# EQUILIBRIUM STATES OF FERROMAGNETIC ABELIAN LATTICE SYSTEMS

by

Jacek Miekisz

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

Joseph Slawny

Joseph Ball

William Greenberg

George Hagedorn

James Thomson

June, 1984
Blacksburg, Virginia

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Committee Chairman: Joseph Slawny

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(ABSTRACT)

Ferromagnetic abelian lattice systems are the topic of this paper. Namely, at each site of $Z^{\nu}$-invariant lattice is placed a finite abelian group. The interaction is given by any real, negative definite, and translation invariant function on the space of configurations. Algebraic structure of the system is investigated. This allows a complete description of the family of equilibrium states for given interaction at low temperatures. At the same time it is proven that the low temperature expansion for Gibbs free energy is analytic. It is also shown that it is not necessary to consider gauge models in the case of $Z_m$ on $Z^{\nu}$ lattice.

## ACKNOWLEDGEMENTS

I want to thank my adviser, for suggesting the subject of this dissertation, for many fruitful discussions, and critical comments. I am indebted to for solving some algebraic problems. I wish to thank for the financial support during summer months and for his assistance during my stay at Virginia Tech. I would like to thank my wife for help in typing the dissertation.

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## Chapter I

## INTRODUCTION

One of the main tasks of statistical mechanics is to describe the family of equilibrium states of a system for a given interaction,external parameters, and temperature.First order phase transition occurs if there is more than one equilibrium state. Lattice systems will be discussed in this paper. In the case of a finite number of ground states there is a complete theory of S.A.Pirogov and Y.G.Sinai [1,2] :the phase diagram at low temperatures is obtained by perturbation of the zero temperature phase diagram.In particular,the number of pure phases at low temperatures is equal to the number of ground states.When the number of ground states is infinite, no general results can be inferred.However,there is one class of models,the translation invariant ferromagnetic spin systems,where the description of all translation invariant equilibrium states is available:W.Holsztynski,J.Slawny [3,4,5].These results are generalized here to an abelian ferromagnetic case.Namely, at each site of the lattice is placed a finite abelian group.The Hamiltonian in finite volume $\Lambda$ can be written as follows:

$$(1.1)\qquad H_\Lambda = -\sum_{\hat{B}\in \hat{X}_\Lambda} J(\hat{B})\hat{B} \qquad J(\hat{B})\geq 0$$

where the summation is over the group dual to the group of configurations in $\Lambda$. In other words, the Hamiltonian is any real, negative definite, translation invariant function on the space of configurations. The problem is to find all equilibrium states [6] in this model.

First consider some examples. The classical one is the ferromagnetic Potts model with m components. The interaction is given by:

$$(1.2) \qquad \delta(x(a),x(b)) = \begin{cases} 1 \text{ if } x(a) = x(b) \\ 0 \text{ otherwise} \end{cases}$$

where a,b is a pair of nearest neighbours on the cubic lattice and x(a),x(b) are configurations at points a and b, respectively. Since

$$(1.3) \qquad \delta(x(a),x(b)) = 1/m \sum_{k=0}^{m-1} \exp\{(2\pi ik/m)[x(a)-x(b)]\}$$

the interaction is of the type (1.1) with $Z_m$ as a group of configurations at one point.

The phase structure of $Z_m$ models was studied by S.Elitzur, R.B.Pearson and J.Shigemitsu [7] because of their role as a bridge between the Ising and XY model behaviour (XY model can be seen as a limit case of $Z_m$ models when m goes to infinity). They have shown that if m is above some critical value, a massless soft phase, characteristic for XY model, will appear between two Ising type ordered and disordered massive phases. When m goes to infinity, the ordered phase disappears except the zero temperature and the remaining two phases are those of the XY model.

$Z_m$ models were used by C.Gruber, A.Hintermann, and D.Merlini [8,9] to investigate higher spin systems. They mapped spin l system into $Z_m$ system with m=2l+1 by a set of local transformations:

$\phi_a : \sigma_a \to n_a \in Z_m$ $a \in L$ (lattice) where $\sigma_a$ is a spin variable at site a. Fourier transformation gave them new coupling constants. They used the group structure to obtain certain results like analyticity of pressure and then they inverted $\{\phi_a\}$ returning to the initial system.

In this paper the structure of the family of equilibrium states in ferromagnetic abelian systems at low temperatures is investigated. The core idea is to use Peierls argument as in [4]. Namely, in the case of spin 1/2 on $Z^\nu$ lattice, finite volume magnetization $\rho_\Lambda^+(\sigma_a)$ where $\rho_\Lambda^+$ is a finite volume $\Lambda$ Gibbs state with "+" boundary condition, is studied.

(1.4) $$\rho_\Lambda^+(\sigma_a) = 1-2 \text{ probability}_\Lambda\{x(a)=-1\}$$

There is a standard Peierls estimation for probability that a configuration at site a is -1 ,

(1.5) $$\text{probability}_\Lambda\{x(a)=-1\} \le \sum_E n_E e^{-\beta E} .$$

where $n_E$ is the number of indecomposable excitations with $x(a)=-1$ and energy E. If probability$_\Lambda\{x(a)=-1\} \to 0$ as $\beta \to \infty$ independent of $\Lambda$ then

$$(1.6) \qquad \rho^+(\sigma_a) = \lim_{\Lambda \to L} \rho^+_\Lambda(\sigma_a) > 0$$

at low temperatures.
Let G belongs to the symmetry group S and let $\rho_G = G\rho^+$ hence

$$(1.7) \qquad \rho_G(\sigma_a) = \sigma_a(G)\rho^+(\sigma_a)$$

$\rho_G$ is an extremal Gibbs state and $\rho_G \neq \rho^+$. Clearly, there are at least $|S|$ pure phases (extremal and periodic Gibbs states). In fact, there are exactly $|S|$ pure phases if $|S|$ is finite. This means that the whole symmetry group is broken in $\rho^+$ state.

On the other hand if there are too many excitations with the same energy, some symmetries can be preserved at low temperatures. It would be best to have a criterion when Peierls argument can be applied. It is known for spin 1/2 on $Z^\nu$ lattice that Peierls argument holds if and only if the greatest common divisor of ideal generated by bonds of the system in certain ring is a unit [4]. If it is not the case, one can pass to the "reduced system" where the standard argument can be used. The structure of the family of equilibrium states of the system can be then established.

The general $Z^{\nu}$-invariant lattice case is more complicated [5].The energy can be invariant under local modifications of configurations.If this is the case,the system is a gauge model.Complete results as in [4] are obtained in [5]. However,more advanced mathematics is required.In place of the ring and its ideals,modules over the ring must be considered.

The notation and the models are introduced in Chapter II of this paper. Basic definitions are given and the basic theorem about the structure of periodic Gibbs states is quoted.It is shown that in some cases the group of configurations can be reduced.Namely,if we have a group G at each site of the lattice then for some interactions the system can be constructed with the group G' whose order is less than the order of G.There is a natural one-to-one correspondence between the family of Gibbs states of these two systems. The second system can be studied by the methods described in the next chapters.

Chapter III deals with the algebraic structure of the system.In particular, the contours and the cycles are defined.

In Chapter IV condition for using Peierls argument, the so called decomposition property is formulated.Also given are criteria for this property to hold.Shown as well is that it is not necessary to consider gauge models on $Z^{\nu}$ in $G=Z_m$ case. Passage to the new system, which is non-gauge, can be achieved as described in the end of Chapter II.

Chapter V is restricted to $Z^{\nu}$ as a lattice - the so called transitive case. In this chapter are generalized results of [4] to $Z_m$ case where m is a product of different primes. Namely, the criterion for the decomposition property to hold and explicit expression for $\mathbf{B}^+$ (the group of characters on which $\rho^+$ is greater than 0) is given in terms of bonds of the system. Also established is the criterion for the decomposition property to hold in $Z_m$ case with arbitrary m. Consider for example the group $Z_pn$ on $Z^{\nu}$ (p-prime). Let $\mathbf{A}$ be the ideal in $Z_pn[Z^{\nu}]$ generated by all bonds of the system. Then it can be shown that the decomposition property holds in system with ideal $\mathbf{A}$ if and only if it holds in system with the ideal $p^{n-1}\mathbf{A}$. Because $p^{n-1}\mathbf{A}$ is a $Z_p[Z^{\nu}]$-module and $Z_p$ is a field ($Z_2$ in spin 1/2) easy generalization of spin 1/2 technique can be applied here.

In Chapter VI the reduced system is constructed for general $Z^{\nu}$-invariant lattice and arbitrary finite abelian group as a configuration at one point. This is a direct generalization of [5]. Also proven is uniquenes of reduction in the transitive $Z_m$ case where m is a product of different primes. The effective way of reduction for any transitive $Z_m$ case is given, as well.

In Chapter VII some necessary and sufficient conditions for the decomposition property to hold on general $Z^{\nu}$-invariant lattices are given.Chapter VIII contains collection of theorems that show the analyticity of pressure at low temperatures.Chapter IX contains examples.In appendices mathematical background is provided.Also some new theorems are proved.

# Chapter II

# NOTATION AND DESCRIPTION OF MODEL

## 2.1 CONFIGURATION SPACE

By lattice L is meant any $Z^{\nu}$-invariant, discrete subset of $R^{\nu}$. A finite abelian group G is placed at each site of the lattice.

$$(2.1) \qquad X = \underset{i \in L}{\times} G$$

is a configuration space of the system

For $A \in X$ $\quad pr_{\{a\}}A = A(a)$ $, a \in L$

If G is equiped with the discrete topology then X becomes the compact abelian group with product topology.

$$(2.2) \qquad X_{\Lambda} = \bigoplus_{i \in \Lambda} G$$

is a finite volume configuration space where $\Lambda$ is any finite subset of L and

$$(2.3) \qquad X_f = \bigoplus_{i \in L} G$$

We provide $X_f$ with the discrete topology.G as a finite abelian group can be decomposed as follows:

$$(2.4) \qquad G \simeq G_{p_1} \oplus \dots \oplus G_{p_k}$$

where $p_i$ are primes and $G_{p_i}$ are $p_i$ primary groups

(2.5) $G_{p_i} \simeq Z_{p_i^{i_1}} \oplus \ldots \oplus Z_{p_i^{i_s}}$

The group dual to X is isomorphic to $X_f$ (Theorem A4)

(2.6) $\hat{X} \simeq X_f$

If $A \in X_f$ then we write $\hat{A}$ for corresponding element in $\hat{X}$.

## 2.2 INTERACTION

The Hamiltonian in finite volume $H_\Lambda$ can be any real, negative definite, and translation invariant function on $X_\Lambda$. By this it is meant that the Fourier decomposition of $H_\Lambda$ is following:

(2.7) $H_\Lambda = - \sum_{B \in X_\Lambda} J(B)\hat{B}$

where $J(B) \geqslant 0$ , $J(B)=J(-B)$ and if $B_2$ can be obtained from $B_1$ by translation then $J(B_1)=J(B_2)$. The family of bonds is defined as follows:

(2.8) $\mathbf{B} = \{B \in X_f \; ; \; J(B)>0\}$

and it is assumed that there is a finite fundamental family of bonds $\mathbf{B}_0$ such that any element of $\mathbf{B}$ can be obtained in a unique way by a translation of an element from $\mathbf{B}_0$.

## 2.3 GIBBS STATES

Let $e_g$ ,the identity of the group G, be placed everywhere outside $\Lambda$.With this as a boundary condition, a finite volume Gibbs state can be constructed.It is denoted traditionally by $\rho_\Lambda^+$. By means of correlation inequalities [10], standard conclusions can be obtained.Namely,Gibbs state $\rho^+$ can be constructed as a limit of $\rho_\Lambda^+$ when $\Lambda\to L$. $\rho^+$ is a translation invariant state, extremal in the set of all Gibbs states,and mixing. For any Gibbs state $\rho$ :

$$(2.9)\qquad \rho^+(\hat{A})\geq|\rho(\hat{A})| \quad \text{for any } A\in X_f$$

$$(2.10)\qquad \rho^+(\hat{B})>0 \qquad \text{for all } B\in \mathbf{B}.$$

Recently Pfister [11] proved some correlation inequalities for $G=Z_m$ case.They can be trivially generalized to arbitrary finite abelian group case.In fact, they can be obtained for any compact abelian group: R.Fernandez [12].

$$(2.11)\qquad \text{If } \rho^+(\hat{A})>0 \text{ and } \rho^+(\hat{B})>0 \text{ then } \rho^+(\hat{A}\hat{B})>0$$

$$(2.12)\qquad \text{If } \rho^+(\hat{A}) = \rho(\hat{A}) \quad \text{and} \quad \rho^+(\hat{B}) = \rho(\hat{B})>0 \quad \text{then}$$

$$\rho^+(\hat{A}\hat{B}) = \rho(\hat{A}\hat{B})$$

$$\text{for any Gibbs state } \rho \text{ and } A,B\in X_f$$

The following definitions are standard:

(2.13a) $\mathbf{A}$ - a subgroup of $X_f$ generated by $\mathbf{B}$

(2.13b) $\mathbf{B}^{+} = \{A\epsilon X_f \,; \rho^{+}(\hat{A})>0\}$

(2.13c) $S = \{G\epsilon X; \hat{B}(G)=1 \text{ for any } B\epsilon\mathbf{B}\}$

(2.13d) $S^{+} = \{G\epsilon S; \rho_G^{+} = \rho^{+}\}$

(2.14) $\Delta^{+} = \{\rho;\rho \text{ is a Gibbs state and } \rho(\hat{A})=\rho^{+}(\hat{A})$
for any $A\epsilon\mathbf{A}\}$.

It is known [3,11,13] that if $\rho\epsilon\Delta^{+}$ then there is a measure $\mu$ on $S/S^{+}$ such that

(2.15) $$\rho = \int_{S/S^{+}} \rho_G^{+}\mu(dG)$$

$\rho$ is translation invariant and ergodic if and only if $\mu$ is. At the temperature for which pressure is differentiable with respect to it, any periodic Gibbs state is in $\Delta^{+}$. This can be proven in a standard way using (2.9) and (2.12) as in [3,11,13,17] The differentiability of pressure at low temperatures is proved in Chapter VIII of this paper. Both $\mathbf{B}^{+}$ and $S^{+}$ depend on the temperature. However, it can be shown that at low enough temperatures $\mathbf{B}^{+}$ and $S^{+}$ are constant. The number of periodic ,extremal Gibbs states at low temperatures (from now on such states will be called pure phases) is equal to $|S/S^{+}|$ at low temperatures. It will be shown later that $\mathbf{B}^{+}/\mathbf{A}$ is isomorphic to the group dual to $S/S^{+}$. This constitutes the following theorem:

Theorem 2.1

The number of extremal,periodic Gibbs states is equal to $|\mathbf{B}^+/\mathbf{A}|=|S/S^+|$ at low temperatures.They are of the form (2.15).

The main goal of this paper is to describe $\mathbf{B}^+$ as completely as possible. This can be used to count pure phases at low temperatures.

## 2.4 REDUCTION OF THE GROUP OF CONFIGURATIONS

In some systems the space of configurations can be reduced.

Let $X=\underset{a\in L}{\times} G$ be the configuration space of the system.

Let $G_a=\mathrm{Ker}(pr_{\{a\}}\hat{B})$ ; $a\in L$ and $X'=\underset{a\in L}{\times} G_a$

Since $X'\subset \mathrm{ClS}_f$, any Gibbs state of the system is $X'$-invariant [5].

Let $X''=X/X'=\underset{a\in L}{\times}(G/G_a)$

To describe the family of Gibbs states of the system it is enough to take $X''$ as the configuration space. In particular it can be assumed without loss of generality that the least common multiple of the orders of elements of $\mathbf{B}$ is equal to the least common multiple of the orders of elements of G. In the case of $L=Z^\nu$ and $G=Z_m$ it means that there is always a non-zero divisor in $\mathbf{B}_0$ (cf.Appendix B).

## Chapter III

## ALGEBRAIC STRUCTURE OF THE SYSTEM

The notions of contours and cycles will be introduced here.These are generalizations of the definitions from [4].Algebraic structure developed in this chapter will allow us to use Peierls argument to prove that for some systems there are at least two pure phases at low temperatures;i.e., phase transition occurs.

### 3.1 CONTOURS AND CYCLES

Let $A\epsilon X_f$ , $\hat{A}(X)$ is a finite group with multiplications of complex numbers as a group action.

Proposition 3.1

$\hat{A}(X)$ is a cyclic group with the order equal to $|A|$,the order of A.In particular for every $A\epsilon X_f$ there is $X_A \epsilon X_f$ ; $\hat{A}(X_A)=\exp\{2\pi i/|A|\}$

Proof: $\hat{A}(X)$ is a subgroup of the group of $|G|$-th roots of unity.Hence $\hat{A}(X)$ is cyclic as a subgroup of a cyclic group $Z_{|G|}$. If for any $a\epsilon\hat{A}(X)$, $a^k=1$ then $A^k=1$.This proves the equality of the orders. ▮

Denote:

$$(3.1) \qquad M = \times_{B\epsilon\mathbf{B}} Z_{|B|}$$

$$(3.2) \qquad M_f = \bigoplus_{B\in\mathbf{B}} Z_{|B|}$$

If each cyclic group is equiped with the discrete topology then M becomes the compact abelian group with the product topology. $M_f$ with the discrete topology is the locally compact abelian group. Let m be the least common multiple of the orders of elements of $\mathbf{B}_0$. By remarks in 2.4 m is equal to l.c.m. of the orders of elements of G. Both M and $M_f$ are $Z_m[Z^\nu]$-modules (cf. Appendix B). Moreover, $M_f$ is a reflexive finitely generated module. X and $X_f$ are also $Z_m[Z^\nu]$-modules. Two useful module homomorphisms can be constructed.

Let

$$(3.3) \qquad \gamma(x)=(\hat{B}(x))_{B\in\mathbf{B}}\ ,x\in X$$

Then by Prop.3.1 it can be written

$\gamma(x)=\alpha\in M$ where $\hat{B}(x)=\exp\{2\pi i\alpha(B)/|B|\}$ so

$\gamma : X\to M$

Let now $\alpha\in M$

$$(3.4) \qquad \varepsilon(\alpha) = \sum_{B\in\mathbf{B}} \alpha(B)B \quad \text{so}$$

$\varepsilon : M\to X$

The sum converges in the topology of X because the interaction is of finite range. It is easy to see that both $\gamma$ and $\varepsilon$ are continuous module homomorphisms.

