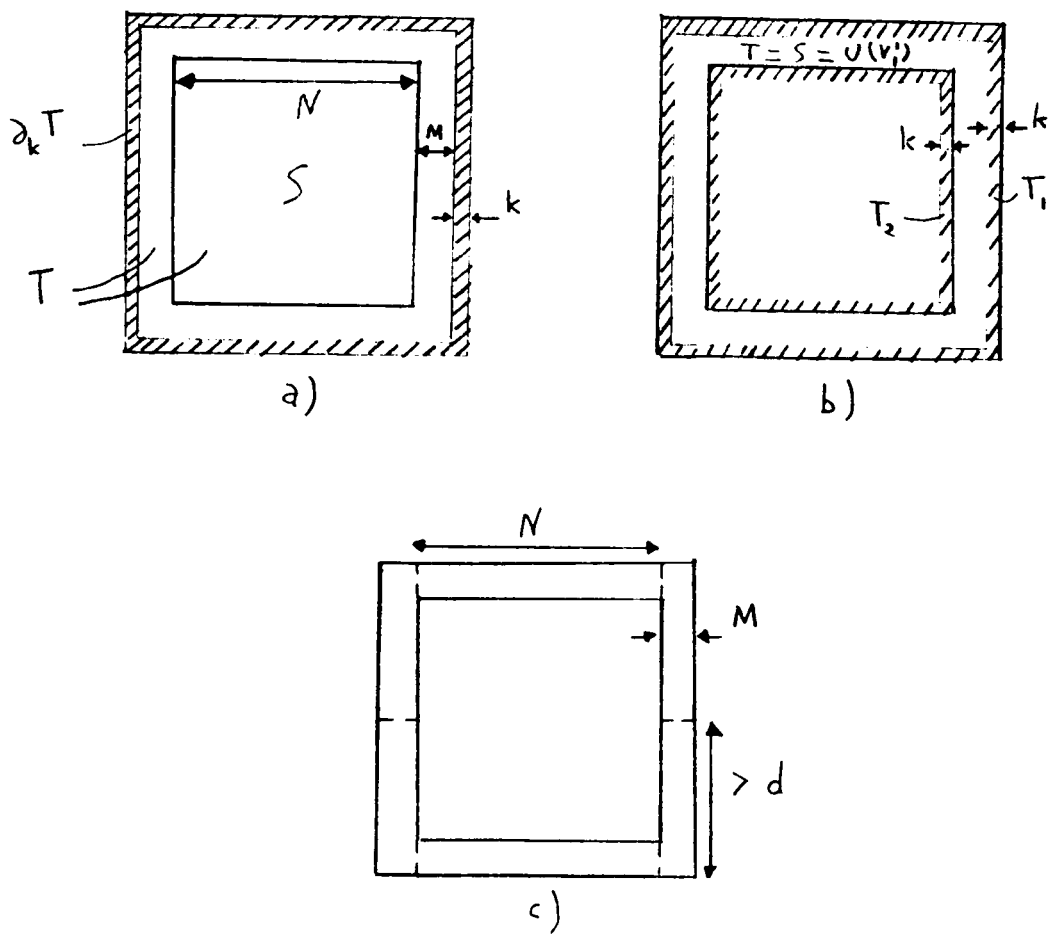

Figure 7: Sets T,S involved in the first (figure a) and second (figure b) steps of the first stage of the reconstruction process. Part c) shows the decomposition of $U(V_1^1) \cup \partial V_1^1$ into a(ν) sets of D(N,k,ν).

After the reconstruction one obtains a mutual realization $(\xi^{(1,1)}_{(Z^\nu\setminus T)\cup S}, \eta^{(1,1)}_{(Z^\nu\setminus T)\cup S})$ such that

$$(\sigma^{\xi^{(1,1)}} \times \sigma^{\eta^{(1,1)}})(\chi_{V_1^1}) \leq C_o A_1 + \delta$$

with (see (VI.11))

$$A_1 = |\partial_k T|[\Xi(M) + \varepsilon|S|] + \delta$$

One has that $|S| = N^\nu$ and $|\partial_k T| = K_1[N+2M]^{\nu-1}$ for some $K_1 = K_1(\nu)$; hence:

$$A_1 = K_1[N+2M]^{\nu-1}[\Xi(M) + \varepsilon N^\nu] + \delta$$

Moreover, $\chi_{V_1^1} \geq \chi_t$ for every $t \in V_1^1$, and the reconstruction does not alter the bounds outside T. Therefore

$$(\text{VI.20}) \quad (\sigma^{\xi^{(1,1)}} \times \sigma^{\eta^{(1,1)}})(\chi_t) \leq \begin{cases} C_o A_1 + \delta & \text{for } t \in V_1^1 \\ C_o & \text{for } t \in Z^\nu \setminus [V_1^1 \cup U(V_1^1)] \end{cases}$$

The fields $\xi^{(1,1)}$, $\eta^{(1,1)}$ are not defined on $U(v_1^1)$, however to apply again lemma VI.6 we need fields defined everywhere. Hence we must take a

ii) Second step (Figure 7 b).

Now we apply a $(U(V_1^1),U(V_1^1))$- reconstruction to the just obtained fields $\xi^{(1,1)}$, $\eta^{(1,1)}$. In this (T,S)-reconstruction S is a union of a(ν) sets in D:

$$(VI.21) \qquad S=U(V_1^1)=\bigcup_{i=1}^{a(\nu)} [V_i \cap U(V_1^1)]$$

where the sts V_i where introduced above. On the other hand $(T\setminus S)\cup\partial S=\partial U(V_1^1)$ can be written as the union of boundaries of the a(ν) sets in the RHS of (VI.21). Each of these sets has 2ν rectangular boundaries with dimension between d and M; hence $\partial U(V_1^1)$ is the union of at most $2\nu a(\nu)$ sets in D. One checks that the conditions defining E are satisfied both by $U(V_1^1)$ and $\partial U(V_1^1)$, thus the pair (T,S) is in Y.

In this case there are two values of c_t on the outside of $T=U(V_1^1)$. On $T_1=[\partial_k U(V_1^1)]\cap V_1^1$ $c_t=C_o A_1+\delta$; while in $T_2=\delta_k U(V_1^1)\setminus T_1$ $c_t=C_o$. Using lemma VI.7 one gets a pair of representants $\xi'^{(1,1)}$, $\eta'^{(1,1)}$ satisfying (VI.20) on points t outside $U(V_1^1)$ and

$$(VI.22) \qquad (\sigma^{\xi'^{(1,1)}} \times \sigma^{\eta'^{(1,1)}})(\chi_{U(V_1^1)}) \leq C_o B_1+\delta'$$

Here

$$B_1=|\partial_k U(V_1^1)|(1+A_1)[\Xi(1)+\varepsilon|U(V_1^1)|]$$

$$\delta'=\delta|\partial_k U(V_1^1)|[\Xi(1)+\varepsilon|U(V_1^1)|]$$

As δ can be chosen as small as wanted, so does δ'. Following DP, no distinction will be made in the sequel among successive δ's, and the same symbol δ will be used for the different leftovers that can be made arbitrarily small. There exist constants $K_1(\nu)$, $K_1(\nu)$ such that

$$|\partial_k U(V_1^1)| = K_1[N+2M]^{\nu-1}$$

$$|U(V_1^1)| = K_1 M[N+2M]^{\nu-1}$$

Therefore

$$(VI.23)\quad B_1 = K_1[N+2M]^{\nu-1}(1+A_1)[\Xi(1)+\varepsilon K_1 M(N+2M)^{\nu-1}],$$

and combining (VI.20) and (VI.21) plus the fact that $\chi_S \geq \chi_t$ for every $t\in S$ one finally gets that the representants $\xi'^{(1,1)}$, $\eta'^{(1,1)}$ satisfy:

$$(VI.24)\quad (\sigma^{\xi'^{(1,1)}} \times \sigma^{\eta'^{(1,1)}})(\chi_t) \leq \begin{cases} C_o A_1 + \delta & t\in V_1^1 \\ C_o B_1 + \delta & t\in U(V_1^1) \\ C_o & \text{otherwise} \end{cases}$$

The same process can be repeated for all the cubes V_i^1 of color 1. As mentioned above, such processes will not change bounds in any other cube of color 1 or its protection zone. Therefore, after treating the cubes $V_1^1,\ldots,V_j^1$ one obtains representants $\xi'^{(1,j)}$, $\eta'^{(1,j)}$ of the physical fields $[\xi]$, $[\eta]$ such that the first two lines of the bound (VI.24) hold

respectively for $t\in \bigcup_{i=1}^{j} V_i^1$ and $t\in \bigcup_{i=1}^{j} [U(V_i^1)]$. The weak limits

$$\xi^{(1)}=\lim_j \xi'(1,j); \quad \eta^{(1)}=\lim_j \eta'(1,j)$$

are representants of $[\xi]$, $[\eta]$ satisfying:

$$(VI.25) \quad (\sigma^{\xi^{(1)}} \times \sigma^{\eta^{(1)}})(\chi_t) \leq \begin{cases} C_o A_1 + \delta & t\in \bigcup_j V_j^1 \\ C_o B_1 + \delta & t\in \bigcup_j U(V_j^1) \\ C_o & \text{otherwise} \end{cases}$$

A glance to (VI.18) shows that for $\varepsilon=0$ A_1 is a damping factor because it is of order $N^{\nu-1}\Xi(N) \underset{N\to\infty}{\to} 0$ (condition (VI.5)). No such factor is present in B_1, hence the bound (VI.25) may be considered -at least for $\varepsilon=0$- as an improvement inside cubes of color 1, but probably a worsening inside the protection zones of such cubes. To improve the bound everywhere more stages are needed.

Second stage of the reconstruction process

Now the bounds inside the cubes of color 2 will be improved starting from the just obtained random fields $\xi^{(1)}$, $\eta^{(1)}$. We do not want to change the already good bound inside the cubes of color 1, hence the protection zones of the cubes V_i^2 will be defined substracting the cubes of color 1:

$U(V_i^2)$=(Cube of side N+2M concentric with V_i^2)$\setminus[V_j^2 \cup (\cup_j (V_j^1))]$ (Figure 8 c).

i) First step (Figure 8 a)

Consider the first cube of color 2 and apply a (T,S) reconstruction to $(\xi^{(1)},\eta^{(1)})$ where $T=V_1^2 \cup U(V_1^2)$, $S=V_1^2$ to obtain a mutual realization $(\xi^{(2,1)}_{(Z^\nu\setminus T)\cup S}, \eta^{(2,1)}_{(Z^\nu\setminus T)\cup S})$
Again $(S,T)\in Y$; in particular $(T\setminus S)\cup\partial S$ can be decomposed into $a(\nu)$ sets of D as in (VI.19). In this case some of the sets -the ones corresponding to points of the boundary of V_1^2 facing cubes of color 1- have "width" zero, while the others have "width" M.

When applying lemma VI.7 one finds that the boundary $\partial_k T$ is divided in three zones with different values of c_t:

T_1=part of $\partial_k T$ intersecting a cube of color 1. There $c_t=C_o A_1+\delta$. This part is at a distance 1 from S.

T_2=part of $\partial_k T$ intersecting the protection zone of cubes of color 1. There $c_t=C_o B_1+\delta$.

T_3=rest of $\partial_k T$, where $c_t=C_o$.

Both T_2 and T_3 are at a distance M from S.
Therefore, from (VI.11) one has that for $t\in V_1^2$

$$(\sigma^{\xi^{(2,1)}}\times\sigma^{\eta^{(2,1)}})(\chi_t)\le(C_oA_1+\delta)|T_1|\Xi(1)+(C_oB_1+\delta)|T_2|\Xi(M) \tag{VI.26}$$
$$+C_o|T_3|\Xi(M)+\varepsilon N^\nu|\partial_k T|\max\{C_oA_1+\delta,C_oB_1+\delta,C_o\}+\delta$$

There exist a constant $K_2(\nu)$ such that

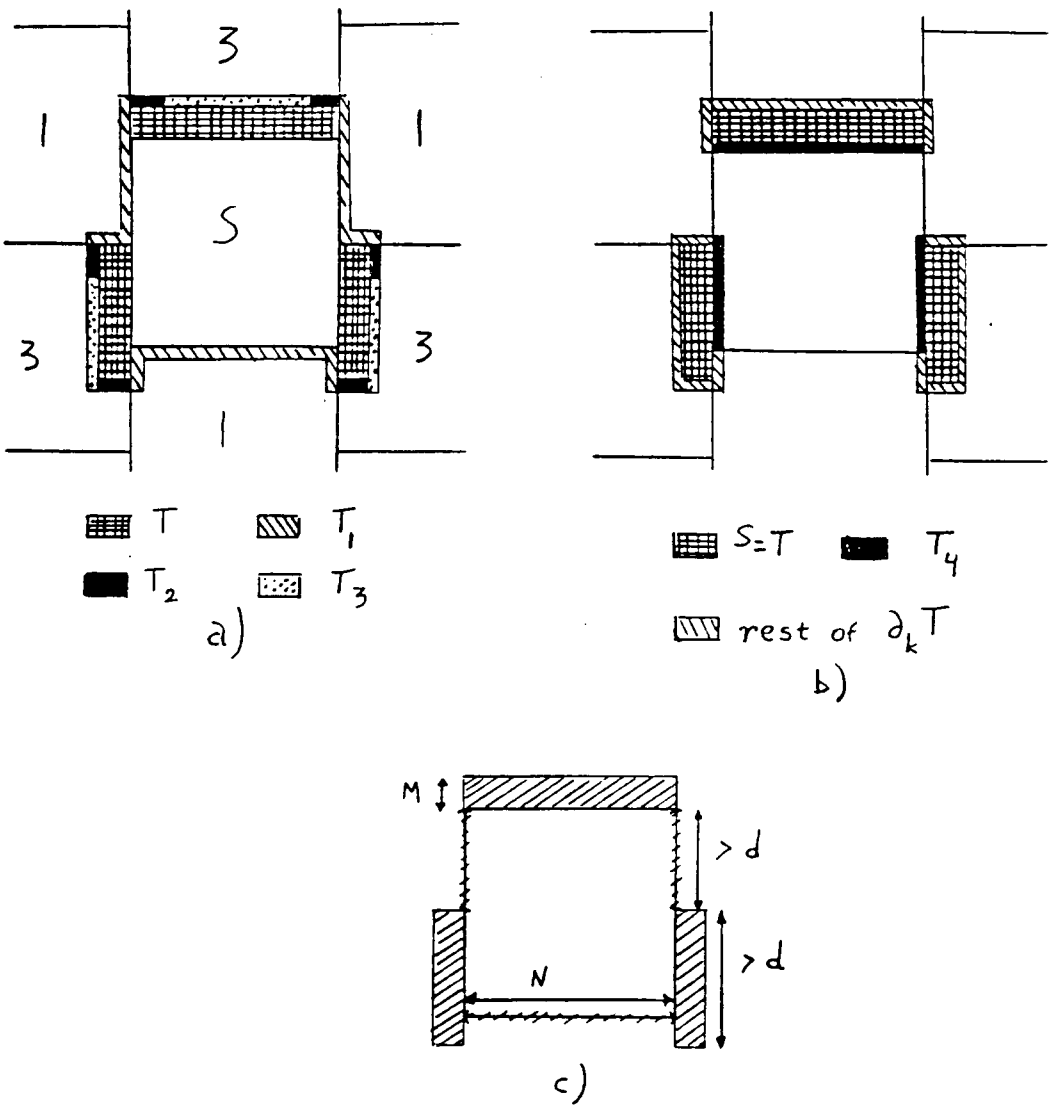

Figure 8: Sets T,S involved in the first(figure a) and second (figure b) steps of the second stage of the reconstruction process. Part c) shows the decomposition of $U(V_1^2) \cup \partial V_1^2$ into a(ν) sets of D(N,k,ν)

$$|T_1|,\ |T_2|,\ |T_3| < |\partial_k T| = K_2[N+2M]^{\nu-1}$$

Moreover

$$\max\{C_o A_1, C_o B_1, C_o\} \le C_o(1+A_1+B_1).$$

With these inequalities and grouping in (VI.26) the terms proportional to δ into a new δ:

$$(VI.27)\quad (\sigma^{\xi^{(2,1)}} \times \sigma^{\eta^{(2,1)}})(x_t) \le \begin{cases} C_o A_2+\delta & \text{for } t\epsilon V_1^2 \\ C_o A_1+\delta & \text{for } t\epsilon C_1 \\ & \\ C_o B_1+\delta & \text{for } t\epsilon P_1(1) \\ C_o & \text{for } t\epsilon R_1 \end{cases}$$

where

$$(VI.28)\quad A_2 = K_2[N+2M]^{\nu-1}\{\Xi(M)(1+B_1)+\Xi(1)A_1+\varepsilon N^{\nu}(1+A_1+B_1)\}$$

