RELIABILITY ESTIMATION USING DOUBLY CENSORED FIELD DATA

by

Arun Kumar

Dissertation submitted to the Graduate Faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Industrial Engineering & Operations Research

APPROVED:

J. A. Nachlas, Chairman

C. P. Koelling

J. D. Tew

G. T. Adel

G. R. Terrell

December 1988

Blacksburg, Virginia

RELIABILITY ESTIMATION USING DOUBLY CENSORED FIELD DATA

by

Arun Kumar

Committee Chairman: Joel A. Nachlas
Industrial Engineering & Operations Research

(ABSTRACT)

Statistical methods for estimating the parameters of a Weibull distribution are developed under the assumption that the available data set is obtained from field performance and is consequently censored on both the left and the right. The extreme lack of data makes estimation very difficult. Estimation equations are defined for both maximum-likelihood and moment based estimates. Simulation results obtained using the defined estimation strategies are not promising but do suggest directions for further study.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to my advisor and dissertation chairman, Dr. Joel A. Nachlas, for his guidance and encouragement throughout the course of this research.

I would also like to acknowledge the advice I received from other members of my dissertation committee, Dr. C. P. Koelling, Dr. J. D. Tew, Dr. G. T. Adel, and Dr. G. R. Terrell, during the course of this study, and I express my sincere thanks to them.

I would also like to thank my parents and other members of my family for their tremendous support and encouragement in this endeavor.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

The assessment of reliability of equipment is of paramount importance in the context of modern technology and its future developments. Special statistical methods have been developed for this purpose during the last few decades but there is no efficient technique available to estimate reliability using certain classes of highly censored field data which represent actual operating experiences.

When we buy equipment or a simple product such as an electric bulb or a calculator, we expect it to function properly for a reasonable period of time. Unexpected failures of equipment are encountered at times. No matter how efficient the manufacturing process is, one or more failures may occur. This failure may be due to any of several sources.

Man-made systems all suffer from imperfections. There are numerous arguments for this: the designer has neglected or has not been aware of some important facts concerning either the environment in which the system is to operate or the operation of the system itself; the manufacturer has introduced defects into the system when producing it; weaknesses are inherent in the materials from which the system is built. These imperfections lead to the failure of the system to perform its function. The second source of failure is natural component deterioration caused by friction and abrasive wear, metal fatigue or corrosion. Very often, system or component failure is the result of many interacting factors.

Another source of failure is chance causes. Random failures occur quite unpredictably at random intervals and cannot be eliminated by taking necessary steps at the planning, production or inspection stage (Sinha, 1986).

When the item fails, a replacement action may be initiated and as a result of the replacement the unit is renewed (renewal models). When the replacement action brings the unit to an as-new-like condition, it is natural to model the failure process as a renewal process where the renewal intervals are the times between successive failures. The Weibull renewal process is often used to model the equipment failure process and to describe the life-time distribution.

Verification of the attained reliability of equipment is often based upon the statistical evaluation of field performance data. In many applications, the process of data reporting or the nature of operation of the equipment cause the accumulated data sets to be highly incomplete. A specific realization of this situation is constructed in this research and an algorithm for obtaining life distribution parameter estimates is developed.

An example of the situation which is studied in this research is monitoring the reliability of a population of computing devices for which the ages of the devices are unknown at the start of the observation interval. This constitutes left censoring of the failure data. During the observation interval, a subset of the devices fail and the balance of the population operates successfully. This constitutes conventional right data censoring. The two forms of data censoring combine to yield a data set that is doubly censored.

Doubly censored data sets provide far less information about device reliability than is available in a life test data set. On the other hand, field data represents actual operating experiences and is often preferred to laboratory data. Consequently, a method for computing failure distribution parameter estimates has substantial practical value.

The general parameter estimation equations for the computation of maximum likelihood estimates and method of moments estimates are defined for the case of doubly censored data. It is shown that when the life distribution is negative exponential, the estimates can be computed readily. The equations are then applied to data from a Weibull life distribution. It is shown that for this case the equations cannot be solved directly but do permit numerical solution (Chapter III). An algorithm for computing the solution estimates is defined for several example data sets. In each case, the accuracy of the estimates obtained is characterized.

2. LITERATURE REVIEW

Information related to Weibull renewal processes is limited and there is no literature available relevant to reliability estimation using doubly censored field data. Several studies have been made by various investigators during the last three decades in the area of reliability and life testing using the Weibull life distribution. All of these investigations describe right or single censoring only.

The Weibull distribution has been extensively used as a model of life length. This leads to its study in life testing and reliability estimation. Weibull (1951) discussed its applicability to a wide variety of failure situations. Kao (1959) used it as a model for vacuum tube failures while Lieblien and Zelen (1956) used it as a model for ball bearing failures. Mann (1968) gives a variety of situations (increasing and decreasing hazard rates) in which the distribution is used for other types of failure data.

Smith and Leadbetter (1963) have developed the renewal function for the Weibull distribution. As one of the early investigations, their study covers the development of a series expansion for the renewal function associated with the Weibull distribution. The expansion is valid for all values of the time t, and the coefficients of the powers of t are easily calculated numerically by a recursive procedure. Lomnicki (1966) studied the Weibull renewal process and expanded the renewal functions of Weibull type not into power series but into infinite series of appropriate Poissonian functions of t^B. An example illustrating the numerical calculations for a given Weibull renewal process is also presented in this paper.

Lehman (1963) has discussed the properties of the Weibull distribution. Given the conditions on the shape parameter B for the existence of a mode and inflection point, the locations of and the values of the function at these points are traced as B grows from zero to infinity. The behavior of the median and first four moments is described and presented in tabular form as a function of B. Other interesting features of the Weibull distribution are noted.

2.1. ESTIMATION OF PARAMETERS

There are several methods by which one can obtain good estimates for the parameters of Weibull distribution. The methods include the iterative solution of the maximum-likelihood equations, moment estimators, and several types of linear estimation techniques. Because of the regularity properties of the Weibull density function maximum-likelihood estimators of Weibull parameters enjoy the properties of consistency, asymptotic efficiency, asymptotic unbiasedness and asymptotic normality (Mann, 1967). The maximum-likelihood estimates cannot, however, be calculated explicitly, but must be determined by iterative procedures applied to sample data.

Cohen (1965) is one of the many individuals who first indicated the maximum-likelihood estimation in the Weibull distribution based on complete and on censored samples. Cohen's study is concerned with the two parameter Weibull distribution which is widely employed as a model in life testing. Maximum likelihood equations are derived for estimating the distribution parameters from (i) complete samples, (ii) singly censored samples, and (iii) progressively (multiple) censored samples. Asymptotic variance-covariance matrices are given for each of these sample types. An illustrative example is also included in his paper.

By using the general theory of Maximum Likelihood Estimation, Cohen (1965) showed that maximum likelihood estimates of parameters $(\hat{\alpha}, \hat{B})$ are bivariate normal with means (α, B) and variance co-variance matrix given by

$$\begin{vmatrix} E\left(-\frac{\partial^2 \ln L}{\partial \beta^2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial \beta \cdot \partial \alpha}\right) \\ E\left(-\frac{\partial^2 \ln L}{\partial \alpha \cdot \partial \beta}\right) & E\left(-\frac{\partial^2 \ln L}{\partial \alpha^2}\right) \end{vmatrix}^{-1}$$

where lnL is the logarithm of the likelihood function defined as:

$$L(\underline{t},\alpha,B) = \prod_{i=1}^{n} f(t_i,\alpha,B)$$

for a sample data set of n observations.

The exact expressions for various expectations above are difficult to obtain. However, in practice they are approximated by using (Cohen, 1965):

$$E\left(-\frac{\partial^2 \ln L}{\partial\beta \cdot \partial\alpha}\right) = -\frac{\partial^2 \ln L}{\partial\beta \cdot \partial\alpha}\bigg|_{\substack{B=\hat{B} \\ \alpha=\hat{\alpha}}} \quad \text{etc.}$$

Accordingly, we have as the approximate variance-covariance matrix

$$\begin{vmatrix} -\frac{\partial^2 \ln L}{\partial\beta^2}\Big|_{\hat{\alpha}\,\hat{\beta}} & -\frac{\partial^2 \ln L}{\partial\beta \cdot \partial\alpha}\Big|_{\hat{\alpha}\,\hat{\beta}} \\ -\frac{\partial^2 \ln L}{\partial\alpha \cdot \partial\beta}\Big|_{\hat{\alpha}\,\hat{\beta}} & -\frac{\partial^2 \ln L}{\partial\alpha^2}\Big|_{\hat{\alpha}\,\hat{\beta}} \end{vmatrix}^{-1} = \begin{vmatrix} \mathrm{Var}(\hat{B}) & \mathrm{Cov}(\hat{\alpha},\hat{B}) \\ \mathrm{Cov}(\hat{\alpha},\hat{B}) & \mathrm{Var}(\hat{\alpha}) \end{vmatrix}$$

Mann (1967) has developed Tables for obtaining the best linear invariant estimates of parameters of the two-parameter Weibull distribution. She considered a censored life test situation and assumed a Weibull failure time distribution. These tables can be used for estimating the logarithms of the reliable life, where the estimator is best among linear estimators with expected loss invariant under translations. According to her study, these best linear invariant estimators have uniformly smaller expected loss than the Gauss-Markov best linear unbiased estimators and are simple linear functions of the best linear invariant estimators.

Mann (1968) has also used moment estimators (based on moments of the distribution of $X = \ln T$) and shows that they are less efficient than estimators based on only a few ordered observations. For this reason, and because of the fact that considerable effort is required in calculating moment estimates for two-parameter Weibull distribution, their use is not recommended (Mann, Schafer, and Singpurwalla, 1974).

Thoman, Bain, and Antle (1969, 1970) present the results of a study of the maximum-likelihood estimator, $\hat{R}(t)$, of the reliability, $R(t)$, when the two-parameter Weibull distribution is assumed. They have shown that the distribution of $\hat{R}(t)$ depends only upon $R(t)$ and the number of samples n. It has been observed in their study that $\hat{R}(t)$ is very nearly unbiased and has a variance that is practically equal to the Cramer-Rao lower bound for the variance of an unbiased estimator. Tables that show lower confidence limits for the reliability are also provided for confidence levels of 0.75, 0.80, 0.85, 0.90, 0.95, and 0.98. For an observed value of $\hat{R}(t)$, the lower confidence limit can be read directly from the table and the large sample normal approximation for $\hat{R}(t)$ is also investigated. The study also included the development of tolerance limits based on the maximum- likelihood estimators of the Weibull parameters.

The method of maximum-likelihood has been used very often by investigators to obtain the analytical estimates of the parameters of the three-parameter Weibull distribution for the case in which all the parameters are unknown. Lemon (1974) has modified the likelihood equations so that one need iteratively solve only two equations for estimates of first two parameters, which together then specify an estimate of the third parameter.

The best linear unbiased estimators of the location and scale parameters of the Weibull distribution when the shape parameter is known has been considered by numerous investigators. Mann (1970) considered warranty estimation and confidence bounds for Weibull parameters based on a few order statistics. Kulldorff (1973) discussed optimum spacings of sample quantiles from six extreme value distributions (of which Weibull is one) when one of the parameters is known. Chan, Cheng and Mead (1974) proposed simultaneous estimation of the location and scale parameters.

Johns and Lieberman (1966) developed a simple method for obtaining exact lower confidence bounds for reliabilities (tail probabilities) for items whose life times follow a Weibull distribution for which the "shape" and "scale" parameters are unknown. The method is used to obtain confidence bounds both for the censored and non-censored cases, and the bounds are asymptotically efficient. They are exact even for small sample sizes in that they attain the desired confidence level precisely. The case of an additional unknown "location" or "shift" parameter is also discussed in their paper in the large sample case. The paper also has tables of exact and asymptotic lower confidence bounds for the reliability for sample sizes of 10, 15, 20, 30, 50 and 100 for various censoring fractions.

White (1969) proposed a method of estimation for the Weibull distribution parameters which is based on a regression approach. His method is applicable to censored samples as well as complete samples.

Sinha and Fu (1977), suggested an estimation procedure which depends on two fundamental concepts, (i) the conditional failure rate, and (ii) the least square principle.

Mann, Schafer, and Singpurwalla, have put together a book entitled Methods of Statistical Analysis of Reliability and Life Data in 1974. The book provides more extensive treatment of numerous methods of estimation.

2.2. BAYES ESTIMATORS AND RELIABILITY ESTIMATION

Susarla and Van Ryzin (1976) gave a class of Bayesian non-parametric estimators for Weibull life time distribution using the Dirichlet distribution as prior.

Breslow and Crowley (1974), Meir (1975), Phadia and Van Ryzin (1980), and Susarla and Van Ryzin (1978, 1979, 1980) studied the large sampling properties of Kaplan-Meir estimator and that of non-parametric Bayesian estimators and found that both are asymptotically equivalent.

Rai, Susarla, and Van Ryzin (1980), have suggested Bayesian non-parametric estimators which shrink the non-parametric estimator towards a prior exponential reliability function. This estimator has been compared with the Kaplan-Meir estimator and maximum-likelihood estimator and is found to be better than the Kaplan-Meir estimator.

Bayes estimators are often obtained as a ratio of two integrals which cannot be expressed in closed forms and numerical approximations are needed. Lindley (1980) developed an asymptotic approximation to the ratio. Sinha (1985, 1986) obtained Bayes estimators using the marginal posteriors and Lindley's approximation. His studies based on Monte Carlo methods reflect that while the maximum-likelihood estimator of R(t) has uniformly smaller squared-error deviation than its Bayesian counterpart, on the basis of the squared-error deviation the efficiency of

Lindley's estimates increases sharply with time compared to the ones obtained by using the marginal posterior.

Sinha and Kale (1980) discuss prior distributions of the unknown parameters B and θ of a Weibull life distribution. They assumed B and θ to be independent. A uniform prior distribution has been chosen for B and a non-informative prior for θ. The joint prior is given by the product of two individual prior distributions.

