QUEUES WITH A MARKOV RENEWAL SERVICE PROCESS

by

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(ABSTRACT)

In the present work, we study a queue with a Markov renewal service process. The objective is to model systems where different customers request different services and there is a setup time required to adjust from one type of service to the next.

The arrival is a Poisson process independent of the service. After arrival, all the customers will be attended in order of arrival. Immediately before a service starts, the type of next customer is chosen using a finite, irreducible and aperiodic Markov chain $\boldsymbol{P}$. There is only one server and the service time has a distribution function F_{ij}, where i and j are the types of the previous and current customer in service, respectively. This model will be called $M/MR/1$.

Embedding at departure epochs, we characterize the queue length and the type of customer as a Markov renewal process. We study a special case where F_{ij} is exponential with parameter μ_{ij}. We prove that the departure is a renewal process if and only if $\mu_{ij} = \mu$, $\forall\, i,j \in E$. Furthermore, we show that this renewal is a Poisson process. The type-departure process is extensively studied through the respective counting processes. The crosscovariance and the crosscorrelation are computed and numerical results are shown. Finally, we introduce several ex-

pressions to study the interdependence among the type-departure processes in the general case, i.e. the distribution function F_{ij} does not have any special form.

For , and .

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Chapter 1

INTRODUCTION AND BACKGROUND

1.1 The Problem

A common assumption in queueing theory has been the independence between the current and the previous service times. This, together with the independence from the arrival process, defines a GI service process, which has been extensively studied. On the other hand, only a few papers have dropped this assumption.

The hypothesis of independence between consecutive services is clearly not adequate if the server needs some setup time to prepare for the next and possibly different type of job. Furthermore, in many real production systems, the server does not know the next type of job until he gets to it. In such cases the sequence of service times is delayed each time there is a change in the type of job to be serviced. Consequently, the service times do not form a sequence of independent and identically distributed random variables, but rather the service times depend on the sequence of the types of customers.

In this paper we will discuss a queueing system where the service process is a Markov renewal process (MRP). As we will see in the next section, the MRP assumption introduces the kind of dependence that we want to study. We retain the assumption of independence between interarrival and service times.

To simplify the analysis, the following additional assumptions are made:

i) The arrival process is a Poisson process with rate λ *and there is no loss so that any arrival to the system will eventually receive service.*

ii) There is just one server and the service times are generated by a MRP.

iii) The service is provided in order of arrival (FCFS).

We shall denote this system by $M/MR/1$. We are interested in studying the effect of the Markov renewal service process on the performance of the queue, in particular, the departure process. In practical situations, the departure process of this queue may be the input for another queue and the knowledge of its characteristics is fundamental to the understanding of the behavior of the entire system.

Some of the questions that we intend to answer are: What is the structure of the departure process? Under what conditions (if any) is the departure process a renewal process? How is the structure of the departure process related to the choice of states in the Markov renewal service process?

This paper is organized as follows. Chapter 1 has two more sections: Background and Literature Review. In Chapter 2 we describe the Markov renewal service process. In Chapter 3 we look at the queue length process of the $M/MR/1$ queue. We obtain the general structure and a characterization of the departure process. Chapter 4 is devoted to a special case: a Markov renewal service process with exponential distributions. We obtain necessary and sufficient conditions for the departure process to be a renewal process. In Chapter 5 the type-departure process is studied in detail. Crosscovariances and crosscorrelations are defined and used to measure the dependence among the counting processes associated with the type-departure processes. A numerical example with several graphs illustrates the behavior of the crosscorrelation. The last chapter, Chapter 6, is dedicated to conclusions and discussion as well as prospectives for future research.

The equations are referenced by two numbers in parenthesis: section number followed by the order of the equation in the section; for instance, the equation (2.7) is the 7th equation in section 2. Theorems, lemmas, corollaries and diagrams are numbered in the same way. For reference to the results in other chapters, we add the chapter number; equation (3.2.7) is the 7th equation in section 2 of Chapter 3.

1.2 Background

We briefly review the definitions of renewal and Markov renewal processes and their properties. We also define the concept of equivalence between a Markov renewal process and a renewal process. The material shown here is an extract from chapters 9 and 10 in Çinlar [1975] and from chapters 1 and 2 in Disney and Kiessler [1987].

We start with the definition of a renewal process. Consider a fixed phenomenon, and let $0 = T_0 < T_1 < T_2 < \cdots$ be the times of its successive occurrences.

Definition (2.1): The sequence $\mathbf{T} = \{T_n : n = 0,1,\ldots\}$ is called a renewal process provided that the interoccurrence times are independent, identically distributed, and non negative random variables. Then each T_n is called a renewal time. ■

One of the most well-known examples of renewal processes is the Poisson process. In this case the inter-renewal times have an exponential distribution.

Let F be the distribution function for the inter-renewal times in a renewal process. Then, by successive convolutions, we obtain

$$P[T_n \leq t] = F^{(*n)}(t)\,, \tag{2.1}$$

where $F^{(*n)}$ represents the convolution of F with itself, n times.

Now we introduce the Markov renewal process. Suppose we have defined for each $n=0,1,\ldots$, a random variable X_n taking values in a countable set E and a random variable T_n taking values in $[0,\infty)$, such that $0 = T_0 < T_1 < T_2 < \cdots$.

Definition (2.2): The stochastic process $(\mathbf{X},\mathbf{T}) = \{(X_n, T_n)\ ;\ n = 0,1,2,...\}$ is said to be a Markov renewal process with state space E provided that:

For all $n = 0,1,2,...$, and $t \in [0, \infty)$

$$P[X_{n+1}, T_{n+1} - T_n \leq t \mid X_0, ...X_n;\ T_0,...,T_n] = P[X_{n+1}, T_{n+1} - T_n \leq t \mid X_n]\ . \blacksquare \quad (2.2)$$

We will only consider time homogeneous processes. That is, for $i, j \in E$,

$$P[X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i] = Q_{ij}(t) \quad \text{independent of n.} \quad (2.3)$$

The matrix Q = $\{Q_{ij}(t); i,j \in E, t \in R_+\}$ is called a semi-Markov kernel over E. Note that the state space E could be multidimensional; the only restriction that we are making here is that E is a countable space.

Let

$$p_{ij} = \lim_{t \to \infty} Q_{ij}(t) \quad , \quad (2.4)$$

$$p_{ij} \geq 0 \quad \text{and} \quad \sum_{j \in E} p_{ij} = 1 \ . \quad (2.5)$$

Then p_{ij} are the transition probabilities for some Markov chain. In fact, from (2.4) and (2.5) we have the following result:

Theorem (2.3) : $\mathbf{X} = \{X_n\ ;\ n = 0,1,...\}$ is a Markov chain with state space E and transition matrix $\mathbf{P} = [p_{ij}]$, $i,j \in E$.

proof :

See proposition (10.1.7) in Çinlar [1975]. ■

Remark (2.4): Note that if the state space E has just one point, the MRP $\{(X_n,T_n)\,;\,n=0,1...\}$ collapses to $\{T_n\,;\,n=0,1...\}$, which is a renewal process. ▪

Remark (2.5): Consider a subset D of the state space. Let $r_0,r_1,r_2,\ldots$ be the successives indices $n \geq 0$ such that $X_n \in D$. For each n=0,1,... define $\hat{X}_n = X_{r_n}$ and $\hat{T}_n = T_{r_n}$. Then by theorem (10.1.13) in Çinlar [1975] the process $(\hat{\mathbf{X}}\ ,\hat{\mathbf{T}}) = \{\hat{X}_n,\hat{T}_n; n \in N\}$ is a Markov renewal process with state space D. ▪

Remark (2.6): The n-step transition in the semi-Markov kernel is given by

$$\begin{aligned} Q_{ij}^{(n)}(t) &= P[X_n = j, T_n \leq t \mid X_0 = i] \\ &= \sum_{k \in E} \int_0^t Q_{kj}^{(n-1)}(t-s)\, dQ_{ik}(s)\ . \end{aligned} \tag{2.6}$$

The last equality gives a recursive expression for $Q_{ij}^{(n)}(t)$. By similarity with the definition of the convolution of functions, the matrix $Q^{(n)}(t) = [Q_{ij}^{(n)}(t)]$, $i,j \in E$, is called the n-th convolution of the kernel Q with itself. ▪

Next, we introduce the concept of equivalence between a Markov renewal process and a renewal process. Let **(X,T)** be a Markov renewal process with semi-Markov kernel Q and state space E. The joint distributions of the increments $\{\, T_n - T_{n-1}\,;\, n=1,2,...\}$ are given by

$$P[T_1 \leq t_1, T_2 - T_1 \leq t_2, \ldots, T_m - T_{m-1} \leq t_m] = \gamma\, Q(t_1)\, Q(t_2) \ldots Q(t_m)\ e\ , \tag{2.7}$$

where γ is some initial state distribution vector with elements $P[X_0 = j]$, $j \in E$ and **e** is a column vector of 1's.

Lemma (2.7): The sequence $\{T_n\,;\, n=0,1,...\}$ is a renewal process if and only if for all $m \in N$ and $t_1,t_2,\ldots,t_m \in R_+$, we have

$$\gamma\, Q(t_1)\, Q(t_2)....Q(t_m)\, e = (\, \gamma\, Q(t_1)\, e)\, (\, \gamma\, Q(t_2)\, e) \dots (\, \gamma\, Q(t_m)\, e)\ . \tag{2.8}$$

proof:

See lemma 2.11.1 in Disney and Kiessler[1987]. ■

Definition (2.8): Let **R** be a renewal process with distribution F. The process **(X,T)** is equivalent to **R** (written **(X,T)** $\sim$ **R**) if,

$$\gamma\, Q(t_1)\, Q(t_2)....Q(t_m)\ e = F(t_1)\, F(t_2)....F(t_m)\ , \tag{2.9}$$

for $m \in N$, and $t_1, t_2, \dots, t_m \in R_+$. ■

Next, we derive two sufficient conditions for **(X,T)** $\sim$ **R**. Other conditions can be found in Chapter 2 of Disney and Kiessler [1987].

Sufficient Condition 1 (2.9):

$$\text{If for } t \in R_+\ , \qquad \gamma\, Q(t) = F(t)\ \gamma\ , \tag{2.10}$$

then we have the equivalence **(X,T)** $\sim$ **R** . ■

Sufficient Condition 2 (2.10):

$$\text{If for } t \in R_+\ , \quad Q(t)\ e = e\, F(t) \tag{2.11}$$

then we have the equivalence **(X,T)** $\sim$ **R**. ■

1.3 *Literature Review*

The first two (and almost simultaneous) contributions to the study of $M/MR/1$ queues are the papers by Neuts [1966] and Çinlar [1967]. In Neuts [1966], Markov renewal theory is used to study the busy period, virtual waiting time, and queue length. The queue length process is studied in continuous time and by embedding just after departure epochs. Conditional expected values for the number of departures are obtained. In Çinlar [1967] the emphasis is on the queue length process. A Markov renewal process is used to characterize the queue length and the type of leaving customer just after a departure. For continuous time the joint distribution of queue length and type in service is presented. The limiting behavior of the probability distribution of the continuous time and the embedded processes are compared.

Purdue [1975] extended the model studied by Neuts and Çinlar, allowing the arrival Poisson process to be dependent on the types. However, he obtained results only for the busy period.

In Neuts [1977a], several computationally useful expressions are given for the expected duration of the busy period, the mean number of customers served, and the lower order moments of the queue length, in both discrete and continuous time. Moreover, Neuts obtained results for the virtual waiting time. The model studied in Neuts [1977a] was a little more general than the one in Neuts [1966] because it is assumed that batch arrivals occur at the epochs of a homogeneous Poisson process. As a special case, Neuts [1977b] considers the model where two kinds of services are provided: ordinary and extraordinary, depending on whether or not the current customer is of the same type as the previous one. Explicit formulas and examples were given.

For a single server queue in which both service and interarrival times may depend on customer type, McNickle [1974a] derived the distribution of the number of departures in terms of the busy period and busy cycle processes. McNickle [1974b] provides more details for this study, but the departure process is not studied any further. Nevertheless, he pointed out the necessity for future research on the dependencies in the departure process. In this dissertation, an attempt is made to fulfill this need.

In 1985, de Smit [1985] studied a large class of Markov renewal queues. He constructed a system of Wiener-Hopf type equations whose solution depends on the ability to obtain explicit factorizations. De Smit and Regterschot [1985] indicated that, these factorizations can be given only in special cases, providing an example where such factorization is possible. They consider that for each customer, an exponential service is supplied whose mean depends on the state of the Markov renewal process at the customer's arrival. Transform expressions for several quantities are computed such as the virtual waiting time, and the number of customers, both in continuous time and at arrival epochs.

The literature associated with flexible manufacturing systems is another source of contributions to the study of queues with changeover times. Eisenberg [1969] considers a system with two queues served by a single server. In each queue, the arrival is a Poisson process and the service has a general distribution.

When the server moves from one queue to another, a setup time with an arbitrary distribution function is required. Two different disciplines are considered: alternating priority and strict priority, with the waiting time distribution computed for each case. In Eisenberg [1970], the results shown in Eisenberg [1969] are extended to M queues with periodic service and alternating priority discipline. Also, Buzacott and Gupta [1986] describe an approximation scheme for the computation of the mean flow time in a network of queues with multiple job classes and changeover times. Their approach is to draw conclusions for the network by using the results obtained in each queue studied in isolation. A numerical example compares the approximation with the simulation results. In Gupta and Buzacott [1988], the results are extended to include feedback. The network approximation for the mean waiting time is made possible by the development of an exact method to obtain the performance indices for single machine models with two queues, setup times, and feedback.

As far as we know, these are the only papers directly related to our work in Markov renewal service queues, although there are other papers in related areas concerned primarily with Markov service queues and, of course, renewal service queues. The number of publications is relatively small, and except for the McNickle work previously cited, the departure process for the $M/MR/1$ queue has not been studied. Of particular interest is the crosscovariance analysis in chapter 5, which constitutes an original contribution to the study of interdependent processes in which category the departure process of $M/MR/1$ is included.

Chapter 2

MARKOV RENEWAL SERVICE PROCESS

2.1 *Introduction*

In this chapter we study the Markov renewal service process and use general results of the theory of Markov renewal processes to obtain some consequences and properties for the service process. A correlation coefficient is used to study the dependence among successive services; the chapter finishes with a numerical example. Our references to well-known results here are found in Çinlar[1975] and Disney and Kiessler [1987].

2.2 The Markov Renewal Structure

Let $\mathbf{S} = \{S_n \,;\, \mathrm{n}=0,1,...\}$ be a sequence of service times, $S_n \in R_+$. For each S_n associate a random variable Z_n , n = 0,1,..., and define $\mathbf{Z} = \{Z_n \,;\, \mathrm{n}=0,1,...\}$. The random process **Z** takes value on a finite set E = {1,2,...,m}.

Assume that the random process (**Z**,**S**) is a time homogenous MRP with semi-Markov kernel **B**. The random variable Z_n will represent the type of the n-th customer. This customer classification is assigned immediately before the service starts.

Let $\boldsymbol{P}$ be the one step transition for the underlying Markov chain **Z** , that is,

$$\boldsymbol{P} = \lim_{t\to\infty} \boldsymbol{B}(t) = \boldsymbol{B}(\infty) = [\, p_{ij} \,] \,, \quad i,j = 1,2, \dots , m. \tag{2.1}$$

In this chapter we assume that $\boldsymbol{P}$ is finite and irreducible. An example with a periodic chain is given in the end of the chapter (section 4). Later in chapter 3, we will also assume that $\boldsymbol{P}$ is aperiodic. That is, from chapter 3 to the end of the dissertation, $\boldsymbol{P}$ will be assumed finite, irreducible and aperiodic.

Define $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_m)$ such that

$$\begin{aligned} \boldsymbol{\eta}\, \boldsymbol{P} &= \boldsymbol{\eta} \,, \\ \boldsymbol{\eta}\, \boldsymbol{e} &= 1 \,, \\ \boldsymbol{\eta} &\geq 0 \,. \end{aligned} \tag{2.2}$$

That is, $\boldsymbol{\eta}$ is the unique stationary probability vector for the Markov chain $\mathbf{Z}$.

For $i,j \in E, t \geq 0$, define $F_{ij}(t)$ as the conditional probability distribution function:

$$F_{ij}(t) = P[S_{n+1} \leq t \,|\, Z_n = i, Z_{n+1} = j] = \frac{B_{ij}(t)}{p_{ij}} \text{ , for } p_{ij} > 0. \tag{2.3}$$

In the case $p_{ij} = 0$ we define $F_{ij}(t) = 1$.

Thus, conditioned on the types of the n-th and (n + 1)-th customers, $F_{ij}(t)$ is the (conditional) service time of the (n + 1)-th customer. This structure permits us to model changeover systems assigning different parameters or distributions for different values of the pair (i, j) , i,j = 1,2,...m .

Since the random variables $\{S_n \,;\, n \geq 0\}$ are, in general, not independent, it is interesting to measure the dependency among the successive service times. We will express the dependency through the correlation coefficient of lag r whose computation requires some preliminary results. In the following we assume that we are in the steady state, that is,

$$P[Z_n = j] = \eta_j \text{ ,} \qquad \text{independent of n .}$$

The distribution function of S_n is given by

$$
\begin{aligned}
P[S_n \le t] &= \sum_{i=1}^{m}\sum_{j=1}^{m} P[S_n \le t,\, Z_n = j,\, Z_{n-1} = i] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} P[Z_n = j,\, S_n \le t \mid Z_{n-1} = i] P[Z_{n-1} = i] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i B_{ij}(t) \ ,
\end{aligned}
$$

or in matrix form

$$P[S_n \le t] = \ \boldsymbol{\eta}\ \boldsymbol{B}(t)\ \boldsymbol{e} \ . \tag{2.4}$$

Then

$$E[S_n] = \ \boldsymbol{\eta}\boldsymbol{B}_1\ \boldsymbol{e} \qquad \text{with} \quad \boldsymbol{B}_1 = [\int_0^{\infty} t\, dB_{ij}(t)] \quad \text{for} \ \ i,j = 1,2, \ldots ,m \tag{2.5}$$

and

$$E[S_n^2] = \ \boldsymbol{\eta}\boldsymbol{B}_2\ \boldsymbol{e} \qquad \text{with} \quad \boldsymbol{B}_2 = \left[\int_0^{\infty} t^2\, dB_{ij}(t)\right] \quad \text{for} \ \ i,j = 1,2, \ldots ,m. \tag{2.6}$$

The lag r joint distribution of two service times is given by

$$
\begin{aligned}
P[S_n \le t, S_{n+r} \le s] &= \sum_{i,j,k,l=1}^{m} P[S_n \le t,\, S_{n+r} \le s,\, Z_{n-1} = i,\, Z_n = j,\, Z_{n-r+1} = k,\, Z_{n+r} = l] \\
&= \sum_{i,j,k,l=1}^{m} \eta_i\, B_{ij}(t)\, [\boldsymbol{P}^{r-1}]_{jk}\, B_{kl}(s) \ ,
\end{aligned}
$$

in matrix notation

$$P[S_n \le t, S_{n+r} \le s] = \eta\, B(t)\, P^{r-1}\, B(s)\, e \qquad \text{for } r = 1,2,\dots \quad . \tag{2.7}$$

Then

$$E[S_n\, S_{n+r}] = \eta\, B_1\, P^{r-1}\, B_1\, e \qquad \text{for } r = 1,2,\dots \quad . \tag{2.8}$$

Note that expressions (2.4) - (2.8) do not depend on n since we have assumed homogeneity of the service process. This implies in particular that $Var[S_n]$ does not depend on n and $E[S_{n+r}]$ depends on neither n nor r. The covariance of lag r of the service time is

$$Cov(S_n, S_{n+r}) = E[S_n S_{n+r}] - E^2[S_n] \quad . \tag{2.9}$$

Finally, the correlation of lag r (in steady state) is obtained by using the previous expressions:

$$corr(S_n, S_{n+r}) = \frac{Cov(S_n, S_{n+r})}{Var(S_n)} \tag{2.10}$$

which depends only on r. Hence the (**Z**,**S**) process is weakly stationary.

2.3 Numerical Example

Consider the MRP $\{(Z_n, S_n)\ ;\ n = 0,1,....\}$ with $Z_n \in E$ and kernel given by

$$B(x) = \begin{bmatrix} (1-a)\, F_{11}(x) & a\, F_{12}(x) \\ b\, F_{21}(x) & (1-b)\, F_{22}(x) \end{bmatrix} ,$$

where $F_{ij}(x) = P[S_n \le x \mid Z_{n-1} = i, Z_n = j]$ for i, j = 1, 2 ;

also , $a = P[Z_n = 2 \mid Z_{n-1} = 1]$

$b = P[Z_n = 1 \mid Z_{n-1} = 2]$.

In this example, for illustration purposes, we will remove the assumption of irreducibility of the matrix $\boldsymbol{P}$. In other words, the parameters a and b are allowed to have value 0 or 1.

Case 1: $F_{ij}(x) = F(x)$ for i, j = 1,2 .

In this case, no matter what the relationship between a and b is, the MRP will model a GI service with distribution F. To see this we observe that the kernel is given by

$$B(x) = \begin{bmatrix} (1-a)\, F(x) & a\, F(x) \\ b\, F(x) & (1-b)\, F(x) \end{bmatrix} .$$

Then

$$B(x)\, e = \begin{bmatrix} F(x) \\ F(x) \end{bmatrix} .$$

Thus the row sums of $B(x)$ are equal. From (1.2.11), we conclude that, in the steady state, $\{Z_n, S_n ; n=0,1,...\}$ is equivalent to a renewal process with distribution F. In other words, the service times $\{S_n ; n = 0,1...\}$ marginally form a sequence of independent and identically distributed random variables. Consequently there is no difference whether or not one assigns the types of service before or after the service.

Case 2 : $F_{ij}(x) = F_j(x)$ for i, j = 1,2 and $F_1(x) \neq F_2(x)$.

In this case the conditional distribution function $F_{ij}(x)$ only depends on the current type in service. The kernel becomes

$$B(x) = \begin{bmatrix} (1-a)\,F_1(x) & a\,F_2(x) \\ b\,F_1(x) & (1-b)\,F_2(x) \end{bmatrix} .$$

Consider the following subcases:

2a) a = 0 .

The Markov chain $\{Z_n , n = 0,1, ... \}$ will have the state 1 as an absorbing state. State 2 will be transient if $0 < b < 1$, or absorbing if b = 0. For both cases, in the long run, the Markov chain is absorbed and the service process becomes a renewal process since there are no more changes in the types.

2b) a = b = 1 .

We have

$$B(x) = \begin{bmatrix} 0 & F_2(x) \\ F_1(x) & 0 \end{bmatrix} .$$

The matrix P is periodic and this model has been used to study replacements in production planning. It is well-known as an alternating renewal process (Heyman and Sobel [1982]).

2c) $a+b=1$,$0<a<1$ and $0<b<1$.

Note that

$$B(x)\ e=\begin{bmatrix} a\,F_2(x)+b\,F_1(x) \\ a\,F_2(x)+b\,F_1(x) \end{bmatrix} .$$

Since the row sums in $B(x)$ are equal, from (1.2.11) we have equivalence between the service process and a renewal process with distribution $F(x)=aF_2(x)+bF_1(x)$. In fact, we have the following result:

Theorem (3.1): The two state Markov renewal process $\{(Z_n, S_n)\,;\,n=0,1,\dots\}$ with $F_{ij}(x)=F_j(x)$ $(F_1(x)\neq F_2(x))$, $0<a<1$ and $0<b<1$ is equivalent to a renewal process if and only if $a+b=1$.

proof:

1st part: $a+b=1 \rightarrow$ renewal process .

This was proved by using the foregoing argument.

2nd part: renewal process $\rightarrow a+b=1$.

A necessary, but not sufficient, condition for a renewal process is the independence between consecutive intervals, that is,

$$\eta\ B(x)\ B(y)\ e=(\ \eta\ B(x)\,e)\,(\ \eta\ B(y)\ e)\quad , \tag{3.1}$$

where η is the stationary distribution for $\{Z_n\,;\,n=0,1,\dots\}$ and is given by

$$\boldsymbol{\eta} = (\eta_1, \eta_2) = \left(\frac{b}{a+b} \; , \; \frac{a}{a+b} \right) \quad .$$

Computing the expressions on both sides of (3.1) and equating the coefficients of $F_1(x)\, F_1(y)$, we have the following condition:

$$\eta_1 \, (1-a)^2 + \eta_2 \, (1-a)b = \eta_1^2 \quad .$$

From this, after some manipulation we conclude $a+b=1$. ■

2d) $a+b \neq 1 \quad 0<a<1$ and $0<b<1$.