C_1=union of cubes of color 1

$P_1(1)$=points in the protection zones of cubes of color 1 that are not in V_1^2 or its protection zone

R_1=remaining points outside $V_1^2 \cup U(V_1^2)$

The realizations $\xi^{(2,1)}$, $\eta^{(2,1)}$ are not defined in $U(V_1^2)$. This gap is filled in the

ii) Second step (Figure VI.3 b)

We now apply a $(U(V_1^2),U(V_1^2))$-reconstruction to the fields $\xi^{(2,1)}\eta^{(2,1)}$. One also has that $(T,S)\epsilon Y$ because by an argument analogous to the one presented in the second step of the first stage $U(V_1^2)$ is the union of $a(\nu)$ parallelepipeds in D while $\partial U(V_1^2)=(T\setminus S)\cup\partial S$ is the union of some of the $2\nu a(\nu)$ rectangular boundaries of such parallelepipeds. Such boundaries are also in D.
Now $\partial_k T$ decomposes into four zones: the zones T_1, T_2, T_3 defined as in the first step and the zone $T_4=(\partial_k T)\cap V_1^2$ where $c_t=C_oA_2+\delta$. From (VI.11) the new representants $\xi'^{(2,1)}$, $\eta'^{(2,1)}$ satisfy for $t\epsilon U(V_1^2)$:

$$(\sigma^{\xi'^{(2,1)}}\times\sigma^{\eta'^{(2,1)}})(\chi_t)\leq\Xi(1)[(c_oA_1+\delta)|T_1|+(C_o+B_1+\delta)|T_2|$$

$$+C_o|T_3|+(C_oA_2+\delta)|T_4|]+\varepsilon|U(V_1^2)||\partial_k T|\max_{t\epsilon\partial T} c_t+\delta$$

Defining as before $K_2(\nu)$, $K_2(\nu)$ by

$$|\partial_k T|=K_2[N+2M]^{\nu-1}$$

$$|U(V_1^2)|=K_2M[N+2M]^{\nu-1},$$

using that

$$\max c_t\leq C_o(1+A_1+A_2+B_1)+\delta,$$

and grouping the terms with δ into a new "δ", one gets

$$(VI.29)\quad (\sigma^{\xi'^{(2,1)}}\times\sigma^{\eta'^{(2,1)}})(\chi_t)\leq C_o+B_2+\delta,\quad t\epsilon U(V_1^2)$$

with

(VI.30) $B_2=K_2[N+2M]^{\nu-1}[1+A_1+A_2+B_1][\Xi(1)+\varepsilon K_2 M]$,

while the estimations (VI.27) still hold for the new representants $\xi'^{(2,1)}$, $\eta'^{(2,1)}$.

Then we repeat the process for all the cubes V_1^2, $i\geq 1$; and take the limit of the successive representants $\xi'^{(2,j)}$, $\eta'^{(2,j)}$ as was done in the first stage. In this fashion one obtains representants $\xi^{(2)}$, $\eta^{(2)}$ such that

$$(\text{VI.31}) \quad (\sigma^{\xi^{(2)}} \times \sigma^{\eta^{(2)}})(x_t) \leq \begin{cases} C_o A_1+\delta & t\in C_1 \\ C_o A_2+\delta & t\in C_2 \\ C_o B_1+\delta & t\in P_1 \\ C_o B_2+\delta & t\in P_2 \\ C_o & \text{otherwise} \end{cases}$$

with:

C_i=points in cubes of color i

P_1=points in the protection zones of cubes of color 1 but not in the cubes of color 2 or their protection zones.

P_2=points in the protection zones of cubes of color 2.

The process continues with a two step reconstruction on cubes of color 3 and their protection zones, then for color

4 and so on. In each step the protection zones have the intersections with cubes of lower color carved out, because the bounds inside these intersctions are already good enough. In all cases the sets S and $(T\setminus S)\cup\partial S$ can be written as a union of parallelepipeds of $D(N,k,\nu)$ in exactly the same way as for the two stages explicity showed above. The boundary $\partial_k T$ decomposes in an increasing number of sets T_i characterized by different values of c_t, hence the bounds become slightly more involved but its calculation is straightforward. The simple but notationally involved inductive step is rigorously formalized in [7] pp 247-249, here it suffices to state the results obtained after the j-th stage.

j-th stage in the reconstruction process

The protection zone of the first cube of color j is of the form

$$U(V_i^j)=(\text{Cube of side } N+2M \text{ concentric with } V_1^j)\setminus \bigcup_{m=1}^{j}\ \bigcup_{i\geq 1} V_i^m$$

The first step involves a $(V_1^j\cup U(V_1^j),V_1^j)$-reconstruction of the fields $(\xi^{(j-1)},\eta^{(j-1)})$. As above $U(V_1^j)\cup\partial V_1^j$ is a union of a(ν) sets of D, hence the pair $(T,S)\in Y$. The boundary $\partial_k T$ decomposes in the form

$$\partial_k T = \bigcup_{i=1}^{2(j-1)+1} T_i$$

where:

For $i=1,\ldots,j-1$ T_i is the part of $\partial_k T$ intersecting a cube of color i, where $c_t=C_o A_i+\delta$. This part is at a distance 1 from $S=V_1^j$.

For $i=j,\ldots,2(j-1)$ T_i is the part of $\partial_k T$ intersecting the protection zone of a cube of color $i-(j-1)$, but not intersecting any cube of color larger than $i-(j-1)$ and smaller than j or its protection zone. Here $c_t=C_o B_i+\delta$; and this part is at a distance M from V_1^j.

$T_{2(j-1)+1}$ is the rest of $\partial_k T$, where $c_t=C_o$. It is also at distance M from V_1^j.

In the second step one performs a $(U(V_1^j),U(V_1^j))$-reconstruction. The set $U(V_1^j)$ is a union of $a(\nu)$ parallelepipeds in D, while $(T\setminus S)\cup\partial S=\partial U(V_1^j)$ is a union of less than $2\nu a(\nu)$ ($(\nu-1)$-dimensional) parallelepipeds of D. One checks that the pair $(T,S)\in Y$. Now $\delta_k T$ decomposes into 2j sets: the first $2(j-1)+1$ are as in the first step, and $T_{2j}=(\partial_k T)\cap V_1^j$ is characterized by $c_t=C_o A_j+\delta$.

Repeating this two-step process for all cubes V_i^j, $i\geq 1$, and going to the limit $i\to\infty$ one obtains representants $\xi^{(j)}$, $\eta^{(j)}$ satisfying:

$$(VI.32)\quad (\sigma^{\xi^{(2)}} \times \sigma^{\eta^{(2)}})(x_t) \leq \begin{cases} C_o A_i + \delta & t \in C_i;\ i \leq j \\ C_o B_i + \delta & t \in P_i;\ i \leq j \\ C_o & \text{otherwise} \end{cases}$$

Here:

$C_i = \bigcup_m V_m^i$ (points in a cube of color i)

$P_i = [\bigcup_m U(V_m^i)] \setminus [\bigcup_{s=i+1}^{j} \bigcup_m (V_m^i \cup U(V_m^i))]$

(points in the protection zone of a cube of color i but not in a cube of color m with $i<m\leq j$ or its protection zone),

and the new constants incorporated in this stage are:

$$(VI.33)\quad A_j = K_j[N+2M]^{\nu-1}\{\Xi(M)(1+\sum_{i=1}^{j-1} B_i) + \Xi(1)\sum_{i=1}^{j-1} A_i + \varepsilon N^{\nu}[1+\sum_{i=1}^{j-1}(A_i+B_1)]\}$$

$$(VI.34)\quad B_j = K_j[N+2M]^{\nu-1}[1+A_j+\sum_{i=1}^{j-1}(A_i+B_i)][\Xi(1)+\varepsilon K_j M]$$

After $\nu+1$ stages, the bounds inside all the cubes have been improved and there are no more B_i's left. In fact, if $A = \max_{1\leq i\leq \nu} A_i$; one has that

$$(\sigma^{\xi^{(\nu+1)}} \times \sigma^{\eta^{(\nu+1)}})(x_t) \leq \begin{cases} C_o A_{\nu+1} + \delta & t \text{ in a cube of color } \nu+1 \\ C_o A + \delta & \text{otherwise} \end{cases}$$

As there is a finite number of geometries involved, parameters r_1, r_2, s_1, s_2, b, c; can be found so all the sets involved in the proof belong to the corresponding set E. Now, one can renumerate the colors so that the color $\nu+1$ becomes the color 1 and viceversa, and apply the whole construction starting from the representants $\xi^{(\nu+1)}, \eta^{(\nu+1)}$. The bound C_o for points outside cubes of color 1 is replaced by $C_oA+\delta$. After grouping the terms involving δ into a new δ, the new reconstruction yields (VI.13).

As remarked above, all the A_j's and hence A are damping factors for $\varepsilon=0$. Indeed, in this situation the dominant term for B_j is of order $N^{\nu-1}B_{j-1}$ and therefore $B_j \sim (N^{\nu-1})^j$. On the other hand

$$A_j \sim N^{\nu-1}\Xi(M)B_j \sim N^{(\nu-1)j}\Xi(N)$$

Therefore

$$A \sim N^{(\nu-1)\nu}\Xi(N) \underset{N\to\infty}{\to} 0.$$

Thus, there exists an $N_o=N_o(\psi)$ such that $A(N,\varepsilon=0)<1$ for $N\geq N_o$. Moreover, A is a continuous function of ε, hence there exists an $\varepsilon_o(N_o)$ such that $A(N,\varepsilon)<1$ for $\varepsilon\leq\varepsilon_o$ and $N\geq N_o$. Then, if $r_o(N_o)$ is defined by condition (VI.15), namely $N_o+2d=r_o$, one has that (VI.13) holds with $A<1$ for specifications satisfying condition $A_Y(\psi,\varepsilon,r)$ with $\varepsilon\leq\varepsilon_o$, $r\geq r_o$. ∎

Corollary VI.11: For each function $\psi \epsilon F_{pow}$ there exist $N_o(\psi)$, $\varepsilon_o(\psi)$, $r_o(\psi)$ such that every k-dependent specification P satisfying condition $A_\gamma(\psi,\varepsilon,r)$ for some $N \geq N_o$, $\varepsilon \leq \varepsilon_o$, $r \geq r_o$; has a unique physical random field consistent with it.

Proof: Pick N_o, ε_o, r_o as defined in the previous theorem. Suppose $[\xi]$ and $[\eta]$ are consistent with P. For a finite $V \subset Z^\nu$ consider an N-partition in $\nu+1$ colors with $N \geq N_o$, such that $V \subset V_1^1$. Apply theorem VI.10 for any two representants ξ, η with $C_o=1$. One obtains new representants ξ', η' satisfying

$$(\sigma^{\xi'} \times \sigma^{\eta'})(x_t) \leq \begin{cases} AA_{\nu+1}{}^{+\delta} & t \text{ in a cube of color 1} \\ A^{2+\delta} & \text{otherwise} \end{cases}$$

No apply again the theorem this time to ξ', η' and with $C_o=A^{2+\delta}$. One obtains a new pair of representants ξ'', η'', such that

$$(\sigma^{\xi''} \times \sigma^{\eta''})(x_t) \leq \begin{cases} A^3 A_{\nu+1}{}^{+\delta} & t \text{ in a cube of color 1} \\ A^{4+\delta} & \text{otherwise} \end{cases}$$

This process can be iterated. After n iterations one obtains representants ξ^n, η^n such that

$$(\sigma^{\xi^n} \times \sigma^{\eta^n})(x_t) \leq \begin{cases} A^{2n-1} A_{\nu+1}{}^{+\delta} & t \text{ in a cube of color } 1 \\ A^{2n}{}_{+\delta} & \text{otherwise} \end{cases}$$

For $\varepsilon \leq \varepsilon_o$ and $r \geq r_o$, $A<1$; moreover δ can be chosen as small as desired, hence this proves that after a sufficient number of iterations $(\sigma^{\xi^n} \times \sigma^{\eta^n})(c_t)$ can be made arbitrarily small for $t \in VII$. This proves (VI.9), then (VI.8) and the uniqueness.∎

Of course, corollary VI.4 can be restated replacing condition $A(\psi,0,\infty)$ by $A_Y(\psi,0,\infty)$.

6.5 CRITERION IN TERMS OF THE DECAYING OF PARTITION FUNCTIONS

The combination of (VI.7) with the modified version of corollary VI.4 involving only condition $A_Y(\psi,0,\infty)$ yields the following sufficient condition for the absence of phase transitions.

Cororllary VI.12: Suppose that the range-k potential ϕ is such that there exist positive C,γ,K so that

$$(VI.35) \qquad |Z_V^y - K| \leq C \exp(-\gamma N)$$

for every $V \in E(N,k,\nu)$, $y \in X_{Z^\nu \setminus V}$; N large enough. Then there is an open neighborhood of ϕ in $\mathcal{B}_k$ where there is a unique Gibbs state.