The following definitions are standard

$$(3.5a) \qquad \Gamma=\mathrm{Im}(\gamma) \quad \Gamma_f=\gamma(X_f)$$

$$(3.5b) \qquad K=\mathrm{Ker}(\varepsilon) \quad K_f=K\cap M_f$$

From the definition $S=\mathrm{Ker}(\gamma)$, $A=\varepsilon(M_f)$ and let $S_f=S\cap X_f$. By continuity of $\varepsilon$ and $\gamma$ , K and S are closed subgroups of M. Because X is compact so is $\Gamma$ and as a subset of a Hausdorff space $M,\Gamma$ is closed. By density of $X_f$ in X, $\Gamma_f$ is dense in $\Gamma$. In general, $K_f$ and $S_f$ are not dense in K and S respectively. It is shown later that density of $K_f$ in K is equivalent to the so called decomposition property and density of $S_f$ in S to the absence of phase transitions at low temperatures. Elements of $\Gamma_f$ are called contours and elements of $K_f$ are called cycles.

## 3.2 BICHARACTERS ON $X\times X_f$ AND $M\times M_f$

It is known that $X,X_f$ and $M,M_f$ are mutually dual groups (cf. Appendix A). For $x\in X$ , $y\in X_f$

$$(3.6)\quad \langle x,y\rangle\equiv\hat{y}(x)=\prod_{a\in L}\exp\{2\pi i(\sum_{i=1} x_i(a)y_i(a))/|G_i|\}$$

where $G=\bigoplus_{i=1}^{r} G_i$ is the decomposition into cyclics

$$X=\bigoplus_{i=1}^{r} X_i \qquad X_i=\times_{a\in L} G_i$$

$$x=\sum_{i=1}^{r} x_i \qquad x\in X_i \quad ;$$

similarly for $\alpha_1\in M$ , $\alpha_2\in M_f$

$$(3.7)\quad \langle\alpha_1,\alpha_2\rangle=_{B\in\mathbf{B}}\exp\{2\pi i\alpha_1(B)\alpha_2(B)/|B|\}$$

Proposition 3.2

(a) $\langle\gamma(x),\alpha\rangle=\langle x,\varepsilon(\alpha)\rangle$ where $x\in X,\alpha\in M_f$

(b) $\langle\alpha,\gamma(x)\rangle=\langle\varepsilon(\alpha),x\rangle$ where $x\in X_f,\alpha\in M$

Proof: Denote $\gamma(x)=\beta\in M$

$$<\gamma(x),\alpha> = \prod_{B\in\mathbf{B}} \exp\{2\pi i\beta(B)\alpha(B)/|B|\} = \prod_{B\in\mathbf{B}} B(x_B)^{\beta(B)\alpha(B)} =$$

$$= \prod_{B\in\mathbf{B}} \hat{B}(x_B)^{\beta(B)\alpha(B)} = \prod_{B\in\mathbf{B}} B(x)^{\alpha(B)} = <x,\varepsilon(\alpha)>$$

The second part of the proposition can be proven in the same manner. ∎

Let $N\subset M_f$ $(X_f)$ then $\mathrm{cl}N \equiv \mathrm{Cl}N\cap M_f$ $(\mathrm{Cl}N\cap X_f)$

Proposition 3.3

(a) $\Gamma_f{}^{\perp}=K$ $\quad K^{\perp}=\Gamma_f$ $\quad K_f{}^{\perp}=\Gamma$ $\quad \Gamma^{\perp}=K_f$

(b) $(\mathrm{Cl}K_f)^{\perp} = \mathrm{cl}\Gamma_f$

(c) $S=\mathbf{A}^{\perp}$

(d) $(\mathrm{Cl}S_f)^{\perp} = \mathrm{cl}\mathbf{A}$

(e) $\mathbf{B}^{+} = (S^{+})^{\perp}$

Proof: Let $\alpha\in M_f$ then by Prop.3.2(a) $\alpha\in\Gamma^{\perp}$ iff $\varepsilon(\alpha)=e_X$ so $\alpha\in K_f$ hence $\Gamma^{\perp} = K_f$. Because $\Gamma$ is closed, by Theorem A2 $K_f{}^{\perp}=\Gamma^{\perp\perp}=\Gamma$. The rest of (a) follows from Prop.3.2 in the same way.

(b) $(\mathrm{Cl}K_f)^{\perp} = K_f{}^{\perp}\cap M_f = \Gamma\cap M_f = \mathrm{cl}\Gamma_f$

(c) follows from prop.3.2(a)

To prove (d) it is enough to notice that by prop.3.2(b) $S_f=(\mathrm{Cl}\mathbf{A})^{\perp}$ so $S_f{}^{\perp}=\mathrm{Cl}\mathbf{A}$

$(\mathrm{Cl}S_f)^{\perp} = S_f{}^{\perp} \cap X_f = \mathrm{cl}\mathbf{A}$

(e) follows directly from the definition of $S^{+},\mathbf{B}^{+}$ and the fact that $B\subset\mathbf{B}^{+}$. ∎

## Chapter IV

## THE DECOMPOSITION PROPERTY AND PEIERLS ARGUMENT

There is a standard Peierls estimation for the probability of occurence of a given contour (Lemma 4.5). It vanishes exponentially when temperature approaches zero. It can be shown that in some systems (the systems which satisfy the decomposition property) contours which are not the sum of two other contours are connected (in the sense described below). If this is the case, then in the estimation of the type (1.5), only connected contours are used.The number of connected contours can be majorized (Proposition 4.4) and the appropriate limit (cf.(1.5)) exists (Theorem 4.6).

### 4.1 THE DECOMPOSITION PROPERTY

Let $B\in X_f$ $\underline{B}=\{a\in L; B(a)\neq e_G\}$

$\alpha\in M$ $\quad \underline{\alpha}=\bigcup_{B;\alpha(B)\neq 0} \underline{B}$

$\mathrm{supp}\,\alpha=\{B;\alpha(B)\neq 0\}$

For $a\in L\subset R^{\nu}$ $\quad a=(a_1,\ldots,a_{\nu})$

$$|a|=\max\{|a_1|,\ldots,|a_{\nu}|\}$$

$$\mathrm{diam}\, B = \max\{|a-b|\ a,b\in\underline{B}\} \qquad \mathrm{diam}\,\alpha=\mathrm{diam}\,\underline{\alpha}$$

For $B_1,B_2 \in X_f$

$\delta(B_1,B_2)=\mathrm{dist}(\underline{B}_1,\underline{B}_2)=\inf\{|a-b|,a\in\underline{B}_1,b\in\underline{B}_2\}$

mesh $\mathbf{B}_0 = \max\{\mathrm{diam}\ \underline{B};B\in\mathbf{B}_0\}$

Let N be a natural number greater than the range of interaction.$\alpha$ can be treated as a graph $(V,E)_N$ where

$V=\{\underline{B};\alpha(B)\neq 0\}$

$E=\{(U,W)\ ;\ U,W\in V\ ,\ \delta(U,W)\leq N\}$.

The components of this graph will be called N-components.

Definition 4.1

The interaction has the decomposition property if there exists a non-negative integer N such that the N-components of contours are again contours.

In other words the decomposition property holds if and only if there exists an integer N such that for each $x\in X_f$ there exists a natural number $n=n(x)$ and $x_1,\ldots,x_n\in X_f$ such that:

(a) $\gamma(x_i)$ is N-connected

(b) $\mathrm{supp}\gamma(x_i)\cap\mathrm{supp}\gamma(x_j)=\phi \quad 1\leq i<j\leq n$

(c) $\sum_{i=1}^{n}\gamma(x_i)=\gamma(x)$

In that case $\gamma(\sum_{i=1}^{n}x_i)=\gamma(x)$ so $x=\sum_{i=1}^{n}x_i+y$ where $y\in S_f$ and

if $pr_\Lambda x\notin pr_\Lambda S_f$ then there is i such that

$pr_\Lambda x_i\notin pr_\Lambda S_f$. If $\alpha_1|_{\mathrm{supp}\alpha_1}=\alpha_2|_{\mathrm{supp}\ \alpha_1}$ then we write $\alpha_1\subset\alpha_2$.

## 4.2 ESTIMATIONS AND THE MAIN THEOREM

For finite $\Lambda\subset L$ let

$\Gamma(N,\Lambda,l)=\gamma(\{x\in X_f;\gamma(x)$ is N-connected; $|\gamma(x)|=l;pr_\Lambda\, x\notin pr_\Lambda S_f\})$

$\Gamma(N,\Lambda)=\bigcup_{l=1}^{\infty}\Gamma(N,\Lambda,l)$.

The upper bound for card $\Gamma(N,\Lambda,l)$ can be found.

Lemma 4.2

Let $l$ be a natural number and let $B\in\mathbf{B}$. The number of $\alpha$'s such that $\alpha\in M_f$ and is N-connected, $|\alpha|=l$ and $B\in\operatorname{supp}\alpha$ is not greater than $[(2b+2N+1)^{\nu}b]^{2l-2}$ where $b=\max(|\mathbf{B}_0||G|,\text{mesh }\mathbf{B}_0)$

This is a straightforward generalization of Lemma 2.3 in [4].

Lemma 4.3

For any $B\in\mathbf{B}$ the number of translates of B contained in $\{\operatorname{supp}\alpha;\alpha\in\Gamma(N,\Lambda,l)\}$ is not greater than $[2(l(b+N)+m)+\operatorname{diam}\Lambda]^{\nu}$

Proof:(due to J.Slawny) First it will be shown that there exists $m>0$ such that if $pr_\Lambda x\notin pr_\Lambda S_f$ then $\operatorname{dist}(\underline{\gamma(x)},\Lambda)\leq m$. Indeed, $\Gamma_f$ is a finitely generated and finitely supported $Z_m[Z^{\nu}]$- module (cf.Appendix C).According to Theorem C13, there exists a set of generators $\gamma_i=\gamma(x_i)$ $x_i\in X_f$ $i=1,\ldots,n$ such that if $\gamma(x)\subset\Lambda'$ , $x\in X_f$ then $\gamma(x)=\sum_{i=1}^{n} g_i\gamma_i$ ; $g_i\in Z^{\nu}$ and $g_i\underline{\gamma}_i\subset\Lambda'$. Now if $\Lambda'$ is far from $\Lambda$

then $pr_\Lambda g_i x_i = pr_\Lambda e_X$ where $e_X$ is a configuration equal to $e_G$ everywhere and $e_G$ is an identity of G

$x - \sum_{i=1}^{n} g_i x_i \in S_f$ so a contradiction arises because

$pr_\Lambda x \notin pr_\Lambda S_f$.

Now because the contours are N-connected and of length l then the number of translates of any $B\epsilon\mathcal{B}$ contained in $\{supp\alpha; \alpha\epsilon\Gamma(N,\Lambda,l)\}$ is not greater than $[2(l(b+N)+m)+diam\Lambda]^\nu$ ∎

Now it follows immediately that

Proposition 4.4

$$\text{Card } \Gamma\ (N,\Lambda,l) \leq b[(2b+2N+1)^\nu b]^{2l-2}[2(l(b+N)+m)+diam\Lambda]^\nu$$

If $\Lambda\subset\Lambda'$ we have the following estimation

Lemma 4.5

$$\rho^+_{\Lambda'}(\{x;\alpha\subset\gamma(x)\}) \leq \exp\{-c \sum_{B\epsilon\, supp\alpha} K\ (B)\}$$

where $c=2-2\cos(1/|G|)$ $\alpha\epsilon\Gamma(N,\Lambda,l)$ and $K(B)>0$ for any $B\epsilon\mathcal{B}$

Proof:

$$\rho^+_{\Lambda'}(\{x;\alpha\subset\gamma(x)\}) = \Big(\sum_{\substack{x;\,x\in X_{\Lambda'}\\ \alpha\subset\gamma(x)}}\exp\{-H^+_{\Lambda'}(x)\}\Big)/\Big(\sum_{x;\,x\in X_{\Lambda'}}\exp\{-H^+_{\Lambda'}(x)\}\Big) \leq$$

$$\leq \Big(\sum_{\substack{x;\,x\in X_{\Lambda'}\\ supp\,\gamma(x)\cap supp\alpha=\emptyset}}\exp\{-H^+_{\Lambda'}(x)\}\exp\{-\sum_{B;\,\alpha(B)\neq0}K(B)+\sum_{B;\,\alpha(B)\neq0}K(B)\exp\{2\pi i\alpha(B)/|B|\}\}\Big)/\Big(\sum_{\substack{x;\,x\in X_{\Lambda'}\\ supp\,\gamma(x)\cap supp\alpha=\emptyset}}\exp\{-H^+_{\Lambda'}(x)\}\Big) \leq$$

$$\leq \exp\{-c \sum_{B;\,\alpha(B)\neq0} K(B)\}$$ ∎

The estimations can be applied to get the following:

Theorem 4.6 [5]

The decomposition property implies that for any finite $\Lambda \subset L$ and $\varepsilon > 0$, for low enough temperatures:

$$\rho^{+}(\{x \in X \; ; \; pr_{\Lambda} x \in pr_{\Lambda} S_f\}) > 1 - \varepsilon$$

Proof: Let N be such that the system has N- decomposition property. Then for any $x \in X_f$ such that $pr_{\Lambda} x \notin pr_{\Lambda} S_f$ there exists $\alpha \in \Gamma(N,\Lambda), \alpha \subset \gamma(x)$

Therefore $\rho^{+}_{\Lambda'}(x; pr_{\Lambda} x \notin pr_{\Lambda} S_f) \leq \sum_{\alpha \in \Gamma(N,\Lambda)} \rho^{+}_{\Lambda'}(\{x; \alpha \subset \gamma(x)\})$

Let $K = \min_{B \in \mathbf{B}} K(B) > 0$ then

$$\rho^{+}_{\Lambda'}(x \in X; pr_{\Lambda} x \notin pr_{\Lambda} S_f) \leq \sum_{l=1}^{\infty} \sum_{\alpha \in \Gamma(N,\Lambda,l)} \exp\{-ckl\} \leq$$

$$\leq \sum_{l=1}^{\infty} b[(2b+2N+1)^{\nu} b]^{2l-2} [2(l(b+N)+m) + \mathrm{diam}\Lambda]^{\nu} \exp\{-ckl\}$$

The series converges for K large enough (at low enough temperatures) and goes to zero uniformly in $\Lambda'$ as temperature goes to zero. $\rho^{+}_{\Lambda'}$ converges weakly to $\rho^{+}$ and the theorem follows. ∎

Corollary 4.7 [5]

For systems with the decomposition property $S^{+} = \mathrm{Cl} S_f$ at low temperatures.

Proof: Obviously $\mathrm{Cl} S_f \subset S^{+}$. Let $y \notin \mathrm{Cl} S_f$, and it will be shown that $y \notin S^{+}$. There exists finite $\Lambda \subset L$ such that $pr_{\Lambda} y \notin pr_{\Lambda} S_f$. Let the temperature be such that Theorem 4.6 holds with $\varepsilon < 1/2$ then

$$\rho^{+}_{y}(x; pr_{\Lambda} x \in pr_{\Lambda} S_f) = \rho^{+}(pr_{\Lambda} x \in \; pr_{L}(-y) + pr_{\Lambda} S_f) \leq$$

$\leq \rho^{+}(x; pr_{\Lambda} x \notin pr_{\Lambda} S_f) < 1/2$ so $\rho^{+}_{y} \neq \rho^{+}$ which means that $y \notin S^{+}$. ∎

Corollary 4.8 [5]

For systems with the decomposition property $\mathbf{B}^+=\mathrm{cl}\mathbf{A}$ at low temperatures.

Proof: $\mathbf{B}^+=(S^+)^\perp=(\mathrm{Cl}S_f)^\perp=\mathrm{cl}\mathbf{A}$ ∎

## 4.3 CRITERIA FOR THE DECOMPOSITION PROPERTY TO HOLD

Theorem 4.9 [5]

The following conditions are equivalent to the decomposition property:

(a) $\mathrm{Cl}K_f=K$

(b) $\mathrm{cl}\Gamma_f=\Gamma_f$

(c) $K_f^\perp \cap M_f=\Gamma_f$

Proof: It is easy to see that (a),(b) and (c) are equivalent one to another. First it will be shown that (a) implies the decomposition property. $K_f$ is a submodule of $M_f$ - finitely generated module over a Noetherian ring $Z_m[Z^\nu]$ so there are $\beta_1,\ldots,\beta_n \in K_f$ such that they generate $K_f$ (B). $\Gamma_f=K^\perp$ and $\mathrm{Cl}K_f=K$ so $\beta \in M_f$ is in $\Gamma_f$ if and only if it is orthogonal to all translates of $\beta_1,\ldots,\beta_n$. Let N=max diam $\underline{\beta}_i$ , $\alpha\in\Gamma_f$ , $\alpha'$ be N-component of $\alpha$. If $\beta$ is a translate of one of $\beta_i$'s and $\underline{\beta}\cap\underline{\alpha}'\neq\Phi$ then $\underline{\beta}\cap(\underline{\alpha-\alpha'})=\Phi$ so $\langle\alpha',\beta\rangle=\langle\alpha,\beta\rangle$ so $\alpha'$ is orthogonal to $\beta$ so $\alpha'\in\Gamma_f$ hence the decomposition property holds.

Now it will be proved by contraposition that the decomposition property implies (b). Suppose $\beta$ is in $\Gamma\cap M_f$ but not in $\Gamma_f$. Since $\Gamma_f$ is dense in $\Gamma$ there exists a net $\beta_i$ of elements of $\Gamma_f$ converging to $\beta$. Fix now N and take R, finite family of elements from $\mathcal{B}$ such that supp $\beta \subset R$ and dist($\underline{\beta}$, L-$\underline{R}$)>N There is $i_R$ such that if $i \geq i_R$ then $\beta_{i|R} = \beta_{|R}$ so $\beta$ is a N-component of $\beta_i$ hence the decomposition property does not hold. ∎

Proposition 4.10 [14]

For every finite $\Lambda \subset L$ there is $n \geq 0$ such that for any $y \in X$ such that $pr_\Lambda y$ is fixed and $pr_\Lambda y \notin pr_\Lambda S$ there is $B \in \mathcal{B}$ with dist($\underline{B}$, $\Lambda$)$\leq$n and $\hat{B}(y) \neq 1$.