In this case, as proved by theorem (3.1), the sequence of service times is dependent and consequently correlated. Using (2.10) we have computed table (3.1) which shows the correlations lag r (r= 1,...5) for F_1 and F_2 exponentials and several values of a and b ($a+b \neq 1$) . We observe positive correlations for all lags when $a+b<1$. However, when $a+b>1$ the lag r correlations can be negative or positive depending on r being odd or even, respectively. The absolute value of the correlation decreases as $a+b$ approaches 1 and increases when $a+b$ becomes larger in absolute value (i.e. near 2). ■

Remark (3.2): For $m \geq 2$, if $\mathbf{Z}$ is a stationary chain and if $\boldsymbol{P}$ is an equilibrium matrix (all rows are equal to the equilibrium distribution $\boldsymbol{\eta}$), then for $r>1$, $\boldsymbol{P}^{r-1} = \boldsymbol{e}\, \boldsymbol{\eta}$ and from expressions (2.4) and (2.7) we conclude that the service times separated by a lag r ($r>1$) are independent. Thus, whether these service times form a renewal process or not depend only on the behaviour of the consecutive service times ($r=1$ in (2.7)). That is, if $\mathbf{Z}$ is a stationary chain and

$\boldsymbol{P}$ is an equilibrium matrix, the sequence $\{S_n ;\ n = 1,2, \dots\}$ is a renewal process if and only if two consecutive intervals are independent. If $m = 1$, the service times form a renewal process by remark (1.2.4). For $m = 2$ and $\boldsymbol{P}$ as an equilibrium matrix (or equivalently $a + b = 1$), example 2c shows a situation where the sequence of service times is a renewal process. Of course, there are cases where $\boldsymbol{P}$ is not an equilibrium matrix but the sequence $\{S_n ;\ n = 1,2, \dots\}$ is a renewal process as our other examples show. ■

2.4 Summary

In this chapter we described the Markov renewal service process that will be used in the next chapters. The structure for the service process discussed here could be used to model systems with changeover where the service depends on the current and the immediately preceeding types of customers. This implies that the sequence of service times is not composed of independent random variables. Expressions for covariance and correlations have been given.

For the special case m = 2 (two types of customers), we discussed conditions on the parameters to obtain a GI service process. Numerical results for the lag r correlations are presented in the case where the services are not independent.

Table (3.1): Correlations for exponential Markov renewal service process .

						service correlations				
a	b	μ_1	μ_2	mean	var	lag-1	lag-2	lag-3	lag-4	lag-5
.6	.6	2	4	.3750	.1719	-.0182	.0036	-.0007	.0001	(-)0+
.6	.3	2	4	.3333	.1389	.0100	.0010	.0001	0+	0+
.6	.05	2	4	.2692	.0814	.0191	.0067	.0023	.0008	.0003
.7	.6	10	6	.1359	.0207	-.0160	.0048	-.0014	.0004	-.0001
.7	.6	15	6	.1205	.0195	-.0382	.0115	-.0034	.0010	-.0003
.7	.1	15	6	.1542	.0260	.0084	.0017	.0830	0+	0+
.1	.1	15	2	.2833	.1742	.2156	.1725	.1380	.1104	.0883
.1	.1	30	1	.5167	.7342	.2546	.2036	.1629	.1303	.1047
.1	.1	100	.5	1.0050	2.9900	.2649	.2119	.1695	.1356	.1085
.9	.9	100	.5	1.0050	2.9900	-.2649	.2119	-.1695	.1356	-.1085

Chapter 3

M/MR/1 QUEUES

3.1 Introduction

In this chapter, we consider a queueing system, denoted by $M/MR/1$, with the following characteristics:

• The arrivals form a Poisson process with a parameter λ, independent of the service process;

• All the arrivals receive service in the order of their arrivals (no loss and FCFS discipline);

• Immediately before a customer enters into the service, his type is chosen from a Markov chain with transition matrix $\boldsymbol{P}$. That is, if the previous customer was type i, the current is type j with probability p_{ij}, $i,j \in E = \{1,2, \ldots ,m\}$. We assume that $\boldsymbol{P}$ is irreducible and aperiodic with unique stationary distribution given by $\boldsymbol{\eta}$ (note that in chapter 2 we only required irreducibility; however, the assumptions in this chapter will be mantained from now on);

- There is only one server;

- The service time distributions are F_{ij} with mean $\bar{s}_{ij}$, where i and j are, respectively, the previous and the current type of customers. Also, we assume that F_{ij} $\forall\, i,j \in E$ is not arithmetic.

As discussed in chapter 2, the service process in this queue could be described as a Markov renewal process with semi Markov kernel

$$\boldsymbol{B}(x) = [p_{ij}\, F_{ij}(x)] \qquad i,j \in E \quad . \tag{1.1}$$

Since $\boldsymbol{P}$ is aperiodic and F_{ij} is not arithmetic, $\boldsymbol{B}(x)$ will be aperiodic.

We define $\bar{s}$ as the unconditional mean service time. From (2.2.5) we have

$$\bar{s} = \boldsymbol{\eta}\, \boldsymbol{B}_1\, \boldsymbol{e}_m \qquad \text{with } \boldsymbol{B}_1 = [\int_0^{\infty} x\, dB_{ij}(x)] = [p_{ij}\, \bar{s}_{ij}\,] \qquad i,j \in E \quad . \tag{1.2}$$

Remark (1.1) : Throughout this paper we assume $\lambda \bar{s} < 1$. That is, the traffic intensity of this queue is less than 1, which implies the existence of a steady state distribution (see Çinlar [1967]). ■

We will study the queue length process embedded at departure epochs. Our interest is to obtain the structure and the pertinent expressions for this queue.

3.2 *Queue length process*

With respect to the queue length process, let

$0 = T_0^d < T_1^d < T_2^d < \cdots$, be a sequence of successives departure times, $n \in N$ and $T_n^d \in R_+$;

N_n^d is the queue length just after T_n^d, $N_n^d \in N$;

Z_n^d is the type of customer that departs at T_n^d, $Z_n^d \in E = \{1, 2, \dots, m\}$;

S_n is the service time of the n-th customer, $S_n \in R_+$.

Denoting $(N^d, Z^d, T^d) = \{(N_n^d, Z_n^d, T_n^d)\,;\, n \in N\}$, we have the following structure for the queue length, embedded at departure times.

Theorem (2.1): The process (N^d, Z^d, T^d), with state space $N \times E$, is a Markov renewal process with kernel $Q(t)$, whose entries are given by

$$Q_{li,kj}(t) = P[N_n^d = k, Z_n^d = j, T_n^d - T_{n-1}^d \le t \mid N_{n-1}^d = l, Z_{n-1}^d = i]$$

$$= \begin{cases} \displaystyle\int_0^t \frac{e^{-\lambda x}(\lambda x)^{k-l+1}}{(k-l+1)!}\, dB_{ij}(x) & \text{for } l > 0 \\ \displaystyle\int_0^t (1 - e^{-\lambda(t-x)})\, \frac{e^{-\lambda x}(\lambda x)^k}{k!}\, dB_{ij}(x) & \text{for } l = 0 \end{cases} \qquad (2.1)$$

where $B_{ij}(x)$ is the element (i, j) in the kernel of the Markov renewal service process.

proof:

To verify that (N^d, Z^d, T^d) is a MRP note:

the successives types $\{Z_n^d ; n \geq 0\}$ form a Markov chain, so Z_n^d only depends on Z_{n-1}^d ;

N_n^d only depends on N_{n-1}^d and on the service time S_n of the n-th customer ;

$T_n^d - T_{n-1}^d$ depends on the service time and on the arrival rate (if the idle time appears);

the service time S_n depends on S_{n-1}, Z_{n-1}^d and Z_n^d .

Then

$$\begin{aligned} &P[N_n^d, Z_n^d, T_n^d - T_{n-1}^d \leq t \mid (N_m^d, Z_m^d, T_m^d ; m = 0,1,...n-1)] = \\ &\quad = P[N_n^d, Z_n^d, T_n^d - T_{n-1}^d \leq t \mid N_{n-1}^d, Z_{n-1}^d] \quad ; \end{aligned}$$

consequently, (N^d, Z^d, T^d) is a Markov renewal process.

To compute the kernel, consider two cases:

Case 1 : $N_{n-1}^d = l > 0$.

In this case $T_n^d - T_{n-1}^d = S_n$, then

$$\begin{aligned} &P[N_n^d = k, Z_n^d = j, T_n^d - T_{n-1}^d \leq t \mid N_{n-1}^d = l, Z_{n-1}^d = i] = \\ &= P[N_n^d = k, Z_n^d = j, S_n \leq t \mid N_{n-1}^d = l, Z_{n-1}^d = i] \\ &= \int_0^t P[N_n^d = k \mid N_{n-1}^d = l, Z_n^d = j, Z_{n-1}^d = i, S_n = x] \\ &\qquad d_x P[Z_n^d = j, S_n \leq x \mid Z_{n-1}^d = i, N_{n-1}^d = l] \\ &= \int_0^t P[N_n^d = k \mid N_{n-1}^d = l, S_n = x] \; d_x P[Z_n^d = j, S_n \leq x \mid Z_{n-1}^d = i] \ . \end{aligned}$$

Then, for $l > 0$,

$$Q_{li,kj}(t) = \int_0^t \frac{e^{-\lambda x}(\lambda x)^{k-l+1}}{(k-l+1)!} \, dB_{ij}(x) \ .$$

Case 2 : $N^d_{n-1} = l = 0$.

In this case $T^d_n - T^d_{n-1} = I_n + S_n$ (I_n: idle time) and, we have

$$\begin{aligned}
&P[N^d_n = k, Z^d_n = j, T^d_n - T^d_{n-1} \le t \mid N^d_{n-1} = 0, Z^d_{n-1} = i] = \\
&= P[N^d_n = k, Z^d_n = j, I_n + S_n \le t \mid N^d_{n-1} = 0, Z^d_{n-1} = i] = \\
&= \int_0^t P[N^d_n = k, Z^d_n = j, S_n \le t - y \mid N^d_{n-1} = 0, Z^d_{n-1} = i] \ d_y P[I_n \le y] = \\
&= \int_0^t \int_0^{t-y} P[N^d_n = k \mid N^d_{n-1} = 0, Z^d_n = j, Z^d_{n-1} = i, S_n = x] \\
&\qquad d_x P[Z^d_n = j, S_n \le x \mid Z^d_{n-1} = i, N^d_{n-1} = 0] \ d_y P[I_n \le y]
\end{aligned}$$

where we used the independence between I_n and S_n, which is true by our assumption of the independence of arrival and service processes; furthermore, since the double integral exists, we can use Fubini's theorem to interchange the order of integration to obtain

$$Q_{0i,kj}(t) = \int_0^t (1 - e^{-\lambda(t-x)}) \ \frac{e^{-\lambda x}(\lambda x)^k}{k!} \ dB_{ij}(x) \ .$$

This completes the proof of the theorem. ■

Remark (2.2): In the proof of the theorem we used a well-known result from M/GI/1 queues; i.e. $P[N^d_n = k \mid N^d_{n-1} = l, S_n = x]$ is the probability of $k - l + 1$ arrivals during the service time x. Since the arrivals form a Poisson process, this probability is equal to $\dfrac{e^{-\lambda x}(\lambda x)^{k-l+1}}{(k-l+1)!}$ $(k \ge l - 1)$. On the other hand,

$P[N_n^d = k \mid N_{n-1}^d = 0, S_n = x]$ is the probability of k arrivals during a service of length x, since one initial arrival is necessary to interrupt the idle time of the server. ■

Corollary (2.3): The process (N^d, Z^d) with state space $N \times E$ is an irreducible and aperiodic Markov chain with one step transition probability matrix $Q(\infty)$ whose entries are given by

$$Q_{li,kj}(\infty) = \begin{cases} \int_0^\infty \dfrac{e^{-\lambda x}(\lambda x)^{k-l+1}}{(k-l+1)!} \, dB_{ij}(x) & \text{for } l > 0 \\ \int_0^\infty \dfrac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB_{ij}(x) & \text{for } l = 0 . \end{cases} \tag{2.2}$$

proof

The result is standard in the theory of Markov renewal processes, which we can see in chapter 10 of Çinlar[1975]. ■

Using a lexicographic order to describe the states (N^d, Z^d) , we could represent $Q(t)$ and $Q(\infty)$ in block matrix form. Define the following $m \times m$ matrices:

$$D_k(t) = \int_0^t (1 - e^{-\lambda(t-x)}) \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x) \quad \text{for } k = 0,1,\dots \quad , \tag{2.3}$$

and

$$C_k(t) = \int_0^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x) \quad \text{for } k = 0,1,\dots \quad . \tag{2.4}$$

Then

$$Q(t) = \begin{bmatrix} D_0(t) & D_1(t) & D_2(t) & D_3(t) & \cdots \\ C_0(t) & C_1(t) & C_2(t) & C_3(t) & \cdots \\ 0 & C_0(t) & C_1(t) & C_2(t) & \cdots \\ 0 & 0 & C_0(t) & C_1(t) & \cdots \\ . & . & 0 & C_0(t) & \cdots \\ . & . & . & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} . \tag{2.5}$$

Analogously, we have

$$Q(\infty) = \begin{bmatrix} C_0 & C_1 & C_2 & C_3 & \cdots \\ C_0 & C_1 & C_2 & C_3 & \cdots \\ 0 & C_0 & C_1 & C_2 & \cdots \\ 0 & 0 & C_0 & C_1 & \cdots \\ . & . & 0 & C_0 & \cdots \\ . & . & . & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} , \tag{2.6}$$

where $C_k = \int_0^\infty \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x)$ for $k = 0,1,\ldots,$ is an $m \times m$ matrix. (2.7)

Remark (2.4) : Note that in the case m = 1, the representations (2.5) and (2.6) became, respectively, the semi-Markov kernel and the embedded Markov chain for M/GI/1 queues (see p. 317 in Çinlar[1975]). For this reason, one can think of (2.5) and (2.6) as matrix versions for the corresponding quantities in the M/GI/1 queue. ∎

Let π be the stationary probability vector of the underlying Markov chain (N^d, Z^d). That is, its components satisfy $0 \le \pi_{k_j} \le 1$ for $\forall k \in N, \forall j \in E$, and π is the solution of the matrix equations

$$\begin{aligned} &(a) \qquad \pi \; Q(\infty) = \pi , \\ &(b) \qquad \pi \; e = 1 \; . \end{aligned} \tag{2.8}$$

We partioned π in the same way as $Q(t)$ and $Q(\infty)$:

$$\pi = (\pi_0, \pi_1, \pi_2, \dots) \;\; , \tag{2.9}$$

where $\quad \pi_k = (\pi_{k_1}, \pi_{k_2}, \dots , \pi_{k_m}) \; .$ $\hfill (2.10)$

Then the equilibrium equations (2.8) become

$$\begin{aligned} &(a) \qquad \pi_n = \pi_0 \; C_n + \sum_{k=1}^{n+1} \pi_k \; C_{n+1-k} \qquad \text{for } n = 0,1, \dots \; , \\ &(b) \quad \pi \; e = 1 \; . \end{aligned} \tag{2.11}$$

Define the following functions

$$\Pi_j(z) = \sum_{n=0}^{\infty} \pi_{n_j} \; z^n \qquad \text{for } |z| \le 1 \;\; , \tag{2.12}$$

and $\Phi_{ij}(z) = \sum_{n=0}^{\infty} C_{n_{ij}} \; z^n \quad \text{for } |z| \le 1 \quad ,$ (2.13)

where $C_{n_{ij}}$ is the element (i, j) of the matrix C_n.

The next theorem introduces a new set of equations. They are equivalent to the equilibrium equations.

Theorem (2.5) : π is a stationary distribution for the Markov chain (N^d, Z^d) if and only if the following set of equations is satisfied for $|z| \le 1$

$$(a) \qquad \Pi_j(z) = \sum_{i=1}^{m} \pi_{0_i} \; \Phi_{ij}(z) \; + \; \sum_{i=1}^{m} \Phi_{ij}(z) \; \left(\frac{\Pi_i(z) - \pi_{0_i}}{z} \right) \quad j = 1,2, \ldots, m \,, \tag{2.14}$$

$$(b) \quad \sum_{j=1}^{m} \Pi_j(1) = 1 \; .$$

proof :

From (2.11-a) we have

$$\pi_{n_j} = \sum_{i=1}^{m} \pi_{0_i} \, C_{n_{ij}} + \sum_{k=1}^{n+1} \sum_{i=1}^{m} \pi_{k_i} \, C_{(n+1-k)_{ij}} \qquad \text{for } j = 1,2,\ldots,m \; \text{ and } \; n = 0,1,2,\ldots.$$

Multiplying by z^n and summing over $n \in N$, we conclude that equation (2.11-a) implies (2.14-a).

To prove the reverse equivalence, note that from (2.14-a) we could obtain the equations (2.11-a) by taking successive derivatives and setting $z=0$. The equivalence between (2.11-b) and (2.14-b) is evident. ■

Theorem (2.6) : The vector $(\pi_{.1}, \pi_{.2}, \dots, \pi_{.m})$ with $\pi_{.j} = \sum_{n=0}^{\infty} \pi_{n_j}$ for $j=1,2,\dots,m$ is the stationary probability vector η for the Markov chain Z^d.

proof :

In (2.14-a) we set $z=1$, since $\Phi_{ij}(1) = p_{ij}$ we obtain m equations of the form

$$\Pi_j(1) = \sum_{i=1}^{m} \left[\pi_{0_i} p_{ij} + \sum_{i=1}^{m} (\Pi_i(1) - \pi_{0_i}) \right] \qquad \text{for } j = 1,2,\dots,m.$$

By (2.12), $\Pi_j(1) = \pi_{.j}$ for $j=1,2,\dots,m$; in matrix notation, with $\Pi(1) = (\pi_{.1}, \pi_{.2}, \dots, \pi_{.m})$, we have

$$\Pi(1)\,(I - P) = 0 \quad \text{or} \quad \Pi(1)\,P = \Pi(1) .$$

Since $\Pi(1)\,e = 1$ by (2.14-b), the theorem is proved since P has a unique stationary distribution η . ■

Note that from our construction, Z^d and (N^d, Z^d) are Markov chains; however, N^d is not. Observe also that, without the information about the types, it is not possible to determine the length of service and consequently the number of arrivals between any two departures. Thus, the knowledge of N^d_{n-1} is not sufficient to compute probabilities for N^d_n .

3.3 *Departure Process*

Definition (3.1): The process $T^d = \{T_n^d \,;\, n \in N\}$ will be called the departure process of the $M/MR/1$ queue. ■

Notice that the structure of the T^d process is not known and we need to use the (N^d, Z^d, T^d) process to have the structure to work it. In this section we initiate the study of the departure process computing joint and one interval distributions. We start with the stationary interdeparture time distribution. In matrix notation we have

$$P[T_n^d - T_{n-1}^d \leq t] = \pi \, Q(t) \, e \ . \tag{3.1}$$

We will work with (3.1) to obtain a simpler expression. The next lemma will be useful.

Lemma (3.2): With the notations of (2.4) and (2.5), we have

$$\sum_{k=0}^{\infty} C_k(t) = B(t) \tag{3.2}$$

and

$$\sum_{k=0}^{\infty} D_k(t) = B(t) - G(t) \tag{3.3}$$

where ,
$$G(t) = \int_0^t e^{-\lambda(t-x)} \, dB(x) \, . \tag{3.4}$$

proof :

From the block matrix form of the kernel $Q(t)$ we have the expressions for the $m \times m$ matrices $C_k(t)$ and $D_k(t)$, k = 0, 1, ... Then ,

$$\sum_{k=0}^{\infty} C_k(t) = \sum_{k=0}^{\infty} \int_o^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x) = \int_o^t \sum_{k=0}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x) = B(t)\,.$$

Also ,

$$\begin{aligned}\sum_{k=0}^{\infty} D_k(t) &= \sum_{k=0}^{\infty} \int_o^t (1 - e^{-\lambda(t-x)}) \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, dB\,(x) \\ &= \int_o^t \sum_{k=0}^{\infty} \frac{e^{-\lambda x}(\lambda x)^k}{k!} (1 - e^{-\lambda(t-x)}) \, dB\,(x) \\ &= B(t) - G(t)\ .\end{aligned}$$

The interchange between sum and integral is justified by Fubini's theorem, since the left hand side of (3.2) and (3.3) exists. ■

Define an infinite column vector of 1's in block form as

$$e = [\ e_m,\ e_m, \cdot, \cdot, \cdot\ ]^T \tag{3.5}$$

where

$$e_m = [\ 1\ , 1\ , \cdot, \cdot, \cdot, \cdot, 1\ ]^T \qquad \text{is an } m \times 1 \text{ vector.} \tag{3.6}$$

Now we return to the simplification of (3.1). The k-th component of $\pi\ Q(t)$ is a 1 × m vector of the form

$$[\ \pi\ Q(t)]_k = \pi_0\, D_{k-1}(t) + \sum_{l=1}^{k} \pi_l\, C_{k-l}(t) \quad \text{for } k = 0,1, \dots\ . \tag{3.7}$$

Then

$$\pi\, Q(t)\, e = \left[\sum_{k=1}^{\infty} \left(\pi_0\, D_{k-1}(t) \;+\; \sum_{l=1}^{k} \pi_l\, C_{k-l}(t) \right) \right] e_m . \tag{3.8}$$

Substituting (3.2) and (3.3) we obtain

$$\pi\, Q(t)\, e = \left[\pi_0\, (\, B(t) - \; G(t)) + \sum_{l=1}^{\infty} \pi_l\, B(t) \right] e_m . \tag{3.9}$$

Finally, using theorem (2.5) we have

$$P[T_n^d - T_{n-1}^d \le t] = [\; \eta\, B(t) - \pi_0\, G(t)]\, e_m . \tag{3.10}$$

Observe that the expression (3.10) operates only with finite square matrices of order m and that the vector π_0 needs to be computed from the equilibrium equations (2.11). Note also that the R.H.S. depends on the equations (2.11) only through π_0 .

We now derive the joint distribution of two consecutive interdeparture times. In matrix notation,

$$P[T_n^d - T_{n-1}^d \le t_1,\, T_{n+1}^d - T_n^d \le t_2] = \; \pi\, Q(t_1)\, Q(t_2)\, e \; . \tag{3.11}$$

As we did for the distribution of a single inter-departure time, we now compute a simpler expression for two consecutive intervals. Using (3.7), we obtain the r-th component of $\pi\, Q(t_1)\, Q(t_2)$,

$$[\,\pi\, Q(t_1)\, Q(t_2)]_r = [\pi_0\, D_0(t_1) + \pi_1\, C_0(t_1)]\, D_{r-1}(t_2) +$$

$$+ \sum_{k=2}^{r+1} \left[\pi_0\, D_{k-1}(t_1) + \sum_{l=1}^{k} \pi_l\, C_{k-l}(t_1) \right] C_{r+1-k}(t_2) \quad . \tag{3.12}$$

Then

$$\pi\, Q(t_1)\, Q(t_2)\, e =$$

$$\left[\sum_{r=1}^{\infty} \left\{ [\pi_0\, D_0(t_1) + \pi_1\, C_0(t_1)] D_{r-1}(t_2) + \sum_{k=2}^{r+1} \left[\pi_0\, D_{k-1}(t_1) + \sum_{l=1}^{k} \pi_l\, C_{k-l}(t_1) \right] C_{r+1-k}(t_2) \right\} \right] e_m \quad . \tag{3.13}$$

This can be simplified to

$$\pi\, Q(t_1)\, Q(t_2)\, e =$$

$$[\,\eta\, B(t_1)\, B(t_2) - \pi_0\, (D_0(t_1)\, G(t_2) + G(t_1)\, B(t_2)) - \pi_1\, C_0(t_1)\, G(t_2)]\; e_m \; . \tag{3.14}$$

From (3.14) we have the joint distribution of two consecutive intervals, using only $m \times m$ matrices. However, we need to know π_0 and π_1.

Remark (3.3) : Writing the equilibrium equation (2.11-a) for n = 0, we have

$$\pi_0 = \pi_0 C_0 + \pi_1 C_0 \qquad \text{then} \quad \pi_1 C_0 = \pi_0\, [I - C_0] \quad .$$

Then if the matrix C_0 has an inverse, we have

$$\pi_1 = \pi_0\, [I - C_0]\, C_0^{-1} \; .$$

Consequently, it is possible to write all $\pi_n(n \geq 1)$ as a function of π_0 only. In M/GI/1 queues this recursive relation also appears; however, no requirements about the existence of inverses are necessary. ∎

With an approach similar to that used here and in chapter 2, it is possible to obtain the joint distribution of two intervals separated by r inter-departures times. The expression is given by

$$P[T_n^d - T_{n-1}^d \leq t_1, T_{n+r}^d - T_{n+r-1}^d \leq t_2] = \pi\, Q(t_1)\, Q(\infty)^{r-1} Q(t_2)\, e \quad , (3.15)$$

for r = 1,2, ... and the convention $Q(\infty)^0 = I$.