Pandey (1987) used Sinha and Kale's (1980) result for the joint prior distribution of (B, θ) and obtained a Bayes estimator of reliability at time t ($0<t<T$). In his study, Pandey observed a fixed number of patients (n) whose times to death had identical Weibull distributions with parameters B and θ. The maximum times of observation for different patients were also assumed independent uniform variables as the patients arrived randomly throughout the trial. The mean of R(t) with respect to the joint posterior distribution was the resulting Bayes estimator.

3. DEVELOPMENT OF MODEL

3.1. ESTIMATION METHODS

There are several methods by which one can obtain good point estimates of the unknown parameters, α and B of the two parameter Weibull distribution. The methods include the iterative solution of the maximum-likelihood equations, moment estimators, and several types of linear estimation techniques. A discussion of the maximum-likelihood method, the moment estimator and the Weibull renewal process are presented here. Subsequently, computer programs that compute the defined estimates are provided.

Presumably, one can construct life distribution parameter estimates by examining the behavior of the corresponding renewal process. A set of doubly censored field data constitutes a set of "snap-shots" of the process or "windows of distribution." The first strategy for parameter estimation developed here is the analysis of the renewal process. The intent is to include the resulting metrics of process behavior in a corresponding likelihood function. This is shown to be analytically intractable. Consequently, an alternate strategy for maximum likelihood estimation is defined and a moment estimation method is also developed.

3.1.1. Weibull Renewal Process:

The Weibull distribution is considered as a model for the failure time distribution. The probability density function of the Weibull distribution is given by:

$$f(t|\alpha,B) = \alpha B\, t^{B-1}\exp(-\alpha t^{B}), \quad t>=0,\ \alpha>0,\ B>0 \quad (1)$$

where B is referred to as a shape parameter and α a scale parameter.

For the Weibull distribution, the distribution function is given by

$$F(t) = 1 - \exp(-\alpha t^{B}), \quad (2)$$

and the reliability function is:

$$R(t) = \exp(-\alpha t^{B}). \quad (3)$$

If t_1, t_2, ------ is a sequence of independent random variables governed by a Weibull distribution, then the random variables $S_K = t_1 + t_2 + ---- + t_k$ are interpreted as the times up to the k^{th} renewal and the probability that $S_K<t$ is given by the k-fold convolution of F(t) (Cox,1962)

$$F_k(t) = \int_0^t F_{k-1}(t-x)dF(x), \quad (4)$$

where $F_0(t) \equiv 1$.

If N_t is the number of renewals in $(0,t)$, then

$$\begin{aligned} \Pr\ (N_t<k) &= \Pr\ (S_k>t) \\ &= 1 - F_k(t), \end{aligned} \tag{5}$$

so that $F_k(t)$ can be regarded as the probability of k or more renewals in the time interval $(0,t)$.

The distribution on the number of renewals is defined by the family of functions $W_k(t)$, where

$$W_k(t) = F_k(t) - F_{k+1}(t), \qquad k=0,1,2,--- \tag{6}$$

is the probability of exactly k renewals in $(0,t)$. The expected number of renewals $E(N_t) = M_t$, is the so-called Renewal Function and is given by:

$$M(t) = \sum_{k=1}^{\infty} k.W_k(t),$$

so that by equation (6)

$$\begin{aligned} M(t) &= \sum_{k=1}^{\infty} k.(F_k(t) - F_{k+1}(t)) \\ &= \sum_{k=1}^{\infty} F_k(t). \end{aligned} \tag{7}$$

If $\alpha=1$, the probability of renewal in (0,t) is equal to

$$F(t) = 1 - \exp(-t^B), \tag{8}$$

and the corresponding probability density function is given by

$$f(t) = Bt^{B-1}\exp(-t^B) \tag{9}$$

In the case of the Weibull distribution, the Laplace Transform approach is not convenient to determine the Renewal Function. In fact, it has been proven that the Moment Generating Function does not exist for the Weibull distribution. However, Lomnicki (1966) evaluated the renewal function by a straight-forward expansion into an infinite series of functions of t^B.

$$M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{s=1}^{\infty} D_S(t^B) \sum_{k=1}^{\infty} \alpha_k(s)$$

$$= \sum_{s=1}^{\infty} c(s) D_S(t^B)$$

$$= \exp(-t^B) \sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{(t^B)^r}{r!} \tag{10}$$

where, $c(s) = \sum_{k=1}^{\infty} \alpha_k(s)$, $\quad s=1,2,$-----

$$\alpha_k(k) = a_k(k)$$

$$\alpha_k(s) = \sum_{r=k}^{s} a_r(s) - \sum_{r=k}^{s-1} a_r(s-1) \qquad \text{for } s>k,$$

$$\alpha_k(s) = \alpha_{k+1}(s) + \{a_k(s) - a_k(s-1)\}$$

$$a_k(s) = \sum_{p=k}^{s} (-1)^{p+k} \binom{S}{p} \frac{b_k(p)}{\gamma(p)} \qquad \text{for } k=0,1,\text{--} \quad s=k,k+1,\text{--}$$

$$b_0(s) = \gamma(s) \qquad \text{for } s=0,1,\text{---}$$

$$b_{k+1}(s) = \sum_{r=k}^{s-1} b_k(r)\gamma(s-r) \qquad \text{for } k=0,1,\text{---} \quad s=k+1,k+2,\text{--}$$

$$\gamma(r) = \Gamma(Br+1)/\Gamma(r+1) \qquad \text{for } r=0,1,\text{---}$$

$$D_k(t) = \exp(-t) \sum_{r=k}^{\infty} \frac{t^r}{r!} \qquad \text{for } k=1,2,\text{---}$$

3.1.1.1. Backward Recurrence-Time:

The backward recurrence-time at a time point t is the length of time measured backwards from t to the last renewal at or before t. Figure 3.1 depicts the backward recurrence-time, U_t, at a fixed time point t. If there have been no renewals up to t, U_t is defined to be t. In this case:

$$Pr\ (U_t=t) = R(t) = \exp(-\alpha t^B). \tag{11}$$

The probability density function of the backward recurrence-time, U_t, is determined as,

$$Pr\ (x<U_t<=x+\Delta x) = M'(t-x).R(x).\Delta x \tag{12}$$

i.e., $x<U_t<=x+\Delta x$ if and only if a renewal occurs in $(t-x,t-x+\Delta x)$ and the new equipment survives until after t.

The probability density function for U_t at time x is:

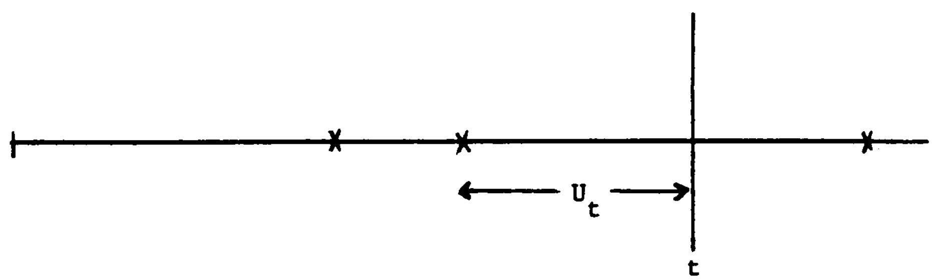

Figure 3.1. Definition of Backward Recurrence-time, U_t.

$$f(u_t | t=x) = \begin{cases} M'(t-x).R(x) & 0<=x<t \\ R(x) & x=t \end{cases} \tag{13}$$

where, $M'(t-x)$ = renewal density function for time $(t-x)$,

$= \frac{d}{dx}[E(N_{t-x})]$

$R(x)$ = reliability function for time x.

Again,

$$f(u_t | t=x) = R(x).M'(t-x), \qquad 0<=x<t \tag{14}$$

for the Weibull failure distribution, and when $\alpha = 1$,

$$f(u_t | t=x) = \exp(-x^B)\{\frac{dM}{dx}(t-x)\}$$

$$= \exp(-x^B)[\frac{d}{dx}\{\sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{(t-x)^{Br}}{r!}\}\exp(-(t-x)^B)]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) + \exp(-(t-x)^B).\{\sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{d}{dx}\frac{(t-x)^{rB}}{r!}\}]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) + \exp(-(t-x)^B).\{\sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} -rB\frac{(t-x)^{rB-1}}{r!}\}]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) + \{-\sum_{s=1}^{\infty} c(s).\sum_{r=s}^{\infty} B\frac{(t-x)^{rB-1}}{(r-1)!}\}\exp(-(t-x)^B)] \tag{15}$$

Now,

$$(t-x)^{rB-1} = (t-x)^{rB-B+B-1} = (t-x)^{(r-1)B}(t-x)^{B-1} \tag{16}$$

Therefore,

$$f(u_t|t=x) = \exp(-x^B)[B(t-x)^{B-1}M(t-x) - \{\sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} B(t-x)^{(r-1)B}\frac{(t-x)^{B-1}}{(r-1)!}\}\exp(-(t-x)^B)]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) - B(t-x)^{B-1}\{\sum_{s=1}^{\infty} c(s). \sum_{r=s}^{\infty} \frac{(t-x)^{(r-1)B}}{(r-1)!}\}\exp(-(t-x)^B)]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) - B(t-x)^{B-1}\{\sum_{s=1}^{\infty} c(s). \frac{(t-x)^{(s-1)B}}{(s-1)!} + \sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{(t-x)^{rB}}{r!}\}\exp(-(t-x)^B)]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) - B(t-x)^{B-1}\{\sum_{s=1}^{\infty} c(s). \frac{(t-x)^{(s-1)B}}{(r-1)!}\}\exp(-(t-x)^B) - B(t-x)^{B-1}\{\sum_{s=1}^{\infty} c(s). \sum_{r=s}^{\infty} \frac{(t-x)^{rB}}{r!}\}\exp(-(t-x)^B)]$$

$$= \exp(-x^B)[B(t-x)^{B-1}M(t-x) - B(t-x)^{B-1}\{\sum_{k=0}^{\infty} c(k+1). \frac{(t-x)^{kB}}{k!}\}\exp(-(t-x)^B) - B(t-x)^{B-1}M(t-x)]$$

$$= \exp(-x^B)[-B(t-x)^{B-1}\{\sum_{k=0}^{\infty} c(k+1).\frac{(t-x)^{kB}}{k!}\}. \exp(-(t-x)^B] \qquad \text{for } 0<=x<t$$

$$= \exp(-x^B) \qquad \text{for } x=t. \tag{17}$$

and,

$$M'(t-x) = B.(t-x)^{B-1}.\exp(-(t-x)^B).\left\{\sum_{k=o}^{\infty} c(k+1).\frac{(t-x)^{kB}}{k!}\right\} \quad (18)$$

This derivative depends upon the statement that:

$$\sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{(t-x)^{(r-1)B}}{(r-1)!} = \sum_{s=1}^{\infty} c(s).\frac{(t-x)^{(s-1)B}}{(s-1)!} + \sum_{s=1}^{\infty} c(s) \sum_{r=s}^{\infty} \frac{(t-x)^{rB}}{r!} \quad (19)$$

The accuracy of the above statement is based on the following lemma:

<u>Lemma</u> 3.1. $\underset{a \to \infty}{\text{Limit}} \frac{(t-x)^{aB}}{a!} = 0 \qquad \cancel{\forall} B, t, x.$ (20)

<u>Proof</u>:

Take $(t-x)^B = y \to$ a fixed value

There will be minimum value of a, say a_0, which will be greater than <u>2y</u>.

$a > a_0 > 2y$

$$\underset{a \to \infty}{\text{Limit}} \frac{(t-x)^{aB}}{a!} = \frac{y^a}{a!}$$

$$= \frac{y^{a-ao}}{a.(a-1).(a-2).....(a_0+1)} . \frac{y^{ao}}{a_o!}$$

fixed value, say A_0

$$= A_0.(y/a).(y/a-1).....(y/a_0+1)$$

$$<= A_0.(1/2)^{a-a_0} = A_0.0$$

$$= 0 \qquad \{\text{Since } a > a_0+1 > 2y$$

$$\text{or, } (1/a)<(1/a_0+1)<1/2y$$

$$\text{or, } (y/a)<(y/a_0+1)<y/2y=1/2\}.$$

The expected value of the backward recurrence-time, U_t, will be:

$$E[U_t] = \int_0^t x.f(x)dx \qquad \text{where } f(x)=\text{p.d.f. of } U_t$$

$$= \int_0^t x.B(t-x)^{B-1}\{\sum_{k=o}^{\infty} c(k+1).\frac{(t-x)^{kB}}{k!}\}.\exp(-(t-x)^B-x^B).dx$$

$$= \int_0^t x.B\{\sum_{k=o}^{\infty} c(k+1).\frac{(t-x)^{kB+B-1}}{k!}\}.\exp(-(t-x)^B-x^B).dx \qquad (21)$$

This form does not have an analytical solution but it can be evaluated numerically.

Cinlar (1975) has developed an equation for the expected value of backward recurrence-time for a general failure interval distribution.

$$E[U_t] = \int_0^t (1 - F(u))du \qquad \text{for all } t>=0 \qquad (22)$$

where, $F(u)$ = failure interval distribution.