Proof: Assume that there is $\Lambda \in L$ such that for every $n \geq 0$ there is $y_n$ such that $pr_\Lambda y_n$ is fixed and $\hat{B}(y_n) = 1$ if dist($\underline{B}$, $\Lambda$)$\leq$n. Because X is compact, $\{ y_n \}$ has a cluster point y and obviously $y \in S$ so a contradiction is obtained. ∎

Theorem 4.11 [5]

If the symmetry group S is finite and the dimension of the lattice is greater or equal than 2 then the decomposition property holds.

Proof: Let $a \in L$ and $[a,m] = \{ b \in L; |a-b| \leq m \}$, $m>0$. By Prop. 4.10 for any m there exists $n>0$ such that for any $x \in X$, $a \in L$ if $pr_{[a,m]} x \in pr_{[a,m]} S$ there is $B \in \mathcal{B}$ such that $\hat{B}(x) \neq 1$ and dist($\underline{B}$, a)$\leq$n. Let $\gamma(x) \in cl\Gamma_f$. It follows that for any m there exists t such that $pr_{[a,m]}x\epsilon pr_{[a,m]}S$ if $|a|\geq t$.On the other hand,it is easy to see that since S is finite for any large enough m the following extension property holds:if $pr_{[a,m]}x=pr_{[a,m]}s_1$ and $pr_{[b,m]}x=pr_{[b,m]}s_2;s_1,s_2\epsilon S$, $|a-b|\leq 1$ then $s_1=s_2$.So there exists $h\epsilon S$ and a number t such that $pr_{L-[0,t]}x=pr_{L-[0,t]}h$.But then $x-h\epsilon X_f$ and $\gamma(x-h)=\gamma(x)$ which shows that $\gamma(x)\epsilon\Gamma_f$.It proves that $cl\Gamma_f=\Gamma_f$ so by Theorem 4.9(b) the decomposition property holds. ▮

For one dimensional systems a stronger version of this theorem is obtained but it is not of interest here (Appendix D).

If $L=Z^\nu$ more specific criterion for the decomposition property to hold can be obtained (Chapter V).

## 4.4 NON-GAUGE MODELS

If $S_f$ is not trivial or, in other words, $\gamma$ is not injective on $X_f$ then the model is called a gauge model.In the case of non-gauge models, the following is an easy consequence of Theorem 4.7.

Theorem 4.12

In non-gauge models with the decomposition property $\mathcal{B}^+=X_f$ at low temperatures.

Proof:The decomposition property implies that $S^+=ClS_f$.$S_f$ is trivial so $S^+$ is trivial.Finally $\mathcal{B}^+=(S^+)^\perp=X_f$. ▮

Take a non-gauge model with the decomposition property. If $s_1, s_2 \in S$ then there exists $B \subset X_f$ $\hat{B}(s_1) \neq \hat{B}(s_2)$ (cf. Theorem A1). So $\hat{B}(s_1)\rho^+(B) \neq \hat{B}(s_2)\rho^+(\hat{B})$ hence $\rho^+_{s_1} \neq \rho^+_{s_2}$ so there are at least $|S|$ pure phases at low temperatures ; in fact exactly $|S|$ pure phases (cf. 2.3).

If $|S| < \infty$ then the assumption of Theorem 4.12 is satisfied.

Theorem 4.13

If the symmetry group S is finite there are $|S|$ pure phases at low temperatures.

Proof: S is finite so the decomposition property holds and $S^+ = \mathrm{Cl}\, S_f$ If $S_f$ is non-trivial then it is infinite so $S_f$ is trivial because S is finite. Finally $S^+$ is trivial. ∎

It is a well known fact that in spin 1/2 case there are no gauge models on $L = Z^\nu$ lattice. This is still true in a more general setting.

Theorem 4.14

There are no gauge models on $L = Z^\nu$ in $G = Z_m$ case.

Proof: First consider the case of $G = Z_p n$. Let $x \in X_f, x \neq e_X$ and $l$ is the biggest number such that $p^l$ divides $pr_{\{a\}} x \equiv x(a)$ for all $a \in \underline{x}$. Let b be the last (in the sense of lexicographic order) element of $\underline{x}$ such that $p^{l+1}$ does not divide $x(b)$. Take now $B \in \mathbf{B}_o$ such that there is $a \in \underline{B}$ and p does not divide B(a) (such B exists by the assumption about interaction, cf. 2.4).Let a' be the first of such x's and $B_{a'}$ is the translate of B by b-a'.It can be shown that $\hat{B}_{a'}(x) \neq 1$

$\hat{B}_{a'}(x)=\exp\{(2\pi i \sum_{a \in Z^\nu} B_{a'}(a)x(a))/p^n\}$ so $\hat{B}_{a'}(x)=1$ if and only if $\sum_{a \in Z^\nu} B_{a'}(a)x(a)=0 \pmod{p^n}$.

$$\sum_{a \in Z^\nu} B_{a'}(a)x(a)=B_{a'}(b)x(b)+\sum_{a \in Z^\nu} B_{a'}(a)x(a)$$

and $p^{l+1}$ divides $\sum_{a \in Z^\nu} B_{a'}(a)x(a)$ and does not divide $B_{a'}(b)x(b)$ so $\sum_{a \in Z^\nu} B_{a'}(a)x(a) \neq 0 \pmod{p^n}$.

This finishes the proof for $G=Z_{p^n}$ case.

Assume now that $G=Z_m$ so

$G=Z_{p_1^{n_1}} \oplus \ldots \oplus Z_{p_k^{n_k}}$ where $p_i$'s are different prime numbers.If $x \in X_f$ , $x \neq e_X$ denote $x=(x_1,\ldots,x_k)$ where $x_i \in X_{fi}$ $X_{fi}=\bigoplus_{j \in Z^\nu} Z_{p_i^{n_i}}$ There is l, $1 \le l \le k$ such that $x_l \neq e_{X_{fl}}$ .From the previous case we know that there is $B \in B_0$ such that

$$\hat{B}(e_{X_1},\ldots,x_l,\ldots,e_{X_k}) \neq 1$$

It is easy to see that $\hat{B}(x) \neq 1$. ∎

## Chapter V

## TRANSITIVE CASE ($L=Z^\nu$)

The results of [4] are generalized here. Namely, the criteria for the decomposition property to hold and the explicit expression for $\mathbf{B}^+$ in terms of bonds of the system are found here.

### 5.1 $G=Z_m$; m IS A PRODUCT OF DIFFERENT PRIME NUMBERS

$X_f$ can be identified with $Z_m[Z^\nu]$ (cf. Appendix B). $Z_m[Z^\nu]$ is isomorphic to the direct product of the rings $Z_{p_l}[Z^\nu]$; $p_l$ are prime numbers, $m=\prod_{l=1}^{\kappa} p_l$. If $P\in Z_m[Z^\nu]$ there is the unique decomposition:

$$P=\sum_{l=1}^{k}(m/p_l)P^l \text{ where } 0\le P^l(a)<p_l \text{ for any } a\in Z^\nu$$

For $B\in\mathbf{B}$ let $\alpha_B\in M_f$ be such that $\alpha_B(A)=\delta_{B,A}$ any $A\in\mathbf{B}$. For any $\alpha\in M_f$ there are unique $P_B\in Z_m[Z^\nu]$, $B\in\mathbf{B}_0$ such that $m/|B|$ divides $P_B(a)$ for any $a\in\underline{P}_B$ and

$$(5.1)\quad \alpha=\sum_{B\in\mathbf{B}_0}(|B|/m)P_B\alpha_B$$

Lemma 5.1

Let $\alpha=\sum_{B\in\mathbf{B}_0}(|B|/m)P_B\alpha_B$

$$\beta=\sum_{B\in\mathbf{B}_0}Q_B\alpha_B$$

be two elements of $M_f$

$\alpha$ is orthogonal to all translates of $\lambda\beta$ for any $\lambda\in Z_m$ if and only if $\sum_{B\in\mathbf{B}_o} P_B I(Q_B)=0$

Proof: $\alpha$ is orthogonal to all translates of $\lambda\beta$ for any $\lambda\in Z_m$ if and only if it is orthogonal to $R\beta$ for any $R\in Z_m[Z^\nu]$.

$<\alpha,R\beta>=\exp\{(2\pi i/m)\sum_{B\in\mathbf{B}_o}\sum_{a\in Z^\nu} P_B(a)(RQ_B)(a)\}=$

$=\exp\{(2\pi i/m)\sum_{B\in\mathbf{B}_o}\sum_{a\in Z^\nu}(PI(Q_B))(a)R(a)\}$

where $I(Q_B)(a)=Q_B(-a)$ hence

$<\alpha,R\beta>=1$ if and only if $\sum_{B\in\mathbf{B}_o} P_B I(Q_B)=0$ ∎

Let $x_l\in X_f$ $x_l(a)=e_G$ if $a\neq 0, a\in Z^\nu$ $x_l(0)=(m/l)t_l$ where $t_l\in Z_{p_l}$ is such that $t_l(m/l)^2=m/l \pmod m$

$\Gamma_f$ as a $Z_m[Z^\nu]$-module is generated by $\{\gamma_l\}$ $1\le l\le k$ where $\gamma_l=\gamma(x_l)$.

Notice that $\Gamma_f$ is also generated by one element, namely $\gamma_0=\sum_{l=1}^{k}\gamma_l$. In fact $\Gamma_f$ is isomorphic to $Z_m[Z^\nu]$. It will be discussed in Chapter VI. It is easy to see that

(5.2) $\gamma_l=\sum_{B\in\mathbf{B}_o}(|B|/p_l)I^l(B)\alpha_B$

so $\gamma_0=\sum_{B\in\mathbf{B}_o}(|B|/m)I(B)\alpha_B$.

Let $\mathbf{B}_0=\{B_1,\ldots,B_n\}$ and if $n>1$ for $1\le i<j\le n$

(5.3) $\beta_{ij}=B_i\alpha_{B_j}-B_j\alpha_{B_i}$

$\varepsilon(\beta_{ij})=B_iB_j-B_jB_i=0$ so $\beta_{ij}\in K_f$.

Lemma 5.2

If the greatest common divisor $\{B^l, B\in\mathbf{B}_0\}$ is a unit in $Z_{p_l}[Z^\nu]$ for any $l$ and $\alpha$ is orthogonal to all translates of $\lambda\beta_{ij}$ for any $\lambda\in Z_m$ then $\alpha\in\Gamma_f$.

Proof: Let $\alpha=\sum_{i=1}^{n}(|B_i|/m)P_i\alpha_i$ where $\alpha_i=\alpha_{B_i}$. By Lemma 5.1 the orthogonality of $\alpha$ to all translates of $\lambda\beta_{ij}$ for any $\lambda\in Z_m$ is equivalent to

(5.4) $P_jI(B_i)=P_iI(B_j)$ $i,j=1,\dots,n$

which means

(5.5) $P_j^lI^l(B_i)=P_i^lI^l(B_j)$ for any $l$

Since $I$ is an automorphism of the ring $Z_{p_l}[Z^\nu]$

g.c.d.$\{I^l(B);B\in\mathbf{B}_0\}$ is a unit in $Z_{p_l}[Z^\nu]$ so it follows from (5.5) that for $l$ such that card$(B^l;B\in\mathbf{B}_0)>1$ there exists $P_i'^l\in Z_{p_l}[Z^\nu]$ such that

(5.6) $P_i'^l=P_i'^lI^l(B_i)$ in $Z_{p_l}[Z^\nu]$.

Hence $(P_i'^l-P_j'^l)I^l(B_i)I^l(B_j)=0$ and since there are no zero divisors in $Z_{p_l}[Z^\nu]$ it follows that $P_i'^l=P_j'^l\equiv P^l$ and

$P_i^l=P^lI^l(B_i)$

For $l$ such that g.c.d.$\{B^l;B\in\mathbf{B}_0\}=B_m^l$ and $B_m^l$ is a unit in $Z_{p_l}[Z^\nu]$ it can be also written:

$P_m^l=P^lI(B_m^l)$ where $P^l=P_m^lI(B_m^l)^{-1}$

Finally $\alpha=\sum_{i=1}^{n}\sum_{l=1}^{k}(m/p_l)(|B_i|/m)P^lI^l(B_i)\alpha_i=$

$=\sum_{l=1}^{k}P^l\sum_{i=1}^{n}(|B_i|/p_l)I^l(B_i)\alpha_i=\sum_{l=1}^{k}P^l\gamma_l$

Let $D^l$ be g.c.d.$\{B^l;B\in\mathbf{B}_0\}$ then $D=\sum_{l=1}^{k}(m/p_l)D^l$ is g.c.d.$(\mathbf{B}_0)$ and vice versa (cf. Appendix B).

Theorem 5.3

The system has the decomposition property if and only if g.c.d.$(\mathbf{B}_0)$ is a unit.

Proof:the if part

Let N=max diam $\beta_{ij}$ It will be shown that N-components of contours are contours.Let $\alpha\epsilon\Gamma_f$ and $\alpha'$ is N-component of $\alpha$.If $\beta$ is a translate of $\lambda\beta_{ij}$ for some $\lambda\epsilon Z_m$, $1\leq i<j\leq n$ and $\underline{\beta}\cap\underline{\alpha'}\neq\phi$ then $\underline{\beta}\cap(\underline{\alpha-\alpha'})=\phi$.This shows that $<\alpha',\beta>=<\alpha,\beta>$ but $\beta\epsilon K_f,\alpha\epsilon\Gamma_f$ so $<\alpha,\beta>=1$ and finally $\alpha'$ is orthogonal to $\beta$.Hence by Lemma 5.2 $\alpha'\epsilon\Gamma_f$.

the only if part

Let $l$ be such that $D^l$=g.c.d$\{B^l;B\epsilon\mathbf{B}_0\}$ is not a unit in $Z_{p_l}[Z^\nu]$

Let $\gamma_l^w=(m/p_l)w_l\sum_{B\epsilon\mathbf{B}_0}(|B|/p_l)I(B^l/D^l)\alpha_B=$

$=(m/p_l)w_l\sum_{B\epsilon\mathbf{B}_0}(|B|/p_l)I^l(B)/I^l(D)\ \alpha_B$ ; $w_l\epsilon Z_{p_l}$

where $B^l/D^l\ \epsilon Z_{p_l}[Z^\nu]$;$(B^l/D^l)D^l=B^l$ $\gamma_l^w\epsilon\Gamma_f$ for any $w_l\epsilon Z_{p_l}$,because if not then it would follow that $\gamma_l^w=P\gamma_l$; $P\epsilon Z_m[Z^\nu]$ (it is easy to see that to generate $\gamma_l^w$ we need only $\gamma_l$)

$(m/p_l)P^l(|B|/p_l)I(B^l)(a)=(m/p_l)w_l(|B|/p_l)(I^l(B)/I^l(D))(a)$ (mod $|B|$)

for any $a\epsilon Z^\nu$ so

$p^1 I^1(B) I^1(D) = I^1(B) w_1$ in $Z_{p_1}[Z^\nu]$

$p^1 I^1(D) = w_1 x^0$ in $Z_{p_1}[Z^\nu]$

where $x^0 \in Z_m[Z^\nu]$ $x^0(a) = \delta_{0,a}$ $a \in Z^\nu$

It contradicts the fact that $D^1$ is not a unit.On the other hand

$(m/p_1) I^1(D)^{p_1^n} \gamma_1^w = (m/p_1) w_1 I^1(D)^{p_1^{n-1}} \gamma_1$ is in $\Gamma_f$.

Now if $I^1(D) = \sum_{a \in Z^\nu} I^1(D)(a)a$ then $I^1(D)^{p_1^n} = \sum_{a \in Z^\nu} I^1(D)(a) p_1^n a$ (cf. Appendix C). so for any fixed N, n can be found such that N-components of $(m/p_1) I^1(D)^{p_1^n} \gamma_0^w$ are translates of $(m/p_1) I^1(D)(a) \gamma_1^w$ hence do not belong to $\Gamma_f$.This means that the decomposition property does not hold. ∎

REDUCTION

Let $\mathbf{B}$ be a translation invariant family of elements of $X_f$ with a finite fundamental subfamily $\mathbf{B}_0$.Let $D \in Z_m[Z^\nu]$ and suppose that D is not a zero divisor.

Let $\mathbf{B}' = \{DB; B \in \mathbf{B}\}$.

Let K be a ferromagnetic translation invariant interaction with bonds $\mathbf{B}$ and let $K'(DB) = K(B); B \in \mathbf{B}$.The theorem below indicates that the systems with interactions K and K' are, in a sense, isomorphic.

Theorem 5.4

If $\rho^+$ and $\rho'^+$ are the equilibrium states corresponding to the interactions K and K' respectively then

(5.7) $\rho'^+(\widehat{DA})=\rho^+(\hat{A})$ any $A\in X_f$

(5.8) $\rho'^+(\hat{A})=0$ if $A\notin(D)$

where $(D)=\{DA;A\in X_f\}$

Proof:The proof of corresponding theorem from [4] will be followed exactly.Some steps will be just sketched.

Let for $A\in X_f$ $A'=DA$ $\mathcal{B}_A=\{aA;a\in Z^\nu\}$ and let $K_A$ be the interaction with bonds $\mathcal{B}_A$ such that $K_A(B)=1$ for any $B\in\mathcal{B}_A$

Step 1

(5.9) $P(K+\lambda K_A)=P(K'+\lambda K_{A'})$ $\lambda\in R$ (equality of pressures)

The formula (5.7) follows from the fact that

$\rho^+(\hat{A})=\lim_{\lambda\to 0^+}\lambda^{-1}(P(K+\lambda K_A)-P(K))$

If $A\notin(D)$ then there is $l,1\le l\le k$ such that $A^l\notin(D^l)$ in $Z_{p_l}[Z^\nu]$

$((m/p_l)D^l)^{p_l^n}=\sum_{a\in Z^\nu}(m/p_l)D^l(a)p_l^n a$ in $Z_{p_l}[Z^\nu]$ (B)

Since $\rho'^+$ is a translation invariant and clustering it follows that

(5.10) $\lim_{n\to\infty}\rho'^+(\widehat{((m/p_l)D^l)^{p_l^n}A})=\prod_{a\in\underline{D}^l}\rho'^+(\widehat{(m/p_l)D^l(a)A})$

On the other hand, obtained by( 5.7) is

$\rho'^+(\widehat{((m/p_l)D^l)^{p_l^n}A})=\rho'^+(\widehat{D^{p_l^n}(m/p_l)A^l})=$

$=\rho^+(\widehat{D^{p_l^{n-1}}(m/p_l)A^l})=\rho^+(\widehat{((m/p_l)D^l)^{p_l^{n-1}}A})$

so to prove( 5.8) it is enough to show that

(5.11) $\rho^{+}(\widehat{((m/p_1)D^1)^{p_1^{n-1}}A})\to 0$ as $n\to\infty$

and use (2.11) to conclude that if $\rho'^{+}(\widehat{kA})=0$; k is a natural number then $\rho'^{+}(\hat{A})=0$

Step 2

$\mathrm{Card}\underline{(D^1)^{p_1^{n-1}}A^1}\to\infty$ as $n\to\infty$

where multiplication is performed in $Z_{p_1}[Z^{\nu}]$.