As expected, the derivation of a simpler expression for (3.14) in terms of $m \times m$ matrices will use $\pi_0, \pi_1, \dots, \pi_{r+1}$. We will not develop this because the final expression is complicated and it is not useful for computation. However, from (2.11) each $\pi_j, j = 1,2, \dots, r+1$ can, in principle, be obtained from π_0. So, these joint distributions are all determined once π_0 is known. Notice that since $Q(\infty)$ is never an equilibrium matrix, the intervals $T_n^d - T_{n-1}^d$, $n = 1,2, \dots$, cannot satisfy remark (2.3.2). Thus, whether these intervals are independent or not requires further investigation, which we give in chapter 4.

3.4 Summary

In this chapter we introduced the M/MR/1 queue with infinite waiting room and FCFS discipline.

We developed the structure to deal with questions about queue length and interdeparture times. We defined the departure process and, in particular, we computed the distributions of a single departure interval and the joint distribution of two consecutive intervals using only finite matrices. Moreover, we investigated the relationship between the stationary distributions of Z^d and (N^d, Z^d). The chapter ended with a discussion of the lag r joint distributions.

In the next chapter we will apply this structure to the special case of exponentially distributed service times and, taking advantage of this, we will discuss the departure process in more detail.

Chapter 4

$M/M^{ij}/1$ QUEUES

4.1 *Introduction*

In this chapter, we maintain the assumptions listed in the beginning of chapter 3 and we consider a special Markov renewal service process where the distribution F_{ij} is exponential with parameter μ_{ij} . Throughout the chapter, we obtain simplified expressions that help us to explore the stationary departure process, T^d, in more detail. Our main result is the establishment of necessary and sufficient conditions for the departure process to be a renewal process. As we will discuss in remark (3.4), except in special cases, the semi-Markov process associated with the service process is not a Markov process. Furthermore, for the (N^d, Z^d, T^d) process, the associated semi-Markov will not be, in any case, a Markov process.

4.2 The queue length process

The kernel $B(t)$ for the Markov renewal service process is given by

$$B(t) = [B_{ij}(t)]_{m \times m} = \left[p_{ij}\ (1 - e^{-\mu_{ij}t})\right]_{m \times m} . \tag{2.1}$$

From (3.1.2) we obtain, in scalar form, the mean service time

$$\bar{s} = \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i p_{ij} \frac{1}{\mu_{ij}} . \tag{2.2}$$

We will assume

$$\mu_{ij} > \lambda \quad \text{for } i,j = 1,2, \dots , m \tag{2.3}$$

implying that $\lambda \bar{s} < 1$, so by remark (3.1.1), the stationary distribution of the queue length exists.

For future use, we obtain a polynomial expression for the inter-departure times in steady state. Recall from (3.3.10) the expression

$$P[T_n^d - T_{n-1}^d \leq t] = [\ \eta\ B(t) - \pi_0\ G(t)]\ e_m ,$$

where now, from (3.3.4)

$$G(t) = \left[\frac{p_{ij}\,\mu_{ij}}{\mu_{ij} - \lambda} \left(e^{-\lambda t} - e^{-\mu_{ij}t}\right)\right]_{m \times m} . \tag{2.4}$$

After some algebric manipulation, we obtain

$$P[T_n^d - T_{n-1}^d \leq t] = 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} \left[\frac{\pi_{0_i} \mu_{ij}}{\mu_{ij} - \lambda} \right] e^{-\lambda t} + + \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} \left[\frac{\pi_{0_i} \mu_{ij}}{\mu_{ij} - \lambda} - \eta_i \right] e^{-\mu_{ij} t} \ . \tag{2.5}$$

The joint distribution of two consecutive intervals was computed earlier in chapter 3, and the final expression is given by (3.3.14), which we reproduce below

$$\pi\, Q(t_1)\, Q(t_2)\, e =$$
$$[\ \eta\, B(t_1)\, B(t_2) - \pi_0(D_0(t_1)\, G(t_2) + G(t_1)\, B(t_2)) - \pi_1 C_0(t_1)\, G(t_2)]\ e_m$$

where ,

$$C_0(t_1) = \left[\frac{p_{ij}\, \mu_{ij}}{\lambda + \mu_{ij}} \left(1 - e^{-(\lambda + \mu_{ij})t_1}\right) \right]_{m \times m} \tag{2.6}$$

$$D_0(t_1) = \left[\frac{p_{ij}}{\lambda + \mu_{ij}} \left(\mu_{ij} - (\lambda + \mu_{ij})\, e^{-\lambda t_1} + \lambda\, e^{-(\lambda + \mu_{ij})t_1}\right) \right]_{m \times m} \tag{2.7}$$

by using expressions (3.2.4) and (3.2.3).

To obtain a polynomial expression for the joint distribution of two consecutive intervals, we use (2.1), (2.4), (2.6) and (2.7) to perform the matrix multiplications. After tedious computation, we obtain

$$
\begin{aligned}
P[T_n^d - T_{n-1}^d \le t_1, T_{n+1}^d - T_n^d \le t_2] = \quad & 1 + \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\left(\pi_{0_i}\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i\right) e^{-\mu_{ij}t_1} \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} p_{ij}\,p_{jk}\left[\frac{\mu_{ij}}{(\mu_{ij}+\lambda)}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}\,(\pi_{0_i}+\pi_{1_i}) - \eta_i\right] e^{-\mu_{jk}t_2} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i}\,p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\,e^{-\lambda t_1} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} p_{ij}p_{jk}\frac{\mu_{ij}}{(\mu_{ij}+\lambda)}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}\,(\pi_{0_i}+\pi_{1_i})\,e^{-\lambda t_2} \qquad (2.8) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \pi_{0_i}\,p_{ij}p_{jk}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}\;e^{-\lambda(t_1+t_2)} \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \pi_{0_i}\,p_{ij}p_{jk}\left(\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \frac{\mu_{jk}}{(\mu_{jk}-\lambda)}\right) e^{-\lambda t_1 - \mu_{jk}t_2} \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{p_{ij}p_{jk}}{(\mu_{ij}+\lambda)}\;\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}\,(\pi_{1_i}\mu_{ij} - \pi_{0_i}\lambda)\;e^{-\mu_{ij}t_1 - \lambda(t_1+t_2)} \\
& + \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} p_{ij}p_{jk}\left(\eta_i - \pi_{0_i}\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\right) e^{-\mu_{ij}t_1 - \mu_{jk}t_2} \;.
\end{aligned}
$$

Note that we have 8 different exponential terms plus a constant term.

Example (2.1): Consider the case where m = 1, that is, we have just one type of customer. In this case, notice that $\boldsymbol{P}$ and $\boldsymbol{\eta}$ are reduced to the scalar 1 and our queue becomes an ordinary M/M/1 . We would expect, as shown in Disney et al. [1973], that (2.8) is the product of two exponential (λ) distributions. Performing the operations suggested in (2.8) we find that indeed this is true. Thus, we have a partial confirmation of our results so far.

4.3 Conditions for equivalence between the departure process and a renewal process

A necessary condition for a stationary process to be a renewal process is that two consecutive intervals be independent. The condition is not sufficient but as we will see, it is the starting point to obtain necessary and sufficient conditions for the equivalence between the (N^d, Z^d, T^d) process and a renewal process.

Theorem (3.1): If two consecutive intervals in the departure process are independent then $\mu_{ij} = \mu$ for any i,j = 1,2,...,m.

proof :

Using (2.5), we obtain

$$
\begin{aligned}
&P[T_n^d - T_{n-1}^d \le t_1] \;\; P[T_{n+1}^d - T_n^d \le t_2] = \qquad 1 \;+ \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i\,)\;\, e^{-\,\mu_{ij}t_1} \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i\,)\;\, e^{-\,\mu_{ij}t_2} \\
&- \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i} p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\;\, e^{-\lambda t_1} \\
&- \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i} p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\;\, e^{-\lambda t_2} \\
&+ \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i} p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\Big)^2\; e^{-\lambda(t_1\,+\,t_2)} \\
&- \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i} p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\Big)\Big(\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i)\;\, e^{-\lambda t_1\,-\,\mu_{ij}t_2}\Big) \\
&- \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} \pi_{0_i} p_{ij}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\Big)\Big(\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i)\;\, e^{-\,\mu_{ij}t_1\,-\lambda t_2}\Big) \\
&+ \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i\,)\, e^{-\,\mu_{ij}t_1}\Big) \\
&\qquad\qquad \Big(\sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}\,(\,\pi_{0_i}\,\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} - \eta_i\,)\, e^{-\,\mu_{ij}t_2}\Big)\;\; .
\end{aligned}
\tag{3.1}
$$

By hypothesis we are assuming that, independent of n,

$$P[T_n^d - T_{n-1}^d \leq t_1, T_{n+1}^d - T_n^d \leq t_2] = P[T_n^d - T_{n-1}^d \leq t_1]\, P[T_{n+1}^d - T_n^d \leq t_2] \,. \quad (3.2)$$

The left hand side of (3.2) was given in its polynomial form by (2.8). Comparing the polynomial expressions in both sides of (3.2), we note that the constant term and the coefficients of $e^{-\mu_{ij}t_1}$ and $e^{-\lambda t_1}$ are equal on both sides. Consequently they are satisfied, independent of any other condition. By identifying the coefficients of the other terms we obtain the following conditions:

Condition I : coefficient of $e^{-\mu_{rs}t_2}$

$$\sum_{i=1}^{m} p_{ir}p_{rs}\left[\frac{\mu_{ir}}{(\mu_{ir}+\lambda)}\frac{\mu_{rs}}{(\mu_{rs}-\lambda)}(\pi_{0_i}+\pi_{1_i}) - \eta_i\right] = p_{rs}\left[\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs}-\lambda)} - \eta_r\right] \quad (3.3)$$

for r,s $= 1,2,\ldots,$ m .

Condition II : coefficient of $e^{-\lambda t_2}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} p_{ij}p_{jk}\frac{\mu_{ij}}{(\mu_{ij}+\lambda)}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}(\pi_{0_i}+\pi_{1_i}) = \sum_{i=1}^{m}\sum_{j=1}^{m}\pi_{0_i}p_{ij}\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} \,. \quad (3.4)$$

Condition III : coefficient of $e^{-\lambda(t_1+t_2)}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\pi_{0_i}p_{ij}p_{jk}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)} = \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_{0_i}p_{ij}\frac{\mu_{ij}}{(\mu_{ij}-\lambda)}\right)^2 \,. \quad (3.5)$$

Condition IV : coefficient of $e^{-\lambda t_1 - \mu_{rs}t_2}$

$$\sum_{i=1}^{m} \pi_{0_i} p_{ir} p_{rs} \left(\frac{\mu_{ir}}{(\mu_{ir} - \lambda)} - \frac{\mu_{rs}}{(\mu_{rs} - \lambda)} \right) = - p_{rs} \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \pi_{0_i} p_{ij} \frac{\mu_{ij}}{(\mu_{ij} - \lambda)} \right) \left(\frac{\pi_{0_r} \mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) \tag{3.6}$$

for r,s = 1,2,..., m .

Condition V : coefficient of $e^{-\mu_{rs}t_1 - \lambda t_2}$

$$0 = - p_{rs} \left(\frac{\pi_{0_r} \mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) \sum_{i=1}^{m} \sum_{j=1}^{m} \pi_{0_i} p_{ij} \frac{\mu_{ij}}{(\mu_{ij} - \lambda)} \tag{3.7}$$

for r,s = 1,2,..., m .

Condition VI : coefficient of $e^{-\mu_{rs}t_1 - \lambda(t_1 + t_2)}$

$$\frac{p_{rs}}{(\mu_{rs} + \lambda)} \sum_{k=1}^{m} p_{sk} \frac{\mu_{sk}}{(\mu_{sk} - \lambda)} (\pi_{1_r} \mu_{rs} - \pi_{0_r} \lambda) = 0 \tag{3.8}$$

for r,s = 1,2,..., m .

Condition VII : coefficient of $e^{-\mu_{rs}t_1 - \mu_{sk}t_2}$

$$- p_{rs} p_{sk} \left(\frac{\pi_{0_r} \mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) = p_{rs} p_{sk} \left(\frac{\pi_{0_r} \mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) \left(\frac{\pi_{0_s} \mu_{sk}}{(\mu_{sk} - \lambda)} - \eta_s \right) \tag{3.9}$$

for r,s,k = 1,2,..., m .

Condition VIII : coefficient of $e^{-\mu_{rs}t_1 - \mu_{jk}t_2}$ for $s \neq j$

$$0 = p_{rs}\, p_{jk} \left(\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) \left(\frac{\pi_{0_j}\, \mu_{jk}}{\mu_{jk} - \lambda} - \eta_j \right) \tag{3.10}$$

for r,s,j,k = 1,2,..., m .

We start by looking at condition V. For fixed i, there is at least one j such that $p_{ij} > 0$, then

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \pi_{0_i} p_{ij} \frac{\mu_{ij}}{(\mu_{ij} - \lambda)} > 0$$

since we are assuming $\mu_{ij} > \lambda$ for any i,j = 1,2,..., m .

Then

$$p_{rs} \left(\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r \right) = 0 \qquad \text{for } r,s = 1,2, \ldots , m.$$

For arbitrary fixed r , by the irreducibility of P, we must have at least one $s \neq r$ such that $p_{rs} > 0$. This is accomplished whether or not $p_{rr} = 0$. Then for any s such that $p_{rs} > 0$, we have

$$\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r = 0 \quad .$$

Then

$$\pi_{0_r} = \eta_r (1 - \rho_{rs}) \quad \text{with} \quad \rho_{rs} = \frac{\lambda}{\mu_{rs}} \ . \tag{3.11}$$

If there is more than one type s such that $p_{rs} > 0$, say $s_1, s_2, \dots, s_n$ for $2 \le n \le m$, then by (3.11)

$$\rho_{rs_1} = \rho_{rs_2} = \cdots = \rho_{rs_n} \ .$$

Consequently ,

$$\mu_{rs_1} = \mu_{rs_2} = \cdots = \mu_{rs_n} \ . \tag{3.12}$$

If for a given r there exists just one s such that $p_{rs} > 0$ (in fact $p_{rs} = 1$), then we could represent μ_{rs} by μ_r since s is unique in this case. In any case the service rate μ_{rs} will not depend on s. To summarize the conclusions of condition V,

$$\mu_{rs} = \mu_r \qquad \text{for any} \ r,s = 1,2, \dots, m \ ; \tag{3.13}$$

$$\pi_{0_r} = \eta_r (1 - \rho_r) \qquad \text{for any} \ r = 1,2, \dots, m \ . \tag{3.14}$$

From condition VI we have

$$(\pi_{1_r}\mu_{rs} - \pi_{0_r}\lambda) \ p_{rs} \left(\frac{1}{(\mu_{rs} + \lambda)} \sum_{k=1}^{m} p_{sk} \frac{\mu_{sk}}{(\mu_{sk} - \lambda)} \right) = 0 \ .$$

The last term is obviously positive. For arbitrary fixed r there exists at least one s with $p_{rs} > 0$. For that s, the condition VI becomes

$$\pi_{1_r}\mu_{rs} - \pi_{0_r}\lambda = 0 \ . \tag{3.15}$$

Under (3.13), the last expression is

$$\pi_{1_r}\mu_r - \pi_{0_r}\lambda = 0 \tag{3.16}$$

or

$$\pi_{1_r} = \pi_{0_r}\rho_r \qquad \text{for } r = 1,2, \dots, m \ . \tag{3.17}$$

From condition I, in (3.3), we have, after using (3.13) and (3.17)

$$\sum_{i=1}^{m} p_{ir}p_{rs}\left(\frac{\mu_i\mu_r\,\pi_{0_i}(1+\rho_i)}{(\mu_i+\lambda)\,(\mu_r-\lambda)} - \eta_i \right) = p_{rs}\left(\frac{\pi_{0_r}\mu_r}{(\mu_r-\lambda)} - \eta_r \right)$$

$$\frac{\mu_r}{(\mu_r-\lambda)}\, p_{rs}\sum_{i=1}^{m}\pi_{0_i}p_{ir} - p_{rs}\sum_{i=1}^{m}\eta_i p_{ir} = p_{rs}\left(\frac{\pi_{0_r}\mu_r}{(\mu_r-\lambda)} - \eta_r \right)$$

$$\frac{p_{rs}}{(1-\rho_r)}\sum_{i=1}^{m}\pi_{0_i}p_{ir} - p_{rs}\eta_r = p_{rs}\left(\frac{\pi_{0_r}\mu_r}{(\mu_r-\lambda)} - \eta_r \right) .$$

Considering only the relevant cases where $p_{rs} > 0$, we obtain, after some cancellation, the equation

$$\sum_{i=1}^{m}\pi_{0_i}p_{ir} = \pi_{0_r} \quad \text{for } r = 1,2, \dots, m \tag{3.18}$$

or in matrix form ,

$$\pi_0 \, P = \pi_0 \; . \tag{3.19}$$

Since P was assumed to be finite, irreducible, and aperiodic, it follows by theorem (7.12) on page 72 in Hunter [1983b] that the vector π_0 must be a scalar multiple of the vector η. From (3.14), we have

$$1 - \rho_r = c \qquad \text{for } r = 1,2, \dots, m \; .$$

Consequently ,

$$\mu_r = \mu \qquad \text{for } r = 1,2, \dots, m \; . \tag{3.20}$$

Under (3.14), (3.17) and (3.20) all the other conditions become identities and the theorem is proved. ■

Theorem (3.2) : The process (N^d, Z^d, T^d) in the $M/M^{ij}/1$ queue is equivalent to a renewal process if and only if $\mu_{ij} = \mu$ for any i,j = 1,2,..., m .

proof :

1st part : renewal process $\rightarrow$ $\mu_{ij} = \mu \quad i,j \in E.$

If the sequence $\{ \, T_n^d \, ; n \geq 0 \, \}$ forms a renewal process, the inter-departure times are independent and identically distributed. In particular, two consecutive intervals must be independent, so by theorem (3.1) this implies $\mu_{ij} = \mu$ for $i,j \in E.$

2nd part : $\mu_{ij} = \mu \quad i,j \in E$ $\rightarrow$ renewal process.

By hypothesis, the sequence of service times are independent and identically exponential distributed random variables. Since the sequence of departure times does not mark the type of customer that is leaving, we could consider, for the study of departure times, the queue without types. In doing so, we have Poisson arrivals with no loss and exponential service in order of arrival. According to Disney et al. [1973], the departure process is a Poisson process which is a special case of a renewal process. ∎

Corollary (3.3) : The renewal process in theorem (3.2) is a Poisson process.

proof :

The second part of the proof of the theorem could be used to prove this corollary. However, as an alternative proof, we find out that the distribution of one interval in the departure process is exponential; this is all we need since theorem (3.2) has estabilished the i.i.d. property.

From (2.5)

$$P[T_n^d - T_{n-1}^d \leq t] = 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} \left[\frac{\pi_{0_i}\mu}{\mu - \lambda} \right] e^{-\lambda t} + \\ + \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij} \left[\frac{\pi_{0_i}\mu}{\mu - \lambda} - \eta_i \right] e^{-\mu t} \quad .$$

Since $\dfrac{\pi_{0_i}\mu}{(\mu - \lambda)} = \eta_i$, by (3.11) and (3.20), we obtain

$$
\begin{aligned}
P[T_n^d - T_{n-1}^d \le t] &= 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i p_{ij} e^{-\lambda t} \\
&= 1 - \sum_{j=1}^{m}\left[\sum_{i=1}^{m} \eta_i p_{ij}\right] e^{-\lambda t} \\
&= 1 - \sum_{j=1}^{m} \eta_j e^{-\lambda t} \\
&= 1 - e^{-\lambda t} .
\end{aligned}
$$

Then the departure process is a renewal process with exponential distribution, that is, it is a Poisson process. ■

Remark (3.4): From Çinlar [1975] (p. 316), if the semi-Markov process associated with a Markov renewal process is a regular Markov process, then the transition between any two states a and b , in the MRP, is given by

$$
Q_{ab}(t) = p_{ab}\,(1 - e^{-\lambda(a)t}) \quad , \tag{3.21}
$$

where p_{ab} is the element (a,b) in the transition matrix of a Markov chain and $\lambda(a)$ is a function of state a.

Let $Y = \{Y(t)\,;\, t \ge 0\}$ be the semi-Markov process associated with the service process (Z, S) . We have $Y(t) = Z_n$ for $\sum_{l=1}^{n} S_l \le t < \sum_{l=1}^{n+1} S_l$. That is, $Y(t)$ is the type of the last customer that completed service on or before t.

In the special case $\mu_{ij} = \mu_i\,, \forall\ i,j \in E$, the Markov renewal service kernel becomes

$$
B_{ij}(x) = p_{ij}(1 - e^{-\mu_i x}) \ , \ \forall \ i,j \in E . \tag{3.22}
$$

Then Y is a Markov process. It seems reasonable that the marginal (Z^d, T^d) process will then be a Markov process, but unless $\mu_{ij} = \mu$, T^d will not be renewal since it depends on Z^d. The (N^d, Z^d, T^d) process will still be fully Markov renewal. A proof of this conjecture relies on $\pi_{0_i} = \eta_i(1 - \frac{\lambda}{\mu_i})$, $i \in E$, a result we have been unable to prove (see chapter 7, section 2).

If $\mu_{ij} = \mu_j$, $\forall\ i,j \in E$, the transition in the Markov renewal service process will depend on j which is the next state in the Y process. The Markovian property for the Y process will not hold in this case.

The semi-Markov process associated with (N^d, Z^d, T^d) is not Markov in any case. To verify this, we use (3.2.1) to compute the transition between the states (1,1) and (1,2). We obtain

$$Q_{11,12}(t) = \frac{\lambda p_{ij}\mu_{ij}}{(\lambda + \mu_{ij})^2}\left[1 - e^{-(\lambda + \mu_{ij})t} - (\lambda + \mu_{ij})te^{-(\lambda + \mu_{ij})t}\right]. \tag{3.23}$$

The third term (in t) will never be zero for finite t, and the kernel $Q(t)$ could not be expressed in the factored form mentioned in (3.21). None the less, if $\mu_{ij} = \mu$, $\forall\ i,j \in E$ (so that Z^d need not be considered in the service distribution) even though $Q_{11,12}(t)$ has the above form, the (N^d, Z^d, T^d) process is equivalent to a renewal process and even to a Poisson process as shown in theorem (3.2).

In this way, the process (N^d, Z^d, T^d) becomes equivalent to a simpler process only when $\mu_{ij} = \mu$, $\forall i,j \in E$. In all of the other cases, we need to deal with the full Markov renewal structure to study the departure process. ■

4.4 *Summary*

In this chapter we particularized our queueing model to an exponential service time with dependence between the previous and the current type of customer. In this way, the Markov renewal service process is characterized by the kernel $\boldsymbol{B}(x) = [p_{ij}\, F_{ij}(x)]$, where F_{ij} is exponential (μ_{ij}) and $\boldsymbol{P}$ is assumed to be finite, irreducible and aperiodic.

We discussed the conditions for the (N^d, Z^d, T^d) process to be equivalent to a renewal process. We proved that this equivalence occurs if and only if $\mu_{ij} = \mu$. That is, marginally, the departure process T^d is a renewal process if and only if $\mu_{ij} = \mu$. The long proof was based on the identification of coefficients in similar terms of polynomial expressions. We also proved that if the departure process is renewal, then it must be a Poisson process with parameter λ.

Thus, our example of the service time including changeover times, as mentioned in chapter 2, indicates that even in the case where service times are exponentially distributed but depend on the changeover from type i to j, the departure process of all customers is a Markov renewal process and is never renewal unless $\mu_{ij} = \mu$, in which case the departure process is a Poisson process. Of course, this latter case is tantamount to saying that there are no changeover times. All customers are of the same type as far as the server is concerned. Hence, the queue acts as if it were a simple $M/M/1$ queue. Things are somewhat different if we look at the departure stream for each type of customer as we will show in the next chapter.

Chapter 5

STUDY OF TYPE-DEPARTURE PROCESS

5.1 Introduction

Our interest in this chapter is the study of departure processes for each type of customer and the relationship among them. The process of interest is $(Z^d, T^d) = \{(Z_n^d, T_n^d) ; n \in N\}$, however, its structure is not known. One approach is to study a larger process, (N^d, Z^d, T^d) , where we have the Markov renewal structure to work with, but we need to deal with an infinite dimensional process. In sections 2, 3 and 4 we study the type-departure processes in $M/M^{ij}/1$ queues. In section 2, instead of working with (N^d, Z^d, T^d) , we construct a finite dimensional Markov renewal process and discuss under what conditions the two processes would be equivalent. Necessary and sufficient conditions are given. Section 3 is devoted to describing properties of each type-departure process. In section 4, the equivalence property proved in section 2 is used to study in detail the dependence among the type-departure processes. In particular, we present expressions to measure the dependence among the type-counting processes. Numerical examples and graphics illustrate the influence of the different parameters. The chapter closes with section 5, which gives results for a general Markov renewal service process.