If failure interval distribution, $F(u)$, is a Weibull distribution with parameters α, and β, then

$$E[U_t] = \int_0^t (1 - 1 + \exp(-\alpha u^B) du$$

$$= \int_0^t \exp(-\alpha u^B) du \qquad (23)$$

[consider,

$$\alpha u^B = x, \quad \text{or } u = (x/\alpha)^{1/B}.$$

$$\alpha B u^{B-1} du = dx$$

$$\text{or, } du = dx/(\alpha B u^{B-1})$$

$$\text{or, } du = dx/(\alpha B(x/\alpha)^{B-1/B}) \quad].$$

Therefore,

$$E[U_t] = \int_0^{\alpha t^\beta} \exp(-x) . \frac{dx}{\alpha\beta(x/\alpha)^{\beta-1/\beta}}$$

$$= \int_0^{\alpha t^\beta} \frac{1}{\alpha^{1/\beta} . \beta} x^{(1/B)-1} . \exp(-x) . dx$$

(consider $1/B=n$)

$$= \frac{1}{\alpha^{1/\beta} . \beta} \int_0^{\alpha t^\beta} x^{n-1} . \exp(-x) . dx$$

$$= \frac{1}{\alpha^{1/\beta} . \beta} [\int_0^\infty x^{n-1} . \exp(-x) . dx - \int_{\alpha t^\beta}^\infty x^{n-1} \exp(-x) dx]$$

$$= \frac{1}{\alpha^{1/\beta} . \beta} [\Gamma(n) - \int_{\alpha t^\beta}^\infty x^{n-1} . \exp(-x) . dx]$$

$$= \frac{1}{\alpha^{1/\beta} \cdot \beta}[\, \Gamma(n) - \{\exp(-\alpha t^{\beta}).(1/(\alpha t^{\beta})^{1-n} - (1-n)/(\alpha t^{\beta})^{1-n+1} + (1-n).(1-n+1)/(\alpha t^{\beta})^{1-n+2} - ---)\}]$$

$$E[U_t] = \frac{1}{\alpha^{1/\beta} \cdot \beta}[\Gamma(n) - \exp(-\alpha t^{\beta})\{1/(\alpha t^{\beta})^{1-n} - (1-n)/(\alpha t^{\beta})^{1-n+1} + (1-n).(1-n+1)/(\alpha t^{\beta})^{1-n+2} - ---)\}] \quad (24)$$

This equation is also an approximation of the series (incomplete Gamma function approximation) and it is easier to compute the expected value of the backward recurrence-time using this equation.

3.1.1.2. <u>Forward Recurrence-Time</u>:

The forward recurrence-time is the time measured from a time point t to the next renewal to occur after t. Figure 3.2 depicts the forward recurrence-time, V_t, at a fixed time point t. Another name for the forward recurrence-time is the residual life-time.

Cox (1962) develops a general equation for the forward recurrence time. His equation is used here to develop the equation for the forward recurrence-time of a Weibull renewal process.

Consider the symbols used in Figure 3.3.

$x = t_0 - t_i$ = backward recurrence time,

$y = t_f - t_0$ = forward recurrence time,

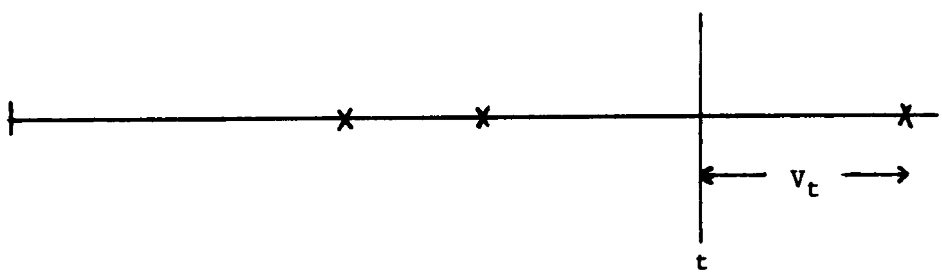

Figure 3.2. Definition of Forward Recurrence-time, V_t.

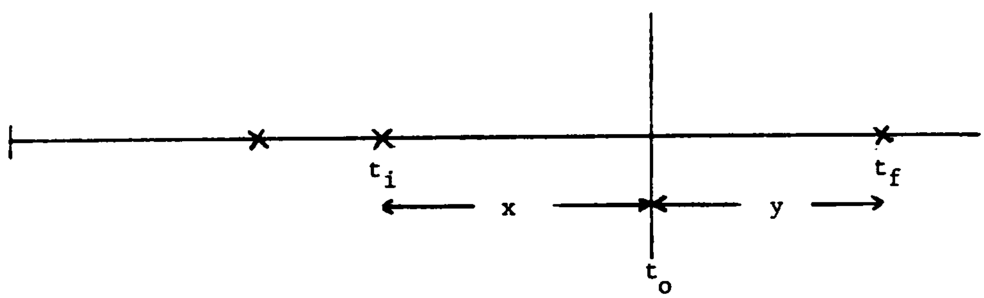

Figure 3.3. Backward and Forward Recurrence-times, x, y.

$f(y)$ = probability density function of V_{t0}.

and,

$$f(y) = f_1(t_0+y) + \int_0^{t_0} M'(t_0-x).f(x+y)dx. \tag{25}$$

where $f_1(t_0+y)$ is said to be the probability of a first failure at t_0+y given no failure before t_0.

Therefore:

$$f_1(t_o+y) = \frac{d}{dy}\left[\frac{F(t_o+y) - F(t_o)}{1 - F(t_o)} \right]$$

$$= \frac{d}{dy}\left[1 - \frac{R(t_o+y)}{R(t_o)} \right]$$

$$= \frac{d}{dy}\left[1 - \exp(-(t_o+y)^B + t_o^B) \right] \tag{26}$$

$$f_1(t_o+y) = B.(t_o+y)^{B-1}.\exp(-(t_0+y)^B+t_o^B) \tag{27}$$

It has already been found in the preceding section that

$$M'(t_o-x) = B.(t_o-x)^{B-1}.\exp(-(t_o-x)^B).\left\{\sum_{k=o}^{\infty} c(k+1).\frac{(t_o-x)^{kB}}{k!}\right\}$$

and for Weibull distribution, we know that

$$f(x+y) = B.(x+y)^{B-1}.\exp(-(x+y)^B) \tag{28}$$

Therefore:

$$f(y) = B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) + \int_0^{t_0} B.(t_o-x)^{B-1}.\exp(-(t_o-x)^B). \left\{\sum_{k=o}^{\infty} c(k+1).\frac{(t_o-x)^{kB}}{k!}\right\} B(x+y)^{B-1}\exp(-(x+y)^B)dx \tag{29}$$

In order to simplify the above function, another form for the summation is found.

Of the terms in the integral, it is considered that

$$A = \exp(-(t_o-x)^B \sum_{k=o}^{\infty} c(k+1).\frac{(t_o-x)kB}{k!} \tag{30}$$

and to simplify the analysis, the quantity (t_o-x) is denoted by Z. The index on the sum is also altered to yield the following equivalent expression:

$$A = \exp(-Z^B).\sum_{s=1}^{\infty} c(s).\frac{(Z^B)^{(S-1)}}{(s-1)!} \tag{31}$$

Next, it has been proved that

$$c(s) = \sum_{i=1}^{s} \binom{s-1}{i-1}.\left\{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\right\} \tag{32}$$

Therefore:

$$A = \exp(-Z^B)\sum_{s=1}^{\infty} \frac{(Z^B)^{S-1}}{(s-1)!} \sum_{i=1}^{s} \binom{s-1}{i-1}. \left\{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\right\}$$

$$A = \exp(-Z^B) \sum_{s=1}^{\infty} \sum_{i=1}^{s} (s-1) \frac{(Z^B)^{S-1}}{(s-1)!} . \{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\}$$

$$A = \sum_{s=1}^{\infty} \sum_{i=1}^{s} \binom{s-1}{i-1} (Z^B)^{S-1} \frac{\exp(-Z^B)}{(s-1)!} . \{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\}$$

and it is noted that in Lomnicki's notation (1966),

$$\exp(-Z^B)(Z^B)^{S-1}/(s-1)! = P_{S-1}(Z^B) \tag{33}$$

and also,

$$\sum_{s=1}^{\infty} \sum_{i=1}^{s} = \sum_{i=1}^{\infty} \sum_{s=i}^{\infty} = \sum_{i=1}^{\infty} \sum_{s-1=i-1}^{\infty} \tag{34}$$

so,

$$A = \sum_{i=1}^{\infty} \{ \sum_{s-1=i-1}^{\infty} \binom{s-1}{i-1} P_{S-1}(Z^B)\}\{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\}$$

$$= \sum_{i=1}^{\infty} \frac{(Z^B)^{i-1}}{(i-1)!} \{\sum_{j=1}^{i} j(-1)^{i+j} b_j(i)/\gamma(i)\}$$

$$A = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \{j(-1)^{i+j} b_j(i)/\gamma(i)(i-1)!\}.Z^{(i-1)B} \tag{35}$$

Substituting A back into the integral will simplify the analysis. Thus,

$$f(y) = B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) + \int_0^{t_0} B.Z^{B-1} \sum_{i=1}^{\infty} \sum_{j=1}^{i} \{j(-1)^{i+j} b_j(i)/\gamma(i)(i-1)!\}. Z^{(i-1)B}.B.(x+y)^{B-1}.\exp(-(x+y)^B).dx$$

$$= B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) + \int_0^{t_o} B.\sum_{i=1}^{\infty}\sum_{j=1}^{i} \{j(-1)^{i+j}b_j(i)/\gamma(i)(i-1)!\}.Z^{iB-1}. B.(x+y)^{B-1}.\exp(-(x+y)^B).dx$$

$$= B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) + \int_0^{t_o} B\sum_{i=1}^{\infty}\sum_{j=1}^{i} \{j(-1)^{i+j}b_j(i)/\gamma(i)(i-1)!\}. (t_o-x)^{iB-1}.B.(x+y)^{B-1}.\exp(-(x+y)^B).dx \tag{36}$$

[Consider, $(x+y)^B = u$

$B.(x+y)^{B-1}.dx = du$

again consider, $\exp(-u) = k$

$\underline{-\exp(-u).du = dk}$

$x = u^{1/B} - y$

or, $x = (-\ln k)^{1/B} - y$

when $x = 0$, $u = y^B$, $k = \exp(-y^B)$

when $x = t_o$, $u = (t_o+y)^B$, $k = \exp(-(t_o+y)^B)$]

Substituting these new variables and limits into the integral equation yields:

$$f(y) = B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) + \int_{y^\beta}^{(t_o+y)^\beta} B \sum_{i=1}^{\infty}\sum_{j=1}^{i} \{j(-1)^{i+j}b_j(i)/\gamma(i)(i-1)!\}. (t_o-x)^{iB-1}.\exp(-u).du \tag{37}$$

or $$f(y) = B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) - \int_{e^{-y^\beta}}^{\exp(-(t_o+y)^\beta} B \sum_{i=1}^{\infty}\sum_{j=1}^{i} \{j(-1)^{i+j}b_j(i)/\gamma(i)(i-1)!\}. \{t_o+y-(-\ln k)^{1/B}\}^{iB-1}.dk$$

Thus, $f(y) = B.(t_o+y)^{B-1}.\exp(-(t_o+y)^B+t_o^B) -$

$$B\sum_{i=1}^{\infty}\sum_{j=1}^{i}\{j(-1)^{i+j}b_j(i)/\gamma(i)(i-1)!\}.\int_{e^{-y\beta}}^{\exp(-(t_o+y)^{\beta})}\{\underline{t_o+y}-(-\ln k)^{1/B}\}^{iB-1}.dk \tag{38}$$

The analytical solution for the above equation is not possible but it can be computed numerically.

Another method to derive the probability density function of the Weibull forward recurrence-time is by using Cinlar's (1975) general equation of the forward recurrence-time probability distribution function. Let V_{to} be the time from t_o until the instant of next renewal in a renewal process S. That is,

$$V_{to}(w) = \begin{cases} S_{n+1}(w) - to & \text{if } S_n(w) <= to < S_{n+1}(w) \\ +\text{Infinity} & \text{otherwise.} \end{cases} \tag{39}$$

Then,

$$\Pr\{V_{to}>z\} = \int_0^{t_o}[1 - F(t_o+z-s)]\underline{dm(s)} \tag{40}$$

$$\text{Let } [1 - F(t_o+z-s)] = H(t_o+z-s), \tag{41}$$

For a Weibull renewal process, let $\alpha = 1$ for simplification. Then,

$$H(t_o+z-s) = \exp(-(t_o+z-s)^B) \tag{42}$$

$$\underline{dm(s)} = B.\exp(-s^B).\sum_{j=0}^{\infty}\frac{c(j+1)}{j!}.s^{jB+B-1}.ds \tag{43}$$

Thus,

$$\Pr\{V_{t_o}>z\} = \int_0^{t_o} H(t_o+z-s)\ \underline{dm(s)}$$

$$= \int_0^{t_o} \exp(-(t_o+z-s)^B) B.\exp(-s^B) \sum_{j=0}^{\infty} \frac{c(j+1)}{j!}.s^{jB+B-1}.ds$$

$$= \sum_{j=0}^{\infty} \frac{c(j+1)}{j!} \int_0^{t_o} B.s^{jB+B-1}.\exp(-(t_o+z-s)^B-s^B).ds \tag{44}$$

This form of the probability also has no analytical solution but can be evaluated numerically.

Therefore, both methods have resulted in numerical solutions only. It is not possible to derive the probability density function of the forward recurrence-time into an analytical form.

The derivation of the Renewal Function for a Weibull renewal process does not help in determining the expected values of backward and forward recurrence-times, and the probability density functions of the recurrence-times into analytical form. Therefore, it appears impractical to attempt to define the parameter estimates for the Weibull life distribution using the renewal function approach when the performance of a device is represented by a set of doubly censored field data. Other strategies, such as the maximum-likelihood method and the moment estimator, are considered to provide better strategies for estimating the parameters of a Weibull life distribution in the case that data is doubly censored.

3.1.2. Maximum-Likelihood Estimation

In a life testing experiment, a number of items are subjected to tests and the data consist of the recorded lives of all or some of the items. Suppose the items are under test with replacements and the failure time distribution is Weibull with parameters α and B. If n items are subject to test and the test is terminated after all the items have failed, the estimation equations defined are for complete information or testing. In the case of censored testing the experiment is terminated either after r failures or after a specified test termination time.