Proof: The RHS of (VI.7) is

$$(VI.36)\quad J=(Z_V \,/\, |Z_U^y\, Z_U^z|)\,|Z^{x\vee y}_{(U\setminus V)\cup\partial V} \,/\, Z_U^z - Z^{x\vee z}_{(U\setminus V)\cup\partial V} \,/\, Z_U^y|$$

Write

$$(VI.37)\qquad Z_V^y = K+\delta(V,y),$$

with, by hypothesis,

$$(VI.38)\qquad |\delta(V,y)|\leq C\,\exp(-\gamma N).$$

If N is large enough so that $C\exp(-\gamma N)<K/2$, then

$$(VI.39)\qquad |Z_V^y|\leq 3K/2,\quad 1/|Z_V^y|\leq 2/K,$$

for every $V\in D$, $y\in X$. Therefore, the first factor in the RHS of (VI.36) is bounded by $6/K$, and the absolute value is the sum of six terms each of them with at least one factor δ. In conclusion

$$(VI.40)\qquad J\leq C'\,\exp(-\gamma' N)$$

with $C'=(K/6)\,\sup(C,C^2)$, $\gamma'=2\gamma$. From (VI.40) and (VI.7) one has that for $(V,U)\in Y(N,k,\nu)$, N large enough:

$$R(P_{V,U}(\bullet|y),P_{V,U}(\bullet|z))\leq C\,\exp(-\gamma N)$$

for some C, γ positive. If $y=z$ the variation distance is zero and there is nothing to prove. If $y_{t_o}\neq z_{t_o}$ for some $t_o\in\partial_k U$, then by the geometry of U there is a $v_o\in V$ with

$$|v_o - t_o| \geq \tfrac{1}{2}[N/2^{\nu} - (1/2^{\nu-1} + k)]$$

Hence if

$$C' = 2C \exp(2 + 2^n k), \qquad \gamma' = 2^{\nu+1},$$

one has for $(U,V) \in A_Y$, N large enough:

$$R(P_{V,U}(\cdot|y), P_{V,U}(\cdot|z)) \leq C' \exp(-\gamma'|v_o - t_o|)$$

$$\leq \sum_{t,v} \psi(t-v)\chi(y_t, z_t)$$

with $\psi(t) = C' \exp(-\gamma'|t|) \in F_{pow}$. The absence of phase transitions follows from the modified version of corollary VI.4.∎

The reasons why an exponential decay for the partition functions can be expected for some systems can be seen from the high temperature expansion. The models considered in this chapter are $|\Omega|$-components systems. Therefore, the algebraic machinery introduced in sections 2.4 and 2.5 is available; in particular the high temperature expansion (II.33). By proposition II.20, if the interaction is real the factors $t(\chi, \alpha(\chi))$ have absolute value strictly smaller than one; hence each non zero $\alpha \in K_\Lambda$ has associated a factor that decreases exponentially with the size of the support of α. One may expect that for systems in which the number of α

with a given support does not increase too fast with the size of the support of α, the exponential damping will prevail and all the terms in the sum (II.33) will disappear exponentially fast except the one for $\alpha=0$. That means that Z_V will tend to $\Pi_\chi f(\chi,0)$ exponentially fast. This is almost condition (VI.35) except that we need a K independent of VII. This in turn can be fixed by cancelling part of the exponents in the numerator and denominator of the expression for $P_{V,U}(\cdot|y)$.

For an interaction J and a finite $V \subset L$ denote:

$$K_V^y = K_V(J^{V,y})$$

$$\tilde{B}_V^y = \tilde{B}_V(J^{V,y})$$

One has that for every $y \in X$:

$$K_V^y = K_V^+$$

$$\tilde{B}_V^y = \tilde{B}_V^+$$

Denote also $t^{y,V}(\chi,l)$ the coefficients (II.35) for the interaction $J^{y,V}$ with "y" boundary conditions outside V . As there is only a finite number of fundamental bonds and Ω is finite, there is only a finite number of coefficients $t^{y,V}(\chi,l)$. Denote

(VI.41) $\tau=\sup\{|t^{y,V}(\chi,l)|:$ finite $V\subset Z^\nu$ $y\in X$, $\chi\in\tilde{B}$, $0\le l\le ord(\chi)-1\}$

One has

(VI.42) $\qquad 0 < \tau < 1$

For a multiplicity function $\alpha \in K_{inf}$ denote

$$\underline{\alpha} = \text{support of } \alpha = \{\chi \in \tilde{B}: \alpha(\chi) \neq 0\}$$

Proposition VI.13: Consider an n-component system and a real interaction $J \in \mathcal{B}_k$ such that there exist positive constants C, γ so that for every $V \in (N,k,\nu)$, N large enough:

$$\text{(VI.43)} \qquad \sum_{\alpha \in K^+, \alpha \neq 0} \tau^{|\underline{\alpha}|} \leq C \exp(-\gamma N).$$

Then there exists an open neighborhood of J in $\mathcal{B}_k$ where the Gibbs state is unique.

Proof: Define an interaction J' so that for every $\chi \in B$

$$\text{(VI.44)} \qquad H'^{(\chi)} = H^{(\chi)} + \ln f(\chi, 0)$$

The interaction J' si equivalent to J in the sense that

$$\text{(VI.45)} \qquad \rho_V^y(J'+I) = \rho_V^y(J+I)$$

for every finite $V \subset Z^\nu$, $y \in X$ and every interaction $I \in \mathcal{B}_k$. That implies that the absence of phase transition under a perturbation of J' is equivalent to absence for perturbations of J. An analogous identity holds if the

interaction is restricted to the component $H^{(x)}$ of the Hamiltonian:

$$(VI.46)\quad \rho_V^y(H^{(x)\prime}) = \rho_V^y(H^{(x)})$$

As a consequence:

$$(VI.47)\quad t^{y,V}(x,l) = \rho_V^y(H^{(x)})(x^l) = (t')^{y,V}(x,l)$$

for finite $V \subset Z^\nu$; $y \in X$. Also, the sets K_V^y; $\tilde{B}_V^y$ are the same for J and J'. The advantage of the interaction J' is its simpler high temperature expansion. In fact from (VI.44)

$$(VI.48)\qquad f'(x,0) = 1 \quad \text{for every } x \in \widetilde{B}$$

and so

$$(VI.49)\qquad (Z')_V^y - 1 = \sum_{\alpha \in K^y, \alpha \neq 0} \prod_{x \in \tilde{B}^y} \left[(t')_{|J|}^{y,V}(x,l)\right]^{\alpha(x)}$$

Hence, from (VI.47),(VI.49) and the hypothesis

$$|(Z')_V^y - 1| \leq C \exp(-\gamma N)$$

for every $V \in D(N,k,\nu)$, N large enough. By corollary (VI.12) this implies the existence of a neighborhood of J' in $\mathcal{B}_k$ without phase transitions. By (VI.45) the same is true for the original J.∎

This proposition is the result needed. The absence of phase transitions under perturbation is linked to an exponential damping of the sum (VI.43). Moreover, this sum does not need to be evaluated for arbitary V but only for the sets in $E(N,k,\nu)$. This criterion is simpler than the original DP result, and it requires a careful study of the cycles $\alpha \in K_V^+$ so as to check if the growth in the number of cycles of a given support is overwhelmed by the exponential damping $\tau^{|\underline{\alpha}|}$. This will be done in the next chapter for a particular family of systems.

Chapter VII

APPLICATION FOR ONE BOND SYSTEMS

7.1 CYCLES FOR FACTORIZABLE ONE-BOND N-TYPE SYSTEMS

Definition VII.1: A one-bond system of basic bond ξ is a translation invariant model with $L=Z^\nu$, for which the state space at each point is the same compact abelian group Ω and such that the set $\widetilde{B}$ is formed by the translated of ξ.

Definition VII.2: A one-bond system of basic bond ξ is an n-type system if $ord(\xi)=n$.

If $\Omega=G_2$ (spin ½) one-bond systems are systems with $|B_o|=1$, and the "basic bond" coincides with the fundamental bond. For n-component systems with n>2, the basic bond need not be a bond and B_o may include many multiples of ξ (and its inverse). Hence in general one-bond systems may have more than one fundamental bond.

One-bond n-type systems are of range $k=diam(\xi)$ and all the cycles can be defined on $(Z_n)^{Z^\nu}$ with

$$\alpha_a = \alpha(\tau_a\xi), \qquad a\epsilon Z^\nu. \tag{VII.1}$$

Let's denote

$$\underline{\alpha} = \text{support of } \alpha = \{a\epsilon Z^\nu: \alpha_a \neq 0\} \tag{VII.2}$$

The cycles K_V^+ for V parallelepiped, are obtained by "cutting" pieces of infinite cycles. For $V\epsilon D(N,k,\nu)$ the cycles are obtained by "patching" cycles in each of the $b(\nu)$ parallelepipeds. Hence we must start analyzing infinite cycles. Denote

$$(VII.3) \qquad X_\xi = \underset{a\epsilon Z^\nu}{X} G_{ord(\xi)}, \qquad R_\xi = \underset{a\epsilon Z^\nu}{X} Z_{ord(\xi)}$$

The group operation of X_ξ will be written multiplicatively, while that of R_ξ will be denoted additively. The elements of R_ξ are called <u>multiplicity functions</u>.

Let us recall that w_1 denotes the neutral element for the ring product in $G_{ord(\xi)}$ (section 2.5). For $\alpha\epsilon R_\xi$, $1_\bullet^\alpha$ will denote the configuration in X_ξ defined as

$$(VI.4) \qquad (1_\bullet^\alpha)_a = (w_1)^{\alpha_a} = \exp(2\Pi i\alpha_a/n) \qquad , a\epsilon Z^\nu.$$

Moreover, let's introduce the map

$$I: \hat{X} \to \hat{X}$$

$$(VII.5) \qquad I(\xi)_a = \xi_{-a}$$

(inversion with respect to $0\epsilon Z^\nu$), and for $\xi\epsilon\hat{X}$ denote

$$(VII.6) \qquad S_\xi = \{x\epsilon X: (\tau_a\xi)(x)=1 \ , \ a\epsilon Z^\nu\} \subset X_\xi.$$

The map $\alpha \mapsto 1_\bullet^\alpha$ is an isomorphism between R_ξ and X_ξ. When restricted to $K_{inf} \subset R_\xi$ it yields the following useful characterization.

Proposition VII.3: Consider a one-bond, n-type system with basic bond ξ. Then: A multiplicity function α is an (infinite) cycle if and only if $1_\bullet^\alpha \in S_{I(\xi)}$.

Proof: The proof is based in the following claim involving the configurations e_s (defined in (II.43)) for $s\in Z^\nu$.

Claim:

$$(VII.7) \qquad [\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](e_s) = [\tau_{-s}I(\xi)](1_\bullet^\alpha)$$

Indeed, as $\hat{w}(1)=1$ for every $\hat{w}\in\hat{\Omega}$ one has that

$$[\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](e_s) = \prod_{a\in Z^\nu : s\in\tau_a\underline{\xi}}[(\tau_a\xi)^{\alpha_a}]_s(w_1)$$

$$= \prod_{a\in Z^\nu : s\in\tau_a\underline{\xi}} \xi_{s-a}^{\alpha_a}(w_1)$$

By linearity of $\xi_{s-a}\in\hat{\Omega}$ and (II.38):

$$\xi_{s-a}^{\alpha_a}(w_1) = \xi_{s-a}(w_1^{\alpha_a})$$

Hence,

$$[\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](e_s) = \prod_{a\in Z^\nu : s\in\tau_a\underline{\xi}} \xi_{s-a}(w_1^{\alpha_a})$$

$$= \tau_{-s}[\prod_{a:-a\in\underline{\xi}} \xi_{-a}(w_1^{\alpha_a})] = [\tau_{-s}I(\xi)](1_\bullet^\alpha)$$

The claim is proved. Now the proof is almost immediate:

(=>) By hypothesis

$$[\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](x) = 1 \ , \quad \text{for every } x\in X$$

Applying this identity to $x=e_s$, $s\in Z^\nu$ and using (VII.7) one gets that

$$[\tau_{-s}I(\xi)](1^\alpha_\bullet) = 1$$

for every $-s\in Z^\nu$. Hence $1^\alpha_\bullet \in S_{I(\xi)}$.

(<=) By density of X_f it is enough to check that

$$[\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](x) = 1$$

for every x of the form $x=\prod_s x_s\bullet e_s$ with $x_s=1$ for all but a finite number of $s\in Z^\nu$. But in this case, by (II.39) and (II.44):

$$[\prod_{a\in Z^\nu}(\tau_a\xi)^{\alpha_a}](x) = \prod_s x_s\bullet\{[\prod_a(\tau_a\xi)^{\alpha_a}](e_s)\}$$

$$= \prod_s x_s\bullet 1 = 1$$

where (VII.7) and the hypothesis were used in the second equality, and (II.37) in the last one.∎

The map

$$\begin{aligned} K_{inf} &\to S_{I(\xi)} \\ \alpha &\longmapsto 1^\alpha_\bullet \end{aligned} \tag{VII.8}$$

is one-one and onto. Moreover, if for $x\in X_\xi$ one defines $\underline{x}=\{a\in Z^\nu: x_a\neq 1\}$, one has that

(VII.9) $\underset{\sim}{1}^{\alpha}_{\bullet} = \underline{\alpha}$

Therefore, with regard to (VI.43) instead of studying the cycles for ξ one can study the configurations of $S_{I(\xi)}$; focussing on the relationship between the number of configurations with a given support and the size of this support. In what follows we will concentrate ourselves in one-bond systems with ord(ξ) prime. Among the reasons for this restriction is the necessity of an structure of unique factorization domain for $\hat{X}$ (corollary II.22). Obviously ord(ξ)=ord(I(ξ)), hence we will study the space S_{ξ} for ord(ξ) prime. The simplifying features of such models are clearly seen in the case ν=1.

Proposition VII.4: Consider a one-bond p-type system (p prime) with basic bond ξ. Then if diam($\underset{\sim}{\xi}$)=m:

i) If x, y belong to S_{ξ} agree on a set of m-1 consecutive points, then x=y.

ii) $|S_{\xi}| = p^{m-1}$

iii) There exists a positive integer q such that every $x \in S_{\xi}$ has period q, that is $x_{a+q}=x_a$ for every $a \in Z$.

Proof: Consider $\xi=\xi_0 \ldots \xi_{m-1}$ with $\xi_i \in \hat{\Omega}$, $\xi_i(w)=w^{l(i)}$ for $w \in \Omega$.

i) Let T be the subset of Z where x and y agree.

Claim: max(T)= ∞.

