A PUBLIC DURABLE GOOD/BAD THEORY

IN AN OVERLAPPING GENERATIONS ECONOMY

by

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(ABSTRACT)

This research analyzes the long-lived public good/bad. The public good/bad is defined to 'live long' in the sense that the external effects of an action persist beyond the decision horizon of the actor. Thus, a very simple overlapping generations economy is modeled in which the agent lives for two periods while the public durable good/bad lasts indefinitely with deterioration/amelioration. Pareto optimality, the Lindahl equilibrium, and the theory of voluntary provision for this overlapping generations model are contrasted with those of the atemporal model.

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CHAPTER 1

INTRODUCTION

An activity by some economic agents often affects other economic agents not directly engaged in the activity. The effects occur inside or outside the normal market process. Externalities are said to exist when the effects occur outside the normal market process.[1] Externalities are classified as external economies or external diseconomies depending on whether the effects are beneficial or harmful. Externalities are also classified as consumption externalities, production externalities or consumption-production externalities depending on whether the effects enter the utility function only, the production function only or both.

A special case, and indeed a logical extreme, of consumptive external economies is traditionally referred to as the public good. Some goods have the property that when one person consumes them, then all other people can physically consume them too. In other words benefits cannot be withheld costlessly by the provider. This property is called **nonexcludability.** Some goods have the property that "each individual's consumption of such a good leads to no subtraction from any other individual's consumption"[Samuelson(1954,p387)]. This property is called **nonrivalry.** The pure public good is characterized by nonexcludable and nonrival external economy.

Most public goods are durable in nature. Many public goods, once provided, yield services over multiple periods of time. There might be some goods whose benefits persist beyond the lifetime, or, in general,

the decision horizon, of the economic agents; that is what Mishan(1981) descriptively called the property of being **"long-lived"**.

The **public durable good** is, then, characterized by nonexcludable, nonrival and long-lived external economy. Even though the public durable good is an abstract construct, close examples of such goods are abundant; defense, light houses, bridges, dams, national parks, knowledge, and so on[Sandler/Smith(1976,pp154-157)].

The other case is the **public durable bad**, the mirror image of the public durable good. The public durable bad is characterized by nonexcludable, nonrival and long-lived external diseconomy. Nuclear waste dumps and toxic wastes are good examples of the public durable bad.

Most of the models in the theoretical works on public goods, with the exception of very few [Sandler/Smith(1976), Mishan(1981), Sandler (1982) and Yoshida(1986)], are atemporal. We can reinterpret these models as dealing with the case where temporal spillovers span over a time period shorter than or, at most, equal to the decision horizon of every economic agent involved. But it seems necessary to also study explicitly the remaining (opposite) case in which intertemporal or intergenerational spillover persist longer than the lifetime (decision horizon) of economic agents. We build a model where we can discuss the public durable good/bad explicitly. This model is based on the overlapping generations model. In the model the public durable good/bad lasts forever with deterioration/amelioration, while the agents live for two periods. After specifying the model, we obtain the steady-state Pareto optimal level of, the steday-state level of the

stationary Lindahl equilibrium, and the noncooperative equilibrium level of the public durable good/bad and compare these levels.

A Public Durable Bad Theory We start our research with the case of reciprocal externalities. The reciprocality means that there are no nonnegativity constraints on values of decision variables. It makes the problem simpler and the implication clearer and provides us good guidelines for the public durable good case where values of decisions are restricted to be nonnegative. We find that the steady-state Pareto optimal level is less than the steady-state level of the stationary Lindahl equilibrium which is less than the steady-state level of any (stationary) noncooperative equilibrium level of the public durable bad.

Public Durable Goods : Pareto Optimality and the Lindahl Equilibrium
We obtain the steady-state Pareto optimal level of the public durable good. This level is positively related to durability. Meanwhile, in Appendix B, the intergenerational version of the Samuelson public good optimality condition is obtained in the more general setting and compared with the (intertemporal) Sandler/Smith(1976) version of the condition in light of Mishan(1981). We also discuss the justification for our simplifying assumptions. Next we obtain the steady-state level of the stationary Lindahl equilibrium of the public durable good. It turns out that the stationary Lindahl equilibrium is **not** Pareto optimal contrary to the atemporal result and that inefficiency may increase as durability increases.

Voluntary Provision Theory for Public Durable Goods After reviewing the atemporal voluntary provision theory very briefly, we ask what the

voluntary provision level of the public durable good would be. We can expect that the young agent might provide more of the publid durable good than he would of the nondurable public good. However the young agent, knowing that the next generation will provide the public durable good anyway, might act the same way as he would when the public good is not durable. We obtain some noncooperative equilibrium results. These results depend upon the parameters of the model. In some cases, there exist an equilibrium where the agent acts as if the durability did not matter. In the other cases, the durability disqualifies this behavior as an equilibrium one. Not surprisingly, every subgame-perfect equilibrium we have found confirms the inefficiency of the voluntary provision.

Finally, we conclude this research and suggest issues for future research.

CHAPTER 2

A Public Durable Bad Theory

We build an overlapping generations economy model where we can discuss the public durable bad explicitly. After specifying the model, we obtain the steady-state Pareto optimal level, the steady-state level of the stationay Lindahl equilibirum, and the steady-state level of any (stationary) noncooperative equilibrium of the public durable bad. We do comparative steady-state analyses and compare these levels.

2.1   The Model

Time is discrete and elapses period by period. All actions occur within a period and will be indexed by the subscript t which takes on integer values. Agents are born at the beginning of each period and live for two periods. Each generation consists of only one agent. There are always two agents in each period; one young and one old. We can interpret this as that each generation is aggregated into one representative agent. [See Wallce(1978).]

An activity of an economic agent brings him not only private nondurable benefits but also produces harmful by-products, say toxic wastes, to the entire economy as a public durable bad. It is analogous to the one-input-two-output technology. The relation between the beneficial activity, x, and its by-products, b, are described by :

$$b = \beta \cdot x$$

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where x is assumed to be in R. It means that we have a case of the reciprocal externality. β is also assumed to be positive. This by-product last forever once produced. Fortunately, mother nature has the capability of purifying the public durable bad at the rate $(1 - k)$ per period where $k \in (0,1)$. Denote as B_t the total accumulated public durable bad in period t and as b_t the newly produced public durable bad in period t. Then the state equation for the public durable bad, B_t, is given by :

$$B_t = kB_{t-1} + b_t.$$

Agents are assumed to be identical except for their periods of birth. The utility functions of the agents are the same and depend on the level of the activity and the total level of the public durable bad. The utility of an agent in generation t is given by :

$$U^t = U^t(x^y_t, B_t, x^o_{t+1}, B_{t+1})$$
$$= x^y_t - u(B_t) + d \cdot [x^o_{t+1} - u(B_{t+1})]$$

where x^y_t denotes the level of the activity the agent t in period t (when the agent is young), x^o_{t+1} the level of the activity of agent t in period $t + 1$ (when the agent is old), and d the time preference rate of agents. It is assumed that $d \in (0,1)$. Note that $(^1)$ the utility function is additively (intertemporally) separable, that $(^2)$ the time preference rate is identical for every agent of every generation, and that $(^3)$ the periodwise utility function is quasi-linear.[3] The

following assumptions on u(·) are maintained throughout :

$$u(B) = 0 \text{ for all } B < 0$$

$$\left.\begin{array}{l} u' > 0 : u'(0) < 1 ; u'(\infty) = \infty \\ \\ u'' > 0 \end{array}\right\} \text{ for all } B > 0.$$

Note that these assumptions implies that $B_t > 0$ for all t.

We can view $u(B)$ as the disutility of the public durable bad in the economy in terms of the private nondurable numeraire. Additive separability and quasi-linearity assumptions are crucial for our overlapping generations economy (infinite horizon) model and seem to be difficult to relax for our study. [See Basar/Olsder(1982).]

Each agent is endowed with the following constraint that

$$\|x^y_t\| < w^y \quad \text{and} \quad \|x^o_t\| < w^o.$$

It is assumed that the agent's endowments are bounded but sufficient enough to allow any desired level of the activity. Our assumptions on utility functions and endowments ensure the interiority of any chosen activity value.

At this point we need to distinguish stationarity from steady-stateness. We simply use 'stationary' when referring to additional production of the public durable bad and 'steady-state' when referring to the total level of the public durable bad. Let $< b_t >$ be the sequence of the additional production of the public durable bad in each period and $< B_t >$ be the associated sequence of the total level of the public durable bad. The sequence $< b_t >$ is stationary if $b_t = b_{t+1}$ and the sequence $< B_t >$ is steady-state if $B_t = B_{t+1}$.

2.2   The Steady-state Pareto Optimal Level

Let $X_t = x^0_t + x^y_t$ be the total activity level in period t. A sequence $< X_t >$ will be Pareto optimal if $< X_t >$ solves the following program :

$$
\begin{array}{l}
\max_{< X_t >} \quad X_t - 2u(kB_{t-1} + \beta X_t) \\
\qquad + d \cdot [X_{t+1} - 2u(k^2 B_{t-1} + k\beta X_t + \beta X_{t+1})] \\
\qquad + d^2 \cdot [X_{t+2} - 2u(k^3 B_{t-1} + k^2 \beta X_t + k\beta X_{t+1} + \beta X_{t+2})] \\
\qquad + \cdots.
\end{array}
$$

(1)

Before we obtain the first-order condition, we need to show the program (1) is well-defined. Invoking our assumptions on endowments, w^y and w^0 are bounded, say, by W. Then $W + kW + k^2 W + \cdots = W/(1 - k)$ is bounded. Hence $W - 2u[W/(1 - k)]$ is bounded, say, by M. Then $M + dM + d^2 M + \cdots = M/(1 - d)$ is bounded. Therefore the above program is well-defined.

The first-order conditions are :

(2a) X_t : $2u'(kB_{t-1} + \beta X_t)\beta$

$\qquad + kd2u'(k^2 B_{t-1} + k\beta X_t + \beta X_{t+1})\beta$

$\qquad + k^2 d^2 2u'(k^3 B_{t-1} + k^2 \beta X_t + k\beta X_{t+1} + \beta X_{t+2})\beta$

$\qquad + \cdots = 1$

(2b) X_{t+1} : $d2u'(k^2 B_{t-1} + k\beta X_t + \beta X_{t+1})\beta$

$\qquad + kd^2 2u'(k^3 B_{t-1} + k^2 \beta X_t + k\beta X_{t+1} + \beta X_{t+2})\beta$

$$+ k^2 d^3 2u'(k^4 B_{t-1} + k^3 \beta X_t + k^2 \beta X_{t+1} + k\beta X_{t+2} + \beta X_{t+3})\beta$$

$$+ \cdots = d$$

(2c) X_{t+2} : $d^2 2u'(k^3 B_{t-1} + k^2 \beta X_t + k\beta X_{t+1} + \beta X_{t+2})\beta$

$$+ kd^3 2u'(k^4 B_{t-1} + k^3 \beta X_t + k^2 \beta X_{t+1} + k\beta X_{t+2} + \beta X_{t+3})\beta$$

$$+ \cdots = d^2$$

and so on for X_{t+3}, $\cdots$. We obtain (3a) from (2a) and (2b), (3b) from (2b) and (2c) :

(3a) $2u'(kB_{t-1} + \beta X_t)\beta = 1 - kd.$

(3b) $2u'(k^2 B_{t-1} + k\beta X_t + \beta X_{t+1})\beta = 1 - kd.$

and similarly for (3c), $\cdots$.

Hence given kB_{t-1} a sequence $\langle X_t \rangle$ is Pareto optimal if its associated sequence $\langle B_t \rangle$ equals $\langle B^P \rangle$ such that :

$$B^P = v\left[\frac{1 - kd}{2\beta}\right]$$

where $v = [u']^{-1}$. We have the following proposition :

PROPOSITION 2.1 : In this overlapping generations economy the steady-state Pareto optimal level of the public durable bad is :

$$B^P = v\left[\frac{(1 - kd)}{2\beta}\right].$$

Remark : $\langle B^P \rangle$ is not a Pareto optimal sequence among the steady-state sequences but the only Pareto optimal sequence.

Note that B^P decreases as k increases from 0+ to 1-, since v is monotonically increasing. Similarly B^P decreases as d increases from 0+ to 1-. Aso note that B^P decreases as β increases.

If the number of the agents in each period is n, then the steady-state Pareto optimal level, $B^P(n)$, of the public durable bad is

$$B^P(n) = v\left[\frac{1}{n} \cdot \frac{(1 - kd)}{2\beta}\right].$$

$B^P(n)$ decreases as n increases.

2.3   The Steady-state Level of the Stationary Lindahl Equilibrium

Lindahl(1958)'s original discussion on public goods was atempotal. It is not easy to guess what would be the proper version of the Lindahl equilibrium for our model according to the spirit of Lindahl, even though the public (durable) bad is the mirror image of the public (durable) good.

The Lindahl equilibrium for the public good is the result of a thought experiment where we design a tax scheme under which an economic agent solves his own utility maximization problem as a tax share taker. The tax scheme requires balanced budget and unanimity. Balanced budget means that costs must be covered by tax revenues. Unanimity means that each agent's demand for the public good given his individual tax share is identical across agents. This tax scheme then achieves two things : optimality and linkage. It provides for a Pareto optimal output of the public good and links agent's taxes to the benefit he recieves.

Linkage is "partly a matter of common sense and partly a matter of justice" [Feldman(1980,p115), see also Wicksell(1958)].

In our model unanimity must mean that $x^o_t = x^y_t$ since the currently alive cannot agree with the future generations yet to be born and the past generations already passed away.

Suppose that somehow X_t is chosen unanimously as the total level of the activity in period t and, thus, βX_t as the new production level of the public durable bad in period t. Then the consumption of X_t has to be shared between the old agent and the young agent. Thus we can define r^o_t and r^y_t as the benefit shares such that $r^o_t X_t$ goes to the old agent and $r^y_t X_t$ to the young agent in period t. Balanced budget simply means that $r^o_t + r^y_t = 1$ for all t.

The real difficulty lies in the linkage issue. Linkage requires that agents have to compensate whole series of external diseconomies caused by him to all others. This linkage issue implies there that must be some kind of intergenerational transfer mechanism that enables transfers from past generations to future generations. That is, the currently old generation, as the representative of all past generations who have have produced the public durable bad, has to pay to the currently young generation, as the representative of all future generations who will suffer. Finding such an intergenerational scheme which achieves Pareto optimality and linkage along with balanced budget and unanimity is a very hard problem which we will leave to future research.

One simple way of approaching this problem is to regard the public durable bad from the previous period as burdensome endowments of the

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public durable bad to the populace. Agents cannot do anything about
it. Then we can have a following version of the Lindahl equilibrium
concept.

DEFINITION 2.1 : An intergenerational Lindahl equilibrium for the
public bad is a sequence $< (r^o_t, r^y_t, b_t^*) >$ of benefit share $(1 - \text{tax}$
share) vectors (r^o_t, r^y_t) and a level of new production schedule b_t^* of
the public durable bad such that $r^o_t + r^y_t = 1$ for all t and the
individually chosen production level of the public durable bad equals
b_t^* for all t.

A (intergenerational) Lindahl equilibrium for the public
durable bad is <u>stationary</u> if $r^o_t = r^o_{t+1} = r^o$, $r^y_t = r^y_{t+1} = r^y$, and
$b_t^* = b_{t+1}^* = b^*(g)$ where g denotes the value of the state, i.e., the
leftover of the public durable bad from the previous period.

We are looking for the stationary Lindahl equilibrium for this
economy. The stationarity constraint can be justified in the sense
that equals are treated equally. Irrespective of generation we treat
the young and old respectively equal.

The old agent's problem is :

$$\max_{x^o} \quad r^o x^o - u(g + \beta x^o).$$

From the first-order condition, we obtain :

$$r^o = u'(g + \beta x^o)\beta \; ;$$

$$x^o(g;r^o) = [v(\frac{r^o}{\beta}) - g]/\beta.$$

The young agent's problem is :

$$\max_{x^y} \quad r^y x^y - u(g + \beta x^y) + d \cdot \Gamma(kg + k\beta x^y; r^o)$$

where $\Gamma(g;r^o) = r^o x^o(g;r^o) - u[g + \beta x^o(g;r^o)]$. From the first-order condition we obtain :

$$r^y - kdr^o = v(g + \beta x^y)\beta$$

$$x^y(g;r^o,r^y) = v(\frac{r^y - kdr^o}{\beta})/\beta.$$

In order to line up the production schedules together, we need :

$$v(\frac{r^y - kdr^o}{\beta}) = v(\frac{r^o}{\beta})$$

$$<=> r^y - kdr^o = r^o$$

since v is strictly monotonic. From $r^y + r^o = 1$ and $r^y - kdr^o = r^o$, we obtain :

$$r^o = \frac{1}{2 + kd} \quad \text{and} \quad r^y = \frac{1 + kd}{2 + kd}.$$

The unanimously chosen production level of the public durable bad is :

$$b^*(g) = v[\frac{1}{(2 + kd)\beta}] - g \quad \text{or} \quad g + b^*(g) = v[\frac{1}{(2 + kd)\beta}].$$

Whatever the level of g is, g + b*(g) is constant. So we define $B^L = v[1/(2 + kd)\beta]$ as the steady-state level of the stationary Lindahl equilibrium of the public durable bad.[4] We have the following proposition :

PROPOSITION 2.2 : In this overlapping generations economy the steady-state level of the stationary Lindahl equilibrium of the public durable bad is not Pareto optimal. That is,

$$B^P = v(\frac{1 - kd}{2\beta}) < B^L = v(\frac{1}{(2 + kd)\beta}).$$

Note that B^L decreases as k increases from 0+ to 1-. Similarly B^L decreases as d increases from 0+ to 1-. Also note that B^L decreases as β increases.

PROPOSITION 2.3 : If u" is nondecreasing, then $B^L - B^P$ increases as k increases.

If the extra marginal disutility from an extra unit of the public durable bad is nondecreasing, then the inefficiency increases as the durability increases.

If the number of the agents in each period is n, then :

$$r^y(n) = \frac{1}{n} \cdot \frac{1 + kd}{2 + kd}$$

$$r^o(n) = \frac{1}{n} \cdot \frac{1}{2 + kd}$$

since $(r^o + r^y)n = 1$ and $r^y - kdr^o = r^o$. Accordingly $B^L(n)$ is :

$$B^L(n) = v[\frac{1}{n} \cdot \frac{1}{(2 + kd)\beta}].$$

PROPOSITION 2.4 : If $u''(x)/x$ is monotonically decreasing/increasing, then $B^L(n) - B^P(n)$ increases/decreases as n increases.

If the average of the extra marginal disutility from an extra unit of the public durable bad is decreasing/increasing, then inefficiency increases/decreases as the number of agents increases.

2.4   The Steady-state Level of Any Noncooperative Equilibrium

Agent t chooses x^y_t units of the activity when he is young and x^o_{t+1} units when old. Given the initial leftover of the public durable bad from the previous period, B_0, we then have the following state equation for the public durable bad :

$$\text{(4)} \qquad \begin{aligned} B_t &= kB_{t-1} + \beta x^o_t + \beta x^y_t \; ; \; B_0 \text{ being given} \\ |x^o_t| &< w^o \text{ and } |x^y_t| < w^y \end{aligned} \Bigg\} \; t > 1.$$

We shall denote by $\langle (x^o_t, x^y_t) \rangle$ a feasible infinite sequence of the activities and by Q the set of all feasible sequences of $\langle (x^o_t, x^y_t) \rangle$ which satisfy (4). Thus under a sequence $\langle (x^o_t, x^y_t) \rangle \in Q$, the lifetime utility of agent t will be :

$$
\begin{aligned}
U^0 &= x^o_1 - u(kB_0 + \beta x^o_1 + \beta x^y_1) &\quad t = 0 \\
(5) \qquad U^t &= x^y_t - u(kB_{t-1} + \beta x^o_t + \beta x^y_t) \\
&\quad + d \cdot [x^o_{t+1} - u(kB_t + \beta x^o_{t+1} + \beta x^y_{t+1}) &\quad t \geqslant 1.
\end{aligned}
$$

As we see in (5), U^t depends not only on his own activity but also on others' activities. This situation, therefore, is a game situation.

Considering the sequential nature of the model, one of the most natural restrictions is that the decisions are made period by period. In each period the old agent and the young agent make their moves simultaneously. Thus the problem which the old agent faces is very different from the one which the young agent faces. The old agent's problem in period t is :

$$
\max_{x^o_t} \quad x^o_t - u(kB_{t-1} + \beta x^o_t + \beta x^y_t).
$$

In general how the old agent will choose x^o_t depends on [1] the initial state of the current period, kB_{t-1}, [2] his expectations of what other agents would choose, [3] the general history of what all the previous players have chosen, and so on.

For the young agent, he has to take into consideration that he is to make one more decision when he becomes old. Since decisions are

made period by period, the young players problem has a character of dynamic programming. The young agent's problem in period t is :

$$\max_{x^y_t} \quad x^y_t - u(kB_{t-1} + \beta x^o_t + \beta x^y_t)$$
$$+ d \cdot [\max_{x^o_{t+1}} x^o_{t+1} - u(k(kB_{t-1} + \beta x^o_t + \beta x^y_t) + \beta x^o_{t+1} + \beta x^y_{t+1})] .$$

Here we assume subgame perfection between the old and young versions of the same person. The young agent has to choose x^y_t provided that he will 'do his best in the next period when he becomes old'. Therefore the young player has to solve first the decision problem as if he were the old agent in the next period in order to derive a decision rule which describes what the young agent would do when old. Only then, given this derived rule, the young agent's problem arrives to the very same dimension in which his contemporary old agent's problem lies.