For a sample of size n, let $t_{(1)} <= t_{(2)} <= t_{(3)}$ -------- $<= t_{(r)}$ denote the failure times arranged in increasing order of magnitude. Then, the likelihood function,

$$L = f_{T(1)----T(r)}(t_{(1)}----t_{(r)}) \tag{45}$$

can be expressed as,

$$L = \frac{n!}{(n-r)!} \left[\prod_{i=1}^{r} f(t_i, \hat{\alpha}, \hat{B}) \right] [1-F(t_r)]^{n-r} \tag{46}$$

where $f(t_i, \hat{\alpha}, \hat{B})$ is the probability density function,

$F(t_r)$ is the cumulative distribution function,

For Type I censoring, $t_r=T$, a specified test termination time, and for Type II censoring $t_r=t_{(r)}$, the observed time of the r^{th} failure at which time testing is terminated.

The logarithm of the likelihood function equation is:

$$\ln L = \ln K + \sum_{i=1}^{r} \ln f(t_i, \hat{\alpha}, \hat{B}) + (n-r)\ln R(t_r) \qquad (47)$$

For the Weibull distribution,

$$f(t_i, \hat{\alpha}, \hat{B}) = \alpha B . t_i^{B-1} . \exp(-\alpha t_i^{B}) \qquad (48)$$

$$R(t_r) = \exp(-\alpha t_r^{B}) \qquad (49)$$

$$K = \frac{n!}{(n-r)!} \qquad (50)$$

so,

$$\ln L = \ln K + r\ln\alpha + r\ln B + (B-1)\sum_{i=1}^{r} \ln t_i - \alpha \sum_{i=1}^{r} t_i^{B} - (n-r)\alpha t_r^{B} \qquad (51)$$

and the maximum likelihood estimates are the solutions of

$$\frac{\sum_{i=1}^{r} t_i^{\hat{B}} \ln t_i + (n-r) t_r^{\hat{B}} \ln t_r}{\sum_{i=1}^{r} t_i^{\hat{B}} + (n-r) t_r^{\hat{B}}} - \frac{1}{\hat{\beta}} = \frac{1}{r}\sum_{i=1}^{r} \ln t_i \qquad (52)$$

and,

$$\hat{\alpha} = \frac{r}{\sum_{i=1}^{r} t_i^{\hat{B}} + (n-r) t_r^{\hat{B}}} \qquad \text{(Cohen, 1965)} \qquad (53)$$

The results for the complete sample can be obtained by setting $r=n$. The above equations may be solved by an iterative method.

It is frequently considered worthwhile to verify the attainment of anticipated reliability using field data. This can lead to difficulties in that the need for adequate sample sizes may impose a need to consider equipment for which the full operating history is not known. In this case, the existing estimation methods break down.

The most typical realization of this situation is the one in which a cohort of equipment is observed over a fixed time interval. During that time interval, some failures occur and some of the devices do not experience failures. The result is a data set that displays conventional Type II right censoring. However, it also occurs that the age of the cohort is not known at the start of the interval so the data is left and right or doubly censored.

An important consideration in the use of field data such as that described above is that it is not a set of observations from a distribution but rather represents the operation of a renewal process (Cox, 1962). Consequently, the data must be interpretted carefully and the estimation procedure must reflect the distinction between a distribution and a renewal process. In the case of life distribution parameter estimation, an additional difficulty is the fact that the renewal function for the Weibull distribution has not been constructed in closed form because the moment generating function does not exist (Nachlas and Kumar, 1988).

Assuming that a cohort of devices is observed and that the data obtained represents a stable renewal process corresponding to a constant device design, there are only three types of sample paths that can occur. These are illustrated in Figure 3.4. In the first type of path a device of unknown age is functioning at the start of the observation interval and fails during the interval. It is replaced by another copy of the device which also fails during the interval thus providing a complete life length reading to the data set. The failed device is again replaced and the same experience may occur or else the new device may survive beyond the end of the interval. In this latter case, the contribution to the data set is a right censored life length reading.

The second type of sample path is the same as the first except that the first replacement device survives the interval so that no complete life length readings are obtained. The result is one left censored and one right censored data reading. The third path occurs when the device that is functioning at the start of the observation interval does not fail during the interval so that the data reading obtained is both left and right censored.

It can be observed that in the first two types of sample paths, the left censored data readings are stochastically equivalent. They are denoted here as the sum of x_i, the censored time, and $z_{1,i}$, the observed operating time. A similar equivalence occurs between the right censored time in the first two types of sample path. These are denoted as $z_{3,i}+y_i$ where y_i is the censored time and $z_{3,i}$ is the observed operating time. The

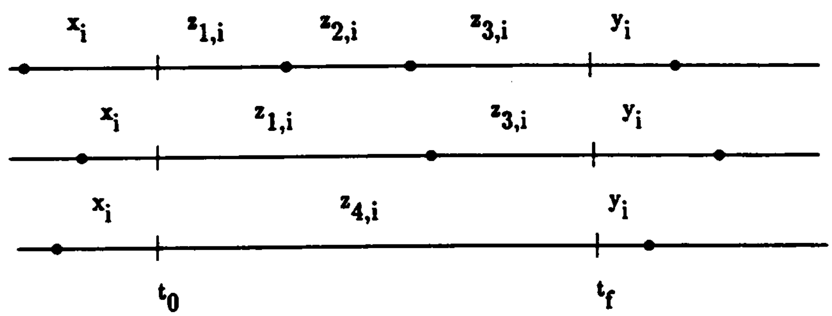

Figure 3.4. Types of Sample Paths.

indexing is defined so that there are "s" sample paths of the first and second types. It is also assumed that there are "r" complete life length readings within the type one sample paths. The complete life lengths are denoted by $z_{2,i}$. The same notational convention leads to the designation of the data readings for the third type of sample path as $x_i+z_{4,i}+y_i$. Of course, the $z_{4,i}$ are all equal to the length of the observation interval. Assume that there are n-s paths of the third type so that the total number of sample paths observed is n.

Maximum Likelihood Approach

Using the above notation, mmaximum likelihood estimation equations can be defined. The equations are defined for complete information and for right censoring as well as for double censoring. This will provide a convenient format for comparing the performance of the estimators based upon doubly censored data.

The general form of the likelihood function in terms of the notation defined is:

$$L=\left[\prod_{i=1}^{s} f(x_i+z_{i,1})\right]\left[\prod_{i=1}^{r} f(z_{2,i})\right]\left[\prod_{i=1}^{s} f(z_{3,i}+y_i)\right]\cdot \left[\prod_{i=n-s+1}^{n} f(x_i+z_{4,i}+y_i)\right] \tag{54}$$

The focus of this research is the two-parameter Weibull distribution. A general form for the Weibull distribution is:

$$F(t) = 1 - \exp(-\alpha t^{B}) \tag{55}$$

and the realization of equation (54) for this function is:

$$L(\alpha,B) = \left[\prod_{i=1}^{s} \alpha B(x_i+z_{1,i})^{B-1}\exp(-\alpha(x_i+z_{1,i})^{B})\right] \cdot \left[\prod_{i=1}^{r} \alpha B\, z_{2,i}^{B-1}\exp(-\alpha z_{2,i}^{B})\right] \cdot \left[\prod_{i=1}^{s} \alpha B(z_{3,i}+y_i)^{B-1}\exp(-\alpha(z_{3,i}+y_i)^{B})\right] \cdot \left[\prod_{i=n-s+1}^{n} \alpha B(x_i+z_{4,i}+y_i)^{B-1}\exp(-\alpha(x_i+z_{4,i}+y_i)^{B})\right] \tag{56}$$

Taking the logarithm of the likelihood function and constructing the partial derivatives with respect to α and B yields the familiar estimation equations:

$$\alpha = (n+r+s)/\left[\sum_{i=1}^{r} z_{2,i}^{B} + \sum_{i=1}^{s} (x_i+z_{1,i})^{B} + \sum_{i=1}^{s} (z_{3,i}+y_i)^{B} + \sum_{i=n-s+1}^{n} (x_i+z_{4,i}+y_i)^{B}\right] \tag{57}$$

and:

$$\left[\sum_{i=1}^{r} z_{2,i}^{B}\ln z_{2,i} + \sum_{i=1}^{s}(x_i+z_{1,i})^{B}\ln(x_i+z_{1,i}) + \sum_{i=1}^{s} (z_{3,i}+y_i)^{B}\ln(z_{3,i}+y_i) + \sum_{i=n-s+1}^{n} (x_i+z_{4,i}+y_i)^{B}\ln(x_i+z_{4,i}+y_i)\right]/$$

$$[\sum_{i=1}^{r} z_{2,i}^B + \sum_{i=1}^{s} (x_i+z_{1,i})^B + \sum_{i=1}^{s} (z_{3,i}+y_i)^B +$$

$$\sum_{i=n-s+1}^{n} (x_i+z_{4,i}+y_i)^B] - (1/B) = [\sum_{i=1}^{r} \ln z_{2,i} +$$

$$\sum_{i=1}^{s} \ln(x_i+z_{1,i}) + \sum_{i=1}^{s} \ln(z_{3,i}+y_i) + \sum_{i=n-s+1}^{n} \ln(x_i+z_{4,i}+y_i)]/$$

$$[n+r+s]. \qquad (58)$$

The calculation of the parameter estimates involves searching for the value of B that satisfies equation (58) and then using equation (57) to compute α.

The modifications to the complete data equations for right censoring are well known. For the notation defined here, the likelihood function is:

$$L = [\prod_{i=1}^{s} f(x_i+z_{1,i})] \cdot [\prod_{i=1}^{r} f(z_{2,i}] \cdot [\prod_{i=1}^{s} R(z_{3,i})] \cdot$$

$$[\prod_{i=n-s+1}^{n} R(x_i+z_{4,i})] \qquad (59)$$

which leads to the revised estimation equations:

$$\alpha = [r+s]/[\sum_{i=1}^{r} z_{2,i}^B + \sum_{i=1}^{s} (x_i+z_{1,i})^B + \sum_{i=1}^{s} z_{3,i}^B +$$

$$\sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B] \qquad (60)$$

and:

$$[\sum_{i=1}^{r} z_{2,i}{}^{B}\ln z_{2,i} + \sum_{i=1}^{s} (x_i+z_{1,i})^B\ln(x_i+z_{1,i}) + \sum_{i=1}^{s} z_{3,i}{}^{B}\ln z_{3,i} + \sum_{i=n-s+1}^{n}(x_i+z_{4,i})^B\ln(x_i+z_{4,i})]/[\sum_{i=1}^{r} z_{2,i}{}^{B} + \sum_{i=1}^{s} (x_i+z_{1,i})^B + \sum_{i=1}^{s} z_{3,i}{}^{B} + \sum_{i=n-s+1}^{n}(x_i+z_{4,i})^B] - (1/B)$$

$$= [\sum_{i=1}^{r} \ln z_{2,i} + \sum_{i=1}^{s} \ln(x_i+z_{1,i}) + \sum_{i=1}^{s} \ln z_{3,i} + \sum_{i=n-s+1}^{n} \ln(x_i+z_{4,i})]/[r+s]. \qquad (61)$$

Equations (60) and (61) clearly contain less information than is included in equations (57) and (58). As a result it is generally expected that the estimates derived from right censored data will be less accurate than those from complete data. This is examined in Chapter IV.

The corresponding equations for the case of doubly censored data sets should be expected to contain correspondingly less information than do the equations for complete or singly censored data. When the values of x_i are known, the general likelihood function can be defined as:

$$L = [\prod_{i=n-s+1}^{n} R(x_i+z_{4,i}|x_i)]\cdot[\prod_{i=1}^{s} R(z_{3,i})]\cdot[\prod_{i=1}^{s} f(x_i+z_{1,i}|x_i]\cdot[\prod_{i=1}^{r} f(z_{2,i})] \qquad (62)$$

Now, for Weibull distribution:

$$F(x_i+z_{4,i}|x_i) = \frac{F(x_i+z_{4,i}) - F(x_i)}{1 - F(x_i)}$$

$$= \frac{R(x_i) - R(x_i+z_{4,i})}{R(x_i)}$$

$$\text{or, } F(x_i+z_{4,i}|x_i) = 1 - R(x_i+z_{4,i})/R(x_i)$$

$$= 1 - \exp(-\alpha(x_i+z_{4,i})^B+\alpha x_i^B) \qquad (63)$$

$$\text{and, } f(x_i+z_{4,i}|x_i) = \alpha B(x_i+z_{4i})^{B-1}\exp(-\alpha(x_i+z_{4,i})^B+\alpha x_i^B) \qquad (64)$$

$$R(x_i+z_{4,i}|x_i) = \exp(-\alpha(x_i+z_{4,i})^B+\alpha x_i^B) \qquad (65)$$

Substituting the values of the R and f functions corresponding to a Weibull distribution into likelihood function yields:

$$L = \{ \prod_{i=n-s+1}^{n} \exp(-\alpha(x_i+z_{4,i})^B+\alpha x_i^B) \} .$$
$$\{ \prod_{i=1}^{s} \exp(-\alpha(z_{3,i})^B) \} .$$
$$\{ \prod_{i=1}^{s} B(x_i+z_{1,i})^{B-1}\exp(-\alpha(x_i+z_{1i})^B+\alpha x_i^B) \} .$$
$$\{ \prod_{i=1}^{r} \alpha Bz_{2,i}^{B-1}\exp(-\alpha z_{2,i}^B\} \qquad (66)$$