Step 3 consists of the lemma which is true in a more more general setting:G is any finite abelian group,L is any $Z^{\nu}$ invariant lattice.

Lemma 5.5

For any ferromagnetic,finite range interaction and for any sequence $A_n\in X_f$ with $\underline{A}_n\to\infty$ as $n\to\infty$ $\rho^{+}(\hat{A}_n)\to 0$ as $n\to\infty$.

This finishes the proof of Theorem 5.4 ∎

Let now K be any ferromagnetic,translation invariant interaction of finite range and let D=g.c.d.($\mathbf{B}$). Let K' be the interaction with bonds $\mathbf{B}'$ obtained by factoring out D from the bonds of $\mathbf{B}$:

$\mathbf{B}=\{DB;B\in\mathbf{B}'\}$ and such that $K'(B)=K(DB)$

Obviously g.c.d.($\mathbf{B}'$) is a unit.Applying Theorem 5.4 and then Theorem 4.12 to the system with interaction K'-the reduced system, the following is obtained:

Theorem 5.6

For any temperature

(5.12) $\rho^{+}(\widehat{DA})=\rho'^{+}(\hat{A})$ for any $A\in X_f$

(5.13) $\rho^{+}(\hat{A})=0$ if $A\notin(D)$

where $\rho^{+},\rho'^{+}$ correspond to K and K' respectively. In particular, for low enough temperatures, $\mathbf{B}^{+}=(D)$.

## 5.2 $G=Z_{p^n}$; P IS A PRIME NUMBER

Let $\mathbf{A}$ be an ideal in $Z_m[Z^\nu]$, m any natural number and let $\mathbf{B}_0=\{B_1,\ldots,B_n\}$ be a finite family of its generators ($Z_m[Z^\nu]$ is a Noetherian ring so there is at least one such family). The lattice system with $\mathbf{B}_o$ as a fundamental family of bonds has the decomposition property if it has this property with respect to any other finite family of generators of $\mathbf{A}$. To see this it is enough to combine Theorem 4.9(b) and the fact that $S=\mathbf{A}^{\perp}$. If this is the case then $\mathbf{A}$ satisfies the decomposition property or simply $\mathbf{A}$ is reduced.

Proposition 5.7

Let $\mathbf{A}_1\subset\mathbf{A}_2$ be the two ideals in $Z_p[Z^\nu]$, p-prime; then if $\mathbf{A}_1$ is reduced then $\mathbf{A}_2$ is reduced.

Proof: If $\mathbf{A}_1$ is reduced then by Theorem 5.3 g.c.d. of its generators is a unit. Then g.c.d. of the generators of $\mathbf{A}_2$ is also a unit so $\mathbf{A}_2$ is reduced by the same theorem. ∎

Let $\mathbf{A}$ be the ideal in $Z_{p^n}[Z^\nu]$, p-prime and $\mathbf{A}^p\equiv\{A\in\mathbf{A};pA=0\}$

$S(\mathbf{A}^p)\equiv(\mathbf{A}^p)^{\perp}$ $S\equiv\mathbf{A}^{\perp}$

Proposition 5.8

$S(\mathbf{A}^p)=pX+S$ where $X=\times_{i\in Z^\nu} Z_{p^n}$

Proof: Let $Y=X/S$ so $\hat{Y}\approx\mathbf{A}$ It will be shown that $\mathbf{A}^p=(pY)^{\perp}$. Really for every $y\in Y$ and $A\in\mathbf{A}$

$\langle pA,y\rangle=\langle A,py\rangle$ Let $A\in(pY)$ then $\langle pA,y\rangle=1$ for every $y\in Y$ so $pA=0$ by Theorem A1 hence $A\in\mathbf{A}^p$. Conversely if $A\in\mathbf{A}^p$ then $\langle A,py\rangle=1$ for every $y\in Y$ so $A\in(pY)^{\perp}$.

Now let $f:X\to Y$ be a canonical homomorphism and $(\mathbf{A}^p)^{\perp}$ the annihilator of $\mathbf{A}^p$ in $X$ then

$(\mathbf{A}^p)^{\perp}=f^{-1}(pY)$ and finally $(\mathbf{A}^p)^{\perp}=pX+S$ ∎

By Theorem 4.9(b) decomposition property holds if and only if $\mathrm{cl}\Gamma_f=\Gamma_f$

This is equivalent to the implication

$\gamma(x)$ is finite $\to$ $x=y+s$ $x\in X, y\in X_f, s\in S$

Now it will be proven for $Z_p\,[Z^\nu]$ case the following:

Lemma 5.9

If $\mathbf{A}^p$ and $p\mathbf{A}$ are reduced then $\mathbf{A}$ is reduced.

Proof: Let $\gamma(x)$ be finite then obviously $\gamma^p(x)$ is finite where $\gamma^p(x)$ is a contour for $\mathbf{A}^p$ system for fixed choice of generators of $\mathbf{A}^p$ as the bonds.

By assumption and Prop.5.8 $x=y_1+s_1+px_1$ where $y_1\in X_f$ $s_1\in S$ $x_1\in X$

By the general assumption (cf.2.4) there is a bond in $\mathbf{A}$ system whose order is equal to $p^n$.Now it is easy to see that $\gamma_p(x)$ is finite where $\gamma_p$is a contour for $p\mathbf{A}$ system.
Hence $x_1=y_2+s_2$ where $y_2\epsilon X_f$;
$s_2\epsilon S(p\mathbf{A})\equiv(p\mathbf{A})^\perp$ and obviously $ps_2\epsilon S$
Finally $x=y_1+py_2+s_1+ps_2$ and hence $\mathbf{A}$ is reduced. ∎

Theorem 5.10

A is reduced if and only if $p^{n-1}\mathbf{A}$ is reduced.

Proof: the if part

Let us introduce $\mathbf{A}^{p^k}\equiv\{A\epsilon p^{k-1}\mathbf{A};pA=0\}$ $k=1,\ldots,n-1$

(5.14) $p^{n-1}\mathbf{A}\subset\mathbf{A}^{p^{n-1}}\subset\mathbf{A}^{p^{n-2}}\subset\ldots\subset\mathbf{A}^p$

It is easy to see that $p^{n-1}\mathbf{A}$ and $\mathbf{A}^{p^k}$ are reduced in $Z_{p^n}[Z^\nu]$ if and only if$(1/p^{n-1})[p^{n-1}\mathbf{A}]$ and $(1/p^{n-1})\mathbf{A}^{p^k}$are reduced in $Z_p[Z^\nu]$

Now if $p^{n-1}\mathbf{A}$ is reduced then by (5.14) and Proposition 5.7 $\mathbf{A}^{p^k}$ is reduced for every $k=1,..,n-1$

$p^{n-1}\mathbf{A},\mathbf{A}^{p^{n-1}}\subset p^{n-2}\mathbf{A}$ so by Lemma 5.9 $p^{n-2}\mathbf{A}$ is reduced.

$p^{n-1}\mathbf{A},\mathbf{A}^{p^{n-1}}\subset p^{n-k-1}\mathbf{A}$ $k=2,\ldots,n-1$ so using the lemma n-2 times more $\mathbf{A}$ is reduced.

the only if part (by contraposition)

Assume that $p^{n-1}B=p^{n-1}(B/D)D$ for $B\in\mathbf{B}_0$-family of generators of $\mathbf{A}$ where $B/D;D\in Z_p[Z^\nu]$ and D is not a unit. Let

(5.15) $\quad \gamma_0'=p^{n-1}\sum_{B\in\mathbf{B}_0}(I(B)/I(D))\alpha_B$

$\gamma_0'\notin\Gamma_f$ because otherwise for $B\in\mathbf{B}_0$ such that $|B|=p^n$

(5.16) $\quad p^{n-1}(I(B)/I(D))=RI(B) \quad$ in $Z_{p^n}[Z^\nu]$

where $R\in Z_{p^n}[Z^\nu]$

Now it is easy to see (cf. the proof of Proposition B2) that $R=p^{n-1}R';R'\in Z_p[Z^\nu]$ so

(5.17) $\quad p^{n-1}(I(B)=p^{n-1}R'I(B)I(D)$

and D is a unit in $Z_p[Z^\nu]$.

$\gamma_0'\in\mathrm{cl}\Gamma_f$ Really if $f\in(\Gamma_f)^0$ then

(5.18) $\quad I(D)f(\gamma_0')=f(I(D)\gamma_0')=f(p^{n-1}\gamma_0)=0$

so $f(\gamma_0')=0$ because I(D) is not a zero divisor as an element of $Z_{p^n}[Z^\nu]$. By Theorem 4.9(b) $\mathbf{A}$ is not reduced.∎

## 5.3 $G=Z_m$; m ARBITRARY

The criterion for the decomposition property to hold in $G=Z_m$ case, m arbitrary, is provided here.

Let $m=\prod_{i=1}^{n}p_i^{k_i}$ $p_i$-prime numbers. If $x\in X$ then

$x=\sum_{i=1}^{n}(m/p_i^{k_i})x_i \;;\; x_i\in \times_{j\in L} Z_{p_i^{k_i}}$

$\mathbf{B}_i=\{(m/p_i^{k_i})B;B\in\mathbf{B}\}$ where $\mathbf{B}$ is the family of bonds.

If $\gamma(x)$ is a contour with respect to $\mathbf{B}$ then $\gamma_i(x)$ is a contour with respect to $\mathbf{B}_i$.

Lemma 5.11

$\gamma(x)\in\mathrm{cl}\Gamma_f$ iff for any i $\gamma_i(x)\in\mathrm{cl}\Gamma_{fi}$.

Proof: It follows easily from the properties of prime numbers.∎

Theorem 5.12

The system $\mathbf{B}$ has the decomposition property iff $\mathbf{B}_i$ has the decomposition property for every i.

Proof: the if part

Assume that $\gamma(x)$ is finite then by Lemma 5.11 $\gamma_i(x)$ is finite $i=1,\ldots,n$. Because $\mathbf{B}_i$ is reduced so there is $y^i\in X_f$ such that $\gamma_i(y^i)=\gamma_i(x)$ and we can choose $(y^i)_j=0$ if $i\neq j$.

Let $y=\sum_{i=1}^{n} y^i \in X_f$

$\gamma(y)=\gamma(x)$ hence $\mathbf{B}$ is reduced.

the only if part

Assume that $\gamma_i(x)$ is finite then $\gamma((m/p_i^{k_i})x_i)$ is finite so there is $y\in X_f$ such that

$\gamma((m/p_i^{k_i})x_i)=\gamma(y)$ and finally

$\gamma_i(x)=\gamma_i(y)$ ∎

$\mathbf{B}_i$ system can be investigated by means of Theorem 5.10. Theorem 5.12 is obviously true in general $Z^{\nu}$-invariant lattices.However, for $Z_p n$ group the characterization of the decomposition property is not available.This problem will be addressed again in Chapter VII.

## Chapter VI

## GENERAL $Z^\nu$-INVARIANT LATTICES

This is a direct generalization of [5]. The expression for $\mathbf{B}^+$ will be found here.First the reduced system (L'K') in which decomposition property holds is constracted. $\mathbf{B}'^+$is found and then by a homomorphism used in the construction of reduced system $\mathbf{B}^+$can be obtained.By Theorem 2.1 all extremal, periodic Gibbs states are found.

### 6.1 REDUCTION

Let (L,K),(L',K') be two lattice systems. Let $\phi$ be a continuous homomorphism of X to X' and let $\hat{\phi}:X_f'\to X_f$ be its dual.$\phi$ is a morphism from (L,K) to (L',K') if it commutes with the action of $Z^\nu$ and satisfies the following three conditions:

(1) $\phi(X_f)\subset X_f'$

(2) $\phi$ intertwines the interactions K and K': $\hat{\phi}$ yields

a bijection $\mathbf{B}'\to\mathbf{B}$ such that $|B'|=|\phi(B')|$ and $K(\phi(B))=K'(B)$ for any $B\in\mathbf{B}'$

(3) If $\phi$ : $M\to M'$ is the isomorphism induced by $\hat{\phi}$ (namely,if $\phi(\alpha)=\alpha'$ then $\alpha(\hat{\phi}(B))=\alpha'(B)$ for any $B\in\mathbf{B}'$) then $\phi(\Gamma_f)$ is dense in $\Gamma_f'$.

Theorem 6.1 [5]

Let (L,K) and (L',K') be two ferromagnetic lattice systems and let $\phi$ be a morphism from (L,K) to (L',K').If the system (L',K') has decomposition property then:

(6.1) $\quad \phi\rho^{+}=\rho'^{+}$ hence

$$\rho'^{+}(\hat{A})=\rho^{+}(\widehat{\hat{\phi}(A)}) \quad \text{for all } A\in X_f$$

(6.2) $\quad \rho^{+}(\hat{A})=0 \quad \text{if } A\notin\hat{\phi}(\mathrm{cl}\mathbf{A}')$

Proof:If $x\in S_f$ then for any $B'\in\mathbf{B}'$

$\langle\phi(x),B'\rangle=\langle x,\hat{\phi}(B')\rangle=1$ so $\phi(x)\in S_f'$ hence $\phi(S_f)\subset S_f'$ and therefore $\phi(\mathrm{Cl}S_f)\subset\mathrm{Cl}S_f'$.

Let $Y=X/\mathrm{Cl}S_f \quad Y'=X'/\mathrm{Cl}S_f'$

and let $\psi : Y\to Y'$ be the homomorphism induced by $\phi$.Now it will be shown that $\psi$ is surjective.Surjectivity of $\psi$ is equivalent to injectivity of the dual map $\hat{\psi}:\widehat{Y'}\to\hat{Y}$. On the other hand,since $Y=X/\mathrm{Cl}S_f$,$\hat{Y}$ is identified with $(\mathrm{Cl}S_f)^{\perp}$ which is equal to $\mathrm{cl}\mathbf{A}$ by Prop.3.3(d).Thus surjectivity of $\psi$ is equivalent to injectivity of the restriction of $\hat{\phi}$ to $\mathrm{cl}\mathbf{A}'$.First it will be shown that $\phi(K)\subset K'$. For $\alpha\in M_f,\beta\in M$

(6.4) $\quad \langle\alpha,\phi(\beta)\rangle=\langle\phi^{-1}(\alpha),\beta\rangle$

Because (L',K') has the decomposition property then by Theorem 4.9 $\mathrm{Cl}K_f=K$.Let $\beta\in K$ then for every $\alpha\in\Gamma_f'$ $\langle\phi^{-1}(\alpha),\beta\rangle=1$ so $\langle\alpha,\phi(\beta)\rangle=1$ for every $\alpha\in\Gamma_f'$ hence $\phi(\beta)\in K'$.The injectivity of the restriction of $\hat{\phi}$ to $\mathrm{cl}\mathbf{A}'$ follows now easily.

Since $\rho'^{+}$ is $ClS_f^{!}$ invariant ($ClS_f^{!}=S'^{+}$) and $cl\mathbf{A}'=(ClS_f^{!})^{\perp}$, to prove (6.1) it is enough to show that there exists a constant c such that for any $A\epsilon cl\mathbf{A}'$, $\lambda>0$

(6.5) $\quad P(K'+\lambda K_A)=c+P(K+\lambda K_{\hat{\phi}(A)})$

Because $\hat{\phi}(A)\epsilon cl\mathbf{A}$, $K+\lambda K_{\hat{\phi}(A)}$ and $K'+\lambda K_A$ define the factor interactions on Y and Y' respectively. These factor interactions are mapped one onto another by $\psi$. Hence (6.5) follows from the results on the transport of pressure under surjective maps [15].

To prove (6.2) it is enough to show that

$\rho^{+}(\hat{A})=0$ if $A\epsilon cl\mathbf{A}-\hat{\phi}(cl\mathbf{A}')$

$\mathbf{B}^{+}\subset cl\mathbf{A}$ always and it is an easy consequence of the fact that $cl\mathbf{A}=(ClS_f)^{\perp}$ and $\rho^{+}$ is $ClS_f$ invariant.

Now, there exists $D\epsilon Z_m[Z^{\nu}]$, $D\neq 0$ such that $Dcl\mathbf{A}\subset\mathbf{A}$ and D is not a zero divisor (Theorem C11). The proof of Theorem 5.4 is now followed.

Let $A\epsilon cl\mathbf{A}-\hat{\phi}(cl\mathbf{A}')$ $\quad m=\prod_{l=1}^{K} p_l^{m_l}$; $p_l$ are all different primes so

$Z_m\simeq \times_{l=1}^{k} Z_{p_l^{m_l}}$ and $D=\sum_{l=1}^{k}(m/p_l^{m_l})D^l$ where $D^l\epsilon Z_{p_l^{m_l}}[Z^{\nu}]$.

Let $A_n^l=[(m/p_l^{m_l})D^l]^{p_l^{nm_l}}A \qquad A_n^l\epsilon\mathbf{A}$

Let $A_n'^l\ \epsilon\mathbf{A}'$ be such that $\hat{\phi}(A_n'^l)=A_n^l$

Now there is $r\epsilon\ Z_{p_l^{m_l}}$ such that there is a sequence of natural numbers $i_n$ such that

$$(m/p_l^{m_l})^{(p_l^{m_l})^{i_n}}=r(m/p_l^{m_l})^{i_n}\ (\mathrm{mod}\ m)$$

and

$$((m/p_l^{m_l})D^l)^{(p_l^{m_l})^{i_n}}=\sum_{j=1}^{t}r(m/p_l^{m_l})D_j^l g_j^{(n)}$$

where t does not depend on n if n is large enough, $g_j^{(n)} \in Z^\nu$ $\mathrm{dist}(g_i^{(n)}, g_j^{(n)}) \to \infty$ as $n \to \infty$ if $i \neq j$; $p_1$ does not divide at least one $D_j^1 \neq 0$, say $D_k^1$ (Lemma B1).