There will be two basic premises in approaching this overlapping generations (dynamic) game. One is that in each period one agent's decision is not affected by his contemporary's decision. This underlying assumption is of Cournot. The other is of Markov type. No matter what has happened in the previous period, the current decision depends only on the value of the state. Under these two premises, the activity depends on nothing but the state variable. Abusing formal expressions, we denote decision rules, which associate activity levels with values of the state, by $x^o_t = O_t(kB_{t-1})$ and $x^y_t = Y_t(kB_{t-1})$. [See Maskin/Tirole(1982,1985).]

Due to our assumption on activity constraints, we simply assume that the agents', young or old, strategy sets are given by :

$$S = \{ \ f : f \text{ is a } \underline{\text{function}} \text{ from } R_+ \text{ to } R \ \}.$$

Thus S is the largest set which is relevant. Note that we do not impose any mathematical structures on it.

We only consider the subgame-perfect equilibrium concept for this overlapping generations game. [See Selten(1975) and van Damme(1983).]

DEFINITION 2.2 : A sequence of strategies $\langle (O_t(\cdot), Y_t(\cdot)) \rangle$ is a subgame-perfect equilibrium if it satisfies the following inequalities simultaneously, for all $t > 1$ and for all $B_0 \ \epsilon \ [0, \infty)$:

$$O_t(kB_{t-1}) - u[kB_{t-1} + \beta O_t(kB_{t-1}) + \beta Y_t(kB_{t-1})]$$

$$> x^o{}_t - u[kB_{t-1} + \beta x^o{}_t + \beta Y_t(kB_{t-1})]$$

for all $x^o{}_t \ \epsilon \ R$

and

$$Y_t(kB_{t-1}) - u[kB_{t-1} + \beta O_t(kB_{t-1}) + \beta Y_t(kB_{t-1})]$$

$$+ d \cdot \Gamma_{t+1}(k(kB_{t-1} + \beta O_t(kB_{t-1}) + \beta Y_t(kB_{t-1})))$$

$$> x^y{}_t - u[kB_{t-1} + \beta O_t(kB_{t-1}) + \beta x^y{}_t]$$

$$+ d \cdot \Gamma_{t+1}[k(kB_{t-1} + \beta O_t(kB_{t-1}) + \beta x^y{}_t)]$$

for all $x^y{}_t \ \epsilon \ R$

where $\Gamma_{t+1}(g) = O_{t+1}(g) - u[g + \beta O_{t+1}(g) + \beta Y_{t+1}(g)]$.

A subgame-perfect equilibrium is $\underline{\text{stationary}}$ if, for all $t > 1$, $O_t(\cdot) = O_{t+1}(\cdot)$ and $Y_t(\cdot) = Y_{t+1}(\cdot)$.

Note that the above inequalities have to hold for all $B_0 \ \epsilon \ [0, \infty)$. We

might consider t = 0 as the beginning of the biological time and set B_0 = 0. [See Leininger (1985) and Bernheim/Ray(1987).]

Obtaining a stationary subgame-perfect equilibrium is finding a pair of functions (O(g),Y(g)) such that for all g ϵ [0,β) :

$$O(g) \text{ solves } \max_{x^o} \quad x^o - u[g + \beta x^o + \beta Y(g)]$$

and

$$Y(g) \text{ solves } \max_{x^y} \quad x^y - u[g + \beta O(g) + \beta x^y]$$
$$+ d \cdot \Gamma[k(g + \beta O(g) + \beta x^y)]$$

where $\Gamma(g) = O(g) - u[g + \beta O(g) + \beta Y(g)]$. Note that Y(g) again appears as a part of the problem.

Let $<$ (O(g),Y(g)) $>$ be a stationary subgame-perfect equilibrium. The first-order condition of the old agent's problem is :

$$1 = \beta u'[g + \beta x^o + \beta Y(g)].$$

Since O(g) belongs to a stationary subgame-perfect equilibrium, we have :

(6) $1 = \beta u'[g + \beta O(g) + \beta Y(g)]$ for all g $\geqslant$ 0.

The first-order condition of the young agent's problem is :

$$1 = \beta u'[g + \beta O(g) + \beta x^y] - k \beta d \cdot [O'[k(g + \beta O(g) + \beta x^y)]$$
$$- u'\{k(g + \beta O(g) + \beta x^y) + \beta O[k(g + \beta O(g) + \beta x^y)]$$

$$+ \beta Y[k(g + \beta O(g) + \beta x^y)]\} \cdot \{1 + \beta O'[k(g + \beta O(g) + \beta x^y)]$$

$$+ \beta Y'[k(g + \beta O(g) + \beta x^y)]\}].$$

Since $Y(g)$ belongs to a stationary subgame-perfect equilibrium, we obtain :

$$1 = \beta u'[g + \beta O(g) + \beta Y(g)] - k\beta d \cdot [O'[k(g + \beta O(g) + \beta Y(g))]$$

$$- \beta u'\{k(g + \beta O(g) + \beta Y(g)) + \beta O[k(g + \beta O(g) + \beta Y(g))]$$

(7)

$$+ \beta Y[k(g + \beta O(g) + \beta Y(g))]\} \cdot \{1/\beta$$

$$+ O'[k(g + \beta O(g) + \beta Y(g))] + Y'[k(g + \beta O(g) + \beta Y(g))]\}]$$

for all $g \geqslant 0.$

(6) and (7) imply :

(8) $Y'[kv(1/\beta)] = - 1/\beta.$

Note that (8) is true irrespective of k and v( ). Hence we have :

$$Y(g) = - g/\beta + \text{const}$$

$$O(g) = v(1/\beta)/\beta - \text{const}$$

$$g + \beta Y(g) + \beta O(g) = v(1/\beta) \text{ for all } g \geqslant 0.$$

That $g + \beta O(g) + \beta Y(g) = v(1/\beta)$ for all $g \geqslant 0$ also comes directly from the fact that any pair of subgame-perfect equilibrium strategies in a certain period must be such that the old player cannot benefit from changing his activity level unilaterally. Whatever g is,

$$g + \beta O_t(g) + \beta Y_t(g) = v(1/\beta)$$

for any subgame-perfect equilibrium. So we define $B^N = v(1/\beta)$ as the steady-state level of any (stationary) subgame-perfect equilibrium of the public durable bad. We have the following proposition :

PROPOSITION 2.5 : In this overlapping generations economy, the steady-state level of any subgame-perfect equilibrium of the public durable bad is not Pareto optimal. That is,

$$B^P = v(\frac{1 - kd}{2\beta}) < B^N = v(\frac{1}{\beta}) \quad (B^P < B^L < B^N).$$

This again confirms the usual Pigouvian conclusion that if there is an externality, then socially optimal level differs from the individually optimal level [Piogu(1920)].

Note that B^N is constant with respect to k. This means that as k increases inefficiency increases. Also note that B^N decreses as β increases.

PROPOSITION 2.6 : $B^N - B^P$ increases as k increases.

Now suppose that the number of the agents in each period is n and $O_i(g)$ denotes the equilibrium strategy of the old agent i and $Y_i(g)$ that of the young agent i, i = 1,···,n. From the first-order condition of the old agent i, we have :

$$1 = u'[g + \sum_{i=1}^{n} O_i(g) + \sum_{i=1}^{n} Y_i(g)] \text{ for all } g > 0.$$

Hence whatever the level of g is, $g + \sum_{i=1}^{n} O_i(g) + \sum_{i=1}^{n} Y_i(g)$ is constant regardless of n and is $v(1/\beta)$.

PROPOSITION 2.7 : $B^N(n) - B^P(n)$ increases as n increases.

CHAPTER 3

PUBLIC DURABLE GOODS :
PARETO OPTIMALITY AND THE LINDAHL EQUILIBRIUM

In this chapter, we deal with a public durable good. The model
itself is similar to the one for the public durable bad except that
agents' actions are restricted to take on only nonnegative values. The
discussions here follow those in Sections 2.3 and 2.4. We obtain the
steady-state Pareto optimal level of the public durable good. A brief
review on the 'atemporal' and 'intertemporal' Samuelson public good
optimality condition and the derivation of and comparison with the
'intergenerational' Samuelson public good optimality condition appear
in Appendix B. We introduce a Lindahl tax scheme for our model. It
turns out that the steady-state level of the stationary Lindahl
equilibrium is not Pareto optimal. We also show the unique existence
of the stationary Lindahl equilibrium in Appendix C. Last, we discuss
the issue of restoring Pareto optimality and linkage through Pigouvian
externality-corrective tax/subsidy in addition to Lindahl tax.

3.1   The Model

Time is discrete and elapses period by period. All actions occur
within a period and will be indexed by t which takes on integer values.
Agents are born at the beginning of each period and live for two
periods. Each generation consists of only one agent. So there are
always 2 agents in each period, one young agent and one old agent.

The model differs from the usual overlapping generations models in

two ways. First, there exists technology for converting private goods into public durable goods. Second, the public durable good lasts forever once provided in a certain point of time. This situation is very similar to the overlapping generations model with storage technology except that the only way we can store the private good is to convert it into the public durable good [Compare with Koda(1984)].

Usually goods are distinguished by their physical characteristics, their spatial locations, and their temporal locations. In the above model, goods are distinguished by their physical characteristics and their temporal locations. Besides the temporal distinction, goods are identical: so there are two goods in the economy; one private good and one public durable good.

The private good is nondurable. Its lifetime is only one period long. It plays the role of the numeraire good in each period and can be interpreted as leisure (negative of labor). The private good will be indexed by t.

There is a technology for converting private goods into public durable goods. This technology is described by a cost function, c, from R_+ to R_+. The cost of producing z units of the public durable good is c(z) units of the private good. In other words, c(z) units of the private good have to be forgone to obtain z units of the public durable good. The following assumption on c(·) is maintained throughout :

$$c(z) = z \text{ for } z \geqslant 0.$$

Notice that the technology is period-insensitive, that is, free of indices meaning, in particular, being free of any technological progress. This assumption can be easily relaxed in the discussion of Pareto optimality.

Here the public durable good lasts forever with depreciation. The newly provided public durable good is, thus, indexed by the period of provision. The depreciation rate is $(1 - k)$, where $0 < k < 1$, and can be described as wearing down or detraction in the good's services due to ageing. If one unit of the public durable good is provided in some period, say period t, then without any additional provision of the public durable good there will be k units of the public durable good in the next period, period t+1, and k^2 units of the public durable good in the following next period, period t+2, and so on. The total level of the public durable good in period t can be expressed by the following state equation :

$$Z_t = kZ_{t-1} + z_t$$

where Z_t denotes the total (service) level of the public durable good in period t and z_t denotes the level of the public durable good newly provided in period t. The total level of the public durable good is the sum of the leftover from the previous period and the newly provided public durable good. Recall that z takes on only nonnegative values.

Agents are considered to be identical except for their periods of birth. The preferences of each agent are the same and are represented by a utility function. The utility of agent t depends on consumption

of the private good and the public durable good :

$$U^t = U^t(Z_t, x^y_t, Z_{t+1}, x^o_{t+1})$$
$$= u(Z_t) + x^y_t + d[u(Z_{t+1}) + x^o_{t+1}]$$

where x^y_t denotes the level of the private good consumed by agent t in period t (when the agent is young), x^o_{t+1} the level of the private good consumed by agent t in period t+1 (when the agent is old), Z_t the total (service) level of the public durable good in period t and d the time preference rate ($0 < d < 1$). The additively separable and periodwise quasi-linear assumption of utility function is crucial for our entire research and it seems to be very difficult to relax this assumption. The underlying reason can be found in Section 3 of Appendix B. The following assumptions on u(·) are maintained throughout :

$u(Z)$ is defined over $Z \geqslant 0$

$u' > 0$; $\lim_{Z \to 0} u'(Z) > 1$; $\lim_{Z \to \infty} u'(Z) = 0$

$u'' < 0$.

We can view $u(Z)$ as the willingness-to-pay for the level of the public durable good, Z, in terms of the private numeraire good.

Each agent is endowed with the private good in both periods ; w^y_t units when young and w^o_{t+1} units when old. It is assumed :

> The agent's per period endowment of the private
> good is bounded above and sufficient to finance
> any desired level of the public durable good.

Our assumption on endowments makes the feasibility constraints redundant and, along with assumptions on preferences, insures the interiority of solutions. We leave the case where the boundaries are binding to future study.

Preferences and technology are sufficiently simple enough to allow for tractable results. The drawback is that some results might hold only in these simplifications. [See Section 3.B of Appendix B.]

3.2   The Steady-state Pareto Optimal Level

With the discussion of Appendix B in mind, we derive the steady-state Pareto optimal level for our model. Note that in our model there is a nice feature that the planner faces the same problem no matter what his reference point of time is.[5]

Given kZ_{t-1}, the planner has to solve the following problem :

$$
\begin{aligned}
(9) \quad \max_{\langle z_t \rangle \geqslant 0} \quad & 2u(kZ_{t-1} + z_t) - z_t \\
& + d[\ 2u(k^2 Z_{t-1} + kz_t + z_{t+1}) - z_{t+1}\ ] \\
& + d^2[\ 2u(k^3 Z_{t-1} + k^2 z_t + kz_{t+1} + z_{t+2}) - z_{t+2}\ ] \\
& + \cdots.
\end{aligned}
$$

The maximand in (9) can be interpreted as the "aggregate net benefits" from the sequence of provisions of the public durable good, $\langle z_t \rangle$, in terms of the private good in period t.

Again we need to show that the above maximization problem is well-defined, before obtaining the first-order condition. Invoking our

assumption on endowments, w^y and w^o are bounded above, say, by B. Then

$B + kB + k^2 B + \cdots = B/(1 - k)$ is bounded. Hence $2u[B/(1 - k)] - B$ is

bounded above, say, by M. Then $M + dM + d^2 M + \cdots = M/(1 - d)$ is

bounded. Therefore the above maximization program is well-defined.

Then the first-order conditions are :

(10a) z_t ; $2u'[kZ_{t-1} + z_t]$

                 $+ kd2u'[k^2 z_{t-1} + kz_t + z_{t+1}]$

                 $+ (kd)^2 2u'[k^3 z_{t-1} + k^2 z_t + kz_{t+1} + z_{t+2}]$

                 $+ \cdots \leq 1$ with = if $z_t > 0$

(10b) z_{t+1} ; $d2u'[k^2 Z_{t-1} + kz_t + z_{t+1}]$

                 $+ dkd2u'[k^3 z_{t-1} + k^2 z_t + kz_{t+1} + z_{t+2}]$

                 $+ d(kd)^2 2u'[k^4 z_{t-1} + k^3 z_t + k^2 z_{t+1} + kz_{t+2} + z_{t+3}]$

                 $+ \cdots \leq d$ with = if $z_{t+1} > 0$

(10c) z_{t+2} ; $d^2 2u'[k^3 Z_{t-1} + k^2 z_t + kz_{t+1} + z_{t+2}]$

                 $+ d^2 kd2u'[k^4 Z_{t-1} + k^3 z_t + k^2 z_{t+1} + kz_{t+2} + z_{t+3}]$

                 $+ d^2 (kd)^2 2u'[k^5 Z_{t-1} + k^4 z_t + k^3 z_{t+1} + k^2 z_{t+2} + kz_{t+3}$

                 $+ z_{t+4}] + \cdots \leq d^2$ with = if $z_{t+1} > 0$

and so on for $z_{t+3}, \cdots$.

Now suppose that $\mu\left(\dfrac{1 - kd}{2}\right) < kZ_{t-1}$. Construct a sequence $< z^1_t >$

and $< z^2_t >$ such that :

z^1_t is chosen so that $kZ_{t-1} + z^1_t > \mu\left(\dfrac{1 - kd}{2}\right)/k$, $z^1_{t+1} = 0$;

$z^2_t = 0$, $z^2_{t+1} = kz^1_t$;

$z^1_{t+2} = z^2_{t+2}$.

Plug $< z^1_t >$ into the maximand in (9). Then we have :

(11a)
$$2u[kZ_{t-1} + z^1_t] - z^1_t$$
$$+ d2u[k^2Z_{t-1} + kz^1_t]$$
$$+ d^2[u[k^3Z_{t-1} + k^2z^1_t + z^1_{t+2}] - z^1_{t+2}]$$
$$+ \cdots.$$

Plug $< z^2_t >$ into the maximand in (9). Then we have :

(11b)
$$2u[kZ_{t-1}]$$
$$+ d[2u[k^2Z_{t-1} + kz^1_t] - kz^1_t]$$
$$+ d^2[u[k^3Z_{t-1} + k^2z^1_t + z^1_{t+2}] - z^1_{t+2}]$$
$$+ \cdots.$$

Subtract (11a) from (11b). Then we obtain :

$$2u[kZ_{t-1}] - (1 - kd)kZ_{t-1}$$
$$- [2u[kZ_{t-1} + z^1_t] - (1-kd)(kZ_{t-1} + z^1_t)]$$
$$> 0.$$

Hence $< z_t > = < \max[\mu(\dfrac{1 - kd}{2}) - kZ_{t-1}, 0] >$ is the only solution to (9) given kZ_{t-1}.

Whatever the level of kZ_{t-1} is, there is an n such that
$$k^nZ_{t-1} < \mu(\dfrac{1 - kd}{2})$$
since $kZ_{t-1} < \infty$. Thus we call $Z^P = \mu(\dfrac{1 - kd}{2})$ as the steady-state Pareto optimal level of the public durable good. We have the following

proposition :

PROPOSITION 3.1 : In this overlapping generations economy the

steady-state Pareto optimal level of the public durable good is :

$$Z^P = \mu \left[\frac{(1 - kd)}{2} \right]$$

where $\mu = (u')^{-1}$.

Remark : Again $< Z^P >$ is not a Pareto optimal sequence among the

steady-state sequences but the only Pareto optimal sequences.

Note that Z^P increases as k increases from 0+ to 1-. Similarly Z^P

increases as k increases from 0+ to 1-.

The new provision level, z^P, for the steady-state Pareto optimal

level will be $(1 - k)\mu[(1 - kd)/2]$ since $Z^P = kZ^P + z^P$.

If the number of the agents in each period is n, then the

steady-state Pareto optimal level, $Z^P(n)$, is :

$$Z^P = \mu \left[\frac{1}{n} \cdot \frac{(1 - kd)}{2} \right]$$

$Z^P(n)$ increases as n increases.

3.3   Stationary Lindahl Equilibrium

As in Section 2.4, one simple way of approaching the problem of

dealing with the leftover of the public durable good from the previous

period is to regard the public durable good from the previous period as

the endowment of the public durable good to the populace. Agents cannot do anything about it. Then we can have a following version of the Lindahl equilibrium concept.

DEFINITION 3.1 : A (intergenerational) Lindahl equilibrium is a sequence $< (p^y_t, p^o_t), 1^*_t >$ of tax share vectors (p^y_t, p^o_t) and a level of new provision 1^*_t of the public durable good such that $p^y_t + p^o_t = 1$ [6] for all $t > 1$ and when the agent t's (the young agent in period t) tax share is p^y_t and the agent (t-1)'s (the old agent in period t) tax share is p^o_t the desired level of public durable good output of each agent in period t equals 1^*_t for all $t > 1$.

A <u>stationary</u> (intergenerational) Lindahl equilibrium is a sequence $< (p^y, p^o), 1^*(q) >$ of tax share vectors (p^y, p^o) and a schedule $1^*(q)$ such that $p^y + p^o = 1$ and, for $i = y, o$, when i's tax share is p^i, his desired level of the public durable good output equals $1^*(q)$ where q is the leftover of the public durable good from the previous period.

Interestingly, in this model there is only one stationary Lindahl equilibrium given k and d and there is no nonstationary Lindahl equilibrium. This nonexistence result even holds for the case where agents are not identical but have additively separable quasi-linear preferences. [See Appendix C.]

$p^y > p^o$ is imposed a priori. This restriction might be justified since one unit of the public durable good will generate more services to the young agent than to the old agent.

The old agent's problem is :

$$\max_{1^o > 0} \; u(q + 1^o) - p^o 1^o.$$

The first-order condition is :

$$u'[q + 1^o] \le p^o \quad \text{with equality if } 1^o > 0.$$

Hence from $1^{o*} = \mu(p^o)$ when $q = 0$,

$$1^o(p^o; q) = \begin{cases} \mu(p^o) - q & \text{if } \mu(p^o) > q \\ 0 & \text{if } \mu(p^o) < q \end{cases}$$

where $\mu = (u')^{-1}$. Let us define :

$$v(p^o; q) = \max_{1^o > 0} u[q + 1^o] - p^o 1^o = u[q + 1^o(p^o; q)] - p^o 1^o(p^o; q)$$

$$= \begin{cases} u[\mu(p^o)] - p^o \mu(p^o) + p^o q & \text{if } \mu(p^o) > q \\ u[q] & \text{if } \mu(p^o) < q. \end{cases}$$

Then the young agent's problem is :

$$\max_{1^y > 0} \; u(q + 1^y) - p^y 1^y + dv[p^o; k(q + 1^y)].$$

Note that $u[q + 1^y] - p^y 1^y + dv[p^o; k(q + 1^y)]$ is unimodal.