Taking the logarithm on both sides yields

$$\begin{aligned}\ln L = & -\alpha \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B + \alpha \sum_{i=n-s+1}^{n} x_i^B - \alpha \sum_{i=1}^{s} (z_{3,i})^B \\ & + s\ln\alpha + s\ln\beta \\ & + (B-1) \sum_{i=1}^{s} \ln(x_i+z_{1,i}) - \alpha \sum_{i=1}^{s} (x_i+z_{1,i})^B \\ & + \alpha \sum_{i=1}^{s} x_i^B + r\ln\alpha + r\ln B \\ & + (B-1) \sum_{i=1}^{r} \ln(z_{2,i}) - \alpha \sum_{i=1}^{r} z_{2,i}^B \end{aligned} \tag{67}$$

or,

$$\begin{aligned}\ln L = & \alpha \sum_{i=1}^{n} x_i^B + (s+r)\ln\alpha + (s+r)\ln B \\ & -\alpha \sum_{i=1}^{s} z_{3,i}^B - \alpha \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B \\ & - \alpha \sum_{i=1}^{s} (x_i+z_{1,i})^B + (B-1) \sum_{i=1}^{s} \ln(x_i+z_{1,i}) \\ & + (B-1) \sum_{i=1}^{r} \ln z_{2,i} - \alpha \sum_{i=1}^{r} z_{2,i}^B \end{aligned} \tag{68}$$

Now, the above equation is differentiated with respect to α, and then with respect to B, and equated to zero for maximization. Thus,

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} x_i^B + (s+r)/\alpha - \sum_{i=1}^{s} z_{3,i}^B - \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B - \sum_{i=1}^{s} (x_i+z_{1,i})^B - \sum_{i=1}^{r} z_{2i}^B = 0 \qquad (69)$$

$$\hat{\alpha} = (s+r)/[\sum_{i=1}^{s} z_{3,i}^B + \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B + \sum_{i=1}^{s} (x_i+z_{1,i})^B + \sum_{i=1}^{r} z_{2,i}^B - \sum_{i=1}^{n} x_i^B] \qquad (70)$$

and,

$$\frac{\partial \ln L}{\partial \beta} = \alpha \sum_{i=1}^{n} x_i^B \ln x_i + (s+r)/B - \alpha \sum_{i=1}^{s} z_{3,i}^B \ln z_{3,i} - \alpha \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^B \ln (x_i+z_{4,i}) - \alpha \sum_{i=1}^{s} (x_i+z_{1,i})^B \ln (x_i+z_{1,i}) + \sum_{i=1}^{s} \ln(x_i+z_{1,i}) + \sum_{i=1}^{r} \ln z_{2,i} - \alpha \sum_{i=1}^{r} z_{2,i}^B \ln z_{2,i} \qquad (71)$$

or,

$$0 = 1/B + \{1/(s+r)\}\cdot[\sum_{i=1}^{s} \ln(x_i+z_{1,i}) + \sum_{i=1}^{r} \ln z_{2,i}]$$

$$- \{ \sum_{i=1}^{s} z_{3,i}{}^{B} \ln z_{3,i}$$

$$+ \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^{B}\ln(x_i+z_{4,i})$$

$$+ \sum_{i=1}^{s} (x_i+z_{1,i})^{B}\ln(x_i+z_{1,i})$$

$$+ \sum_{i=1}^{r} z_{2,i}{}^{B}\ln z_{2,i} - \sum_{i=1}^{n} x_i{}^{B}\ln x_i\}/$$

$$\{ \sum_{i=1}^{s} z_{3,i}{}^{B} + \sum_{i=n-s+1}^{n} (x_i+z_{4,i})^{B}$$

$$+ \sum_{i=1}^{s} (x_i+z_{1,i})^{B} + \sum_{i=1}^{r} z_{2,i}{}^{B} - \sum_{i=1}^{n} x_i{}^{B}\} \qquad (72)$$

Equation (72) is the realization of equation (61) under the notation defined above. The above equations are incorporated into a computer program and the values of α and B are determined by an iterative procedure.

Since the left censoring time x_i is selected randomly for the field data, the individual values of x_i are unknown. This makes our problem more difficult. When the values of the x_i are not known, the density function terms in the maximum likelihood equations must be modified. There are several promising strategies for doing this. The approach taken here is described as follows:

1. Let $\hat{L}$ be likelihood function which represents an inaccurate realization of L.

2. The likelihood function of equation (59) is assumed to apply but to be inaccurate.
3. An arbitrary set of values is used in place of the various x_i and these are taken to be the source of the error in $\hat{L}$.
4. In the partial derivative equations for $\hat{L}$, a correction term is applied.
5. The adjusted equations are used to compute the parameter estimates.

To simplify the algebra, let:

$$T_i = x_i + z_{1,i} \qquad i=1,s$$
$$= x_i + z_{4,i} \qquad i=s+1,n$$

and let $\hat{T}_i$ represent the value of T_i that is used in the likelihood equation. That is, assume that T_i is the correct value that would be used if the x_i were known and let the $\hat{T}_i$ be the values that are actually included in the computations. The result is that the likelihood function that is computed, $\hat{L}$, is also an inaccurate realization of L. Further, assume that the correct and inaccurate values of the failure times are related by:

$$\hat{T}_i = \delta_i T_i \tag{73}$$

Then the likelihood function can be stated as:

$$\hat{L} = [\prod_{i=1}^{s} f(\hat{T}_i)][\prod_{i=1}^{r} f(z_{2,i})][\prod_{i=1}^{s} R(z_{3,i})][\prod_{i=s+1}^{n} R(\hat{T}_i)] \tag{74}$$

and substituting the relation in equation (73) yields:

$$\ln \hat{L} = \ln L + (B-1) \sum_{i=1}^{s} \ln \delta_i - \alpha \sum_{i=1}^{n} (\delta_i^B - 1) T_i^B. \tag{75}$$

Next, taking partial derivatives of the logarithm of the likelihood function yields:

$$\partial \ln \hat{L}/\partial\alpha = \partial \ln L/\partial\alpha - \sum_{i=1}^{n} (\delta_i^B - 1) T_i^B \tag{76}$$

and:

$$\partial \ln \hat{L}/\partial\beta = \partial \ln L/\partial\beta + \sum_{i=1}^{s} \ln \delta_i - \alpha \sum_{i=1}^{n} (\delta_i^B - 1) T_i^B \ln T_i - \alpha \sum_{i=1}^{n} \delta_i^B T_i^B \ln \delta_i \tag{77}$$

Denote the non-differential terms in equations (76) and (77) as K_α, and K_B, respectively. Then, the revised estimation equations are:

$$\partial \ln L/\partial\alpha = \partial \ln \hat{L}/\partial\alpha - K_\alpha \tag{78}$$

$$\partial \ln L/\partial B = \partial \ln \hat{L}/\partial B - K_B. \tag{79}$$

The simultaneous solution of these equations yields the estimation equations:

$$\alpha = (r+s)/[\sum_{i=1}^{r} z_{2,i}^B + \sum_{i=1}^{n} T_i^B + \sum_{i=1}^{s} z_{3,i}^B - K_\alpha] \tag{80}$$

and:

$$[\sum_{i=1}^{r} z_{2,i}^B \ln z_{2,i} + \sum_{i=1}^{n} T_i^B \ln T_i + \sum_{i=1}^{s} z_{3,i}^B \ln z_{3,i} - (K_B - \sum_{i=1}^{s} \ln \delta_i)/\alpha] / [\sum_{i=1}^{r} z_{2,i}^B + \sum_{i=1}^{s} z_{3,i}^B + \sum_{i=1}^{n} T_i^B - K_\alpha]$$

$$-(1/B) = [\sum_{i=1} \ln z_{2,i} + \sum_{i=1} \ln T_i + \sum_{i=1} \ln \delta_i]/(r+s). \quad (81)$$

The remaining issue is how the arbitrary values of the x_i should be selected. An associated question is how the δ_i should be determined. There are actually very many strategies that could be used for choosing these quantities. The approach taken here is to select values of x_i at random from a uniform distribution over a multiple of the observed complete life lengths, $z_{2,i}$. The largest of the $z_{2,i}$ is assumed to correspond to $F(t) = (r-.3)/(n+r+.4)$ so the range of the randomly selected x_i is taken to be $z_{2,r}(n+.4)/(r-.3)$. Then it is assumed that this imposes an error of δ_i where:

$$\delta_i = 1 + \varepsilon_i \quad (82)$$

where ε_i is a normally distributed random quantity with a mean of zero and a standard deviation of .3. The resulting ε_i are used to construct the δ_i and the δ_i are used to set the values of the $\hat{T}_i$ in equations (80) and (81). The parameter estimates can then be computed.

The defined approach to estimating the parameters of a Weibull life distribution in the case that data is doubly censored is highly heuristic and is not claimed to possess any desirable statistical properties. Instead, a simulation experiment is used to explore the apparant performance of the method. The equations that represent the estimates in terms of errors is considered a worthwhile approach to the estimation problem. It is likely that more effective strategies for selecting the x_i and δ_i can be defined. In addition, if any method can be shown to display a consistent bias, an additional set of correction factors can be created to compensate for this. The above estimation methods are implemented using simulated data as are a set of estimators based on the method of moments that are constructed next.

3.1.3. Moment Estimator

The method of moments provides a promising alternative to the likelihood approach. It is generally easier to use but has only been applied previously to complete data sets. Since the Weibull life distribution has two parameters that are to be estimated, two moment equations are required to construct the estimates. The obvious approach in this case is to use the mean and variance of the distribution in concert with the corresponding sample measures to obtain the parameter estimates.

It happens that it is actually more convenient to use the mean and coefficient of variation. The coefficient of variation is the ratio of the standard deviation to the mean. For the Weibull distribution, the mean and variance are given by:

$$\mu = \alpha^{-1/B}\Gamma(1+1/B)$$

and, $$\sigma^2 = \alpha^{-2/B}\Gamma(1+2/B) - \mu^2$$

Consequently, the coefficient of variation is:

$$\Phi = [\{\Gamma(1+2/B)/\Gamma^2(1+1/B)\} - 1]^{1/2} \tag{83}$$

When complete data sets are available, the use of the method of moments involves the direct application of equation (83). The mean and standard deviation of the observed life lengths are computed. For the above defined notation, these are:

$$\bar{t} = [\sum_{i=1}^{s}(x_i+z_{1,i}) + \sum_{i=1}^{r} z_{2,i} + \sum_{i=1}^{s}(z_{3,i}+y_i) + \sum_{i=n-s+1}^{n}(x_i+z_{4,i}+y_i)]/(n+r+s) \tag{84}$$

and:

$$s = [\{\sum_{i=1}^{s}(x_i+z_{1,i}-\bar{t})^2 + \sum_{i=1}^{r}(z_{2,i}-\bar{t})^2 + \sum_{i=1}^{s}(z_{3,i}+y_i-\bar{t})^2 + \sum_{i=n-s+1}^{n}(x_i+z_{4,i}+y_i-\bar{t})^2\}/\{n+r+s-1\}]^{1/2} \tag{85}$$

so:

$$\hat{\Phi} = s/\bar{t} \tag{86}$$

and the parameter estimate for B is the solution to:

$$\hat{\Phi} - [\{\Gamma(1+2/B)/\Gamma(1+1/B)\} - 1]^{1/2} = 0 \tag{87}$$

and the estimate for α is:

$$\alpha = \{\Gamma(1+1/B)/\bar{t}\}^{B} \tag{88}$$

A pertinent point here is the fact that equation (87) can be solved efficiently using a well known approximation for the gamma functions (Abramowitz and Stegun, 1965).

A significant issue in the use of the method of moments is the fact that the sample statistics for censored data sets are not obvious. Noting the fact that y_i is a forward recurrence time, the approach taken here is to replace the y_i in equation (84) by the expected value of the forward recurrence time. Even for the Weibull distribution, this quantity can be determined as Cox (1962) proves that in general it is equal to:

$$E[y_i] = \{(\mu^2+\sigma^2)/\mu\}/2 \qquad (89)$$

In the case of the Weibull, this becomes:

$$E[y_i] = \{\Gamma(1+2/B)/2\alpha^{1/B}\Gamma(1+1/B)\} \qquad (90)$$

By replacing y_i with $E[y_i]$ in equation (84), the estimates can be constructed for the right censored case. However, equation (87) will include both parameters.

To resolve this problem, note that the defined realizations of equations (84) and (85) are:

$$\bar{t} = \{\sum_{i=1}^{k} T_i + n\,E[y_i]\}/k \qquad (91)$$

and:

$$s = [\{\sum_{i=1}^{k} T_i^2 - (\sum_{i=1}^{k} T_i)^2/k + 2E[y_i]\{\sum_{i=s+r+1}^{k} T_i - (n/k)\sum_{i=1}^{k} T_i\} + E^2[y_i](1-(n/k))\}/(k-1)]^{1/2} \qquad (92)$$

so that:

$$\hat{\Phi} = [k^2\{\sum_{i=1}^{k} T_i^2 + 2E[y_i]\sum_{i=s+r+1}^{k} T_i + n\,E^2[y_i]\}/(k-1).$$

$$\{(\sum_{i=1}^{k} T_i)^2 + 2nE[y_i]\sum_{i=1}^{k} T_i + n^2E^2[y_i]\} - \{k/(k-1)\}]^{1/2} \qquad (93)$$

In the above expressions, the notation has been simplified by using:

$$T_i = \begin{cases} x_i + z_{1,i}, & i=1,s \\ z_{2,i} & i=s+1,s+r \\ z_{3,i} & i=s+r+1,2s+r \\ x_i + z_{4,i} & i=2s+r+1,k \end{cases}$$

where k=n+r+s. By substituting equation (90) into equation (91) and solving for $E[y_i]$:

$$E[y_i] = [\Gamma(1+2/B)\sum_{i=1}^{k} T_i]/[2k\Gamma^2(1+1/B) - n\,\Gamma(1+2/B)] \qquad (94)$$

and using this expression in equation (93) permits the computation of an estimate for B. Then the estimate for α is defined by equation (88).