As in Theorem 5.4 we get

$$\lim_{n\to\infty} \rho^+(\widehat{((m/p_1^m 1)D^1)(p_1^m 1)^{i_n}} A) = \prod_{j=1}^{t} \rho^+(\widehat{r(m/p_1^m 1)D_j^1 A})$$

$$\rho^+(\widehat{((m/p_1^m 1)D^1)(p_1^m 1)^{i_n}} A) = \rho^+(\hat{\phi}(\widehat{A_{i_n}^{1\prime}})) = \rho'^+(\widehat{A_{i_n}^{1\prime}}) \quad \text{by (6.1).}$$

As before (2.11) can be used to show that if $\rho^+(\widehat{kA})=0$ then $\rho^+(\hat{A})=0$; k is a natural number. Now to finish the proof it is enough to show that card $\underline{A}_n^{1\prime} \to \infty$ as $n \to \infty$.

Therem 4.9(b) applied to $\mathrm{cl}\mathbf{A}'$ in place of $\Gamma_f$ shows that $\mathrm{cl}\mathbf{A}'$ has the decomposition property; i.e., there exists N such that for any $A \in \mathrm{cl}\mathbf{A}'$ its N-components belong to $\mathrm{cl}\mathbf{A}'$. Let $C_i; i \in I$ be the N-components of $A_n^{1\prime}$. It can be assumed that N is so large that for any $x, y \in X_f^{!}$ such that $\delta(\underline{x}, \underline{y}) \geq N$, $\underline{\hat{\phi}(x)} \cap \underline{\hat{\phi}(y)} = \Phi$. Then $\underline{\hat{\phi}(C_i)} \cap \underline{\hat{\phi}(C_j)} = \Phi$ $i \neq j$ $\bigcup_i \hat{\phi}(c_i) = A_n^1$

$$A_n^1 = \bigcup_{j=1}^{t} (m/p_1^m 1) D_j^1 A_j \quad \text{where } A_j = g_j^{(n)} A$$

Let $\quad I(j) = \{ i \in I; \underline{\hat{\phi}(C_i)} \cap \underline{A_j} \neq \Phi \}$

$\quad I(k) \cap I(j) = \Phi$ for each $1 \leq j \leq t$ $k \neq j$ then

$(m/p_1^m 1) D_k^1 g_k^{(n)} A = \sum_{i \in I(k)} \hat{\phi}(C_i)$ and obviously there is $r \in Z_m$ such that

$(m/p_1^m 1) D_k^1 r = 1 \pmod m$ so finally

$$g_k^{(n)} A = \sum_{i \in I(k)} \hat{\phi}(rC_i) \in \hat{\phi}(\mathrm{cl}\mathbf{A}')$$

contrary to what was assumed. Hence there exists $i_0 \in I(k)$ and $j; 1 \le j \le t$ such that $\hat{\phi}(C_{i_0})$ intersects both $\underline{A}_j$ and $\underline{A}_k$. It follows that $\hat{\phi}(C_{i_0})$ has diameter which goes to infinity as $n \to \infty$. On the other hand it is not hard to see that because of translation invariance there exists a constant c such that for any $a, b \in L'$ and any $x, y; \underline{x} = \{a\}, \underline{y} = \{b\}$

$\delta(\hat{\phi}(x), \hat{\phi}(y)) \le c\,\mathrm{dist}(a,b)$.

Therefore the diameter of $C_{i_0}$ goes to infinity as $n \to \infty$. Since $C_{i_0}$ is N-connected, card $\underline{C}_{i_0}$ goes to infinity as $n \to \infty$ and finally card $\underline{A}_n^{1}{}' \to \infty$ as $n \to \infty$. ∎

In particular, Theorem 6.1 says that $\mathbf{B}^+ = \hat{\phi}(\mathbf{B}'^+)$ and equivalently $S^+ = \phi^{-1}(S'^+)$. Since the system $(L', K')$ has the decomposition property, at low enough temperatures $\mathbf{B}^+ = \hat{\phi}(\mathrm{cl}\mathbf{A}')$. From Proposition C8 we get that $\hat{\phi}$ is an isomorphism between $\mathrm{cl}\mathbf{A}'$ and $\mathbf{A}^{**}$ (It will be proved later (Theorem C18) that $i^{**}: \mathbf{A}^{**} \to X_f$ is injective hence $\mathbf{A}^{**}$ can be identified with its image in $X_f$.). Finally $\mathbf{B}^+ = \mathbf{A}^{**}$ at low temperatures so the number of pure phases is equal to $|\mathbf{A}^{**}/\mathbf{A}|$ at low temperatures.

## 6.2 CONSTRUCTION OF THE REDUCED SYSTEM

Proposition 6.2

Let $\phi : \bigoplus_{i\in L} G \to \bigoplus_{i\in L'} G'$ be a $Z_m[Z^\nu]$-module homomorphism then $\phi$ extends to a unique continuous module homomorphism

$\phi : \times_{i\in L} G \to \times_{i\in L'} G'$

Proof: It is enough to consider the case $L,L'=Z^\nu$. The proposition follows from the following fact: for any finite subset $A\subset L'$ there exists a finite subset $B\subset L$ such that for any $x\in \bigoplus_{i\in Z^\nu} G$ $\underline{x}\cap B=\Phi$ we have

$\underline{\phi(x)}\cap A=\Phi$ (continuity at $e_X$).

Let $y_k\in \bigoplus_{i\in Z^\nu} G$ $\underline{y}_k=\{0\}$

$y_k(0)$ is a generator of G and k runs over all generators of G in decomposition of G into cyclic groups.

For a finite $A\subset Z^\nu=L'$

let $B=\{a\in Z^\nu; (a\cup_k \underline{\phi(y_k)})\cap A\neq\Phi\}$.

Clearly, B has the required property. ∎

Let $(L,K), k\neq 0$ be any lattice system. The reduction of $(L,K)$ can be constructed as follows.

Let F be a finite family of generators of $cl\Gamma_f$ and let $G'=\bigoplus_{\alpha\in F} Z_{|\alpha|}$ where $|\alpha|$ is the order of $\alpha$ in M. Such a family exists since $cl\,\Gamma_f$ is a submodule of a finitely generated module $M_f$ over a Noetherian ring $Z_m[Z^\nu]$ (Theorem C1).

Let $X'=\times_{i\in Z^\nu} G'$ and let

$\eta : X_f^{\prime} \to cl\Gamma_f$ be the $Z_m[Z^\nu]$ homomorphism which extends the inclusion map $\{y_k^{\prime}\} \to cl\Gamma_f$ where $y_k^{\prime}$ are from Prop.6.2.By Prop.6.2 $\eta$ has unique extension to a continuous homomorphism $\eta : X' \to M$

Proposition 6.3

$\eta(X') = \gamma(X)$

Proof:By construction $\eta(X') = cl\Gamma_f$ Let $\alpha = \gamma(x)$ $x = \sum_i x_i$ $x_i \in X_f$ $\underline{x}_i \cap \underline{x}_j = \Phi$ $i \neq j$ $\gamma(x) = \sum_i \gamma(x_i)$.Let $x_i^{\prime}$ be such that $\eta(x_i^{\prime}) = \gamma(x_i)$ then $\eta(\sum_i x_i^{\prime}) = \gamma(x)$ ∎

$\hat{\eta}$ denotes the dual map $\hat{\eta} : M_f \to X_f^{\prime}$ define

$B' = \hat{\eta}(\alpha_B)$;$B \in \mathbf{B}$, $\alpha_B(A) = \delta_{A,B}$, $A \in \mathbf{B}$

$\mathbf{B}' = \{B' ; B \in \mathbf{B}\}$

The interaction $K'$ on $\mathbf{B}'$ is defined by $K'(B') = K(B)$ for any $B \in \mathbf{B}$.

Proposition 6.4

The order of $B'$ is equal to the order of $B$ for any $B$ from $\mathbf{B}_0$.

Proof:Obviously $|B'|$ divides $|B|$.For $s, 0 < s < |B|$ there is $x \in X$ such that $\langle x, \varepsilon(s\alpha_B) \rangle \neq 1$.

$\langle x, \varepsilon(s\alpha_B) \rangle = \langle \gamma(x), s\alpha_B \rangle = \langle \eta(x'), s\alpha_B \rangle =$

$= \langle x', s\hat{\eta}(\alpha_B) \rangle = \langle x', sB' \rangle$

where $x' \in X'$ so $|B'| \neq s$ and finally $|B'| = |B|$ ∎

Proposition 6.5

$B \to B'$ is a bijection of $\mathcal{B}$ onto $\mathcal{B}'$.

Proof:Let $\hat{\eta}(\alpha_{B_1} = \hat{\eta}(\alpha_{B_2})$ $B_1, B_2 \in \mathcal{B}$ then for any $x' \in X'$

$<x', \hat{\eta}(\alpha_{B_1})> = <x', \hat{\eta}(\alpha_{B_2})>$ so

$<\eta(x'), \alpha_{B_1}> = <\eta(x'), \alpha_{B_2}>$ so

$<\gamma(x), \alpha_{B_1}> = <\gamma(x), \alpha_{B_2}>$ for any $x \in X$ by Prop. 6.3

so $\hat{B}_1(x) = \hat{B}_2(x)$ for any $x \in X$ hence $B_1 = B_2$ ∎

Proposition 6.6

$\Gamma_f'$ is isomorphic to $\mathrm{cl}\Gamma_f$

Proof:Proposition follows from the equations:

$$<x', \hat{\eta}(\alpha_B)> = <\eta(x'), \alpha_B> = <\gamma(x), \alpha_B> =$$
$$= <x, \varepsilon(\alpha_B)> = <x, B> \quad \text{where } x \in X, x' \in X_f'$$ ∎

By Theorem 4.9(b) the system $(L', K')$ has the decomposition property.Take $\{y_k^i\}_{i \in L}$ from Prop.6.2 ($y_k$ for each copy of $Z^\nu$in L)

Let $\{\gamma(y_k^i), \beta_j\}$ be the family of generators of $\mathrm{cl}\Gamma_f$.
Homomorphism $\phi : X_f \to X_f'$ is constructed to make the following diagram to commute.

$$\begin{tikzcd} X_f \arrow[r, "\gamma"] \arrow[d, "\phi"'] & \Gamma_f \arrow[d, "\text{inclusion}"] \\ X'_f \arrow[r, "\eta"'] & cl\Gamma_f \simeq \Gamma'_f \end{tikzcd} \qquad \eta\phi=\gamma$$

Let $\phi : X \to X'$ be the extension which exists by Prop.6.2.

$\phi$ defines a reduction of $(L,K)$

$\phi(X_f) \subset X'_f$ is satisfied by construction.

$\eta\phi=\gamma$ $\hat{\gamma}=\varepsilon_{|M_f}$ so $\hat{\phi}\hat{\eta}=\varepsilon$ hence $\hat{\phi}$ yields the bijection $B' \to B$ as needed.Obviously $\phi(\Gamma_f)$ is dense in $\Gamma'_f$.

## 6.3 UNIQUENESS OF REDUCTION

Determination of $B^+$ and $S^+$ in Theorem 6.1 is non-unique.Though $B^+$ depends on L,B only, $\hat{\phi}(cl\mathbf{A}')$ depends on the generally non-unique choice of L' and $\phi$.However,there is a natural reduction in the transitive case ($L=Z^\nu$) and $G=Z_m; m=\prod_{l=1}^{\kappa} p_l$ where $p_l$ are different prime numbers.The following proposition is true for arbitrary m.

Proposition 6.7

$\Gamma_f$ is a free $Z_m[Z^\nu]$-module generated by one element.

Proof:Let $\gamma_0=\gamma(x^0)$ where $x^0(a)=\delta_{0,a}$ $0,a\in Z^\nu$ $\gamma_0$ generates $\Gamma_f$.If $R\in Z_m[Z^\nu]$, $R\neq 0$, then $R\gamma_0=\gamma(R)\neq 0$ because by Theorem 4.14 $\gamma$ is injective when restricted to $X_f$. ∎

For m described as above is

Proposition 6.8

$cl\Gamma_f$ is a free $Z_m[Z^\nu]$-module generated by one element.

Proof: $\gamma_0=\sum_{B\in\mathbf{B}_0}(|B|/m)I(B)\alpha_B$ (cf.5.1) Now introduced is

$\gamma'_0=\sum_{B\in\mathbf{B}_0}(|B|/m)I(B/D)\alpha_B$

where $D=$g.c.d.$(\mathbf{B}_0)$ and D is not a zero divisor. To show that

$\gamma'_0\in cl\Gamma_f$ let $f\in(\Gamma_f)^0$ then

$I(D)f(\gamma'_0)=f(I(D)\gamma'_0)=f(\gamma_0)=0$

hence $f(\gamma'_0)=0$. Let $\alpha\in cl\Gamma_f$

$\alpha=\sum_{i=1}^{n}(|B_i|/m)P_i\alpha_i$ so $\langle\beta_{ij},\alpha\rangle=0$ (see 5.2 for notation)

hence as before

(6.6) $P_iI(B_j)=P_jI(B_i)$ $1\le i<j\le n$

(6.7) $P_i^l(I(B_j^l)/I(D^l))=P_j^l(I(B_i^l)/I(D^l))$

in $Z_{p_l}[Z^\nu]$ for every $l$ $1\le l\le k$ so

$P_i^l=P_j^l\equiv P^l$ hence

(6.8) $P_i^l=P^l(I(B_i^l)/I(D^l))$

(6.9) $\alpha=\sum_{i=1}^{n}(|B_i|/m)\sum_{l=1}^{k}(m/p_l)P^l(I(B_i^l)/I(D^l))\alpha_i$ so

(6.10) $\alpha=P\sum_{i=1}^{n}(|B_i|/m)(I(B_i)/I(D))\alpha_i=P\gamma'_0$ where

$P=\sum_{l=1}^{k}(m/p_l)P^l$

Hence $cl\Gamma_f$ is generated by $\gamma_0^{\prime}$. To see that $cl\Gamma_f$ is free let $R\epsilon Z_m[Z^{\nu}]$. $R\gamma_0^{\prime}=0$ means $RI(D)\gamma_0^{\prime}=R\gamma_0=0$ so $R=0$. ∎

The conclusion is that in the transitive case and $G=Z_m$ with m described as above, there exists a unique transitive lattice L' which gives reduction. Generally the following proposition can be proved:

Proposition 6.9

$cl\Gamma_f$ is a free module if and only if the reduced system is non-gauge.

Proof: $cl\Gamma_f$ is a free module if and only if $\eta$ is injective on $X_f^{\prime}$. Proposition follows now from the equality:

$<\eta(x'),\alpha_B>=<x',\hat{\eta}(\alpha_B)>=<x',B'>$ where $x'\epsilon X_f^{\prime}$ $B\epsilon\mathbf{B}$ ∎

## 6.4 EFFECTIVE WAY OF REDUCTION

The main concern here is to find generators of $cl\Gamma_f$ in the cases where there is no unique reduction like in 6.3. First is discussed the case of $L=Z^{\nu}, G=Z_{p^2}$. By Proposition 6.8 $cl\Gamma_f^{A^p}$ is generated by $\gamma^{A^p}(x)$ and $cl\Gamma_f^{pA}$ is generated by $\gamma^{pA}(y)$ where $x,y\epsilon \underset{i\epsilon Z^{\nu}}{\times} Z_p$ and both modules are $Z_p[Z^{\nu}]$-modules.

Lemma 6.10

If there is a generator of $cl\Gamma_f^{A^p}$ $\gamma^{A^p}(x)$ such that $\gamma(x)$ is finite than $cl\Gamma_f$ is generated by $\gamma(x)$ and $\gamma(py)$ where

$\gamma(y)$ is a generator of $cl\Gamma_f^{pA}$.The coefficients can be taken from $Z_p[Z^\nu]$.

Proof:Let $\gamma(z)$ be finite so $\gamma^{A^p}(z)$ is finite hence

$\gamma^{A^p}(z)=R\gamma^{A^p}(x)$ ; $R\epsilon Z_p[Z^\nu]$

By Proposition 5.8

$z=Rx+s_1+px_1$ ; $s_1\epsilon S(A),x_1\epsilon X$

Because $\gamma(px_1)$ is finite then $\gamma^{pA}(x_1)$ is finite so

$\gamma^{pA}(x_1)=P\gamma^{pA}(y)$ ; $P\epsilon Z_p[Z^\nu]$ hence

$x_1=Py+s_2$ ; $s_2\epsilon S(pA)$ finally

$z=Rx+Py+s_1+ps_2$ so

$\gamma(z)=R\gamma(x)+P\gamma(py)$ ∎

The following two propositions describe the cases when the assumption of Lemma 6.10 is satisfied.

Proposition 6.11

If $A^p$ is reduced then $cl\Gamma_f$ is generated by $\gamma(x)$ and $\gamma(py)$ as in Lemma 6.10.

Proof:If $A^p$ is reduced then $cl\Gamma_f^{A^p}$ is equal to $\Gamma_f^{A^p}$ and is generated by $\gamma^{A^p}(x^0)$ where $x^0(a)=\delta_{0,a};0,a\epsilon Z^\nu$. ∎

Proposition 6.12

If $pA$ is the principal ideal in $Z_p\ [Z^\nu]$ then $cl\Gamma_f$ is generated by $\gamma(x)$ and $\gamma(py)$ as in Lemma 6.10.