Consider $q = 0$ in particular. See Figure I. Given p^o and, thus, p^y, there are two possible cases depending on [1] the magnitude of k, [2] the magnitude of d, and [3] the shape and slope of $u(\cdot)$. One is

that if $u'[\mu(p^o)/k] - p^y + dp^o < 0$, then $u[1^y] - p^y \cdot 1^y + dv(p^o;k1^y)$

achieves its maximum between $\mu(p^y)$ and $\mu(p^o)/k$ and the other is that

if $u'[\mu(p^o)/k] - p^y + dp^o > 0$, then its maximum occurs at somewhere

beyond $\mu(p^o)/k$. The lining-up of 1^y and 1° given q is possible only

when the equilibrium levels of p^y and p^o satisfy :

(12) $u'[\mu(p^o)/k] - p^y + dp^o < 0.$

Thus the corresponding relevant problem should be :

$$\max_{1^y > 0} \; u[q + 1^y] - p^y 1^y + d\{u[\mu(p^o)] - p^o\mu(p^o) + p^o kq\}$$

From the first-order condition, $1^{y*} = \mu(p^y - kdp^o)$ when $q = 0$. Thus

$$1^y(p^y,p^o;q) = \begin{cases} \mu(p^y - kdp^o) - q & \text{if } \mu(p^y - kdp^o) > q \\ 0 & \text{if } \mu(p^y - kdp^o) < q. \end{cases}$$

In order to line up $1^o(p^o;q)$ and $1^y(p^y,p^o;q)$ together, we need :

$$\mu(p^y - kdp^o) = \mu(p^o)$$

Hence $\mu(p^y - kdp^o) = \mu(p^o)$ means $p^y - kdp^o = p^o$, since μ is strictly

monotonic. From $p^y + p^o = 1$ and $p^y - kdp^o = p^o$, we obtain :

$$p^y = \frac{1 + kd}{2 + kd} \quad \text{and} \quad p^o = \frac{1}{2 + kd}.$$

Note that the equilibrium values of p^y and p^o satisfy our restriction

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FIGURE I.   Lining Up of $1^o(p^o;q)$ and $1^y(p^y,p^o;q)$

$q = 0$

$u(1) - p^o 1$

$u(1) - p^y 1$

$\mu(p^y)$

$\mu(p^o) = 1^{o*}$

$u(1) - p^y + dv(p^o;k1)$

$d[u[\mu(p^o)] - p^o\mu(p^o) + dp^ok1]$

$u(k1)$

$\mu(p^y)$ 1^{y*} $\mu(p^o)/k$

(12) :
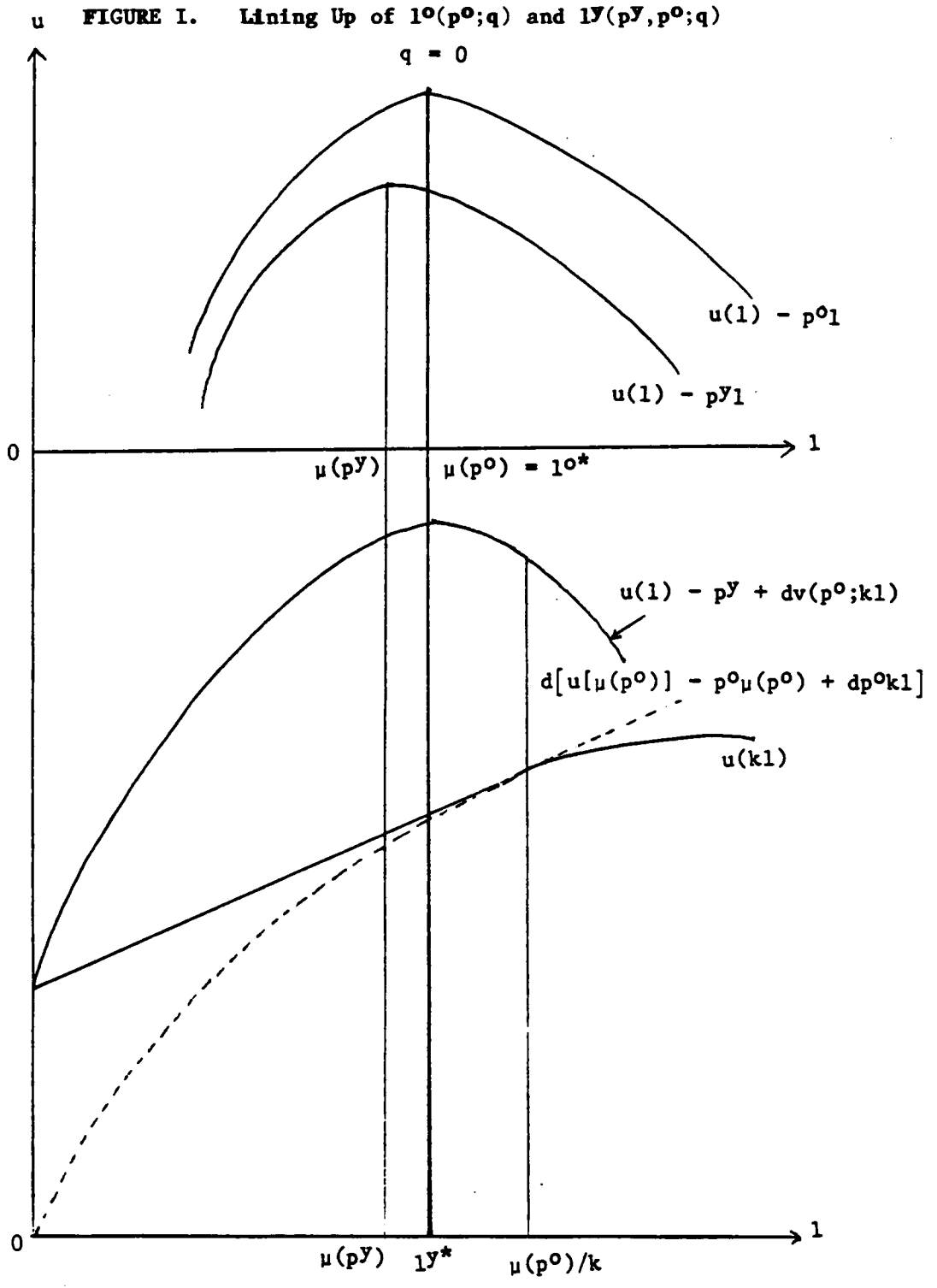
$$u'[\mu(\frac{1}{2 + kd})/k] - \frac{1 + kd}{2 + kd} + d \frac{1}{2 + kd}$$

$$< u'[\mu(\frac{1}{2 + kd})] - \frac{1 + kd}{2 + kd} + d \frac{1}{2 + kd} = 0.$$

The stationary Lindahl equilibrium schedule of the public durable good with respect to q is :

$$1^*(q) = \max[\mu(\frac{1}{2 + kd}) - q, 0].$$

Whatever the level of q is, there is an n such that $k^n q < \mu[1/(2 + kd)]$ since $q < \infty$. $Z^L = \mu[1/(2 + kd)]$ is the steady-state level of the stationary Lindahl equilibrium level of the public durable good.

PROPOSITION 3.2 : In this overlapping generations model the steady-state level of the stationary Lindahl equilibrium is not Pareto optimal. That is,

$$Z^L = \mu(\frac{1}{2 + kd}) < Z^P = \mu(\frac{1 - kd}{2}).$$

However this is not surprising at all since the agents are only concerned about their lifetimes while the planner is concerned about entire generations involved.

Note that Z^L increases as k increases from 0+ to 1-. Similarly Z^L increases as d increases from 0+ to 1-.

$\lim_{k \to 0+} Z^L = \lim_{k \to 0+} Z^P$, meaning that when depreciation goes to 100% the

intergenerational externalities goes to zero and a Lindahl equilibrium

is Pareto optimal. $\lim_{k \to 1-} Z^L < \lim_{k \to 1-} Z^P$; $\lim_{d \to 1-} Z^L < \lim_{d \to 1-} Z^P$.

PROPOSITION 3.3 : If u" is nonincreasing, then $Z^P - Z^L$ increases as k

increases from 0+ to 1-.

If the extra marginal utility from an extra unit of the public durable

good is nonincreasing, then the inefficiency increases as the

durability increases.

If the number of agents in each generation is n, then :

$$p^o(n) = \frac{1}{n} \cdot \frac{1}{2 + kd} \quad \text{and} \quad p^y(n) = \frac{1}{n} \cdot \frac{1 + kd}{2 + kd}$$

since $(p^o + p^y)n = 1$ and $p^y - kdp^o = p^o$. Accordingly $Z^L(n)$ will be :

$$Z^L(n) = \mu\left(\frac{1}{n} \cdot \frac{1}{2 + kd}\right).$$

$Z^L(n)$ increases as n increases.

PROPOSITION 3.4 : If u"(x)/x is monotonically decreasing/increasing,

then $Z^P(n) - Z^L(n)$ increases/decreases as k increases from 0+ to 1-.

If the average of the extra marginal utility from an extra unit of the

public durable good is monotonically decreasing/increasing, then the

inefficiency increases/decreases as the number of agents in each period

increases.

3.4 Pigou-Lindahl Tax/Subsidy Scheme

Contrary to the usual (atemporal) result that the Lindahl equilibrium is Pareto optimal [e.g., Wicksell(1958), Lindahl(1958), Foley(1970), Feldman(1980,pp114-119), Tresch(1981,p119,fn11), and Cornes/Sandler(1986,pp98-102)], the Lindahl equilibrium of Definition 3.1 is not Pareto optimal. Relying only on unanimity and balanced budget together will not achieve Pareto optimality and linkage.

Linkage requires that the cost of producing the public durable good in a certain period must be covered by every agent who enjoys the extra benefits from that public durable good. Thus linkage implies that there must be some kind of intergenerational transfer mechanism that enables transfers from future generations to past generations. That is, the currently young generation, as the representative of all the future generations who will enjoy the extra benefits from the public durable good left over from the previous period, has to pay to the currently old generation, as the representative of all the past generations who has been contributed to the public durable good accumulated up to the current period.

Let us consider a (stationary) linear transfer payment schedule, $r \cdot q$, on the existing public durable good from the young agent to the old agent. Under this intergenerational transfer mechanism, the old agent's problem is :

$$\max_{1^o > 0} \quad u[q + 1^o] - p^o 1^o + rq$$

From the first-order condition,

$$1^o(p^o;q) = \max[\mu(p^o) - q, 0].$$

The young agent's problem is :

$$\max_{1^y > 0} \quad u[q + 1^y] - p^y 1^y - rq$$
$$+ d\left[\max_{1^o > 0} \ u[k(q + 1^y) + 1^o] - p^o 1^o + rk(q + 1^y) \right].$$

From the lining-up condition,

$$1^y(p^y, p^o; q) = \max[\mu(p^y - kdp^o - kdr) - q, 0].$$

Thus $p^o = p^y - kdp^o - kdr$. The balanced budget condition, $p^y + p^o = 1$, and the lining-up condition, $p^o = p^y - kdp^o - kdr$, together do not solve for r a priori. We cannot determine p^o, p^y, and r without an additional condition. So we apply the externality-corrective tax/subsidy idea[Pigou (1920)] to the Lindahl conditions. This means pinning down the unanimously chosen demand schedule on the Pareto optimal level; that is, $\mu(p^o) = \mu(p^y - kdp^o - kdr) = \mu[(1 - kd)/2]$. Then :

$$p^{o*} = \frac{1 - kd}{2} \quad \text{and} \quad p^{y*} = \frac{1 + kd}{2}$$

$$r^* = \frac{1 + kd}{2}.$$

Note that $r^* = p^{y*}$.

Hence Lindahl taxes, p^{o*} and p^{y*}, and Pigou tax/subsidy, r^*, achieve unanimity, balanced budget and Pareto optimality with improved linkage.

CHAPTER 4

VOLUNTARY PROVISION THEORY FOR PUBLIC DURABLE GOODS

First, we very briefly review the literature on (atemporal) voluntary provision theory of public goods. Second, we explain the game-theoretic nature of the problem, strategy sets, and equilibrium concepts for our model. Third, we obtain a few noncooperative equilibria. In general, these equilibria turn out to be not Pareto optimal. Finally, we discuss general implications of the voluntary provision theory for the public durable goods.

4.1   Review

Olson(1965) started the theory of voluntary provision of public goods[See Chamberlin(1974) and McGuire(1974)]. He concluded [Olson(1965,p2,p36)] :

> Unless the number of individuals in a group is quite small, or unless there is coercion or some other special device to make individuals act in their common interest, rational, self-interested individuals will not achieve their common or group interests. ... The larger a group is, the farther it will fall short of obtaining an optimal supply of any collective good, and the less likely that it will act to obtain even a minimal amount of such a good. In short, the larger the group, the less it will further its common interests.

Consider an economy with a public good where individual preferences are given by $U^i = u^i(Z) + x^i$, $i = 1, \cdots, n$. Each individual i contributes z^i to produce Z such that $Z = \sum_i z^i$. Then the utility

level of individual i is $U^i = u^i(\sum_j z^j) - z^i + w^i$. The strategy of

individual i is $z^i(> 0)$. The payoff to individual i is :

$$U^i(z^i, \sum_{j \neq i} z^j) = u^i(\sum_j z^j) - z^i.$$

Let $(z^{1*}, \cdots, z^{n*})$ be a Nash equilibrium. Then :

$$U^i(z^{i*}, \sum_{j \neq i} z^{j*}) = \max_{z^i > 0} U^i(z^i, \sum_{j \neq i} z^{j*})$$

which is equivalent to :

$$\frac{\partial u^i(z^i, \sum_{j \neq i} z^{j*})}{\partial z^i} \Big]_{z^i = z^{i*}} < 1 \quad \text{with} = \text{if } z^{i*} > 0 .$$

Consider the following individual optimization problem :

$$\max_{z^i > 0} u^i(z^i) - z^i + w^i$$

The optimal solution for this problem is denoted by z^{i+}. Then the

total level of contribution, $Z^* = \sum_i z^{i*}$, will be $\max\{ z^{i+} \}$ and z^{i*}

will be zero for all i such that $z^{i+} < \max\{ z^{i+} \}$. Let Z^P be the

Samuelson public good optimal level; $\sum_i \partial u^i(Z^P)/\partial Z = 1$. Clearly, Z^P is

greater than Z^*. Voluntary provision equilibrium of the public good in

the Nash sense is suboptimal.

Now assume further that $u^i(Z) = u(Z)$ for all i. The Samuelson

public good optimal level is $\text{argmax } nu(Z) - Z$; $Z^P = \mu(1/n)$ where

$\mu = (u')^{-1}$. As n inceases, Z^P increases. The quasi-linear utility

function implies that the Nash equilibrium level is independent of n,

while the optimal level rises with n. Thus $Z^P - Z^*$ increases as n

increases.

The following results do not depend on quasi-linearity of

preferences. In a model of a binary public good with binary

participation, Palfrey/Rosenthal(1984) concluded that Nash equilibria

are inefficient. In a model of a continuous public good with

continuous contributions, Cornes/Sandler(1986) confirmed that the Nash
equilibrium is typically not Pareto optimal and questioned the idea
that the higher the expected contribution by others the lower will be
the individual's own contribution. In a model of a continuous public
good with binary participation, Gradstein/Nitzan (1987) concluded not
only that voluntary participation is suboptimal but also that as the
number of potential participants increases the inefficiency increases.
Andreoni(1987) showed that as the size of the whole group increases to
infinity, the size of the contributors and the average contribution
decreases to zero.[7]

All theoretical works mentioned above have been developed using
models where time is neglected (atemporal models). Yet we can
interpret those models as dealing with the situation in which the
temporal spillover of an agent's decision lasts a certain number of
periods shorter than or at most equal to the decision horizon of
agents. However, in our model, the intergenerational spillover of
agent's decisions persists longer than the agent's decision horizon
(agent's lifetime). Then we ask what the voluntary provision level
would be in this new situation. We might expect that the young agent
would provide more of the public durable good than he would of the
nondurable public good. However the young agent, knowing that next
generation will provide the public durable good anyway, might act the
same way as he would when the public good is not durable. Our results
indicate that both cases may prevail and confirm that voluntary
provisions would be suboptimal in general.

4.2   Intergenerational Game

4.2.1   The nature of the problem

We use the same model in Chapter 3 :

$$U^t = u(Z_t) + x^y_t + d[u(Z_{t+1}) + x^o_{t+1}] \quad : \text{preferences}$$

$$Z_t = kZ_{t-1} + z_t \qquad\qquad\qquad : \text{state equation}$$

$$c(z_t) + x^o_t + x^y_t = w^o_t + w^y_t \qquad : \text{budget constraint}$$

$$c(z) = z \qquad\qquad\qquad\qquad : \text{technology.}$$

Now agent t voluntarily provides z^y_t units of the public durable good when the agent is young and z^o_{t+1} units when old. The choice of z^y_t and z^o_{t+1} is constrained : $0 \leqslant z^y_t \leqslant w^y_t$ and $0 \leqslant z^o_{t+1} \leqslant w^o_{t+1}$. Given the leftover of the public durable good from the previous period, kZ_{t-1}, we then have the following state equation :

$$(13) \quad \begin{aligned} &Z_t = kZ_{t-1} + z^o_t + z^y_t; \; Z_0 \text{ being given} \\ &\text{with } 0 \leqslant z^o_t \leqslant w^o_t \, , \; 0 \leqslant z^y_t \leqslant w^y_t \end{aligned} \Bigg\} \quad \text{for } t = 1, 2, \cdots$$

We shall denote by $\langle (z^o_t, z^y_t) \rangle$ a nonnegative infinite sequence of voluntary provisions, and by Q the set of all infinite sequences $\langle (z^o_t, z^y_t) \rangle$ that satisfies (13). Q is, therefore, the set of all feasible voluntary provision sequences. Because of our assumptions on utility functions and endowments, we are only concerned with <u>the net</u> <u>(lifetime) benefit</u> from voluntary provisions. Then under a sequence

$< (z^{o}{}_{t}, z^{y}{}_{t}) >$ ε Q, the agents will enjoy :

$$\Pi^{0} = u(kZ_{0} + z^{o}{}_{1} + z^{y}{}_{1}) - z^{o}{}_{1} \qquad\qquad t = 0$$

(14) $\qquad \Pi^{t} = u(kZ_{t-1} + z^{o}{}_{t} + z^{y}{}_{t}) - z^{y}{}_{t} \qquad\qquad t \geqslant 1$

$$\qquad\qquad + d[u(kZ_{t} + z^{o}{}_{t+1} + z^{y}{}_{t+1}) - z^{o}{}_{t+1}]$$

So agent 0 has to choose $z^{o}{}_{1}$ to maximize Π^{0} and each and every agent t,

t ⩾ 1, has to choose $(z^{y}{}_{t}, z^{o}{}_{t+1})$ to maximize Π^{t}.

As we can see in (14), the net lifetime benefit of agent t, Π^{t},

from voluntary provisions, also depends on other players' actions, $z^{o}{}_{t}$

and $z^{y}{}_{t+1}$, as well as his own actions, $z^{y}{}_{t}$ and $z^{o}{}_{t+1}$. The situation,

therefore, is a dynamic game with an infinite horizon.

4.2.2   Strategy set and equilibrium concept

Consider the decision problem of the old player in period t :

$$\max_{z^{o}{}_{t} \geqslant 0} \quad u(kZ_{t-1} + z^{o}{}_{t} + z^{y}{}_{t}) - z^{o}{}_{t}$$

In general the old player's decision rule of choosing $z^{o}{}_{t}$ depends on

$(^{1})$ the state value, kZ_{t-1}, $(^{2})$ his expectations of the other player's

choice, $(^{3})$ the general history of what all the previous players have

chosen, and so on.

The decision problem of the young player in period t, however, is

very different from that of the old player in that period. Considering

the sequential nature of the model, the most natural restriction is

that the <u>decisions</u> <u>are</u> <u>made</u> <u>period</u> <u>by</u> <u>period.</u> In period t the old player (agent t-1) and the young player (agent t) make their moves simultaneously. Since the young player has to take into consideration that he is to make one more decision when he becomes old, periodwise decision-making involves a certain nature of dynamic programming. The young player has to choose $z^y{}_t$ provided that he will 'do his best in the next period'. The decision problem of the young player is :

$$\max_{z^y{}_t \geqslant 0} u(kZ_{t-1} + z^o{}_t + z^y{}_t) - z^y{}_t$$

$$+ d\left[\max_{z^o{}_{t+1} \geqslant 0} u[k(kZ_{t-1} + z^o{}_t + z^y{}_t) + z^o{}_{t+1} + z^y{}_{t+1}] - z^o{}_{t+1}\right].$$

Hence the decision-making of the young player carries more complexity than that of the old player. Only after he solves the decision problem as if he were the old player in the next period in order to derive the decision rule of his old age, can the young player worry about how to choose $z^y{}_t$. This amounts to assuming the subgame perfection between the old and young versions of the agent.

In order to pull the dimension of the decision problem of the young player down to that of the old player, we need two assumptions. One is that in each period one player's decision is not affected by his contemporary's decisions. The other is that no matter what happened in the previous period, the current decisions depend only on the initial state of the current period out of the entire history. These two assumptions, Cournot and Markov, implies that voluntary provision decisions depend on nothing but the state variable and allow us to

denote decision rules, which associate voluntary provision levels with values of state, by $z^o{}_t = 0_t(kZ_{t-1})$ and $z^y{}_t = Y_t(kZ_{t-1})$ for all t. [See Maskin/Tirole(1982, 1985).]

The strategy sets for both the old player and the young player in a certain period consist of such decision rules. The strategy set, $S^o{}_t$, of player t-1 when he is old consists of all functions from the set of values which kZ_{t-1} possibly takes to the interval $[0,w^o{}_t]$ and the strategy set, $S^y{}_t$, of player t when he is young consists of all functions from the set of values which kZ_{t-1} possibly takes to the interval $[0,w^y{}_t]$. We simply assume [8]:

$$S^o{}_t = S^y{}_t = S = \{ f : f \text{ is a function from } [0,\infty) \text{ to } [0,\infty) \}.$$

Thus S will be the largest set which is relevant, since "there is nothing a priori to limit the functional form" for strategy sets [Dasgupta (1984,p420)].

We only consider the subgame-perfect equilibrium concept. [See Selten(1975) and van Damme(1983).]