For the case of doubly censored data, the same approach yields very similar expressions that can be used to obtain parameter estimates. In this analysis, the expected value of the backward recurrence time, $E[x_i]$, is used in place of the x_i as was done with the y_i for the right censored case. Note that Cox (1962) shows that $E[x_i]$ is equal to $E[y_i]$. By redefining the T_i as:

$$T_i = \begin{cases} z_{1,i}, & i=1,s \\ z_{2,i} & i=s+1,s+r \\ z_{3,i} & i=s+r+1,2s+r \\ z_{4,i} & i=2s+r+1,k \end{cases}$$

the equations for $\bar{t}$ and $\hat{\Phi}$ are:

$$\bar{t} = \{ \sum_{i=1}^{k} T_i + 2n\, E[y_i]\}/k \tag{95}$$

$$\hat{\Phi} = \{k^2[\sum_{i=1}^{k} T_i^2 + 2E[y_i](\sum_{i=1}^{s} T_i + \sum_{i=s+r+1}^{2s+r} T_i + 2 \sum_{i=2s+r+1}^{k} T_i) +3n\ E^2[y_i]]/[(\sum_{i=1}^{k} T_i)^2 + 4nE[y_i] \sum_{i=1}^{k} T_i + 4n^2E^2[y_i]] - [k/(k-1)]\}^{1/2} \tag{96}$$

and the associated expression for $E[y_i]$ is:

$$E[y_i] = \{\Gamma(1+2/B) \sum_{i=1}^{k} T_i\}/\{2k\Gamma^2(1+1/B) - 2n\Gamma(1+2/B)\}. \tag{97}$$

Once again, the estimate for B is the solution to equation (87) with $\hat{\Phi}$ defined by equation (96) and the corresponding estimate for α is given by equation (88).

The above equations define strategies for obtaining parameter estimates for the Weibull life distribution when the performance of a device is represented by a set of doubly censored field data. In addition, an approach to using the method of moments for right censored data is constructed. Taken together, the six sets of equations provide the ingredients for a comparison of the methods and for examining the effect of field data censoring on the accuracy of parameter estimates.

4. NUMERICAL EVALUATION

There are six sets of estimation equations defined in section 3.2 and they are listed as follows:

Method of Moments

-- complete data

-- right censoring

-- double censoring

Maximum-Likelihood

-- complete data

-- right censoring

-- double censoring

These methods are applied to several sets of simulated failure data. For each data set, all six estimates are computed. The simulation trials are undertaken for two values of n, the number of observed sample paths, and for several sets of assumed underlying parameter values. The values of n are 40, and 75 while the assumed values of B are 0.75, 1.00, 1.25, 1.66, and 2.00. In each case two different values of the mean for the life distribution of 16000, and 20000 hours are assumed. Then for each assumed value of B, the corresponding value of α is computed and used to drive the random problem generator. For each combination of assumed parameter values and number of sample paths, 20 simulated data sets are generated. In every case the observation window is assumed to be from 10000 to 14000 hours. The results obtained are summmarized in Table 4.1 through Table 4.10.

The entries of the table are the arithmetic averages of the estimates for B and α obtained over the randomly created data sets. The first set of entries corresponds to the case in which n=75 sample paths. The second set corresponds to n=40 sample paths.

All of the estimation methods perform rather poorly. In general, increasing the number of sample paths provides only small improvements, if any. More interesting is the fact that the maximum likelihood estimates based upon doubly censored data sets are not worse than those obtained using either complete or right censored data sets. This is unexpected. The same is not true for the method of moments estimates which degrade as the data sets are increasingly censored. In view of the fact that maximum likelihood estimates are generally more accurate than moment-based estimates, the results suggest that further study is warranted.

As the value of mean for the life distribution is increased from 16,000 to 20,000 hours, the estimates deteriorate for censored data sets. This indicates that the observation interval or window location does have an effect on the estimates. If the observation interval is located very close to the expected life length of the device, the life distribution parameters can be estimated

Table 4.1.

T_m = 20,000 hours

B = 0.75

Approach	B N = 75	B N = 40
method of moments		
complete data	0.9841422	1.0187280
right censoring	2.9447682	2.5058436
double censoring	2.8491797	2.7413173
maximum likelihood		
complete data	1.0849811	1.0898476
right censoring	1.2668771	1.3978946
double censoring	1.0184412	1.4907438

Table 4.2.

T_m = 20,000 hours

B = 1.00

Approach	B N = 75	B N = 40
method of moments		
complete data	1.3396172	1.3078545
right censoring	3.6050939	3.3979018
double censoring	3.7547409	3.4797595
maximum likelihood		
complete data	1.4389002	1.5013184
right censoring	2.1354561	2.5204850
double censoring	1.5454468	1.4293840

Table 4.3.

T_m = 20,000 hours

B = 1.25

Approach	B N = 75	B N = 40
method of moments		
complete data	1.6577720	1.6265473
right censoring	3.9029604	3.6829104
double censoring	3.9614258	3.6666583
maximum likelihood		
complete data	1.7670275	1.7512907
right censoring	2.8538437	3.2035644
double censoring	1.5366547	1.2050730

Table 4.4.

T_m = 20,000 hours

B = 1.66

Approach	B N = 75	B N = 40
method of moments		
complete data	2.1451143	2.1256827
right censoring	4.3239503	4.1097658
double censoring	4.1699341	3.9097340
maximum likelihood		
complete data	2.2405700	2.3000575
right censoring	4.0918852	4.4801991
double censoring	1.3681631	1.4611520

Table 4.5.

T_m = 20,000 hours

B = 2.00

Approach	B N = 75	B N = 40
method of moments		
complete data	2.5026248	2.4385590
right censoring	4.5055909	4.1497268
double censoring	4.2833000	3.8995160
maximum likelihood		
complete data	2.6084869	2.6595920
right censoring	4.1842193	4.2767610
double censoring	1.3775525	1.3989040

Table 4.6.

T_m = 16,000 hours

B = 0.75

Approach	B N = 75	B N = 40
method of moments		
complete data	1.0445932	1.0863322
right censoring	2.6256793	2.5821553
double censoring	2.5804755	2.5281048
maximum likelihood		
complete data	1.1133017	1.1461618
right censoring	1.2016194	1.2392211
double censoring	1.0898981	1.5671184

Table 4.7.

T_m = 16,000 hours

B = 1.00

Approach	B N = 75	B N = 40
method of moments		
complete data	1.4015658	1.3580447
right censoring	3.2386755	3.0711876
double censoring	3.3367824	3.0428660
maximum likelihood		
complete data	1.4922831	1.4588583
right censoring	2.1933797	2.1864664
double censoring	1.5027585	1.4288399

Table 4.8.

T_m = 16,000 hours

B = 1.25

Approach	B N = 75	B N = 40
method of moments		
complete data	1.6718552	1.6445022
right censoring	3.5480042	3.2451804
double censoring	3.4832940	3.3538060
maximum likelihood		
complete data	1.7528604	1.7481540
right censoring	2.8132585	2.5469394
double censoring	1.3278389	1.5886655

Table 4.9.

T_m = 16,000 hours

B = 1.66

Approach	B N = 75	B N = 40
method of moments		
complete data	2.1837130	2.1183033
right censoring	3.9531008	3.7265042
double censoring	3.7570803	3.4794727
maximum likelihood		
complete data	2.2673905	2.2577185
right censoring	3.9400453	3.8859414
double censoring	1.2728726	1.5107028

Table 4.10.

T_m = 16,000 hours

B = 2.00

Approach	B N = 75	B N = 40
method of moments		
complete data	2.4350010	2.4140940
right censoring	3.9361593	3.8668220
double censoring	3.5882475	3.6335270
maximum likelihood		
complete data	2.5192751	2.5935200
right censoring	4.0928522	4.0347820
double censoring	1.3828031	1.2309750

more accurately using the doubly censored field data.

The results were first obtained using a personal computer with its internal random number generator routine (Linear Congruential Generator, IBM). Then the observed results were confirmed using a mainframe based random number generator (Linear Congruential Generator, IMSL). There are no significant variations in the results. The accuracy of the estimates do not improve or deteriorate when a mainframe based random number generator is used.

5. CONCLUSIONS

A reliability model and computer program have been developed to facilitate life distribution parameter and reliability estimation from doubly censored field data. The available data set is obtained from field performance and is consequently censored on both the left and the right.

Estimation equations have been defined for both maximum likelihood and moment based estimates. Each method obtains estimates using complete, right censored, and doubly censored data sets.

All of the estimation methods performed rather poorly. In general, increasing the number of sample paths provided only small improvements if any. More interesting is the fact that the maximum likelihood estimates based upon doubly censored data sets were not worse than those obtained using either complete or right censored data sets. The same is not true for the method of moments estimates which deteriorate as the data sets are increasingly censored.

The estimates deteriorate for censored data sets with increasing value of the mean for life distribution and a fixed observation interval. It is also concluded that an observation interval which extends very close to the expected life length of the device will improve the accuracy of parameter estimates.

The maximum likelihood estimates are generally more accurate than moment based estimates. The width of the observation interval has virtually no influence on the estimates. The results obtained do not vary significantly.

The results were obtained using a personal computer with its internal random number generation routine and also using a mainframe based random number generator. Both the random number generators produced similar values of estimates. The results did not vary significantly.

The development of moment based estimation equations for censored data is new. Also, the strategy suggested for treating double censoring in maximum likelihood estimation has not been defined previously. The problem addressed is one of considerable interest. The methods suggested here for treating the problem lead to an effective approach to obtaining accurate estimates using highly censored field data.

Recommendations for Future Research:

The selection rules used to set the unknown backward recurrence times may not have been the best ones available. Other rules should be tested in future trials.

The methods suggested here for treating the problem should be examined further. They may prove to be better than they appear or they may lead to an effective approach to obtaining accurate estimates using highly censored field data. In view of the fact that maximum likelihood estimates are generally more accurate than moment based estimates, the results suggest that further study is warranted.

REFERENCES

Abramowitz, M., and Stegun, I. A., Handbook of Mathematical Functions, New York, Dover, 1965.

Breslow, N., and Crowley, J., "A Large Sample Study of the Life Table and Product Limit Estimates Under Random Censorship," Ann. of Statistics, Volume 2, pp. 437-453, 1974.

Chan, L. K., Cheng, S. W., and Mead, E. R., "Simultaneous Estimation of Location and Scale Parameters of the Weibull Distribution," IEEE Transactions on Reliability, Volume R-23, pp. 335-341, 1974.

Cinlar, Erhen, Introduction to Stochastic Processes, Prentice Hall, Inc., Englewood, New Jersey, 1975.

Cohen, A. C., "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples," Technometrics, Volume 7, Number 4, pp. 579-588 November 1965.

Cox, C. R., Renewal Theory, New York, John Wiley & Sons, 1962.

Johns, M. V., and Lieberman, G. J., "An Exact Asymptotically Efficient Confidence bound for Reliability in the Case of Weibull Distribution," Technometrics, Volume 8, Number 1, pp. 135-171, February 1966.

Kao, J. H. K., "A Graphical Estimation of Mixed Weibull Parameters in Life Testing Electron Tubes," Technometrics, Volume 1, pp. 389-407, 1959.

Kapur, K. C., and Lamberson, L. R., Reliability in Engineering Design, New York, John Wiley & Sons, 1977.

Kulldorff, G., "A Note on the Optimum Spacing of Sample Quantiles From the Six Extreme Value Distributions," Ann. of Statistics, Volume 1, Number 3, pp. 562-567, 1973.

Lehman, E. H., "Shapes, Moments, and Estimators of the Weibull Distribution," IEEE Transactions on Reliability, September 1963.

Lemon, G. H., "Maximum Likelihood Estimation for the Three-Parameter Weibull Distribution Based on Censored

Samples," Technometrics, 1974.

Lieblen, J., and Zelen, M., "Statistical Investigation of the Fatigue Life of Deep Groove Ball Bearings", Journal of Res. Natl. Bureau of Standard 57, pp. 273-315, 1956.

Lindley, D. V., "Approximate Bayesian Methods", Trabajos Estadistica 31, pp. 223-237, 1980.

Lomnicki, Z. A., "A Note on the Weibull Renewal Process", Biometrika 53, 3 and 4, pp. 375, 1966.

Mann, N. R., "Tables for Obtaining Linear Invariant Estimate of Parameters of the Weibull Distribution", Technometrics, vol. 9, no. 4, pp. 629-645, Nov. 1967.

Mann, N.R., "Results on Statistical Estimation and Hypothesis Testing with Application to the Weibull and Extreme Value Distribution", Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio", 1968.

Mann, N. R., "Point and Interval Estimation Procedures for the Two-Parameter Weibull and Extreme-Value Distribution", Technometrics, Vol. 10, No. 2, pp. 231-255, May 1968.

Mann, N. R., "Warranty Periods Based on Three Ordered Sample Observations from a Weibull Population", IEEE Transactions on Reliability, Vol. R - 19, pp. 167-171, 1970.

Mann, N. R., "Estimation and Exact Confidence Bounds for Weibull Parameters Based on a Few Ordered Observations", Technometrics 12, pp. 345-361, 1970.

Mann, N. R., Schafer, R.E., and Sinpurwalla, N. D., Methods for Statistical Analysis of Reliability and Life Data, John Wiley & Sons, 1974.

Nachlas, Joel A., and Kumar, A., "Reliability Estimation Using Doubly Censored Field Data," Proceedings of the SRE Symposium on Reliability, Sweden, September 1988.

Pandey, M., "Bayes Estimator of Survival Probability from Randomly Sensored Observations with With Weibull Distribution of Time to Death", Biometric Journal, Vol. 29, pp. 491-496, 1987.

Phadia, E. and Van Ryzin, J., "A Note on Convergence Rates for the Product Limit Estimators", Ann. of Statistics, Vol. 8, pp. 673-678, 1980.

Rai, K., Susarla, V. and Van Ryzin, J., "Shrinkage Estimation in Nonparametric Bayesian Survival Analysis: A Simulation Study", Comm. Statistics - Simulation Comput. B9(3), pp.271-298, 1980.