5.2 *Equivalence*

Recall from chapters 3 and 4 that the Markov renewal process (N^d, Z^d, T^d) in the $M/M^{ij}/1$ has semi Markov kernel $Q(t)$ whose elements are

$$Q_{il,jk}(t) = P[N_n^d = k, Z_n^d = j, T_n^d - T_{n-1}^d \leq t \mid N_{n-1}^d = l, Z_{n-1}^d = i]$$

$$= \begin{cases} \int_0^t \dfrac{e^{-\lambda x}(\lambda x)^{k-l+1}}{(k-l+1)!} \; p_{ij}\,\mu_{ij}\, e^{-\mu_{ij}x}\, dx & \text{for } l > 0 \\ \int_0^t (1 - e^{-\lambda(t-x)}) \; \dfrac{e^{-\lambda x}(\lambda x)^k}{k!} \; p_{ij}\,\mu_{ij}\, e^{-\mu_{ij}x}\, dx & \text{for } l = 0 \;. \end{cases} \tag{2.1}$$

Assuming stationary conditions, that is,

$$\begin{aligned} &\text{(a)} \quad P[N_n^d = l, Z_n^d = i] = \pi_{l_i} \quad l \in N, i \in E \;, \\ &\text{(b)} \qquad\quad P[Z_n^d = i] = \eta_i \quad i \in E \;, \end{aligned} \tag{2.2}$$

we marginal out the queue length in (2.1) to obtain, for $i,j \in E$ and $t \geq 0$,

$$K_{ij}(t) = P[Z_n^d = j\,,\, T_n^d - T_{n-1}^d \le t \mid Z_{n-1}^d = i]$$
$$= p_{ij}\,[\,1 - a_{ij}\,e^{-\lambda t} - (1 - a_{ij})\,e^{-\mu_{ij} t}\,]\,, \tag{2.3}$$

where $$a_{ij} = \frac{\pi_{0_i}\,\mu_{ij}}{\eta_i\,(\mu_{ij} - \lambda)} > 0\,. \tag{2.4}$$

Theorem (2.1) : For $0 < a_{ij} \le 1$, $i,j \in E$, the matrix $K(t) = [K_{ij}(t)]$ is a kernel for some Markov renewal process with a state space E .

proof:

We need to verify that $K_{ij}(t)$ satisfies all the properties of a kernel:

i) $K_{ij}(t) \ge 0$, $t \ge 0$;

ii) $K_{ij}(t)$ is monotone non decreasing ;

iii) $\sum_{j \in E} K_{ij}(\infty) = 1$.

Expression (2.3) could be re-written as

$$K_{ij}(t) = p_{ij}\,[\,a_{ij}\,(1 - e^{-\lambda t}) + (1 - a_{ij})\,(1 - e^{-\mu_{ij} t})]\,. \tag{2.5}$$

Then, for $0 < a_{ij} \le 1$, (i) and (ii) follow easily since $\lambda > 0$ and $\mu_{ij} > 0$. To prove (iii), note that

$$K_{ij}(\infty) = p_{ij}\,, \quad \text{consequently} \quad K(\infty) = P\,. \;\blacksquare \tag{2.6}$$

Observe the difference between $K(t)$ and the kernel for the Markov renewal service process given by $B(t)$. The queue length influences $K(t)$ through π_0 in the expression for a_{ij}. In other words, if a zero queue length is left behind by a departure, then the next interdeparture interval is the sum of an idle time plus a service time. Expression (2.5) reflects this situation by weighting the arrival and service distributions by a factor which depends on the stationary probability of zero customers in the system.

Definition (2.2) : The Markov renewal process (Z^*,T^*) with state space E and semi Markov kernel $K(t) = [K_{ij}(t)]$, $i,j \in E$, is called the one-step projection of the process (N^d,Z^d,T^d) . ■

According to our construction, the probabilistic structure of (Z^*,T^*) comes from the one step transition in the (N^d,Z^d,T^d) process. The Markov chains (Z^d) and (Z^*) have the same state space E and transition matrix P. Assuming that they have the same initial distribution, we see that they will be the same process and have the same stationary distribution η. However, (Z^*,T^*) and (Z^d,T^d) are different processes. The latter is the projection of (N^d,Z^d,T^d) after each transition, in this way all its properties are connected to the original process. On the other hand, the process (Z^*,T^*) , once the kernel $K(t)$ is computed, proceeds independent of (N^d,Z^d,T^d) .

An interesting question to answer is under what conditions are the two Markov renewal processes (N^d,Z^d,T^d) and (Z^*,T^*) equivalent. In other words, we would like to know when we could substitute one process for the other in order to study properties related to the interdeparture times. When this interchange is possible, we have the advantage of working with a finite dimensional process

(Z^*,T^*) instead of an infinite dimensional one. The next theorems will address this question.

Theorem (2.3): The processes (Z^*,T^*) and (N^d,Z^d,T^d) have the same inter-departure distributions.

proof:

For the process $(Z^*,T^*) = \{Z_n^*, T_n^*; n \in N\}$ we have

$$\begin{aligned}
P[T_n^* - T_{n-1}^* \le t] &= \eta\, K(t)\, e \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i p_{ij} [\, 1 - a_{ij}\, e^{-\lambda t} - (1 - a_{ij})\, e^{-\mu_{ij} t}] \\
&= 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i p_{ij} [a_{ij}\, e^{-\lambda t} + (1 - a_{ij})\, e^{-\mu_{ij} t}] \\
&= P[T_n^d - T_{n-1}^d \le t]
\end{aligned}$$

by using the value of a_{ij} and expression (4.2.5) . ■

From theorem (2.3) we note that single intervals have the same distribution in (Z^*,T^*) and (N^d,Z^d,T^d) . Consequently, any quantity computed from this distribution will be the same for the two processes. This could give someone the false idea that the processes are identical, as we will see, this is not the case unless some additional assumptions are made.

The joint distribution of two consecutive intervals in the (Z^*,T^*) process is given, in polynomial form, by

$$P[T_n^* - T_{n-1}^* \le t_1 , T_{n+1}^* - T_n^* \le t_2] = \eta \, K(t_1) \, K(t_2) \, e$$

$$= 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i \, p_{ij} \, (1 - a_{ij}) \, e^{-\mu_{ij} t_1}$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, (1 - a_{jk}) \, e^{-\mu_{jk} t_2}$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_i \, p_{ij} \, a_{ij} \, e^{-\lambda t_1}$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, a_{jk} \, e^{-\lambda t_2} \qquad (2.7)$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, a_{ij} \, a_{jk} \, e^{-\lambda (t_1 + t_2)}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, a_{ij} \, (1 - a_{jk}) \, e^{-\lambda t_1 - \mu_{jk} t_2}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, (1 - a_{ij}) \, a_{jk} \, e^{-\mu_{ij} t_1 - \lambda t_2}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \eta_i \, p_{ij} \, p_{jk} \, (1 - a_{ij}) \, (1 - a_{jk}) \, e^{-\mu_{ij} t_1 - \mu_{jk} t_2} ,$$

where a_{ij} with $i,j \in E$ was given in (2.4).

Theorem (2.4): If the processes (N^d, Z^d, T^d) and (Z^*, T^*) have the same joint distribution of two consecutive intervals then $\mu_{ij} = \mu$, $i,j \in E$.

proof :

Equating (4.2.8) to (2.7) above, the following conditions must be satisfied:

Condition I : coefficient of $e^{-\mu_{rs}t_2}$

$$\sum_{i=1}^{m} p_{ir}p_{rs}\left[\frac{\mu_{ir}}{(\mu_{ir}+\lambda)}\frac{\mu_{rs}}{(\mu_{rs}-\lambda)}(\pi_{0_i}+\pi_{1_i})-\eta_i\right]=p_{rs}\left[\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs}-\lambda)}-\eta_r\right] \tag{2.8}$$

for r,s = 1,2,..., m .

Condition II : coefficient of $e^{-\lambda t_2}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} p_{ij}p_{jk}\frac{\mu_{ij}}{(\mu_{ij}+\lambda)}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}(\pi_{0_i}+\pi_{1_i})=\sum_{i=1}^{m}\sum_{j=1}^{m}\pi_{0_i}p_{ij}\frac{\mu_{ij}}{(\mu_{ij}-\lambda)} \tag{2.9}$$

Condition III : coefficient of $e^{-\lambda(t_1+t_2)}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\pi_{0_i}p_{ij}p_{jk}\frac{\mu_{jk}}{(\mu_{jk}-\lambda)}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\eta_i p_{ij}p_{jk}\left(\frac{\pi_{0_i}\mu_{ij}}{\eta_i(\mu_{ij}-\lambda)}\right)\left(\frac{\pi_{0_j}\mu_{jk}}{\eta_j(\mu_{jk}-\lambda)}\right). \tag{2.10}$$

Condition IV : coefficient of $e^{-\lambda t_1-\mu_{rs}t_2}$

$$\sum_{i=1}^{m}\pi_{0_i}p_{ir}p_{rs}\left(\frac{\mu_{ir}}{(\mu_{ir}-\lambda)}-\frac{\mu_{rs}}{(\mu_{rs}-\lambda)}\right)=-p_{rs}\left(1-\frac{\pi_{0_r}\mu_{rs}}{\eta_r(\mu_{rs}-\lambda)}\right)\left(\sum_{i=1}^{m}\eta_i p_{ir}\frac{\pi_{0_i}\mu_{ir}}{\eta_i(\mu_{ir}-\lambda)}\right) \tag{2.11}$$

for r,s = 1,2,..., m .

Condition V : coefficient of $e^{-\mu_{rs}t_1 - \lambda t_2}$

$$0 = \eta_r p_{rs} \left(1 - \frac{\pi_{0_r} \mu_{rs}}{\eta_r (\mu_{rs} - \lambda)}\right) \left(\sum_{k=1}^{m} p_{sk} \frac{\pi_{0_s} \mu_{sk}}{\eta_s (\mu_{sk} - \lambda)}\right) \tag{2.12}$$

for r,s = 1,2,..., m .

Condition VI : coefficient of $e^{-\mu_{rs}t_1 - \lambda(t_1 + t_2)}$

$$\frac{p_{rs}}{(\mu_{rs} + \lambda)} \sum_{k=1}^{m} p_{sk} \frac{\mu_{sk}}{(\mu_{sk} - \lambda)} (\pi_{1_r}\mu_{rs} - \pi_{0_r}\lambda) = 0 \tag{2.13}$$

for r,s = 1,2,..., m .

Condition VII : coefficient of $e^{-\mu_{rs}t_1 - \mu_{sk}t_2}$

$$- p_{rs}p_{sk} \left(\frac{\pi_{0_r}\mu_{rs}}{(\mu_{rs} - \lambda)} - \eta_r\right) = \eta_r p_{rs}p_{sk} \left(1 - \frac{\pi_{0_r} \mu_{rs}}{\eta_r (\mu_{rs} - \lambda)}\right) \left(1 - \frac{\pi_{0_s} \mu_{sk}}{\eta_s (\mu_{sk} - \lambda)}\right) \tag{2.14}$$

for r,s,k = 1,2,..., m .

Since $\boldsymbol{P}$ is irreducible and aperiodic, the arguments used in the proof of theorem (4.3.1) could be repeated here. From condition V we obtain

$$1 - \frac{\pi_{0_r} \mu_{rs}}{\eta_r (\mu_{rs} - \lambda)} = 0 \qquad \therefore \quad \pi_{0_r} = \eta_r (1 - \rho_{rs})$$

$$\text{with} \quad \rho_{rs} = \frac{\lambda}{\mu_{rs}} \qquad r,s \in E \ .$$

in the above expression the left hand side only depends on r, whereas the right hand side can not depend on s. This implies that

$$\mu_{rs} = \mu_r \tag{2.15}$$

and

$$\pi_{0_r} = \eta_r(1 - \rho_r) \qquad \text{for } r \in E \ . \tag{2.16}$$

Applying these to condition IV yields

$$\sum_{i=1}^{m} \pi_{0_i} p_{ir} \left(\frac{\mu_i}{\mu_i - \lambda} - \frac{\mu_r}{\mu_r - \lambda} \right) = 0$$

then $\pi_{0_r} = \sum_{i=1}^{m} \pi_{0_i} p_{ir}$ for each $r \in E$ or, in matrix form

$$\pi_0 \ \boldsymbol{P} = \pi_0 \ . \tag{2.17}$$

By Hunter [1983b] the vector π_0 is a scalar multiple of η, then

$$1 - \rho_r = \text{constant} \ , \qquad \text{for } r \in E,$$

consequently , $\mu_r = \mu$, for $r \in E$.

(2.18)

From condition VI, after substitution of (2.18) we obtain

$$\pi_{1_r} = \pi_{0_r} \rho \qquad \text{with } \rho = \frac{\lambda}{\mu} \ , \quad \text{for } r \in E. \tag{2.19}$$

Under (2.16), (2.18) and (2.19) all the other conditions become identities and the theorem is proved. ■

Next we move to the direction of proving the converse of theorem (2.4). In fact, we will prove a more general converse, i.e., if $\mu_{ij} = \mu$, for $i,j \in E$, then (N^d, Z^d, T^d) and (Z^*, T^*) are equivalent.

Re-ordering the states of (N^d, Z^d, T^d) such that we collect the states with the same type of departure, instead of queue length as we did before, the kernel $Q(t)$ becomes

$$Q(t) = \begin{bmatrix} p_{11}Q_{11}(t) & p_{12}Q_{12}(t) & \cdots & \cdots & p_{1m}Q_{1m}(t) \\ p_{21}Q_{21}(t) & p_{22}Q_{22}(t) & \cdots & \cdots & p_{2m}Q_{2m}(t) \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ p_{m1}Q_{m1}(t) & p_{m2}Q_{m2}(t) & \cdots & \cdots & p_{mm}Q_{mm}(t) \end{bmatrix} . \quad (2.20)$$

Here $Q_{ij}(t)$ is an infinite dimensional matrix representing the kernel of the queue length process embedded at departure times analogous to an $M/M/1$ queue with arrival λ and service rate of μ_{ij}. To represent $Q_{ij}(t)$ in a compact form we will define two auxiliary functions:

for $k \in N$
$$f_k(t) = \int_0^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} \mu_{ij} e^{-\mu_{ij} x} dx \quad (2.21)$$
$$g_k(t) = \int_0^t \lambda e^{-\lambda x} f_k(t-x) \mu_{ij} e^{-\mu_{ij} x} dx , \quad (2.22)$$

where we have suppressed the superscripts to keep the notation simple.

Then, the expression for $Q_{ij}(t)$ is

$$Q_{ij}(t) = \begin{bmatrix} g_0(t) & g_1(t) & g_2(t) & g_3(t) & \cdots \\ f_0(t) & f_1(t) & f_2(t) & f_3(t) & \cdots \\ 0 & f_0(t) & f_1(t) & f_2(t) & \cdots \\ 0 & 0 & f_0(t) & f_1(t) & \cdots \\ . & . & 0 & f_0(t) & \cdots \\ . & . & . & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} . \tag{2.23}$$

The embedded Markov chain (N^d, Z^d) has transition matrix $Q(\infty)$ which is given in block matrix form by

$$Q(\infty) = \begin{bmatrix} p_{11}Q_{11}(\infty) & p_{12}Q_{12}(\infty) & \cdots & \cdots & p_{1m}Q_{1m}(\infty) \\ p_{21}Q_{21}(\infty) & p_{22}Q_{22}(\infty) & \cdots & \cdots & p_{2m}Q_{2m}(\infty) \\ . & . & \cdots & \cdots & . \\ . & . & \cdots & \cdots & . \\ . & . & \cdots & \cdots & . \\ p_{m1}Q_{m1}(\infty) & p_{m2}Q_{m2}(\infty) & \cdots & \cdots & p_{mm}Q_{mm}(\infty) \end{bmatrix} . \tag{2.24}$$

Remark (2.5) : When $\mu_{ij} = \mu$, $i,j \in E$, we represent $Q_{ij}(t)$ by $Q_\mu(t)$, and the expressions (2.20) - (2.25) change accordingly. Then, from $M/M/1$ theory, the stationary distribution of $Q_\mu(\infty)$ is

$$v = (1-\rho)\,(1, \rho, \rho^2, \dots) \qquad \text{with} \quad \rho = \frac{\lambda}{\mu} \ (<1) \ . \ \blacksquare \tag{2.26}$$

Theorem (2.6) : In the case $\mu_{ij} = \mu$, $i, j \in E$, the stationary distribution of the Markov chain (N^d, Z^d) is given by

$$\pi = (\pi_{\bullet 1}, \pi_{\bullet 2}, \dots, \pi_{\bullet m}) \qquad \text{where } \pi_{\bullet i} = \eta_i \, \nu \quad \text{for } i \in E \ . \tag{2.27}$$

proof :

First, we check $\pi \, e = 1$.

$$\pi \, e = \sum_{i=1}^{m} \pi_{\bullet i} \, e_\infty = \sum_{i=1}^{m} \eta_i \, \nu \, e_\infty = \sum_{i=1}^{m} \eta_i = 1$$

where $e = [e_\infty, e_\infty, \dots, e_\infty]^T$ is an $m \times 1$ block vector and e_∞ is an infinite vector of ones.

To complete the proof we verify the equation $\pi \, Q(\infty) = \pi$. For $j \in E$,

$$[\ \pi \, Q(\infty)]_j = \sum_{i=1}^{m} \pi_{\bullet i} \, p_{ij} \, Q_\mu(\infty) = \sum_{i=1}^{m} \eta_i \, p_{ij} \, \nu \, Q_\mu(\infty) = \eta_j \, \nu = \pi_{\bullet j} \ . \ \blacksquare$$

In the case of $\mu_{ij} = \mu$, there is independence between the service process and the types, which implies by (2.27) that the stationary distribution of (N^d, Z^d) becomes the product of the two marginal distributions or, in other words, the independence between queue length and type. So, there is no difference in assigning the types before or after the service.

Remark (2.7) : The following formula establishes a sufficient condition for the Markov renewal departure process of the M/M/1 queue to be equivalent to a renewal (Poisson) process. Denoting by $Q_{\mu s}$, F_s and K_s the Laplace Stieltjes transforms of $Q_\mu(t)$, $F(t) = 1 - e^{-\lambda t}$, and $K(t)$, respectively, we have

$$v\, Q_{\mu s} = F_s\, v\,, \quad \text{for } s \geq 0\,. \tag{2.28}$$

The proof is given in Disney and Kiessler [1987], pages 68-69. ■

The theorem (2.6) and remark (2.7) will be used in the proof of the next result. We obtain in theorem (2.8) a (more general) converse of theorem (2.4).

Theorem (2.8) : If $\mu_{ij} = \mu$, $i,j \in E$, then the Markov renewal processes (N^d, Z^d, T^d) and (Z^*, T^*) are equivalent.

proof :

By the definition of equivalence we need to check that

for all $s_1, s_2, \ldots, s_n \geq 0$ and $n \in N$,

$$\pi\, Q_{s_1}\, Q_{s_2} \cdots\, Q_{s_n}\, e = \eta\, K_{s_1}\, K_{s_2} \cdots\, K_{s_n}\, e\,. \tag{2.29}$$

We start by looking at the left hand side

$$\begin{aligned}
\pi\, Q_s &= [\eta_1\, v, \eta_2\, v, \ldots, \eta_m\, v]\; Q_s \\
&= [\sum_{i=1}^{m} \eta_i\, v\, p_{i1}\, Q_{\mu s}, \sum_{i=1}^{m} \eta_i\, v\, p_{i2}\, Q_{\mu s}, \ldots, \sum_{i=1}^{m} \eta_i\, v\, p_{im}\, Q_{\mu s}] \\
&= [\eta_1\, v\, Q_{\mu s}, \eta_2\, v\, Q_{\mu s}, \ldots, \eta_m\, v\, Q_{\mu s}] \\
&= F_s[\eta_1\, v, \eta_2\, v, \ldots, \eta_m\, v] \\
&= F_s\, \pi\,,
\end{aligned} \tag{2.30}$$

where the fourth equality follows from expression (2.28) and the last equality follows from (2.27). Then successive applications of (2.30) in the left hand side of (2.29) gives

$$LHS = \prod_{l=1}^{n} F_{s_l}\, \pi\, e = \prod_{l=1}^{n} F_{s_l} \quad \text{for } n \in N \text{ and } s_l \geq 0 \ . \tag{2.31}$$

On ther other hand, note that in the special case $\mu_{ij} = \mu,\ i,j \in E$, $K(t)$ becomes

$$K(t) = (1 - e^{-\lambda t})\, P \tag{2.32}$$

or in LS transforms ,

$$K_s = F_s\, P \tag{2.33}$$

consequently

$$\eta\, K_s = \eta F_s\, P = F_s\, \eta \ . \tag{2.34}$$

Now substituting successively (2.34) on the right hand side of (2.29), we get

$$RHS = \prod_{l=1}^{n} F_{s_l}\, \eta\, e = \prod_{l=1}^{n} F_{s_l} \qquad \text{for } n \in N \text{ and } s_l \geq 0 \ . \tag{2.35}$$

From the equality between (2.31) and (2.35) the theorem is proved. ■

Corollary (2.9) : If $\mu_{ij} = \mu$, $i,j \in E$ the Markov renewal process (N^d, Z^d, T^d) is equivalent to a Poisson process (λ) .

proof :

Expression (2.30) is a sufficient condition for the equivalence (see (1.2.11)). Note that this was proven before in theorem (4.3.2), here we only have the confirmation of the result by using a different technique. ■

Corollary (2.10) : If $\mu_{ij} = \mu$, $i,j \in E$, the Markov renewal process (Z^*,T^*) is equivalent to a Poisson process (λ) .

proof :

Also from (1.2.11), the expression (2.34) is a sufficient condition for the equivalence. ■

We close this section with a important result about equivalence. In the following theorem, the conditions for equivalence between (N^d,Z^d,T^d) and its one step projection are stabilished.

Theorem (2.11) : The Markov renewal processes (N^d,Z^d,T^d) and (Z^*,T^*) are equivalent if and only if $\mu_{ij} = \mu$ $i,j \in E$.

proof :

1st part : equivalence $\rightarrow$ $\mu_{ij} = \mu$ $i,j \in E$.

Equivalence implies in particular that the joint distribution of two consecutive intervals are equal in both processes. However, by theorem (2.4) this implies that $\mu_{ij} = \mu$, $i,j \in E$.

2nd part : $\mu_{ij} = \mu$, $i,j \in E$ $\rightarrow$ equivalence .

This was proved in theorem (2.8). ■

Theorem (2.11) was our main objective in this section. We created a finite process (Z^*,T^*), called the one step projection of (N^d,Z^d,T^d), to imitate the process of interest (Z^d, T^d) whose probabilistic structure is not known. Note that, by theorem (2.3), the steady state interdeparture distribution for one interval is identical in both processes. Consequently, quantities such as mean and variance of a single interval are the same. However as we proved in this section, theorems like (2.3) which look only at a single interval are not enough to justify the equality between (Z^d, T^d) and (Z^*, T^*). Theorem (2.11) specifies the only situation where the one step projection could substitute the type-departure process in the $M/M^{ij}/1$ queues. In the next section, we will obtain properties for the individual type-departure processes of these queues, and in section 4 we will use theorem (2.11) to study the dependency among the type-departure processes, by using a finite and equivalent Markov renewal process.

5.3 *Individual type-departure process*

In this section we restrict ourselves to the case $\mu_{ij} = \mu$, $i,j \in E$ to study in detail the departure process for each type of customer. We will study, through filtering techniques, the structure of the queue length process for each fixed type of departure and we will prove the equivalence between this process and a renewal process. In case of two types of customers, detailed expressions are given.