DEFINITION 4.1 : A sequence of strategies $\langle (0_t{}^*(\cdot),Y_t{}^*(\cdot)) \rangle$ is a subgame-perfect equilibrium if it satisfies the following inequalities simultaneously, for all $t > 1$ and for all $Z_0 \in [0,\infty)$:

$$u[kZ_{t-1} + 0_t{}^*(kZ_{t-1}) + Y_t{}^*(kZ_{t-1})] - 0_t{}^*(kZ_{t-1})$$

(15a) $$> u[kZ_{t-1} + z^o{}_t + Y_t{}^*(kZ_{t-1})] - z^o{}_t$$

for all $z^o{}_t > 0$

$$u[kZ_{t-1} + O_t{}^*(kZ_{t-1}) + Y_t{}^*(kZ_{t-1})] - Y_t{}^*(kZ_{t-1})$$

$$+ dV_{t+1}\big[k[kZ_{t-1} + O_t{}^*(kZ_{t-1}) + Y_t{}^*(kZ_{t-1})]\big]$$

(15b) $$\geqslant u[kZ_{t-1} + O_t{}^*(kZ_{t-1}) + zY_t] - zY_t$$

$$+ dV_{t+1}\big[k[kZ_{t-1} + O_t{}^*(kZ_{t-1}) + zY_t]\big]$$

for all $zY_t \geqslant 0$

where $V_{t+1}[q] := u[q + O_{t+1}{}^*(q) + Y_{t+1}{}^*(q)] - O_{t+1}{}^*(q)$.

Note first that the decision problem of the old agent in period t is a very simple optimization problem given kZ_{t-1} and, thus, $Y^*(kZ_{t-1})$. However the decision problem of the young agent in period t requires internalizing all the externalities imposed on him by the next young player's behavior, $Y_{t+1}{}^*(\cdot)$, as well as the old version of himself, $O_{t+1}{}^*(\cdot)$. Only then, as we have seen in (15b), the objective function of the young agent in period t has the single decision variable given kZ_{t-1} and, thus, $O_t{}^*(kZ_{t-1})$. Secondly, from (15b) existence of a subgame-perfect equilibrium requires that there is a solution to the following maximization problem, for all $q \geqslant 0$:

$$\max_{x \geqslant 0} u[q + O_t{}^*(q) + x] - x + dV_{t+1}[k(q + O_t{}^*(q) + x)].$$

Clearly, the existence of a solution to the above problem depends on the structure of $V_{t+1}(\cdot)$, which depends on $O_{t+1}{}^*(\cdot)$ and $Y_{t+1}{}^*(\cdot)$, which[9] depend on the mathematical structures of the strategy sets.

Generally the decisions of the young agent in period t will have the form :

$$z^y{}_t = f[q \; ; \; Y_{t+1}{}^*(\cdot)]$$

Thus the optimal decision rule, $Y_t{}^*(\cdot)$, will be functionally related to $Y_{t+1}{}^*(\cdot)$ and so on recursively, for instance :

$$Y_t{}^*(\cdot) = g[Y_{t+1}{}^*(\cdot)]$$

So for an infinite horizon problem, a fixed point of mapping q, if any, will be a natural object to concentrate on. [See Phelps/Pollack(1968) and Kydland/Prescott(1977).]

DEFINITION 4.2 : An identity sequence of a pair of provisions $\langle\, (0^*(),Y^*()) \,\rangle$ is a <u>stationary</u> subgame-perfect equilibrium if it satisfies the following inequalities simultaneously, for all $q > 0$:

$$u[q + 0^*(q) + Y^*(q)] - 0^*(q) > u[q + z^o + Y^*(q)] - z^o$$

for all $z^o > 0$

$$u[q + 0^*(q) + Y^*(q)] - Y^*(q) + dV[k[q + 0^*(q) + Y^*(q)]]$$

$$> u[q + 0^*(q) + z^y] - z^y + dV[k[q + 0^*(q) + z^y]]$$

for all $z^y > 0$

where $V[q] = u[q + 0^*(q) + Y^*(q)] - 0^*(q)$.

Obtaining a stationary subgame-perfect equilibrium is finding a pair of functions $(0^*(q),Y^*(q))$ such that, for all $q > 0$,

$$0^*(q) : \max_{z^o > 0} \quad u[q + Y^*(q) + z^o] - z^o$$
solves

$$Y^*(q) : \max_{z^y > 0} \quad u[q + 0^*(q) + z^y] - z^y + d\big[u[k(q + 0^*(q) + z^y)$$
$$+ 0^*(k(q + 0^*(q) + z^y)) + Y^*(k(q + 0^*(q) + z^y))]$$
$$- 0^*(k(q + 0^*(q) + z^y))\big].$$

Note that the solution $Y^*(\cdot)$, if any, appears in the problem itself.

"Although this [identity] restriction might appear "natural" since all generations have the same preferences, and face the same technology, there is no demonstration of the fact that along [an] $\cdots$ equilibrium, the ... schedules would have to be identical" [Lane/Mitra (1981,p322)]. There may exist some non-identical (cyclical) equilibria. [See McTaggart/Salant(1986) for an example.]

DEFINITION 4.3 : A subgame-perfect equilibrium is **periodic** if there exists a integer H and H pair of functions
($(\tilde{0}_1(q),\tilde{Y}_1(q))$, $\cdots$,$(\tilde{0}_H(q),\tilde{Y}_H(q))$) such that $0_{h+Ht}^* = \tilde{0}_h(q)$ and $Y_{h+Ht}^* = \tilde{Y}_h(q)$, for $t \geqslant 1$ and $h = 1,2, \cdots ,H$. The integer H is the period of the equilibrium.[See Bernheim/Ray(1987).]

4.3. **Noncooperative Equilibria**

4.3.1 **Stationary and periodic equilibrium : examples**

There are three parameters in this intergenerational game : $u(\cdot)$, d and k. Any noncooperative equilibrium level of the net lifetime benefit depends on these parameters crucially. In the examples provided below we show some relations between equilibria and

parameters.

Consider an overlapping generations economy with the public durable good where $u(z) = 2\sqrt{z}$. Suppose the public good is not durable ($k = 0$). The individual optimization problem, regardless of age, is to $\max_{z > 0} 2\sqrt{z} - z$. The individual optimum is 1. In fact, 1 is the total provision level (the simple Nash equilibrium level) when $k = 0$.

Stationary subgame-perfect equilibrium

We might expect that the young agent would provide more of the public durable good than he would of the nondurable public good. However, for relatively small k, the agent, knowing that the next generation might provide the public durable good anyway, seemingly acts the same way as he would when the public good is not durable. In the following example we show that this scenario holds.

EXAMPLE A : $d = 0.8$ and $k = 0.25$.

Claim 1 : $\langle\, (0, \max[1 - q, 0])\, \rangle$ is a stationary subgame-perfect equilibrium.

Simply consider the following optimization problem :

$$\max_{z \geq 0} \quad 2\sqrt{q + z} - z$$
$$+ (0.8)2\sqrt{0.25(q + z) + \max[1 - 0.25(q + z), 0]}$$

This problem achieves its maximum if z is chosen according to

max[1 - q,0] :

$$2\sqrt{1} - 1 + q + (0.8)2\sqrt{1} = 2.6 + q \qquad \text{for } 0 < q < 1$$

$$2\sqrt{q} + (0.8)2\sqrt{1} = 1.6 + 2\sqrt{q} \qquad \text{for } 1 < q < 4$$

$$2\sqrt{q} + (0.8)2\sqrt{0.81q} = 2.8\sqrt{q} \qquad \text{for } 4 < q.$$

Since q + max[1 - q,0] > 1 for all q > 0, we know that

< (0,max[1 - q,0]) > is a stationary subgame-perfect equilibrium

The total level of the public durable good under this stationary

subgame-perfect equilibrium is 1, which is also the simple Nash

equilibrium level when k = 0.

Is < (0,max[1 - q,0]) > the only stationary equilibrium? The

answer is no and, in fact, we can find another kind of stationary

subgame-perfect equilibrium other than < (0,max[1 - q,0]) >. Consider

the following decision rule :

$$v(q) = \begin{cases} \dfrac{1}{4}\left[\dfrac{3.2 - 2\sqrt{\dfrac{q}{0.25}} + \dfrac{q}{0.25}}{0.8}\right]^2 - q & \text{if } q < 1 \\ 0 & \text{if } 1 < q \end{cases}$$

This decision rule says that if the value of the state is less than 1,

then provide in such a way that $2\sqrt{q/0.25} - q/0.25 + (0.8)2\sqrt{q + v(q)}$

achieve 3.2. Otherwise, do not provide. We call these kinds of

decision rules "cooperative" ones.

<u>Claim 2</u> : < (0,v(q)) > is a stationary subgame-perfect equilibrium.

Consider the following optimization problem :

$$\max_{z > 0} \quad 2\sqrt{q + z} - z$$
$$+ (0.8)2\sqrt{0.25(q + z) + \nu[0.25(q + z)]}$$

This problem achieves its maximum :

$3.2 + q$ if z is chosen among $[0, 4 - q]$ for $q < 4$

$2.8\sqrt{q}$ if $z = 0$ is chosen   for $q > 4$

Note that $1.96 = \text{argmax}_{z > 0} \ 2\sqrt{z} - z + (0.8)2\sqrt{0.25z}$. It is true that

(i) $q + \nu(q) > 1$ for all $q > 0$

(ii) $q + \nu(q) < 4$ for all $q \in [0, 1]$,

since :

$$q + \nu(q) = 1/4\left[\dfrac{3.2 - 2\sqrt{\dfrac{q}{0.25}} + \dfrac{q}{0.25}}{0.8}\right]^2 > 1$$

$$\iff \ 1.6 - 2\sqrt{\dfrac{q}{0.25}} + \dfrac{q}{0.25} > 0 \quad \iff \quad \left[\sqrt{\dfrac{q}{0.25}} - 1\right]^2 + 0.6 > 0$$

which is true for all $q > 0$, and

$$q + \nu(q) < 4 \quad \iff \quad 1/4\left[\dfrac{3.2 - 2\sqrt{\dfrac{q}{0.25}} + \dfrac{q}{0.25}}{0.8}\right]^2 < 4$$

$$\iff \ 0 < 2\sqrt{q} - q$$

which is true for all $q \in [0, 4]$. Hence the old agent provides nothing

when the young agent provides more than 1. Given the next young agent

providing according to $\nu(q)$, the presently young agent is indifferent

to providing among $[0,4 - q]$ when the value of the state is q. So he

chooses the "cooperative" one $\nu(q)$. Hence $< (0,\nu(q)) >$ is a stationary

subgame-perfect equilibrium. Agents can do better by choosing z

according to $\nu(q)$ rather than according to $\max[1 - q,0]$. Under $\nu(q)$,

agents achieve their net lifetime benefit $3.2 + q$ for $q < 4$ and $2.8\sqrt{q}$

for $q > 4$. Here every agent Pareto-improves.

Claims 1 and 2 together demonstrate the multiplicity of equilibria

for the same parameter values.

Periodic subgame-perfect equilibrium : example

For relatively large k, it might be the case that durability

disqualifies $< (0,\max[1 - q,0]) >$ as a stationary subgame-perfect

equilibrium. Furthermore there are no "cooperative" equilibria for

certain pairs of d and k. Yet we find some periodic subgame-perfect

equilibria.

EXAMPLE B : d = 0.8 and k = 0.81.

<u>Claim 3</u> : $< \cdots,(0,\max[1 - q,0]),(0,\max[2.9584 - q,0]),\cdots >$ is a

periodic subgame-perfect equilibrium of period 2.

Consider the following optimization problem :

$$\max_{z>0} \quad 2\sqrt{q + z} - z$$
$$+ (0.8)2\sqrt{0.81(q + z)} + \max[1 - 0.81(q + z),0].$$

This problem achieves its maximum if z is chosen according to
max[2.9584 - q,0] :

$$2\sqrt{2.9584} - 2.9584 + q + (0.8)2\sqrt{(0.81)2.9584}$$

$$= 2.9584 + q \qquad\qquad \text{for} \quad 0 < q < 2.9584$$

$$2\sqrt{q} + (0.8)2\sqrt{(0.81)q} = 3.44\sqrt{q} \qquad \text{for} \quad 2.9584 < q.$$

Note that $2.9584 = \text{argmax}_{z>0}\ 2\sqrt{z} - z + (0.8)2\sqrt{0.81z}$ and, therefore,

might be called the guaranteed (maximin) net lifetime benefit level

for this set of parameters. Thus < (0,max[1 - q,0]) > cannot be a

stationary subgame-perfect equilibrium when d = 0.8 and k = 0.81.

Now consider another optimization problem :

$$\max_{z>0}\ 2\sqrt{q + z} - z$$
$$+ (0.8)2\sqrt{0.81(q + z)} + \max[2.9584 - 0.81(q + z),0].$$

The previous problem achieves its maximum if z is chosen according to
max[1 - q,0] :

$$2\sqrt{1} - 1 + q + (0.8)2\sqrt{2.9584} = 3.752 + q \quad \text{for } 0 < q < 1$$

$$2\sqrt{q} + (0.8)2\sqrt{2.9584} = 2.752 + 2\sqrt{q} \qquad \text{for } 1 < q < \frac{2.9584}{0.81}$$

$$2\sqrt{q} + (0.8)2\sqrt{0.81q} = 3.44\sqrt{q} \qquad \text{for } \frac{2.9584}{0.81} < q.$$

Since q + max[2.9584 - q,0] > q + max[1 - q,0] > 1 for all q > 0,

< ···,(0,max[1 - q,0]),(0,max[2.9584 - q,0]),··· > is a periodic

subgame-perfect equilibrium of period 2 when d = 0.8 and k = 0.81.

Remark :

$$< (0, \{ \begin{array}{l} \max[1 - q, 0] \text{ with prob } 1/2 \\ \max[2.9584 - q, 0] \text{ with prob } 1/2 \end{array}) >$$

is a stationary subgame-perfect equilibrium, if the strategy set is enlarged properly.

Let

$$\tau(q) = \begin{cases} 1/4 \left[\dfrac{3.2 - 2\sqrt{\dfrac{q}{0.25}} + \dfrac{q}{0.25}}{0.8} \right]^2 - q & \text{if } q \leqslant 3.24 \\ 0 & \text{if } 3.24 < q. \end{cases}$$

Claim 4 : $< (0, \tau(q)) >$ is a subgame-perfect equilibrium.

Claims 3 and 4 together demonstrate that a periodic subgame-perfect equilibrium and a "cooperative" stationary subgame-perfect equilibrium coexist for the same set of parameters. [We will not prove Claim 4 here since it is supported by Proposition 4.1.]

Note that not every agent Pareto-improves under $< (0, \tau(q)) >$ over $< \cdots, (0, \max[1 - q, 0]), (0, \max[2.9584 - q, 0]), \cdots >$. The 'optimistic' generations, who play $(\max[1 - q, 0], 0)$, do worse and the 'pessimistic' generations, who play $(\max[2.9584 - q, 0], 0)$, do better under $< (0, \tau(q) >$.

The next question naturally arises : Can we always construct such a "cooperative" stationary subgame-perfect equilibrium for any pair of d and k. The answer is no.

EXAMPLE C : d = 0.4 and k = 0.81. Let us construct a "cooperative" decision rule based on Proposition 4.1 (which appears later) :

$$\chi(q) = \begin{cases} 1/4\left[\cfrac{1.6 - 2\sqrt{\cfrac{q}{0.25}} + \cfrac{q}{0.25}}{0.4}\right]^2 - q & \text{if } q < 3.24 \\ 0 & \text{if } 3.24 < q. \end{cases}$$

Suppose that q = 0. The agent can do better by choosing z = 1.8496 since $\max_{z > 0} 2\sqrt{z} - z + (0.4)2\sqrt{0.81z} = 1.8496$ when z = 1.8496 which is greater than 1.6.

Through examples we have seen how k and d mold subgame-perfect equilibria. Next we obtain a few general results.

4.3.2   Existence propositions

Let us go back to the general model. Define the following :

$$\omega = \text{argmax}_{z > 0}\; u[z] - z \qquad\qquad ; \quad \Omega = u[\omega] - \omega + du[\omega];$$

$$\theta = \text{argmax}_{z > 0}\; u[z] - z + du[kz] \quad ; \quad \Xi = u[\theta] - \theta + du[k\theta];$$

$$e(>0) \text{ solves } u[z] - z = u[0] \qquad ; \quad E = du[e] + u[0].$$

PROPOSITION 4.1 : If $(d,k,u(\cdot))$ satisfies $E > \max[\Xi,\Omega]$ and $e > \max[\theta, \omega/k]$, then there exists a stationary subgame-perfect equilibrium $< (0, \upsilon(q;ke,E)) >$ where $\upsilon(q;ke,E)$ is defined by :

$$\upsilon(q;ke,E) = \begin{cases} u^{-1}\left[\cfrac{E - u[q/k] + (q/k)}{d}\right] - q & \text{if } q < ke \\ 0 & \text{if } ke < q \end{cases}$$

where E is a constant and ke is a fixed critical level of the argument

of $y(\cdot)$ which enables $u[q/k] - (q/k) + du[q + y(q)]$ to achieve E for all $q < c$ and becomes zero for all $q > c$.

$\upsilon(q;ke,E)$ guarantees that every agent will achieve a net lifetime benefit of E plus the value of the state, i.e., $\Pi_t = E + kZ_{t-1}$ for all t.

PROPOSITION 4.2 : If $(d,k,u(\cdot))$ satisfies $\Omega > \Xi$ and $\theta < \omega/k$, then there exists a stationary subgame-perfect equilibrium. That is, $< (0,\zeta(q)) >$ where $\zeta(q) = \max[\omega - q,0]$.

PROPOSITION 4.3 : If $(d,k,u(\cdot))$ satisfies $\Xi > \Omega$, then there exist two periodic subgame-perfect equilibria of period 2. These are $< (0,\phi(q)),(0,\psi(q)),\cdots >$ and $< (0,\psi(q)),(0,\phi(q)),\cdots >$ where $\psi(q) = \max[\theta - q,0]$ and $\phi(q) = \max[\omega - q,0]$.

Remark : These two periodic equilibria are not essentially different; the amplitudes and the frequencies are the same and only the phases differ.

Equilibrium strategies we have found so far are special cases in the sense that there may be other equilibrium strategies such that the strategy of the old age is not zero for all $q > 0$ but a non-zero function of the state. In general if $< (0_t^*(q),Y_t^*(q)) >$ is a subgame-perfect equilibrium, then the following must be satisfied, at least, for all t :

$$q + O_t^*(q) + Y_t^*(q) > \omega \text{ for all } q > 0$$

(16) $\quad \{ \; O_t^*(q) = 0 \text{ for all } q > \omega$

If $O_t^*(\overline{q}) > 0$ for some $\overline{q} < \omega$, then $\overline{q} + O_t^*(\overline{q}) + Y_t^*(\overline{q}) = \omega.$

The set of equilibria is restricted by the fact that the decision rule of the old player must be such that it is not advantageous to deviate from it unilaterally.

In the following discussion we demonstrate that a pair of strategies satisfying the conditions in (16) can not be an stationary subgame-perfect equilibrium. In general, the characteristics of $O_t^*(q)$ and $Y_t^*(q)$ are not known yet.

Recall Example A where $u(z) = 2\sqrt{z}$, $d = 0.8$ and $k = 0.25$. We know that $o^*(q) = 0$ and $y^*(q) = \max[1 - q, 0]$ are a stationary subgame-perfect equilibrium strategy pair. Note that $o^*(q) + y^*(q) = \max[1 - q, 0]$. Now consider a strategy pair such as $\max[t^o \cdot (1 - q), 0]$ and $\max[t^y \cdot (1 - q), 0])$ where $t^o + t^y = 1$, $t^o, t^y \in \mathbf{R_+}$. This strategy satisfies the conditions in (16). We show that this strategy pair violates the assumption of subgame-perfection between the old and young versions of the agent. Suppose $q < \omega$. If the agent follows this strategy pair, then he would achieve :

$$u(1) - t^y(1 - q) + d[u(1) - t^o(1 - k)].$$

While if the agent chooses $t^y(1 - q) + \varepsilon$ $(\varepsilon > 0)$ instead, then he will achieve :

$$u(1) - t^y(1 - q) - \varepsilon + d[u(1) - t^o(1 - k) + t^o k \varepsilon].$$

For $(\max[t^o \cdot (1 - q), 0], \max[t^y \cdot (1 - q), 0])$ to be a stationary subgame-perfect equilibrium strategy, the following must hold, for all

$\epsilon > 0$:

$$u(1) + \epsilon - u(1 + \epsilon) - kdt^o\epsilon > 0$$

which is equivalent to :

$$t^o < \frac{u(1) + \epsilon - u(1 + \epsilon)}{kd\epsilon}.$$

Thus t^o must be zero.

The previous three propositions do not cover the case where $\Omega > \theta$ and $\omega < k\theta$. In the following examples, we are only able to show that there exist some equilibria for some parameter values which satisfy $\Omega > \Xi$ and $\omega < k\theta$. This incompleteness seems to rise from the fact that the old agent and the young agent make their moves simultaneously. For the intergenerational altruism model where each information set contains only one decision node (no simultaneous moves) there exists a general existence result. [See Hellwig/Leininger(1985) and Leininger(1985).]

Example D : Let $u(z) = 2\sqrt{z}$, $d = 0.4$ and $k = 0.64$. Define

$$\zeta^+(q) = \begin{cases} 1 - q & 0 < q < 1 \\ 0 & 1 < q < 1.5376 \\ 1.7424 - q & 1.5376 < q < 1.7424 \\ 0 & 1.7424 < q \end{cases}$$

where $1.7424 = \text{argmax}_{z > 0}\ 2\sqrt{z} - z + (0.4)2\sqrt{0.64z}$ and 1.5376 solves $2\sqrt{z} - z + (0.4)2\sqrt{1} = 2\sqrt{1.7424} - 1.7424 + (0.4)2\sqrt{0.64 \cdot 1.7424}$.