Proof: Let $pA$ be generated by $pA$. By Proposition 6.8 there is $z \in X$ such that $pA(z)=\exp\{2\pi i/p\}$, $pA_i(z)=1$ for any translate $pA_i$ of $pA$. Let $x_1$ be such that $\gamma^{A^p}(x_1)$ is a generator of $cl\Gamma_f^{A^p}$ There is finite $\Lambda$ such that if $B \in \mathcal{B}$, $pB \neq 0$ and $\underline{B} \cap \Lambda = \phi$ then $\gamma(x_1)(B)=p$. Now using the fact that $\Gamma$ is closed can be found $z_1 \in pX$ such that $\gamma(x_1+z_1)$ is finite. Let $x=x_1+z_1$ Obviously $\gamma^{A^p}(x)$ is a generator of $cl\Gamma_f^{A^p}$ ∎

In Chapter IX an example is given of the model which does not satisfy the assumption of Lemma 6.10.

The above ideas can be easily applied to the case of an arbitrary finite abelian group on $Z^\nu$-invariant lattices.

# Chapter VII

## THE DECOMPOSITION PROPERTY FOR ARBITRARY LATTICES

Some necessary and sufficient conditions for the decomposition property to hold in the case of arbitrary $Z^{\nu}$-invariant lattices are described in this chapter. Because of Theorem 5.12 it is enough to consider the case $G=Z_{p^n}$ where p is a prime number.

Let $\mathbf{A}^{p^k}=\{A\epsilon p^{k-1}\mathbf{A};pA=0\}$ $k=1,...,n$.

The following theorem can be extracted from the proof of Theorem 5.10

Theorem 7.1

If $\mathbf{A}^{p^k}$ is reduced for any k then $\mathbf{A}$ is reduced.

Because $\mathbf{A}^{p^k}$ are $Z_p[Z^{\nu}]$-modules it is now enough to consider $Z_p$ case or equivalently

$G=\bigoplus_{i=1}^{r} Z_p$ on $Z^{\nu}$–invariant lattice.

Let A be a subgroup of $\bigoplus_{j\epsilon Z^{\nu}}\bigoplus_{i=1}^{r} Z_p$ generated by bonds of the system.

Now the configuration space is decomposed: $X=\bigoplus_{i=1}^{r} X_i$ and projections $P_i$ $i=1,...,r$ introduced such that

$P_i x=x_i$ where $x=\sum_{i=1}^{r} x_i$ $x_i\epsilon X_i$ $P_k^{\prime}=\sum_{i\leq k} P_i$.

For any subgroup H of $\bigoplus_{j\epsilon Z^{\nu}}\bigoplus_{i=1}^{k} Z_p$ ; $H_k=\{h\epsilon H;P_{k-1}^{\prime}h=0\}$

Proposition 7.2

$$S(\mathbf{A}_r)=S+\bigoplus_{i=1}^{r-1} X_i$$

Proof:Let $Y=X/S$ so $Y\cong\hat{\mathbf{A}}$.For every $y\in Y$ and $A\in\mathbf{A}$

(7.1) $\langle P'_{r-1}A, y\rangle=\langle A, P'_{r-1}y\rangle$

hence $(P'_{r-1}Y)^{\perp}=\mathbf{A}_r$

Let $f:X\to Y$ be a canonical homomorphism and $\mathbf{A}_r^{\perp}$ the annihilator of $\mathbf{A}_r$ in X ,then

(7.2) $S(\mathbf{A}_r)=f^{-1}(P'_{r-1}Y)$ and finally

(7.3) $S(\mathbf{A}_r)=\bigoplus_{i=1}^{r-1}X_i+S$ ∎

Lemma 7.3

If $\mathbf{A}_r$ and $P'_{r-1}\mathbf{A}$ are reduced then $\mathbf{A}$ is reduced.

Proof:Let $\gamma(x)$ be finite;$x\in X$ then $\gamma^r(x)$ is finite where $\gamma^r(x)$ is a contour corresponding to $\mathbf{A}_r$ system.Because $\mathbf{A}_r$ is reduced then by Theorem 4.9(b)

(7.4) $x=y+s_r$ where $y\in X_f$ and $s_r\in S(\mathbf{A}_r)$

By Prop.7.2:

(7.5) $s_r=s+x_1$

where $s\in S(A)$ and $x_1\in\bigoplus_{i=1}^{r-1}X_i$

$\gamma^{r-1}(x_1)$ is finite where $\gamma^{r-1}(x_1)$ is a contour corresponding to $P'_{r-1}\mathbf{A}$ system.

(7.6) $x_1=y_1+s_1$

where $y_1\in(\bigoplus_{i=1}^{r-1}X_i)_f, s_1\in S(P'_{r-1}\mathbf{A})$

where the annihilator is taken in $\bigoplus_{i=1}^{r-1}X_i$ obviously $s_i\in S$ and finally

(7.7) $x=y+y_1+s+s_1$

hence $\mathbf{A}$ is reduced. ∎

Using Lemma 7.3 r-1 times we get the following theorem

Theorem 7.4

If $[P_k^{\prime}\mathbf{A}]_k$ k=1,...,r are reduced then $\mathbf{A}$ is reduced.

$[P_k^{\prime}\mathbf{A}]_k$ are $Z_p$ systems on $Z^{\nu}$ so Theorem 5.3 can be applied to check if the decomposition property holds.

It was proven that for systems with the decomposition property $\mathbf{B}^+=\mathrm{cl}\mathbf{A}$ (Corollary 4.7) and $S^+=\mathrm{ClS}_f$(Corollary 4.8) at low temperatures.For spin 1/2 on general $Z^{\nu}$-invariant lattices there is the criterion for the above properties to hold:J.Slawny [5].It can be immediately generalized to $Z_p$ case where p is a prime number.

Theorem 7.5

The following conditions are equivalent:

(a) the greatest common divisor of the determinantal ideal $\Delta(A,X_f)$ is a unit

(b) $\mathbf{B}^+=\mathrm{cl}\mathbf{A}$ at low temperatures

(c) $S^+=\mathrm{ClS}_f$ at low temperatures.

# Chapter VIII

## ANALYTICITY OF PRESSURE AT LOW TEMPERATURES

Analyticity of pressure at low temperatures for spin 1/2 was shown by J.Slawny in [16] and for higher spins using Griffiths' representation in [3]. C.Gruber et. al. [8,9] attempted this for $Z_m$ case.They decomposed the system in a finite volume into subsystems. However, their subsystems did not have uniformly bounded size so they could not use the Ruelle theorem on zeros of partition function [18].This difficulty is removed here by a theorem on precise generators in certain modules:W.Holsztynski [20].

Consider the low temperature expansion for a partition function in the finite volume $\Lambda$. Standard transformations yield:

$$(8.1)\quad Z_\Lambda=|S_\Lambda|\prod_{B\in X_\Lambda}\exp\{\beta J(B)\}\sum_{\alpha\in\Gamma_\Lambda}\prod_{B\in\operatorname{supp}\alpha}\exp\{\beta J(B)[\cos(2\pi\alpha(B)/|B|)-1]\}$$

Let us introduce the reduced pressure:

$$(8.2)\quad p_\Lambda^0=(1/|\Lambda|)\log\sum_{\alpha\in\Gamma_\Lambda}\prod_{B\in\operatorname{supp}\alpha}\exp\{\beta J(B)[\cos(2\pi\alpha(B)/|B|)-1]\}$$

and the variables: $z_{B,\alpha(B)}$ $\quad z^\alpha\equiv\prod_{B\in\operatorname{supp}\alpha}z_{B,\alpha(B)}$

$$(8.3)\quad p_\Lambda^0=(1/|\Lambda|)\log\sum_{\alpha\in\Gamma_\Lambda}z^\alpha\Big|_{z_{B,\alpha(B)}=\exp\{\beta J(B)[\cos(2\pi\alpha(B)/|B|)-1]\}}$$

$\lim_{\Lambda\to L}(1/|\Lambda|)\log|S_\Lambda|$ and $\lim_{\Lambda\to L}(1/|\Lambda|)\sum_{B\in X_\Lambda}\beta J(B)$ exist so

$p^0=\lim_{\Lambda\to L}p_\Lambda^0$

Consider now the group of cycles. $K_f$ is a finitely generated and finitely supported $Z_m[Z^\nu]$-module (cf. Appendix C). By Theorem C13 there is a set of generators $\alpha_1,\dots,\alpha_m \in K_f$ such that if $\alpha \in K_f$ $\underline{\alpha} \subset \Lambda$ then $\alpha = \sum_{i=1}^{m} g_i \alpha_i$ where $g_i \in Z^\nu$ and $\underline{g_i \alpha_i} \subset \Lambda$. In other words, there exists $\Lambda_0$ such that for every finite large enough $\Lambda$, $K_\Lambda$ is generated by $\bigcup_i K_{\Lambda_i}$ where summation is over all translates of $\Lambda_0$ contained in $\Lambda$.

(8.4) $M(Z_{\mathbf{B}_{\Lambda}}) = \sum_{\alpha \in \Gamma_\Lambda} z^\alpha$

(8.5) $M(Z_{\mathbf{B}_{\Lambda_i}}) = \sum_{\alpha \in \Gamma_{\Lambda_i}} z^\alpha$

Proposition 8.1

$M(z_{\mathbf{B}_\Lambda})$ is the Asano contraction of $\{M(z_{\mathbf{B}_{\Lambda_i}})\}$ if and only if $K_\Lambda$ is generated by $\bigcup_i K_{\Lambda_i}$.

This is a straightforward generalization of Proposition 2.1 [16].

$M(z_{\mathbf{B}_{\Lambda_i}})$ are polynomials in number of variables independent of $\Lambda$ and with free term equal to 1. Hence there exists $r_0$ independent of $\Lambda$ such that $M(z_{\mathbf{B}_{\Lambda_i}}) \neq 0$ if $z_{B,\alpha(B)} < r_0$ for every $\alpha \in \Gamma_{\Lambda_i}$. On the other hand each $B \in \mathbf{B}$ is contained in no more then $|\Lambda_0|$ of $\{\mathbf{B}_{\Lambda_i}\}$. The theorem of Ruelle on zeros of polynomials contracted according to Asano [16,18] can be used to conclude that

$\mathcal{M}(z_{\mathbf{B}_\Lambda})\neq 0$ if $z_{B,\alpha(B)}<r_0^{|\Lambda_0|}$ for every $\alpha\epsilon\Gamma_\Lambda$. The Vitali theorem can be used as in [16,19] to obtain that

$f(z_{\mathbf{B}_o})=\lim_{\Lambda\to L}(1/|\Lambda|\log M(z_{\mathbf{B}_\Lambda})$ is an analytic function in variables

$z_{B,\alpha(B)}$ $B\epsilon\mathbf{B}_0$ $\alpha(B)=1,\ldots,|B|-1$ if $z_{B,\alpha(B)}<r_0^{|\Lambda_0|}$. Finally

$p_0=f(z_{\mathbf{B}_0})|z_{B,\alpha(B)}=\exp\{\beta J(B)[\cos(2\pi\alpha(B)/|B|)-1]\}$

so $p_o$ is an analytic function if $\beta J(B);B\epsilon\mathbf{B}_0$ is large enough (low temperatures in ferromagnetic systems).

# Chapter IX

## EXAMPLES

In this chapter, the number of extremal,periodic Gibbs states is evaluated in several models.It will be convenient to use the following "polynomial" notation for the bonds of the system.Let $L=L_1 \cup \ldots \cup L_l$ $l=|L_0|$ be the decomposition of L into a sum of $Z^\nu$ lattices.Thus any element $A \in X_f = \bigoplus_{i \in L} Z_m$ will be identified with a sequence of l elements from $\bigoplus_{i \in Z^\nu} Z_m$
$A=(A_1,\ldots,A_l)$
On the other hand, elements of $\bigoplus_{i \in Z^\nu} Z_m$ are identified with polynomials in variables
$x_1,\ldots,x_\nu,x_1^{-1},\ldots,x_\nu^{-1}$ $x_i x_i^{-1}=1$ with coefficients in $Z_m$
(cf. Appendix B).
Let $B \in \bigoplus_{i \in Z^\nu} Z_m$ ; $\underline{B}=\{b_1,\ldots,b_k)$
$B(b_i) \equiv B_i$ $b_i=(b_i^1,\ldots,b_i^\nu) \in Z^\nu$
then we can write:

$$B=\sum_{i=1}^{k} B_i \prod_{j=1}^{\nu} x_j^{b_i^j}$$

In all examples below $Z_4$ is placed at each site of $Z^2$ lattice.Several examples with spin 1/2 on general $Z^\nu$-invariant lattices were discussed by J.Slawny [5].From now on x denotes $x_1$ and y denotes $x_2$.

Example 9.1

$\mathbf{B}_0=\{B_1,B_2,B_1^{-1},B_2^{-1}\}$

$B_1=2+2x$

$B_2=1+x+y+xy$

$2\mathbf{A}$ is generated by $2+2x+2y+2xy$

so the system is not reduced.

$\mathbf{A}^2$ is generated by $2+2x$

$\mathrm{cl}\Gamma_f^{\mathbf{A}^2}$ is generated by $\gamma^{\mathbf{A}^2}(z)$ where

$$z=\sum_{n=0}^{\infty} x^{2n}+3\sum_{n=1}^{\infty} x^{2n-1}$$

$\mathrm{cl}\Gamma_f^{2\mathbf{A}}$ is generated by $\gamma^{2\mathbf{A}}(w)$ where

$$w=\sum_{n,m=0}^{\infty} x^n y^m$$

By Lemma 6.10 $\mathrm{cl}\Gamma_f$ is generated by $\gamma(z)$ and $\gamma(2w)$

The lattice of the reduced system has a fundamental family which consists of two elements. $Z_4$ is placed on one $Z^2$ sublattice ; $Z_2$ on the other one.

New bonds can be obtained by method described in 6.2

$B_1'=(2,0)$

$B_2'=(1+y,1)$

$(L',K')$ system is reduced hence

$\mathbf{B}'^{+}=\mathrm{cl}\mathbf{A}'$

It is easy to see that $\mathrm{cl}\mathbf{A}'=\mathbf{A}'$

$\mathbf{B}^{+}=\hat{\phi}(\mathbf{B}'^{+})=\hat{\phi}(\mathbf{A}')=\mathbf{A}$

Hence at low temperatures there is unique periodic Gibbs state.

Example 9.2

$\mathbf{B}_0=\{B_1,B_2,B_3,B_1^{-1},B_2^{-1},B_3^{-1}\}$

$B_1=1+x$

$B_2=1+3x$

$B_3=2+2y$

$2\mathbf{A}$ is generated by $2+2x$ so the system is not reduced.

$\mathbf{A}^2$ is generated by 2 so $\mathbf{A}^2$ is reduced.

$\mathrm{cl}\Gamma_f^{\mathbf{A}^2}$ is generated by $\gamma^{\mathbf{A}^2}(x^0)$

$\mathrm{cl}\Gamma_f^{2\mathbf{A}}$ is generated by $\gamma(z)$ where

$$z=\sum_{n=0}^{\infty} x^n$$

$\mathrm{cl}\Gamma_f$ is generated by $\gamma(x^0)$ and $\gamma(2z)$.Again, in the reduced system we have $Z_4$ on one $Z^2$ sublattice and $Z_2$ on the other one.

$B_1'=(1+x,1)$

$B_2'=(1+3x,1)$

$B_3'=(2+2y,0)$

$\mathbf{B}'^+$ is generated by $(2,0)$ and $(1+x,1)$

$\hat{\phi}((2,0))=2 \qquad \hat{\phi}((1+x,1))=1+x$

hence $\mathbf{B}^+=\mathbf{A}$ so there is unique periodic Gibbs state at low temperatures.The reason here is that $\mathbf{A}$ can be generated by $B_1,B_2$ so the system is essentialy one dimensional.

Example 9.3

$\mathbf{B}_0=\{b_1,B_2,B_1^{-1},B_2^{-1}\}$

$B_1=1+x$

$B_2=2+2y$

$2\mathbf{A}$ is generated by $2+2x$ so the system is not reduced.

$\mathbf{A}^2$ is generated by $2+2x$ and $2+2y$ so $\mathbf{A}^2$ is reduced.

$\mathrm{cl}\Gamma_f$ is generated by $\gamma(x^0)$ and $\gamma(2z)$ like in Example 9.2

$B_1'=(1+x,1)$

$B_2'=(2+2y,0)$

$\mathbf{B}'^+$ is generated by $(2,0)$ and $(1+x,1)$ hence $\mathbf{B}^+$ is generated by $2$ and $1+x$.

It is easy to see that $|\mathbf{B}^+/\mathbf{A}|=2$ so there are two extremal,periodic Gibbs states at low temperatures.

Example 9.4

$\mathbf{B}_0=\{B_1,B_2,B_1^{-1},B_2^{-1}\}$

$B_1=1+x+2x^2+2y+xy+x^2y$

$B_2=1+2x+y+xy+2y^2+xy^2$

$2\mathbf{A}=\mathbf{A}^2$ is generated by $2+2x+2xy+2x^2y$ and $2+2y+2xy+2xy^2$

g.c.d.$[(1/2)2\mathbf{A}]=1+xy$ in $Z_2[Z^\nu]$ so the system is not reduced.

$\mathrm{cl}\Gamma_f^{2\mathbf{A}}$ is generated by $\gamma^{2\mathbf{A}}(z)$ where

$$z=3\sum_{n=0}^{\infty} x^n y^n+2\sum_{n=1}^{\infty} x^{2n-1}y^{2n}+2\sum_{n=1}^{\infty} x^{2n}y^{2n-1}$$

$\mathrm{cl}\Gamma_f$ is generated by $\gamma(z)$

We can choose a fundamental family of new bonds as

$B_1'=2+3x+3x^2+2xy+2x^2y$

$B_2'=2x+2y+3xy+2y^2+3xy^2$

$\mathbf{B}'^+=\mathrm{cl}\mathbf{A}'=X_f$

$\mathbf{B}^+$ is generated by

$\hat{\phi}(1)=1+x^{-1}y^{-1}+2x^{-2}y^{-1}+2x^{-1}y^{-2}$.

After some algebra $|\mathbf{B}^+/\mathbf{A}|=2$ so there are two extremal, periodic Gibbs states at low temperatures.

Example 9.5

$\mathbf{B}_0=\{B_1,B_2,B_1^{-1},B_2^{-1}\}$

$B_1=1+x+y^2+xy^2$

$B_2=1+x^2+y+x^2y$

$2\mathbf{A}$ is generated by $2+2x+2y^2+2xy^2$ and $2+2x^2+2y+2x^2y$.

g.c.d.$[(1/2)2\mathbf{A}]=1+x+y+xy$ so the system is not reduced.