Indeed, if $\max(T)=c<\infty$, take the translated $\xi'=\tau_{c-m+1}\xi$ stretching from c-m+1 to c+1. As by hypothesis $|T|\geq m-1$,

$$1=\xi'(x)=\xi'(y) \Rightarrow (x_{c+1})^{l(m-1)}=(y_{c+1})^{l(m-1)}$$

$$\Rightarrow (x_{c+1}\ y_{c+1}^{-1})^{l(m-1)}=1$$

Therefore $(x_{c+1}\ y_{c+1}^{-1}) \in G_{l(m-1)}\cap G_p=\{1\}$ because p is prime. This implies $x_{c+1}=y_{c+1}$ against the maximality of c.
The proof that $\min(T)=-\infty$ is analogous.

ii) By i) every $x\in S_\xi$ is uniquely determined by its projections on m-1 consecutive points. As there are up to p choices per point, $|S_\xi|\leq p^{m-1}$. To see the opposite inequality note that every configuration x on [0,...,m-2] can be uniquely extended to an element of S_ξ. For instance x_{m-1} is defined as the unique solution in G_p of

$$(x_{m-1})^{l(m-1)} = 1 \Big/ \prod_{i=0}^{m-2} (X_i)^{l(i)}$$

iii) For each interval [j,j+m], $j\in Z$ there are only p^{m-1} possibles projections for a configuration $x\in S_\xi$. Therefore after a number $d\leq p^{m-1}$ of such intervals, there are two identical projections. By i) the configuration is then periodic with period d. One can choose q = l.c.m.{periods of the p^{m-1} configurations of S_ξ} ∎

Using parts ii) and iii) one can easily prove (VII.3) for the one dimensional one-bond p-type systems. Indeed, while the number of configurations -i.e. of cycles of the inverted bond- is bounded independently of the length of the support, the length of the support increases with N due to the periodicity. The systems in higher dimensions with similar properties are the factorizable systems.

For $\bar{a}\in Z^{\nu-1}$, $1\leq i\leq \nu$, denote $L(\bar{a},i)$ the line in Z^{ν} parallel to the i-th axis that crosses the "plane" $\{b\in Z^{\nu}: b_i=0\}$ at the point $(a_1,a_2,\ldots,a_{i-1},0,a_i,\ldots,a_{\nu-1})$, i.e.:

(VII.10) $$L(\bar{a},i)=\{b\in Z^{\nu}: b_1=a_1,\ldots,b_{i-1}=a_{i-1},b_{i+1}=a_i,\ldots,b_{\nu}=a_{\nu}\}$$

A character is said factorizable if it is the $\bullet$-product in X (defined in (II.45)) of one dimensional characters.

Definition VII.5: A character ξ is said factorizable if it is of the form

(VII.11) $$\xi=\xi^{(1)}\bullet\xi^{(2)}\bullet\ldots\bullet\xi^{(\nu)}$$

where each character $\xi^{(i)}$ satisfies

(VII.12) $$[\xi^{(i)}]_b = 1 \qquad \text{if } b\notin L(\bar{0},i)$$

A factorizable one-bond system is a one bond system with a factorizable basic bond. For such a system the basic bond ξ can be chosen so to "start" at 0. More precisely $0\in Z^{\nu}$ is the

first point $a\epsilon Z^\nu$ -in lexicographic order- such that $\xi_a \neq 1$. Such choice will be supposed in the sequel.

It is simple to check that the map (VII.5) commutes with "•":

$$(VII.13) \qquad I(\xi)=I(\xi^{(1)})\bullet \ \ldots \ \bullet I(\xi^{(\nu)})$$

Thus, the study of cycles for a factorizable one bond system with basic bond ξ is equivalent -via the bijection (VII.8)-, to the study of configurations in $S_\xi, \subset X_\xi$, for a factorizable bond ξ' with $diam(\underline{\xi}')=diam(\underline{\xi})$, $ord(\xi')=ord(\xi)$.

For a factorizable bond ξ as in (VII.11), denote:

$X_\xi^{(i)}=\{x\epsilon X:\ x_a=1 \text{ if } a\notin L(\bar{0},i)\}$
(configurations along the i-th axis)

$\bar{B}_i=\{$subgroup of X_ξ generated by the translatted of $\xi^{(i)}$ along the i-th axis, i.e. by $\tau_a\xi^{(i)};\ a\epsilon L(\bar{0},i)\}$

$S_i = \bar{B}_i^{\perp} = S_{\xi(i)}$.

Proposition VII.6: For a factorizable one-bond p-type system (p prime) with basic bond ξ:

i) If $\chi\epsilon\hat{X}_\xi$ is such that for every $i\leq i\leq\nu$, $a\epsilon Z^\nu$, $[t_a\chi\restriction X^{(i)}]\epsilon\bar{B}_i$, then $\chi\epsilon\bar{B}$. (The opposite is true even for p not prime).

ii) S_ξ is generated by the translated of $S_1,\ldots,S_\nu$.

Proof: i) For each $1\le i\le\nu$ $\bar{a}\in Z^{\nu-1}$ denote $\chi^{(i)}_{(\bar{a})}$ the "restriction" of χ to $L(\bar{a},i)$:

$$\chi^{(i)}_{(\bar{a})}=\begin{cases}\chi_b \text{ if } a_1=b_1,\dots a_{i-1}=b_{i-1}, a_i=b_{i+1},\dots,a_{\nu-1}=b_\nu\\ \\ 1 \quad \text{otherwise.}\end{cases}$$

If

$$a'=(a_1,\dots,a_{i-1},0,a_i,\dots,a_{\nu-1}) \tag{VII.14}$$

then $\chi^{(i)}_{(\bar{a})}=(\tau_{-a'}\chi)\restriction X^{(i)}$; hence by hypothesis $\chi^{(i)}_{(\bar{a})}\in B_i$. Therefore there exists a set of $\alpha(j)\in Z_p$, $j\in L(0,i)$ such that

$$\chi^{(i)}_{(\bar{a})} = \prod_j [\tau_j \xi^{(i)}]^{\alpha(j)} = \prod_j (e_j)^{\alpha(j)\bullet\xi^{(i)}}$$

The last equality is due to (II.47) and (II.48). Hence there exist $Q^{(i)}_{(\bar{a})}\in X_\xi$ such that $\chi^{(i)}_{(\bar{a})}=Q^{(i)}_{(\bar{a})}\bullet\xi^{(i)}$. Therefore

$$\chi=\prod_{\bar{a}\in Z^{\nu-1}}\chi^{(i)}_{(\bar{a})}=\prod_{\bar{a}\in Z^{\nu-1}}Q^{(i)}_{(\bar{a})}\bullet\xi^{(i)} = Q^{(i)}\bullet\xi^{(i)}$$

The ring X_ξ is a unique factorization domain (corollary II.22). It can be proved that if $i\neq j$ $\xi^{(i)}$ and $\xi^{(j)}$ are coprime in this domain (this can be seen for instance using polynomical notation). Therefore, the fact that

$$Q^{(1)}\bullet\xi^{(1)} = \chi = Q^{(2)}\bullet\xi^{(2)}$$

implies that $\xi^{(1)}$ divides $Q^{(2)}$, i.e. $Q^{(2)}=Q'\xi^{(1)}$ for some $Q'\epsilon\hat{X}_\xi$. Proceeding inductively one concludes that there exists a $P\epsilon\hat{X}_\xi$ such that

$$\chi = P\bullet\xi^{(1)}\bullet\ldots\bullet\xi^{(\nu)}$$

Thus, $\chi\epsilon\bar{B}$

ii) Denote $\widetilde{S}$ the group generated by the translated of $S_1,\ldots,S_\nu$. Pick $\chi\epsilon\widetilde{S}^\perp$. As S is translation invariant $\tau_a\chi\epsilon\widetilde{S}^\perp$ for every $a\epsilon Z^\nu$. In particular $\tau_a\chi(x)=1$ for every $a\epsilon Z^\nu$, $x\epsilon S_i\subset S$, $1\leq i\leq\nu$. Thus $\tau_a\chi\epsilon\widetilde{S}_i^\perp=\bar{B}_i$ for every $a\epsilon Z^\nu$, $1\leq i\leq\nu$. By i) this means that $\chi\epsilon\bar{B}=S^\perp$, hence $\widetilde{S}^\perp\subset S^\perp$ what implies that $S\subset\widetilde{S}$.∎

Part ii) of the previous proposition tells us that every $x\epsilon S_\xi$ is of the form

$$x = \prod_{1\leq i\leq\nu}\ \prod_{\bar{a}\epsilon Z^{\nu-1}} X^{(i)}_{(\bar{a})} \tag{VII.15}$$

where $X^{(i)}_{(\bar{a})}\epsilon\tau_{a'}S_i$ (a' defined in (VII.14)). Combining this result with part iii) of proposition VII.4 and the bijection (VII.8) one gets the following important result.

In the sequel, whenever a multiplicity function on Z^ν is denoted as $\alpha^{(i)}_{(\bar{a})}$ it will imply that its support is contained in $L(\bar{a},i)$. Moreover an $\alpha\epsilon R_\xi$ will be said periodic in the i-th direction with period q if -denoting $\hat{\imath}$ the versor in the i-th direction-, $\alpha_{b+q\hat{\imath}}=\alpha_b$ for every $b\epsilon Z^\nu$.

Proposition VII.7: For a factorizable one-bond p-type system (p prime), there exists an integer $q>0$ such that every $\alpha\in K_{inf}$ can be written as

$$\alpha = \sum_{1\le i\le \nu}^{(p)} \sum_{\bar{a}\in Z^{\nu-1}}^{(p)} \alpha_{(\bar{a})}^{(i)} \tag{VII.16}$$

where each $\alpha_{(\bar{a})}^{(i)}$ is periodic in the i-th direction with period q. The notation $\Sigma^{(p)}$ indicates sum in Z_p.

The decomposition (VII.16) is far from unique.

7.2 CYCLES INSIDE PARALLELEPIPEDS

When considering cycles in K_V^+ -V a subset of Z^ν- the bonds are of the form

$$\chi = \tau_a \xi \restriction X_V \tag{VII.17}$$

For each χ, the $a\in Z^\nu$ satisfying (VII.17) is uniquely defined, hence we will identify χ with the (possibly "truncated") $\tau_a\chi$ and denote

$$\alpha(\chi) = \alpha_a \tag{VII.18}$$

With this identification every cycle in K_V^+ is given by an $\alpha\in R_\xi$ with support in

$$\tilde{V} = \{a\in Z^\nu: \tau_a \underline{\chi} \cap V \neq \phi\}$$

If V is a union of parallelepipeds, so is $\widetilde{V}$ by the regularity of χ. For a parallelepiped V denote

$$\Pi(V)=\{\alpha\in R_\xi:\ \alpha=\sum_{1\le i\le\nu}{}^{(p)}\ \sum_{\bar a\in Z^{\nu-1}}{}^{(p)}\ \alpha_{(\bar a)}^{(i)}$$

with $\underline{\alpha}_{(\bar a)}^{(i)}\subset V$, and such that $\alpha_{(\bar a)}^{(i)}\restriction V$ is periodic in the i-th direction with period q}

One has that

$$\text{(VII.20)}\qquad \sum_{\alpha\in K_V^+}\tau^{|\underline{\alpha}|}\le\sum_{\alpha\in\Pi(\widetilde{V})}\tau^{|\underline{\alpha}|}$$

If $\widetilde{V}$ is a union of disjoints ν-dimensional parallelepipeds, $(V_j)_{j=1}^b$, then each $\alpha\in\Pi(\widetilde{V})$ can be decomposed as

$$\text{(VII.21)}\qquad \alpha=\sum_{j=1}^{b}\alpha(j)$$

where $\alpha(j)$ is the restriction of α to V_j. Hence $\alpha(j)\in\Pi(V_j)$.

If $V=U\cup W$, with U being a union of ν-dimensional parallelepipeds and W a union of ν-1-dimensional parallelepipeds, then every cycle in V decompose in a unique way as

$$\text{(VII.22)}\qquad \alpha=\alpha(1)+\alpha(2)$$

with $\alpha(1)\in\Pi(\widetilde{U})$ and $\alpha(2)$ a cycle in $W\setminus\widetilde{U}$.

For a parallelepiped V that has one side of size zero; some cycles in V may correspond to α's that do not belong to $\Pi(\widetilde{V})$. This can happen when the basic bond has two parallel sections, neither of them trivial, such that one is not the translated of the other. In such case the system of bonds obtained for "+" boundary conditions on a rectangle parallel to these sections may have more than one basic bond and hence all the previous discussion does not apply. In particular there may be cycles that are not a sum of periodic unidirectional cycles. As periodicity is crucial for our argument, we must restrict ourselves to a family of factorizable one-bond systems such that when restricted to "flat" parallelepipeds the system remains a one-bond system.

Definition VII.8: A character $\xi\epsilon\hat{X}$ is regular if it is factorizable and for each $1\leq i\leq\nu$ there exists $s^{1(i)}\epsilon\hat{\Omega}$ such that $[\xi^{(i)}]_a$ is either 1 or $s^{1(i)}$. For example, for spin 1/2 ($\Omega=G_2$) all factorizable characters are regular.

A regular one-bond system is a one-bond system whose basic bond is regular.

For $1\leq i\leq\nu$, $b\epsilon Z$, denote M(b,i) as the "plane" perpendicular to the i-th axis of height b:

$$M(b,i) = \{a\epsilon Z^{\nu}: a_i=b\}$$

Proposition VII.9: If ξ is regular, then for every $1\leq i\leq\nu$, $b\in Z$, $\xi\restriction X_{M(b,i)}$ is either trivial or it is equal to $\xi\restriction X_{M(0,i)}$

Proof: The proof is a consequence of the following identity:

$$\xi_a=[\xi^{(1)}]_{a_1\tilde{1}}\bullet \ldots \bullet[\xi^{(\nu)}]_{a_\nu\tilde{\nu}} \tag{VII.23}$$

where $a\in Z^\nu$ and "$\bullet$" is the product in Ω. This identity can be proved straightforwardly from (II.45) and the fact that $[\xi^{(i)}]_a=1$ unless $a_j=0$ for $j\neq i$.

As the "$\bullet$" product is commutative, it is enough to prove the proposition for i=1. If $a\in M(b,1)$:

$$\xi_a=[\xi^{(1)}]_{b\tilde{1}}\bullet \ldots \bullet[\xi^{(\nu)}]_{a_\nu\tilde{\nu}}$$

If $[\xi^{(1)}]_{b\tilde{1}}=1$, by (II.40) $\xi_a=1$ and $\xi\restriction M(b,1)$ is trivial. If not,

$$\xi_a=s^{1(1)}\bullet \ldots \bullet[\xi^{(\nu)}]_{a_\nu\nu} \tag{VII.24}$$

independent of b. In particular $[\xi^{(1)}]_0=s^{1(1)}$ by the convention discussed below (VII.12), hence for all $b\in Z$ with $[\xi^{(1)}]_{b\tilde{1}}\neq 1$:

$$\xi_{(b,a_2,\ldots,a_\nu)} = \xi_{(0,a_2,\ldots,a_\nu)}.\blacksquare$$

7.3 THE MAIN RESULT

The main result of this chapter is the proof that regular one-bond systems of type p (p prime) and real interactions have an unique Gibbs state even when perturbed by sufficiently small finite range interactions. We will start probing the bound (VI.43) for parallelepipeds in D. As mentioned above this is very easy to prove for one dimensional one-bond systems. In fact, for one dimension one has the freedom of multiplying each factor $\tau^{|\underline{\alpha}|}$ for $\alpha\neq 0$, by a polynomial in N and the sum still tends to one exponentially fast with N. The extension to higher dimensions is based in the following inductive step.