Claim 5 : $\langle \cdots,(0,\max[1 - q,0]),(0,\max[\frac{1.5376}{0.64} - q,0]),(0,\zeta^+(q)),\cdots \rangle$ [12]
is a periodic subgame-perfect equilibrium of period 3.

Consider the following optimization prblems (17), (18) and (19) :

(17)
$$\max_{z \geqslant 0} \quad 2\sqrt{q + z} - z$$
$$+ (0.4)2\sqrt{0.64(q + z)} + \max[1 - 0.64(q + z), 0]$$

This problem achieves its maximum if z is chosen according to $\zeta^+(q)$:

$$2\sqrt{1} - 1 + q + (0.4)2\sqrt{1} = 1.8 + q \qquad 0 \leqslant q \leqslant 1$$

$$2\sqrt{q} + (0.4)2\sqrt{1} = 0.8 + 2\sqrt{q} \qquad 1 \leqslant q \leqslant 1.5376$$

$$2\sqrt{1.7424} - 1.7424 + q \qquad 1.5376 \leqslant q \leqslant 1.7424$$
$$+ (0.4)2\sqrt{0.64 \cdot 1.7424} = 1.7424 + q$$

$$2\sqrt{q} + (0.4)2\sqrt{0.64q} = 2.64\sqrt{q} \qquad 1.7424 \leqslant q.$$

(18)
$$\max_{z \geqslant 0} \quad 2\sqrt{q + z} - z$$
$$+ (0.4)2\sqrt{0.64(q + z)} + \zeta^+[0.64(q + z)]$$

This problem achieves its maximum if z is chosen according to
$\max[\frac{1.5376}{0.64} - q,0]$:

$$2\sqrt{\frac{1.5376}{0.64} - \frac{1.5376}{0.64} + q} + (0.4)2\sqrt{1.7424} \qquad 0 \leqslant q \leqslant \frac{1.5376}{0.64}$$
$$= 1.7535 + q$$

$$2\sqrt{q} + (0.4)2\sqrt{1.7424} = 1.056 + 2\sqrt{q} \qquad \frac{1.5376}{0.64} \leqslant q \leqslant \frac{1.7424}{0.64}$$

$$2\sqrt{q} + (0.4)2\sqrt{0.64q} = 2.64\sqrt{q} \qquad \frac{1.7424}{0.64} < q.$$

(19) $\max\limits_{z \geqslant 0} \dfrac{2\sqrt{q+z} - z}{+ (0.4)2\sqrt{0.64(q+z)} + \max[\frac{1.5376}{0.64} - 0.64(q+z), 0]}$

This problem achieves its maximum if z is chosen according to

max[1 - q,0] :

$$2\sqrt{1} - 1 + q + (0.4)2\sqrt{\frac{1.5376}{0.64}} = 2.24 + q \qquad 0 < q < 1$$

$$2\sqrt{q} + (0.4)2\sqrt{\frac{1.5376}{0.64}} = 1.24 + 2\sqrt{q} \qquad 1 < q < \frac{1.5376}{(0.64)^2}$$

$$2\sqrt{q} + (0.4)2\sqrt{0.64q} = 2.64\sqrt{q} \qquad \frac{1.5376}{(0.64)^2} < q.$$

Since $q + \zeta^+(q) > 1$ for all $q \geqslant 0$ and

$q + \max[1.5376/0.64 - q,0] > q + \max[1 - q,0] \geqslant 1$ for all $q \geqslant 0$,

$< \cdots,(0,\max[1 - q,0]),(0,\max[\frac{1.5376}{0.64} - q,0]),(0,\zeta^+(q)),\cdots >$ is a

periodic subgame-perfect equilibrium of period 3.

Remark : Is

$$< \cdots,(0,\left\{ \begin{array}{l} \max[1\ 5376/0\ 64 - q,0] \text{ with prob 1/3} \\ \max[1 - q,0] \qquad\quad \text{with prob 1/3} \\ \zeta^+(q) \qquad\qquad\quad \text{with prob 1/3} \end{array} \right\}),\cdots >$$

a stationary subgame-perfect equilibrium, if the strategy set is

enlarged properly.

In the following example we recapitulate the subgame-perfect

equilibria based on the general results.

EXAMPLE E : Consider $u(z) = 2\sqrt{z}$. Then $e = 4$, $E = 4d$; $\omega = 1$,

$\Omega = 1 + 2d$; $\theta = (1 + \sqrt{kd})^2$, $\Xi = (1 + \sqrt{kd})^2$.

Let $\zeta(q) = \max[1 - q, 0]$, $\psi(q) - \phi(q) = \max[\theta - q, 0] - \max[1 - q, 0]$

and $\nu(q) = \begin{cases} 1/4[(4 - 2\sqrt{q/k} + q/k)/d]^2 - q & \text{if } q < 4k \\ 0 & \text{if } q > 4k \end{cases}$

Given $(k,d) \epsilon (0,1)^2$, see Table I and Figure II. The regions in

Figure II are distinguished by the conditions in Table I accordingly.

Points A, B, C and D in Figure II correspond to Cases A, B, C and D

respectively. Equilibrium strategies we have found for Example E are

based on our existence propositions.

Region III_c demands special remarks. We might approximate an

equilibrium by a stationary strategy of type $\zeta(q)$ for we can justify

this a little more. Suppose $kZ_0 \epsilon [0,1]$. Then any sequence generated

by $< (0,\zeta(q)) >$ given kZ_0 is a Nash equilibrium of voluntary provisions

since $u[\omega] - \omega + du[\omega]$ is greater than $u[\theta] - \theta + du[k\theta]$. For the

general discussion on the parameter values which satisfy $\Omega > \Xi$ and

$\omega < k\theta$, see Appendix D.

4.4   Suboptimality of Voluntary Provisions

Let $< z^P >(kZ_0)$ denote the solution to the following problem,

given kZ_0 :

(20) $\underset{< z_t > \geqslant 0}{\max}$ $2u(kZ_0 + z_1) - z_1$

$+ d[2u(k^2Z_0 + kz_1 + z_2) - z_2]$

$+ d^2[2u(k^3Z_0 + k^2z_1 + kz_2 + z_3) - z_3]$

$+ \cdots.$

TABLE I. Equilibrium Strategies for Example E

REGION	CONDITION		STRATEGY
Ia		$\omega/k > e > \theta$	$\zeta(q)$
Ib	$E > \Omega > \theta$	$e > \omega/k > \theta$	$\nu(q)$, $\zeta(q)$
Ic		$e > \theta > \omega/k$	$\nu(q)$
IIa		$\omega/k > e > \theta$	$\zeta(q)$
IIb	$\Omega > E > \theta$	$e > \omega/k > \theta$	$\zeta(q)$
IIc		$e > \theta > \omega/k$	$\nu(q)$
IIIa		$\omega/k > e > \theta$	$\zeta(q)$
IIIb	$\Omega > \theta > E$	$e > \omega/k > \theta$	$\zeta(q)$
IIIc		$e > \theta > \omega/k$	?
IV	$E > \theta > \Omega$	$e > \theta > \omega/k$	$\nu(q)$, $\psi(q) \rightarrow \phi(q)$
V	$\theta > E > \Omega$	$e > \theta > \omega/k$	$\psi(q) \rightarrow \phi(q)$
VI	$\theta > \Omega > E$	$e > \theta > \omega/k$	$\psi(q) \rightarrow \phi(q)$
VII	$\theta \begin{smallmatrix} > \\ < \end{smallmatrix} \begin{smallmatrix} E \\ < \\ \Omega \end{smallmatrix}$	$e > \theta > \omega/k$	$\psi(q) \rightarrow \phi(q)$

FIGURE II. Equilibria for Example E
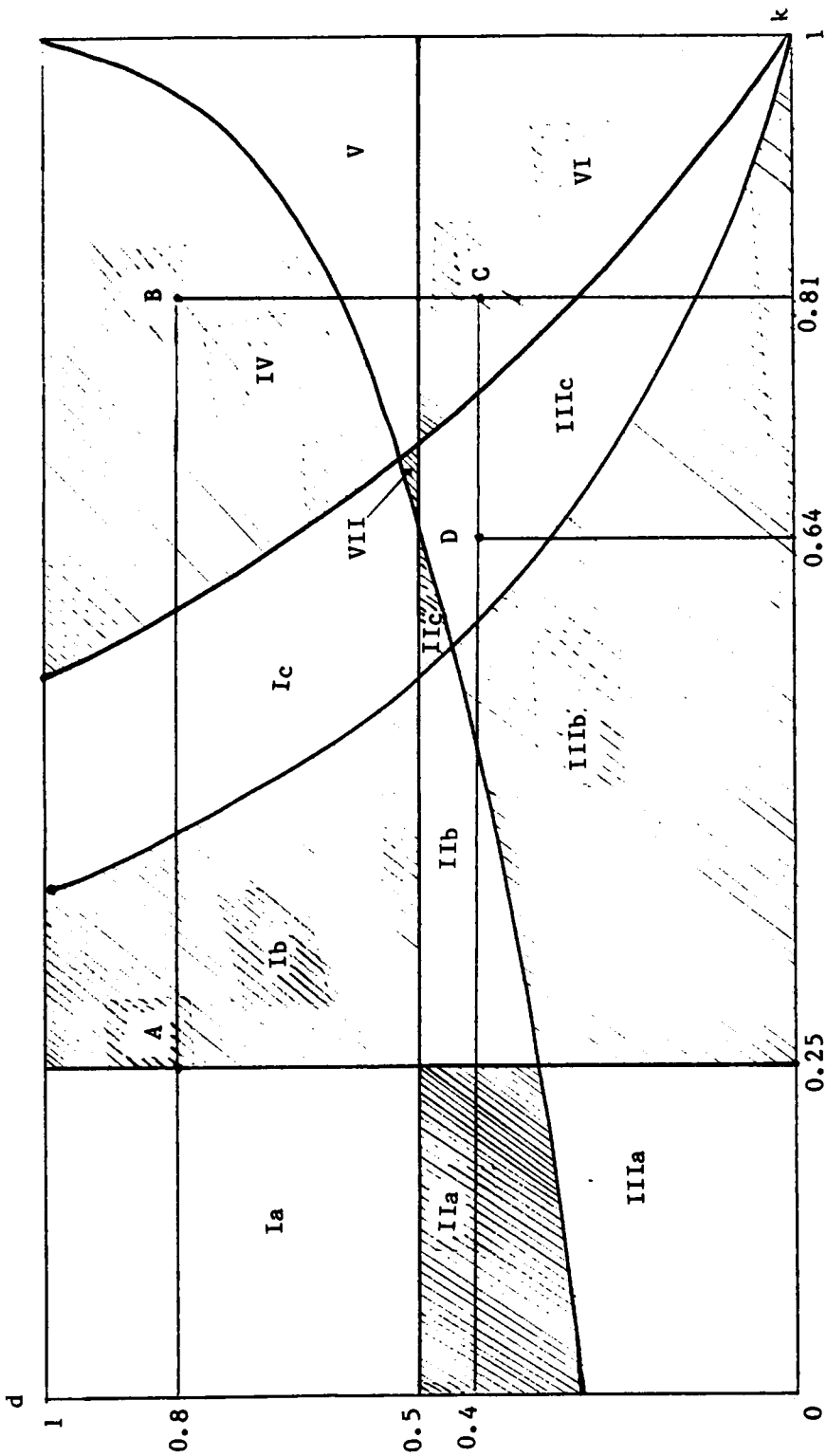
$< Z^P >(kZ_0)$ denotes the sequence of the total level associated with
$< z^P >(kZ_0)$, $< Z^\zeta >(kZ_0)$ the sequence of the total level associated
with the stationary subgame-perfect equilibrium $< (0,\zeta(q)) >$,
$< Z^\psi \leftrightarrow Z^\phi >(kZ_0)$ the one associated with the periodic subgame-perfect
equilibrium $< \cdots,(0,\psi(q)),(0,\phi(q)),\cdots >$, and $< Z^\upsilon >(kZ_0)$ the one
associated with the stationary subgame-perfect equilibrium $< (0,\upsilon(q)) >$
given kZ_0. In general $< Z^\zeta >(kZ_0)$, $< Z^\psi \leftrightarrow Z^\phi >(kZ_0)$, and $< Z^\upsilon >(kZ_0)$
are very different from $< Z^P >(kZ_0)$.

CONJECTURE 4.4 : Voluntary provisions of the public durable good are
not Pareto optimal in general.

It is a conjecture simply because we have not found and
characterized every existing equilibrium. We only provide an example :

EXAMPLE F : Let $u(z) = 2\sqrt{z}$ and d = 0.4. k only varies. Suppose
k = 0.25. Then we can predict $Z^\zeta = 1$ and $Z^P(0.25) \simeq 4.9383$. If k =
0.3, then $Z^\zeta = 1$ and $Z^P(0.3) \simeq 5.1653$. If k = 0.81, then $Z^\phi = 1.4982$,
$Z^\psi = 1.8496$ and $Z^P(0.81) \simeq 8.7532$. In this example we observe that
inefficiency increases as k increases.

The theory of voluntary provision for the public good has been
criticized because of its limited applicability. [See Margolis(1982)
and Sugden(1982).] However the theory of voluntary provision for the
public durable good seems to be very relevant in this overlapping
generations economy in the sense that what people of one generation

provide for themselves can be understood as voluntary provisions on their part towards peoples of other generations. If we accept this relevancy, then the suboptimality of voluntary provision might be an enormous phenomenon especially when the durability is relatively large, which cannot be detected explicitly in the atemporal model.

CHAPTER 5

CONCLUSIONS AND QUESTIONS FOR FUTURE RESEARCH

Here we present the conclusions we have drawn from the analyses in this research. We also raise some questions for future research.

1   Conclusions

1.   The stationary Lindahl equilibrium [Definition 2.1] of the public durable bad is not Pareto optimal. However this is not surprising at all since the agents are only concerned about their lifetimes while the planner is concerned about entire generation involved. The noncooperative voluntary provision equilibrium [Definition 2.2] of the public durable bad is not Pareto optimal. This is due to the fact that any pair of equilibrium strategies in a certain period must be such that the old players are on their best responses.

2.   There does not exist any non-stationary Lindahl equilibrium [Definition 2.2] of the public durable good. There exists only one unique stationary Lindahl equilibrium which is unstable. The stationary Lindahl equilibrium of the public durable good is not Pareto optimal. This is again due to the difference of decision horizons between the agents and the planner. Our Lindahl/Pigou scheme seems to work. But this scheme has an innate instability caused by the temporal separation of costs and benefits of the public durable good to the agents involved under the scheme. [See Samuelson(1958) and Sjoblom(1985).] In other words, the Lindahl/Pigou scheme is not

time-consistent. [See Arrow(1974), Dasgupta(1974a,b),
Kydland/Prescott(1977) and Stutzer(1984).]

3. Any noncooperative voluntary provision equilibrium of the public
durable good we have found so far is not Pareto optimal. This again
comes from the fact that any pair of equilibrium strategies in a
certain period must be such that the old players are on their best
responses. The atemporal theory of voluntary provision for the public
good has been criticized of its limited applicability. However the
intergenerational theory of voluntary provision of the public durable
good must be immune to this criticism in the sense that what people of
one generation provide for themselves can be understood as voluntary
provisions on their part towards peoples of other generations. This
aspect can be fortified since if a democratic government represents
only the currently alive, then it is hard to imagine a government which
caters fully to the preferences of all generations to come. Especially
note that the subgame-perfect equilibrium concept in the intertemporal
theory of voluntary provosion for the public durable good is
'time-consistent' or 'conjecture-consistent'. [See Peleg/Yaari(1973),
Goldman(1980), Lane/Mitra(1981) and Lane/Leininger(1984).]

4. Concludingly, if the public durable good lasts longer than the
agents live, then the level of inefficiency will be higher than the
level of inefficiency we analyze in the atemporal model.

5.2   Questions for Future Research

5.2.1   Length of decision horizon

The inefficiency is due to the fact that the agents only care about their lifetimes while the planner cares about indefinite future. Various introductions of altruism between generations might reduce the ineffciency or even restore the efficiency. First of all we point out that this overlapping generations model is mathematically equivalent to the simple paternalistic model of intergenerational altruism where a certain generation's utility depends on its own consumption and the consumption of the immediate next generation. [See Kohlberg(1976), Lane/Mitra(1981), Lane/Leininger(1984), Leininger(1985), Bernheim/Ray(1987), and Ray(1987).] Hence this short-range altruism will not do the job. It will be interesting that we introduce the 'long-range' paternalistic model of intergenerational altruism where a certain generation's utility depends on its own consumption and the consumptions of entire future generations. [See Barro(1974) and Gale(1985).] Or we can introduce the non-paternalistic model of intergenerational altruism where each generation's utility depends on its own consumption and the utility of the immediate next generation. Thus one agent's decision horizon is, in fact, an infinite one. [See Ray(1987).] We may consider allowing the agents to live for, say m periods, and check whether the inefficiency disappears as m tends to infinity. However we doubt this approach since as m increases the population also increases - there will be m agents in each period.

5.2.2   Alternatives for Lindal equilibrium

Lindahl equilibrium is a unanimity decision rule. We might

develop a model where we can discuss a majority decision rule. [See

Kaneko(1977a,b).]

Suppose that the government or the planner must be chosen among

the currently alive. Then any tax/subsidy policy must be

time-consistent or subgame-perfect. [See Phelps/Pollack(1968),

Kydland/Prescott(1977) and McTaggart/Salant(1986).]

5.2.3   Suboptimality of voluntary provisions

In general the overlapping generations economy only with the

private good allows Pareto improvement through very simple

punishment-reward scheme. [See Hammond(1975), Shell(1975) and

Salant(1988).] That this economy does not have any public (durable)

good makes the punishment-reward scheme (trigger strategy) rather

simple. [See Friedman(1971,1985).] However if there is any public

(durable) good at all, then this fact brings one more difficulty in

constructing the trigger strategy. In the overlapping generations

model, the equilibrium must be such that the oldest agent should be on

his best response. Hence the hardest punishment will be no provision

of the public (durable) good on all the younger agents' parts. This

implies these younger agents' actions should be rewarded in their later

lives. But these punishments and rewards will prevent arriving at the

Pareto frontier. However it will be interesting to check how far we

can push out towards the Pareto frontier by constructing trigger

strategy type equilibria. The possibility of success might be greater under no discounting and longer life.

5.2.4   Relaxation of quasi-linearity and additive separability

The complete existence result and characterization of the subgame-perfect equilibria for our model have to be done. There is a trade-off of simplicity of structure between the strategy sets and the utility (payoff) functions. Since we have the simplest form of the utlity functions, we might impose more on the strategy set side.

FOOTNOTES

1. In general externalities are distinguished between pecuniary externalities and nonpecuniary (technological) externalities depending on whether the effects (beneficial or harmful) are properly reflected in the market system. Here externalities mean only nonpecuniary externalities.

2. It is difficult to imagine the 'scrap' market for public goods. In the private good case, even though the lifetime of the good is longer than the lifetime of the economic agent, we can resolve this problem by modeling under existence of the scrap market for the private good.

3. See Varian(1983) pp. 278-283 and Green/Laffont(1979) pp. 27-34. Bewley(1977) analyzed the underlying rationale for the quasi-linear utility function. For criticism of quasi-linearity, see Bergstrom/Cornes(1983) and the references therein.

4. The uniqueness result of the stationary Lindahl equilibrium can be inferred from Appendix C.

5. The social planner's decision, i.e. the solution to (9) can be considered to be parameterized with respect to the leftover of the public good from the previous period, kZ_{t-1}. $\langle z_t^* \rangle = \langle z_t^* \rangle (kZ_{t-1})$. If $kZ_{\tau-1} = kZ_{s-1}$, then $\langle z_t^* \rangle_{t=\tau} = \langle z_t^* \rangle_{t=s}$ for all τ and s.

6. See Kaneko(1977) for the ratio equilibrium concept and Mas-Colell(1980) and Mas-Colell/Silverstre(1985) for the nonlinear technology case in the atemporal setting.

7. See Guttman(1978) and Bagnoli/Lippman(1987) for the opposite

   result.

8. There is an upper bound up to which kZ_{t-1} possibly takes values

   since endowments are bounded from above. For convenience we

   choose the strategy space S. Compare with strategy sets used in

   Leininger(1985) or Bernheim/Ray(1987).

9. Yet we do not know ideal restrictions - the so called preservation

   property restrictions - on strategy sets. See Leininger (1985).

10. an agents do better than these levels? Let $\nu_W(q)$ be $\nu(q)$ with 3.2

    being replaced by some number W which is greater than 3.2. Suppose

    $q = 0$. Then

    $$2\sqrt{\nu_W(0)} - \nu_W(0) + (0.8)2\sqrt{0.25\nu_W(0)} + \nu_W[(0.25\nu_W(0))]$$

    becomes :

    $$2\sqrt{\nu_W(0)} - \nu_W(0) + (0.8)2\sqrt{0.25\nu_W(0)} = \frac{7}{4}\cdot W - \frac{25}{64}\cdot W^2$$

    since $\nu_W(0) = 1/4[\,[W - 2\sqrt{0/0.25} + 0/0.25]/0.8]^2 - 0 = W^2/(0.64\cdot 4)$

    and $0.25\nu_W(0) = W^2/(0.64\cdot 16) > (3.2)^2/(0.64\cdot 16) = 1$. Given that

    the next generation's decision rule is $\nu_W(q)$, the current agent can

    achieve W by choosing z from $[0,4]$ instead $\nu_W(0) = \frac{25}{64}\cdot W^2 > 4$ since

    1.6 is greater than $\frac{7}{4}\cdot W - \frac{25}{64}\cdot W^2$. Thus $< (0,C_W(q)) >$ cannot be a

    stationary subgame-perfect equilibrium. So we cannot "push up too

    far."