$\mathbf{A}^2$ is generated by $2+2x^2y^2$ $2x^2+2y^2$ $2x+2y+2x^2y+2xy^2$ $2x+2y^2+2xy^2+2x^2y^2$ $2+2y+2x^2y+2y^2$.

g.c.d.$[(1/2)\mathbf{A}^2]=1$ so $A^2$ is reduced.

$\mathrm{cl}\Gamma_f^{2\mathbf{A}}$ is generated by $\gamma^{2\mathbf{A}}(z)$ where

$$z=\sum_{n,m=0}^{\infty} x^n y^m$$

$\mathrm{cl}\Gamma_f$ is generated by $\gamma(x^0)$ and $\gamma(2z)$.

In the reduced system $Z_4$ is on the $Z^2$ sublattice and $Z_2$ on the other one.

$B_1'=(1+x^{-1}+y^{-2}+x^{-1}y^{-2},1+y)$

$B_2'=(1+x^{-2}+y^{-1}+x^{-2}y^{-1},1+x)$

$\mathbf{B}'^{+}$ is generated by $B_1', B_2'$ and $(2,0)$

$\hat{\phi}((2,0))=2$

After some algebra it can be shown that $|\mathbf{B}^{+}/\mathbf{A}|=249$ so there are 249 extremal, periodic Gibbs states at low temperatures.

Example 9.6

$\mathbf{B}_0=\{B_1, B_2, B_1^{-1}, B_2^{-1}\}$

$B_1=1+2x+x^2+y+2xy+x^2yu$

$B_2=1+x+2y+y^2+xy^2$

$2\mathbf{A}$ is generated by $2+2x^2+2y+2x^2y$ and $2+2x+2y^2+2xy^2$

g.c.d.$[(1/2)2\mathbf{A}]=1+x+y+xy$ so the system is not reduced.

$\mathbf{A}^2$ is generated by $2+2x$

$\mathrm{cl}\Gamma_f^{\mathbf{A}^2}$ is generated by $\gamma^{\mathbf{A}^2}(z)$ where

$$z=\sum_{n=0}^{\infty} x^{2n}+3\sum_{n=1}^{\infty} x^{2n-1}$$

It is easy to see that it is impossible to find a generator of $\mathrm{cl}\Gamma_f^{\mathbf{A}^2}$, $\gamma^{\mathbf{A}^2}(w)$ such that $\gamma(w)$ is finite.Therefore the assumption of Lemma 6.10 is not satisfied.

$\mathrm{cl}\Gamma_f$ is generated by $\gamma(v)$ and $\gamma(x^0)$ where

$$v=\sum_{n=0}^{\infty} x^{2n}+3\sum_{n=1}^{\infty} x^{2n-1}$$
$$+3\sum_{n=0}^{\infty} x^n y+\sum_{m=2}^{\infty}\sum_{n=0}^{\infty} x^{2n}y^m$$

In the reduced system $Z_4$ is placed on both $Z^2$ sublattices.

$B_1'=(1+x^{-1}+2x^{-1}y^{-1}+y^{-2}+x^{-1}y^{-2}, 1+2x+x^2+y+2xy+x^2y)$

$B_2'=(x^{-1}y+3x^{-1}+3x^{-1}y^{-1}+x^{-1}y^{-2}, 1+x+2y+y^2+xy^2)$

It can be checked that $\hat{\phi}(\mathrm{cl}\mathbf{A}')=\mathbf{A}$ so there is unique periodic Gibbs state at low temperatures.

In the above examples the number of extremal, periodic Gibbs states is evaluated by the purely algebraic methods.Namely, it is enough to find the second dual $\mathbf{A}^{**}$ of the module $\mathbf{A}$ generated by the bonds of the system. At low enough temperatures the number of extremal, periodic Gibbs states is equal to $|\mathbf{A}^{**}/\mathbf{A}|$.This is an illustration of the reduction described in Chapter VI of this paper.

# Appendix A

## CHARACTERS

A character of a locally compact abelian group G is a continuous function $\hat{x}$ on G such that $\hat{x}(x)\in\mathbf{C}; |\hat{x}(x)|=1;$ $\hat{x}(x_1x_2)=\hat{x}(x_1)\hat{x}(x_2)$ for any $x,x_1,x_2\in G$. [21]

Theorem A1

Given any $a\in G; a\neq e$ there is a character $\hat{x}$ such that $\hat{x}(a)=1$

The dual group $\hat{G}$ is the set of all characters of G with multiplication as group action. Topology in $\hat{G}$ is introduced in the following way. Let C be a compact set in G and $\varepsilon>0$ then $U(C,\varepsilon)=\{\hat{x}\in\hat{G}; |\hat{x}(x)-1|<\varepsilon$ for all $x\in C\}$. Let the family of $U(C,\varepsilon)$ for all compact sets C in G and $\varepsilon>0$ be a basis of neighborhoods of the unit character. $\hat{G}$ is locally compact abelian topological group.

The group to $Z_n$ is isomorphic to $Z_n$

$\hat{Z}_n=\{\hat{x}_k; \hat{x}_k(a)=\exp\{2\pi ika/n\}\ a\in z_n; k=0,\ldots,n-1\}$

The group dual to the direct product of two groups $G_1,G_2$ is isomorphic to the direct product of their duals

$\widehat{G_1 \oplus G_2} \simeq \hat{G}_1 \oplus \hat{G}_2$

If H is a subgroup of G then

$H^{\perp}=\{\hat{g}\in\hat{G}; \hat{g}(h)=1$ for all $h\in H\}$

If F is a subgroup of $\hat{G}$ then

$F^{\perp}=\{g\in G; \hat{f}(g)=1 \text{ for all } \hat{f}\in F\}$

Theorem A2

If H is a closed subgroup of G then $H^{\perp\perp}=H$.

Theorem A3

If G is a locally compact abelian group then $\hat{\hat{G}}$ is isomorphic to G.

Theorem A4

Let $G_a; a\in L$ be compact abelian groups then $H=\times_{a\in L} G_a$ is compact in product topology by the Tychonoff theorem and $\hat{H}\simeq \bigoplus_{a\in L}\hat{G}_a$ where the topology of $\hat{H}$ is discrete.

Theorem A5

Let G be a locally compact abelian group with the Haar measure dx and $f\in L^{1'},(G)$ is a continuous function. The Fourier transform is defined as follows:

$\hat{f}(\hat{x})=\int f(x)\overline{\hat{x}}(x)dx$ for any $\hat{x}\in\hat{G}$ then $\hat{f}$ is continuous on $\hat{G}$. If $\hat{f}\in L^1(\hat{G})$ then there is a normalization of Haar measure on $\hat{G}, d\hat{x}$ such that

$f(x)=\int \hat{f}(\hat{x})\hat{x}(x)d\hat{x}$ for any $x\in G$.

## Appendix B

## GROUP RINGS

Let R be a commutative ring with identity and G an abelian group.Denote by R[G] the set of all functions from G to R with finite support. Functions are added pointwise and multiplication can be introduced in the following way:if $F,H\in R[G]$ then

$$FH(a)=\sum_{b\in G} F(a-b)H(b)$$

With this addition and multiplication R[G] becomes a group ring. If R=k ia a field then k[G] is a k-vector space with pointwise multiplication so k[G] is a commutative algebra.From now on $G=Z^{\nu}$.

If $a\in Z^{\nu}$ $x^a\in R[Z^{\nu}]$ $x^a(b)=\delta_{a,b};b\in Z^{\nu}$

$x_i=x^{e_i}$ $e_i=(\delta_{1i},\ldots,\delta_{\nu i})\in Z^{\nu}$

$x^a=x_1^{a_1}\ldots x_{\nu}^{a_{\nu}}$ $a=(a_1,\ldots,a_{\nu})\in Z^{\nu}$

The ring of polynomials in $\nu$ variables with coefficients in R; $R[x_1,\ldots,x_{\nu}]$ (if all $a_i\geq 0$) is obtained.

Each element of $R[Z^{\nu}]$ can be written as

$x^aP;a\in Z^{\nu},P\in R[x_1,\ldots,x_{\nu}]$

### Greatest Common Divisors

If R=k is a field then $k[x_1,\ldots,x_{\nu}]$ is a unique factorization domain.$\lambda x^a;\lambda\in k,\lambda\neq 0,a\in Z^{\nu}$ are the only units in

$k[Z^\nu]$ so $k[Z^\nu]$ is also a unique factorization domain. Hence for any family T of elements of $k[Z^\nu]$ there is a unique up to a unit greatest common divisor of T. If $R=Z_m$ where $m=\prod_{l=1}^{K} p_l$; $p_l$ are different primes then $Z_m[Z^\nu]$ is isomorphic to the direct product of the rings $Z_{p_l}[Z^\nu]$. Let $P\in Z_m[Z^\nu]$ then there is a unique decomposition

$P=\sum_{l=1}^{k}(m/p_l)P^l$ where $0\le P^l(a)<p_l$ for any $a\in Z^\nu$

Let B be any family of elements in $Z_m[Z^\nu]$ such that for every l there is at least one element B in this family such that $B^l\neq 0$. Let $D^l=c.d.\{B^l;B\in \mathbf{B}\}$ then

$\sum_{l=1}^{k}(m/p_l)D^l=D\in Z_m[Z^\nu]$ is c.d.$(\mathbf{B})$ and vice versa. Hence g.c.d.$(\mathbf{B})$ is well defined in $Z_m[Z^\nu]$ and is not a zero divisor. F is a unit in $Z_m[Z^\nu]$ if and only if $F^l$ is a unit in $Z_{p_l}[Z^\nu]$ for any $l; 1\le l\le k$.

## POWERS OF ELEMENTS IN $Z_{p^n}[Z^\nu]$

Let $A\in Z_p[Z^\nu]$ where p is a prime number

$A=\sum_{a\in Z^\nu} A(a)a$ then

(B1) $A^{p^n}=\sum_{a\in Z^\nu} A(a)p^n a$

It follows readily from the fact that if $u\in Z_p$ then $u^{p^n}=u \pmod p$ and $pu=0 \pmod p$.

The second equality yields in particular

$(u+w)^p=u^p+w^p$ in $Z_p$

It would be best to have an analog of (B1) in the $Z_{p^n}[Z^\nu]$ case.

Lemma B1

Let $u,w\in Z_{p^n}$; p is a prime number then

$$(u+w)^{(p^n)^m}=u^{(p^n)^m}+w^{(p^n)^m}+\sum_{k=1}^{r} c_k u^{u(m,k)} w^{w(m,k)}$$ where r does not depend on m if m≥n and p divides $c_k$.

Proof: $$(u+w)^{p^n}=u^{p^n}+w^{p^n}+\sum_{k=1}^{p^{n-1}-1} d_k u^{p^n-pk} w^{pk}$$

$d_k=p^n!/(pk)!(p^n-pk)!$ so p divides $d_k$.

It is easy to see that if m≥n then

$$(u+w)^{(p^n)^m}=[u^{p^n}+w^{p^n}]^{(p^n)^{m-1}}$$ so let now m=n+r ; r≥0 then

$$(u+w)^{(p^n)^m}=[u^{(p^n)^{r+1}}+w^{(p^n)^{r+1}}]^{(p^n)^{n-1}}=$$

$$=u^{(p^n)^m}+w^{(p^n)^m}+\sum_{k=1}^{m(p^n)^{n-1}-1}\binom{(p^n)^{n-1}}{pk}u^{(p^n)^{r+1}[(p^n)^{n-1}-k]}w^{(p^n)^{r+1}k} \quad ∎$$

Let $A\in Z_{p^n}[Z^\nu]$ $A=\sum_{a\in Z^\nu}A(a)a$ and there is $a\in Z^\nu$ such that $A(a)\neq0$ and p does not divide $A(a)$

(by Proposition B2 A is not a zero divisor)

then from Lemma B1 follows that

$$A^{(p^n)^m}=\sum_{a\in Z^\nu}A(a)^{(p^n)^m}(p^n)^m a+\sum_{k=1}^{v}A_{k,m}$$

where v is independent of m if $m \geq n$ and $A_{k,m} \in Z_{p^n}[Z^\nu]$ and are all zero divisors. The sequence of natural numbers $i_m$ can be chosen such that

$$A^{(p^n)^{i_m}} = \sum_{i=1}^{t} A_i g_i^{(m)}$$

where t does not depend on m if m is large enough and $\mathrm{dist}(g_i^{(m)}, g_j^{(m)}) \to \infty$ as $m \to \infty; i \neq j$.

At least one $A_i \neq 0$ because there is $a \in Z^\nu$ such that $A(a)^{(p^n)^m} \neq 0$ and p does not divide $A(a)^{(p^n)^m}$ and all $A_{k,m}$ are such that if $a \in A_{k,m}$ ; $A_{k,m}(a) \neq 0$ then p divides $A_{k,m}(a)$.

Zero Divisors

Proposition B2

If $D \in Z_{p^n}[Z^\nu]$ is such that there is $a \in Z^\nu$ $D(x) \neq 0$ and p does not divide $D(a)$ ($D(a)$ is not a zero divisor in $Z_{p^n}$) then D is not a zero divisor in $Z_{p^n}[Z^\nu]$.

Proof: Let A be any non-zero element from $Z_{p^n}[Z^\nu]$. It will be shown that $DA \neq 0$. Let z be the least (in the sense of lexicographic order) $a \in Z^\nu$ such that $D(a) \neq 0$ and p does not divide $D(a)$. Let k be the least number such that there is $b \in Z^\nu$ such that $A(b) = sp^k$ and p does not divide s. Let w be the least of such b's. It will be shown that $(DA)(z+w) \neq 0$

$$(DA)(z+w) = D(z)A(w) + \sum_{\substack{a_1, a_2 \\ a_1 + a_2 = z + w \\ a_1 \neq z \quad a_2 \neq w}} D(a_1)A(a_2) =$$

$$=D(z)A(w)+\sum_{\substack{a_1,a_2\\ a_1<z\\ a_2>w\\ a_1+a_2=z+w}} D(a_1)A(a_2)+\sum_{\substack{a_1,a_2\\ a_1>z\\ a_2<w\\ a_1+a_2=z+w}} D(a_1)A(a_2)$$

Now it is easy to see that $(DA)(z+w)$ is not a multiple of $p^n$ because it is not a multiple of $p^{k+1}$ ∎

NOETHERIAN RINGS

A ring is Noetherian if every ascending chain of ideals eventually stabilizes, i.e., there is a n such that $I_n=I_{n+i}$ for every $i\geq 0$ if $I_1 \subset I_2 \subset \ldots$

Theorem B3 [22]

If G is an abelian group then R[G] is a Noetherian ring if and only if R is a Noetherian ring and G is a finitely generated group.

For example $Z_m[Z^\nu]$ is a Noetherian ring.

## Appendix C

## MODULES

Let R be a commutative ring with identity (1). R-module M is an abelian group under addition (+) which is also equiped with a scalar multiplication $R \times M \to M$ such that

$(r+s)m=rm+sm$

$r(m+n)=rm+rn$

$(rs)m=r(sm)$

$1m=m$

For $r,s \in R$ and $m,n \in M$

Theorem C1

Any submodule of a finitely generated module over a Noetherian ring is finitely generated.

M is finite free module if it is isomorfic to $R^n$ for a finite n.Homomorphism of M to R form the dual module $M^*$ of M.One has homomorphism $C_M \to M^{**} \equiv (M^*)^*$ which to $x \in M$ assigns $C_M(x): f \to f(x)$ ; $f \in M^*$

M is a reflexive module if $C_M$ yields an isomorphism of M onto $M^{**}$.

Finite free modules are reflexive.M is torsionless if $C_M$is injective.Submodules of free modules are torsionless.

Let G be a finite abelian group and m=least common multiple of orders of all elements of G. $T=\times_{i\in Z^\nu} G$

$T_f=\bigoplus_{i\in Z^\nu} G$ are $Z_m[Z^\nu]$-modules with action of $Z^\nu$ as translation and $Z_m$ as addition.

$T_f$ is reflexive module (in fact $T_f^\star \simeq T_f$). Generally $T_f$ is not a finite free module but it is "almost free".

Namely, let $\{g_i\}$ $i=1,\ldots,k$ be the generators of G in the decomposition of G into cyclic groups and let

$y_i \in \bigoplus_{i\in Z^\nu} G$ $y_i(0)=g_i$ $y_i(a)=e\in G$ if $a\neq 0$ $a\in Z^\nu$

$\{y_i\}$ generates $T_f$ and if $\sum_{i=i}^{k} r_i y_i=0$ then $r_i y_i=0$.

For every $i, r_i\in Z_m[Z^\nu]$ although $r_i$ does not need to be zero for every i.

For any homomorphism $\phi:M\to N$ one has the transposed homomorphism $\phi^\star:N^\star\to M^\star$

$\phi^\star(f)(m)=f(\phi(m));f\in N^\star;m\in M$

For a submodule M of amodule P one defines

$M^O=\{f\in P^\star;f(m)=0, \text{for any } m\in M\}$

and the closure of M in P

$cl_P M=\{p\in P;f(p)=0, \text{for any } f\in M^O\}$

Proposition C2 [5]

Let $\varepsilon:P\to Q, \mathbf{A}=\mathrm{Im}(\varepsilon), C=\mathrm{Im}(\varepsilon^\star)$ where P and Q are reflexive modules, then

(a) the modules: $\mathbf{A}^\star/\mathrm{Im}(Q^\star\to\mathbf{A}^\star)$ and $cl_{P^\star}(C)/C$ are isomorphic.

(b) if C is closed in $P^\star$ then

$$Im(A^{**} \to Q) = cl_Q A.$$

Suppose that in addition to $\varepsilon$ one has $\varepsilon':P' \to Q'$ where $P'$ and $Q'$ are reflexive modules. Furthermore, let $\sigma:Q' \to Q$ be a homomorphism and $\rho:P \to P'$ an isomorphism such that $\sigma\varepsilon'\rho = \varepsilon$. Assuming $\rho(Ker(\varepsilon)) = Ker(\varepsilon')$, $\sigma$ yields an isomorphism of $A' \approx P'/Ker(\varepsilon')$ and $A \approx P/Ker(\varepsilon)$. Moreover:

Proposition C3 [5]

If $C'$ ($=Im\ \varepsilon'^{*}$) is closed in $P'^{*}$ then $\sigma$ yields an isomorphism of $cl_{Q'}A'$ onto $Im(A^{**} \to Q)$.