For a given $\alpha_{(\bar{a})}^{(i)}$ denote

$$\delta(\alpha_{(\bar{a})}^{(i)})=\begin{cases} 1 & \text{if } \alpha_{(\bar{a})}^{(i)} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and for $\alpha\in\Pi(V)$, $V\subset Z^{\nu}$ finite, define

$$g(\alpha) = \min\{\sum_{\bar{a},i} \delta(\alpha_{(\bar{a})}^{(i)}):\ \Sigma^{(p)}\ \alpha_{(\bar{a})}^{(i)} = \alpha\}$$

($g(\alpha)$ is the minimun number of non zero one dimensional cycles needed to construct α)

Theorem VII.10: Consider a parallelepiped $V=[0,L_1]\times\ldots\times[0,L_{\nu}]$ and denote $V(\nu)=V\cap M(0,\nu)$, then, if $\kappa>0$:

$$\text{(VII.25)}\qquad \sum_{\alpha\in\Pi(V)} \kappa^{g(\alpha)_\tau|\underline{\alpha}|} \le \Big[\sum_{\alpha\in\Pi([0,L_\nu])} \kappa^{g(\alpha)_\tau|\underline{\alpha}|/3}\Big]^{L_\nu+1} \times \Big[\sum_{\alpha\in\Pi(V(\nu))} \kappa^{g(\alpha)_\tau|\underline{\alpha}|/3}\Big]^{|V(\nu)|}$$

Proof: If the terms for $i\neq\nu$ are grouped in the decomposition (VII.16), one obtains that each $\alpha\in\Pi(V)$ can be written as

$$\text{(VII.26)}\qquad \alpha = \sum_{0\le j\le L_\nu}{}^{(p)}\ \alpha(j) +{}^{(p)} \sum_{\bar{a}\in V(\nu)}{}^{(p)}\ \alpha^{(\nu)}_{(\bar{a})}$$

with $\alpha(j)\in\Pi(V(\nu))$ ("sections"), and $\alpha^{(\nu)}_{(\bar{a})}\in\Pi([0,L_\nu])$ ("columns"). One has that:

$$\text{(VII.27)}\qquad g(\alpha) \le \sum_j g(\alpha(j)) + \sum_{\bar{a}} g(\alpha^{(\nu)}_{(\bar{a})})$$

for any decomposition $\{\alpha(j),\alpha^{(\nu)}_{(\bar{a})}\}$ of α. To prove (VII.25) one needs to express the sum over $\alpha\in\Pi(V)$ as a sum over independent sections and columns. One can not just majorize the sum over α's by the sum over the summands in the RHS of (VII.26) assuming them independent because the decomposition (VII.26) is far from unique and such assumption can incorporate factors growing exponentially with L_i.

To illustrate the argument, a very simple example will be presented. Consider the case ord(ξ)=2, ν=2, q=2; i.e. a spin 1/2 bond in Z^2 such that its one directional cycles have period 2 in each direction. (This happens for the model defined by $\Omega=G_2$, and the the square bond $\xi=s_{(0,0)}\ s_{(0,1)}\ s_{(1,0)}\ s_{(1,1)}$ as fundamental bond). In such case the multiplicity functions have values 0 or 1. An $\alpha(j)$ of the decomposition (VII.26) is a column at a distance j of the origin formed all by 1's or all by 0's. The function $\alpha^{(2)}_{(\bar{a})}$ is a row at a height a formed also either all by 1's or all by 0's. In figure 9 a) and b) it is shown how an $\alpha\epsilon\Pi(V)$ can admit two different decompositions (VII.26). Every α in this example can be decomposed in exactly two ways; one obtained from the other by taking the complementary rows and columns. The problem can be traced back to the fact that there are two ways of decomposing $0\epsilon\Pi(V)$: the trivial one with all rows and columns zero, and the nontrivial one with all rows and columns equal to one (figure 9 c).

Given one decomposition of any $\alpha\epsilon\Pi(V)$, the other is obtained by adding (module 2) to each column and each row the corresponding column or row of the nontrivial decomposition of $0\epsilon\Pi(V)$ (figure 10). This double degeneracy would introduce a factor 2 inside each bracket in (VII.25) if the

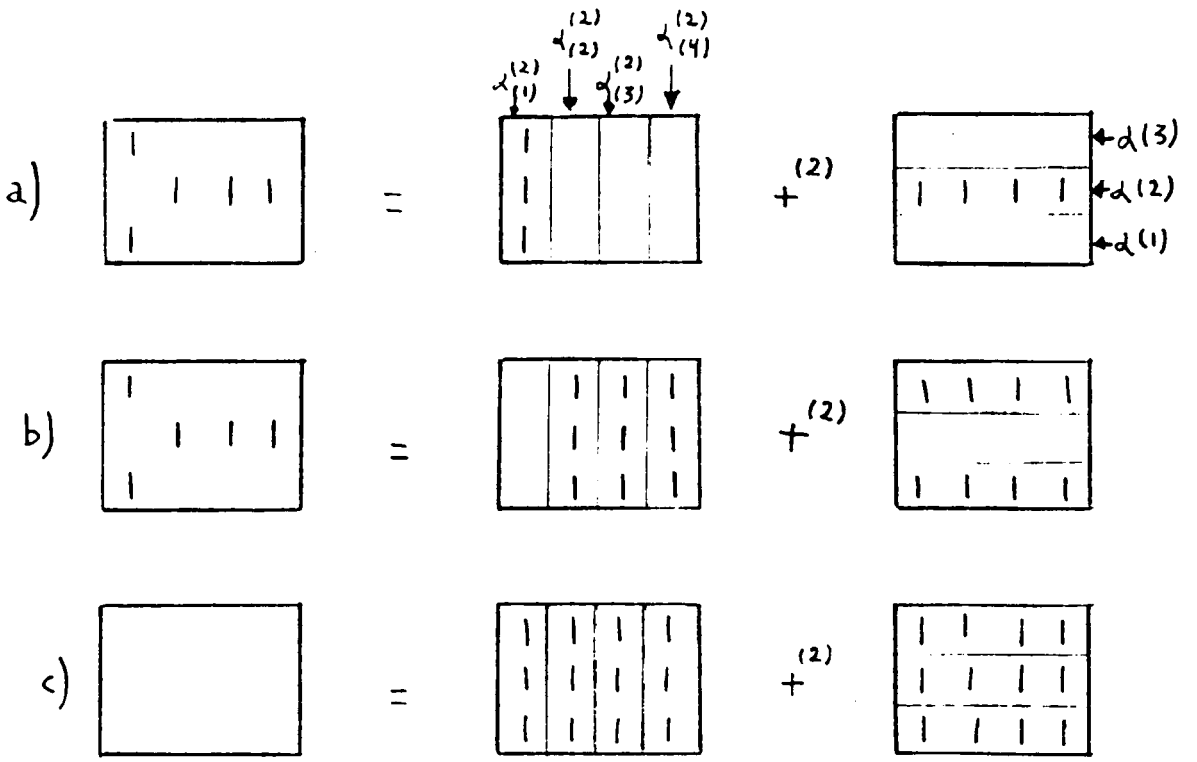

Figure 9: Parts a) and b) show two different decompositions (VII.26) of the same $\alpha \in \Pi(V)$. Part c) pictures the nontrivial decomposition of $0 \in \Pi(V)$

rows and columns were summed independently; which would yield an unacceptable factor 2^N for the parallelepipeds of D. Therefore, before switching to independent sums over rows and columns, it is necesary to restrict the choice in the decomposition so that this degeneracy is prevented.

There is also another aspect to take care of. To prove the factorization (VII.25) one has to be able to relate the support of α with the support of the sections and columns in the sum (VII.26). In this regard one would like to avoid situations like the one in figure 9 b) in which there are many "anihilations" in the intersections of rows and columns and the support of the resulting α is widely different from the sum of the supports of the rows and columns. Hence the restriction in the choice of decompositions must be so to single out decompositions with few anihilations. The number of these is bounded by the number of intersections which in turn can not exceed the sum of the supports of the columns. This motivates the following definition.

Definition: A decomposition

$$\{(\alpha(j), \alpha^{(\nu)}_{(\bar{a})}) : j \in [0, L_\nu],\ \bar{a} \in V(\nu)\}$$

of an $\alpha \in \Pi(V)$ is economical if every other decomposition $\{(\alpha'(j), \alpha'^{(\nu)}_{(\bar{a})})\}$ of the same α verifies

$$\text{(VII.28)} \qquad \sum_{\bar{a} \in V(\nu)} |\underline{\alpha}^{(\nu)}_{(\bar{a})}| \leq \sum_{\bar{a} \in V(\nu)} |\underline{\alpha}'^{(\nu)}_{(\bar{a})}|$$

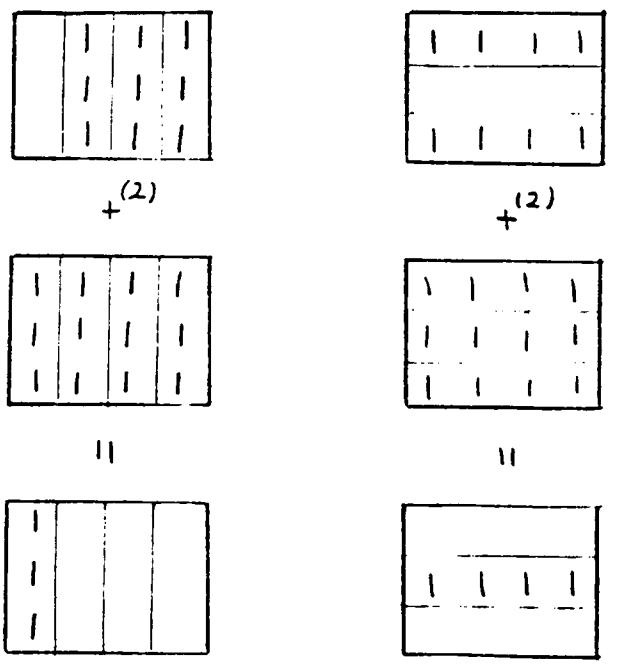

Figure 10: The decomposition of part a) of figure 9 (bottom line) is obtained from that of part b) of the same figure by adding columnwise and rowwise the nontrivial decomposition of zero

Denoting Π^* the set of economical decompositions of $\alpha\in\Pi(V)$, one has, on account of (VII.27):

$$(VII.29)\qquad \sum_{\alpha\in\Pi(V)} \kappa^{g(\alpha)}\,\tau^{|\underline{\alpha}|} \le$$

$$\le \sum_{\Pi^*} \kappa^{[\Sigma g(\alpha(j))+\Sigma g(\alpha^{(\nu)}_{(\bar a)}]}\,\tau^{|\underline{\Sigma^{(p)}\alpha(j)\ +^{(p)}\ \Sigma^{(p)}\alpha^{(\nu)}_{(\bar a)}}|}$$

The restriction to economical decompositions does not eliminate completely the above discussed degeneracy. Figure 11 shows an α in our example with two different economical decompositions. However, the remaining degeneracy is harmless, and there is no need for additional restrictions in the choice of the decompositions.

From (VII.29) one sees that to prove the proposition it is enough to prove the following

Claim 1: For an economical decomposition

$$(VII.30)\qquad |\underline{\Sigma^{(p)}\alpha(j)\ +^{(p)}\ \Sigma^{(p)}\alpha^{(\nu)}_{(\bar a)}}| \ge (1/3)\Sigma|\underline{\alpha(j)}| + (1/3)\ \Sigma|\underline{\alpha}^{(\nu)}_{(\bar a)}|$$

Indeed, if for $\bar a\in V(\nu)$, $j\in[0,L_\nu]$; $n(\bar a,j)$ denotes the number of anihilations at the poin $(a,j)\in V$:

$$n(\bar a,j)=\begin{cases}1 \text{ if } [\alpha(j)+\alpha^{(\nu)}_{(\bar a)}]_{(\bar a,j)}\neq 0 \text{ but } [\alpha(j)+^{(p)}\alpha^{(\nu)}_{(\bar a)}]_{(\bar a,j)}=0\\ 0 \text{ otherwise}\end{cases}$$

then

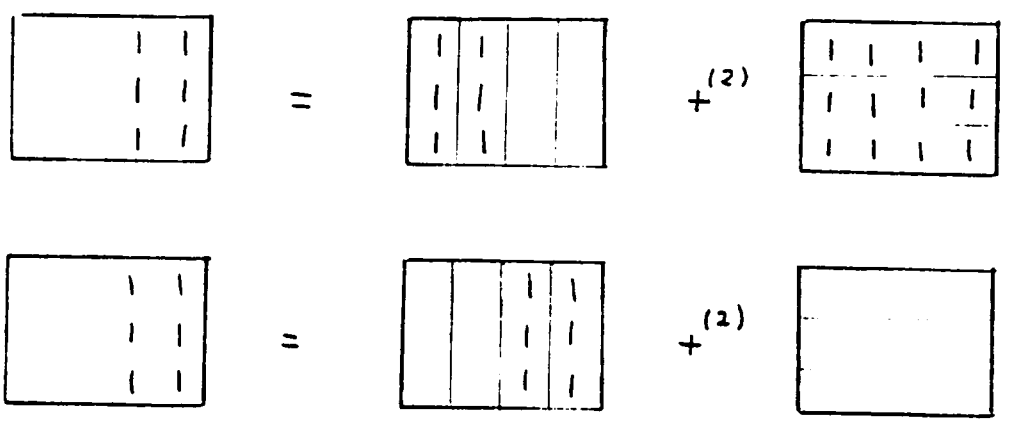

Figure 11: Example of a multiplicity function with two economical decompositions

$$(VII.31)\quad |\underline{\Sigma^{(p)}\alpha(j) + {}^{(p)}\Sigma^{(p)}\alpha^{(\nu)}_{(\bar{a})}}| = \sum_j |\underline{\alpha(j)}| + \sum_{\bar{a}} |\underline{\alpha}^{(\nu)}_{(\bar{a})}| - 2 \sum_{\bar{a},j} n(\bar{a},j)$$

This expresion can be written more simply introducing the following notation:

$$(VII.32)\quad N(j) = \sum_{\bar{a}} n(\bar{a},j)$$

(number of "anihilations" in the plane $M(j,\nu)$)

$$(VII.33)\quad N'(j) = |\alpha(j)| - N(j)$$

(number of "surviving bonds" in the support of $\alpha(j)$)

With this notation the RHS of (VII.31) becomes

$$\sum_j (N'(j)-N(j)) + \sum_{\bar{a}} |\underline{\alpha}^{(\nu)}_{(\bar{a})}|,$$

and (VII.30) is equivalent to

$$\sum_j (N'(j)-N(j)) + \sum_{\bar{a}} |\underline{\alpha}^{(\nu)}_{(\bar{a})}| \geq (1/3)\sum_j (N'(j)+N(j)) + (1/3)\sum_{\bar{a}} |\underline{\alpha}^{(\nu)}_{(\bar{a})}|$$

or

$$(VII.34)\quad (2/3)\Sigma N'(j) - (4/3)\sum_j (N'(j)-N(j)) + (2/3)\sum_{\bar{a}} |\underline{\alpha}^{(\nu)}_{(\bar{a})}| \geq 0$$

But $\Sigma |\underline{\alpha}^{(\nu)}_{(\bar{a})}| \geq \Sigma N(j)$ because each anihilation occurs at one point of the support of some column $\alpha^{(\nu)}_{(\bar{a})}$. Hence the LHS of (VII.34) is larger or equal than $(2/3)\Sigma(N'(j)-N(j))$ and the claim is proven if

$$(VII.35)\quad \Sigma N'(j) \geq \Sigma N(j)$$

But the fact that economical decompositions keep the number of intersections to a minimun combined with the periodicity of the columns implies even more than (VII.35):

Claim 2: For an economical decomposition, $N'(j) \geq N(j)$ for every $j \in [0, L_\nu]$.