11. However consider the following optimization problem :

    $$\max_{z \geqslant 0} \quad 2\sqrt{q + z} - z + (0.4)2\sqrt{0.81(q + z)} + \chi[0.81(q + z)]$$

    Suppose $q = 0$. Given that the generation's decision rule is $\chi(\ )$,

if the current agent chooses z according to $\chi(\ )$, then his net

lifetime benefit will be 1.6. Instead assume that the agent

chooses z = 1.8496. The next generation is supposed to choose z

accoding to $\chi(\ )$:

$\chi(0.81\cdot1.8496)$

$= 1/4[\,[1.6 - 2\sqrt{1.8496} + 1.8496]/0.4]^2 - 0.81\cdot1.8496$

$= - 0.64207793975 < 0$

This means that the next generation's decision rule cannot

be realized. They are confined to choose a nonnegative level.

Thus the worst thing that can happen to the current agent is that

the next generation chooses zero level. This implies the current

agent's guaranteed (maximin) net lifetime benefit is :

$2\sqrt{1.8496} - 1.8496 + (0.4)2\sqrt{0.81\cdot1.8496} = 1.8496$

which is greater than 1.6. Thus $\chi(q)$ cannot be a stationary

subgame-perfect equilibrium. If a decision rule is a stationary

subgame-perfect equilibrium, then it must satisfy β-individual

rationality constraint.

12. There also exist a periodic subgame-perfect equilibrium of period

3 for each and every k between 0.64 and 0.71 when d = 0.4 and

$u(z) = 2\sqrt{z}$.

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APPENDIX A : PROOFS OF PROPOSITIONS

● Proof of Proposition 2.2 : Since v is monotonically increasing, we need to show :

$$\frac{1-kd}{2} < \frac{1}{2+kd}$$

which is true because $2 - kd - kd^2 < 2$.

● Proof of Proposition 2.3 :

$$\frac{\partial(B^L - B^P)}{\partial k}\Big]_{k \,\epsilon\, (0,1)}$$

$$= \frac{d}{\beta} \cdot \left[v'\left(\frac{1-kd}{2\beta}\right)\cdot\frac{1}{2} - v'\left(\frac{1}{(2+kd)\beta}\right)\cdot\frac{1}{(2+kd)^2}\right]$$

$$= \frac{d}{\beta} \cdot \left[\frac{1}{u''\left(\frac{1-kd}{2\beta}\right)}\cdot\frac{1}{2} - \frac{1}{u''\left(\frac{1}{(2+kd)\beta}\right)}\cdot\frac{1}{(2+kd)^2}\right]$$

$$> \frac{d}{\beta}\cdot\frac{1}{u''\left(\frac{1}{(2+kd)\beta}\right)}\left[\frac{1}{2} - \frac{1}{(2+kd)^2}\right] > 0.$$

Remark : The nondecreasing assumption of u'' is sufficient for our proposition.

● Proof of Proposition 2.4 :

$$
\frac{\partial[B^L(n) - B^P(n)]}{\partial n}]
$$

$$
= v'(\frac{1}{n} \cdot \frac{1}{(2+kd)\beta}) \cdot \frac{1}{(2+kd)\beta} \cdot \frac{-1}{n^2} - - v'(\frac{1}{n} \cdot \frac{1-kd}{2\beta}) \cdot \frac{1-kd}{2\beta} \cdot \frac{-1}{n^2}
$$

$$
= \frac{1}{n} \cdot [v'(\frac{1}{n} \cdot \frac{1-kd}{2\beta}) \cdot \frac{1}{n} \cdot \frac{1-kd}{2\beta} - v'(\frac{1}{n} \cdot \frac{1}{(2+kd)\beta}) \cdot \frac{1}{n} \cdot \frac{1}{(2+kd)\beta}]
$$

$$
= \frac{1}{n} \cdot [\frac{\frac{1}{n} \cdot \frac{1-kd}{2\beta}}{u''(\frac{1}{n} \cdot \frac{1-kd}{2\beta})} - \frac{\frac{1}{n} \cdot \frac{1}{(2+kd)\beta}}{u''(\frac{1}{n} \cdot \frac{1}{(2+kd)\beta})}].
$$

● Proof of Proposition 2.5 : v is monotonically increasing.

● Proof of Proposition 2.6 :

$$
\frac{\partial(B^N - B^P)}{\partial k}]_{k \, \epsilon \, (0,1)} = \frac{d}{\beta} \cdot [v'(\frac{1-kd}{2\beta}) \cdot \frac{1}{2} > 0.
$$

● Proof of Proposition 2.7 :

$$
\frac{\partial[B^L(n) - B^P(n)]}{\partial n}] = \frac{1}{n} \cdot [v'(\frac{1}{n} \cdot \frac{1-kd}{2\beta}) \cdot \frac{1}{n} \cdot \frac{1-kd}{2\beta} > 0.
$$

● Proof of Proposition 3.1 : μ is monotonically decreasing.

● Proof of Proposition 3.2 :

$$\frac{\partial(z^L - z^P)}{\partial k}\Big]_{k \,\epsilon\, (0,1)}$$

$$= [\mu'(\frac{1 - kd}{2})\cdot\frac{-d}{2} - \mu'(\frac{1}{(2 + kd)})\cdot\frac{-1}{(2 + kd)^2}]$$

$$= -d\cdot[\frac{1}{u''(\frac{1 - kd}{2\beta})}\cdot\frac{1}{2} - \frac{1}{u''(\frac{1}{(2 + kd)})}\cdot\frac{1}{(2 + kd)^2}]$$

$$> -d\cdot\frac{1}{u''(\frac{1}{(2 + kd)})}[\frac{1}{2} - \frac{1}{(2 + kd)^2}] > 0.$$

Remark : The nonincreasing assumption of u" is sufficient for our proposition.

● Proof of Proposition 3.3 :

$$\frac{\partial[z^L(n) - z^P(n)]}{\partial n}\Big]$$

$$= \mu'(\frac{1}{n}\cdot\frac{1 - kd}{2})\cdot\frac{1 - kd}{2}\cdot\frac{-1}{n^2} - \mu'(\frac{1}{n}\cdot\frac{1}{2 + kd})\cdot\frac{1}{2 + kd}\cdot\frac{-1}{n^2}$$

$$= \frac{1}{n}\cdot[\frac{\frac{1}{n}\cdot\frac{1 - kd}{2}}{u''(\frac{1}{n}\cdot\frac{1 - kd}{2})} - \frac{\frac{1}{n}\cdot\frac{1}{(2 + kd)}}{u''(\frac{1}{n}\cdot\frac{1}{(2 + kd)})}].$$

● Proof of Proposition 4.1 :

Instead of proving directly, we show that $< (0.\upsilon(q;ke,E)) >$ is a stationary subgame-perfect equilibrium by constructing $\upsilon(q;ke,E)$ from a family, Y, of functions :

$$
Y = \{ \; y(q) : y(q;c,W) = \begin{cases} u^{-1}\left[\dfrac{W - u[q/k] + (q/k)}{d}\right] - q & q < c \\[2em] 0 & q > c \end{cases} \; \}
$$

where W is a constant and c is a fixed level of the argument of $y(\cdot)$ which enables $u[q/k] - (q/k) + du[q + y(q)]$ to achieve W for all $q < c$ and becomes zero for all $q > c$.

$y(q)$ must be restricted to be nonnegative for all $q > 0$. In particular,

$$
y(q) = u^{-1}\left[\dfrac{W - u[q/k] + (q/k)}{d}\right] - q > 0 \quad \text{for all } q < c
$$

which is equivalent to :

$$
W > u[q/k] - (q/k) + du[q] \quad \text{for all } q < c
$$

If c is chosen so that $c < k\theta$, then W must be greater than or equal to $u[c/k] - (c/k) + du[c]$ and if c is chosen so that $c > k\theta$, then W must be greater than or equal to $u[\theta] - \theta + du[k\theta]$.

Now we want the following to hold true, for all $q > 0$:

(A1) $u[q + y(q)] - y(q) + du[k(q + y(q)) + y(k(q + y(q)))]$

   $> u[q + z] - z + du[k(q + z) + y(k(q + z))]$ for all $z > 0$.

Firstly, for all $q > c/k$, $y(q) = 0$ and $k(q + y(q)) > c$. Hence, the left-hand side (LHS) of (A1) becomes $u(q) + du[kq]$. Suppose $c/k < \theta$. Then the right-hand side (RHS) of (A1) is less than or equal to :

$$\sup_{z>0} u[q + z] - z + du[k(q + z)] = \Xi + q.$$

It is possible that LHS < RHS. So we need to have the restriction :

(A2) $c/k > \theta$.

Then (A1) holds true as an equality.

   Secondly, for all $q < c/k$, consider the following problem :

(A3) $\sup_{z>0} u[q + z] - z \, du[k(q + z) + y(k(q + z))]$

If z is chosen so that $k(q + z) < c$, then (A3) becomes $W + q$. If z is chosen so that $k(q + z) > c$, then (A3) is equivalent to :

$$\sup_{z>(c/k)-q} u[q + z] - z + du[k(q + z)] < \Xi + q.$$

Meanwhile for q such that c/k > q > c, y(q) = 0 and k(q + y(q)) < c; hence LHS of (A1) becomes W + q. For q such that q < c, there are two possibilities : (^{1}) k(q + y(q)) < c or (^{2}) k(q + y(q)) > c. If k(q + y(q)) < c, then LHS of (A1) becomes W + q. If k(q + y(q)) > c, then LHS of (A1) becomes u[q + y(q)] − y(q) + du[k(q + y(q))] < Ξ + q. Note that LHS of (A1) should be not less than max[W + q, Ξ + q]. Therefore we need the follwing restriction :

(A4)   W > Ξ.

To guarantee (A4), the following must hold, for all q < c :

$$y(q) = u^{-1}\left[\frac{W - u[q/k] + (q/k)}{d}\right] - q < c/k - q$$

which is equivalent to :

$$W < u[q/k] - (q/k) + du[c/k]$$

Note that $\min_{q<c} u[q/k] - (q/k)$ occurs at either (^{1}) q = 0 or (^{2}) q = c only when u[0] + du[c/k] > u[c/k] − (c/k) + du[c/k]. Since u[e] − e = u[0], c = $\text{argmin}_{q<c}$ u[q/k] − (q/k) only when c > ke. Thus :

$$W < du[c/k] + u[0] \qquad\qquad \text{if } c < ke$$
$$W < u[c/k] - (c/k) + du[c/k] < du[c/k] + u[0] \quad \text{if } c > ke$$

Therefore, by choosing c = ke, W can achieve, at most, du[c/k] + u[0] which is less than or equal to du[e] + u[0]. Hence from (A2), (A4) and (A5) :

(A6)
$$\begin{cases} \Xi < W = E \\ \theta < c/k = e. \end{cases}$$

Now we need :

(A7)   c = ke > ω.

and since :

q + υ(q) > ω   for all q > ω

<=> $u^{-1}\big[[E - u(q/k) + q/k]/d \big]$ > ω   for all q > ω

<=> E > u(q/k) - q/k + du(ω)   for all q > ω

which will be guaranteed by :

(A8)   E > u(ω) - ω + du(ω) = Ω.

From (A6), (A7), and (A8) :

$$\begin{cases} \max[\Omega, \Xi] < E = W \\ \max[\omega/k, \theta] < e = c/k. \end{cases}$$

● Proof of Proposition 4.2 : We first show that $q + \zeta(q) + 0 > \omega$ for all $q > 0$:

$$q + \zeta(q) + 0 = q + \max[\omega - q, 0] > \omega$$

Now we have to show that the following is true, for all $q > 0$:

$$u[q + \zeta(q)] - \zeta(q) + du[k(q + \zeta(q)) + \zeta(k(q + \zeta(q)))]$$
$$> u[q + z] - z + du[k(q + z) + \zeta(k(q + z))] \quad \text{for all } z > 0.$$

Firstly, for $q > \omega/k$, $\zeta(q) = 0$, $\zeta(kq) = 0$, and $\zeta(k(q + z)) = 0$.
LHS $= u[q] + du[kq]$. RHS $= u[q + z] - z + du[k(q + z)] < u[q] + du[kq]$
since $\omega/k > \theta$.

Secondly, for $\omega/k > q > \omega$, $\phi(q) = 0$ and $\zeta(kq) = \omega - kq$.
LHS $= u[q] + du[\omega]$. RHS $= u[q + z] - z + du[k(q + z) + \zeta(k(q + z))]$
becomes :

$$u[q + z] - z + du[\omega] < u[q] + du[\omega] \quad \text{if } k(q + z) < \omega$$
$$u[q + z] - z + du[k(q + z)] \quad\quad\quad \text{if } k(q + z) > \omega.$$
$$< u[\omega/k] - \omega/k + q + du[\omega]$$

Finally, for $q < \omega$, $\zeta(q) = \omega - q$ and $\zeta(k\omega) = \omega - k\omega$. LHS becomes $u[\omega] - \omega + q + du[\omega]$. RHS $= u[q + z] - z + du[k(q + z) + \zeta[k(q + z)]$ becomes either (1) $u[q + z] - z + du[\omega] < u[\omega] - \omega + q + du[\omega]$ if $k(q + z) < \omega$ or (2) $u[q + z] - z + du[k(q + z)]$ which is less than $u[\omega/k] - \omega/k + q + du[\omega]$ if $k(q + z) > \omega$.

● Proof of Proposition 4.2 : Firstly, we show that the following holds, for all q > 0 :

$$u[q + \phi(q)] - \phi(q) + du[k(q + \phi(q)) + \psi(k(q + \phi(q)))]$$

$$\geq u[q + z] - z + du[k(q + z) + \psi(k(q + z)] \text{ for all } z > 0$$

For q > θ/k, since φ(q) = 0, ψ(kq) = 0, and ψ(k(q + z)) = 0 :

$$RHS = u[k + q] - z + du[k(q + z)] < u[q] + du[kq] = LHS.$$

For θ/k > q > ω, since φ(q) = 0 and ψ(kq) = θ - kq :

$$RHS = u[q + z] - z + du[k(q + z) + \psi(k(q + z))]$$

$$= \begin{cases} u[q + z] - z + du[\theta] < u[q] + du[q] = LHS \text{ if } k(q + z) < \theta \\ u[q + z] - z + du[k(q + z)] \\ < u[\theta/k] - \theta/k + q + du[\theta] < LHS \qquad \text{if } k(q + z) > \theta. \end{cases}$$

For q < ω, since φ(q) = ω - q and ψ(kω) = θ - kω :

$$RHS = \begin{cases} u[\omega] - \omega + q + du[\theta] = LHS \qquad \text{if } k(q + z) < \theta \\ u[q + z] - z + du[k(q + z)] \\ < u[\theta/k] - \theta/k + q + du[\theta] \\ < u[\omega] - \omega + q + du[\theta] = LHS \text{ if } k(q + z) > \theta. \end{cases}$$

Secondly, we show that the following holds, for all $q > 0$:

$$u[q + \psi(q)] - \psi(q) + du[k(q + \psi(q)) + \phi(k(q + \psi(q)))]$$

$$\geq u[q + z] - z + du[k(q + z) + \phi(k(q + z)] \quad \text{for all } z > 0$$

Note that $k\theta > \omega$. For $q > \theta$, since $\psi(q) = 0$ and $\phi(kq) = 0$:

$$\text{RHS} = u[q + z] - z + du[k(q + z)] < u[q] + du[kq] = \text{LHS}.$$

For $\theta \geq q > \omega$, since $\psi(q) = \theta - q$ and $\phi(k\theta) = 0$:

$$\text{RHS} = \begin{cases} u[q + z] - z + du[\omega] < u[q] + du[\omega] \\[4pt] \quad < u[\theta] - \theta + q + du[k\theta] = \text{LHS} \quad \text{if } k(q + z) < \omega \\[8pt] u[q + z] - z + du[k(q + z)] \\[4pt] \quad < u[\omega/k] - \omega/k + q + du[\omega] \\[4pt] \quad < u[\omega/k] - \omega/k + q + du[\omega] = \text{LHS if } k(q + z) > \omega. \end{cases}$$

For $q < \omega$, since $\psi(q) = \theta - q$ and $\phi(k\theta) = 0$:

$$\text{RHS} = \begin{cases} u[q + z] - z + du[\omega] < u[\omega] - \omega + q + du[\omega] \\[4pt] \quad < u[\theta] - \theta + q + du[k\theta] = \text{LHS} \quad \text{if } k(q + z) < \omega \\[8pt] u[q + z] - z + du[k(q + z)] \\[4pt] \quad < u[\omega/k] - \omega/k + q + du[\omega] \\[4pt] \quad < u[\theta] - \theta + q + du[k\theta] = \text{LHS} \quad \text{if } k(q + z) > \omega. \end{cases}$$

APPENDIX B : THE SAMUELSON PUBLIC GOOD OPTIMALITY CONDITION

First, we very briefly review the Samuelson public good optimality condition derived from a typical atemporal model. Second, we introduce a finite horizon model and obtain the intertemporal version of the Samuelson public good optimality condition. Since this model is different from the Sandler/Smith(1976) model, the result needs a different interpretation. Third, we introduce a more general version of our original overlapping generations model. Our intent is not only to extend a finite horizon model but also to show the differences between the original overlapping generations model and the general one and the difficulties therein. This section naturally explains why we have very restrictive assumptions on utility functions - additive separability and quasi-linearity. The derivations are gathered in the last section.

1   Review

Samuelson(1954,1955) obtained a formal optimality condition for the provision of public goods - now known as the Samuelson public good optimality condition - from a typical atemporal model.

Atemporal model : There are two goods in the economy; one is public, z, and the other is private, x. There are n agents in the economy, whose utility functions are represented by $U^i = U^i(z,x^i)$, $i = 1,\cdots,n$. The technology in the economy is given by $F(z,X) = 0$, where $\sum_i x^i = X$. $U^i(\ )$ and $F(\ )$ are assumed to satisfy the second-order conditions.

The Samuelson public good optimality condition can be obtained by

solving the following problem :[1]

$$\max_{z>0,\ x^i>0} \quad U^1(z,x^1)$$

$$\text{subject to } \begin{cases} U^i(z,x^i) = \overline{U}^i \quad i = 2,\cdots,n \\ F(z,X) = 0 \text{ where } X = \sum_i x^i. \end{cases}$$

The well-known (atemporal) Samuelson public good optimality condition

is [See Section 4] :

(B1) $\quad MRT_{zX}(z,\sum_i x^i) = \sum_i MRS^i(z,x^i)$

Any allocation $(z,x^1,\cdots,x^n)$ satisfying (B1) is Pareto optimal.

Note that the Pareto optimal level of the public good, z, is always

jointly determined with $(x^1,\cdots,x^n)$ rather than with $\sum_i x^i$ as we can

see from (B1). In this sense the determination of the Pareto optimal

level of the public good is not free of distributional issue.[2]

One way of allowing the issue of efficiency to be independent from

the issue of distribution is the quasi-linearity assumption.[3]

2 The Intertemporal Samuelson Public Good Optimality Condition

We will introduce a finite horizon model.

T period model : The model is T periods long. In each period, there

are two goods in the economy; one is public, Z, and the other is

private, x. There are n agents in this economy whose utility functions

are given by $U^i = U^i(Z_1, x^1_1, \cdots, Z_T, x^i_T)$, $i = 1, \cdots, n$, where Z_t, $t = 1, \cdot$ $\cdots, T$, denotes the total (service) level of the public good in period t and x^i_t denotes the consumption level of the private good of agent i in period t. The technology in this economy is given by $G_t(z_t, X_t) = 0$, $t = 1, \cdots, T$, where $X_t = \sum_i x^i_t$, z_t denotes the new provision level of the public durable good in period t and X_t the total supply of the private good in period t. This periodwise independent technology means that there are limited resources for the production of the private good and the public durable good in each period. This limitation is independently given period by period. In other words there is no saving technology which allows a larger pie for tomorrow.[4] The relation between the total (service) level of the public good in period t, Z_t, and the previous and current provisions, z_s, $s < t$, are given by $Z_t = Z_t(z_1, \cdots, z_t)$, $t = 1, \cdots, T$. The functional form $Z_t( )$ implicitly describes deterioration due to ageing and the like. $U^i( )$, $G_t( )$, and $Z_t( )$ are assumed to satisfy the second-order conditions.

The intertemporal Samuelson public good optimality condition for this model can be obtained by solving the following problem :

$$\max_{z_t > 0, x^i_t > 0} U^1[Z_1(z_1), x^1_1, \cdots, Z_T(z1, \cdots, z_T), x^1_T]$$

$$\text{subject to} \begin{cases} U^i[Z_1(z_1), x^i_1, \cdots, Z_T(z1, \cdots, z_T), x^i_T] = \bar{U}^i, \quad i \neq 1 \\ G_t(z_t, \sum_i x^i_t) = 0, \quad t = 1, \cdots, T. \end{cases}$$

The intertemporal Samuelson public good optimality condition for

our T period model is [See Section 4] :

$$(B2) \qquad MRT_{z_s X_s} = \sum_{t=s}^{T} \sum_i MRS^i_{z_t x_t} \frac{\partial z_t}{\partial z_s} MRS^i_{x_t x_s} \qquad s = 1,\ldots,T.$$

Note that in this model it is meaningless to define the marginal rate

of transformation between the numeraire good in period t and in period

s, since there is no relation between G_t and G_s for $t \neq s$. Therefore

we obtain a corollary to Cabe's theorem [Cabe(1982,Theorem I)]:

Corollary   Pareto optimal allocation of resources requires
discounting the value of future services of a
public good newly provided in period t at a
discount rate corresponding to the marginal rate
of substitution between the numeraire good in
period t and the numeraire good in the period
in which services of the public good accrue.