Recall that (2.20)-(2.24) give the expressions in block (type) form for the kernel $Q(t)$ of the Markov renewal process (N^d, Z^d, T^d) . Re-writing these expressions for $\mu_{ij} = \mu$, $i,j \in E$ we obtain

$$Q(t) = \begin{bmatrix} p_{11}Q_\mu(t) & p_{12}Q_\mu(t) & \cdots & \cdots & p_{1m}Q_\mu(t) \\ p_{21}Q_\mu(t) & p_{22}Q_\mu(t) & \cdots & \cdots & p_{2m}Q_\mu(t) \\ . & . & \cdots & \cdots & . \\ . & . & \cdots & \cdots & . \\ . & . & \cdots & \cdots & . \\ p_{m1}Q_\mu(t) & p_{m2}Q_\mu(t) & \cdots & \cdots & p_{mm}Q_\mu(t) \end{bmatrix} \tag{3.1}$$

with

$$Q_\mu(t) = \begin{bmatrix} g_0(t) & g_1(t) & g_2(t) & g_3(t) & \cdots \\ f_0(t) & f_1(t) & f_2(t) & f_3(t) & \cdots \\ 0 & f_0(t) & f_1(t) & f_2(t) & \cdots \\ 0 & 0 & f_0(t) & f_1(t) & \cdots \\ . & . & 0 & f_0(t) & \cdots \\ . & . & . & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix}, \tag{3.2}$$

where for $k \in N$ $\quad f_k(t) = \int_o^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} \mu\, e^{-\mu x}\, dx \qquad (3.3)$

$$g_k(t) = \int_o^t \lambda e^{-\lambda x} f_k(t-x)\, \mu\, e^{-\mu x}\, dx\,. \tag{3.4}$$

Define $(N^j, T^j) = \{(N_n^j, T_n^j)\,;\, n \in N\}$ as the queue length process left behind by a departure of type j . In this way, (N^j, T^j) is a filtered process obtained from (N^d, Z^d, T^d). In other words, it is the process that would be observed if only the states and times of jump into the subset $\{(N_n^d, Z_n^d)\,;\, Z_n^d = j,\, n \in N\}$ are observable.

Note that we are assuming the origin as an epoch of transition in the process (N^j,T^j). If this assumption is dropped then the first transition is different from the others and so we have a delayed process. We will prove several results for a generic member of the family $\{ (N^j,T^j) ; j \in E \}$.

Theorem (3.1) : The process (N^j,T^j) with state space N is a Markov renewal process with the following semi-Markov kernel

$$Q^j(t) = \sum_{n=1}^{\infty} f_{jj}^{(n)} [Q_\mu(t)]^{(*n)} , \tag{3.5}$$

where $f_{jj}^{(n)}$ is the probability that starting in j , the first return to j is on the n-th step in the Markov chain (Z^d) .

proof :

That (N^j,T^j) also is a Markov renewal process is a simple consequence of its definition (see page 22 in Disney and Kiessler [1987]). To compute the kernel $Q^j(t)$ we note that, starting in any state in block j, we return for the first time to j after n steps with probability $f_{jj}^{(n)}$ and for each state inside of the blocks, the transition time is given by $Q_\mu(t)$. Expression (3.5) follows since the number of transitions is arbitrary (n = 1, 2,....) . ■

Theorem (3.2) : The process $N^j = \{N_n^j ; n \in N\}$ is a Markov chain with stationary distribution

$$\pi^j = \nu , \tag{3.6}$$

where ν is the stationary distribution for M/M/1 queue.

proof :

The process N^j is a Markov chain as a consequence of (N^j,T^j) being a Markov renewal process. Its transition matrix is given by

$$Q^j(\infty) = \lim_{t\to\infty} \sum_{n=1}^{\infty} f_{jj}^{(n)} [\, Q_\mu(t)]^{(*n)} \; . \tag{3.7}$$

Since the right hand side of (3.7) is the limit of the sum of a convergent series of bounded functions, we could interchange the limit and the sum. Also from lemma (4.1.2) in Simon [1980], we have

$$\lim_{t\to\infty} [\, Q_\mu(t)]^{(*n)} = [\, Q_\mu(\infty)]^n \; .$$

Then (3.7) becomes

$$Q^j(\infty) = \sum_{n=1}^{\infty} f_{jj}^{(n)} [\, Q_\mu(\infty)]^n \tag{3.8}$$

and ,

$$v\, Q^j(\infty) = \sum_{n=1}^{\infty} f_{jj}^{(n)}\, v\, [\, Q_\mu(\infty)]^n = \sum_{n=1}^{\infty} f_{jj}^{(n)}\, v\, [\, Q_\mu(\infty)]^{n-1} = \cdots = \sum_{n=1}^{\infty} f_{jj}^{(n)}\, v \, .$$

Then $\quad v\, Q^j(\infty) = \; v \, ,\;$ since $\displaystyle\sum_{n=1}^{\infty} f_{jj}^{(n)} = 1.$

Finally we observe that $ve_{(\infty)} = 1$ and v satisfies the equilibrium equations, which was shown above. We conclude that v is the stationary distribution for N^j . ■

Theorem (3.3) : The process (N^j, T^j) is equivalent to a renewal process with L-S distribution function

$$F_s^j = \sum_{n=1}^{\infty} f_{jj}^{(n)}\, F_s^n \;, \qquad s \ge 0 \; . \tag{3.9}$$

proof :

Define Q_s^j as the L-S transform of $Q^j(t)$. Using (3.9), we will verify that $v\, Q_s^j = F_s^j\, v$, which is a sufficient condition for the equivalence to be satisfied. We have

$$v\, Q_s^j = v \sum_{n=1}^{\infty} f_{jj}^{(n)}\ Q_{\mu s}^n = \sum_{n=1}^{\infty} f_{jj}^{(n)}\ v\ Q_{\mu s}^n\ .$$

By applying successively expression (2.28) we obtain

$$v\, Q_s^j = \Big(\sum_{n=1}^{\infty} f_{jj}^{(n)}\, F_s^{\,n} \Big)\, v = F_s^j\, v\ . \ \blacksquare$$

In theorems (3.2) and (3.3) we observed that the process (N^j,T^j) has similar characteristics to the queue length process (embedded at departure epochs) of a M/M/1 queue. They have the same state space and stationary distribution, however, the one step transitions are different. Obviously they will coincide if $m = 1$ (only one type of customer) as we easily see from (3.1).

In order to compute explicitly the distribution function $F^j(t)$, we need to have an expression for $f_{jj}^{(n)}$ and from it, we compute and invert F_s^j. For $m > 2$ it is difficult to find explicit expressions for $f_{jj}^{(n)}$. We illustrate the computation for the case $m = 2$.

Example (3.4) : Computation of $F^j(t)$ in the special case $m = 2$.

The matrix for change types could be written as

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \tag{3.10}$$

where $0 < a, b < 1$ since we assumed that P is irreducible and aperiodic.

Theorem (5.1.8) in Hunter [1983-a], gives the n-step first passage time:

$$(a)\quad f_{11}^{(n)} = \begin{cases} ab(1-b)^{n-2} & n \geq 2 \\ 1-a & n=1\,; \end{cases}$$
$$(b)\quad f_{22}^{(n)} = \begin{cases} ab(1-a)^{n-2} & n \geq 2 \\ 1-b & n=1\,. \end{cases} \tag{3.11}$$

Then applying (3.9) with $F_s = \dfrac{\lambda}{\lambda + s}$ we obtain

$$(a)\quad F_s^{\,1} = \frac{\lambda s(1-a) + \lambda^2 b}{(\lambda + s)(\lambda b + s)}\;;$$
$$(b)\quad F_s^{\,2} = \frac{\lambda s(1-b) + \lambda^2 a}{(\lambda + s)(\lambda b + s)}\;. \tag{3.12}$$

Inversion of (3.12) yields

$$(a)\quad F^1(t) = \left(\frac{1-a-b}{1-b}\right)(1 - e^{-\lambda t}) + \left(\frac{a}{1-b}\right)(1 - e^{-\lambda b t})\,;$$
$$(b)\quad F^2(t) = \left(\frac{1-a-b}{1-a}\right)(1 - e^{-\lambda t}) + \left(\frac{b}{1-a}\right)(1 - e^{-\lambda a t})\,. \tag{3.13}$$

Note that both F_s^1 and F_s^2 are hyperexponential distributions or, in other words, convex combinations of two exponential distributions .

Theorem (3.5) : The distribution functions $F^1(t)$ and $F^2(t)$ in (3.13) are exponentials if and only if the matrix P has identical rows.

proof :

We prove the theorem for $F^1(t)$. For $F^2(t)$ the proof is similar .

1st part : $F^1(t)$ exponential $\rightarrow$ P has identical rows .

By hypothesis $F_s^1 = \dfrac{\lambda_1}{\lambda_1 + s}$ and equating coefficients of a polynomial in s on both sides of (3.12), we have the following conditions:

condition 1 : coefficient of s

$$\lambda_1(1+b) = \lambda_1(1-a) + \lambda b$$
$$\therefore \quad \lambda_1 = \left(\frac{b}{a+b}\right)\lambda \ . \tag{3.14}$$

condition 2 : coefficient of s^2

$$\lambda_1 = \lambda(1-a) \ ,$$

applying (3.14) we obtain $\quad \dfrac{b}{a+b} = 1 - a \ \therefore \ a = 1 - b \quad .$ (3.15)

Under (3.15) , the matrix $\boldsymbol{P}$ becomes $\boldsymbol{P} = \begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$

that is, the change of types is independent of previous type. This is known in the literature as Bernoulli switching.

2nd part: $\boldsymbol{P}$ has identical rows $\rightarrow$ F_l is exponential .

Without loss of generality, we can assume that $\boldsymbol{P}$ has the form

$$\boldsymbol{P} = \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_1 & \eta_2 \end{bmatrix} .$$

Consequently, $a = \eta_2$ and $b = \eta_1$ where the η's are the elements of the stationary probability vector for $\boldsymbol{P}$. Substituting in (3.12) yields

$$F_s^1 = \frac{\lambda s(1-\eta_2) + \lambda^2\eta_1}{(\lambda+s)(\lambda\eta_1+s)} = \frac{\lambda\eta_1(\lambda+s)}{(\lambda+s)(\lambda\eta_1+s)} = \frac{\lambda\eta_1}{(\lambda\eta_1+s)} \ .$$

Then F^1 is exponential with parameter $\lambda\eta_1$. ∎

In the last theorem we obtained a Poisson process for each of the type-departure processes when P is an equilibrium matrix. From remark (2.3.2), with the same structure for P, the service process is renewal if and only if consecutive intervals are independent. In this sense, it becomes simpler to verify the occurrence of a renewal process.

Theorem (3.6) : For $d \geq 0$, the following order relations are observed:

(a) If $\eta_1 = \eta_2$, then $F^1(t) = F^2(t)$, $t \geq 0$;

(b) If $\eta_1 > \eta_2$, then $F^1(t) > F^2(t)$, $t > 0$;

(c) If $\eta_2 > \eta_1$, then $F^2(t) > F^1(t)$, $t > 0$.

proof :

Note that in our case $\eta_1 = \eta_2$ is equivalent to b = a. To prove (a), note that when $t = 0$, $F^1(t) = F^2(t)$. For $t > 0$, the result follows by putting a = b in (3.13).

To verify (b) we compute the difference $F^1(t) - F^2(t)$,

$$\begin{aligned} F^1(t) - F^2(t) &= \frac{a}{1-b}(e^{-\lambda t} - e^{-\lambda bt}) + \frac{b}{1-a}(e^{-\lambda at} - e^{-\lambda t}) \\ &> \frac{a}{1-b}(e^{-\lambda t} - e^{-\lambda bt}) + \frac{b}{1-a}(e^{-\lambda bt} - e^{-\lambda t}) \\ &= \frac{e^{-\lambda bt} - e^{-\lambda t}}{(1-a)(1-b)}(b(1-b) - a(1-a)) \\ &\geq 0 \ , \end{aligned}$$

since for $0 < b < 1$ and $t > 0$ we have $e^{-\lambda bt} > e^{-\lambda t}$ and also

$$b(1-b) - a(1-a) = (b-a)(1-a-b) = (b-a)d \geq 0 \ .$$

Then we conclude that $F^1(t) > F^2(t)$ for $t > 0$ and part (b) is proved.

To verify (c), we apply the argument used in (b) to obtain

$$F^2(t) - F^1(t) > \frac{e^{-\lambda at} - e^{-\lambda t}}{(1-a)(1-b)}(a(1-a) - b(1-b)) \geq 0 \quad ,$$

by using $e^{-\lambda at} > e^{-\lambda t}$ for $0 < a < 1$ and $t > 0$. This concludes the proof. ■

Note that, so far, our analysis looked at each element in the family $\{(N^j,T^j)\ ; j \in E\}$ in isolation. Also we required that the initial occurrence was type j. If this assumption is dropped, theorems (3.3) and (3.5) continue to be valid but the results are for delayed renewal processes. For instance, suppose that the initial state is i, then some time is expended until the first visit to j. This initial time distribution does not have, in general, the same distribution as the other intervals formed by successive returns to j.

If the matrix $\boldsymbol{P}$ is a Bernoulli switching, theorem (3.5) shows that both renewals are Poisson processes. Furthermore, from theorem (4.3.2) in chapter 4, we know that the superposition of the type-departure processes is a Poisson process and in the case of Bernoulli decomposition, it is well-known (c.f. Cooper [1981] page 59) that they are independent Poisson processes. If we do not have a Bernoulli switching, theorem (3.3) provides an interesting situation where a superposition of renewal processes gives a Poisson process. From point process theory (see e.g. Lewis [1972]), these renewal processes must be dependent. Figure (3.1) illustrates this occurrence for $m = 3$. We explore this dependence in the next section.

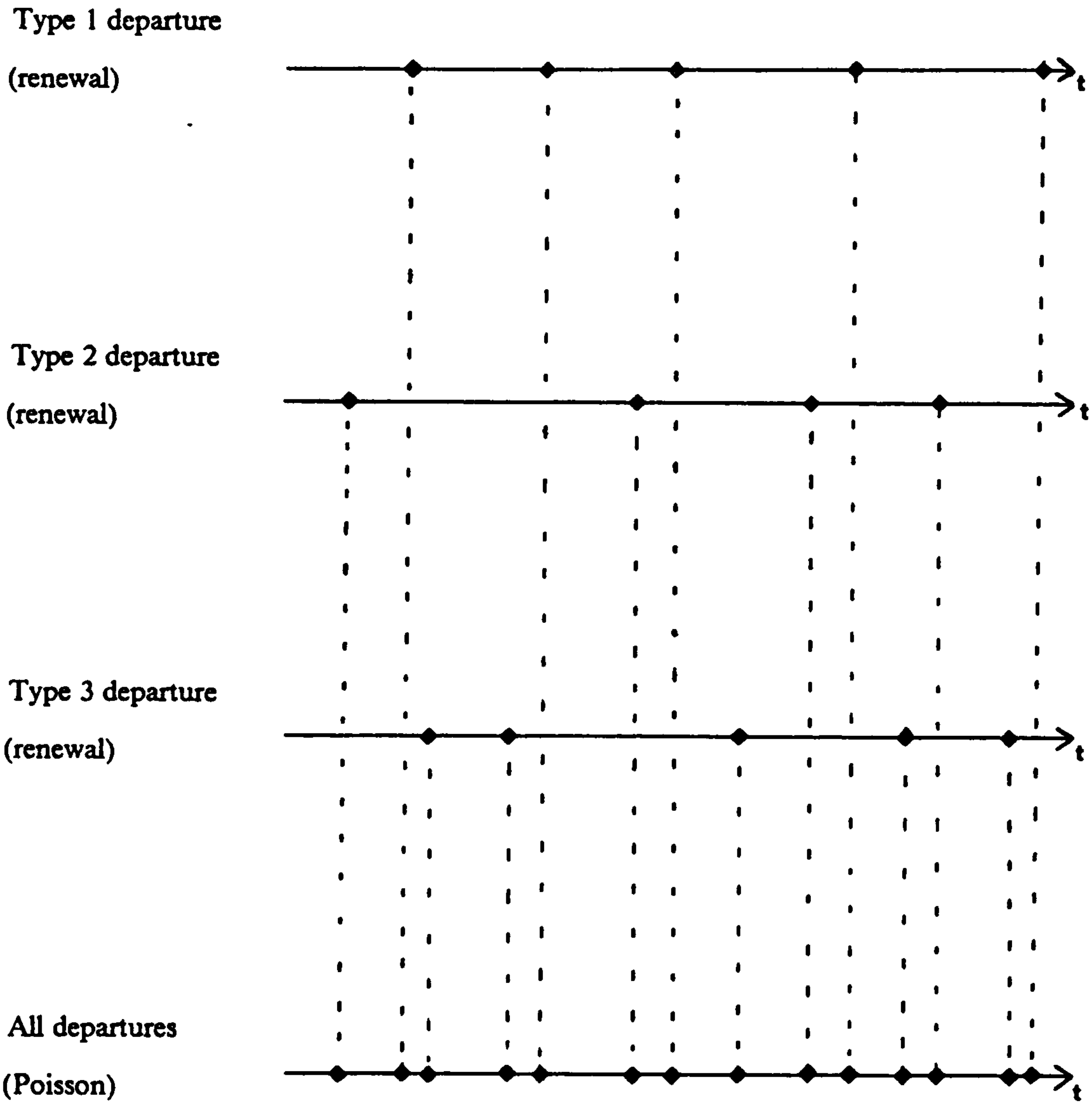

Figure 3.1: Superposition of departure processes (m = 3).

5.4 Dependent type-departure processes

We will devote this section to the study of the dependence among the renewal type-departure processes introduced in the last section. From theorem (2.11), we know that we could study these type-departure processes by using a finite (and equivalent) Markov renewal process (Z^*,T^*). Our analysis will use the crosscovariance and crosscorrelation (to be defined) to measure dependency among the counting processes associated with each renewal type-departure process. To simplify the notation, from now on, we supress the " * " in the representation of the process (Z^*,T^*) studied in section 2 of this chapter. Recall that, from (2.32), the semi-Markov kernel of (Z,T) is

$$K(t) = (1 - e^{-\lambda t})\, P .$$

Associated with $K(t)$ we have the Markov renewal function $R_{ij}(t)$ which is the expected number of visits in (0,t] to state j, starting in i. From Disney and Kiessler [1987], we get

$$R_{ij}(t) = \sum_{n=1}^{\infty} K_{ij}{}^{(*n)}(t) , \tag{4.1}$$

where $K_{ij}{}^{(*n)}$ is the (i,j) entry of the matrix $K^{(*n)}$. Analogously, the Markov renewal kernel is

$$R(t) = \sum_{n=1}^{\infty} K^{(*n)}(t) \quad . \tag{4.2}$$

Define the counting processes:

$M = \{M(t)\,;\, t \in R_+\}$, the number of all departures in (0 , t], $t \in R_+$; (4.3)

$M_j = \{M_j(t)\,;\, t \in R_+\}$, the number of type j departures in (0 , t],$t \in R_+$. (4.4)

From our definition, note that we will not count the occurrence at the origin. This is a common convention in the study of counting processes. From (4.1) and (4.4) we immediately conclude that in steady state,

$$E[M_j(t)] = \sum_{i=1}^{m} \eta_i \, R_{ij}(t) , \qquad \text{for each } \; j \in E. \tag{4.5}$$

The conditional expected value of the product of two counting variables was given in Çinlar [1969] by his expression (8.14). On the other hand, lemma (7.7.2) in Disney and Kiessler [1987] gives a similar result for the case where the counting is made on a partition of the state space and one variable is shifted from the other (lag r expectation). The next theorem combines both results since it computes the conditional expectation for any two counting variables associated with the state space and there is a shift of r units of time from one counting to the other.

Theorem (4.1) : For $i, j, k \in E$, and $t, r \in R_+$,

$$E[M_j(t)\, M_k(t+r) \,|\, Z_0 = i] = \delta_{jk} R_{ij}(t) + R_{ik} {*} R_{kj}(t) + \int_0^t R_{jk}(t+r-u)\, dR_{ij}(u) , \tag{4.6}$$

where $\delta_{jk} = 1$ for $j = k$, and zero otherwise.

proof :

Note that by the definition of the counting processes, we have

$$M_j(t) M_k(t+r) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} 1_{\{Z_n = j\}}\, 1_{\{Z_l = k\}}\, 1_{\{0< T_n \le t\}}\, 1_{\{0< T_l \le t+r\}} \; .$$

Then, by applying conditional expectation we obtain

$$
\begin{aligned}
E[M_j(t)M_k(t+r)\,|\,Z_0=i] &= \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} P[Z_n=j\,,\,Z_l=k\,,\,T_n\le t\,,\,T_l\le t+r\,|\,Z_0=i] \\
&= \sum_{n=2}^{\infty}\sum_{l=1}^{n-1} P[Z_n=j\,,\,Z_l=k\,,\,T_n\le t\,,\,T_l\le t+r\,|\,Z_0=i] \\
&\quad + \sum_{n=l=1}^{\infty} P[Z_n=j\,,\,Z_l=k\,,\,T_n\le t\,,\,T_l\le t+r\,|\,Z_0=i] \\
&\quad + \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty} P[Z_n=j\,,\,Z_l=k\,,\,T_n\le t\,,\,T_l\le t+r\,|\,Z_0=i]\,,
\end{aligned}
\tag{4.7}
$$

where we divided the sum according to $l<n\,,\, l=n$, and $l>n$ respectively.

For $l<n$ then $T_l<T_n$. Then $\{T_n\le t\}\rightarrow\{T_l\le t\}\rightarrow\{T_l\le t+r\}$ and

$$
\begin{aligned}
&P[Z_n=j,\,Z_l=k,\,T_n\le t,\,T_l\le t+r\,|\,Z_0=i]= \\
&\qquad = P[Z_n=j,\,Z_l=k,\,T_n\le t\,|\,Z_0=i] \\
&\qquad = P[Z_n=j,\,Z_l=k,\,T_n\le t,\,T_l\le t\,|\,Z_0=i] \\
&\qquad = \int_0^t P[Z_n=j,\,T_n\le t\,|\,Z_l=k,\,T_l=u]\;dP[Z_l=k,\,T_l\le u\,|\,Z_0=i] \\
&\qquad = K_{ik}^{(*l)} * K_{kj}^{(*(n-l))}(t)\,.
\end{aligned}
$$

Then, for $l<n$ we have

$$
\sum_{n=2}^{\infty}\sum_{l=1}^{n-1} K_{ik}^{(*l)} * K_{kj}^{(*(n-l))}(t) = R_{ik} * R_{kj}(t) \tag{4.8}
$$

by applying (4.1) and some manipulation with the convolution exponents.

For $l=n$, then $T_l=T_n$ and

$$
\begin{aligned}
P[Z_n=j,\,Z_l=k,\,T_n\le t,\,T_l\le t+r\,|\,Z_0=i] &= \delta_{jk}P[Z_n=j,\,T_n\le t\,|\,Z_0=i] \\
&= \delta_{jk}K_{ij}^{(*n)}(t)\quad .
\end{aligned}
$$

From expression (4.1) there follows

$$\sum_{n=1}^{\infty} \delta_{jk} K_{ij}^{(*n)}(t) = \delta_{jk} R_{ij}(t) \ . \tag{4.9}$$

For $l > n$, then $T_l > T_n$ and

$$\begin{aligned}
P[Z_n = j, Z_l = k, T_n \le t, T_l \le t + r \mid Z_0 = i] = \\
&= \int_0^t P[Z_l = k, T_l \le t + r \mid Z_n = j, T_n = u] \ dP[Z_n = j, T_n \le u \mid Z_0 = i] \\
&= \int_0^t P[Z_l = k, T_l - T_n \le t + r - u \mid Z_n = j] \ dP[Z_n = j, T_n \le u \mid Z_0 = i] \\
&= \int_0^t K_{jk}^{(*(l-n))}(t + r - u) \, dK_{ij}^{(*n)}(u) \ ,
\end{aligned}$$

by standard arguments in conditional probability. In order to perform the double summation in n and l, note that for fixed i,j, and k, the sums $\sum_{l=n+1}^{L_1} K_{jk}^{(*(l-n))}(.)$ and $\sum_{n=1}^{L_2} K_{ij}^{(*n)}(.)$ are nondecreasing functions of L_1 and L_2, respectively. The monotone convergence theorem could be applied to interchange the summation with the integral to obtain

$$\sum_{n=1}^{\infty} \sum_{l=n+1}^{\infty} \int_0^t K_{jk}^{(*(l-n))}(t + r - u) \, dK_{ij}^{(*n)}(u) = \int_0^t R_{jk}(t + r - u) \, dR_{ij}(u) \ . \tag{4.10}$$

Finally, substituting (4.8), (4.9) and (4.10) in (4.7) we obtain the expression (4.6) and the theorem is proved. ■

In steady state, the unconditional expected value is

$$E[M_j(t)\, M_k(t+r)] = \sum_{i=1}^{m} \eta_i[\delta_{jk} R_{ij}(t) + R_{ik} {*} R_{kj}(t) + \int_0^t R_{jk}(t+r-u)\, dR_{ij}(u)] \; . \tag{4.11}$$

From (4.5) and (4.11) the variance is obtained as

$$Var[M_j(t)] = E[M_j^2(t)] - E^2[M_j(t)] \quad , \qquad \text{for} \quad j \in E \quad , \tag{4.12}$$

and $E[M_j^2(t)]$ is given in (4.11) in the case $r = 0$ and $j = k$.