Indeed, suppose there exists a j_o such that

(VII.36) $$N'(j_o) < N(j_o)$$

One such example is given in figure 9 b) for which every non zero section (rows $\alpha(1)$ and $\alpha(3)$) has $N(j)=3$ anihilated positions, while only $N'(j)=1$ position survives to form the α of the left. The idea is that if such thing happens, then as in figure 9 one can find a nontrivial decomposition of zero such that when added to the original decomposition yields another decomposition with columns with less support (figure 9 a). This violates the supposed economicity of the original decomposition. The way the new decomposition is defined in general is as follows: In the new decomposition the position j_o of each column is defined to be equal to the value of the function α at this position. In particular, the $N'(j_o)$ positions formerly anihilated are eliminated from the support of the columns, while the number of positions added to the support of the columns can not exceed $N(j_o)$. Hence,

the support of the columns decreases at least by an amount $N'(j_o)-N(j_o)>0$, and the new decomposition is more economical than the original one. The formalization is as follows. Consider the following decomposition of $O\epsilon\Pi(V)$:

$$O = \sum_j \beta(j) + \sum_{\bar{a}} \beta^{(\nu)}_{(\bar{a})} \tag{VII.37}$$

with

$$[\beta(j)]_{(\bar{a},j)} = \begin{cases} [-^{(p)}(\alpha(j_o))]_{(\bar{a},j_o)} & \text{if } j=j_o+mq \text{ for } m\epsilon Z \\ 0 & \text{otherwise} \end{cases}$$

(VII.38)

$$[\beta^{(\nu)}_{(\bar{a})}]_{(\bar{a},j)} = \begin{cases} [\alpha(j_o)]_{(\bar{a},j_o)} & \text{if } j=j_o+mq \text{ for } m\epsilon Z \\ 0 & \text{otherwise} \end{cases}$$

I.e. the sections $\beta(j)$ for $j=j_o$(module the period q), are all copies of the Z_p-inverse of the section $\alpha(j_o)$. The columns make up for the term needed for the whole sum to be zero. Now, consider the decomposition

$$\alpha'(j) = \alpha(j) + \beta(j)$$

$$\alpha'^{(\nu)}_{(\bar{a})} = \alpha^{(\nu)}_{(\bar{a})} + \beta^{(\nu)}_{(\bar{a})} \tag{VII.39}$$

By (VII.37) the original and primed decomposition are decompositions of the same $\alpha\in\Pi(V)$. From (VII.38) the support of the columns $\alpha^{(\nu)}_{(\bar{a})}$ and $\alpha'^{(\nu)}_{(\bar{a})}$ differs only at the intersection of each column with an hyperplane $M(j_o+mq,\nu)$, $m\in Z$. Hence, if one denotes

$$\delta_j(\alpha^{(\nu)}_{(\bar{a})})=\begin{cases}1 & \text{if } [\alpha^{(\nu)}_{(\bar{a})}]_{(\bar{a},j)}\neq 0\\ 0 & \text{otherwise}\end{cases}$$

one has

$$\sum_{\bar{a}}[|\underset{\sim}{\alpha}^{(\nu)}_{(\bar{a})}| - |\underset{\sim}{\alpha}'^{(\nu)}_{(\bar{a})}|]=\sum_{j=j_o+mq}\sum_{\bar{a}}[\delta_j(\alpha^{(\nu)}_{(\bar{a})}) - \delta_j(\alpha'^{(\nu)}_{(\bar{a})})]$$

or, by periodicity of the columns in the ν-th direction

$$\text{(VII.40)}\qquad \Sigma[|\underset{\sim}{\alpha}^{(\nu)}_{(\bar{a})}| - |\underset{\sim}{\alpha}'^{(\nu)}_{(\bar{a})}|]=|C|\sum_{\bar{a}}[\delta_{j_o}(\alpha^{(\nu)}_{(\bar{a})}) - \delta_{j_o}(\alpha'^{(\nu)}_{(\bar{a})})]$$

where $C=\{m\in Z:\ j_o+mq\in[0,L_\nu]\}$. But from (VII.38) and (VII.39)

$$\text{(VII.41)}\qquad [\alpha'^{(\nu)}_{(\bar{a})}]_{(\bar{a},j_o)} = [\alpha^{(\nu)}_{(\bar{a})} +^{(p)} \alpha(j_o)]_{(\bar{a},j_o)}$$

We can classify the $\bar{a}\in M(j_o,\nu)$ in three categories:

a) $C_1=\{\bar{a}:\ \delta_{j_o}(\alpha^{(\nu)}_{(\bar{a})}) = \delta_{j_o}(\alpha'^{(\nu)}_{(\bar{a})})\}$. They drop from the sum of the RHS of (VII.40).

b) $C_2=\{\bar{a}:\ [\alpha^{(\nu)}_{(\bar{a})}]_{(\bar{a},j_o)}\neq 0$ but $[\alpha'^{(\nu)}_{(\bar{a})}]_{(\bar{a},j_o)}=0\}$. By (VII.41) each of the elements of C_2 correspond to an anihilation in the original decomposition. Hence there are exactly $N(j_o)$ of them, for each of which the square bracket on the RHS of (VII.40) is equal to +1.

c) $C_3=\{\bar{a}:\ [\alpha^{(\nu)}_{(\bar{a})}]_{(\bar{a},j_o)}=0$ and $[\alpha'^{(\nu)}_{(\bar{a})}]_{(\bar{a},j_o)}\neq 0\}$. They are some of the $N'(j_o)$ surviving positions of the original decomposition. Each of them contributes with -1 to the sum of the RHS of (VII.40).

In conclusion in (VII.40)

$$\Sigma[|\underline{\alpha}^{(\nu)}_{(\bar{a})}| - |\underline{\alpha}'^{(\nu)}_{(\bar{a})}|]=|C|[|C_2|-|C_3|]\geq|C|[N(j_o)-N'(J_o)]>0$$

This contradicts the fact that the original decomposition is economical. This contradiction proves claim 2 and hence claim 1 and the theorem.∎

Corollary VII.11: Consider a one-bond system whose basic bond is regular and of prime order and whose interaction J is real. Then if $\kappa=\kappa(N)$ is a polynomial in N with positive coefficients, there exist positive C, γ (depending on κ, τ, N, ν, r_1, r_2, s_1, s_2) such that

$$\sum_{\alpha\in\Pi(V),\,\alpha\neq 0} \kappa^{g(\alpha)}\,\tau^{|\underline{\alpha}|} \leq C\, e^{-\gamma N}$$

for every $V \in D$.

Proof: By induction in ν. For $\nu=1$, $V=[0,L] \in D$ there are at most p^{m-1} multiplicity functions $\alpha \in \Pi(V)$ (proposition VII.4), where p is the order and m the diameter of the basic bond. One of these functions is the zero function (for which $g(\alpha)=0$) and the other are periodic with period q, hence they at least have L/q points in its support. For the latter $g(\alpha)=1$. Then

$$\sum_{\alpha \in \Pi(V), \alpha \neq 0} \kappa^{g(\alpha)} \tau^{|\underline{\alpha}|} \leq p^{m-1} \kappa \tau^{L/q}$$

$$\leq p^{m-1} \kappa \tau^{(r_1 N+s_1)/q} \leq C_1(\tau) \exp(-\gamma_1(\tau)N)$$

Suppose that the exponential decay is true for $\nu-1$. From theorem VII.10:

$$\sum_{\alpha \in \Pi(V), \alpha \neq 0} \kappa^{g(\alpha)} \tau^{|\underline{\alpha}|} \leq [1+C_1(\tau/3) \exp(-\gamma_1(\tau/3)N)]^{c(N)}$$

$$\times \ [1+C_{\nu-1}(\tau/3) \exp(-\gamma_{\nu-1}(\tau/3)N)]^{c(N)^{\nu-1}} - 1 \leq C \exp(-\gamma N)$$

with $c(N)=r_2 N+s_2$ and for suitable $C(\tau)$, $\gamma(\tau)$. ∎

Corollary VII.12: Consider a one-bond system whose basic bond is regular and of prime order and whose interaction J is real. Then for any positive integer k there exists and open interval in $\mathcal{B}_k$ around J where there is a unique Gibbs state.

Proof: By proposition VI.13 and (VII.22), it is enough to prove that

$$(VII.42) \qquad S_1 = \sum_{\alpha\in\Pi(U),\alpha\neq 0} \tau^{|\underline{\alpha}|} \leq C\, e^{-\gamma N}$$

and

$$(VII.43) \qquad S_2 = \sum_{\alpha\in K^+_{W\setminus\widetilde{U}},\alpha\neq 0} \tau^{|\underline{\alpha}|} \leq C\, e^{-\gamma N}$$

The bound (VII.42) is a consequence of the fact that by part ii) of the definition of E and (VII.21), S_1 factorizes in a finite product of sums over parallelepipeds in D. This product tends to zero exponentially fast with N because of the previous corollary (for $\kappa=1$). The argument to prove the bound (VII.43) can be understood from the simple case in wich $W\setminus\widetilde{U}=W_1\cup W_2$ with W_1, W_2 ν-1-dimensional parallelepipeds sharing a common edge. For the sake of the argument assume

$$W_1 = \{a\in Z^\nu:\ a_\nu=0,\ 0\leq a_i\leq L_i,\ i=1,\ldots,\nu-1\}$$
$$W_2 = \{a\in Z^\nu:\ a_1=0,\ 0\leq a_i\leq L_i,\ i=2,\ldots,\nu\}$$

The common edge is

$$E_{1,2} = \{a\in Z^\nu:\ a_1=0=a_\nu,\ 0\leq a_i\leq L_i,\ i=2,\ldots,\nu-1\}$$

With the identification (VII.1) the cycles in W_1 are defined by the Z^p-sum of one-dimensional cycles $\alpha^{(i)}_{(\underline{a})}$ with $1\leq i\leq\nu-1$ and some a_ν in the range $0\leq a_\nu\leq k_\nu$; where k_ν is the length of the basic bond in the ν-th direction. If W_2 were absent,

then one may fix $a_\nu=0$ because, by the regularity of the basic bond (definition VII.8), all the a_ν such that $\xi \restriction X_{M(a_\nu,\nu)}$ is not trivial produce the same cycles in W_1. However, the presence of W_2 produces a different pattern of bonds for each a_ν. Indeed, the value of a_ν determines which bond "links" W_1 with W_2 across the edge. The same argument holds for the cycles in W_2. More precisely, if

$$\bar{W}_1 = \{a\epsilon W_1 : a_1 \geq k_1\}$$
$$\bar{W}_2 = \{a\epsilon W_2 : a_\nu \geq k_\nu\}$$

each $\alpha\epsilon K^+_{W_1 \cup W_2}$ can be decomposed as the sum of three functions of disjoint support:

(VII.44) $$\alpha = \beta + \gamma + \iota$$

where $\beta\epsilon\Pi(\bar{W}_1)$, $\gamma\epsilon\Pi(\bar{W}_2)$ and ι can be written (with the identification (VII.1)) as:

$$\iota = \Big[\sum_{a_1=-k_1+1}^{0}(p) \sum_{a_\nu=-k_\nu+1}^{0}(p) \sum_{a\in Z^{\nu-3}}(p) \sum_{i=2}^{\nu-1}(p) \iota^{(i)}_{(a_1,a,a_\nu)} \Big]_{+(p)}$$

$$\Big[\sum_{a_\nu=-k_\nu+1}^{0}(p) \sum_{a\in Z^{\nu-3}}(p) \iota^{(1)}_{(a,a_\nu)} \Big]_{+(p)}$$

$$\Big[\sum_{a_1=-k_1+1}^{0}(p) \sum_{a\in Z^{\nu-3}}(p) \iota^{(\nu)}_{(a_1,a)} \Big]$$

or

$$\iota = \Big[\sum_{a_1=-k_1+1}^{0}(p) \sum_{a_\nu=-k_\nu+1}^{0}(p) \hat{\iota}_{(a_1,a_\nu)} \Big]_{+(p)}$$

$$\Big[\sum_{a_\nu=-k_\nu+1}^{0}(p) \hat{\iota}^{(1)}_{(a_\nu)} \Big]_{+(p)}$$

$$\Big[\sum_{a_1=-k_1+1}^{0}(p) \hat{\iota}^{(\nu)}_{(a_1)} \Big]$$

where $\hat{\iota}_{(a_1,a_\nu)}$ is a cycle "parallel" to E_{12}. In fact if $Q(a_1,a_\nu)$ is the parallelepiped

$$Q(a_1,a_\nu) = \{b\in Z^\nu : b_1=a_1,\ b_\nu=a_\nu,\ 0\le b_i\le L_i \quad i=2,\dots,\nu-1\}$$

then

$$\hat{\iota}_{(a_1,a_\nu)} \in Q(a_1,a_\nu) \tag{VII.45}$$

The cycles $\hat{\imath}^{(1)}_{(a_\nu)}$ and $\hat{\imath}^{(\nu)}_{(a_1)}$ are "perpendicular" to E_{12}. One has:

$$(VII.46) \qquad |\underline{\imath}| \geq \Sigma\,\Sigma\, |\underline{\hat{\imath}}_{(a_1,a_\nu)}| - \Sigma\, |\underline{\hat{\imath}}^{(1)}_{(a_\nu)}| - \Sigma\, |\underline{\hat{\imath}}^{(\nu)}_{(a_1)}]|$$

where the sums range over $-k_i+1 \leq a_i \leq 0$; $i=1,\nu$. But each non zero $\imath^{(1)}_{(a,a_\nu)}$ is the continuation of a non zero one dimensional cycle in the decomposition of β. If this decomposition is taken to be the one with the minimum number of non zero one dimensional cycles, one has that the number of such $\imath^{(1)}_{(a,a_\nu)}$ is at most

$$(VII.47) \qquad p^{k_1} g(\beta)$$

Therefore:

$$(VII.48) \qquad |\hat{\imath}^{(1)}_{(a_\nu)}| \leq k_1\, g(\beta)$$

Analogous inequalities hold in relation with the functions $\hat{\imath}^{(\nu)}_{(a_1)}$. From (VII.44)-(VII.48) and using that k_1, $k_\nu \leq k$:

$$(VII.49) \qquad \sum_{\alpha \in K^+_{W_1 \cup W_2}} \tau^{|\underline{\alpha}|} \leq [\sum_{\beta \in \Pi(\bar{W}_1)} \kappa^{g(\beta)}\, \tau^{|\underline{\beta}|}]$$

$$[\sum_{\gamma \in \Pi(\bar{W}_2)} \kappa^{g(\gamma)}\, \tau^{|\underline{\gamma}|}][\sum_{\imath \in \Pi(E_{12})} \tau^{|\underline{\imath}|}]^{k^2}$$

with

$$\kappa = p^{k_1 - k^2}.$$

The three factors tend to one exponentially fast with N by the previous corollary. The estimation also holds if there are more than two sets W_i sharing the same edge, except that one obtains more factors corresponding to sums over additional $\Pi(\bar{W}_i)$.