Now consider the following model based on Sandler/Smith(1976,1977,

1982), Bishop(1977) and Cabe(1982), which is the same as our T period

model except for the technology.

Sandler/Smith model : Instead the technology is given by $F(z_1,\cdots,z_T,$

$X_1,\cdots,X_T)$ = 0. This depicts the technological relation between the

periodwise public good production and the periodwise total supply of

private good across all periods.

For this model, the intertemporal Samuelson public good optimality

condition is :

$$(B3) \qquad MRT_{z_s X_s} = \sum_{t=s}^{T} \sum_i MRS^i_{z_t x_t} \frac{\partial z_t}{\partial z_s} MRT_{X_t X_s} \qquad s = 1,\cdots,T.$$

Thus Cabe's theorem is obtained [Cabe(1982,Theorem I)]:

> Pareto optimal allocation of resources requires
> discounting the value of future services of a
> public asset(valued in terms of a numeraire) at
> a discount rate corresponding to the <u>marginal
> rate of transformation</u> between the numeraire good
> in the current period and the numeraire good in
> the period in which services of the public asset
> accrue.

With the numeraire assumption[Sandler/Smith(1976,p156)], the technology

is simply given by $F(z_1, \cdots z_T, X_1 + \cdots + X_T) = 0$. Then $MRT_{X_t X_s} = 1$ for

all $t, s = 1, \cdots, T$ and (B3) becomes :

$$(B4) \qquad MRT_{z_s X_s} = \sum_{t=s}^{T} \sum_i MRS^i_{z_t x_t} \frac{\partial z_t}{\partial z_s} \qquad s = 1, \cdots, T.$$

Thus Sandler/Smith's theorem is obtained [Sandler/Smith(1977)] :

> Discounting the estimates of the marginal value
> of the services of a public asset in each period
> over the life of the asset will lead to a Pareto-
> inefficient allocation of resources.

This theorem implies that we are required to "treat each person's

incremental benefits from the public good in question equally

regardless of the time they receive the benefits"

[Sandler/Smith(1977,p255)]. We conclude this section pointing

out ([1]) that Sandler/Smith's theorem is entirely based on the numeraire

assumption they apply and ([2]) that our corollary directly applies to

Sandler/Smith model since $MRT_{X_t X_s} = MRS^i_{x_t x_s}$, $i = 1, \cdots, n$, at any Pareto

optimal allocation while Cabe's theorem does not directly apply to our

model because $MRT_{X_t X_s}$ is not well defined in our model.

3   Intergenerational Pareto Optimality in an Infinite Horizon

3.1   Derivation

In the previous discussion, we have derived a intertemporal
version of Samuelson public good optimality condition. However the
derivation was based on a finite horizon model. Here we will consider
an infinite horizon model. This change brings one major restriction
that we cannot have an intertemporal transformation function such as
Sandler/Smith(1976) has - so their model cannot be extended to an
infinite horizon model - and one difficult problem of grouping the
relevant agents on which the public (durable) good, provided in some
period, has impacts, when the lifetime of the public (durable) good is
finite but fairly long - not shorter than the lifetime of agents
involved.

Consider the following general overlapping generations model which
differs from the original overlapping generations model on three
accounts :

General OG model : The utility functions are more general and given by:

$$U^0 = U^0(Z_1, x^0{}_1) = u^0{}_1(Z_1, x^0{}_1) \qquad t = 0$$
$$U^t = U^t(Z_t, x^y{}_t, Z_{t+1}, x^0{}_{t+1}) \qquad t \geqslant 1.$$

The technologies are more general and given by:

$$G_t(z_t, x^0{}_t + x^y{}_t) = 0 \qquad t \geqslant 1.$$

The relation between the total (service) level of the public durable good, Z_t, and the previous and the current provisions, z_s, $s < t$, are given by:

$$Z_t = Z_t(z1,\ldots,z_t) \qquad\qquad 1 < t < p$$

$$Z_t = Z_t(z_{t-p+1},\cdots,z_{t-1},z_t) \qquad t \geqslant p.$$

These relations depict that the public durable good lasts for p ($p \geqslant 2$) periods once provided and that the total (service) level of the public good in period t is determined by the previous $p-1$ provisions which still 'exist' in period t and the new provision in period t. Overall the general overlapping generations model is exactly identical to the original overlapping generations model we have except utility functions, technologies, and the nature of the public durable good. $U^t(\ )$, $G_t(\ )$, and $Z_t(\ )$ are assumed to satisfy the second-order conditions.

The intergenerational Samuelson public good optimality condition can be obtained by solving the following problem:

$$\begin{aligned}
&\underset{\langle(z_t,x^o_t,x^y_t)\rangle\geqslant 0}{\max} && U^0 = u^o{}_1(Z_1,x^o{}_1) \\
&\text{subject to} && U^t(Z_t,x^y_t,Z_{t+1},x^o_{t+1}) = \overline{U}^t && t \geqslant 1 \\
& && G_t(z_t,x^o_t + x^y_t) = 0 && t \geqslant 1 \\
& && Z_t = Z_t(z_1,\cdots,z_t) && t < p \\
& && Z_t = Z_t(z_{t-p+1},\cdots,z_{t-1},z_t) && t \geqslant p.
\end{aligned}$$

The intergenerational Samuelson public good optimality condition is [See Section 4]:

$$
(B5) \qquad MRT_{z_s x_s} = \sum_{t=s}^{s+p-1} \left[MRS^{t-1}_{z_t x_t} \frac{\partial Z_t}{\partial z_s} \left[\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a} \right] \right.
$$

$$
\left. + MRS^t_{z_t x_t} \frac{\partial Z_t}{\partial z_s} \left[\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a} \right] \right] \qquad s \geqslant 1.
$$

Note that our original model has $p = \infty$ in (B5). Note also that $\left[\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a} \right]$ is meaningful because agents are connected. That is, "there is ••• a common point of time at which each person can attach an equivalent value"[Mishan(1981,p199)]. This is why we chose an overlapping generations model. If there is a point of time where agents are not connected, then we cannot apply the Pareto criterion[See Mishan(1981) for a detailed discussion].

3.2   Simplification

Every < (z_t, x^o_t, x^y_t) > satisfying (B5) is Pareto optimal for our general overlapping generations economy. However (B5) consists of an infinite number of equations. This difficulty is dicussed in the following section.

Besides the infinity problem in the general overlapping generations model, z_t's and, thus, Z_t's are determined together with x^o_t's, and x^y_t's . But we want to have z_t's and, thus, Z_t's determined independently from x^o_t's and x^y_t's as in the quasi-linear case in the atemporal model. Thus we simplify the general overlapping generations model as follows ; the utility function are simplified to :

$$U^0 = u^o{}_1(Z_1) + x^o{}_1 \qquad\qquad t = 0$$

$$U^t = u^y{}_t(Z_t) + x^y{}_t + d^t[u^o{}_t(Z_{t+1}) + x^o{}_{t+1}] \quad t \geqslant 1.$$

The technologies to :

$$z_t + x^o{}_t + x^y{}_t = w^o{}_t + w^y{}_t \qquad\qquad t \geqslant 1.$$

The relations between the total (service) level of the public durable good in period t, Z_t, and the previous and current provisions of the public durable good, z_s's, $s \leqslant t$, to :

$$Z_1 = z_1 \qquad\qquad t = 1$$

$$Z_2 = kz_1 + z_2 \qquad\qquad t = 2$$

$$\cdots$$

$$Z_{p-1} = k^{p-2}z_1 + \cdots + z_{p-1} \qquad\qquad t = p-1$$

$$Z_t = k^{p-1}z_{t-p+1} + \cdots + kz_{t-1} + z_t \qquad\qquad t \geqslant p.$$

While the public durable good lasts forever with the deterioration rate $(1 - k)$ in our original model, the public durable good lasts for p $(p \geqslant 2)$ periods and vanishes completely $(p + 1)$ periods later in this general model.

In the simplified model one unit of the private good in period t will be exchanged at the rate of d^t with the private good in period t+1 since agent t is willing to change x_t for x_{t+1} at the rate of d^t, i.e., agent t's time preference rate. To increase one unit of the public durable good newly provided in period t, we have to forgo one unit of

the private good in period t. One additional unit of the public

durable good newly provided in period t will generate extra benefits

from period t to period t+p-1.

In period τ such that $t < \tau < t+p-1$, the extra benefits are, in

terms of the private good in period τ, worth

$$\left\{ \left[\frac{\partial u^o_\tau}{\partial Z_\tau} \right]_{Z_\tau} + \left[\frac{\partial u^y_\tau}{\partial Z_\tau} \right]_{Z_\tau} \right\} k^{\tau-t}$$

which is, in terms of the private good in period t, worth

$$\left\{ \left[\frac{\partial u^o_\tau}{\partial Z_\tau} \right]_{Z_\tau} + \left[\frac{\partial u^y_\tau}{\partial Z_\tau} \right]_{Z_\tau} \right\} k^{\tau-t} \left[\prod_{a=t}^{\tau-1} d^a \right]$$

where $\left[\prod_{a=t}^{\tau-1} d^a \right]$ should be understood as $\left[\prod_{a=t}^{\tau-1} d^a \right] = 1$ when $\tau = t$. Thus

the values of z_t's and, thus, Z_t's are determined by :

$$(B6) \qquad 1 = \sum_{s=t}^{t+p-1} \left\{ \left[\frac{\partial u^o_\tau}{\partial Z_\tau} \right]_{Z_\tau} + \left[\frac{\partial u^y_\tau}{\partial Z_\tau} \right]_{Z_\tau} \right\} k^{\tau-t} \left[\prod_{a=t}^{\tau-1} d^a \right] \qquad t > 1.$$

This is a simplified version of the intergenerational Samuelson public

good optimality condition. Now compare (B6) with (B5). $\partial u^a_s / \partial Z_s]_{Z_s}$ in

(B6) amounts to $MRS^h_{Z_s x_s}$ for h = o,y in (B5); $\left[\prod_{}^{\tau-1} d^a \right]$ in (B6)

corresponds to $\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a}$ in (B5).

As we see from (B6), we can, in principle, determine z_t's and,

thus, Z_t's independently from x^o_t's and x^y_t's even though there are

infinite number of equations. Note that as t varies, $\left[\prod_{a=t}^{\tau-1} d^a \right]$ varies.

Thus $< z_t >$ satisfying (B6) can be considered to be parameterized by

$< d^t >$ and $< (u^o_t(\cdot), u^y_t(\cdot)) >$. We also assume further that

(^{1}) $d^t = d$ and (^{2}) $u^o{}_t(\cdot) = u^y{}_t(\cdot) = u(\cdot)$ for all $t \geqslant 1$ in order to

avoid $\langle z_t \rangle$'s depending on the beginning of the biological time.

We conclude that (^{1}) quasi-linearity and additive separability is

necessary for z_t's to be determined independently from distribution of

x_t's and (^{2}) that the stationarity of the discount factor is necessary

for us to have some time-free results.

4   Derivations of the Pareto Optimal Conditions

● The atemporal Samuelson public good optimality condition can be

obtained by solving the following problem :

$$\max_{z>0,\ x^1>0} U^1(z,x^1)$$

$$\text{subject to } \begin{cases} U^i(z,x^i) = \overline{U}^i & i = 2, \cdots, m \\ F(z,X) = 0 \text{ where } X = \sum_i x^i. \end{cases}$$

The corresponding Lagrangian is :

$$L = \sum_i \lambda^i \{U^i(z,x^i) - \overline{U}^i\} + \eta F(z,X).$$

The first-order conditions are :

$$z\ ;\ \sum_i \lambda^i U^i{}_z = \eta F_z$$

$$x^i;\ \lambda^i U^i{}_x = \eta F_X \quad i = 1,\cdots,n$$

where λ^i and η are the Lagrangian multipliers. Then the first-order conditions become :

$$\frac{F_z}{F_X} = \frac{\eta F_z}{\eta F_X} = \frac{\sum_i \lambda^i U^i_z}{\eta F_X} = \sum_i \frac{\lambda^i U^i_z}{\lambda^i U^i_x} = \sum_i \frac{U^i_z}{U^i_x}.$$

(B1) $MRT_{zX}(z, \sum_i x^i) = \sum_i MRS^i(z, x^i).$

●   The intertemporal Samuelson public good optimality condition for this model can be obtained by solving the following problem :

$$\begin{array}{l} \max_{z_t > 0, x^i_t > 0} \quad U^1[Z_1(z_1), x^1_1, \cdots, Z_T(z1, \cdots, z_T), x^1_T] \\[2mm] \text{subject to} \begin{cases} U^i[Z_1(z_1), x^i_1, \cdots, Z_T(z1, \cdots, z_T), x^i_T] = \overline{U}^i, \quad i \neq 1 \\[2mm] G_t(z_t, \sum_i x^i_t) = 0, \quad t = 1, \cdots, T \end{cases} \end{array}$$

The corresponding Lagrangian is :

$$L = \sum_i \lambda^i \{U^i[Z_1(z_1), x^i_1, \cdots, Z_T(z_1, \cdots, z_T), x^i_T] - \overline{U}^i\}$$
$$- \sum_{t=1}^{T} \eta_t G_t(z_t, \sum_i x^i_t).$$

Then the first-order conditions are :

$$z_s \; ; \quad \sum_{t=s}^{T} \sum_i \lambda^i \frac{\partial U^i}{\partial Z_t} \frac{\partial Z_t}{\partial z_s} - \eta_s \frac{\partial G}{\partial z_s} = 0 \qquad s = 1, \cdots, T$$

$$x^i_t \; ; \quad \lambda^i \frac{\partial U^i}{\partial x^i_t} - \eta_t \frac{\partial F}{\partial X_t} = 0 \qquad i = 1, \cdots, n \; ; \; t = 1, \cdots, T.$$

With $(\partial U^i/\partial Z_t)/(\partial U^i/\partial x^i_t) = MRS^i_{Z_t x_t}$ and $(\partial G/\partial z_t)/(\partial G/\partial X_t) = MRT_{z_t X_t}$, the first-order conditions reduce to :

$$MRT_{z_s X_s} = \frac{\eta_s \dfrac{\partial F}{\partial z_s}}{\eta_s \dfrac{\partial F}{\partial X_s}} = \frac{\displaystyle\sum_{t=s}^{T} \sum_i \lambda^i \dfrac{\partial U^i}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s}}{\lambda^i \dfrac{\partial U^i}{\partial x^i_s}} = \sum_{t=s}^{T} \sum_i \frac{\dfrac{\partial U^i}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s} \dfrac{\partial U^i}{\partial x^i_t}}{\dfrac{\partial U^i}{\partial x^i_t} \dfrac{\partial U^i}{\partial x^i_s}}$$

$$= \sum_{t=s}^{T} \sum_i MRS^i_{Z_t x_t} \frac{\partial Z_t}{\partial z_s} MRS^i_{x_t x_s}.$$

The intertemporal Samuelson public good optimality condition for our T period model is :

$$(B2) \qquad MRT_{z_s X_s} = \sum_{t=s}^{T} \sum_i MRS^i_{Z_t x_t} \frac{\partial Z_t}{\partial z_s} MRS^i_{x_t x_s} \qquad s = 1, \ldots, T.$$

● The intergenerational Samuelson public good optimality condition can be obtained by solving the following problem:

$$\begin{array}{ll}
\max\limits_{\langle(z_t,x^o{}_t,x^y{}_t)\rangle \gg 0} & U^0 = u^o{}_1(Z_1,x^o{}_1) \\
\text{subject to} & U^t(Z_t,x^y{}_t,Z_{t+1},x^o{}_{t+1}) = \overline{U}^t \qquad t > 1 \\
& G_t(z_t,x^o{}_t + x^y{}_t) = 0 \qquad t > 1 \\
& Z_t = Z_t(z_1,\cdots,z_t) \qquad t < p \\
& Z_t = Z_t(z_{t-p+1},\cdots,z_{t-1},z_t) \qquad t > p.
\end{array}$$

The corresponding Lagrangian is :

$$L = \lambda^0[u^o{}_1(Z_1,x^o{}_1) - \overline{u}^o{}_1]$$

$$+ \sum_{t=1}^{\infty} \lambda^t \{ U^t[Z_t,x^y{}_t,Z_{t+1},x^o{}_{t+1}] - \overline{U}^t \}$$

$$- \sum_{t=1}^{\infty} \eta_t G_t(z_t,x^o{}_t + x^y{}_t).$$

The first-order conditions are:

$$z_s \;\; ; \;\; \sum_{t=s}^{s+p-1} \left[\lambda^{t-1} \frac{\partial U^{t-1}}{\partial Z_t} \frac{\partial Z_t}{\partial z_s} + \lambda^t \frac{\partial U^t}{\partial Z_t} \frac{\partial Z_t}{\partial z_s} \right] - \eta_s \frac{\partial G_s}{\partial z_s} = 0 \qquad s > 1$$

$$x^o{}_t \;\; ; \;\; \lambda^{t-1} \frac{\partial U^{t-1}}{\partial x^o{}_t} - \eta_t \frac{\partial G_t}{\partial X_t} = 0 \qquad t > 1$$

$$x^y{}_t \;\; ; \;\; \lambda^t \frac{\partial U^t}{\partial x^y{}_t} - \eta_t \frac{\partial G_t}{\partial X_t} = 0 \qquad t > 1.$$

With $(\partial G_t/\partial z_t)/(\partial G_t/\partial X_t) = MRT_{z_t X_t}$, $(\partial U^t/\partial Z_t)/(\partial U^t/\partial x^y_t) = MRS^t_{z_t x_t}$,

and $(\partial U^{t-1}/\partial Z_t)/(\partial U^{t-1}/\partial x^o_t) = MRS^{t-1}_{z_t x_t}$, the first-order conditions

becomes :

(i)
$$MRT_{z_s X_s} = \frac{n_s \dfrac{\partial G_s}{\partial z_s}}{n_s \dfrac{\partial G_s}{\partial X_s}} = \frac{\displaystyle\sum_{t=s}^{s+p-1} \left[\lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s} + \lambda^t \dfrac{\partial U^t}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s}\right]}{n_s \dfrac{\partial G_s}{\partial X_s}}$$

$$= \sum_{t=s}^{s+p-1} \left[\frac{\lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s}}{\lambda^{s-1} \dfrac{\partial U^{s-1}}{\partial x^o_s}} + \frac{\lambda^t \dfrac{\partial U^t}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s}}{\lambda^s \dfrac{\partial U^s}{\partial x^y_s}} \right]$$

$$= \sum_{t=s}^{s+p-1} \left[\frac{\lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s} \Big/ \lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial x^o_t}}{\lambda^{s-1} \dfrac{\partial U^{s-1}}{\partial x^o_s}} + \frac{\lambda^t \dfrac{\partial U^t}{\partial Z_t} \dfrac{\partial Z_t}{\partial z_s} \Big/ \lambda^t \dfrac{\partial U^t}{\partial x^y_t}}{\lambda^s \dfrac{\partial U^s}{\partial x^y_s}} \right]$$

$$= \sum_{t=s}^{s+p-1} \left[MRS^{t-1}_{z_t x_t} \dfrac{\partial Z_t}{\partial z_s} \frac{\lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial x^o_t}}{\lambda^{s-1} \dfrac{\partial U^{s-1}}{\partial x^o_s}} + MRS^t_{z_t x_t} \dfrac{\partial Z_t}{\partial z_s} \frac{\lambda^t \dfrac{\partial U^t}{\partial x^y_t}}{\lambda^s \dfrac{\partial U^s}{\partial x^y_s}} \right] .$$

If $t \neq s$, there is no ready interpretation for

$$\lambda^{t-1} \dfrac{\partial U^{t-1}}{\partial x^o_t} \Big/ \lambda^{s-1} \dfrac{\partial U^{s-1}}{\partial x^o_s} \quad \text{and} \quad \lambda^t \dfrac{\partial U^t}{\partial x^y_t} \Big/ \lambda^s \dfrac{\partial U^s}{\partial x^y_s}$$

since t and s belong to different generations. However, defining

$MRS^{t-1}_{x_t x_{t-1}} = (\partial U^{t-1}/\partial x^o_t)/(\partial U^{t-1}/\partial x^y_{t-1})$ we obtain :

(iia)
$$
\dfrac{\dfrac{\partial U^{t-1}}{\partial x^o_t}}{\dfrac{\partial U^{s-1}}{\partial x^o_s}} = \dfrac{\dfrac{\partial U^{t-1}}{\partial x^y_{t-1}}\dfrac{\partial U^{t-1}}{\partial x^o_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^y_{t-2}}}{\dfrac{\partial U^{t-1}}{\partial x^y_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^o_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^y_{t-2}}\dfrac{\partial U^{t-3}}{\partial x^o_{t-2}}} \cdots \dfrac{\dfrac{\partial U^{s+1}}{\partial x^o_{s+2}}\dfrac{\partial U^{s+1}}{\partial x^y_{s+1}}\dfrac{\partial U^s}{\partial x^o_{s+1}}\dfrac{\partial U^s}{\partial x^y_s}}{\dfrac{\partial U^{s+1}}{\partial x^y_{s+1}}\dfrac{\partial U^s}{\partial x^o_{s+1}}\dfrac{\partial U^s}{\partial x^y_s}\dfrac{\partial U^{s-1}}{\partial x^o_s}}
$$

$$
= MRS^{t-1}_{x_t x_{t-1}} \frac{\lambda_{t-2}}{\lambda_{t-1}} MRS^{t-2}_{x_{t-1} x_{t-2}} \frac{\lambda_{t-3}}{\lambda_{t-2}} \cdots
$$

$$
\cdots MRS^{s+1}_{x_{s+2} x_{s+1}} \frac{\lambda_s}{\lambda_{s+1}} MRS^s_{x_{s+1} x_s} \frac{\lambda_{s-1}}{\lambda_s}
$$

$$
= MRS^{t-1}_{x_t x_{t-1}} MRS^{t-2}_{x_{t-1} x_{t-2}} \cdots MRS^{s+1}_{x_{s+2} x_{s+1}} MRS^s_{x_{s+1} x_s} \frac{\lambda_{s-1}}{\lambda_{t-1}}
$$