Sinha, S. K. and Fu, J. C., "Least Square Estimators of Weibull Parameters", Proceedings of the 41st Session of the International Statistical Institute, pp. 535-537, 1977.

Sinha, S. K., and Kale, B. K., "Life Testing and Reliability Estimation", Wiley Eastern Limited, New Delhi, India, 1980.

Sinha, S. K., "Bayes Estimation of the Reliability Function and Hazard Rate of a Weibull Failure time Distribution", Revista, Trabojos Estadistica, 1985.

Sinha, S. K., "Reliability and Life Testing", John Wiley & Sons, 1986.

Smith, W. L. and Leadbetter, M. R., "Notes on the Renewal Function for the Weibull Distribution", Technometrics, Vol. 5, pp. 393-396, August 1963.

Susarla, V. and Van Ryzin, J., "Large Sample Theory for a Bayesian Nonparametric Survival Curve Estimator Based on Sensored Samples, Ann. of Statistics, Vol. 6, pp. 755-768, 1978.

Susala, V., and Van Ryzin, J. "Large Sample Theory for Survival Estimators Under Variable Sensoring", Optimizing Methods in Statistics, Vol. II, Academic Press, 1980.

Thoman, D. R., Bain, L. J. and Antle, C. E., "Inferences on the Parameters of the Weibull Distribution", Technometrics, Vol. 11, No. 3, pp. 445-460, 1969.

Thoman, D. R., Bain, L. J., and Antle, C. E., "Maximation Likelihood Estimation, Exact Confidence Intervals for Reliability, and Tolerance Limits in the Weibull Distribution", Technometrics, Vol. 12, No. 2, pp. 363-371, 1970.

Weibull, W., "A Statistical Distribution Function of Wide Applicability", Journal of Applied Mechanics 18, pp. 293-297, 1951.

White, J. S., "The Moments of Log-Weibull Order

Statistics", Technometrics, Vol. 11, pp. 373-386, 1969.

APPENDIX A.