Next we will define crosscovariance and crosscorrelation between two counting variables. They will be our measure of dependence between the renewal processes through their respective counting processes.

Definition (4.2) : For all $t, r \in R_+$ the crosscovariance of lag r of the processes M_j and M_k , $j, k \in E$ is defined by

$$ccov[M_j(t) \, , M_k(t+r)] = E[M_j(t)\, M_k(t+r)] - E[M_j(t)]\, E[M_k(t+r)] \; . \; \blacksquare \tag{4.13}$$

Definition (4.3) : For all $t, r \in R_+$ the crosscorrelation of lag r of the processes M_j and M_k , $j, k \in E$ is defined by

$$ccorr[M_j(t) \, , M_k(t+r)] = \frac{ccov[M_j(t) \, , M_k(t+r)]}{\sqrt{Var[M_j(t)]\; Var[M_k(t+r)]}} \; . \; \blacksquare \tag{4.14}$$

The expressions above give us the elements to compute a variance-covariance matrix for the m dependent renewal processes. The computation is based on $\boldsymbol{R}(t)$ which exists for all finite t (see prop.9.1.14 in Çinlar [1975]), however, its explicit expression for $m > 2$ is complicated.

The L-S transform of $R(t)$ is represented by R_s. A useful expression for numerical evaluation of the Markov renewal kernel is given by

$$R_s = K_s (I - K_s)^{-1}, \quad s \geq 0 \tag{4.15}$$

which is proved, with small changes, in proposition (10.2.20) in Çinlar [1975].

We will restrict ourselves to the case m=2 and will compute detailed expressions for the crosscovariance and crosscorrelation. From the representation (3.10) we have $d = 1 - a - b \neq \pm 1$, that is, $-1 < d < 1$. By Hunter [1983-b], (p. 8), we have

$$P = W D W^{-1} \tag{4.16}$$

where $W = \begin{bmatrix} 1 & -a \\ 1 & b \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$.

By applying the L-S transform in expression (4.2) and using (4.16), we get

$$R_s = \sum_{n=1}^{\infty} K_s^n = W \Big(\sum_{n=1}^{\infty} \big(\frac{\lambda}{\lambda + s} \big) D^n \Big) W^{-1} .$$

After computation we get

$$R_s = \frac{1}{1-d} \begin{bmatrix} b\alpha_s + a\beta_s & a\alpha_s - a\beta_s \\ b\alpha_s - b\beta_s & a\alpha_s + b\beta_s \end{bmatrix} \tag{4.17}$$

with $\alpha_s = \frac{\lambda}{s}$ and $\beta_s = \frac{\lambda d}{\lambda(1-d) + s}$.

Taking the inverse,

$$R(t) = \frac{1}{1-d}\begin{bmatrix} b\lambda t + \frac{ad}{1-d}H(t) & a\lambda t - \frac{ad}{1-d}H(t) \\ b\lambda t - \frac{bd}{1-d}H(t) & a\lambda t + \frac{bd}{1-d}H(t) \end{bmatrix} \tag{4.18}$$

where $H(t) = 1 - e^{-\lambda(1-d)t}$, $t \geq 0$.

Recall that $\eta = (\frac{b}{1-d} , \frac{a}{1-d})$, then applying (4.5) we obtain

$$\begin{aligned} &(a) \quad E[M_1(t)] = \eta_1 \lambda t , \quad t \geq 0 ; \\ &(b) \quad E[M_2(t)] = \eta_2 \lambda t , \quad t \geq 0 . \end{aligned} \tag{4.19}$$

Note that the process M defined by (4.3) is a Poisson process as we proved in theorem (4.3.2). Then, we obtain a confirmation of the above expression since

$$E[M_t] = E[M_1(t) + M_2(t)] = E[M_1(t)] + E[M_2(t)] = \lambda t \quad ,$$

which is the expected value of a Poisson random variable with parameter λt.

Substituting the elements of $R(t)$ (given by (4.18)) in the expression (4.11), we obtain the expected value of the product of two counting variables. After considerable algebraic manipulation, we get

$$\begin{aligned} E[M_1(t)M_2(t+r)] = \eta_1\eta_2\Big(&\frac{d}{(1-d)^2} - \frac{2d}{1-d}\lambda t + \lambda^2 tr + \lambda^2 t^2 \Big) \\ &- \frac{\eta_1\eta_2 d}{(1-d)^2}\left(e^{-\lambda(1-d)t} + e^{-\lambda(1-d)(t+r)} - e^{-\lambda(1-d)r}\right) . \end{aligned} \tag{4.20}$$

For r = 0 and j = k = 1, 2 , we substitute in (4.11) the value of $R(t)$ to obtain

$$(a)\quad E[M_1^2(t)] = \eta_1 R_{11}(t) + \eta_2 R_{21}(t) + 2\eta_1 R_{11}^{(*2)}(t) + 2\eta_2 R_{21} * R_{11}(t)$$
$$= \frac{b}{1-d}\lambda t + \frac{2abd}{(1-d)^4}\left(-1 + \lambda(1-d)t + \frac{b}{2ad}(\lambda(1-d)t)^2 + e^{-\lambda(1-d)t}\right);$$
$$(4.21)$$
$$(b)\quad E[M_2^2(t)] = \eta_1 R_{12}(t) + \eta_2 R_{22}(t) + 2\eta_2 R_{22}^{(*2)}(t) + 2\eta_1 R_{12} * R_{22}(t)$$
$$= \frac{a}{1-d}\lambda t + \frac{2abd}{(1-d)^4}\left(-1 + \lambda(1-d)t + \frac{a}{2bd}(\lambda(1-d)t)^2 + e^{-\lambda(1-d)t}\right).$$

Then the expressions for the variances are

$$(a)\quad Var[M_1(t)] = \frac{b}{1-d}\lambda t + \frac{2abd}{(1-d)^4}\left(-1 + \lambda(1-d)t + e^{-\lambda(1-d)t}\right);$$
$$(4.22)$$
$$(b)\quad Var[M_2(t)] = \frac{a}{1-d}\lambda t + \frac{2abd}{(1-d)^4}\left(-1 + \lambda(1-d)t + e^{-\lambda(1-d)t}\right).$$

Remark (4.4) : Expressions (4.22) are non-negative for all $t \in R_+$. Observe that the first term is the respective expected value and it is non-negative for $t \geq 0$. For $0 \leq d < 1$, the second term is positive since the term inside the parenthesis is

$$-1 + \lambda(1-d)t + e^{-\lambda(1-d)t} \geq 0 \quad , \qquad \text{for } t \geq 0 \quad .$$

To verify this, note

$$-1 + \lambda(1-d)t + e^{-\lambda(1-d)t} = -1 + E[\tilde{M}(t)] + P[\tilde{M}(t) = 0] \, ,$$

where $\tilde{M}(t)$ is a Poisson r.v. with mean $E[\tilde{M}(t)] = \lambda(1-d)t$. Also we have

$$E[\tilde{M}(t)] = \sum_{n=1}^{\infty} P[\tilde{M}(t) \geq n] = 1 - P[\tilde{M}(t) = 0] + \sum_{n=2}^{\infty} P[\tilde{M}(t) \geq n] \; .$$

Then ,

$$-1 + \lambda(1-d)t + e^{-\lambda(1-d)t} = \sum_{n=2}^{\infty} P[\tilde{M}(t) \geq n] \ \geq 0 \quad , \quad t \geq 0 \ .$$

When $d < 0$, we re-write $Var[M_1(t)]$ as

$$Var[M_1(t)] = (\frac{-2abd}{(1-d)^4})(1 - e^{-\lambda(1-d)t}) + (1 + \frac{2ad}{(1-d)^2})(\frac{b}{1-d}\lambda t) \ .$$

Now observe that $1 - d = a + b$ and

$$1 + \frac{2ad}{(a+b)^2} = \frac{(a+b)^2 + 2a(1-a-b)}{(a+b)^2} = \frac{2a - a^2 + b^2}{(a+b)^2} = \frac{a(2-a) + b^2}{(a+b)^2} > 0.$$

Since all the other terms are clearly non-negatives, we conclude that $Var[M_1(t)] \geq 0$. The same analysis could be applied to type 2 process to obtain $Var[M_2(t)] \geq 0 \, , \ t \geq 0$. ∎

Theorem (4.5) : For $a \leq b$,

(*a*) $E[M_1(t)] \geq E[M_2(t)] \quad , \quad t \geq 0 \quad ;$
(*b*) $Var[M_1(t)] \geq Var[M_2(t)] \quad , \quad t \geq 0 \quad .$

proof

Recall that the stationary vector η for the matrix $\boldsymbol{P}$ is given by

$$\eta = \left[\ \frac{b}{a+b} \quad , \quad \frac{a}{a+b} \ \right] \ .$$

From (4.19), the inequality in (a) is immediate.

To prove (b) we compare (4.22-a) and (4.22-b) and, after observing that several terms are identicals, we obtain the result. ∎

Since $E[M_2(t+r)] = \eta_2\lambda(t+r)$ by (4.19), the expression for the crosscovariance of lag r becomes

$$ccov[M_1(t), M_2(t+r)] = \eta_1\eta_2\left(\frac{d}{(1-d)^2} - \frac{2d}{1-d}\lambda t\right) - \frac{\eta_1\eta_2 d}{(1-d)^2}\left(e^{-\lambda(1-d)t} + e^{-\lambda(1-d)(t+r)} - e^{-\lambda(1-d)r}\right) . \tag{4.23}$$

Expressions (4.22) and (4.23) could be used to obtain, through (4.14), the expression for $ccorr[M_1(t), M_2(t+r)]$. The final expression does not have an apparent simplification and we do not show it, however, at the end of the section, we will present a numerical example where the crosscorrelation is computed for several values of the parameters.

Remark (4.6) : We could use again the fact that M is a Poisson process to verify our formulas so far. We have

$Var[M(t)] = Var[M_1(t) + M_2(t)] = Var[M_1(t)] + Var[M_2(t)] + 2ccov[M_1(t), M_2(t)].$

Putting r = 0 in (4.23) we obtain

$$ccov[M_1(t), M_2(t)] = \frac{2\eta_1\eta_2 d}{(1-d)^2}\left(1 - \lambda(1-d)t - e^{-\lambda(1-d)t}\right) . \tag{4.24}$$

Applying (4.22) and (4.24) , we obtain $Var[M(t)] = \lambda t$, which indeed is the variance of a Poisson r.v. with parameter λt . ■

Next we prove a theorem given in Disney and Kiessler [1987] which has an arithmetic mistake in the crosscovariance expression. Note that the theorem was applied to the case where a Markov chain is thinning a renewal process. In our case, this also happens because we proved in theorem (2.11) the equivalence between (N^d, Z^d, T^d) and (Z^*, T^*) . Despite the error in the proof, the theorem remains true.

Theorem (4.7) : Let $F(t) = 1 - e^{-\lambda t}$, and let $\eta = (\eta_1, \eta_2)$ be the stationary distribution for the Markov chain P . Then the following are equivalent :

(*a*) $ccov[M_1(t), M_2(t)] = 0$, $t \in E$;
(*b*) P is a Bernoulli switch ;
(*c*) M_1 and M_2 are two independent Poisson processes .

proof :

From point process theory (see Lewis [1972]), $b \rightarrow c$ and $c \rightarrow a$ are true. We will prove $a \rightarrow b$.

Since η_1, η_2 and $1 - d$ are non zero by the irreducibility of P , using (4.24) the hypothesis (a) implies $d = 0$ and consequently $b = 1 - a$. In this case P has identical rows and it is a Bernoulli switching, so (b) holds. ■

From the last theorem, when $d = 0$, the processes M_1 and M_2 are independent Poisson processes. In addition, the next theorems help to explain the behavior of the crosscovariance and crosscorrelation as function of d, t and r. In particular, we computed the limit of the crosscorrelation when $t \rightarrow \infty$ (r fixed) and when $r \rightarrow \infty$ (t fixed).

Theorem (4.8) : For $d > 0$, the crosscovariance is monotone decreasing in r.

proof :

Let $r_2 \geq r_1$. Using (4.23) we look at the difference

$$ccov[M_1(t), M_2(t+r_1)] - ccov[M_1(t), M_2(t+r_2)] =$$
$$= -\frac{\eta_1\eta_2 d}{(1-d)^2}\left(e^{-\lambda(1-d)(t+r_1)} - e^{-\lambda(1-d)(t+r_2)} - e^{-\lambda(1-d)r_1} + e^{-\lambda(1-d)r_2}\right)$$
$$= -\frac{\eta_1\eta_2 d}{(1-d)^2}(e^{-\lambda(1-d)t} - 1)(e^{-\lambda(1-d)r_1} - e^{-\lambda(1-d)r_2})$$
$$\geq 0$$

by noting that the first two factors are negative and the last one is positive since $r_2 \geq r_1$. ∎

Theorem (4.9) : For fixed t ,

$$\lim_{r\to\infty} ccorr[M_1(t), M_2(t+r)] = 0 \ .$$

proof: If t is fixed, by (4.23)

$$\lim_{r\to\infty} ccov[M_1(t), M_2(t+r)] = \frac{\eta_1\eta_2 d}{(1-d)^2}(1 - 2\lambda(1-d)t - e^{-\lambda(1-d)t}) < \infty \ .$$

On the other hand,

$$\lim_{r\to\infty} Var[M_2(t+r)] = \infty \quad , \quad \text{by (4.22)} \ .$$

Then using (4.14), the theorem is proved. ∎

Theorem (4.10) : For fixed r ,

$$\lim_{t\to\infty} ccorr[M_1(t), M_2(t+r)] = \frac{-2\eta_1\eta_2 d}{\sqrt{ab\left(1+\frac{2ad}{(a+b)^2}\right)\left(1+\frac{2bd}{(a+b)^2}\right)}} \ . \qquad (4.25)$$

proof:

From (4.23),

$$\frac{1}{t}\, ccov[M_1(t), M_2(t+r)] = \frac{-2\eta_1\eta_2 d\lambda}{(1-d)} + A(t)$$

also, from (4.22),

$$\frac{1}{t}\, Var[M_1(t)] = \frac{b}{(1-d)}\lambda + \frac{2abd}{(1-d)^3}\lambda + B(t)$$

$$\frac{1}{t}\, Var[M_2(t+r)] = \frac{a}{(1-d)}\lambda + \frac{2abd}{(1-d)^3}\lambda + C(t)$$

where A(t), B(t) and C(t) go to zero when $t \to \infty$.

Applying the above expressions in (4.14) and passing to the limit when $t \to \infty$, we obtain the result after performing some algebraic simplifications. ∎

The expressions above show us that the characteristics of the dependence between the two counting processes M_1 and M_2 relies on the structure of the matrix $\boldsymbol{P}$. Note that the limit in (4.25) does not depend on λ or r and its sign could be positive or negative according to -d. The next theorem permits us to predict the sign of the crosscovariance and crosscorrelation by looking at the behavior of the parameters a and b.

Theorem (4.11) : For $t > 0$, the crosscovariance and crosscorrelation of lag zero are positive or negative depending on $a + b > 1$ or $a + b < 1$, respectively.

proof : Note that from definitions (3.8) and (3.9), the crosscovariance and crosscorrelation have the same sign. Expression (4.14) gives the crosscovariance of lag zero and remark (4.4) showed that the term inside the parenthesis is always less than zero. Consequently, the sign of the crosscovariance will be the sign of -d. Since $d = 1 - a - b$, the theorem is proved. ■

Example (4.12) : Using expression (4.14), we compute the lag zero crosscorrelation for the special case described in this section. In order to show the results in a 3-dimensional graph we define a new parameter $u = \lambda t$, so the crosscorrelation becomes function of a, b and u .

Figures 4.1 - 4.5 represent the behaviour of the crosscorrelation for fixed a and different values of b and u. Observe that there are positive and negative values depending on the parameters. Also if we fix a and b and look at the crosscorrelation for different u, we observe that the influence of u practically disappears when it becomes larger. In fact, these graphs are showing that the convergence of the crosscorrelation to its limit value (when $t \rightarrow \infty$) could be very fast. From the sequence of graphics 4.1 - 4.5 we note that the minimum and the maximum values of the crosscorrelation increase when we move from $a = 0.1$ to $a = 0.9$.

Figures 4.6 - 4.10 show the crosscorrelation for fixed u and different values of a and b . The crosscorrelation is symmetric in a and b , that is, interchanging the values of a and b will not alter its value. Furthermore, the crosscorrelation increases with $a + b$ and can be either negative or positive depending on $a + b < 1$ or $a + b > 1$, respectively. It is possible to observe from fig 4.6 - 4.10 that the range of the crosscorrelation enlarges with the value of u.