(iib)
$$
\dfrac{\dfrac{\partial U^t}{\partial x^y_t}}{\dfrac{\partial U^s}{\partial x^y_s}} = \dfrac{\dfrac{\partial U^t}{\partial x^y_t}\dfrac{\partial U^{t-1}}{\partial x^o_t}\dfrac{\partial U^{t-1}}{\partial x^y_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^o_{t-1}}}{\dfrac{\partial U^{t-1}}{\partial x^o_t}\dfrac{\partial U^{t-1}}{\partial x^y_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^o_{t-1}}\dfrac{\partial U^{t-2}}{\partial x^y_{t-2}}} \cdots \dfrac{\dfrac{\partial U^{s+2}}{\partial x^y_{s+2}}\dfrac{\partial U^{s+1}}{\partial x^o_{s+2}}\dfrac{\partial U^{s+1}}{\partial x^y_{s+1}}\dfrac{\partial U^s}{\partial x^o_{s+1}}}{\dfrac{\partial U^{s+1}}{\partial x^o_{s+1}}\dfrac{\partial U^{s+1}}{\partial x^y_{s+1}}\dfrac{\partial U^s}{\partial x^o_s}\dfrac{\partial U^s}{\partial x^y_s}}
$$

$$
= \frac{\lambda_{t-1}}{\lambda_t} MRS^{t-1}_{x_t x_1} \frac{\lambda_{t-2}}{\lambda_{t-1}} MRS^{t-2}_{x_{t-1} x_{t-2}} \cdots
$$

$$
\cdots \frac{\lambda_{s+1}}{\lambda_{s+2}} MRS^{s+1}_{x_{s+2} x_{s+1}} \frac{\lambda_s}{\lambda_{s+1}} MRS^s_{x_{s+1} x_s}
$$

$$
= MRS^{t-1}_{x_t x_{t-1}} MRS^{t-2}_{x_{t-1} x_{t-2}} \cdots MRS^{s+1}_{x_{s+2} x_{s+1}} MRS^s_{x_{s+1} x_s} \frac{\lambda_s}{\lambda_t}
$$

Substituting (iia) and (iib) into (i), we can derive the intergenerational Samuelson public good optimality condition :

$$
\text{(B5)} \qquad MRT_{z_s X_s} = \sum_{t=s}^{s+p-1} \left[MRS^{t-1}_{z_t x_t} \frac{\partial z_t}{\partial z_s} \left[\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a} \right] \right.
$$

$$
\left. + MRS^t_{z_t x_t} \frac{\partial z_t}{\partial z_s} \left[\prod_{a=s}^{t-1} MRS^a_{x_{a+1} x_a} \right] \right] \qquad s > 1.
$$

Footnotes of Appendix B

1. "Let us suppose that this will be supplied later and that we know in advance it will have the following special individualistic property: leaving each person on the same indifference level will leave social welfare unchanged; at any point, a move of each man to a higher indifference curve can be found that will increase social welfare." [Samuelson(1955)]

2. Bergstrom/Cornes(1983) identified the restrictions on utility functions under which allocative efficiency is independent from distribution in the theory of public goods. The condition is : $U^i(z,x^i)$ is of the form $A^i(z) + B(z)x^i$. Also see Cornes/Sandler (1986) pp. 95-98.

3. Quasi-linearity is more than neccesary. It is a special case where $B(z) = 1$.

4. See Cabe(1982) and Sandler/Smith(1982) for this concern.

APPENDIX C : NON—EXISTENCE OF THE NONSTATIONARY LINDAHL EQUILIBRIUM

Recall that the old agent's problem is :

$$\max_{1^o > 0} \quad u(q + 1^o) - p^o 1^o$$

and we get the old agent's demand schedule :

$$1^o(p^o; q) = \max[\mu(p^o) - q, 0]$$

where $\mu = [u']^{-1}$. Let us define :

$$v(p^o; q) = u[q + 1^o(p^o; q)] - p^o 1^o(p^o; q).$$

Then the young agent's problem is :

$$\max_{1^y > 0} \quad u(q + 1^y) - p^y 1^y + dv[p^o; k(q + 1^y)].$$

Thus the young agent's demand schedule is :

$$1^y(p^y, p^o; q) = \max[\mu(p^y - kdp^o) - q, 0].$$

Then the lining-up condition is, since μ is monotonic :

$$p^o = p^y - kdp^o.$$

However for the nonstationary Lindahl equilibrium, it becomes

$$p^o{}_t = p^y{}_t - kdp^o{}_{t+1} \quad \text{or} \quad p^o{}_{t+1} = \frac{(1 - 2p^o{}_t)}{kd}$$

with $1 - p^o{}_t - kdp^o{}_{t+1} > 0$. For the above equation to be well-defined, we need :

$$1/4 < (2 - kd)/4 < p^o{}_t < 1/2 \quad \text{for all t.}$$

Since $\left| \frac{-2}{kd} \right| > 1$, there is no nonstationary Lindahl equilibrium. The intersection between $p^o{}_{t+1} = \frac{(1 - 2p^o t)}{kd}$ and $p^o{}_t = p^o{}_{t+1}$ is the only stationary Lindahl equilibrium given k and d. See Figure C.

In general, the lining-up condition for the additively separable quasi-linear case is :

$$\mu^y{}_t(1 - p^o{}_t - kdp^o{}_{t+1}) = \mu^o{}_t(p^o{}_t)$$

where $\mu^a{}_t = [u^a{}_t{}']^{-1}$ for a = y,o. If we totally differentiate the previous equation, then :

$$\frac{dp^o{}_{t+1}}{dp^o{}_t} = \frac{\mu^o{}_t{}' + \mu^y{}_t{}'}{- kd \cdot \mu^y{}_t{}'} < -1$$

since $\mu^a{}_t{}' < 0$ for a = y,o. Thus the non-existence result holds again.

FIGURE C : Non-existence of the Nonstationary Lindahl Equilibrium

$$p^o_{t+1} = \frac{1 - 2p^o_t}{kd}$$

$$p^o_{t+1} = p^o_t$$
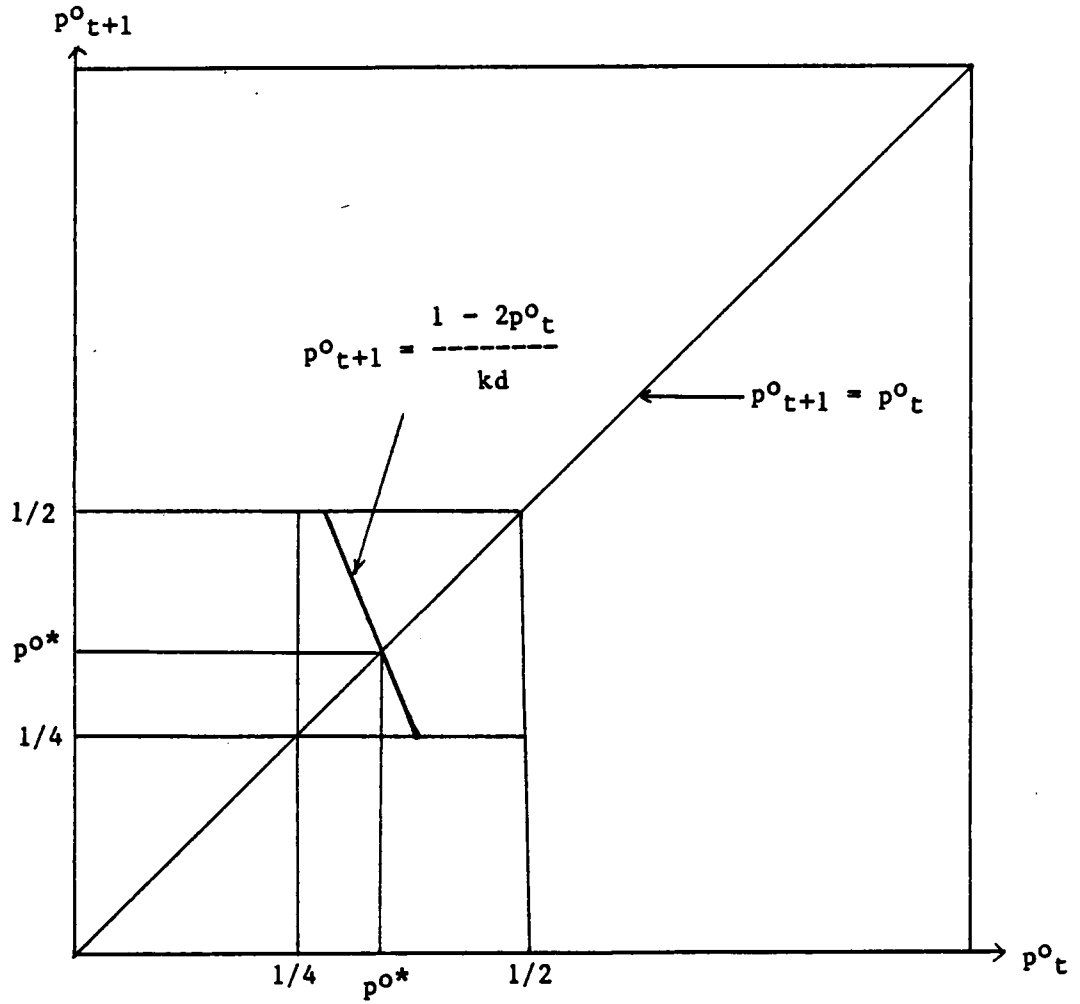
APPENDIX D : DISCUSSION OF THE CASE WHERE

$(\mathbf{u,d,k})$ IS SUCH THAT $\Omega > \Xi$ AND $\omega < k\theta$

Define τ^0 as $u[\tau^0] - \tau^0 + du[\omega] = u[\theta] - \theta + du[k\theta]$. Note that $\omega < \tau^0 \leqslant \omega/k$. Then there are three subcases :

    (1) $u[\tau^0/k] - \tau^0/k + du[\theta] \geqslant u[\omega] - \omega + du[\omega]$

    (2) $u[\omega] - \omega + du[\omega] > u[\theta] - \theta + du[k\theta]$

        $\geqslant u[\tau^0/k] - \tau^0/k + du[\theta]$

    (3) $u[\omega] - \omega + du[\omega] > u[\tau^0/k] - \tau^0/k + du[\theta]$

        $> u[\theta] - \theta + du[k\theta].$

For Case (1), there are three periodic equilibria of period 3. Define

$$\zeta^0(q) = \begin{cases} 1 - q & 0 \leqslant q \leqslant \omega \\ 0 & \omega \leqslant q \leqslant \tau^0 \\ \theta - q & \tau^0 \leqslant q \leqslant \theta \\ 0 & \theta \leqslant q \end{cases}$$

and $\xi^0(q) = \max[\tau^0/k - q, 0]$. Then $\zeta^0(q)$ is the best response to $\zeta(q)$, $\xi^0(q)$ to $\zeta^0(q)$, and $\zeta(q)$ to $\xi^0(q)$.

First, consider the following maximization problem :

$$\max_{z > 0} u[q + z] - z + du[k(q + z) + \zeta(k(q + z))]$$

	$0 < q + z < \omega/k$	$\omega/k < q$
$0 < q < \omega$	$z = \omega - q$	$z = \theta - q$
	$u[\omega] - \omega + du[\omega] + q$ *	$u[\theta] - \theta + du[k\theta] + q$
$\omega < q < \tau^0$	$z = 0$	$z = \theta - q$
	$u[q] + du[\omega]$ *	$u[\theta] - \theta + du[k\theta] + q$
$\tau^0 < q < \omega/k$	$z = 0$	$z = \theta - q$
	$u[q] + du[\omega]$	$u[\theta] - \theta + du[k\theta] + q$ *
$\omega/k < q < \theta$	$z = 0$	$z = \theta - q$
	$u[q] + du[kq]$	$u[\theta] - \theta + du[k\theta] + q$ *
$\theta < q$	$z = 0$	$z = 0$
	$u[q] + du[kq]$ *	$u[q] + du[kq]$ *

where * denotes the corresponding maximum.

Hence $\zeta^0(q)$ is the best response to $\zeta(q)$.

Secondly, consider the following maximization problem :

$$\max_{z \geqslant 0} u[q + z] - z + du[k(q + z) + \zeta^0(k(q + z))]$$

There are two cases : either $\tau^0/k < \theta$ or $\tau^0/k > \theta$.

Case : $\theta < \tau^0/k$

	$0 < q + z$ $< \omega/k$	$\omega/k <_0 q + z$ $< \tau^0/k$	$\tau^0/k < q$ $< \theta/k$	$\theta/k < q + z$
$0 < q < \omega$	$z = \omega - q$ $u[\omega] - \omega$ $+ du[\omega] + q$	$z = \theta - q$ $u[\theta] - \theta$ $+ du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k$ $+ du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k$ $+ du[\theta] + q$
$\omega < q$ $< \omega/k$	$z = 0$ $u[q] + du[\omega]$	$z = \theta - q$ $u[\theta] - \theta$ $+ du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k$ $+ du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k$ $+ du[\theta] + q$
$\omega/k < q$ $< \theta$		$z = \theta - q$ $u[\theta] - \theta$ $+ du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k$ $+ du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k$ $+ du[\theta] + q$
$\theta <_0 q$ $< \tau^0/k$		$z = 0$ $u[q] + du[kq]$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k$ $+ du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k$ $+ du[\theta] + q$
$\tau^0/k < q$ $< \theta/k$			$z = 0$ $u[q] + du[\theta]$	$z = \theta/k - q$ $u[\theta/k] - \theta/k$ * $+ du[\theta] + q$
$\theta/k < q$			$z = 0$ $u[q] + du[kq]$*	$z = 0$ $u[q] + du[kq]$*

The same result holds for Case : $\tau^0/k < \theta$. Hence $\xi^0(q)$ is the best response to $\zeta^0(q)$.

Thirdly, consider the following maximization problem :

$$\max_{z \geqslant 0} u[q + z] - z + du[k(q + z) + \xi^0(k(q + z))]$$

	$0 \leqslant q + z \leqslant (\tau^0/k)/k$	$(\tau^0/k)/k \leqslant q$
$0 \leqslant q \leqslant \omega$	$z = \omega - q$	$z = (\tau^0/k)/k - q$
	$u[\omega] - \omega + du[\tau^0/k] + q$ *	$u[\theta] - \theta + du[k\theta] + q$
$\omega \leqslant q \leqslant \omega/k$	$z = 0$	$z = (\tau^0/k)/k - q$
	$u[q] + du[\tau^0/k]$ *	$u[\theta] - \theta + du[k\theta] + q$
$\omega/k \leqslant q \leqslant (\tau^0/k)/k$	$z = 0$	$z = (\tau^0/k)/k - q$
	$u[q] + du[\tau^0/k]$ *	$u[\theta] - \theta + du[k\theta] + q$
$(\tau^0/k)/k \leqslant q$	$z = 0$	$z = 0$
	$u[q] + du[kq]$ *	$u[q] + du[kq]$ *

Hence $\zeta(q)$ is the best response to $\xi^0(q)$.

For Case (2), $u[\theta] - \theta + du[k\theta] > u[\tau^0/k] - \tau^0/k + du[\theta]$ implies $\theta < \tau^0/k$ since $\theta > k\theta$. Consider the following maximization problem :

$$\max_{z>0} u[q + z] - z + du[k(q + z) + \zeta^0(k(q + z))]$$

	$0 < q + z < \omega/k$	$\omega/k < q + z < \tau^0/k$	$\tau^0/k < q < \theta/k$	$\theta/k < q + z$
$0 < q < \omega$	$z = \omega - q$ $u[\omega] - \omega + du[\omega] + q$ *	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\omega < q < \tau^0$	$z = 0$ $u[q] + du[\omega]$ *	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\tau^0 < q < \omega/k$	$z = 0$ $u[q] + du[\omega]$	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$ *	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\omega/k < q < \theta$		$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$ *	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\theta < q < \tau^0/k$		$z = 0$ $u[q] + du[kq]$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\tau^0/k < q < \theta/k$			$z = 0$ $u[q] + du[\theta]$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\theta/k < q$				$z = 0$ $u[q] + du[kq]$ *

Note that, for $\theta < q < \tau^0/k$, we do not know which is larger,

$u[q] - q + du[kq]$ or $u[\tau^0/k] - \tau^0/k + du[\theta]$. However, since

$u[\tau^0/k] - \tau^0/k + du[\tau^0] < u[\tau^0/k] - \tau^0/k + du[\theta]$, we can define :

$$\tau^1 \; \epsilon \; (\theta, \tau^0/k) \; : \; u[\tau^1] - \tau^1 + du[k\tau^1] = u[\tau^0/k] - \tau^0/k + du[\theta]$$

and, thus,

$$\zeta^1(q) = \begin{cases} 1 - q & 0 < q < \omega \\ 0 & \omega < q < \tau^0 \\ \theta - q & \tau^0 < q < \theta \\ 0 & \theta < q < \tau^1 \\ \tau^0/k - q & \tau^1 < q < \tau^0/k \\ 0 & \tau^0/k < q \end{cases}$$

Consider the following maximization problem :

$$\max_{z \geqslant 0} u[q + z] - z + du[k(q + z) + \zeta^1(k(q + z))]$$

Then there are three subcases for Case (2) :

(2.1) $u[\tau^1/k] - \tau^1/k + du[\tau^0/k] \geqslant u[\omega] - \omega + du[\omega]$

(2.2) $u[\omega] - \omega + du[\omega] > u[\theta] - \theta + du[k\theta]$

$\qquad \geqslant u[\tau^1/k] - \tau^1/k + du[\tau^0/k]$

(2.3) $u[\omega] - \omega + du[\omega] > u[\tau^1/k] - \tau^1/k + du[\tau^0/k]$

$\qquad > u[\theta] - \theta + du[k\theta]$

For Case(2.1), there are four periodic equilibria of period 4. Define

$\xi^1(q) = \max[\tau^1 - q, 0]$. $\zeta(q)$ is the best response to $\xi^1(q)$, $\xi^1(q)$ to

$\zeta^1(q)$, $\zeta^1(q)$ to $\zeta^0(q)$, and $\zeta^0(q)$ to $\zeta(q)$.

For Case (2.2), there will be three subcases in the similar way.

For Case (3), consider the following maximization problem :

$$\max_{z > 0} \; u[q + z] - z + du[k(q + z) + \zeta^0(k(q + z))]$$

There are two cases : either $\tau^0/k \leq \theta$ or $\tau^0/k > \theta$.

Case (3a) : $\theta < \tau^0/k$

	$0 \leq q + z \leq \omega/k$	$\omega/k \leq q + z \leq \tau^0/k$	$\tau^0/k \leq q \leq \theta/k$	$\theta/k \leq q + z$
$0 \leq q \leq \omega$	$z = \omega - q$ $u[\omega] - \omega + du[\omega] + q$ *	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\omega \leq q \leq \tau^0$	$z = 0$ $u[q] + du[\omega]$	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\tau^0 \leq q \leq \omega/k$	$z = 0$ $u[q] + du[\omega]$	$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\omega/k \leq q \leq \theta$		$z = \theta - q$ $u[\theta] - \theta + du[k\theta] + q$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\theta \leq q \leq \tau^0/k$		$z = 0$ $u[q] + du[kq]$	$z = \tau^0/k - q$ $u[\tau^0/k] - \tau^0/k + du[\theta] + q$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\tau^0/k \leq q \leq \theta/k$			$z = 0$ $u[q] + du[\theta]$ *	$z = \theta/k - q$ $u[\theta/k] - \theta/k + du[\theta] + q$
$\theta/k \leq q$				$z = 0$ $u[q] + du[kq]$ *

Define $\ell^1_1 \in (\omega, \tau^0)$ such as $u[\ell^1_1] - \ell^1_1 + du[\omega] = u[\tau^0] - \tau^0 + du[\theta]$

and, thus,

$$\rho^1_1(q) = \begin{cases} 1 - q & 0 < q < \omega \\ 0 & \omega < q < \ell^1_1 \\ \tau^0/k - q & \ell^1_1 < q < \tau^0/k \\ 0 & \tau^0/k < q \end{cases}$$

There will be three subcases for Case (3a) again.

For Case (3b) : $\theta > \tau^0/k$, define similarly $\ell^1_2 \in (\omega, \tau^0)$ such as $u[\ell^1_2] - \ell^1_2 + du[\omega] = u[\tau^0] - \tau^0 + du[\theta]$ and, thus,

$$\rho^1_1(q) = \begin{cases} 1 - q & 0 < q < \omega \\ 0 & \omega < q < \ell^1_2 \\ \tau^0/k - q & \ell^1_2 < q < \tau^0/k \\ 0 & \tau^0/k < q \end{cases}$$

There will be three subcases for Case (3b).

Therefore, given $(u,d,k) \in \{ (u,d,k) : W > \Omega$ and $\omega < k\theta \}$, we can continue this process on and on. In every step, either we will find periodic equilibria or we will continue again. However as the process goes on, the number of constraints increases and the difference of the lifetime net benefit from approximation becomes insignificant. Yet we have not found any general result for the case where $W > \Omega$ and $\omega < k\theta$.