## C.1 HERMITIAN FORMS

If $u, w \in T_f$

$$u = \sum_{i=1}^{k} A_i y_i$$

$$w = \sum_{i=1}^{k} B_i y_i$$

where $A_i, B_i \in Z_m[Z^\nu]$. Let

$$(u,w) = \sum_{i=1}^{k} (m/n_i) A_i I(B_i)$$

where $n_i = |y_i|$ and $I(A)(a) = A(-a)$ if $A \in Z_m[Z^\nu]$ then

$$(ru+sw, v) = r(u,v) + s(w,v)$$

$$(u,w) = I((w,u)) \quad \text{for any } r,s \in Z_m[Z^\nu]; u,w,v \in T_f$$

For $u \in T_f$ define $j(u) \equiv j_{T_f}(u) \in T_f^{*}$ through

$$j(u)(w) = (w,u); w \in T_f$$

It is easy to see that j defines an I-isomorphism of $T_f$ onto $T_f^*$ i.e. a bijection $j:T_f \to T_f^*$ such that

$j(ru+sw)=I(r)j(u)+I(s)j(w)$ ; $r,s \in Z_m[Z^\nu], u,w \in T_f$

j is obviously injective. Let $f \in T_f^*$ and $I(A_i)=f(y_i)/(m/n_i)$

and $u=\sum_{i=1}^{k} A_i y_i$

then $j(u)=f$ so j is surjective.

The elements of $T_f$ can be identified with characters on T, (cf. Appendix A) so for $u,w \in T_f$

(C1) $\hat{w}(u)=\prod_{i=1}^{K} \exp\{(2\pi i \sum_{a\in Z^\nu} A_i(a)B_i(a))/n_i\}=$

$=\exp\{2\pi i(u,w)(0)/m\}$.

Proposition C4

If N is a submodule of $T_f$ then

$N_0 \equiv N^\perp \cap T_f = \{u \in T_f ; (u,w)=0 \text{ for any } w \in N\}$

Proof: If $u \in N_0$ then for $w \in N$ $\langle u, x^b w\rangle = 1$ where $x^b(a)=\delta_{a,b}$

$a,b \in Z^\nu$

$\langle u,x^b w\rangle=(u,x^b w)(0)=[\sum_{i=1}^{k} A_i I(x^b B_i)](0)=$

$=[I(x^b)\sum_{i=1}^{k} A_i I(B_i)](0)=[x^{-b}(u,w)](0)=(u,w)(b)$.

Hence if $\langle u,x^b w)=1$ for any $b \in Z^\nu$ then $(u,w)=1$. ∎

Corollary C5

For any submodule N of $T_f$ $cl_{T_f} N = N_{00}$.

Corollary C6

For any submodule N of $T_f$ $cl_{T_f^*}j(N)=j(cl_{T_f}N)$.

If we equip G with discrete topology then T becomes the compact abelian group with the product topology. For $N\subset T_f$ denote $clN\equiv ClN\cap\Gamma_f$

Proposition C7

For any submodule N of $T_f$ $clN=cl_{T_f}N$

Proof: It will be shown that $clN=N_{00}$.

$ClN=(ClN)^{\perp\perp}$ by Theorem A2 but $(ClN)^{\perp}=N_0$ so $ClN=(N_0)^{\perp}$ and finally $clN=N_{00}$

Now the proposition follows from Corollary C5. ∎

If we set $T_f=X_f$ or $T_f=M_f$ we obtain hermitian forms on $X_f$ and $M_f$ (see 3.1 for notation).

For the homomorphism

$\varepsilon:M_f\to X_f$ there exists unique

$\varepsilon^+:X_f\to M_f$ such that

$(\varepsilon^+(x),\alpha)=(x,\varepsilon(\alpha))$ ;$x\in X_f,\alpha\in M_f$.

Obviously $\varepsilon^+=j_{M_f}^{-1}\varepsilon^*j_{X_f}$.

Propositions C4 and 3.2 yield : $\gamma_{|X_f}=\varepsilon^+$ and therefore

$C=j_{M_f}(\Gamma_f)$

From Corollary C6 we obtain that C is closed in $M_f^*$ if and only if $\Gamma_f$ is closed in $M_f$ which by Theorem 4.9 is equivalent to the decomposition property. It will be proven later (Theorem C18) that $i^{**}:\mathbf{A}^{**}\to X_f$ is injective where $i^{**}$ is the double dual of the inclusion map, so $\mathbf{A}^{**}$ can be identified with its image in $X_f$. Under the assumption of 6.1 we have the following:

Proposition C8

$\hat{\phi}$ is an isomorphism between $cl\mathbf{A}'$ and $\mathbf{A}^{**}$.

Proof: If we set $P=M_f, Q=X_f, P'=M_f^!, Q'=X_f^!$, $\rho=\phi, \sigma=\hat{\phi}$ then the result follows from Proposition C3 and the fact that C' is closed in $M_f^{!*}$ because of the decomposition property in (L',K') system. ∎

## C.2 CLOSURE OF MODULES

Let R be an integral domain and k is the field of fractions of R. Let $P=R^n, V=k^n$. V is the vector space of the field k. It is also S-submodule for any subring S of k. For any subset M of V, SM is the S-submodule of V generated by M

$SM=\{s_1a_1+\ldots+s_na_n; \text{any } n, s_i\in S, a_i\in M\}$

In particular V=kP. Let M be a subset of V then define

$S:M=\{f\in V^*; f(M)\subset S\}$

S:M is S-submodule of $V^*$. If M is S-submodule of V then $f\to f_{|M}$ is a surjective S-homomorphism from S:M to $Hom_S(M,S)$.

For any S-linear form $g$ on $M$ has unique extension to k-linear form $g'$ on $kM$

$g'((x/y)a)=(x/y)g(a)$ ; $x,y\in R, a\in M$

Since $kM$ is a subspace of the vector space $V$, $g'$ extends to a linear form on $V$.

Proposition C9 [5]

If $M$ is $R$ submodule of $P=R^n$ then $cl_P M=(kM)\cap P$.

Proof:By the preceding discussion any R-form on $P$ which is zero on $M$ is zero on $(kM)\cap P$ so $(kM)\cap P\subset cl_P M$.On the other hand if $N$ is R-submodule of $P$ then $kN$ is strictly larger than $kM$.Therefore there exists $f\in V^*$ which is zero on $kM$ and non-zero on $kN$.Let $\{e_i\}$ $i=1,\dots,n$ be the canonical basis of $V$ and let

$f(e_i)=x_i/y_i$ ; $x_i,y_i\in R$

let $y=y_1\dots y_n$ then $yf\in R{:}P$ is zero on $M$ and non-zero on $M$ ∎

Corollary C10

If $M$ is R-submodule of $P=R^n$ then there is $D\in R, D\neq 0$ such that $D\,cl_P M\subset M$.

Proof:$cl_P M$, as submodule of finitely generated module over a Noetherian, ring is finitely generated (Theorem C1).Let $\{m_i\}$ be a family of generators of $cl_P M$.By Proposition C9

$m_i=\sum_j (x_i^j/y_i^j)a_j$

where $x_i^j, y_i^j\in R, a_j\in M$

Let $D=\prod_{i,j} y_i^j$

. then $Dcl_pM\subset M$. ∎

It would be best to extend the corollary to the case of $R=Z_m[Z^\nu]$ and $M=\mathbf{A}$, the submodule generated by the bonds (characters) of the system. Let $R=Z_{p^n}[Z^\nu]$

Let $\mathbf{A}'^{p^i}=\{A\epsilon\mathbf{A}; p^iA=0\}$

$\mathbf{A}_{p^i}=\mathbf{A}_{|p^iX}$ ; characters restricted to $p^iX$

If $A\epsilon cl\mathbf{A}$ then it can be decomposed as follows:

$A=A_1+\ldots+A_n$ where

$A_k\epsilon cl[(\mathbf{A}'^{p^{n-k+1}})_{p^{n-k}}]$ ; $k=1,\ldots,n$

and $cl[(\mathbf{A}'^{p^{n-k+1}})_{p^{n-k}}]$ are $Z_p[Z^\nu]$-modules.

By Corollary C10 there are $D_k$ ; $k=1,\ldots,n$ such that

$D_k \circ cl[(\mathbf{A}'^{p^{n-k+1}})_{p^{n-k}}]\subset(\mathbf{A}'^{p^{n-k+1}})_{p^{n-k}}$

where scalar multiplication o is performed in $Z_p[Z^\nu]$ modules.

Let $D=\prod_{i=1}^{n} D_i$

$DA_k=(\prod_{\substack{i=1\\i\neq k}}^{n} D_i)D_kA_k=(\prod_{\substack{i=1\\i\neq k}}^{n} D_i)[D_k\circ A_k+B_k]$

where $B_k$ is defined by the above equation and $B_k\epsilon cl\mathbf{A}'^{p^{n-k}}$. Now continuing the process on $B_k$ by $D_{k+1}$, $DA_k\subset\mathbf{A}$ and finally $DA\epsilon\mathbf{A}$.

Using decomposition $m=\prod_{i=1}^{n} p_i^{k_i}$; $p_i$ are prime numbers, it is easy to extend the above result to the $Z_m[Z^\nu]$ case with arbitrary m.It constitutes the following theorem:

Theorem C11

There is non-zero divisor $D\in Z_m[Z^\nu]$ such that $D\subset 1\mathbf{A}\subset\mathbf{A}$.

## C.3 DETERMINANTAL IDEAL

With any pair $(\mathbf{A},P)$ where P is a finite free R-module and $\mathbf{A}$ is a submodule of P,an ideal $\Delta(\mathbf{A},P)$ of R, the so called determinantal ideal of $(\mathbf{A},P)$ can be associated.

Let $e_1,\ldots,e_n$ be a basis of P and

$a_1,\ldots,a_m$ be generators of $\mathbf{A}$

$a_i=\sum_{j=1}^{n} x_{ij}e_j$ ; $x_{ij}\in R, i=1,\ldots,m.$

Let $\Delta_k$, $k=1,\ldots,\min(m,n)$ be the ideal of R generated by the minors of the rank k of the matrix

$(x_{ij})$ $i=1,\ldots,m$ $j=1,\ldots,n.$

$\Delta_k$ depends neither on the choice of the basis of P nor of the generators of $\mathbf{A}$.Let $k_0$ be the largest k for which $\Delta_k\neq 0$.Define determinantal ideal $\Delta(\mathbf{A},P)$ as $\Delta_{k_0}$. Since $\Delta_{k+1}\subset\Delta_k$ $\Delta(\mathbf{A},P)=\bigcap_{k\le k_0}\Delta_k$.

## C.4 PRECISE GENERATORS OF MODULES

The theorem of W.Holsztynski is quoted here [20].It is used in the proof of analyticity of pressure and to obtain a locality property in energy excitations in Peierls argument.

Let A be a left Noetherian ring and G a set with a binary associative operation defined in it and with a neutral element.M is a finitely generated and finitely supported A[G] module [20].If some technical conditions are satisfied (they are trivially satisfied in $Z_m[Z^\nu]$-modules) then we have

Theorem C12 [Theorem II.4.9 in 20]

There exists a finite set $\Delta \subset M_0$ where $M_0$ is a A[G] submodule of M such that every A module $M_{0|\Lambda}$;$\Lambda \in P$ is generated by

$\{gx; x\in\Delta, g\in G, gs(x)\subset\Lambda\}$ where

$s:M\to 2^T$ is a support of M,T is a G-set and P is a boxing family in T.

This theorem can be applied to $Z_m[Z^\nu]$-module $M_f = \bigoplus_{B\in\mathbf{B}} Z_{|B|}$.As a $Z^\nu$ set take $Z^\nu$ itself.Theorem 7.10 in [20] states that $P=\{(a,b);a,b\in Z^\nu\}$ is a boxing family in the $Z^\nu$ set $Z^\nu$. Construct a map $w:L\to Z^\nu$ by

$w(0)=0, w(a_i)=0$ where $a_i \in L_0$ - fundamental family of L such that $0\in L_0$,L is a lattice and then extend this to L in $Z^\nu$-invariant way.

Define the support of $M_f$

$s:M_f \to$ finite subsets of $Z^\nu$

$s(\alpha)=w(\underline{\alpha});\alpha\in M_f$.

$M_f$ is then finitely supported $Z_m[Z^\nu]$-module.It is also finitely generated so as a special case of Theorem C12 we get

Theorem C13

For any submodule of $Z_m[Z^\nu]$ module $M_f=\bigoplus_{B\in\mathcal{B}}|B|$ there exist n and $\beta_1,\ldots,\beta_n\in M_f$ such that for any parallelepiped $\Lambda\subset L$ the submodule $M_\Lambda$ of $M_f$

$M_\Lambda=\{\alpha\in M_f;w(\underline{\alpha})\subset\Lambda\}$

is generated by translates of $\beta_1,\ldots\beta_n$ with support contained in $\Lambda$.

## C.5 INJECTIVITY OF THE DOUBLE DUAL OF INJECTIVE MAP

The following theorems are the work of D.Farkas [23]

Let $R=Z_m[Z^\nu]$, C(R) be the set of non-zero divisors in R, Q be the ring R localized at C(R).From Proposition B2 we get

Lemma C14

If $r\in R$ is a zero divisor then there is a positive integer n properly dividing m such that nr=0.

Lemma C15

Every non-zero element in $R$ can be written in the form $nc$ where $n$ is a positive integer properly dividing $m$ and $c \in C(R)$.

Proof: Let $r \in R$ and have $n$ denote the largest divisor of $m$ dividing every coefficient appearing in $r$. Then $r=nc$ for some $c \in R$. If $c \notin C(R)$ then by Lemma C14 there is an $n'<m$ such that $n'$ divides $m$ and $n'c=0$. Thus every coefficient appearing in $c$ is divisible by $m/n'$. This contradicts the maximality of $n$. In conclusion, $c \in C(R)$. ∎

Theorem C16

$Q$ is an injective $R$-module.

Proof: We use Baer's criterion- given a nonzero ideal $I \subset R$ and an $R$-module map $\phi: I \to Q$ we must complete

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\phi"'] & R \arrow[dl, dashed] \\ Q & \end{tikzcd}$$

Choose $n$ minimal among the proper divisor of $m$ such that some $nc \in I$ with $c \in C(R)$. If $n'c' \in I$ with $n'$ dividing $m$ and $c' \in C(R)$ then both $ncc'$ and $n'cc'$ lie in $I$. Hence $\text{g.c.d.}(n,n')cc' \in I$ where $cc' \in C(R)$. It follows from the minimality of $n$ that $n$ divides $n'$. Suppose $\phi(nc)=q \in Q$. By Lemma C15 we may write $q=ld/e$ where $l$ is a proper divisor of $m$ and $d,e \in C(R)$. Now

$(m/n)q=(m/n)\phi(nc)=\phi(mc)=0$, hence $(m/n)l=0$ so $n$ divides $l$

Define $\theta:R\to Q$ by $\theta(r)=r(ld/nce)$.To show that $\theta$ extends $\phi$,consider $w=n'c'\epsilon I$.Certainly $wc\epsilon I$

$\phi(wc)=\phi((n'/n)c'nc)=(n'/n)c'\phi(nc)=$

$=(n'/n)c'(ld/e)=w(ld/nc)$.

That is,$c\phi(w)=cw(ld/nce)$.Divide by $c$. ∎

Lemma C17

Suppose $M,N$ are finitely generated $R$-modules and $i:M\to N$ is an injective module map.Then for each $f\epsilon M^*$ there is an $r\epsilon C(R)$ and $g\epsilon N^*$ such that $rf=i^*(g)$

Proof:Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[ddl, dashed, "h"] \\ & R \arrow[d] & \\ & Q & \end{tikzcd}$$

which can be completed by the injectivity of $Q$.

Let $r\epsilon C(R)$ be a common denominator for the images in $Q$ of the finitely many generators of $N$ (under $h$).If we set $g=rh$ then $g\epsilon N^*$.Moreover,

$hi=f \to rhi=rf \to i^*(g)=rf$. ∎

Theorem C18

If $i:M \to N$ is an injective map of finitely generated R-modules then $i^{**}:M^{**} \to N^{**}$ is injective.

Proof: Suppose $\alpha \in M^{**}$ and $i^{**}(\alpha)=0$. That means $(\alpha i^{*})(g)=0$ for all $g \in N^{*}$. If $f \in M^{*}$ is arbitrary choose $r \in C(R)$ and $g \in N^{*}$ as in Lemma C17

$\alpha(i^{*}(g))=0 \to \alpha(rf)=0 \to r\alpha(f)=0$

but $r \in C(R)$. Thus $\alpha(f)=0$. This proves that $\alpha=0$. ∎

## Appendix D

## ONE DIMENSIONAL SYSTEMS

Proposition D1

In the case of $G=Z_m$ the decomposition property holds if and only if $|S|=1$

Proof:If the symmetry group is trivial then obviously the decomposition property holds.For the other direction assume that $|S|>1$.Let $x\epsilon X_f$ be such that $x(a)=0$ if $b_1<a<b_2$ $pr_{[b_1,b_2]}x=pr_{[b_1,b_2]}s_1$ and $s_1$ is not an identity in S.If $b_2-b_1>N+2\text{mesh}\mathbf{B}_0$ then using the method from Proposition 4.14 it is easy to see that $\gamma(x)$ has two N-components.Let $y\epsilon X_f$ be such that $\gamma(y)$ consists of just one N-component of $\gamma(x)$, say the right one. $\gamma(x-y)$ consists of the other one so if $a>b_1+\text{mesh}\mathbf{B}_0$ then $y(a)=x(a)$ so $\underline{\gamma(y)}\cap(-\infty,b_1+\text{mesh}\mathbf{B}_0)\neq\phi$ and it contradicts the assumption about y.Hence the decomposition property does not hold. ∎

Proposition D2

There is a unique pure phase at low temperatures in the case of $G=Z_p$ where p is a prime number.

Proof:$X_f$ is a free module over principal ideal domain $Z_p[Z]$.$\mathbf{A}$ as a submodule of $X_f$ is also free so in particular $\mathbf{A}^{**}=\mathbf{A}$ hence $|\mathbf{B}^+/\mathbf{A}|=|\mathbf{A}^{**}/\mathbf{A}|=1$. ∎

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