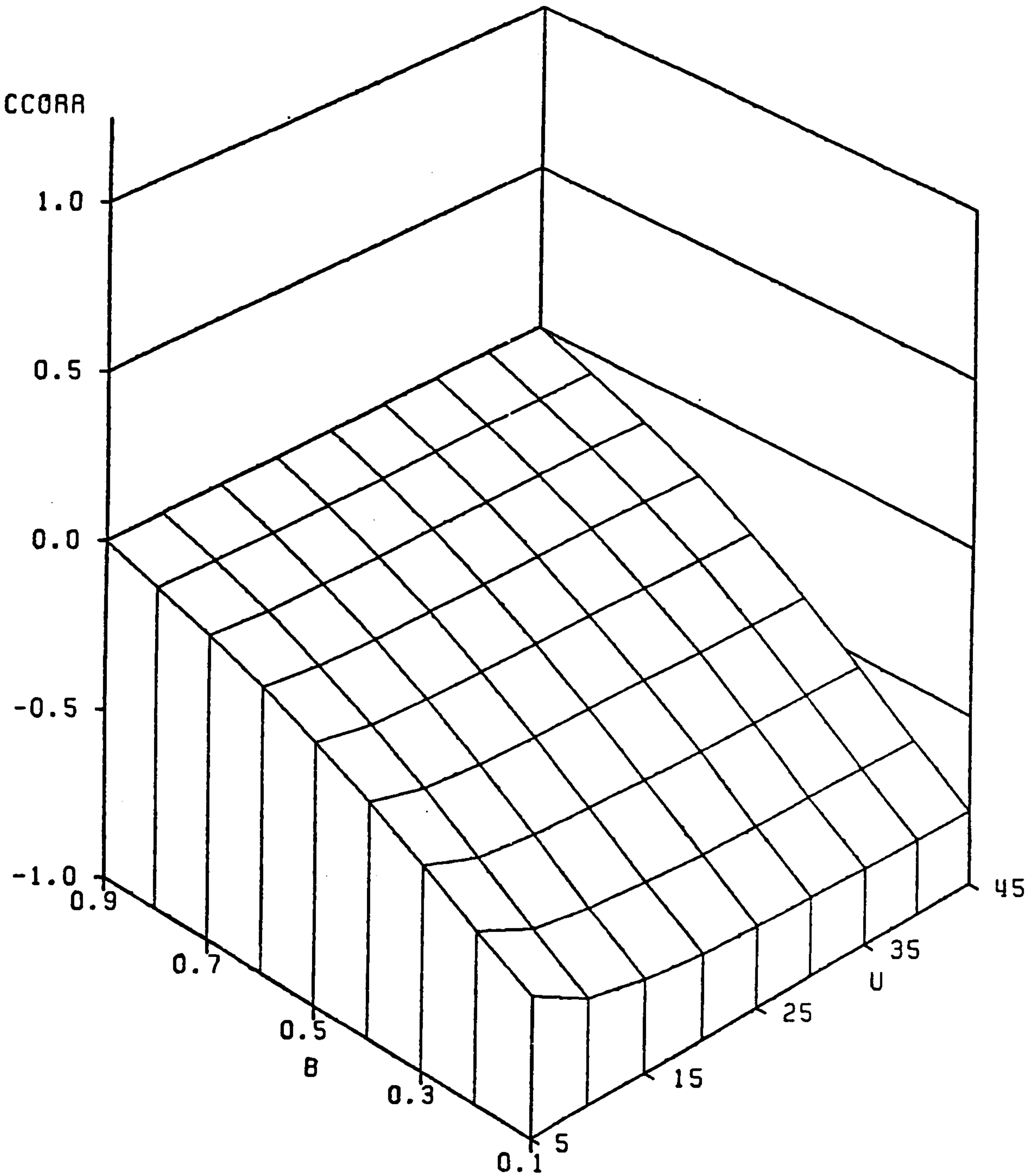

Fig.4.1 : a=0.1 (fixed), varying b and u

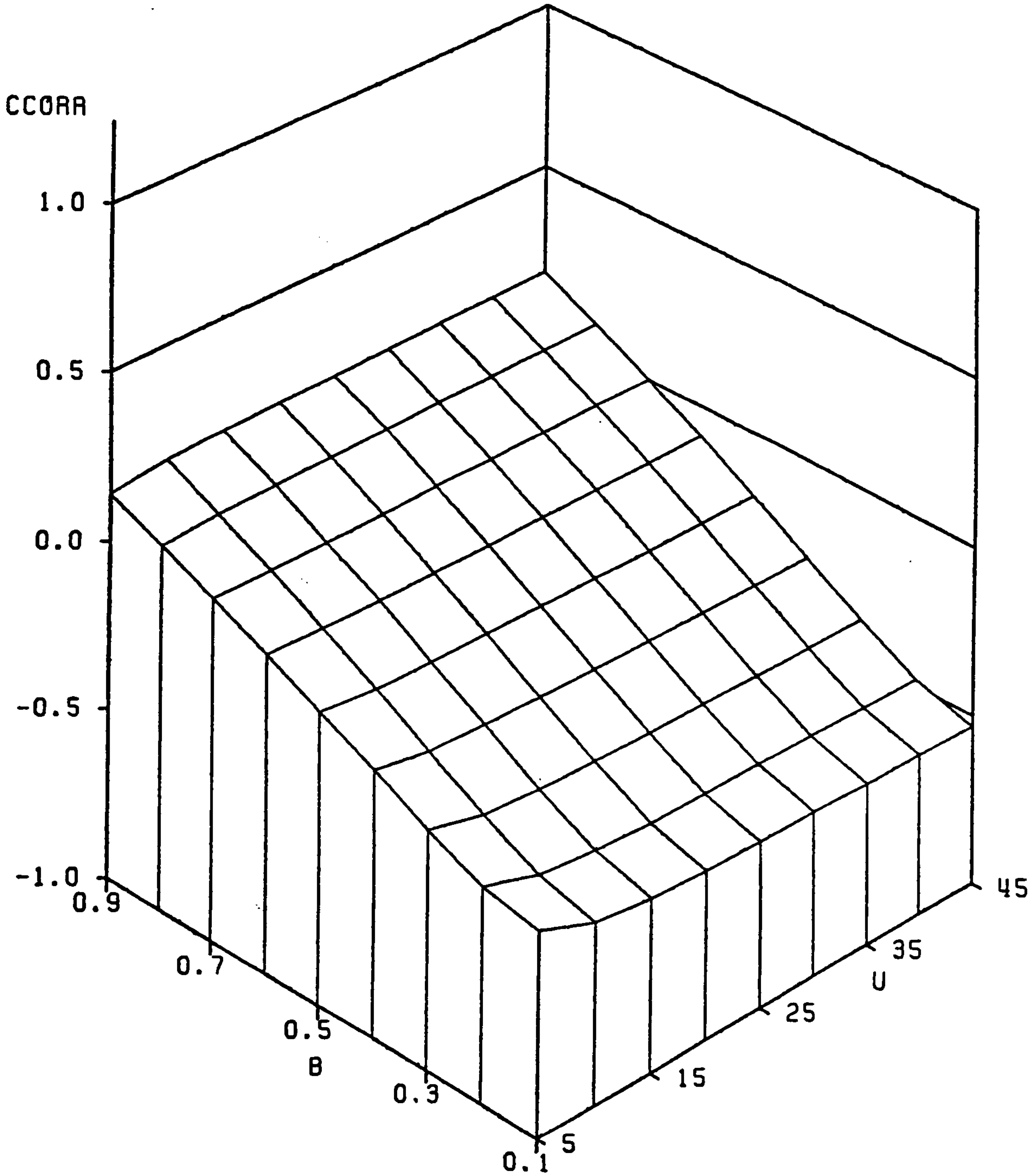

Fig. 4.2 : a=0.3 (fixed), varying b and u

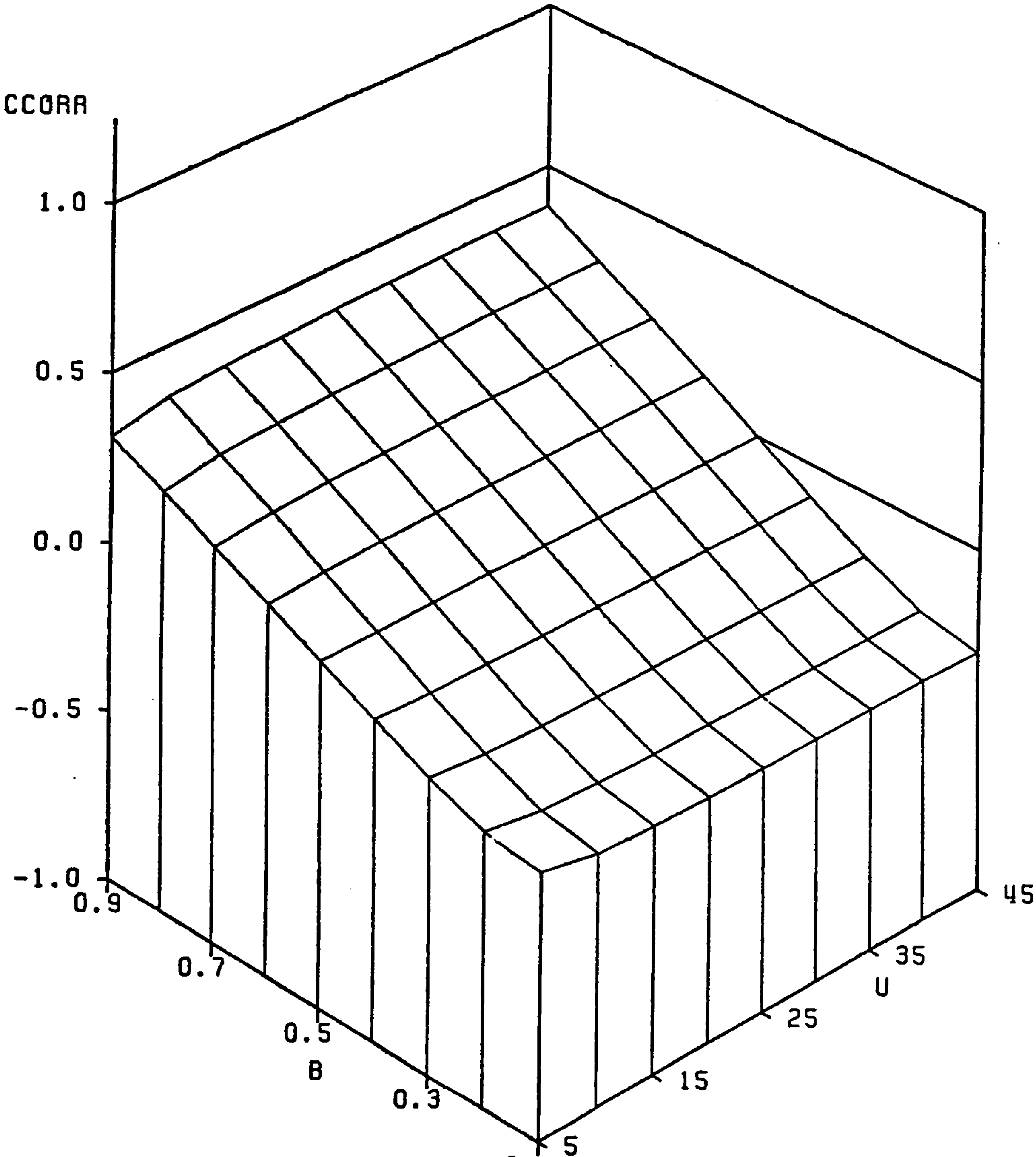

Fig. 4.3 : a=0.5 (fixed), varying b and u

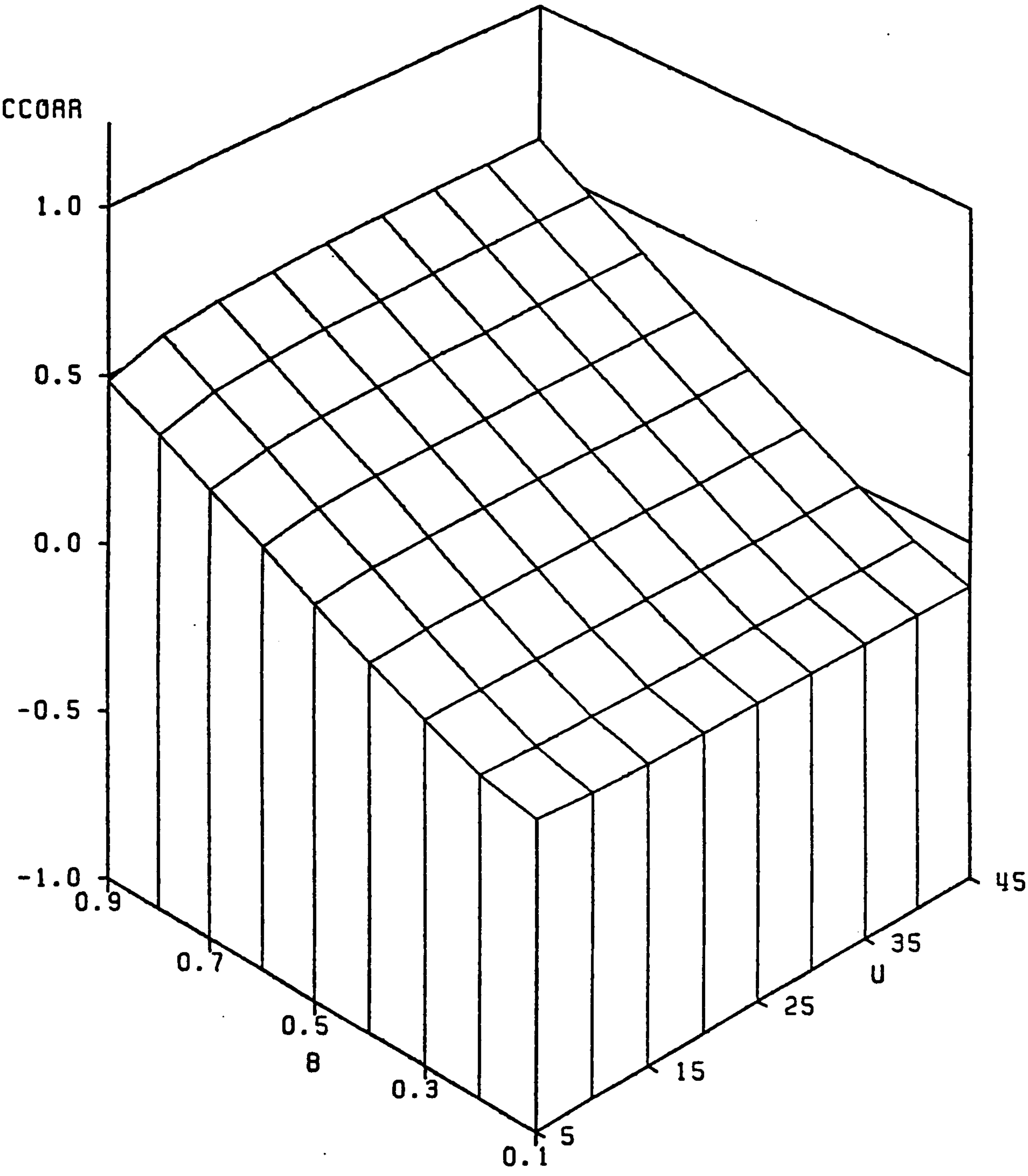

Fig. 4.4 : a=0.7 (fixed), varying b and u

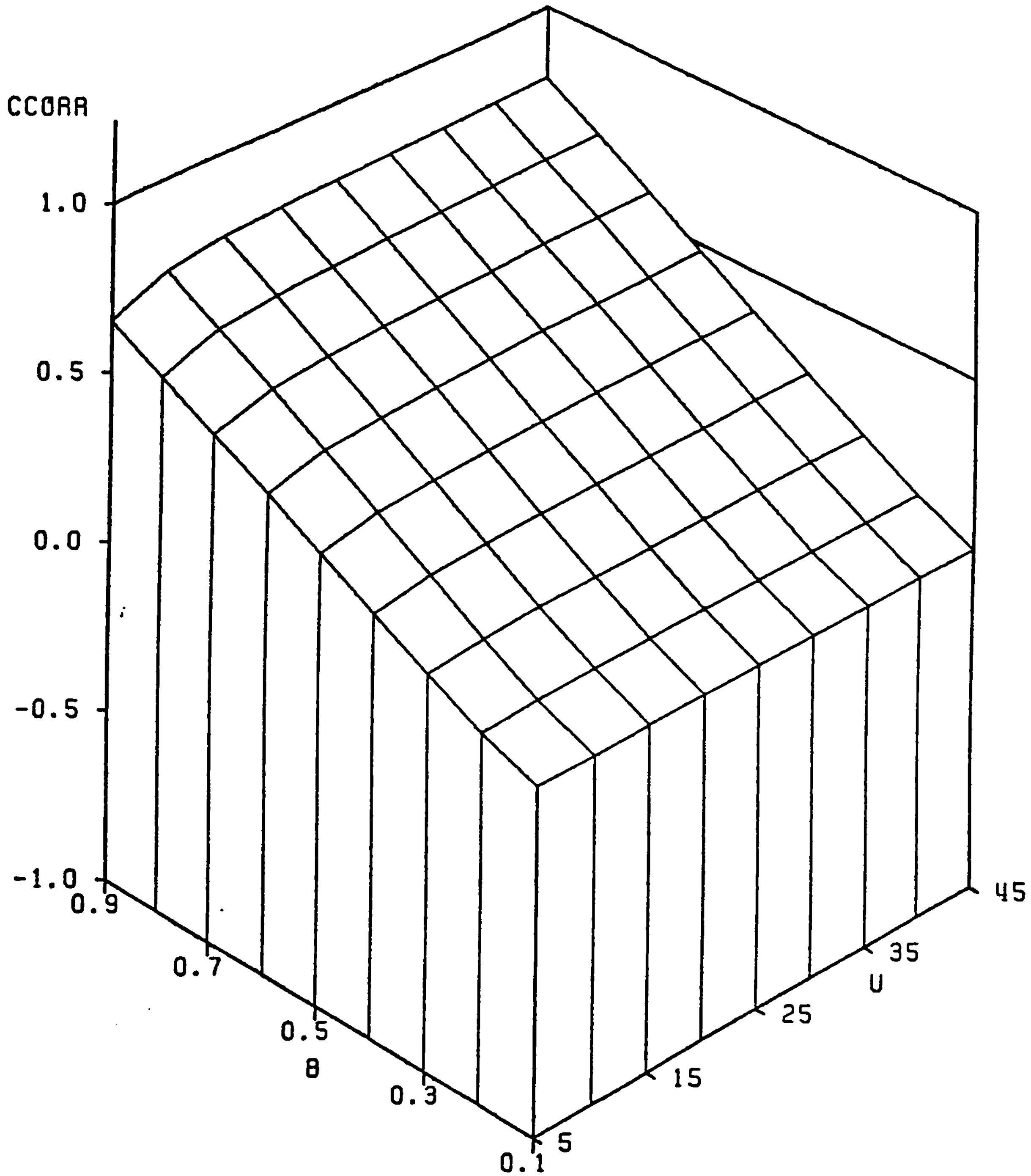

Fig. 4.5 : a=0.9 (fixed), varying b and u

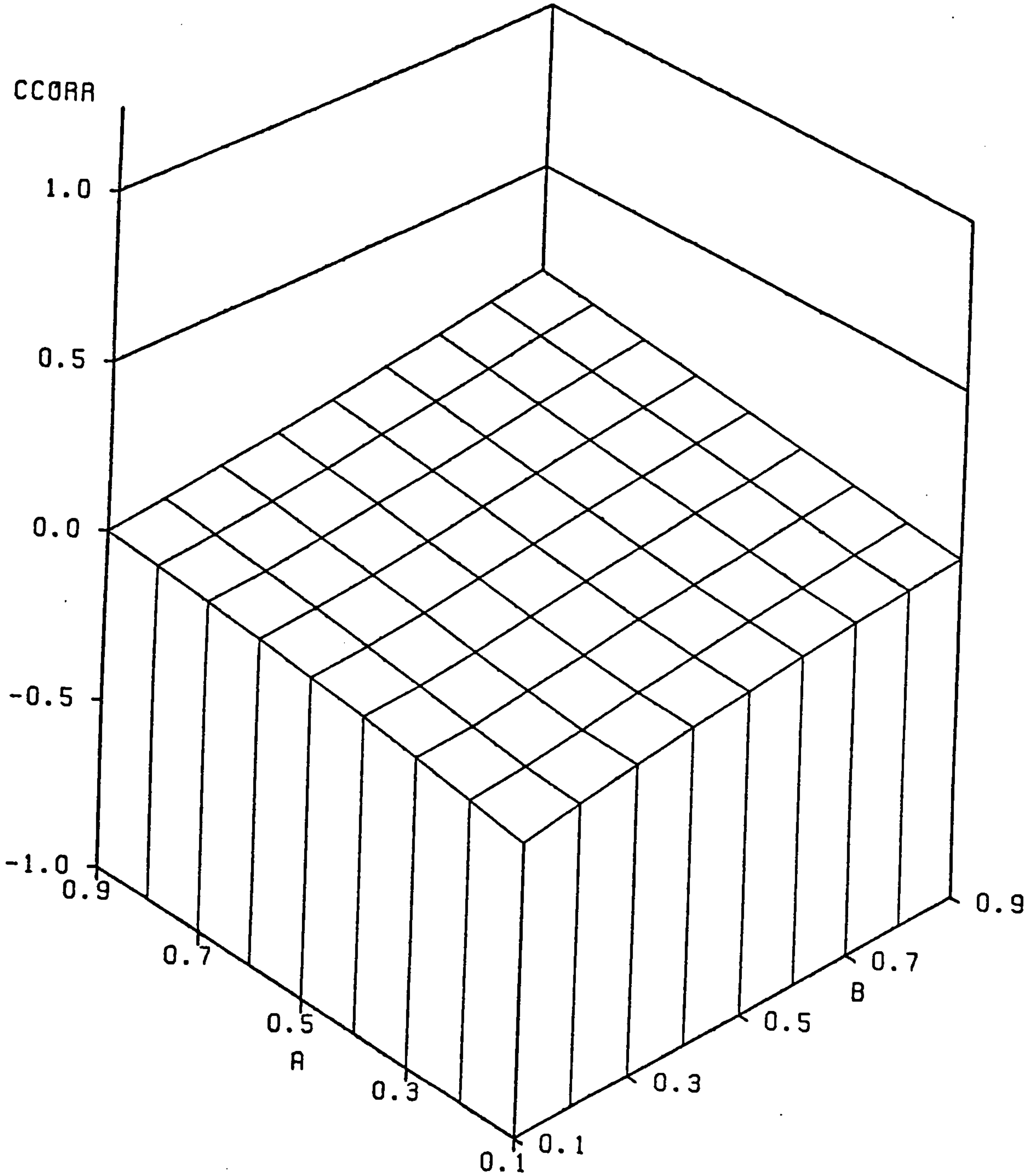

Fig. 4.6 : u=0.5 (fixed), varying a and b

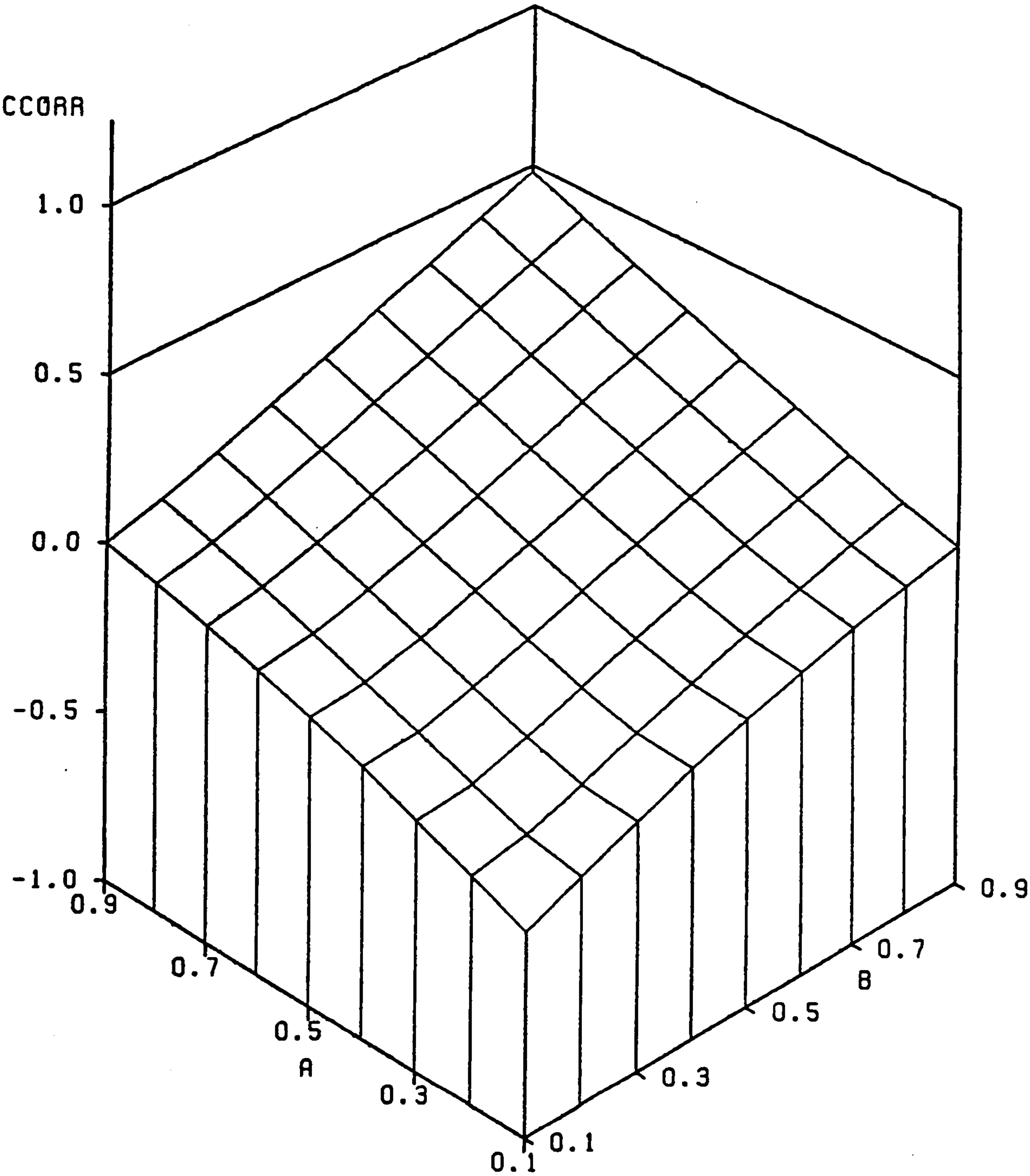

Fig. 4.7 : u=2 (fixed), varying a and b

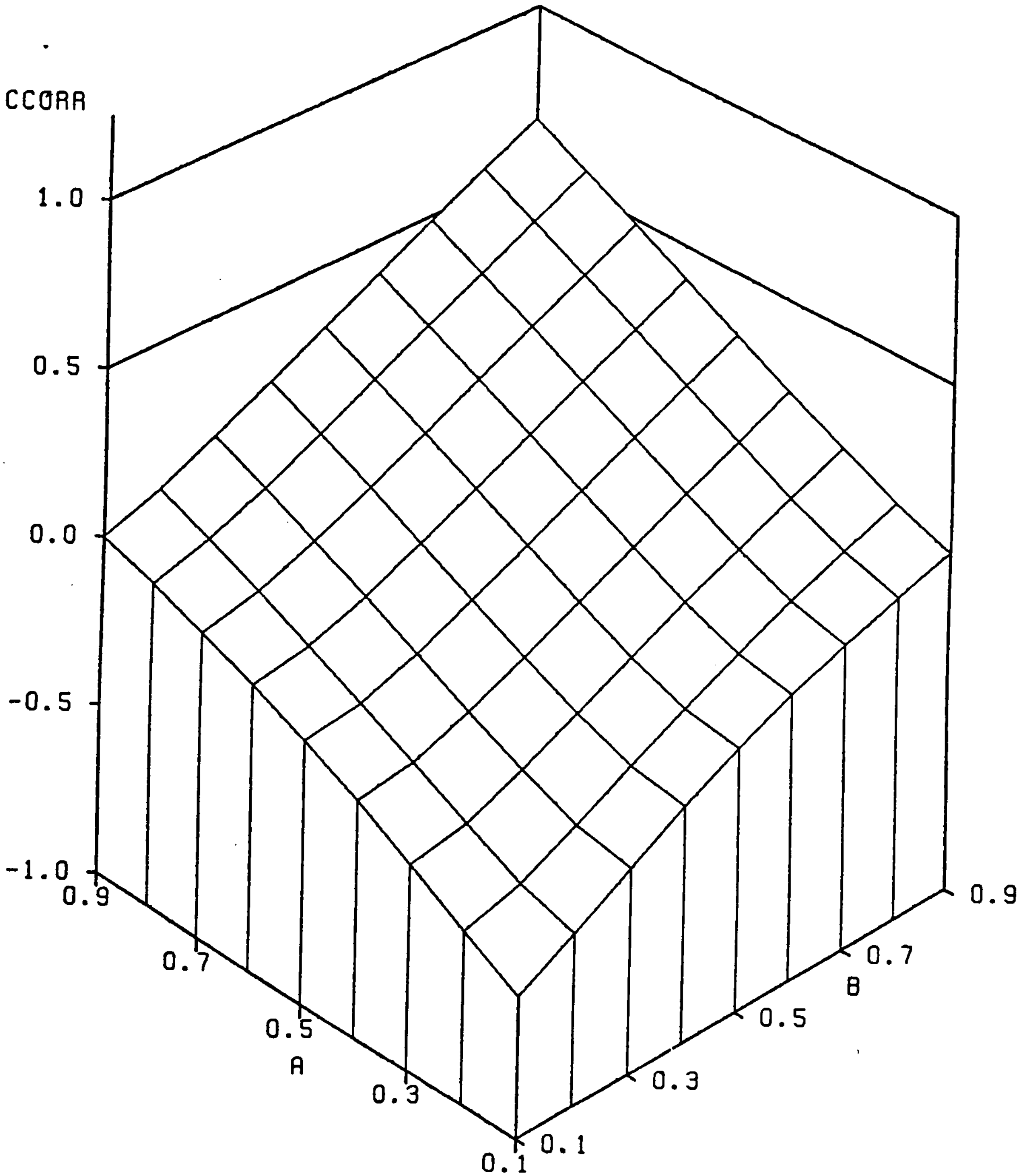

Fig. 4.8 : u=5 (fixed), varying a and b

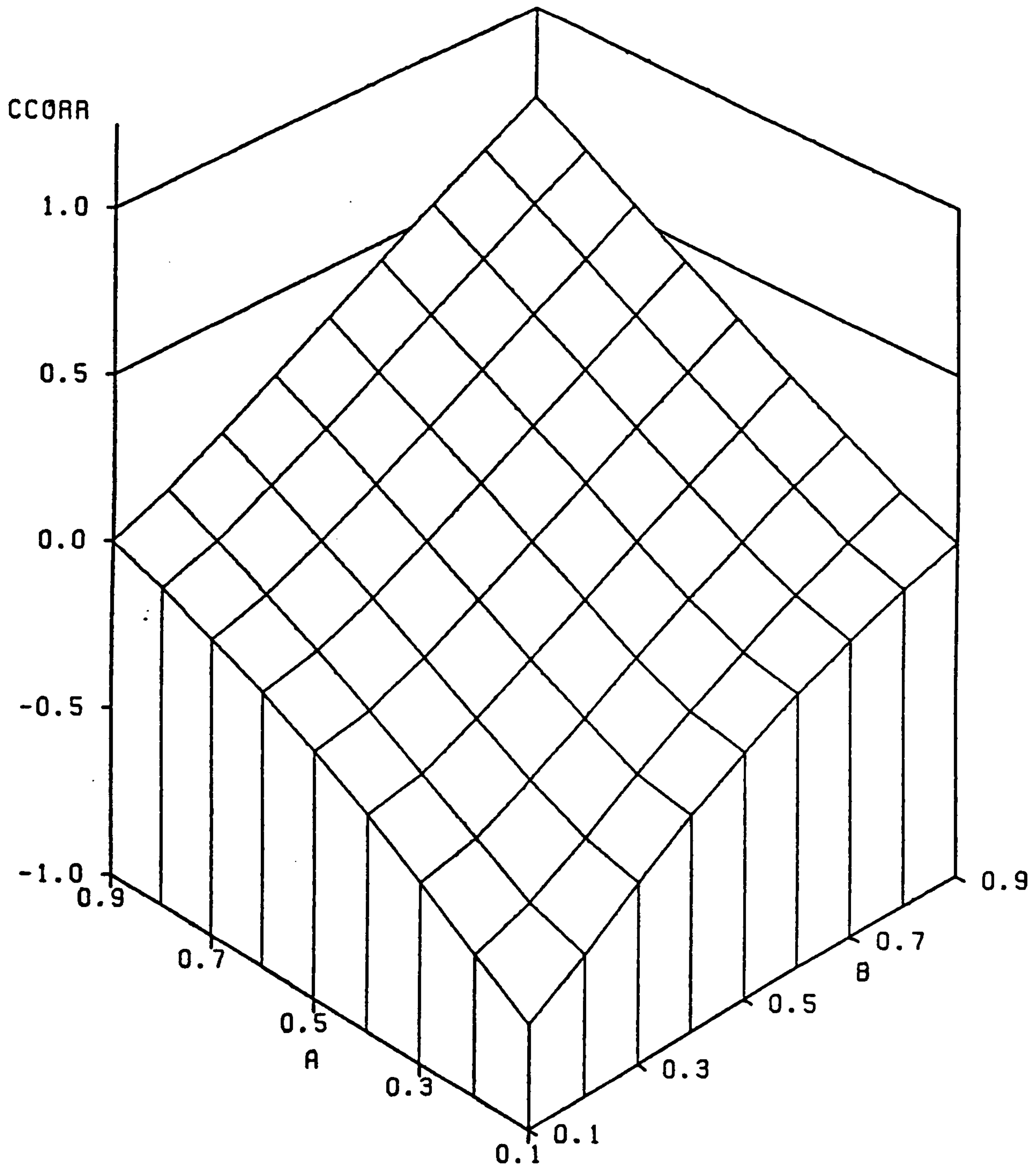

Fig. 4.9 : u=10 (fixed), varying a and b

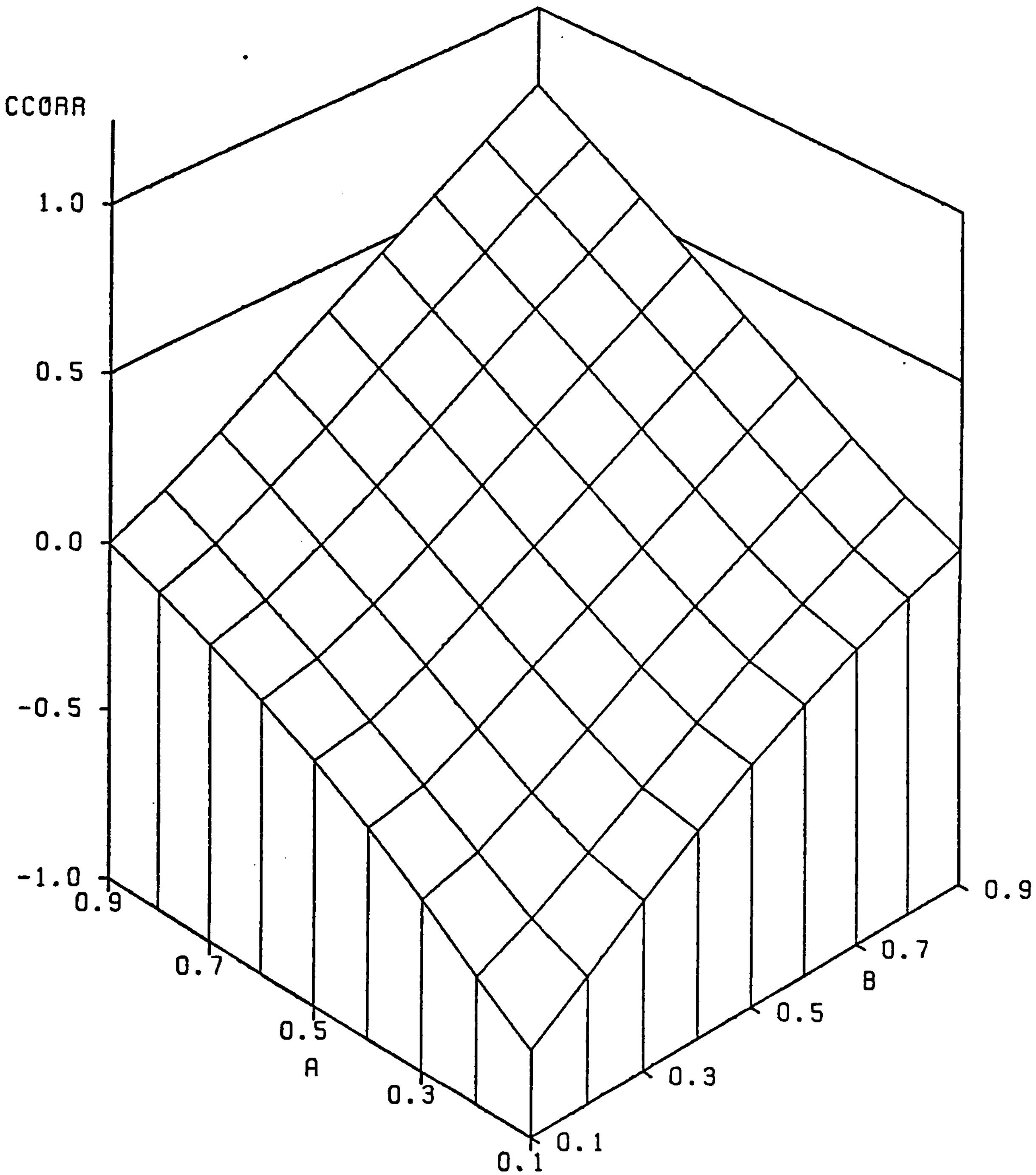

Fig. 4.10 : u=20 (fixed), varying a and b

5.5 General case

In this section, we return to the general Markov renewal service process and continue the study initiated in chapter 3. Recall that the semi Markov kernel for the service process is given by $B(x) = [p_{ij} F_{ij}(x)]$ where F_{ij} is a general distribution function and P is finite, irreducible and aperiodic. As we discussed before, for the general case we need all the structure of the process (N^d, Z^d, T^d) to characterize the departure process since there is no equivalence to a lower dimension process. As we did in section 4, we will use the crosscorrelation associated with each type-counting process to study the dependence among the successives departures for each type of customer.

The kernel for (N^d, Z^d, T^d) was given, in its general form, by (3.2.1). As we did in (2.20), we re-order the states such that we collect states with the same type of departure. We have

$$Q(t) = \begin{bmatrix} p_{11}Q_{11}(t) & p_{12}Q_{12}(t) & \cdots & \cdots & p_{1m}Q_{1m}(t) \\ p_{21}Q_{21}(t) & p_{22}Q_{22}(t) & \cdots & \cdots & p_{2m}Q_{2m}(t) \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ p_{m1}Q_{m1}(t) & p_{m2}Q_{m2}(t) & \cdots & \cdots & p_{mm}Q_{mm}(t) \end{bmatrix}, \qquad (5.1)$$

where $Q_{ij}(t)$ is an infinite dimensional matrix representing the kernel of the queue length process embedded at departure times for an M/GI/1 queue with arrival λ and service time distribution of F_{ij}. Again, by using two auxiliary functions f and g (the subscripts were suppressed), we represent $Q_{ij}(t)$ as

$$Q_{ij}(t) = \begin{bmatrix} g_0(t) & g_1(t) & g_2(t) & g_3(t) & \cdots \\ f_0(t) & f_1(t) & f_2(t) & f_3(t) & \cdots \\ 0 & f_0(t) & f_1(t) & f_2(t) & \cdots \\ 0 & 0 & f_0(t) & f_1(t) & \cdots \\ . & . & 0 & f_0(t) & \cdots \\ . & . & . & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix}, \tag{5.2}$$

where for $k \in N$

$$f_k(t) = \int_0^t \frac{e^{-\lambda x}(\lambda x)^k}{k!} \, p_{ij} \, dF_{ij}(x) \tag{5.3}$$

$$g_k(t) = \int_0^t \lambda e^{-\lambda x} f_k(t-x) \, p_{ij} \, dF_{ij}(x) \, . \tag{5.4}$$

The expected number of visits to a state in (0 , t], conditional on the initial state, is given by the Markov renewal matrix $R(t)$ whose value is

$$R(t) = \sum_{n=1}^{\infty} Q^{(*n)}(t) \quad . \tag{5.5}$$

Since $R(0) < \infty$, theorem (2.23) in Çinlar [1969] proves that $R(t)$ exists for every $t \in R_+$.

Partitioning the state space as we did in (2.1), we could write

$$R(t) = \begin{bmatrix} R_{11}(t) & R_{12}(t) & \dots & \dots & R_{1m}(t) \\ R_{21}(t) & R_{22}(t) & \dots & \dots & R_{2m}(t) \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ . & . & \dots & \dots & . \\ R_{m1}(t) & R_{m2}(t) & \dots & \dots & R_{mm}(t) \end{bmatrix}, \tag{5.6}$$

here $R_{ij}(t)$ is an infinite dimensional matrix whose entries are

$$R_{li,kj}(t) = \sum_{n=1}^{\infty} Q_{li,kj}^{(*n)}(t) \quad , \tag{5.7}$$

for $i, j \in E$ and $l, k = 0, 1, \dots$.

Definition (5.1) : For $\forall\ (k,j) \in N \times E$, define the counting processes:

$$M_{kj} = \{M_{kj}(t)\ ;\ t \in R_+\} \tag{5.8}$$

where $M_{kj}(t)$ is the number of departures of type j in (0, t] which left behind a queue length of k customers. Also define

$$M_{.j} = \{M_{.j}(t)\ ;\ t \in R_+\} \tag{5.9}$$

where $M_{.j}(t)$ is the total number of departures of type j in (0, t] . ■

It is clear from the above definition that

$$M_{.j}(t) = \sum_{k=0}^{\infty} M_{kj}(t) \quad , j \in E \ . \tag{5.10}$$

Furthermore, recall from chapter 3, that the stationary distribution of the embedded Markov chain (N^d, Z^d) is given by π whose elements π_{l_i} represent the stationary probability of l customers left behind by a departure of type i. From (5.7) the unconditional expected value of $M_{kj}(t)$ is

$$E[M_{kj}(t)] = \sum_{l=0}^{\infty}\sum_{i=1}^{m} \pi_{l_i} R_{li,kj}(t) \ . \tag{5.11}$$

Noting that for fixed $j \in E$ and $t \in R_+$, $\sum_{k=1}^{K} M_{kj}(t)$ is a non-decreasing function of K, then we could apply the monotone convergence theorem to interchange sum and expectation in order to obtain

$$E[M_{.j}(t)] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=1}^{m} \pi_{l_i} R_{li,kj}(t) \ . \tag{5.12}$$

Defining a m-vector $EM(t)$ whose elements are $E[M_{.j}(t)]$, we could write in matrix form

$$EM(t) = \pi \, R(t) \, e \ , \tag{5.13}$$

with $e^T = [e_\infty, e_\infty, \dots, e_\infty]$.

The next result is similar to theorem (4.1). It computes the lag r conditional expected value of the product of two counting variables in the (N^d, Z^d, T^d) process.

Theorem (5.2) : For $z_0, i, j \in E$, $n_0, l, k \in N$ and $t, r \in R_+$,

$$E[M_{li}(t)\, M_{kj}(t+r) \mid (N_0, Z_0) = (n_0, z_0)] =$$
$$\delta_{li,kj} R_{n_0 z_0,\, li}(t) + R_{n_0 z_0,\, kj} * R_{kj,li}(t) + \int_0^t R_{li,kj}(t+r-u)\, dR_{n_0 z_0,\, li}(u) \ . \tag{5.14}$$

proof :

The proof follows from theorem (4.1), extending the unidimensional state space to the bidimensional in our case. Despite the fact that we do not have a finite state space, there is no change in the proof since we worked with three arbitrary, but fixed states (n_0, z_0) , (l, i) and (k, j). ■

Now, the unconditional lag r expected value for the product of the two counting variables could be computed. The expression is

$$E[M_{li}(t)\, M_{kj}(t+r)] = \sum_{n_0=0}^{\infty} \sum_{z_0=1}^{m} \pi_{n_0 z_0}\, E[M_{li}(t)\, M_{kj}(t+r) \,|\, (N_0, Z_0) = (n_0, z_0)]\,. \quad (5.15)$$

To study the dependency between the type-counting processes, we need the expression of the lag r expected value for the product of two type-counting variables. We start by using (5.10) to obtain

$$\begin{aligned}
&E[M_{.i}(t)\, M_{.j}(t+r) \,|\, (N_0, Z_0) = (n_0, z_0)] = \\
&\quad = E\Big[\Big(\sum_{l=0}^{\infty} M_{li}(t)\Big)\Big(\sum_{k=0}^{\infty} M_{kj}(t+r)\Big) \,|\, (N_0, Z_0) = (n_0, z_0)\Big] \\
&\quad = E\Big[\Big(\lim_{L\to\infty} \sum_{l=1}^{L} M_{li}(t)\Big)\Big(\lim_{K\to\infty} \sum_{k=1}^{K} M_{kj}(t+r)\Big) \,|\, (N_0, Z_0) = (n_0, z_0)\Big] \\
&\quad = \lim_{L\to\infty} \lim_{K\to\infty} E\Big[\Big(\sum_{l=1}^{L} M_{li}(t)\Big)\Big(\sum_{k=1}^{K} M_{kj}(t+r)\Big) \,|\, (N_0, Z_0) = (n_0, z_0)\Big]\,,
\end{aligned}$$

where we used the monotone convergent theorem to interchange limit and expectation. Finally, noting that the conditional expectation is a linear operator, we obtain

$$E[M_{.i}(t)\, M_{.j}(t+r) \mid (N_0, Z_0) = (n_0, z_0)] =$$
$$= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} E[M_{li}(t)\, M_{kj}(t+r) \mid (N_0, Z_0) = (n_0, z_0)] \quad . \tag{5.16}$$

Multiplying by the stationary distribution, we could remove the conditional in the above expectation to obtain:

$$E[M_{.i}(t)\, M_{.j}(t+r)] = \sum_{n_0=0}^{\infty} \sum_{z_0=1}^{m} \pi_{n_0 z_0}\, E[M_{.i}(t)\, M_{.j}(t+r) \mid (N_0, Z_0) = (n_0, z_0)] \,. \tag{5.17}$$

For $i = j$ and $r = 0$ we obtain the expression for $E[M_{.i}^2(t)]$ which is necessary to compute:

$$Var[M_{.i}(t)] = E[M_{.i}^2(t)] - E^2[M_{.i}(t)] \quad . \tag{5.18}$$

Analogously as we did in section 3 (see (3.8) and (3.9)), we define, for all $t, r \in R_+$, the crosscovariance and crosscorrelation of lag r of the processes $M_{.i}$ and $M_{.j}$, $i,j \in E$, by the expressions:

$$ccov[M_{.i}(t)\, , M_{.j}(t+r)] = E[M_{.i}(t)\, M_{.j}(t+r)] - E[M_{.i}(t)]\, E[M_{.j}(t+r)] \tag{5.19}$$

and $$ccorr[M_{.i}(t)\, , M_{.j}(t+r)] = \frac{ccov[M_{.i}(t)\, , M_{.j}(t+r)]}{\sqrt{Var[M_{.i}(t)]\; Var[M_{.j}(t+r)]}} \quad . \tag{5.20}$$

Notice that (5.19) and (5.20) are computed by using the previous expressions in this section.

5.6 Summary

In this chapter we studied the type-departure processes for the $M/MR/1$ queues. We started with the special case where the service is exponential with parameter μ_{ij}. This queue was studied in chapter 4 and named $M/M^{ij}/1$. For these queues, we obtained conditions under which the infinite queue length process is equivalent to its finite dimensional one step projection. The equivalence was used to study the dependence among the type-departure processes and to compute the crosscovariance and crosscorrelation between the associated counting processes. A numerical example illustrated this computation.

For each single type-departure process, we obtained, through filtering techniques, the structure and properties for the queue length process. In a special case, detailed expressions were computed.

The last section was devoted to a general case. We computed the expressions to measure the dependency between type-departure processes in $M/MR/1$ queues. Again, we used the crosscovariance and crosscorrelation between the associated counting processes to explain the interdependence among the type-departure processes.

Chapter 6

CONCLUSIONS AND EXTENSIONS

6.1 Summary

In this paper, we have studied the departure process for $M/MR/1$ queues with FCFS discipline. These queues can be used to model systems where the service needs some time to adjust between different tasks, or in other words, systems with changeover times. In our study we considered that the arrival is a Poisson process and immediately before the service starts, a type is assigned to the customer (or job) that will initiate the service. A m-dimensional Markov chain with transition matrix $\boldsymbol{P}$ is used in this assignment. The service time has distribution F_{ij} where i and j are the types of previous and current customers in service. A Markov renewal service process was constructed, including the change in types and the dependency in the service time distribution. In chapter 2 we discussed its characteristics and in particular, we were interested in the dependency between consecutive service times caused by this structure (see section 3 for a numerical example).