Another situation that may happen is that two edges intercept. For instance, consider the previous example but adding a parallelepiped

$$W_3 = \{a\epsilon Z^\nu:\ a_2=0;\ 0\le a_i\le L_i,\ i\neq 2\}$$

This incorporates two more edges:

$$E_{13} = \{a\epsilon Z^\nu:\ a_1=0=a_2;\ 0\le a_i\le L_i,\ i\neq 1,2\}$$
$$E_{23} = \{a\epsilon Z^\nu:\ a_2=0=a_\nu;\ 0\le a_i\le L_i,\ i\neq 2,\nu\}$$

Working as above one obtains a bound like (VII.49) with the following characteristics: There are three factors corresponding to sums over $\Pi(\bar{W}_i)$, i=1,2,3; with

$$\bar{W}_1 = \{a\epsilon W_1:\ a_1\ge k_1,\ a_2\ge k_2\}$$
$$\bar{W}_2 = \{a\epsilon W_2:\ a_\nu\ge k_\nu,\ a_2\ge k_2\}$$
$$\bar{W}_3 = \{a\epsilon W_3:\ a_1\ge k_1,\ a_\nu\ge k_\nu\}$$

The cycles $\hat{\imath}_{(a_1,a_\nu)}$ parallel to E_{12} intercept the cycles $\hat{\imath}_{(a_1,a_2)}$ parallel to E_{23} and both types of cycles intercept the cycles $\hat{\imath}_{(a_2,a_\nu)}$ parallel to E_{13}. However, the number of intersections is less than

$$k(g(\hat{\imath}_{(a_1,a_\nu)}) + g(\hat{\imath}_{(a_1,a_2)}) + g(\hat{\imath}_{(a_2,a_\nu)}))$$

Hence one obtains three factors

$$[\sum_{\imath\in\Pi(E_{ij})} (\tau^{-k})^{g(\imath)}\ \tau^{|\underline{\imath}|}]k^2$$

In total, the analogous to (VII.49) for this stuation has therefore six factors, all of them converging to one exponentially fast with N.

The general geometry is just a combination of the situations above discussed. If $W\setminus\tilde{U}$ is formed by the union of n ν-1-dimensional parallelepipeds $W_1,\ldots,W_n$; and it presents m (ν-2-dimensional) edges $E_1,\ldots,E_m$ which intercept at j ν-3-dimesional parallelepipeds; then working as in (VII.49):

$$\sum_{\alpha\in K^+_{W\setminus\tilde{U}}} \tau^{|\underline{\alpha}|} \leq \prod_{i=1}^{n} [\sum_{\beta\in\Pi(\bar{W}_i)} \kappa^{g(\beta)}\ \tau^{|\underline{\beta}|}]$$

$$\prod_{i=1}^{m} [\sum_{\imath\in\Pi(E_i)} (\tau^{-mk})^{g(\imath)}\ \tau^{|\underline{\imath}|}]k^2$$

The RHS tends to one exponentially fast with N by the previous corollary.∎

Chapter VIII

CONCLUSIONS

Corollary VII.12 in conjunction with the construction of theorem V.7 yield a large family of interactions with at least n phase transitions, for any fiven n. For instance the system of chapter IV, which exhibits two phase transitions, can be extended to a system exhibiting at least n phase transitions considering

$$B_n = \{B,C\}$$

$$B_i = D^{i-1} \bullet A \qquad 1 \leq i \leq n-1$$

A construction like this, with $D_k = \xi^k$ for some regular ξ can be applied only if $ord(\xi)=2$, because otherwise it is simple to check that the bond ξ^k is not regular. To be sure, ξ^k is factorizable if ξ is, but if $ord(\xi)>2$ the one dimensional characters in which ξ^k decomposes have more than one non trivial $s^{l(i)}$ against definition V.8. On the other hand , the greater common divisor of a set of regular bonds is itself regular. Hence, to construct an interaction in the conditions of theorem V.7 one can for example start with the sets B_i formed by regular bonds.

Independently of its application in Slawny's construction, corollary V.13 provides an example of a system

that is not unidimensional and for which perturbative results are obtained for all temperatures, not only for low or high temperatures.

The technique used in the proof of corollary VII.13 can perhaps be extended to arbitrary trivial systems, i.e. systems with $K=\phi$. However, it certainly can not be used for systems that have non empty finite cycles. Indeed, for such systems for a given parallelepiped V there are plenty of cycles of length not related with the size of V -in fact this number is at least of the order of $|V|$. Therefore, the sum (VI.43) diverges when $n \to \infty$.