Listing of Computer Program (Software).

```
10   DIM BES(7,25),ALES(7,25),ACUM(7)
20   DIM X(2,200),Z(2,200),WT(200),WRK(200),WY(200),WZ(200)
25   OPEN "a:ppr9.dat" FOR OUTPUT AS 1
30   ALPHA=1.96E-09
40   BETA=2!
50   N=75
60   TM=20000
70   T0=10000
80   TF=14000
90   KLIM=200
92   FOR I=1 TO 7
93   ACUM(I)=0
94   NEXT
100  FOR ITER=1 TO 20
110  FOR I=1 TO KLIM
120  X(1,I)=0
130  X(2,I)=0
140  Z(1,I)=0
150  Z(2,I)=0
160  WT(I)=0
170  NEXT
200  GOSUB 9000
210  FOR I=1 TO KLIM
220  WRK(I)=0
230  NEXT
240  FOR I=1 TO N-KOUNT1
250  WRK(I)=X(1,I)+Z(1,I)
260  WRK(N-KOUNT1+I)=X(2,I)+Z(2,I)
270  NEXT
280  K=2*(N-KOUNT1)
290  FOR I=1 TO KOUNT1
300  WRK(K+I)=TF-T0+X(1,I)+X(2,I)
310  NEXT
320  K=K+KOUNT1
330  FOR I=1 TO KOUNT2
340  WRK(K+I)=WT(I)
350  NEXT
360  K=K+KOUNT2
370  EPS=.001
400  PRINT "iteration:";ITER
401  LPRINT "iteration:";ITER
410  PRINT
411  LPRINT
420  GOSUB 5000
430  PRINT "parameter check"
431  LPRINT "parameter check"
440  PRINT "          beta=";Q1,"alpha=";Q2
441  LPRINT "          beta=";Q1,"alpha=";Q2
450  PRINT
451  LPRINT
460  BES(1,ITER)=Q1
```

```
470  ALES(1,ITER)=Q2
480  ACUM(1)=ACUM(1)+Q1
490  PRINT #1, Q1,Q2
500  PRINT "method of moments"
501  LPRINT "method of moments"
510  PRINT "    complete data"
511  LPRINT "    complete data"
520  GOSUB 3000
530  PRINT "        ";"beta=";Q1,"alpha=";Q2
531  LPRINT "        ";"beta=";Q1,"alpha=";Q2
550  BES(2,ITER)=Q1
560  ALES(2,ITER)=Q2
580  ACUM(2)=ACUM(2)+Q1
590  PRINT #1, Q1,Q2
600  PRINT "    right censoring"
601  LPRINT "    right censoring"
610  GOSUB 2000
620  PRINT "        ";"beta=";Q1,"alpha=";Q2
621  LPRINT "        ";"beta=";Q1,"alpha=";Q2
650  BES(3,ITER)=Q1
660  ALES(3,ITER)=Q2
680  ACUM(3)=ACUM(3)+Q1
690  PRINT #1, Q1,Q2
700  PRINT "    double censoring"
701  LPRINT "    double censoring"
710  GOSUB 6000
720  PRINT "        ";"beta=";Q1,"alpha=";Q2
721  LPRINT "        ";"beta=";Q1,"alpha=";Q2
750  BES(4,ITER)=Q1
760  ALES(4,ITER)=Q2
780  ACUM(4)=ACUM(4)+Q1
790  PRINT #1, Q1,Q2
800  PRINT "maximum likelihood method"
801  LPRINT "maximum likelihood method"
810  PRINT "    complete data"
811  LPRINT "    complete data"
820  GOSUB 4000
830  PRINT "        ";"beta=";Q1,"alpha=";Q2
831  LPRINT "        ";"beta=";Q1,"alpha=";Q2
850  BES(5,ITER)=Q1
860  ALES(5,ITER)=Q2
880  ACUM(5)=ACUM(5)+Q1
890  PRINT #1, Q1,Q2
900  PRINT "    right censoring"
901  LPRINT "    right censoring"
910 GOSUB 7000
920  PRINT "        ";"beta=";Q1,"alpha=";Q2
921  LPRINT "        ";"beta=";Q1,"alpha=";Q2
950  BES(6,ITER)=Q1
960  ALES(6,ITER)=Q2
980  ACUM(6)=ACUM(6)+Q1
```

```
990  PRINT #1, Q1,Q2
1000 PRINT "    double censoring"
1001 LPRINT "    double censoring"
1010 GOSUB 8000
1020 PRINT "            ";"beta=";Q1,"alpha=";Q2
1021 LPRINT "           ";"beta=";Q1,"alpha=";Q2
1030 PRINT
1031 LPRINT
1050 BES(7,ITER)=Q1
1060 ALES(7,ITER)=Q2
1080 ACUM(7)=ACUM(7)+Q1
1090 PRINT #1, Q1,Q2
1100 NEXT
1110 PRINT
1111 LPRINT
1120 PRINT
1121 LPRINT
1130 PRINT "averages of the estimates"
1131 LPRINT "averages of the estimates"
1140 PRINT
1141 LPRINT
1150 FOR I=1 TO 7
1160 ACUM(I)=ACUM(I)/20
1170 PRINT #1,ACUM(I)
1180 PRINT "i=";I,"avg beta=";ACUM(I)
1181 LPRINT "i=";I,"avg beta=";ACUM(I)
1190 NEXT
1200 FOR I=1 TO 60
1210 LPRINT
1220 NEXT
1230 CLOSE 1
1240 END
2000 JK=N-KOUNT1
2010 FOR I=1 TO JK
2020 WZ(I)=X(1,I)+Z(1,I)
2030 WZ(JK+I)=Z(2,I)
2040 NEXT
2050 FOR I=1 TO KOUNT2
2060 WZ(2*JK+I)=WT(I)
2070 NEXT
2080 FOR I=1 TO KOUNT1
2090 WZ(2*JK+KOUNT2+I)=X(1,JK+I)+TF-TO
2100 NEXT
2110 A1=-.57486
2120 A2=.95124
2130 A3=-.69986
2140 A4=.42455
2150 A5=-.10107
2160 L=2*N-KOUNT1+KOUNT2
2170 SUM1=0
2180 FOR I=1 TO L
```

```
2190 SUM1=SUM1+WZ(I)
2200 NEXT
2210 Q1=.15
2220 Y=Q1
2230 GOSUB 2500
2240 F1=GAM
2250 Q2=4.8
2260 Y=Q2
2270 GOSUB 2500
2280 F2=GAM
2290 Y=(Q1+Q2)/2
2300 GOSUB 2500
2310 IF(GAM<0)GOTO 2360
2320 Q2=Y
2330 F2=GAM
2340 IF(F2-F1<EPS)GOTO 2390
2350 GOTO 2290
2360 Q1=Y
2370 F1=GAM
2380 IF(F2-F1>EPS)GOTO 2290
2390 Q1=(Q1+Q2)/2
2400 Q2=((F3^.5)*L/(SUM1+N*TI))^Q1
2410 RETURN
2500 Q3=1/Y
2510 Q4=2/Y
2520 JJ=INT(Q3)
2530 JK=INT(Q4)
2540 F3=1
2550 IF(JJ<1)GOTO 2600
2560 FOR I=1 TO JJ
2570 F3=F3*(Q3+1-I)
2580 NEXT
2590 Q3=Q3-JJ
2600 F3=F3*(1+A1*Q3+A2*(Q3^2)+A3*(Q3^3)+A4*(Q3^4)+A5*(Q3^5))
2610 F3=F3^2
2620 F4=1
2630 IF(JK<1)GOTO 2680
2640 FOR I=1 TO JK
2650 F4=F4*(Q4+1-I)
2660 NEXT
2670 Q4=Q4-JK
2680 F4=F4*(1+A1*Q4+A2*(Q4^2)+A3*(Q4^3)+A4*(Q4^4)+A5*(Q4^5))
2690 TI=(F4*SUM1)/(2*L*F3-N*F4)
2700 SUM2=0
2710 FOR I=1 TO L
2720 SUM2=SUM2+(WZ(I)^2)
2730 NEXT
2740 JJ=L-KOUNT1
2750 Q3=0
2760 FOR I=1 TO KOUNT1
2770 Q3=Q3+WZ(JJ+I)
```

```
2780 NEXT
2790 JK=N-KOUNT1
2800 FOR I=1 TO JK
2810 Q3=Q3+WZ(JK+I)
2820 NEXT
2830 Q4=(SUM2+2*TI*Q3+N*(TI^2))/(L-1)
2840 Q4=Q4*(L^2)/((SUM1^2)+2*N*TI*SUM1+(N^2)*(TI^2))
2850 Q4=(Q4-(L/(L-1)))^.5
2860 DEL=((F4/F3)-1)^.5
2870 GAM=Q4-DEL
2880 RETURN
3000 SUM1=0
3010 SUM2=0
3020 FOR I=1 TO K
3030 SUM1=SUM1+WRK(I)
3040 SUM2=SUM2+(WRK(I)^2)
3050 NEXT
3060 SUM1=SUM1/K
3070 SUM2=((SUM2/(K-1))-(SUM1^2))^.5
3080 DEL=SUM2/SUM1
3090 A1=-.57486
3100 A2=.95124
3110 A3=-.69986
3120 A4=.42455
3130 A5=-.10107
3140 TI=SUM1
3150 Y=.15
3160 GOSUB 3700
3170 Q1=Y
3180 F1=GAM
3190 Y=4.8
3200 GOSUB 3700
3210 Q2=Y
3220 F2=GAM
3230 Y=(Q1+Q2)/2
3240 GOSUB 3700
3250 IF(GAM>0)GOTO 3300
3260 Q2=Y
3270 F2=GAM
3280 IF(F1-F2<EPS)GOTO 3330
3290 GOTO 3230
3300 Q1=Y
3310 F1=GAM
3320 IF(F1-F2>EPS)GOTO 3230
3330 Q1=(Q1+Q2)/2
3340 Q2=((1+A1*Q1+A2*(Q1^2)+A3*(Q1^3)+A4*(Q1^4)+A5*(Q1^5))/TI)^Q1
3350 RETURN
3700 Q3=1/Y
3710 Q4=2/Y
3720 JJ=INT(Q3)
3730 JK=INT(Q4)
```

```
3740 F3=1
3750 IF(JJ<1)GOTO 3800
3760 FOR I=1 TO JJ
3770 F3=F3*(Q3+1-I)
3780 NEXT
3790 Q3=Q3-JJ
3800 F3=F3*(1+A1*Q3+A2*(Q3^2)+A3*(Q3^3)+A4*(Q3^4)+A5*(Q3^5))
3810 F3=F3^2
3820 F4=1
3830 IF(JK<1)GOTO 3880
3840 FOR I=1 TO JK
3850 F4=F4*(Q4+1-I)
3860 NEXT
3870 Q4=Q4-JK
3880 F4=F4*(1+A1*Q4+A2*(Q4^2)+A3*(Q4^3)+A4*(Q4^4)+A5*(Q4^5))
3890 GAM=(((F4/F3)-1)^.5)-DEL
3900 RETURN
4000 DEL=0
4010 FOR I=1 TO K
4020 DEL=DEL+LOG(WRK(I))
4030 NEXT
4040 DEL=DEL/K
4050 Q1=.15
4060 Y=Q1
4070 GOSUB 4500
4080 F1=GAM
4090 Q2=4.8
4100 Y=Q2
4110 GOSUB 4500
4120 F2=GAM
4130 Y=(Q1+Q2)/2
4140 GOSUB 4500
4150 IF(GAM<0)GOTO 4200
4160 Q2=Y
4170 F2=GAM
4180 IF(F2-F1<EPS)GOTO 4230
4190 GOTO 4130
4200 Q1=Y
4210 F1=GAM
4220 IF(F2-F1>EPS)GOTO 4130
4230 Q1=(Q1+Q2)/2
4240 Y=0
4250 FOR I=1 TO K
4260 Y=Y+(WRK(I)^Q1)
4270 NEXT
4280 Q2=K/Y
4290 RETURN
4500 SUM1=0
4510 SUM2=0
4520 FOR I=1 TO K
4530 SUM1=SUM1+(WRK(I)^Y)
```

```
4540 SUM2=SUM2+((WRK(I)^Y)*LOG(WRK(I)))
4550 NEXT
4560 GAM=(SUM2/SUM1)-(1/Y)-DEL
4570 RETURN
5000 FOR I=1 TO JK
5010 WZ(I)=WY(I)
5020 NEXT
5400 DEL=0
5410 FOR I=1 TO JK
5420 DEL=DEL+LOG(WZ(I))
5430 NEXT
5440 DEL=DEL/JK
5450 Q1=.15
5460 Y=Q1
5470 GOSUB 5800
5480 F1=GAM
5490 Q2=4.8
5500 Y=Q2
5510 GOSUB 5800
5520 F2=GAM
5530 Y=(Q1+Q2)/2
5540 GOSUB 5800
5550 IF(GAM<0)GOTO 5600
5560 Q2=Y
5570 F2=GAM
5580 IF(F2-F1<EPS)GOTO 5630
5590 GOTO 5530
5600 Q1=Y
5610 F1=GAM
5620 IF(F2-F1>EPS)GOTO 5530
5630 Q1=(Q1+Q2)/2
5640 Y=0
5650 FOR I=1 TO JK
5660 Y=Y+(WZ(I)^Q1)
5670 NEXT
5680 Q2=JK/Y
5690 RETURN
5800 SUM1=0
5810 SUM2=0
5820 FOR I=1 TO JK
5830 SUM1=SUM1+(WZ(I)^Y)
5840 SUM2=SUM2+((WZ(I)^Y)*LOG(WZ(I)))
5850 NEXT
5860 GAM=(SUM2/SUM1)-(1/Y)-DEL
5870 RETURN
6000 JK=N-KOUNT1
6010 FOR I=1 TO JK
6020 WZ(I)=Z(1,I)
6030 WZ(JK+I)=Z(2,I)
6040 NEXT
6050 FOR I=1 TO KOUNT2
```

```
6060 WZ(2*JK+I)=WT(I)
6070 NEXT
6080 FOR I=1 TO KOUNT1
6090 WZ(2*JK+KOUNT2+I)=TF-T0
6100 NEXT
6110 A1=-.57486
6120 A2=.95124
6130 A3=-.69986
6140 A4=.42455
6150 A5=-.10107
6160 L=2*N-KOUNT1+KOUNT2
6200 SUM1=0
6210 SUM2=0
6220 FOR I=1 TO L
6230 SUM1=SUM1+WZ(I)
6240 SUM2=SUM2+WZ(I)^2
6250 NEXT
6300 Q1=.15
6310 Y=Q1
6320 GOSUB 6500
6330 F1=GAM
6340 Q2=4.8
6350 Y=Q2
6360 GOSUB 6500
6370 F2=GAM
6380 Y=(Q1+Q2)/2
6390 GOSUB 6500
6400 IF(GAM<0)GOTO 6450
6410 Q2=Y
6420 F2=GAM
6430 IF(F2-F1<EPS)GOTO 6480
6440 GOTO 6380
6450 Q1=Y
6460 F1=GAM
6470 IF(F2-F1>EPS)GOTO 6380
6480 Q1=(Q1+Q2)/2
6490 Q2=((F3^.5)*L/(SUM1+2*N*TI))^Q1
6495 RETURN
6500 Q3=1/Y
6510 Q4=2/Y
6520 JJ=INT(Q3)
6530 JK=INT(Q4)
6540 F3=1
6550 IF(JJ<1)GOTO 6600
6560 FOR I=1 TO JJ
6570 F3=F3*(Q3+1-I)
6580 NEXT
6590 Q3=Q3-JJ
6600 F3=F3*(1+A1*Q3+A2*(Q3^2)+A3*(Q3^3)+A4*(Q3^4)+A5*(Q3^5))
6610 F3=F3^2
6620 F4=1
```

```
6630 IF(JK<1)GOTO 6680
6640 FOR I=1 TO JK
6650 F4=F4*(Q4+1-I)
6660 NEXT
6670 Q4=Q4-JK
6680 F4=F4*(1+A1*Q4+A2*(Q4^2)+A3*(Q4^3)+A4*(Q4^4)+A5*(Q4^5))
6690 TI=(F4*SUM1)/(2*L*F3-2*N*F4)
6700 Q3=0
6710 JK=N-KOUNT1
6720 FOR I=1 TO JK
6730 Q3=Q3+WZ(I)+WZ(JK+I)
6740 NEXT
6750 FOR I=1 TO KOUNT1
6760 Q3=Q3+2*(TF-TO)
6770 NEXT
6780 Q4=(SUM2+2*TI*Q3+(2*N+2*KOUNT1)*(TI^2))/(L-1)
6790 Q4=Q4*(L^2)/((SUM1^2)+4*N*TI*SUM1+4*(N^2)*(TI^2))
6800 Q4=(Q4-(L/(L-1)))^.5
6810 DEL=((F4/F3)-1)^.5
6820 GAM=Q4-DEL
6830 RETURN
7000 JK=N-KOUNT1
7010 FOR I=1 TO JK
7020 WZ(I)=X(1,I)+Z(1,I)
7030 WZ(JK+KOUNT2+I)=Z(2,I)
7040 NEXT
7050 FOR I=1 TO KOUNT2
7060 WZ(JK+I)=WT(I)
7070 NEXT
7080 JJ=2*JK+KOUNT2
7090 FOR I=1 TO KOUNT1
7100 WZ(JJ+I)=X(1,JK+I)+TF-TO
7110 NEXT
7150 DEL=0
7160 JJ=JJ+KOUNT1
7170 FOR I=1 TO JJ
7180 DEL=DEL+LOG(WZ(I))
7190 NEXT
7200 DEL=DEL/(JK+KOUNT2)
7210 Q1=.15
7220 Y=Q1
7230 GOSUB 7500
7240 F1=GAM
7250 Q2=4.8
7260 Y=Q2
7270 GOSUB 7500
7280 F2=GAM
7290 Y=(Q1+Q2)/2
7300 GOSUB 7500
7310 IF(GAM<0)GOTO 7360
7320 Q2=Y
```

```
7330 F2=GAM
7340 IF(F2-F1<EPS)GOTO 7390
7350 GOTO 7290
7360 Q1=Y
7370 F1=GAM
7380 IF(F2-F1>EPS)GOTO 7290
7390 Q1=(Q1+Q2)/2
7400 Q2=JK/F3
7410 RETURN
7500 F3=0
7510 F4=0
7520 FOR I=1 TO JJ
7530 F3=F3+(WZ(I)^Y)
7540 F4=F4+((WZ(I)^Y)*LOG(WZ(I)))
7550 NEXT
7560 GAM=(F4/F3)-(1/Y)-DEL
7570 RETURN
8000 SUM1=0
8010 SUM1=KOUNT2
8020 JK=N-KOUNT1
8030 TI=(N+SUM1+.4)*WT(KOUNT2)/(SUM1-.3)
8040 FOR I=1 TO JK
8050 Y=RND
8060 WZ(I)=Z(1,I)+Y*TI
8070 WZ(JK+KOUNT2+I)=Z(2,I)
8080 NEXT
8090 FOR I=1 TO KOUNT2
8100 WZ(JK+I)=WT(I)
8110 NEXT
8120 JJ=2*JK+KOUNT2
8130 FOR I=1 TO KOUNT1
8140 Y=RND
8150 WZ(JJ+I)=TF-TO+Y*TI
8160 NEXT
8170 K=2*N-KOUNT1+KOUNT2
8180 GOSUB 8750
8190 JJ=JK+KOUNT2
8200 DEL=0
8210 FOR I=1 TO JJ
8220 DEL=DEL+LOG(WZ(I))
8230 NEXT
8240 SUM2=0
8250 FOR I=1 TO JK
8260 SUM2=SUM2+LOG(WY(I))
8270 DEL=DEL+LOG(WY(I))
8280 NEXT
8290 DEL=DEL/JJ
8300 Q1=.15
8310 Y=Q1
8320 GOSUB 8500
8330 F1=GAM
```

```
8340 Q2=4.8
8350 Y=Q2
8360 GOSUB 8500
8370 F2=GAM
8380 Y=(Q1+Q2)/2
8390 GOSUB 8500
8400 IF(GAM<0)GOTO 8450
8410 Q2=Y
8420 F2=GAM
8430 IF(F2-F1<EPS)GOTO 8480
8440 GOTO 8380
8450 Q1=Y
8460 F1=GAM
8470 IF(F2-F1>EPS)GOTO 8380
8480 Q1=(Q1+Q2)/2
8490 Q2=JK/F3
8495 RETURN
8500 F3=0
8510 F4=SUM2
8520 Q3=0
8530 Q4=0
8540 FOR I=1 TO JK
8550 F3=F3+(WZ(I)^Y)
8560 F4=F4+((WZ(I)^Y)*LOG(WZ(I)))
8570 F3=F3+(WZ(JK+KOUNT2+I)^Y)
8580 F4=F4+((WZ(JK+KOUNT2+I)^Y)*LOG(WZ(JK+KOUNT2+I)))
8590 GAM=((WY(I)^Y)-1)*(WZ(I)^Y)
8600 Q3=Q3+GAM
8610 Q4=Q4+GAM*LOG(WY(I))
8620 NEXT
8630 JJ=2*JK+KOUNT2
8640 FOR I=1 TO KOUNT1
8650 F3=F3+(WZ(JJ+I)^Y)
8660 F4=F4+((WZ(JJ+I)^Y)*LOG(WZ(JJ+I)))
8670 GAM=((WY(JJ+I)^Y)-1)*(WZ(JJ+I)^Y)
8680 Q3=Q3+GAM
8690 Q4=Q4+GAM*LOG(WY(JJ+I))
8700 NEXT
8710 GAM=((F4-Q4)/(F3-Q3))-(1/Y)-DEL
8720 RETURN
8750 FOR I=1 TO K
8755 WY(I)=0
8760 NEXT
8765 B1=.34802
8770 B2=-.09588
8775 B3=.74786
8780 B4=.47047
8785 FOR I=1 TO JK
8790 S=RND
8795 IF(S<.5) GOTO 8815
8800 GOSUB 8900
```

```
8805 WY(I)=1+.3*GAM
8810 GOTO 8830
8815 S=1-S
8820 GOSUB 8900
8825 WY(I)=1-.3*GAM
8830 NEXT
8835 FOR I=1 TO KOUNT1
8840 S=RND
8845 IF(S<.5) GOTO 8865
8850 GOSUB 8900
8855 WY(JJ+I)=1+.3*GAM
8860 GOTO 8880
8865 S=1-S
8870 GOSUB 8900
8875 WY(JJ+I)=1-.3*GAM
8880 NEXT
8885 RETURN
8900 Q3=.0001
8905 Q4=3.099
8910 TI=(Q3+Q4)/2
8915 F3=EXP(-(TI^2))
8920 F4=1/(1+B4*TI)
8925 GAM=1-(B1*F4+B2*(F4^2)+B3*(F4^3))*F3
8930 IF(GAM<S) GOTO 8950
8935 IF(GAM-S<EPS) GOTO 8965
8940 Q4=TI
8945 GOTO 8910
8950 IF(S-GAM<EPS)GOTO 8965
8955 Q3=TI
8960 GOTO 8910
8965 GAM=(Q3+Q4)/2
8970 RETURN
9000 FOR K=1 TO 55
9010 Y=RND
9020 NEXT
9030 K=0
9040 KOUNT1=0
9050 KOUNT2=0
9100 JK=0
9110 S=0
9120 FOR I=1 TO 25
9130 Y=RND
9140 TI=((-LOG(Y))/ALPHA)^(1/BETA)
9150 IF(I>1)GOTO 9180
9160 JK=JK+1
9170 WY(JK)=TI
9180 S=S+TI
9190 WRK(I)=S
9200 IF(S>TM)GOTO 9250
9210 NEXT
9220 PRINT "random time generation error"
```

```
9230 STOP
9250 IF(I>1)GOTO 9350
9260 X(1,N-KOUNT1)=TO
9270 X(2,N-KOUNT1)=S-TF
9280 KOUNT1=KOUNT1+1
9290 IF(K+KOUNT1<N)GOTO 9110
9300 RETURN
9350 K=K+1
9360 FOR J=1 TO I
9370 IF(WRK(J)<TO)GOTO 9830
9380 IF(J>1)GOTO 9500
9390 IF(WRK(1)>TF)GOTO 9450
9400 X(1,K)=TO
9410 Z(1,K)=WRK(1)-TO
9420 GOTO 9600
9450 X(1,N-KOUNT1)=TO
9460 X(2,N-KOUNT1)=WRK(1)-TF
9470 K=K-1
9480 GOTO 9280
9500 IF(WRK(J)<TF)GOTO 9550
9510 X(1,N-KOUNT1)=TO-WRK(J-1)
9520 X(2,N-KOUNT1)=WRK(J)-TF
9530 K=K-1
9540 GOTO 9280
9550 X(1,K)=TO-WRK(J-1)
9560 Z(1,K)=WRK(J)-TO
9600 IF(WRK(J+1)<TF)GOTO 9700
9610 Z(2,K)=TF-WRK(J)
9620 X(2,K)=WRK(J+1)-TF
9630 GOTO 9290
9700 FOR L=1 TO I-J
9710 IF(WRK(J+L)>TF)GOTO 9800
9720 KOUNT2=KOUNT2+1
9730 WT(KOUNT2)=WRK(J+L)-WRK(J+L-1)
9740 NEXT L
9750 GOTO 9220
9800 X(2,K)=WRK(J+L)-TF
9810 Z(2,K)=TF-WRK(J+L-1)
9820 GOTO 9290
9830 NEXT J
9840 GOTO 9220
```

The vita has been removed from the scanned document