We discussed the general structure of the $M/MR/1$ queues in chapter 3. We denoted the process observed just after a departure instant, T_n^d, by $(N^d, Z^d, T^d) = \{(N_n^d, Z_n^d, T_n^d) ; n = 0,1, \dots \}$, in which the information about the queue length and type of departure is represented by N_n^d and Z_n^d, respectively. We proved that this process is a Markov renewal process and we used its structure to study the departure process T^d. The equilibrium equations for the embedded Markov chain, (N^d, Z^d), were computed and its stationary distribution was related with the stationary distribution for the matrix P. We also obtained simpler expressions for the single and joint intervals in the departure process (see (3.3.10) and (3.3.14)). It is important to emphasize that these expressions are in the time-domain (not in transform- space) and they only include finite $m \times m$ matrices, in contrast to the usual definition which requires operations with infinite matrices.

In chapter 4, we examined the special case where $F_{ij}(t) = 1 - e^{-\mu_{ij} t}$. We found that $\mu_{ij} = \mu$ for $i,j = 1,2, \dots, m$ was a necessary and sufficient condition for the (N^d, Z^d, T^d) process to be equivalent to a renewal process (theorem (4.3.2)). We also proved that this renewal process is, in fact, a Poisson process.

To study the departure process in more detail, we looked at consecutive departures for a fixed type of customer. These processes were called type-departure processes, as presented in chapter 5. In the case of exponential service time distribution (μ_{ij}), we computed the marginal queue length in the joint queue length-type Markov renewal process, (N^d, Z^d, T^d), to obtain its one-step projection, (Z^*, T^*), a finite Markov renewal process (see definition (5.2.2)). We proved in theorem (5.2.11) that these two Markov renewal processes are equivalent if and only if $\mu_{ij} = \mu$ for $i,j = 1,2, \ldots, m$. Under this condition, we studied each type-departure process (in isolation) and we proved that they are renewal processes. Furthermore, except in special cases, these renewal processes are never Poisson. Since $\mu_{ij} = \mu$ for $i,j = 1,2, \ldots, m$. is also the condition for Poisson departure, we obtained a situation where a superposition of renewal processes gives a Poisson process. From a known result in point process theory (e.g. Lewis [1972]), these renewal processes must be dependent on each other; our interest was to investigate the relationship among them. For this purpose, we defined in (5.4.4) the counting process associated with each type-departure process and computed the lag r crosscorrelation between any two of these processes. In the special case m = 2, detailed results were given and a numerical example with several graphs illustrated the behavior of the crosscorrelation as a function of the parameters of the model.

To complete the analysis, in section 5.5, we investigated the type-departure processes for a general Markov renewal service process. In this case, the service time distribution was a general function F_{ij}. Expressions for the analysis of the type-departure process were computed, including the lag r crosscorrelation between any two type-counting processes.

One should note that the problem here could be structured as a queueing system with vacations as is done in Eisenberg [1969]. However, such a structuring necessarily loses the dependence between the several queueing systems, a loss that is irretrievable and unfortunate. It is the dependence which might be the essential element in these systems, especially if one is to add a controller. Hence, our approach to the problem seems to be more reasonable and thorough than all of the related existing models of queues with vacations.

6.2 Discussion and Extensions

Our purpose in this paper was to study analytical properties of the departure process and in order to keep the probabilistic structure as clear as possible, we avoided the traditional use of transforms. The advantage of this approach is that we work with time domain expressions whose interpretation is, in general, easier than the respective transforms.

The dependence on the service process causes the interdeparture times to be correlated. In this way, if a queue, such as the one studied in this paper, is used to feed customers into other queues (or a network in general), we could not consider the arrival process as GI. Even in the case of exponentially identical rates which give an overall Poisson departure, the dependence appears if the different types are assigned to different queues. As we discussed in chapter 5 (section 3), the renewal processes which constitute the arrival process to these queues will be interdependent on each other. Note that we would not have a Poisson process being thinned by a Bernoulli-type switch but by a Markov chain. In other words, the decomposition does not produce independent Poisson processes.

In the course of this research, several topics for future investigation have arisen and we will briefly discuss them. In chapter 5, to study the type-departure processes, we created the one step projection process of the (N^d, Z^d, T^d) for the $M/M^{ij}/1$ queue. In order to prove theorem (5.2.1), we had to impose $0 < a_{ij} \leq 1$. Since a_{ij} is obviously larger than zero (see (5.2.4)), the only part to verify would be $a_{ij} \leq 1$, which is equivalent to $\pi_{0_i} \leq \eta_i(1 - \frac{\lambda}{\mu_{ij}})$, $\forall\ i,j \in E$. However, except in the special case $\mu_{ij} = \mu$, $\forall\ i,j \in E$, we were not able to verify the above inequality. Furthermore, another open question related to the above topic is to know whether it is possible to have exponential interdeparture times without independence of the consecutive intervals. Note that exponential interdepartures are obtained when $a_{ij} = 1$, or equivalently $\pi_{0_i} = \eta_i(1 - \frac{\lambda}{\mu_i})$, $i \in E$ (see expression (3.3.10)). The difficulty appears when we try to verify, the equilibrium equations (3.2.8) for this value of π_{0_i}. We would like to address this question in future research.

When we created the one step projection process, we intended to substitute an infinite dimensional by a finite dimensional Markov renewal process. For the general Markov renewal (MR) service, we would like to discuss how good the substitution is. Recall that by definition the one-step projection and the original processes have the same interdeparture time distribution. A criterion based on the joint distribution of consecutive intervals could be used to define the approximation. Furthermore, a simulation study would compare the performance of the approximation. In a theoretical sense, the discussion of substitution of one process for another must include a norm (e.g., Levi distance) but it seems to be a very difficult problem.

Disney et al. [1973] proved that, among all $M/GI/1$ queues with FCFS discipline, the $M/M/1$ is the only queue with a Poisson departure process. It seems possible to extend this result to the MR-service class (which include GI as a special case). Theorem (4.3.2) was one step in this direction, but it was restricted to the exponential (μ_{ij}) MR-service. The technique used in its proof must be modified, since the scalar form of the interdeparture distribution is not available in the general case. However, it may be possible to apply some of the ideas developed in section 4.3. For instance, imposing the independence between consecutive intervals had a key role in the proof of theorem (4.3.2). Perphaps, we could repeat this approach and obtain a restriction on the behavior of F_{ij}, $i,j = 1,2, \ldots, m$, and from this the desired generalization.

As an extension of this work, the study of sojourn times for the $M/MR/1$ queues could be accomplished. Different names have been used to define the variables describing waiting times of a customer in the system. For our purposes, the sojourn time is the total time spent in the system, that is, the waiting plus the service time. The study of waiting and sojourn times has been one of the most commonly pursued topic in queueing theory literature. Indeed, for some applications, this is the most important characteristic to know. For Markov renewal service queues, Neuts [1966 and 1976] obtained the Laplace-Stieltjes transform of the waiting time distribution. However, as far as we know, the sojourn times of this queue have never been studied. In the special case of Markov renewal exponential service, we would expect some simplifications and, perhaps, we could obtain time domain expressions for the sojourn time distribution.

In some practical applications, the capacity of the queue is finite. For instance, no more than L customers could wait for service and some of them might leave the queue without receiving any service. The lost customers form the overflow process, and the characteristics of this process are not known in the case of queues with a Markov renewal service process.

The influences of more servers and/or different disciplines in the queues with a Markov renewal service are two other new points of interest. The results could be especially useful to evaluate the performance of production systems.

Another extension would be the introduction of feedback in the $M/MR/1$ queue. In this case, after the completion of service, the customer decides to feedback or not, according to a probability law which could depend on its type. In manufacturing systems, the feedback of a customer is very common and usually represents extra work to correct mistakes or to improve the quality. There are several options in the study of this feedback queue. For instance, the customer could change its type after feedback, and it could return to the beginning or to the end of the queue. Also, the mechanism of the decision could be made to be dependent on the type of the previous departure or feedback. This seems to be a promising topic for future research.

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