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Appendix A

LISTING OF THE CODE SPIN

```
      DOUBLE PRECISION DS1,DS2
      LOGICAL SPIN(102,104)
      INTEGER SITE,CHK1,CHK10,ISPN(104),ITIT(20),ADIC,LAST(11)
C11   INTEGER ISPL(102,104)
      DIMENSION BETV(42),FENY(42),FEND(42),FOP1(42),FP1D(42)
      DIMENSION FOP2(42),FP2D(42),LCTN(10000)
      DIMENSION ENERGY(10000),ORD1(10000),ORD2(10000),XAXIS(10000)
      DATA IPLS,MINS,IDECO,LAST / '+   ','-   ','0   ',11*'****' /
C     DATA DS1,DS2,DS3/ 12345.67D0 ,47892.5D0,53894.3D0/
      DATA DS1,DS2,DS3/ 12.03D0 ,4.5D0 ,53.4D0/
      COMMON /AREA1/ AA,BB,CC,LATV,LATV1
      COMMON /AREA2/ PROB(23),DELE(22),BETA,IPY,ILIM
      COMMON /AREA3/ SPIN
      COMMON /AREA4/ ISPN
C
      PROB(23) = 1.00000
C
      READ ( 5 , 16) (ITIT(I0),I0=1,20)
      READ ( 5 , 17) ST,SP,CT,FRAC
      READ ( 5 , 18) MCSS,LATS,ICONFC,ADIC,CHK1,IFRQ
      READ ( 5 , 19) AX , BX , CX
   16 FORMAT (20A4)
   17 FORMAT (4F5.2)
   18 FORMAT (6I5)
   19 FORMAT (3E15.4)
C------------AX++++++++++VERTICAL RECTANGLE
C------------BX++++++++++LONG RECTANGLE
C------------CX++++++++++++++BOX
      NTEMP = ABS((SP-ST)/CT)+ 1.1
      BASE  = ST
      IF (CT .LE. 0.) BASE = SP
      WRITE(6,1000) (ITIT(I0),I0=1,20)
C
C     PROTECTIONS
C
      IF(MOD(LATS,2) .EQ. 0) GOTO 30
      LATS = LATS + 1
      WRITE(6,1050) LATS
   30 MCST = MCSS + CHK1
      IF (MCST .LE. 9998) GOTO 35
      MCSS = 9998 - CHK1
      WRITE(6,1060) MCSS
      MCST  = 9998
   35 MCST1 = MCST + ADIC
      IF (ADIC .LE. 9998) GOTO 36
```

```
      ADIC  = 9998 - MCST
      MCST1 = 9998
      WRITE(6,1070) ADIC
   36 NBLK  = MCSS/IFRQ
      IREL1 = CHK1 + ADIC
      LAT2  = LATS * LATS
      LATV1 = LATS + 1
      LATV  = LATS + 2
      LATH  = LATS + 4
      LAGC  = CHK1 - IFRQ + 1
      LATR1 = IFRQ * LAT2
      LAGC1 = LAGC + ADIC
      LGATR1= LATR1 * NBLK
      BETC  = 0.5 * ALOG(1.00 + SQRT(2.0))
C
C                    DEFINE INITIAL SPIN CONFIGURATION
C
      AA    = 2 * AX
      BB    = 2 * BX
      CC    = 2 * CX
      MSTOTO = MCST1 + 1
      DO 44 IN=1,MSTOTO
   44 XAXIS(IN) = IN - 1
      DELE(1) = 4*(AX+BX+2*CX)
      DO 40 IR = 1,102
      DO 40 IC = 1,104
C11   ISPL(IR,IC) = 0
   40 SPIN(IR,IC) = .FALSE.
      IFRAC = FRAC * LATS + 1
      IF (IFRAC .LT. 2) GOTO 52
      DO 50 IR =2,IFRAC
      DO 50 IC = 3,LATV
      IF (RAN3(DS3) .GT. 0.5) GOTO 50
      SPIN(IR,IC) = .TRUE.
   50 CONTINUE
   52 IF (FRAC .EQ. 0.) GOTO 55
      CALL EVAL (ENER, OP1, OP2)
      GOTO 57
   55 ENER  = LAT2*(-AX-BX-CX)
      OP1   = LAT2
      OP2   = LAT2
   57 CALL PLOTS(0,0,50)
      CALL PLOT(0.5,0.,-3)
C
C          TEMPERATURE CHANGE (BETA) VALUES
C
      DO 900 IB = 1,NTEMP
      GAVE = 0.0
      GAVD = 0.0
      GOP1 = 0.0
      GP1D = 0.0
      GOP2 = 0.0
      GP2D = 0.0
      BETA = BETC * (BASE+(IB-1)*CT)
      ILIM = 1
```

```
      PROB(1) = EXP(-DELE(1) * BETA)
C
C
C                    DEFINE INITIAL SPIN VALUES
      CALL TIMEON
      BBTC = BETA/BETC
      ENERGY(1) = ENER/LAT2
      ORD1(1) = ABS(OP1/LAT2)
      ORD2(1) = ABS(OP2/LAT2)
      WRITE(6,1200) LATS,LATS,AX,BX,CX,BBTC,MCSS
      WRITE(6,1250)
      CALL CONFOU(0,LATV,LATH,IPLS,MINS,IDECO)
      IF (IB .GT. 2) GOTO 60
      MCST0 = MCST
      CHK10 = CHK1
      LAGC0 = LAGC
      IF (IB .EQ. 2) GOTO 60
      MCST0 = MCST1
      CHK10 = IREL1
      LAGC0 = LAGC1
C
C-----------------------------------------------------------------
   60 DO 800 MCS = 1,MCST0
      MCS1 = MCS + 1
C00   CALL RANPER(LAT2,LCTN)
C01   DO 500 LSI = 1,LAT2
C00   SITE = LCTN(LSI) - 1
C01   SITE = LAT2*RAN2(DS2)
C01   IV   = SITE/LATS
C01   IR   = 2 + IV
C01   IC   = 3 + SITE - IV*LATS
      DO 500 IC=3,LATV
      DO 500 IR=2,LATV1
C11   ISPL(IR,IC) = ISPL(IR,IC) + 1
      SPIN(IR,IC) = .NOT. SPIN(IR,IC)
C
C         VARIANCE REDUCTION CALCULATION
C
      CALL HAM ( IR , IC , ECGE , PAR2 )
      CALL W12 ( ECGE )
      IF ( PROB(IPY) .LT. RAN1(DS1) ) GO TO 100
      ENER = ENER + ECGE
      OP1  = OP1 + 2*ISN1(IR,IC)
      OP2  = OP2 + PAR2
      GO TO 500
  100 SPIN(IR,IC) = .NOT. SPIN(IR,IC)
  500 CONTINUE
C
C                  STATISTICS ACCUMULATION
C
      IF(MOD(MCS,ICONFC).EQ.0) CALL CONFOU(MCS,LATV,LATH,IPLS,MINS,IDECO
     Z )
      ENERGY(MCS1) = ENER/LAT2
      ORD1(MCS1) = ABS(OP1/LAT2)
      ORD2(MCS1) = ABS(OP2/LAT2)
```

```
      IF ( MCS .LE. CHK10 ) GO TO 800
      LMCS = MCS - LAGCO
      IF ( MOD(LMCS,IFRQ) .NE. 0 ) GO TO 640
      NATR = 0
      AAE  = 0.0
      AVE  = 0.0
      OD1  = 0.0
      OD2  = 0.0
      OS1  = 0.0
      OS2  = 0.0
  640 NATR = NATR + 1
      AAE  = AAE + ENER
      AVE  = AVE + ENER*ENER
      OD1  = OD1 + ABS(OP1)
      OD2  = OD2 + ABS(OP2)
      OS1  = OS1 + OP1*OP1
      OS2  = OS2 + OP2*OP2
      IF ( NATR .NE. IFRQ ) GO TO 800
C
C      BLOCK STATISTICS OUTPUT
C
      AVEN = AAE/LATR1
      OA1  = OD1/LATR1
      OA2  = OD2/LATR1
      OSD1 = SQRT(ABS(IFRQ*OS1-OD1*OD1))/LATR1
      OSD2 = SQRT(ABS(IFRQ*OS2-OD2*OD2))/LATR1
      SDVE = SQRT(ABS(IFRQ*AVE-AAE*AAE))/LATR1
      WRITE(6,1300) MCS,MCSTO,CHK10,AVEN,SDVE,OA1,OSD1,OA2,OSD2
      GAVE = GAVE + AAE
      GAVD = GAVD + AAE*AAE
      GOP1 = GOP1 + OD1
      GP1D = GP1D + OD1 * OD1
      GOP2 = GOP2 + OD2
      GP2D = GP2D + OD2 * OD2
      IF ( MCS .NE. MCSTO ) GO TO 800
C
C                        FINAL STATISTICS OUTPUT
      GAVD = SQRT(ABS(NBLK*GAVD-GAVE*GAVE))/LGATR1
      GP1D = SQRT(ABS(NBLK*GP1D-GOP1*GOP1))/LGATR1
      GP2D = SQRT(ABS(NBLK*GP2D-GOP2*GOP2))/LGATR1
      BETV(IB) = BBTC
      FENY(IB) = GAVE/LGATR1
      FEND(IB) = GAVD
      FOP1(IB) = GOP1/LGATR1
      FP1D(IB) = GP1D
      FOP2(IB) = GOP2/LGATR1
      FP2D(IB) = GP2D
  800 CONTINUE
C
      IF (MOD(MCSTO,ICONFC) .NE. 0)
     ZCALL CONFOU(MCSTO,LATV,LATH,IPLS,MINS,IDECO)
C
C11   WRITE(6,1549)
C11   DO 830 II = 1,LATV
C11   WRITE(6,1550) (ISPL(II,JJ),JJ=1,LATH)
```

```
C1830 CONTINUE
C11    DO 850 II=1,LATV
C11    DO 850 JJ=1,LATH
C1850 ISPL(II,JJ) = 0
C-----PLOTTING--------------------------------------------------------
      MTOT = MCSTO + 1
      N1   = MTOT + 1
      N2   = MTOT + 2
C  INCLUSION OF (0,0) IN THE AXIS
      ENERGY(N1) = 0.
      ORD1(N1) = 0.
      ORD2(N1) =0.
      IF (IB .GT. 2) GOTO 930
      XAXIS(N1) = 0.
      CALL SCALE (XAXIS,10.,N1,1)
      XAXIS(N1) = XAXIS(N1+1)
      XAXIS(N1+1) = XAXIS(N1+2)
  930 CALL SCALE (ENERGY,10.,N1,1)
      CALL SCALE (ORD1,10.,N1,1)
      CALL SCALE (ORD2,10.,N1,1)
C  ELIMINATION OF (0,0) IN THE PLOT
      DO 940 I7 =1,2
      ENERGY(MTOT+I7) = ENERGY(MTOT+I7+1)
      ORD1(MTOT+I7) = ORD1(MTOT+I7+1)
  940 ORD2(MTOT+I7) = ORD2(MTOT+I7+1)
C HEADING
      CALL SYMBOL(0.,9.,0.5,5HB/BC=,0.,5)
      CALL NUMBER(2.5,9.,0.5,BBTC,0.,3)
C ENERGY PLOT
      CALL PLOT(7.,0.125,-3)
      CALL AXIS (0.,ORX(ENERGY(N1),ENERGY(N2)),26HMONTE CARLO STEPS PER
     ZSPIN,26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0.,15HENERGY PER SPIN,15,10.,90.,ENERGY(N1)
     Z           ,ENERGY(N2))
      CALL LINE (XAXIS,ENERGY,MTOT,1,0,0)
C MAGNETIZATION PLOT
      CALL PLOT (12.,0.125,-3)
      CALL AXIS (0.,ORX(ORD1(N1),ORD1(N2)),26HMONTE CARLO STEPS PER SPIN
     Z           ,-26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0.,22HMAGNETIZATION PER SPIN,22,10.,90.,ORD1(N1)
     Z           ,ORD1(N2))
      CALL LINE (XAXIS,ORD1,MTOT,1,0,0)
C PSEUDOMAGNETIZATION PLOT
      CALL PLOT (12.,0.,-3)
      CALL AXIS (0.,ORX(ORD2(N1),ORD2(N2)),26HMONTE CARLO STEPS PER SPIN
     Z           ,-26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0. 28HPSEUDOMAGNETIZATION PER SPIN,28,10.,90.,
     Z          ORD2(N1),ORD2(N2))
      CALL LINE (XAXIS,ORD2,MTOT,1,0,0)
      IF (IB .EQ. NTEMP) GOTO 941
      CALL PLOT(16.,-0.25,-3)
C-----------------------------------------------------------------------
C
C                  TIME COMPUTATION
C
```

```
C1830 CONTINUE
C11    DO 850 II=1,LATV
C11    DO 850 JJ=1,LATH
C1850 ISPL(II,JJ) = 0
C-----PLOTTING-------------------------------------------------------
      MTOT = MCSTO + 1
      N1   = MTOT + 1
      N2   = MTOT + 2
C  INCLUSION OF (0,0) IN THE AXIS
      ENERGY(N1) = 0.
      ORD1(N1) = 0.
      ORD2(N1) =0.
      IF (IB .GT. 2) GOTO 930
      XAXIS(N1) = 0.
      CALL SCALE (XAXIS,10.,N1,1)
      XAXIS(N1) = XAXIS(N1+1)
      XAXIS(N1+1) = XAXIS(N1+2)
  930 CALL SCALE (ENERGY,10.,N1,1)
      CALL SCALE (ORD1,10.,N1,1)
      CALL SCALE (ORD2,10.,N1,1)
C  ELIMINATION OF (0,0) IN THE PLOT
      DO 940 I7 =1,2
      ENERGY(MTOT+I7) = ENERGY(MTOT+I7+1)
      ORD1(MTOT+I7) = ORD1(MTOT+I7+1)
  940 ORD2(MTOT+I7) = ORD2(MTOT+I7+1)
C HEADING
      CALL SYMBOL(0.,9.,0.5,5HB/BC=,0.,5)
      CALL NUMBER(2.5,9.,0.5,BBTC,0.,3)
C ENERGY PLOT
      CALL PLOT(7.,0.125,-3)
      CALL AXIS (0.,ORX(ENERGY(N1),ENERGY(N2)),26HMONTE CARLO STEPS PER
     ZSPIN,26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0.,15HENERGY PER SPIN,15,10.,90.,ENERGY(N1)
     Z          ,ENERGY(N2))
      CALL LINE (XAXIS,ENERGY,MTOT,1,0,0)
C MAGNETIZATION PLOT
      CALL PLOT (12.,0.125,-3)
      CALL AXIS (0.,ORX(ORD1(N1),ORD1(N2)),26HMONTE CARLO STEPS PER SPIN
     Z          ,-26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0.,22HMAGNETIZATION PER SPIN,22,10.,90.,ORD1(N1)
     Z          ,ORD1(N2))
      CALL LINE (XAXIS,ORD1,MTOT,1,0,0)
C PSEUDOMAGNETIZATION PLOT
      CALL PLOT (12.,0.,-3)
      CALL AXIS (0.,ORX(ORD2(N1),ORD2(N2)),26HMONTE CARLO STEPS PER SPIN
     Z          ,-26,10.,0.,XAXIS(N1),XAXIS(N2))
      CALL AXIS (0.,0. 28HPSEUDOMAGNETIZATION PER SPIN,28,10.,90.,
     Z          ORD2(N1),ORD2(N2))
      CALL LINE (XAXIS,ORD2,MTOT,1,0,0)
      IF (IB .EQ. NTEMP) GOTO 941
      CALL PLOT(16.,-0.25,-3)
C--------------------------------------------------------------------
C
C                  TIME COMPUTATION
C
```

```
  941 CALL TIMECK(ITIM2)
      TIME = ITIM2/100.0
      TMCS = TIME/(MCSTO*LAT2)
      WRITE(6,1400) TIME,      TMCS
      WRITE(6,1450) (LAST(IO),IO=1,11)
  900 CONTINUE
C.....................................................................
      CALL PLOT(100.,-2.,999)
      WRITE(6,1604)
      DO 950 IT = 1,NTEMP
      CALL ONS(BETV(IT),EONS,OMAG,FEND(IT))
      WRITE(6,1605) BETV(IT),FENY(IT),FEND(IT),FOP1(IT),FP1D(IT)
     Z             ,FOP2(IT),FP2D(IT),EONS,OMAG
  950 CONTINUE
      STOP
 1000 FORMAT (////17X,20A4)
 1050 FORMAT (//' NOTE: LATTICE SIZE CHANGED TO',I3,' , ONLY EVEN SIZES
     ZARE ALLOWED')
 1060 FORMAT(' NOTE: MCSS REDUCED TO',I5,'  (PROGRAM ADMITS A MAXIMUN OF
     Z9998 TOTAL TRIALS)')
 1070 FORMAT(' NOTE: INITIAL RELAXATION (ADIC) REDUCED TO',I5,'  (PROGRA
     ZM ADMITS A MAXIMUN OF 9998 TOTAL TRIALS)')
 1100 FORMAT(2I5,1P(E12.5))
 1200 FORMAT(/////30X,'MONTE CARLO SIMULATION: LATTICE SPIN SYSTEMS'//
     Z41X,'LATTICE SIZE=',I2,' X',I3,/21X,'      HAMILTONIAN      CONSTA
     ZNTS=',3F7.4,/24X,'INVERSE TEMPERATURE/BCRITICAL=',F8.5/32X,
     Z'        MCS/SPIN SITE=',I6///)
 1250 FORMAT(////4X,'MCS/SPIN',5X,'CONFIGURATION')
 1300 FORMAT(///45X,'BLOCK#',I4,' STATISTICS'/34X,'TOTAL NUMBER OF TRIAL
     ZS=',I7/36X,'STATISTICS STARTED AFTER ',I4,' MCS/SPIN'/
     Z 40X,'ENERGY: MEAN-STD=',2(1PE13.5)/
     Z 40X,'ORDER1: MEAN-STD=',2(1PE13.5)/
     Z 40X,'ORDER2: MEAN-STD=',2(1PE13.5)///)
 1400 FORMAT(///30X,'PROCESSOR TIME ELAPSED=',F7.1,'SEC',5X,
     Z                    //,40X,'TIME PER MCS = ',1PE9.3)
 1450 FORMAT(/////30X,11A4)
 1500 FORMAT(15X,102A1)
 1549 FORMAT (///' ATTEMPTED FLIPPINGS PER SITE'/)
 1550 FORMAT(1X,3014/1X,3014)
 1604 FORMAT(///30X,'SUMMARY TABLE'//2X,'BETA/BCRIT',10X,'ENERGY',10X,'S
     ZD',10X,'OP1',10X,'SD',10X,'OP2',10X,'SD',15X,'T/TC',5X,'SP HT')
C    Z,5X,
 1605 FORMAT(4X,F5.3,12X,3(0PF8.5,5X,1PE10.3,2X),5X,3(1PE10.3))
      END
      FUNCTION ORX(FVAL,DV)
      IF (FVAL .LT. 0.) ORX=-FVAL/DV
      IF (FVAL .GE. 0.) ORX = 0.
      RETURN
      END
      SUBROUTINE ONS(BR,EGY,OMAG,DEN)
      DATA A0,A1,A2,B0,B1,B2/1.3862944,0.1119723,0.0725296
     Z                      ,0.50,0.1213478,0.0288729 /
      EGY  = 1/BR
      OMAG = BR*BR*DEN*DEN
      RETURN
```

```
   10 BT    =0.4406868*BR
      TBT=2*BT
      SH    = SINH(TBT)
      C1    =2*SH/COSH(TBT)**2
      C2    = SQRT(ABS(1.0 - C1*C1))
      IF ( BR .LT. 1.000 ) C2 = -C2
      ETA= ABS(1.0 - C1*C1)
      AETA = -ALOG(ETA)
      ETA2=ETA*ETA
      EITG = A0+A1*ETA+A2*ETA2+(B0+B1*ETA+B2*ETA2)*AETA
      F1    = 1.0 + 2 * C2 * EITG/3.1415926
      EGY   = -COSH(TBT)/SH * F1
      OMAG = 0.0
      IF (BR.GT.1.0) OMAG  = (1-1/SH**2)**0.125
      RETURN
      END
      SUBROUTINE W12 ( DH )
      COMMON/AREA2/ PROB(23),DELE(22),BETA,IPY,ILIM
      IF ( DH .LE. 0.0 ) GOTO 100
      DO 50 IPZ = 1,ILIM
      IF ( DH .NE. DELE(IPZ) ) GO TO 50
      IPY  = IPZ
      RETURN
   50 CONTINUE
      ILIM = ILIM + 1
      IF (ILIM .GT. 22) GO TO 60
      DELE(ILIM) = DH
      PROB(ILIM) = EXP(-DH*BETA)
      IPY  = ILIM
      RETURN
   60 WRITE (6,70)
   70 FORMAT (///' ERROR: TOO MANY ENERGY CHANGES')
      STOP
  100 IPY = 23
      RETURN
      END
      SUBROUTINE HAM(I,J,ERGY,PAR2)
      COMMON/AREA1/ AA,BB,CC,LATV,LATV1
      IM    = I - 1
      JM    = J - 1
      IP    = I + 1
      JP    = J + 1
      KR    = J + 2
      KL    = J - 2
C-----------PERIODIC BOUNDARY CONDITIONS ----------------------------
      IF ((I.GT.2).AND.(I.LT.LATV1).AND.(J.GT.4).AND.(J.LT.LATV1))
     Z         GOTO 40
      IF (I.EQ.2) IM=LATV1
      IF (I.EQ.LATV1) IP=2
      IF ((J.GT.4).AND.(J.LT.LATV1)) GOTO 40
      IF (J.GT.4) GOTO 20
      IF (J.EQ.4) GOTO 10
      JM    = LATV
      KL    = LATV1
      GOTO 40
```

```
   10 KL    = LATV
      GOTO 40
   20 IF (J.EQ.LATV1) GOTO 30
      JP    = 3
      KR    = 4
      GOTO 40
   30 KR    = 3
C-----------------------------------------------------------------------
   40 IB1   = ISN2 (  IM,J,I,J  )
      IB2   = ISN2 (  I,J,IP,J  )
      IB3   = ISN2 (  IM,JM,I,JM  )
      IB4   = ISN2 (I,JM,IP,JM  )
      IB5   = ISN2 (  IM,JP,I,JP  )
      IB6   = ISN2 (  I,JP,IP,JP  )
C
      IR1   = ISN2 (  I,J,I,KL  )
      IR2   = ISN2 (  I,J,I,KR  )
C
      ERGY = -CC * (IB1*(IB3 + IB5) + IB2 * (IB4 + IB6))
     Z         -BB*(IR1 + IR2) - AA*(IB1 + IB2)
      PAR2 = 2 * (  ISN2(I,J,I,JP) + ISN2(I,J,I,JM)  )
      RETURN
      END
      FUNCTION ISN2 (I1,J1,I2,J2)
      LOGICAL SPIN (102,104),SP1,SP2
      COMMON /AREA3/ SPIN
C        .FALSE.  IS POSITIVE
      SP1   = SPIN(I1,J1)
      SP2   = SPIN(I2,J2)
      ISN2 = -1
      IF (  (SP1 .AND. .TRUE.) .AND. (SP2.AND. .TRUE.) )ISN2 = +1
      IF (  (.NOT.SP1 .AND. .TRUE.) .AND. (.NOT.SP2.AND. .TRUE.) )
     Z                      ISN2 = +1
      RETURN
      END
      FUNCTION ISN1(I1,J1)
      LOGICAL SPIN(102,104)
      COMMON /AREA3/ SPIN
      ISN1 = +1
      IF (  SPIN(I1,J1) ) ISN1 = -1
      RETURN
      END
      SUBROUTINE RANPER(N,LOC)
      DOUBLE PRECISION DS2
      DIMENSION  LOC(1)
      DO 10 I= 1,N
   10 LOC(I) = I
      N1   = N + 1
      DO 20 M= 1,N
      L     = M+RAN2(DS2) * (N1 - M)
      L1   = LOC(L)
      LOC(L) = LOC(M)
   20 LOC(M) = L1
      RETURN
      END
```

```
     Z    ISPN(JJ) = IDECO
 840 CONTINUE
     WRITE(6,1500) (ISPN(JJ),JJ=1,LATH)
 850 CONTINUE
     RETURN
1500 FORMAT(15X,102A1)
1501 FORMAT (///,5X,I6)
     END
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